Remembering M. S. Narasimhan (1932–2021)

Arnaud Beauville, Oscar García-Prada, Nigel Hitchin, Chandrashekhar Khare, Shrawan Kumar (Editor), Herbert Lange, Nitin Nitsure, M. S. Raghunathan, T. R. Ramadas, and S. Ramanan

Narasimhan was born on June 7, 1932 in Tandarai, a small village in Tamil Nadu with hardly any infrastructure. After his early education in a rural part of the country, he joined Loyola College in Madras for his undergraduate education. Narasimhan joined the Tata Institute of Fundamental Research (TIFR), Bombay, for his graduate studies in 1953. He obtained his PhD in 1960 (granted by Bombay University) under the supervision of K. Chandrasekharan (a well-known number theorist).

His initial area of focus at TIFR was partial differential operators. During this time, he visited France under the invitation of Laurent Schwartz and was exposed to the works of other French mathematicians including Jean-Pierre Serre, Claude Chevalley, Élie Cartan, and Jean Leray. Following some fundamental works of D. Mumford, he started to study the moduli of vector bundles on curves (with Seshadri and later with G. Harder and S. Ramanan) where he made pioneering contributions. His most important ground-breaking work jointly with C. S. Seshadri is known as the *Narasimhan-Seshadri theorem* giving a topological characterization of stable vector bundles on smooth projective curves, thus building a bridge between topology and algebraic geometry. With Harder, he introduced what is known as the *Harder-Narasimhan filtration of vector bundles*, which is of fundamental importance in a variety of topics. Some of his later works are explained in the following pages.

On the administrative side, he became the first chair of the National Board for Higher Mathematics and introduced many initiatives. In the early 1990s, after retiring from TIFR, he decided to join the International Centre for Theoretical Physics in Trieste and worked for over ten years as the head of its mathematical division.

He was an inspiring advisor and guided a number of graduate students including: K. Gowrisankaran, M. S. Raghunathan, S. Ramanan, M. K. V. Murthy, V. K. Patodi, G. A. Swarup, R. R. Simha, R. Parthasarathy, S. Kumaresan, T. R. Ramadas, N. Nitsure, S. Subramanian, and F. Coiai; many of whom went on to become outstanding mathematicians in their own right.

He won several awards in India as well as internationally. Among others, he was recipient of the Shanti Swarup Bhatnagar Award (1975); Third World Academy of Sciences Prize for Mathematics (1987); Srinivasa Ramanujan Medal (1988); Chevalier de Ordre National du Mérite of France (1989); and King Faisal International Prize for Science (2006) (jointly with S. Donaldson). He was a Fellow of all three of the Indian Academies of Science. He was elected Fellow of the Royal Society in 1989. He was awarded Padma Bhushan in 1990 (India's third highest civilian honor).

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Figure 1. Family photo: with son Mohan, daughter Shobhana, and wife Sakuntala, January 2020.

Apart from mathematics, he liked Modern Art (particularly Impressionism), Carnatic music, and modern Tamil literature.

He continued to be mathematically active till literally the last day of his life. He wrote a paper with Gallego and García-Prada which appeared on the ArXiv on May 13, 2021. Battling with cancer, he passed away on May 15, 2021.

S. Ramanan

Recently M. S. Narasimhan, one of the finest mathematicians in India, passed away. Coming on the heels of the sad demise of another stalwart C. S. Seshadri, it has been a big loss. I had the good fortune to have had close friendship with both, academically as well as socially. Narasimhan was my PhD advisor, colleague, and collaborator for over three decades.

S. Ramanan is a retired professor of mathematics at the Tata Institute of Fundamental Research, Mumbai, and currently an adjunct professor at the Chennai Mathematical Institute, Chennai. His email address is pudavai@gmail.com.

Narasimhan was born in a small village with hardly any infrastructure. There was no high school there and not even any bus service to the nearest town where he went to study. Narasimhan often recalled how he had to travel to school by bullock cart. Although his family was reasonably well to do, his father passed away early and they had to manage with some difficulty.

His performance at the final examination at school was excellent, and he got into Loyola College, one of the highly rated colleges in Madras at the time. He was very happy with the standard of teaching he received there. The head of the mathematics department was a Jesuit, Father Racine. Unlike most professors of those days, Fr. Racine was au courant with contemporary mathematics. Although he was not rated a great lecturer, he took keen interest in students of high calibre and encouraged them to go into research. There was another lecturer, Professor Krishnamurthy who taught Real Analysis, and Narasimhan was appreciative of his efforts to inculcate a deep interest in the subject.

Fr. Racine was in touch with the development of the incipient Tata Institute of Fundamental Research (TIFR) in Bombay, and he recommended bright students to do their research at the fledgling institution. Accordingly, some students like Seshadri and Narasimhan, became graduate students at TIFR. The school of mathematics there had as its head, Professor K. Chandrasekharan (KC), who realized the best way to introduce modern mathematics to the Indian scene. KC invited top mathematicians from all over the world, and asked each to give introductory lectures on a current subject. One of the graduate students was assigned the task of writing up the notes. The Fields medalist Laurent Schwartz, known for his theory of distributions, was one of the invitees. The note taker of his lectures was Narasimhan. This proved to be fortuitous, for Schwartz was impressed with the keenness and ability of Narasimhan, as well as a few other graduate students. When Schwartz returned to France, he persuaded some of his students, like Jacques Lions, Bernard Malgrange, etc. to visit TIFR and give courses. On his visit, Lions posed a question connected with the limits of partial differential operators on manifolds, and Narasimhan solved the question in the affirmative.

KC soon realized the high quality of the graduate students and felt that there was no one to mentor them in India. Consequently, with the help of Schwartz, he delegated Narasimhan, among a couple of others, to visit France for a few years and work on such topics. In Paris, Narasimhan came in contact with a Japanese graduate student, Keisuke Kotake, and proved a nice result on elliptic operators, in collaboration with him.

Unfortunately, he contracted pleurisy while in Paris and had to be hospitalized. He looked upon it not as a disaster, but as an opportunity to be with "real Parisians." He also told me that his spoken French improved manyfold as a consequence.

On his return to India, he obtained his doctorate and soon was appointed Associate Professor. It is remarkable that he could start advising students for their doctorate so early in his career.

I was then a graduate student at TIFR and he warmed up to me when he came to know that I was conversant with the work of Kodaira and Spencer. I became his first student and we soon wrote up a paper, "Universal Connections," where we proved that the classifying space for principal bundles with compact structure group also had a connection which was universal for bundles with connections. This was very well received, and later became useful for many developments in differential geometry as well as in theoretical physics.

Seshadri and he jointly wrote a few papers in which the theory of vector bundles over a smooth projective algebraic curve was the main thrust and soon it culminated in what is now known as "the Narasimhan-Seshadri theorem." This provided a deep understanding of the classification of vector bundles over curves and led to many later developments. Narasimhan occupied himself with the study of the moduli space so constructed, and it was my good fortune to be able to work jointly with him in this enterprise.

A distinguishing feature of Narasimhan's research was his ability to come to grips with questions, even in an area in which he had no previous expertise, and bring new ideas to the questions, and then solve them. He would fill in the details later, often running a seminar on the topic. This explains the versatility of his research, and his propensity to collaborate with different types of mathematicians, from young graduate students to famous achievers. Thus, apart from his work with Indian colleagues like Ramadas, Seshadri, Simha, and myself, he worked jointly with Kotake, Harder, Beauville, Hirschowitz, Lange, Okamoto, and a host of others. Much of this research spawned new directions, which are still of value decades later. But even more remarkable is the fact that these collaborations span a wide range of fields including Number theory, Differential Equations, Differential Geometry, Lie groups, Algebraic Geometry, and even Theoretical Physics.

