COMMUNICATION

Ars Mathemalchemica: From Math to Art and Back Again

Susan Goldstine, Elizabeth Paley, and Henry Segerman

Mathemalchemy was on display at the National Academy of Sciences in Washington, DC, from mid-January to mid-June, 2022. For information about other venues, as well as supplementary information about the history, development, and mathematical and artistic content of the installation, see `mathemalchemy.org`.

Mathemalchemy is a collaborative multimedia art installation that celebrates the creativity and beauty of mathematics (Figure 1). The project is headed by mathematician Ingrid Daubechies, a professor at Duke University, and artistic director Dominique Ehrmann, a fiber artist based in Québec (Figure 2A). Twenty-four core mathematicians and artists created the installation, with additional contributions from "adjuvant," "coset," "adjoint," and "apprentice" mathemalchemists.

Reveling in both mathematics and art, *Mathemalchemy* depicts multiple narrative scenes. Among them, a cat and mouse prepare tessellating cookies in a bakery; a bird gathers shiny mathematical treasures in a curio shop; herons tag a new species of aquatic knots in the bay; a tortoise slowly but surely ambles toward the end of Zeno's path with her Sierpiński kite in tow; chipmunks explore number games in the playground; squirrels chatter about prime number algorithms in the garden; and an octopus slips between two- and three-dimensional space to paint graffiti under a lighthouse whose beams project into the sky. The installation is fabricated with myriad media: beadwork, ceramics, crochet, embroidery, knitting, leatherwork, needlefelting, origami, painting, polymer clay, 3D printing, quilting, sewing, stained glass, steel welding, light, temari, weaving, wire bending, and woodworking.

Figure 1. The installation at its first public venue.

The artwork developed out of extended conversations between our large group of collaborators, who brought their different backgrounds and perspectives from mathematics, art, and the sciences. A recurrent theme of these discussions was the relationship between mathematics and art and how they would intersect in the fantasia we were constructing. Indeed, the authors of this paper are two professional mathematicians and one professional

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(A) Ingrid Daubechies and Dominique Ehrmann. (B) A spherical photograph of the installation surrounded by a subset of the mathemalchemists.

Figure 2. The installation at the completion of its construction at Duke University.

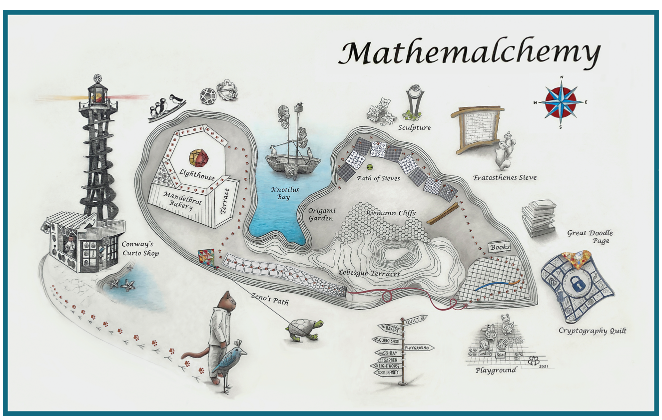

Figure 3. A map of the installation, by Bronna Butler. Compare with Figure 2B. Pawprints indicate the path that Harriet and Arnold take.

artist, though all three of us create art and explore mathematical ideas.

As we talked over how to present *Mathemalchemy* in this article, we kept returning to how our experiences within our respective fields had shaped our perceptions of this fabricated realm. In particular, the artist noted several colleagues in the visual and musical arts who self-deprecatingly deny having any aptitude for mathematics—despite using mathematics in their disciplines. Our discussions coalesced into an imagined dialogue between two of the characters that inhabit the installation, one responding to their world through a mathematical lens, the other through an artistic one. To highlight attitudes that we have internalized, we toy with some common tropes about mathematics and art—for example that art is accessible to all, while math is accessible only to some. Our characters interact with other creatures to reflect the challenges and pleasures of collaboration and communication across diverse fields of expertise. Their discussion also gives us an opportunity to tour parts of the exhibit. The map in Figure 3 shows the route that our protagonists take.

* * *

Harriet,[1] a bowerbird, has channeled her propensity for collecting mathematically intriguing objects into a thriving business. Her fine-math gallery, Conway's Curios,[2] is located in the heart of Mathemalchemy's vibrant downtown (Figure 4).

