Most mathematicians probably view mathematics as a subject in which we establish eternal truths with absolute certainty. What is there for philosophers to do, when contemplating mathematics, other than stand back and admire its beautiful perfection? And yet, when one looks more closely, one discovers that mathematics is full of philosophical mysteries.

For example, what are numbers? No one has ever seen one. (Of course, you can see the mark “5” on a piece of paper, but that is no more the number five than I am the sequence of letters “Dan.”) G. H. Hardy, in his *A Mathematician’s Apology*, famously claimed that mathematical objects do not belong to physical reality, but rather to “another reality, which I will call ‘mathematical reality’” [Har92, p. 123]. What is the nature of this reality?

If mathematical objects belong to a reality that is separate from physical reality, how is it that their properties are so useful for understanding physical reality? As Eugene Wigner put it, how do we explain “the unreasonable effectiveness of mathematics in the natural sciences” [Wig85]?

How do we discover facts about mathematical reality? It appears that we do it by pure thought. But how is it that, just by thinking, we can discover properties of mathematical reality? Perhaps mathematical objects exist only in our minds. But if so, does each of us have our own, private...
version of mathematical reality, and are they different? And if not—if, as Hardy believed, “mathematical reality lies outside us” [Har92, p. 123]—how does thought give us access to that reality?

And what are we to make of mathematical statements, such as the continuum hypothesis, that are known to be neither provable nor disprovable? Does the continuum hypothesis have an objective truth value? If not, does that mean that mathematical reality has an indeterminate quality? But if it has an objective truth value, then it seems to be a truth value that neither our senses nor our thoughts give us access to. Belief in such truth values begins to seem more like religious faith than science; perhaps Calvin is right about mathematics!

In the preface to his book Lectures on the Philosophy of Mathematics, Joel Hamkins says his aim is “to present a mathematics-oriented philosophy of mathematics.” This orientation is reflected in the organization of the book into eight chapters, each focusing on a different mathematical topic: numbers, rigor, infinity, geometry, proof, computability, incompleteness, and set theory. In each chapter, Hamkins presents mathematical results of foundational significance, but these mathematical results lead naturally to discussions of a variety of philosophical issues and positions.

For example, in the chapter on numbers, Hamkins sketches constructions of the different number systems of mathematics. He introduces the logicist program of Gottlob Frege and explains how Frege tried to define the natural numbers. For the case of the real numbers, he describes constructions using both Dedekind cuts and equivalence classes of Cauchy sequences. Of course, these two constructions yield isomorphic structures. One way to see this is to observe that both constructions lead to a complete ordered field, and all complete ordered fields are isomorphic. These mathematical results motivate the philosophical position of structuralism, according to which, for example, there is nothing more to be said about what the real numbers are other than that they are a complete ordered field. This characterizes the real numbers up to isomorphism, which is all that mathematics requires. For the structuralist, the purpose of the constructions of the real numbers is not to identify what the real numbers “really” are—there is no such thing—but rather to give the existence half of the proof that there exists a unique (up to isomorphism) complete ordered field. Hamkins discusses several versions of structuralism, advocated by various philosophers. Most mathematicians will find some version of structuralism appealing, but in my view it requires a background theory, such as set theory, that is strong enough to allow us to prove the necessary existence and uniqueness theorems. Thus, it does not completely resolve the philosophical problems associated with defining numbers; rather, it pushes those problems back to the background theory.

In Chapter 2, Hamkins tells the story of how calculus was made rigorous by the introduction of the \( \epsilon-\delta \) definition of limits, and how this led to the understanding of important subtle distinctions in analysis, such as the difference between convergence and uniform convergence. The fact that calculus is used in almost all branches of science leads naturally to the philosophical argument, put forward by Hilary Putnam and Willard Van Orman Quine, that the indispensability of mathematical objects for science is evidence for the existence of those objects. Hamkins then describes the opposing argument by Hartry Field that mathematical objects are not actually indispensable for science, but rather can be considered to be convenient fictions.