Mathematics was not his only interest. He liked Modern Art, particularly Impressionism, thanks to his French connection. He was also fond of books. We used to frequent the Strand Book Stall and buy books. He liked to read contemporary Tamil books. Even after he retired and settled in Bengaluru, he tried to time his visits to Chennai in order to buy books at the Annual Tamil Book Fair.

On the administrative side, when I was the dean of the school of mathematics at TIFR, there were efforts elsewhere to institute a mathematical establishment along the lines of the Council of Scientific and Industrial Research. Since 'Higher Mathematics' came under the Department of Atomic Energy, with the active help of the then Chair of the Atomic Energy, the National Board for Mathematics (NBHM) was established. Narasimhan became the first Chair of NBHM and it was my pleasure to be its first secretary and collaborate with him in this endeavor as well. He introduced many initiatives and NBHM has now become the principal funding agency for mathematical research in India. In the 90s, after decades of research and mentoring, he decided to join the International Centre for Theoretical Physics in Trieste and worked for over ten years as the head of its mathematical division. There he mentored students which, he mentioned to me, gave him great satisfaction as he always wished to make some contribution to science in the developing world.

He returned to India in the late 1990s. Although he had retired, he was full of ideas pertaining to moduli space and its generalizations. Apart from mathematics, he was interested in Carnatic (South Indian classical) music, as well as modern Tamil literature. However, his deep interest in mathematics never waned. A few months after he contracted the cursed disease, he wrote to me that our work had new ramifications and that they are now talking about 'Narasimhan-Ramanan branes'! Very close to his final days, he even helped a student of our Spanish friend Oscar García-Prada with some ideas.

Fortunately, his stature did not go unrecognized, in India as well as internationally. He was awarded Padma Vibhushan, the Bhatnagar prize and was a Fellow of all three Institutes of Science. He was elected Fellow of the Royal Society, was honored with Chevalier d'Ordre du Merite of France, the Abdul Kalam prize, and so on.

I am fortunate to have been his close friend till the very end.

Nigel Hitchin

My first encounter with Narasimhan was at a conference in July 1974 in Durham, England. The bus taking us on an excursion had broken down outside a pub (not serving beer because of restricted licensing hours in those days) so we sat at tables outside. I had given a talk about vanishing theorems in differential geometry which are often called Weitzenböck formulae and he asked me if I had read Weitzenböck's 1923 book *Invariantentheorie*. I had

Nigel Hitchin is an emeritus Savilian Professor of Geometry at the University of Oxford. His email address is hitchin@maths.ox.ac.uk.

skimmed through it in the library but he told me to look at the Foreword and take the first letter from each sentence. It spells

N-I-E-D-E-R-M-I-T-D-E-N-F-R-A-N-Z-O-S-E-N.

"Down with the French!" Not that he shared that view (Weitzenböck was living in the French-occupied Rhineland at the time) because he followed the comment with interesting accounts of his experiences in the Paris of Serre, Cartan, and Schwartz which was clearly a formative experience for him. His talk at the symposium was about moduli of vector bundles on curves. At that time I took little notice, but somewhat later our interests converged much more.

In the intervening years it was more a question of action at a distance. When I got interested with Michael Atiyah in Yang-Mills theory and instantons in the late 1970s, he would send his younger collaborators to Oxford to find out what was going on, and then I got to know his entourage better. But it was the 1983 paper of Atiyah and Bott [AB] and Simon Donaldson's first paper [D] in the same year which attracted my attention. It brought gauge theory down from four to two dimensions and applied it to the famous theorem of Narasimhan and Seshadri about stable vector bundles on curves. Framed as it was in terms of moment maps and symplectic geometry it led to my investigation into what are now called Higgs bundles. I sent Narasimhan a preprint of my paper [H] and I subsequently found that we shared a common view on what were the interesting offshoots from this, though our motivations might well have been different.

To put it in context, the 1965 theorem of Narasimhan and Seshadri asserts that if a holomorphic bundle E on a compact Riemann surface C is stable—an open condition relating to its subbundles—then it admits a projectively flat unitary connection. A Higgs bundle is an extension of this idea, considering a pair of holomorphic objects (E, Φ) where $\Phi : E \to E \otimes K$ is holomorphic, K being the canonical line bundle, and the stability condition must hold only for Φ-invariant subbundles. In this case a theorem asserts that there is a unitary connection A with curvature F_A such that $F_A + [\Phi, \Phi^*] = 0$. When $\Phi = 0$ one recovers the original theorem but for other choices one can prove, for example, the uniformization theorem for Riemann surfaces, and also describe flat non-unitary connections with this data. My approach was differential-geometric but Narasimhan sent his young colleague Nitin Nitsure to Oxford who helped develop an algebro-geometric version, and the interaction was most useful for us both.

One feature, which had little to do with the equations, but arose naturally when I was writing the paper, was the use of the spectral curve – the covering of C defined by the characteristic equation $\det(x - \Phi) = 0$. It provided a way of constructing a Higgs bundle from a line bundle on the spectral curve. My treatment in the paper was rather ad hoc (though I had read about direct images as a student) but, together with Beauville and Ramanan, Narasimhan gave a much better account [BNR] which led to the consideration of non-abelian theta functions through spectral curves. Subsequent work in India especially, with Narasimhan looking on, clarified many aspects of the theory.

Over the past 30 years we saw each other on many occasions. A memorable one was his joint 60th birthday conference with Seshadri at the Tata Institute. But we met in Trieste for committees, which often ended in mathematical discussions in a restaurant overlooking the sea, and also amongst the community of researchers into vector bundles on curves who held conferences around the world. Many of these encounters were in Spain where Oscar García-Prada organized meetings to capitalize on the expertise from the Indian school of algebraic geometry. The familiar dark-suited figure was always a welcome sight, often accompanied by his long-time collaborator Ramanan. I valued both his knowledge about specific points and his view of the changing trends in mathematics and how one could react to them. His presence will be greatly missed.

Chandrashekhar Khare

The Narasimhan-Seshadri theorem as inspiration. I met Professor M. S. Narasimhan just a few times spread out over a couple of decades. The last few times were in the city of Bangalore where he then lived, at the prize ceremonies of the Infosys Science Foundation, and also at a Commonwealth Science Congress. He had a natural charisma, presence, intensity, an aura around him for me because of his renown.

Much to my regret, I did not get to know him well, either personally or mathematically. In this tribute to him, I will focus on his role as an inspiring figure—because of the importance of the work he had done, where he had done it, and when he had done it—to a young person (like myself) trying to do research in pure mathematics in India in the 1990s.

TIFR had established itself in the world of pure mathematics through important theorems proved by mathematicians working there, through the decades from the 1950s onwards, and of these there was none more celebrated than the Narasimhan-Seshadri theorem. It was in an area of algebraic geometry and differential geometry that was

Chandrashekhar Khare is a professor of mathematics at the University of California, Los Angeles. His email address is shekhar@math.ucla.edu.

Figure 2. Passport photo of Narasimhan, around 1974.

far away from my area of work which was in number theory.

I had come back to India immediately after my thesis in 1995, and joined TIFR as a Visiting Fellow (the entry level postdoctoral position available to someone after finishing their PhD).

I did not know the mathematical content of the theorem. Many of my senior colleagues at TIFR worked in different aspects of the mathematical specialty—vector bundles on curves—that had been decisively impacted by the Narasimhan-Seshadri theorem. It continued to be the focus of much of the research done at TIFR decades after the theorem had been proven.