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[1] *Harriet is named after the Harriss Spiral, discovered by Edmund Harriss* [4]. *The spiral adorns her feathers and features prominently on the Curio Shop sign and wall.*

[2] *Named after John H. Conway, with the permission of his family. Like Conway's office, the shop is full of mathematical curiosities.*

(A) Harriet in her fine-math gallery, Conway's Curios.

(B) Arnold in his bakery, with his assistant Mo[u]se.[3]

(C) Harriet's brother Horton[4] runs the terrace café, accessed via the lighthouse ramp.

Figure 4. Harriet and Arnold meet on the terrace.

One evening, Harriet meets her friend Arnold,[5] a cat, on the terrace above the shop. Arnold is the chef at the Mandelbrot Bakery next door, and he has brought some award-winning cookies to share.

ARNOLD: *Bon soir*, bowerbird. How's the new installation progressing?

HARRIET: Felicitations, my favorite feline; thanks for asking. I've just hung a new Alexander Horned Sphere next to the Möbius Strip in the window, opposite the Hopf Fibration of the Three-Sphere. You must have caught a glimpse on your way to the café. What do you think?

ARNOLD: Captivating! I don't know math, but I know what I like. You sure have an eye for it though, Harriet.

HARRIET: Why, you do too, Arnold. Consider your cookies—don't they tessellate?

ARNOLD: Oh, that's art, not math. I created the intertwining forms as a metaphor for the intertwining ingredients, plus it prevents overworking the dough. Artistic intuition and a practiced paw shaped these delectable *objets d'art*.

HARRIET: Form follows flavor! You know, I wish I were better at art. I used to be pretty good at it, but after I fledged, my art teacher told me my "little bird brain" was better suited to math, and my art ability simply evaporated.

ARNOLD: Alas, I've heard many a mathematician say the same thing. Yet you obviously understand art: your gallery is overflowing with it!

HARRIET: Art? You've seen my inventory—Klein bottles, interlocking sliced tori, rhombicosidodecahedra, double helices, fractal trees, borromean rings...These are all *objets de mathématique*. I fill my shop with reflections of fundamental truths, Arnold—that is what math is all about.

* * *

Mathemalchemy began when Ingrid saw Dominique's quilted installation, *Time to Break Free* [6], in which a fantastical machine converts the flat figures on a traditional quilt into fully three-dimensional characters (Figure 5). This inspired Ingrid to wonder whether something like this could similarly bring the beauty and creativity of mathematical ideas to life.

Figure 5. *Time to Break Free*, an installation by Dominique Ehrmann.

[3] *The unusually punctuated name of Arnold's assistant is a tip of the hat to Vladimir Arnold's research collaborator Jürgen Moser, recognized in the name of the KAM (Kolmogorov-Arnold-Moser) Theorem* [5].

[4] *Conway's middle name.*

[5] *The folding and stretching of pastry dough brought to mind the stretching and compressing of dynamical systems; thus Arnold was named after mathematician Vladimir Arnold* [1].

Figure 6. Mathemalchemists, from left to right, top to bottom: Emily Baker, Bronna Butler, Edmund Harriss, Elizabeth Paley, Kimberly Roth, Edward Vogel; Dorothy Buck, Rochy Flint, Li-Mei Lim, Kathy Peterson, Henry Segerman, Jake Wildstrom; Ingrid Daubechies, Faye Goldman, Sabetta Matsumoto, Samantha Pezzimenti, Jessica K. Sklar, Mary William; Dominique Ehrmann, Susan Goldstine, Vernelle A. A. Noel, Tasha Pruitt, Daina Taimina, Carolyn Yackel.

Ingrid and Dominique presented at the Joint Mathematics Meetings in January 2020, and a group coalesced around the idea (Figure 6). When COVID preempted the first planned in-person workshop in March 2020, the participants began meeting regularly via Zoom. From an initial cornucopia of concepts, Dominique created a vividly detailed quarter-scale maquette that served as the basis for further discussion, revisions, prototyping, adjustments, and collaborative construction from afar (facilitated by the United States and Canadian postal services).

After 18 months of planning and long-distance making, participants came together in July 2021 at Duke University to assemble, trouble-shoot, and fine-tune the installation.

* * *

Arnold and Harriet continue their discussion with a stroll, exiting the terrace via the lighthouse ramp (Figure 4C). Between them and the ground are a pair of puffins who tend the lighthouse, absorbed in conversation.