In the next chapter Hamkins presents Cantor’s theory of different sizes of infinity, the infinite ordinal and cardinal numbers, and the continuum hypothesis. Cantor’s work brings up two philosophical questions. The first is whether infinity should be regarded as actual or potential; that is, whether infinite collections can be treated as completed totalities, or whether it is better to think of the infiniteness of a collection as merely the unending potential to produce more and more elements of the collection. This distinction was introduced by Aristotle, who believed all infinity was potential. Cantor’s work was initially controversial, in part because he treated infinity as actual. The second question is whether a proof of the existence of a mathematical object should actually produce the object asserted to exist; that is, whether existence proofs should be constructive. It is sometimes claimed that Cantor’s proof of the existence of transcendental numbers is nonconstructive, but, as Hamkins explains, a careful analysis of the proof shows that it is actually constructive.

Chapter 4 describes Euclid’s axiomatic approach to geometry. What is the purpose of geometry? Is it to study what can be proven from Euclid’s axioms, or perhaps what can be constructed by straightedge and compass? Or is it to discover the truth about space? Hamkins discusses the views of Kant and Hume about geometry. The story of the discovery of non-Euclidean geometry leads to the modern view, as expressed by Poincaré, that all geometries are equally mathematically correct. No geometry can be picked out as the one true geometry, although physical experiments might determine that one geometry is most useful as a representation of physical space.

The subject of mathematical proof is taken up in the next chapter. Hamkins explains several theorems of formal logic, such as the soundness and completeness of classical first-order logic. And he addresses a number of philosophical questions about proofs. What is the purpose of a mathematical proof? Is it merely to certify that a theorem is true, or is it to explain why it is true? Which is more convincing, a formal proof of a theorem, every step of which can (and must!) be checked, or an informal proof that conveys an understanding of why the theorem is true? As
Hamkins points out, the answer to this last question may change as proof assistant software becomes more powerful and its use becomes more widespread. (Proof assistants are programs, such as Isabelle, Coq, and Lean, that can help a mathematician write a formal proof and verify that the proof is correct. Further information about proof assistants can be found in [Avi18, Hal08].) Hamkins also discusses alternatives to classical logic, such as intuitionistic logic, although he does not explain in detail how one must revise many mathematical theorems if one pursues alternative logical systems, together with Wiles's proof, would constitute a contradiction in infinitary mathematics by proving the consistency of infinitary mathematics.

It might have been helpful if Hamkins had given a little more detail about how such a consistency proof would have guaranteed the reliability of infinitary methods for establishing finitary truths. To take an example, consider Wiles's proof of Fermat's last theorem. If, following Hilbert, we view this proof as just a meaningless arrangement of symbols that obeys the formal rules of infinitary mathematics, why should we trust the conclusion that Fermat's last theorem is true? One answer is that if there were a counterexample to the theorem, then that counterexample, together with Wiles's proof, would constitute a contradiction in infinitary mathematics. Thus, if we knew that infinitary mathematics was consistent, then Wiles's proof, even when viewed as a meaningless arrangement of symbols, would assure us that no such counterexample could exist. (Hilbert used Fermat's last theorem as an example in his 1927 lecture Die Grundlagen der Mathematik, an English translation of which can be found in [vH67, pp. 464–479]. For more details, see [Vel97].)

Unfortunately, Hilbert's hopes were dashed by Gödel's incompleteness theorems. Hamkins sketches the proofs of the incompleteness theorems and explains related results concerning decidability and definability.

Finally, in Chapter 8, Hamkins explains how set theory can serve as a foundation for all of mathematics. He presents the cumulative hierarchy of sets and the ZFC axioms (the Zermelo-Fraenkel axioms, including the axiom of choice) that describe that hierarchy. Then he explains that much of modern research in set theory is concerned with showing that various statements cannot be proven from the ZFC axioms (assuming those axioms are consistent), or that they are independent of the axioms—that is, they are neither provable nor disprovable. This includes, for example, the continuum hypothesis, as well as a long list of statements asserting the existence of various kinds of large infinite cardinal numbers. This raises a host of philosophical questions. Should any of these statements be accepted as new axioms? How do we choose axioms? Must axioms be statements that are intuitively evident, or can the fruitful consequences of a statement count as evidence that it should be accepted as an axiom? Which is a better foundation for mathematics, a weak theory that assumes only those axioms in which we have the most confidence, or a strong theory that describes mathematical reality more fully through the inclusion of additional axioms and therefore allows us to prove more? Is there a single "real" universe of sets, in which independent statements have their "correct" truth values? Or are there multiple universes of set theory—what Hamkins calls a "multiverse"—some in which the continuum hypothesis is true and some in which it is false, some containing various kinds of large cardinals and some with none?