For me, the influence of the Narasimhan-Seshadri theorem was more indirect but still psychologically quite important. The discovery of such an influential theorem by two young brilliant Indian mathematicians, in their early thirties, working in Bombay at TIFR in the 1960s, led to putting TIFR on the world map of mathematics. It also made one feel that as someone working at the same Institute, one had to try and live up to the high repute of the place, live up to that theorem in a way. Further it gave a sense that it was possible to do first-class mathematics working at a place somewhat distant from the traditional, mainly Western, centers of mathematical research, especially as their work had been done in Bombay in times when the world was far less connected than now.

I read later in an interview with Narasimhan that he felt it could be advantageous to work somewhere at a distance from the main mathematical centers, and follow the latest developments from this distance. One could then work on one's own ideas, partly inspired by the work happening at these centers, without being overwhelmed by the influence of the leaders in the subject. Such direct influence, while it could be greatly beneficial for some, might dissuade others from trying out ideas that could seem unpromising to the experts, but that one could not give up on internally.

Although I did not know Professor Narasimhan's quote at that time, I found for myself that this awareness from a distance of important developments in one's area, while working independently on one's own, worked quite well for me in the roughly 10 years I spent working at TIFR.

After arriving in India in 1995, I worked on trying to generalize the work I had done in my thesis, which used ingredients that overlapped with the astonishing work of Andrew Wiles on Fermat's Last Theorem. I arrived in a faltering way at a satisfactory generalization of my thesis work over a period of a few years after I returned to India. Temperamentally I am drawn to tilting at windmills, and I started doing, more or less simultaneously, more open-ended work inspired by Serre's modularity conjecture. Particular cases of this conjecture had been used by Wiles in his work on Fermat's Last Theorem, but the general case of the conjecture was wide open.

I kept musing about questions suggested by Serre's conjecture, carrying them in my mind, experiencing mainly frustration, but also small eureka moments, making small observations that I wrote up as short papers. It was a little like kicking the ball around on a field, with the goalposts obscured by a thick fog. As there was no way of reasonably aiming to kick at the goal which was smothered in the fog, one just kicked the ball around and chased after it in bursts of somewhat random, sporadic, but still intense, activity.

Later, I read another piece of Professor Narasimhan's advice to young people, which was to work "off the top": work on something without necessarily knowing precisely all the background required, getting by on a sense of the subject, impressionistic to begin with, which could be deepened as one continued thinking (continually!) about the subject. This way one would not get bogged down and overwhelmed by the myriad technical details right at the beginning, which could have a paralyzing effect on a novice, and instead learn them as one needed to.

On reading Professor Narasimhan's interview, I realized that I had been unconsciously following his advice of working "off the top" all along. In my work, I reached my "natural boundary" (as defined by Eilenberg, and referenced by Narasimhan in the interview) very quickly, and could go past it only by obsessing about a piece of mathematics and living with it in my mind.

The fact that the Narasimhan-Seshadri theorem had been proven when India was still young as an independent country (not yet twenty years old) was also fascinating. TIFR was founded by Homi Bhabha who had filled it with paintings by contemporary Indian artists (through the 1950s and 1960s), many of them working in Bombay, not so far away from Navy Nagar where TIFR is located. The works of the members of the Bombay Progressive Artists' Group were amply present on the walls of the Institute. The modernity the paintings represented, made in a newly independent country, which had its own ancient culture and tradition of art, architecture, music, and dance, married these civilizational influences with what was happening in the contemporary art world then. Many of the artists whose paintings Bhabha collected (with discernment and a remarkable sense or intuition for what was vital in the art made in India then) had spent time in Paris and returned to produce work which was influenced by what they had absorbed in their time there. Narasimhan and Seshadri also had been deputed to Paris, from 1957 to 1960 as I learnt from interviews of Professor Narasimhan, absorbed new ideas and influences there, and after returning to India proved their landmark theorem.

It was a different India that I lived in during my years working at TIFR (post the economic liberalization of 1991), but the example of Professor Narasimhan, Professor Seshadri, and their colleagues, who had worked at TIFR and proved path breaking theorems decades earlier, lived on as an inspiration, present in the air, setting a certain tone, holding me and my colleagues accountable, pushing us to try and live up to their formidable legacy.

Nitin Nitsure

I knew M. S. Narasimhan for nearly four decades, first as my thesis advisor, and then as a friend. The first volume of the collected papers of Narasimhan [N] begins with two excellent review articles on his research corpus—an unsigned article, and an article by C. S. Seshadri. I have addressed the story of the Narasimhan-Seshadri theorem in my recent memorial article on Seshadri [Ni] where there is also some material on Narasimhan. The videos of the memorial meetings for Narasimhan in Mumbai and Bangalore [NMM] in June 2021 have a lot of interesting material, including reminiscences by many mathematicians from different parts of the world, together with a rapid account of Narasimhan's entire mathematical career. But beyond his celebrated research contributions and his substantial role in nurturing and running institutions, Narasimhan had a deep impact on a large number of younger people with whom he interacted, and I am going to focus on the aspects of his persona and his behaviour that helped bring this about.

When Narasimhan became my thesis advisor, he was about twice my age, and was a distant and formidable figure. This was in early 1983, when I was a graduate student in the School of Mathematics, Tata Institute of Fundamental Research (TIFR), Mumbai. The first problem he gave me was to construct a relative Picard space in the holomorphic category—a problem that had been posed by Grothendieck—and then left me alone till I came back with a solution two months later. Unfortunately, my effort was wasted as it turned out that the result had already been proven by Bingener. This was indeed a disappointment for me, but it must have given Narasimhan some confidence in my ability. After that, our interactions became more frequent, and he gave me another interesting problem. My thesis was completed in 1986, but we continued to discuss mathematics. Before I went to Oxford for a postdoc in 1987, he asked me to look at Hitchin's latest papers. Somewhat later he said that I must study Grothendieck's FGA. These suggestions turned out to be very important for me. The student-teacher relationship was central and lifelong in the ancient Indian scholarly tradition (as in Indian musical, spiritual, or artisanal traditions), and traces of that attitude have clearly endured. Over the decades as the ratio between our ages improved, we became good friends, which continued till the end.

In the 1980s, Narasimhan was much feared not just by most students and postdocs, but also by some of the younger faculty members. He did not mix easily with others, and even a casual observer would have noticed his unrelenting single-minded seriousness, and refusal to indulge in any small talk. So it may appear surprising at first sight that a continuous stream of talented students did their PhD with Narasimhan, and became his lifelong well-wishers and friends. I will say more about this later, but let me note that for all his apparent lack of sociability, he was perfectly well mannered when he actually engaged with anybody, if the other party kept to the business at hand. He was a patient listener who rarely interrupted others, but allowed others to interrupt him quite easily, which is certainly unusual among people of his eminence. When I asked him about it many years later, he said that it is because he already knows his own thoughts, but is curious to know what the other party thinks! I never saw him either raise his voice or lose his composure in the nearly four decades that I knew him. Once when a disgruntled faculty member known for a bad temper raised his voice while arguing with Narasimhan, who was then the Dean, Narasimhan is known to have calmly said "You do not appear to be in the right frame of mind for a discussion now, please come back later." The other person left without further argument.

The rapid emergence of modern mathematics in TIFR in the 1950s owes a lot to the active support of French mathematicians such as Laurent Schwartz and like-minded colleagues. These were idealistic leftists, some of whom had strong revolutionary beliefs as followers of Lenin and Trotsky. They believed not just in universal functorial properties within mathematics but in universal human values, and took the trouble to spend months at a time visiting India, which must surely have appeared as a backward, poor, dirty, and inconvenient place to them, to help spread modern mathematics. (Interestingly, the undergraduate teacher who introduced Narasimhan and Seshadri to modern mathematics was a French Jesuit missionary called

Nitin Nitsure is a retired professor of mathematics at the Tata Institute of Fundamental Research, Mumbai. His email address is `nitsure@gmail.com`.