HARRIET: I do enjoy a helical hop around the lighthouse.

ARNOLD: I keep thinking I should make a lighthouse-shaped pastry to celebrate this edifice.

HARRIET: Flavor follows form this time? Tell me more.

ARNOLD: Naturally, the flavor and texture must inspire thoughts of light. I've tried my paw at an occasional phyllo tower, but it's a challenge to get free-standing pastry to support itself.

DEL: Pardon, friends—could not help hearing
That pastry treats need engineering.

NABLA: Could steel beams here perhaps inspire
A means to make dough towers higher?

DEL: The long straight girders 'round axis Z
are shaped like Ts from sky to sea.[6]

NABLA: Attach the ramp with L-shaped brace
and phyllo dough should stay in place.

ARNOLD: Your thoughts for food are food for thought...I'll play around with that perpendicularity idea to see if it helps with the strength.

HARRIET: Aha! Geometry *does* inform your art.

DEL: He's won awards for pastries fine
when art, taste, math all intertwine!

ARNOLD: *Touché*, Harriet. Can you puffins also help this bowerbird recognize the art in her math?

NABLA: We'd stay and chat, but must away;
We're changing shifts, *adieu*, good day!

[6] *With apologies to non-American readers.*

(A) The surface S, with lines of principal curvature γ drawn in red.

(B) Strips meet along the curves γ, forming T- or L- shaped cross-sections.

(C) The lighthouse stands nine feet tall.

(D) The steel frame was cut and assembled at the University of Arkansas.

(E) A Fresnel lens focuses a beam of light through the stained glass dodecahedron.

Figure 7. Design and construction of the lighthouse. The renders in Figures 7A and 7B are by Sabetta Matsumoto.

* * *

We designed the art in the installation to incorporate mathematical details that would appeal as much to inquisitive non-mathematicians as to those devoted to the field. The mathematics includes expositions of classical results, demonstrations of recent mathematical innovations,[7] and mathematics developed specifically in response to the challenges of constructing our vision.

The construction of the lighthouse (Figure 7) is an example of this last connection. The key idea comes from Edmund Harriss and Emily Baker [3]. We want to represent a surface S (the shape of the lighthouse) by a grid of curves on S. Each such curve γ is built by bending then welding together two strips of metal, forming either a T- or L-shaped cross-section. Thin sheets of metal can be bent out of plane without too much effort, but twisting them is very difficult. After welding, each strip reinforces the bends we make in its partner, resulting in a rigid structure. Deforming the welded strips would require twisting the metal (or bending within the plane of the metal sheet).

One of the line segments of the T- or L- shaped cross-section is tangent to S while the other is perpendicular. These two directions together with the tangent to the curve γ form the *Darboux frame* for γ. Keeping the metal from twisting corresponds to this frame having no torsion. It turns out that this restricts γ to being a *line of principal curvature* of S.

Sabetta Matsumoto designed a surface S with interesting lines of principal curvature that would produce an attractive lighthouse. It is a perturbed hyperboloid of one sheet, with one of the lines of principal curvature giving a gentle spiral ramp. The lighthouse was physically built by Emily Baker and her students.

[7] *For example, the stained glass dodecahedron (Figure 7E) atop the lighthouse illustrates that unlike the other four platonic solids, the dodecahedron admits geodesic paths from a vertex back to itself that don't cross any other vertices, a result that was published in May 2020* [2].

(A) Visiting knot theorists Adélie, Chinstrap, and Gentoo in the penguin's nest.

(B) The herons lead a knot tag-and-release program.

Figure 8. Teamwork is essential at sea (and in many other contexts).

* * *

Having reached the base of the lighthouse, Harriet and Arnold are delighted to hear music coming from the bay (Figure 8).

HERONS [SINGING]: Come put these knots in order
'cause they've tangled in the water,
Heave away, me jollies, heave away;
Come put these knots before us, hyperbolic, sat'lite, torus,
Heave away, me jolly birds, we'll all heave away!

PENGUINS [SHOUTING]: Ahoy, cousin bird, ahoy friend cat!

HARRIET: Look Arnold! Up in the crow's nest—penguins!

ADÉLIE[8]: Where we come from, we call it a penguin's nest, thank you very much.

ARNOLD: Have you traveled far, friend birds?