Each chapter ends with a list of "questions for further thought." Some of these questions ask for solutions to mathematical problems, and some ask for philosophical reflection.

In the preface, Hamkins says that he has "tried to provide something useful for everyone, always beginning gently but still reaching deep waters." My impression is that, for the most part, Hamkins has student readers in mind; he often devotes considerable space to explaining ideas, such as the construction of the number systems, rigor in calculus, or the distinction between countable and uncountable sets, that will be familiar to all professional mathematicians. But there is still plenty of material that professionals will find interesting. To give just one example, he presents the standard proof of the irrationality of √2, which will be familiar to all mathematicians, but he follows that with a lovely pictorial proof, due to Stanley Tenenbaum, that I had not seen before. And, of course, even readers who are already familiar with the mathematical topics will find Hamkins's philosophical discussions stimulating and thought-provoking.

As the summary above makes clear, the book discusses a very wide range of topics. For most topics, Hamkins summarizes the main ideas and then provides suggestions for
further reading, and this approach usually works well. For example, when discussing the construction of the number systems, Hamkins explains how rational numbers can be defined as equivalence classes of pairs of integers (the second of which is nonzero) and how the basic arithmetic operations can be defined on these equivalence classes, but he doesn’t go through the tedious verification that these operations satisfy the field axioms. Similarly, the explanation of the incompleteness theorems gives a very good idea of how the proofs go without getting bogged down in the details of Gödel numbering.

However, occasionally Hamkins skips over enough details that readers may struggle a bit. For example, in the chapter on infinity, Hamkins introduces the infinite cardinal numbers $\beth_\omega$ by saying that “at each stage we apply the power set operation $\beth_{\alpha+1} = 2^{\beth_\alpha}$,” but as far as I can tell, the exponentiation operation on cardinal numbers was never defined. Will readers figure out that $\beth_{\omega+1}$ is the cardinality of the power set of a set of cardinality $\beth_\omega$?

Another example occurs when Hamkins is describing the cumulative hierarchy of sets. He says that the hierarchy is built up in stages, with each stage containing sets whose elements were constructed at earlier stages, but he doesn’t say that the stages are numbered by the ordinal numbers, he doesn’t introduce the notation $V_\alpha$ for the $\alpha$th stage of the construction, and he doesn’t introduce the terminology that the rank of a set is the stage at which it is constructed. As a result, readers may have a hard time understanding what he means a few pages later when he refers to “the set $V_{\omega+\omega}$, the rank-initial segment of the cumulative hierarchy up to rank $\omega+\omega$.” Hamkins also sometimes makes use of the distinction between sets and proper classes, but I was unable to find a place where this distinction was explained.

While most of the book is written at a level that an undergraduate student will understand, Hamkins occasionally refers to more advanced topics. For example, when explaining why some geometric constructions cannot be done with straightedge and compass, he uses some ideas from the theory of field extensions. And every so often he can’t resist quoting an interesting advanced idea that will go over the heads of many student readers. Thus, students will need to be willing to skip over the occasional passage that is not explained at their level. The book would work well as a set of readings for a seminar course, where the professor could help students deal with these occasional advanced topics.

The book is written in an engaging, conversational style, with numerous well-drawn figures. There are very few typographical errors. (The only one I found that could cause any confusion was the use of the word “transfinite” on p. 258 where Hamkins clearly meant “transitive.”)

Of course, Hamkins does not resolve all of the philosophical puzzles he discusses. So it is appropriate that he ends his book, not with a conclusion, but with an invitation: "Please join us in what I find to be a fascinating conversation." The book can serve as an interesting and enjoyable introduction to this conversation for a wide range of readers.

References


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