Father Racine, another practitioner of universal values.) Narasimhan said that not just his mathematics, but his understanding of political matters, developed rapidly during his three year postdoctoral visit to Paris during 1958–1961. He became a lifelong leftist.

What practical shape did Narasimhan's socialist ideology take? This has a simple answer: he made it his mission to nurture mathematical excellence at the highest level in all parts of humanity. He believed in the universality of mathematical talent, and was certain that among the teeming millions of the underprivileged all over the world, of all races, nationalities, religions, and genders, there were a huge number of potential top mathematicians, who needed to be detected and educated. But he did not think that a talented mathematician should, for example, go to a slum to teach school dropouts, since that job can be done as well by many others. What only a front line mathematician can do is to mentor talented young mathematicians who have somehow survived and risen from backward places or countries. They need help at the doctoral and postdoctoral stage to get into the really interesting directions of research and make a lasting contribution to mathematics, otherwise their talent is in danger of being wasted. Consistent with this, soon after his retirement from TIFR in 1992 at the age of 60, Narasimhan took up the position of the Head of Mathematics at the International Center for Theoretical Physics (ICTP), Trieste, at the invitation of its founder-director Abdus Salam. Numerous workshop participants and postdoctoral visitors come to ICTP every year from all over the developing world. Narasimhan engaged with a large number of them, and he had the uncanny ability to suggest interesting problems and directions which would suit their ability and inclination. With age, his social style also changed—he became a relaxed, reassuring figure that the young would gather around (which surprised some TIFR people who only knew his younger avatar). Finding doable interesting problems for the young from very diverse areas is not easy. Narasimhan made the necessary effort to read current literature in different areas and consulted experts. You will find many first-rate mathematicians from all over the world who say they owe him a lot for giving them a research problem, a useful idea, or simply a nudge in the right direction at a crucial stage in their career.

When it came to institutions, Narasimhan was not a revolutionary, but a true conservative. He always emphasized how difficult it is to build institutions and how easy it is to destroy them. So even here, stability (or at least semi-stability) was of paramount importance to him. Narasimhan put up with the many imperfections that he saw around him, and kept his focus on his chosen mission: to do top-level mathematical research himself and to help others do it. His integrity shone through all of his actions. As one of the large number of young mathematicians that he helped and influenced, I will say that he has earned our undying admiration and gratitude.

M. S. Raghunathan

I saw Narasimhan for the first time some time in my first year as a graduate student at TIFR in 1960–1961. He was dressed formally in a suit, which to a student of my background, suggested a stiff, unapproachable persona. I was soon disabused of that perception, when one day he stopped me in the corridor to compliment me on my proof of a result that was making the rounds in the School of Mathematics.

In my third year, I attended a seminar in Differential Geometry run by Narasimhan and Ramanan (then a senior student). Towards the end of the seminar Narasimhan, Ramanan, and I had frequent informal discussions. In these interactions, many of them over coffee, I learnt a great deal, and not just mathematics. My understanding of political and social issues came to acquire some sophistication. Narasimhan had an abiding interest in Tamil literature and I learnt many things about contemporary Tamil writing from him. In the course of a few walks along the seashore in TIFR, he explained to me the entire Kodaira Spencer Deformation Theory of Complex Structures. He was an outstanding teacher, excelling, especially in one-on-one communication of mathematical ideas.

He suggested a problem (connected with the Kodaira Spencer theory) for me to work on. When I eventually solved it he told me that the work was adequate for a PhD thesis and asked me to register for the degree with him as the thesis advisor, which I did. He interceded with the University of Bombay to get the waiting time for the submission of the thesis reduced to two years from the mandatory three. He got me invited to speak at the prestigious International Mathematical Colloquium (on Differential Analysis) held in Bombay in January 1964, and made me rehearse my talk with him. That resulted in a well-received lecture, surprising colleagues, who knew of my poor track record as a speaker.

Around this time I toyed with the idea of quitting mathematics to join my father in the family business. Narasimhan got wind of this and when he happened to meet a friend of my family, told her that my quitting would be a loss for mathematics. This reached my family soon,

M. S. Raghunathan is a retired eminent professor of mathematics at the Tata Institute of Fundamental Research, Mumbai, and currently Distinguished Professor of Mathematics at the Centre of Excellence in Basic Sciences, Mumbai. His email address is `madabusisr@gmail.com`.

Figure 3. With M. S. Raghunathan in Bangalore, January 2019.

and they promptly put an end to my idea of changing my vocation. Narasimhan's role in my career thus went well beyond that of a mentor in mathematics. And this would also be true for many of his other students.

After I wrote my thesis, my mathematical interests moved away from his and our mathematical interactions were of a general nature and not intense. Nevertheless each of us had a pretty good idea of what the other was doing in mathematics. Narasimhan's approach to mathematics was very French, Bourbakian, in fact. Riemann and Poincaré headed the pantheon of the greats whom he admired. Among his contemporaries Kodaira held pride of place in his admiration and somewhat later Grothendieck joined him, or maybe, even displaced him. He once told me that the idea of the proof of the Narasimhan-Seshadri theorem was inspired by Poincaré's method of "balayage." He told me that Grothendieck was amazing, saying something to the effect that any time you come up with a problem in the subject, you find that Grothendieck has already said something profound about it.

In 1966, I was inducted into the Mathematics Faculty of the Tata Institute of Fundamental Research. From day one, Narasimhan treated me as an equal, but it was necessarily an asymmetric relationship; there was no way I could forget that Narasimhan was my teacher. Our views on most issues were almost identical, but when we differed, I would invariably defer to his views. Our mathematical interactions continued, but were less frequent and less intense as the focus of my mathematical interests had drifted away from his.

Narasimhan took considerable interest in promoting mathematics in the country at large. Already in the sixties he served on many committees of universities and other institutions of higher learning. He was of course one of the principal architects of the rise of the School of Mathematics of TIFR from a minor player to international eminence in mathematics. In the early seventies, Jawaharlal Nehru University wanted to start a Department of Mathematics with him as the Head. He was not averse to the idea, perhaps because he wanted to build another centre rivaling the TIFR school in the country. Fortunately for TIFR, the project did not materialize and Narasimhan continued to guide the School for another 20 years.

When in 1983, the DAE set up the National Board of Higher Mathematics, an agency for the promotion of mathematics, he was the natural choice to head it. Under his leadership, the Board took many initiatives which went a long way towards fulfilling its mandate. As a fellow faculty member and a member of NBHM, I had the good fortune of observing him at close quarters and I learnt a great deal about science administration. He was a stickler for dignified and correct conduct on all occasions; he would visibly wince when someone failed on that front. He planned and conducted meetings meticulously. His decisions were always taken after considerable thought. His letters were written with great care; he not only wanted to say the right things but wanted them said the right way.

During his ICTP days, my contacts with him were few and far between. I paid a few short visits to ICTP at his invitation and they were very enjoyable. After he retired to Bangalore, the contacts, which were mostly on the telephone, became more regular.

Narasimhan—unlike many men of comparable stature—was accessible; and interaction with him, whether professional or otherwise, was easy and pleasant; his style though was not the deliberate informality of the corporate world. He had wide interests, beyond mathematics, and had interesting things to say on a variety of subjects. His sartorial preferences were conservative: I have never seen him in jeans and he was seen in a T-shirt only rarely. He was not averse to dressing formally if the occasion demanded it. But he was no conservative on social and political issues. He was of a strongly leftist persuasion and was unequivocal in his condemnation of the caste system and the Hindu right.