HERONS: I wrote me knot a letter, 'twas a Theta sweet and smooth,

[8] *Many of our characters are named after species. Chinstrap, Adélie, and Gentoo are species in the genus Pygoscelis.*

Heave away, me jollies, heave away;
Then along came Reidemeister—twist, poke, slide—he made his move.
Heave away, me jolly birds, we'll all heave away!

CHINSTRAP: Quite far! We're visiting scholars, knot theorists by trade. We're here to assist the herons with their wavebreaking baywork—

ADÉLIE: —That's groundbreaking fieldwork, as you land mammals say—

CHINSTRAP: They've started a critical tag-and-release program to study the behavior of—

HERONS: Sometimes a spotted Cinquefoil, sometimes striped Stevedore,
Heave away, me jollies, heave away;
But now we've found empirically what theorists thought were lore!
Heave away, me jolly birds, we'll all heave away!

GENTOO: What the herons sing is true. We did not expect that *amphibichiral* theta-curves[9] could realistically survive in brackish water, so when the herons notified us of their newly discovered invertiblate[10]—

ADÉLIE: —well naturally, we came to assist. They have a completely different perspective on the deck down below than we have up here from this tower. Thanks to their preliminary data—

CHINSTRAP: —we're now able to computationally model the spawning locations—

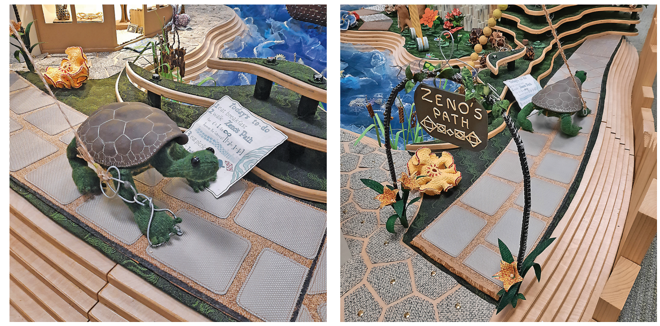
GENTOO: —and the herons are classifying new knots by the netload!—

HERONS: Heave away, me jolly birds, we'll all heave away!

* * *

From concept to realization, we benefited immensely from the diverse skills and expertise of our twenty-four core collaborators. The seeds planted through *Mathemalchemy*'s regular large and small group discussions germinated into playful layerings of mathematical concepts within scenes, as well as intersections and recurring themes across scenes. Mathematics inspired narrative elements that inspired artistic design. In tandem, artistic vision inspired narrative elements that shaped mathematical design.

At the same time, we faced real challenges in how to fit the contributions from so many different perspectives together. The mathematical content had to be correct and mathematically interesting, and the assembled work had to be structurally sound. But a cohesive whole also required making aesthetic and materials-related judgment calls. We learned to appreciate—or at least have patience with—one anothers' different disciplinary priorities and modes of communication. Compromise, flexibility, and trust were essential from conception to fabrication to installation. We embraced variety and abundance as part of our aesthetic, and relied on Dominique's keen artistic eye to keep us on track.



(A) Tess the tortoise and her to-do list. (B) Zeno's path.

Figure 9. Tess keeps herself busy by walking Zeno's Path every day.

* * *

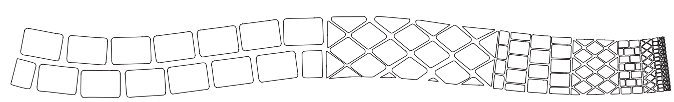
The two friends arrive at Zeno's Path and enter through the ornamental gate (Figure 9).

HARRIET: Oh, look, Tess is up ahead! I haven't talked to her in ages.

ARNOLD: Let's catch up!

HARRIET: Arnold, you try racing on Zeno's Path every time we come here, and it never works.

ARNOLD: I'm sure it will work this time—I've been training, and Tess walks so much slower than we do. Ready, set...



ARNOLD: Well, that's odd. We've been almost to the end for a while now, and we still haven't caught up.

HARRIET: This trail always takes forever. Why don't we just aim for the playground instead?

Some time later, Harriet and Arnold arrive at the edge of the garden, where chipmunks are enthusiastically arranging acorns on the playground (Figure 10).

HARRIET: How delightful—the chipmunks are playing their prime numbers game.

ARNOLD: Evening, kids! Who's winning?

UINTA: We both are. We're practicing for a doubles tournament, working on technique and artistry.