In recent years he was much distressed at the decline, in our country, of the values that he held dear; mathematics, his magnificent obsession, kept him from sinking into greater despondency. The manner in which he kept up with recent developments in mathematics right till the end, was truly amazing. He was of course a professional mathematician, yet his excitement with mathematics was reminiscent of the little boy on the seashore in Newton's famous self-assessment.

His passing away is a great personal loss to me: he was a close friend and mentor.

T. R. Ramadas

Narasimhan had an abiding curiosity about most aspects of mathematics. He firmly believed that mathematics has an internal dynamic, and that intelligent and insightful interrogation yields the deepest insights of the subject. But he was very open to insights that came from allied fields, particularly Physics.

I joined TIFR in 1977 as a graduate student in theoretical physics, with a good training from the Indian Institute of Technology, Kanpur. I knew of the work of Aharanov-Bohm, Wu-Yang, and others on the geometric aspects of gauge theories. The quantization of these theories involves an integration over the space of "gauge potentials," i.e., space of connections on a principal bundle on space-time. This space is infinite-dimensional, and this leads to the usual, and still largely unresolved, mathematical problems of quantum field theory. In the case of gauge-theories, there is an additional complication, due to the fact that the integrand is invariant under the infinite-dimensional group of gauge-transformations, i.e., automorphisms of the principal bundle. This is dealt with by "gauge-fixing." The Russian physicist V. N. Gribov had pointed out that there were ambiguities in this procedure, and speculated on physical consequences of this.

Although Narasimhan is best known as an algebraic geometer, his early training was as a complex geometer and analyst, and he had a deep knowledge of both the analytical and formal aspects of differential geometry, in particular the theory of connections on bundles. The theorem on universal connections due to Narasimhan and S. Ramanan is a fundamental insight. (The second of their joint papers on the subject contains an elegant proof of the basic result in Chern-Weil theory. This is now the standard proof, and rarely credited to its discoverers.)

I explained to Narasimhan my rather naive understanding of these matters. Narasimhan very quickly brought all the geometry into focus. He insisted on the "correct" analytical setting for infinite-dimensional geometry, and our work contained the earliest construction of the space of connections as an infinite-dimensional principal bundle modeled on a suitable Sobolev space. "Gauge-fixing," taken literally, would mean choosing a section for this bundle, and the point is that it is rarely trivial, except possibly in the case of connections on a principal bundle with abelian structure group. This had been proved independently and earlier by I. M. Singer, whose paper appeared after we finished our manuscript.

In the early sixties, Narasimhan and Seshadri had made the fundamental discovery that stable vector bundles (an algebro-geometric notion introduced by D. Mumford) on a compact Riemann surface correspond to irreducible unitary representations of its fundamental group (possibly with a "puncture," depending on the degree of the bundle). The study of these moduli spaces was a major preoccupation of Narasimhan in the next two decades. His work with S. Ramanan and later with G. Harder provided a template for much of the later work on moduli.

In the early eighties, Atiyah and Bott realized that the Theorem of Narasimhan and Seshadri could be interpreted as an infinite-dimensional version of Kempf-Ness theory, with the curvature of a connection interpreted as the moment map. Their starting point was an investigation of a toy version of gauge theory, with a Riemann surface replacing space-time, and the norm (squared) of the moment map playing the role of "action functional."

All this was "classical mechanics." In the late eighties and nineties, physicists realized that certain quantum field theories in two and three space-time dimensions had as their quantum "state spaces," spaces of (holomorphic) sections of line bundles on moduli spaces of vector bundles on Riemann surfaces. It is fair to say that this revealed aspects of linear series on these spaces that were entirely new. In analogy with the classical case of Jacobians, these are called generalized theta functions.

Narasimhan was particularly intrigued by the beautiful formulae derived by E. Verlinde for the dimensions of these linear series on these moduli spaces. These matters remained a major preoccupation from then on.

Narasimhan and J.-M. Drézet developed the basic theory of the theta bundle on moduli spaces of vector bundles of arbitrary rank and degree. Narasimhan and I then worked out a proof of the Verlinde formula in purely algebro-geometric terms. We had to give a careful construction of parabolic moduli spaces on singular curves, (which we did à la Simpson), definition of theta bundle thereon, a vanishing theorem (in a context where Kodaira vanishing could not be immediately applied), and finally a geometric proof of "factorization."

It is worth taking stock of what the physics ingredient was in these matters. In general terms, the realization that there was a rich structure to the theory of generalized theta functions. Specifically, first the insight that degeneration techniques and the incorporation of "parabolic structures" made possible an inductive expression for the dimensions. Second, Verlinde's ingenious introduction of his algebra

T. R. Ramadas is a professor of mathematics at the Chennai Mathematical Institute, Chennai. His email address is ramadas@cmi.ac.in.

Figure 4. With wife Sakuntala and daughter Shobhana, Bombay, 1964.

and its use in deriving an explicit formula. Third, the role of Kac-Moody groups (clarified largely through the work of Tsuchiya-Ueno-Yamada) in the context.

Narasimhan enlisted S. Kumar and A. Ramanathan in the first of a series of works that carefully elucidated the relationship between the definitions of conformal blocks in algebra-geometric terms and in terms of loop groups. These papers, technically difficult and carefully written, remain standard references.

In more recent times, Narasimhan remained engaged in developments in moduli theory, and followed the work on Bridgeland stability particularly closely. His last works were devoted to derived categories of coherent sheaves on his beloved moduli spaces, using the Hecke transform, a pioneering tool invented with Ramanan decades earlier.

Arnaud Beauville

My interest in vector bundles on curves was triggered by the work of Narasimhan and Ramanan. In the early 80s, I was interested in the Schottky problem. Recall that one associates to a curve C of genus g a complex torus J of dimension g, the *Jacobian variety*, which one can view as parameterizing line bundles L of degree $g-1$ on C (by choosing one of these as the origin). Then the locus Θ of those line bundles which admit a nonzero section is a hypersurface in J, the *Theta divisor*. The pair (J, Θ) is what we call a *principally polarized abelian variety* – p.p.a.v. for short.

As soon as $g \geq 4$, the p.p.a.v.'s depend on more parameters than the Jacobians; the *Schottky problem* asks for a characterization of Jacobians among all p.p.a.v.'s (A, Θ). There was a flurry of activity around this in the early 80s; most approaches involve the linear system $|2\Theta|$ (that is, the projective space of divisors linearly equivalent to 2Θ). For $a \in A$, the divisor $\kappa(a) := (\Theta + a) + (\Theta - a)$ belongs to this linear system; the map $\kappa : A \to |2\Theta|$ embeds the *Kummer variety* $\mathrm{Km}(A) := A/i$ into $|2\Theta|$, where i is the involution $a \mapsto -a$ of A. One of the approaches to the Schottky problem characterizes Jacobians by the existence of trisecants to their Kummer variety in $|2\Theta|$.

I was studying these questions when I discovered that Narasimhan and Ramanan had found a remarkable connection between $|2\Theta|$ and rank 2 vector bundles. Let $\mathcal{M}$ be the moduli space of semi-stable, rank 2 vector bundles on C with trivial determinant. Given $E \in \mathcal{M}$, the locus of line bundles $L \in J$ such that $E \otimes L$ admits a nonzero section is an element $\theta(E)$ of $|2\Theta|$; we thus get a map $\theta : \mathcal{M} \to |2\Theta|$, which maps the singular locus of $\mathcal{M}$ exactly onto the Kummer variety. In the beautiful paper [NR], Narasimhan and Ramanan work out completely the genus 3 case: θ is an embedding, and its image turns out to be the unique quartic hypersurface in $|2\Theta| \cong \mathbf{P}^7$ singular along $\mathrm{Km}(J)$. This hypersurface had been discovered by Coble long ago through algebraic manipulations; its geometric interpretation via vector bundles was entirely new.

This result inspired me to study the map $\theta : \mathcal{M} \to |2\Theta|$ in higher genus [B]. I noticed that there is a kind of duality between J and $\mathcal{M}$: given $L \in J$, one defines a divisor Θ_L on $\mathcal{M}$ as the locus of vector bundles E such that $E \otimes L$ has a nonzero section; the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{M}}(\Theta_L)$ does not depend on L, and there is a natural isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |2\Theta|$ which identifies θ with the map $\varphi_{\mathcal{L}}$ defined by the global sections of $\mathcal{L}$.

For vector bundles of rank $r \geq 3$, the analogous statements make sense (with a rational map $\mathcal{M} \dashrightarrow |r\Theta|$), but the proof in rank 2 is not directly adaptable. After a few months I found a way to do it. Shortly after I met Narasimhan and Ramanan at the AMS Summer conference on theta functions (in 1987), we realized we had had the same ideas—using the Hitchin fibration and the notion of very stable vector bundle introduced by Drinfeld. So we decided to write the joint paper [BNR]. Narasimhan invited me to the Tata Institute; I spent one month there in 1988. By that time, we had essentially finished the paper, but I had many lively discussions with Narasimhan, both mathematical and non-mathematical—he was a very cultured man, with interesting points of view on a wide range of subjects.

The key ingredient in [BNR] was the computation of the dimension of the space of global sections $\Gamma(\mathcal{L})$—often called "generalized theta functions." At about that time, word spread among mathematicians that physicists had a formula for the dimension of $\Gamma(\mathcal{L}^{\otimes k})$ for all k, in a much more general setting including for instance moduli spaces of G-bundles for all semi-simple groups G. This

Arnaud Beauville is an emeritus professor at Université Côte d'Azur, Nice. His email address is beauville@unice.fr.

Verlinde formula soon became a challenge for algebraic geometers, and half a dozen proofs appeared in the following years, including one by Narasimhan with Kumar and Ramanathan [KNR] and one by Laszlo and myself [BL]. Our (independent) proofs were actually quite close: both papers used infinite-dimensional algebraic geometry, with the language of infinite-dimensional manifolds in [KNR] and of algebraic stacks in [BL].

I had no other opportunity to collaborate with Narasimhan. We met briefly in a few conferences; on one occasion he came to Nice, we had a nice lunch with a very pleasant conversation. Narasimhan liked good food, art, and music. He was a great mathematician and a colleague of high human quality.

Shrawan Kumar

I am saddened by Professor Narasimhan's passing away. It is a personal loss to me and a great loss for the mathematical community and all associated with him. I hold him in the utmost respect not only for all the beautiful mathematics he created (of which I have learnt only a tiny bit), but his integrity, administrative capability, and care for all those associated with him.

On a personal note, I vividly remember that one day while I was a graduate student, he called me to his office and explained a problem to me. But more importantly, he told me how to go about working on a problem. To this day, I try to follow his advice when I think about a problem. He was the one who introduced me to the problems surrounding the Verlinde formula, which became among one of the important projects I pursued for a while (mostly jointly with him and also some with A. Ramanathan and A. Boysal) and recently I wrote a book on the subject. I visited Professor Narasimhan at I.C.T.P. during 1994. He was very warm and caring.

I had the privilege of writing two papers with Professor Narasimhan, which I briefly describe here. Let Σ be a smooth projective irreducible s-pointed ($s \geq 1$) curve of any genus $g \geq 0$ with marked points $\vec{p} = (p_1, \ldots, p_s)$ and let G be a simply connected simple algebraic group with Lie algebra $\mathfrak{g}$. We fix a positive integer ℓ called the *level* and let P_ℓ be the set of dominant integral weights of $\mathfrak{g}$ of level at most ℓ. We attach weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_s)$ (each $\lambda_i \in P_\ell$) to the marked points $\vec{p}$ respectively. Associated to the triple $(\Sigma, \vec{p}, \vec{\lambda})$, there is the space $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ of *conformal blocks* (also called *space of vacua*), which is a certain finite dimensional space of $\mathfrak{g} \otimes \mathbb{C}[\Sigma \setminus \vec{p}]$-invariants of a tensor product of s-copies of integrable highest weight modules with highest weights $\vec{\lambda}$ and level ℓ of the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. This space is a basic object in Rational Conformal Field Theory arising from the Wess-Zumino-Witten model associated to G. Now, E. Verlinde gave a remarkable conjectural formula for the dimension of $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ in 1988. This conjecture was "essentially" proved by a pioneering work of Tsuchiya-Ueno-Yamada [TUY], wherein they proved the *Factorization Theorem* and the *invariance of dimension* of the space of conformal blocks under deformations of the curve Σ, which allow one to calculate the dimension of the space of conformal blocks for a genus g curve from that of a genus $g-1$ curve. Thus, the problem gets reduced to a calculation on a genus 0 curve, i.e., on $\Sigma = \mathbb{P}^1$. The corresponding algebra for $\Sigma = \mathbb{P}^1$ is encoded in the fusion algebra associated to $\mathfrak{g}$ at level ℓ, which gives rise to a proof of an explicit Verlinde dimension formula for the space $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$.

Classical theta functions can be interpreted in geometric terms as global holomorphic sections of a certain determinant line bundle on the moduli space $\mathrm{Pic}^{g-1}(\Sigma)$ of line bundles of degree $g-1$ on Σ (a smooth curve of genus g). This has a natural non-abelian generalization, where one replaces the line bundles on Σ by principal G-bundles on Σ to obtain the parabolic moduli space (or stack) $\mathrm{Parbun}_G(\Sigma)$ and certain determinant line bundles over $\mathrm{Parbun}_G(\Sigma)$. Holomorphic sections of these determinant line bundles over $\mathrm{Parbun}_G(\Sigma)$ are called the generalized theta functions (generalizing the classical theta functions). The Verlinde dimension formula attracted considerable further attention from mathematicians and physicists when it was realized that the space of conformal blocks admits an interpretation as the space of generalized theta functions. This interpretation was rigorously established in the "non-parabolic" case in my joint work with Narasimhan-Ramanathan (for general G) [KNR]. It was independently established around the same time by Faltings (for general G) and Beauville-Laszlo (for the special case $G = \mathrm{SL}_n$); and in the case of parabolic space by Pauly (for the special case $G = \mathrm{SL}_n$) and for parabolic stacks by Laszlo-Sorger (for general G).

In a second joint paper with Professor Narasimhan, we proved that the moduli *space* $M_G(\Sigma)$ of semistable G-bundles over a smooth irreducible projective curve Σ (of any genus) is Gorenstein and has its Picard group isomorphic with the group of integers [KN], thus generalizing the corresponding result for $G = \mathrm{SL}_n$ by Drezet-Narasimhan. We also proved the vanishing of higher cohomology of $M_G(\Sigma)$ with coefficients in positive line bundles.

It is simply amazing that inspite of all the pain and suffering he was going through towards the end of his life,

Shrawan Kumar is John R. and Louise S. Parker Distinguished Professor of Mathematics at the University of North Carolina, Chapel Hill. His email address is shrawan@email.unc.edu.

he wrote a paper with Gallego and García-Prada which appeared on the ArXiv on May 13, 2021, just a couple of days before he passed away. It is a testimony to his utmost devotion to mathematics!