DURANGO: You might think Pickle-Prime is all about the numbers—

[9] *Knotted theta-curves* [8] *are a current topic of study. "Amphibichiral" is a made up word.*
[10] *This is also made up.*

Figure 10. Uinta and Durango play a game with Babylonian cuneiform numeral tiles and acorns.

UINTA: —determining whether a game tile is prime or composite—

DURANGO: —but acorn placement is becoming increasingly important to the judges.

UINTA: We've decided freestyle layout is just too …distracting. Makes us want to eat all the acorns. So we're going with this parallel placement instead.

HARRIET: And the braided rings?

DURANGO: They remind us not to eat the remainders.

* * *

The pandemic forced us to substantially extend the initial timeline for completing *Mathemalchemy*, as six months stretched into almost eighteen. That extra time allowed us to refine our underlying story-telling. The initial proposal for *Mathemalchemy* depicted studious mice at work, not play, and woe to a tearful mouse who struggled to understand prime and composite numbers. Our Garden subteam set about changing the narrative: now, two inquisitive chipmunks actively engage in collaborative, hands-on play with tangible three-dimensional objects, reveling in their discoveries as much as in the game's outcomes. We similarly hope *Mathemalchemy* itself offers viewers a vehicle for joyfully engaging with mathematics.

* * *

Harriet and Arnold leave the chipmunks to their practice and continue through the garden, stopping at the gaussian-integer stepping stones to watch the squirrels (Figure 11).

ARNOLD: Looks like the squirrels are getting ready for their annual Sieving of the Primes!

HARRIET: Gets bigger every year.

ARNOLD: How many sieves do they have? 2…3…5, and now 7 and 11 are rolling in…Where do they store them?

HARRIET: Behind the Riemann Cliffs. I used to lend them storage space in Conway's Curios, but they kept adding sieves. Even without the sieves, I hardly have room to turn around.

Figure 11. 3D printed polyhedra appear as stamens and pistils for origami flowers in the garden and as sea creatures in the reef. In the background, two squirrels discuss prime number algorithms in front of their Sieve of Eratosthenes.

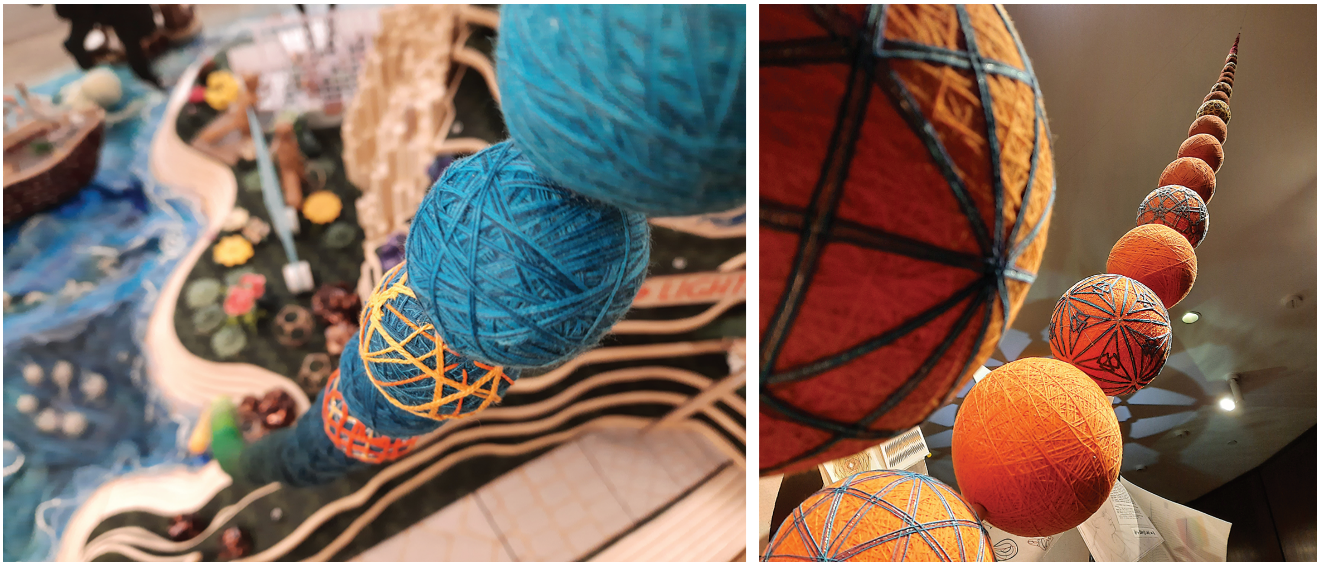
* * *

Just like Harriet, we had to make some decisions about what to include in the installation. Our criteria included mathematical and artistic considerations. Despite our extra months, we were also restricted by time.