Oscar García-Prada

As a graduate student in Oxford in the late 1980s, working under the supervision of Nigel Hitchin and Simon Donaldson, I was very much influenced by the theorem of Narasimhan and Seshadri, and the important generalizations that were inspired by this theorem around that time. Published in 1965, this theorem captures the interconnection between various branches of geometry, topology, and theoretical physics, and was the basis for later fundamental works by numerous mathematicians, including Atiyah, Bott, Donaldson, Uhlenbeck, Yau, Hitchin, Simpson, and many others. After briefly recalling the theorem of Narasimhan and Seshadri, I will comment here on some generalizations related to representations of the fundamental group of a compact Riemann surface, with particular reference to works that are close to my own interests and research.

Upon his return from Paris to the Tata Institute for Fundamental Research (TIFR) in Bombay in 1960, Narasimhan embarked on an intense collaboration with Seshadri that resulted in their famous theorem. Inspired by some remarks in the 1938 paper of A. Weil on "Généralisation des fonctions abéliennes," in 1961–1962, Narasimhan and Seshadri started looking at unitary vector bundles. A unitary representation ρ of dimension n of the fundamental group of a compact Riemann surface X defines a holomorphic vector bundle E_ρ of rank n and degree 0, which is referred to as a *unitary vector bundle*. This is called an *irreducible unitary vector bundle* if ρ is irreducible. They showed that the infinitesimal deformations of a unitary vector bundle E_ρ as a holomorphic bundle can be identified with the infinitesimal deformations of the representation ρ. From this, they deduced that the set of equivalence classes of unitary vector bundles had a natural structure of a complex manifold, and were able to compute the expected dimension.

A breakthrough came with the work of Mumford on Geometric Invariant Theory. In the 1962 International Congress in Stockholm, he introduced the notion of stability of a vector bundle on a compact Riemann surface, and proved that the set of equivalence classes of stable bundles of fixed rank and degree has a natural structure

Oscar García-Prada is professor of mathematics at the Instituto de Ciencias Matemáticas, Madrid. His email address is oscar.garcia-prada@icmat.es.

Figure 5. Narasimhan with Oscar García-Prada at ICMAT, Madrid, 2017.

of a non-singular quasi-projective algebraic variety, projective if the rank and degree are coprime. After Narasimhan and Seshadri became aware of Mumford's work, the relation with unitary bundles was clear to them. They proved that an irreducible unitary bundle is stable. For arbitrary degree they showed that the stable vector bundles on X are precisely the vector bundles on X which arise from certain irreducible unitary representations of suitably defined Fuchsian groups acting on the unit disc and having X as quotient.

The result that they proved in [NS] can be easily reformulated as saying that a holomorphic vector bundle over X is stable if and only if it arises from an irreducible projective unitary representation of the fundamental group of X. From this, one deduces that a reducible projective unitary representation of the fundamental group corresponds to a direct sum of stable holomorphic vector bundles of the same slope, where the slope of a vector bundle is the quotient of its degree by its rank, (what is nowadays referred as a *polystable* vector bundle). One can observe that the projective unitary representations lift to unitary representations of a certain *central extension* of the fundamental group of X. The gauge-theoretic point of view of Atiyah and Bott [AB], using the differential geometry of connections on holomorphic bundles, and the new proof of the Narasimhan–Seshadri theorem given by Donaldson [D] following this approach, brought new insight and new analytic tools into the problem. In this approach, a projective unitary representation of the fundamental group is the holonomy representation of a unitary projectively flat connection.

A very natural question to ask is whether there is a holomorphic interpretation of representations of the fundamental group of X in $\mathrm{GL}(n,\mathbb{C})$ that are not unitary. The answer to this required the introduction of new holomorphic objects on the Riemann surface X called *Higgs bundles*. These objects, introduced by Hitchin in [H], are pairs (E, Φ) consisting of a holomorphic vector bundle E over X and a homomorphism $\Phi : E \to E \otimes K$, where K is the canonical bundle of X. There is a notion of stability similar to that of vector bundles, and corresponding moduli spaces. The correspondence between stable Higgs bundles and irreducible representations of the fundamental group of X (or its universal central extension if the degree is different from zero) in $\mathrm{GL}(n, \mathbb{C})$ was proved in the above mentioned paper by Hitchin [H] for $n = 2$ and by Simpson (1988) for arbitrary n (and in fact, for higher dimensional Kähler manifolds). The correspondence in the case of $\mathrm{GL}(n, \mathbb{C})$ needed an extra ingredient—not present in the compact case—having to do with the existence of twisted harmonic maps from X into the symmetric space $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$. This theorem was provided by Donaldson (1987) for $n = 2$ and by Corlette (1988) for arbitrary n (who also proved it for higher dimensional compact Riemannian manifolds).

It turns out that the theory of Higgs bundles is also central in the study of representations of the fundamental group of X in non-compact real forms $G_{\mathbb{R}} \subset \mathrm{GL}(n, \mathbb{C})$. Indeed, the case of the *split* real form $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ (more precisely $\mathrm{SL}(n, \mathbb{R})$) was studied by Hitchin (1992). Using Morse-theoretic techniques he counted the number of connected components of the moduli space. He also identified special components, now known as *Hitchin components*, whose representations, as shown by Labourie (2006) using concepts from dynamical systems, have similar properties to those in the Teichmüller space of the surface, regarded (Goldman, 1980) as a topological component of the moduli space of representations in $\mathrm{SL}(2, \mathbb{R})$.

The case of the pseudo-unitary groups $G_{\mathbb{R}} = \mathrm{U}(p, q)$ with $p \neq 0 \neq q$ and $p + q = n$ is in a sense closer to the case of the unitary group $\mathrm{U}(n)$ studied by Narasimhan and Seshadri, since these real forms are inner equivalent to $\mathrm{U}(n)$. This situation was investigated by Bradlow and Gothen in collaboration with the author [BGG]. Here, a Higgs bundle (E, Φ) corresponding to a representation in $\mathrm{U}(p, q)$ is of the form $E = V \oplus W$, where V and W are holomorphic vector bundles of rank p and q respectively, and Φ, in terms of this decomposition, has zeros in the diagonal. There is a topological invariant, called the *Toledo invariant*, defined as $\tau = 2\frac{aq-bp}{p+q}$, where a and b are the degrees of V and W respectively, for which the semistability of (E, Φ) implies the so-called *Milnor–Wood inequality* $0 \leq |\tau| \leq 2\min\{p, q\}(g - 1)$. In [BGG] it is proved that for any value of the degrees a and b so that τ satisfies the Milnor–Wood inequality the moduli space of stable $\mathrm{U}(p, q)$-Higgs bundles is non-empty and connected. This is very much in contrast with the case of $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$, for which there are no constraints on the topological invariant, and for which for any value of the degree there exists a non-empty connected component. Moreover, when n is even and $p = q$, the representations in the component with maximal Toledo invariant have properties similar to those in the Hitchin component, as shown by Burger–Iozzi–Labourie–Wienhard (2005). The study of these components and the Hitchin components is part of the content of the recent field of *higher Teichmüller theory*.

To complete the list of real forms of $\mathrm{GL}(n, \mathbb{C})$, when n is even one can consider the group $\mathrm{U}^*(n)$, the non-compact dual of $\mathrm{U}(n)$. This has been treated by Oliveira and the author (2011). In this case a Higgs bundle (E, Φ) is such that E is equipped with a holomorphic symplectic structure, with respect to which Φ is symmetric. Using the Morse-theoretic techniques introduced by Hitchin, the main result proved here is that the moduli space of $\mathrm{U}^*(n)$-Higgs bundles is non-empty and connected.

I first met Narasimhan quite soon after having completed my doctoral thesis in 1991. From the very beginning, he was very kind to me, and extremely generous in the exchange of ideas. Our mathematical and personal friendship grew over the years and we had the opportunity to meet many times in Europe and India. He visited our institute in Madrid on several memorable occasions, including Nigel Hitchin's 60th birthday conference and S. Ramanan's 70th birthday conference, as well as a conference in his honor on the occasion of his 80th birthday.

In addition to discussing mathematics and Indo-European collaboration schemes, Narasimhan and I very much liked to enjoy a glass (or two!) of good red wine, very often in company of our common friend and collaborator Ramanan, and other good friends. I last saw Narasimhan in person in Bangalore in February 2020, during a meeting at the International Centre for Theoretical Sciences (ICTS). After the ICTS meeting, I went for few days to Chennai for a visit at the Chennai Mathematical Institute (CMI), where as a matter of fact I saw C. S. Seshadri for the last time. During the last year of Narasimhan's life we were very actively in contact working on a joint project with him and my student Guillermo Gallego on a generalization of the Hitchin system. A paper on this work [GGN] saw the light just a few days before his passing.

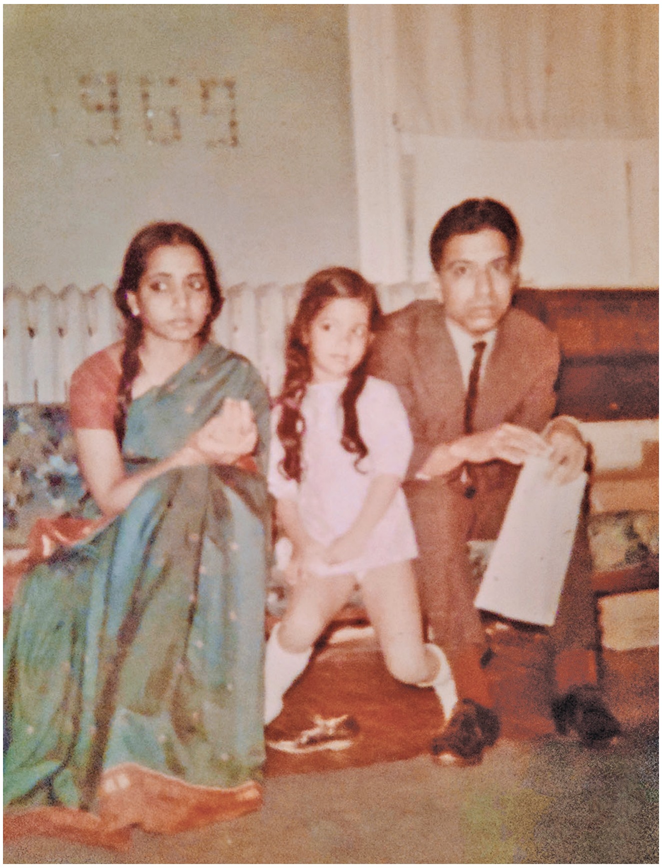

Figure 6. With wife Sakuntala and daughter Shobhana, 1969.

Herbert Lange

First of all I would like to say that it always was a great pleasure to work with Narasimhan. We met many times, at TIFR, in Chennai and Bangalore, at ICTP, in Erlangen, and at many conferences. Whenever we met, of course we mainly discussed Mathematics, but also other subjects, even personal ones.

I got to know Narasimhan as a very generous person, not only in mathematics. He was not just a colleague, but also a friend. Although he certainly was the better mathematician, he never let me feel it. Apart from publishing two papers together, the main results of which I will describe below, he also had an influence on some of my other papers. Moreover, some of the results of our discussions we did not publish.

First we met in Nice, where I spent some months and he a whole year and where we shared an office. Soon we found a problem, on vector bundles on curves, which led to our joint paper [LN1], the main results of which are as follows:

Let X be a smooth irreducible curve of genus g over an algebraically closed field of characteristic 0. For any vector bundle E of rank 2 over X define the invariant $s(E) := \deg E - 2\max\deg(L)$ where the maximum is taken over all line subbundles of E. E is stable if and only if $s(E) \geq 1$ and it is well known that $s(E) \leq g$. Denote by $M(E)$ the subscheme of $\mathrm{Pic}(X)$ formed by the maximal subbundles of E. Maruyama proved that $\dim M(E) = 1$ if $s(E) = g$ and conjectured that $M(E)$ is finite whenever E is not of the form $L \oplus L$ and $s(E) \leq g-1$. Our main result is the following

Theorem.

$s(E) = 2$: *If* $\deg E \equiv 0 \mod 2$, *then* $\dim M(E) = 0$ *for every* E *of rank* 2 *if and only if* X *is not double elliptic. In the double elliptic case every double elliptic cover yields a* g*-dimensional subspace of the moduli space of stable* E *of rank* 2 *with* $\dim M(E) = 1$.

$s(E) = 3$: *For every* $g \geq 4$ *there is a curve* X *of genus* g *which admits a vector bundle* E *with* $s(E) = 3$ *and* $\dim M(E) = 1$.

In each case, explicit examples are given. The method of proof is to translate the problem into a problem of projective geometry: namely to determine curves of genus g and degree 2g in the projective space $\mathbb{P}^g$ and points in $\mathbb{P}^g$, not on the curve, through which infinitely many secant lines of the curve pass. The maximal subbundles correspond to the secant lines passing through the point.

Our second paper was written during a visit of Narasimhan in Erlangen. This time the subject was not vector bundles, but abelian varieties.

According to a classical theorem of Lefschetz the n-th power of an ample line bundle of an abelian variety is very ample for any $n \geq 3$. The paper [LN2] deals with the analogous question for the second power of an ample line bundle. The results are not new, however, most of the proofs are. So for an ample line bundle L on X consider the map

$$\Phi = \Phi_{L^2} : X \to \mathbb{P}^N = P(H^0(L^2)).$$

To state the main theorem, let

$$(X, L) = (X_1, L_1) \times \cdots \times (X_s, L_s)$$

denote the decomposition of the polarized abelian variety (X, L) into a product of irreducible polarized abelian varieties. Suppose that (X_ν, L_ν) for $\nu = 1, \dots, r$ are principally polarized and for $\nu = r+1, \dots, s$ are not principally polarized. For $\nu \leq r$ let $K_\nu = X_\nu/\pm id$ denote the Kummer variety of X_ν with canonical projection $p_\nu : X_\nu \to K_\nu$.

Herbert Lange is a professor of mathematics at Universität Erlangen, Germany. His email address is `lange@mi.uni-erlangen.de`.

Define $K = K_1 \times \ldots K_r \times X_{r+1} \times \cdots \times X_s$ and $p = p_1 \times \cdots \times p_r \times id_{X_{r+1}} \times \cdots \times id_{X_s}$. So Φ factorizes as

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \mathbb{P}^N \\ & K \arrow[ur, "\psi"'] & \end{tikzcd}$$

with a holomorphic map ψ. The main result is,

Theorem. *ψ is an embedding.*

So Φ is of degree 2^r onto its image. In particular, if none of the (X_i, L_i) is principally polarized, Φ is an embedding. On the other hand, in the case of an irreducible principally polarized abelian variety, Φ embeds the Kummer variety. The main contributors of this topic are Wirtinger, Andreotti-Mayer, Sasaki, Ramanan and Ohbuchi.

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Credits

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