Over one hundred small, 3D printed models of polyhedra adorn *Mathemalchemy* (Figure 11). We have the platonic, archimedean, and Johnson solids, and the first few prisms and antiprisms, totalling over a hundred pieces. These could have been made in ceramics, paper, or wood, but it would have taken far longer. By 3D printing them, we were able to save time on the manufacturing end, which allowed more time for Tasha Pruitt to envision how they could be incorporated into the exhibit. Supplemented with paint, embroidery, and beadwork, they manifest as stamens and pistils, boulders, cushions, planters, jellyfish, and other imaginative adornments.

Decisions about how to optimize quality and quantity also played a role in the two arches of thread-wrapped balls that rise above the garden (Figure 12). One arch illustrates a geometric series, while the other illustrates a diverging series.[11] The visible portions of these series involve over 120 woven balls, a selection of which are embroidered following a form of Japanese folk art called *temari*. Early on, it became clear that embellishing all of the balls was neither feasible nor visually desirable, so the team faced the

[11] *The ratio of the* $(n+1)$*st ball diameter to the nth is* $(n/(n+1))^{2/3}$.



(A) The diverging ball arch plunges into the bay.

(B) The converging ball arch reaches upwards.

Figure 12. The ball arches alone required hundreds of hours of work.

mathematical, practical, and aesthetic question of which to embellish and which to leave plain. The solution was to embroider the 22 balls at the indices of the twin primes. This choice extends the theme of primes associated with the garden below, yields a manageable number of temari, and gives a pleasingly irregular arrangement.

* * *

ARNOLD: You do have a lot of mathematical pieces in the gallery. Is it difficult to curate such a large collection?

HARRIET: I try to make objective decisions, so I rely on basic principles of the field. Simplicity, clarity, parsimony. . . universal facts—I always try to have some of the classics on hand. Of course, it's also nice to have contemporary work; unexpected connections between distant fields. And elegance; I'm so fond of elegant proofs. . .

ARNOLD: What makes something elegant?

HARRIET: Something is elegant when. . . well, when. . . it's. . . elegant. Hmm. On reflection, elegance is not so easy to define. I know it when I see it.

ARNOLD: That doesn't sound terribly objective.

HARRIET: Now you mention it, topics do go in and out of vogue in mathematics. . . Rigor is particularly popular these days, possibly at the expense of intuitive understanding. . .

ARNOLD: Wait wait wait, aren't those all matters of aesthetics?

HARRIET: Aesthetics? Aesthetics. I suppose so...

ARNOLD: Why Harriet, I believe you have a mind for art after all.

* * *

In our imagined narrative, Harriet is about to rediscover her confidence in matters of art, overcoming the internalized lessons of her youth. In the world the authors inhabit, it is mathematics, not art, whose students often receive the message that this domain is not meant for them. *Mathemalchemy* harnesses art and narrative to invite viewers to participate in the joy of mathematics. This invitation is embodied by three human silhouettes experiencing mathematics at different stages of life: a child improvises music with her trumpet; a teenager surfs across a swirling vortex of ideas, trailing a pennant that whips in the wind; and an adult releases an exuberant cavalcade of mathematical notes and jottings into the world (Figures 13A and 13B). Whether they are observing *Mathemalchemy* or imagining it into existence is up to the viewer to decide.

The three silhouettes are deliberately coded as female. This choice stems from a desire on the part of the core mathemalchemists, twenty of whom are women, to support the cultural shift that is making mathematics a more gender-inclusive field. In the installation, we celebrate the mathematical work of many women. A "Great Doodle Page" quilt floats with the cavalcade above the playground (Figure 13C); it features drawings by women mathematicians, including Maryam Mirzakhani, to date the only woman awarded a Fields Medal. On the artistic side, *Mathemalchemy* incorporates a variety of art forms such as crochet, knitting, and cross stitch, that the fine arts community has traditionally dismissed as "crafts" or "women's work" but that are weaving their way into fine art. The quilted bakery floor is appropriately adorned with a

(A) Perched on a stack of books, the child blows into a trumpet.

(B) A cavalcade of mathematical pages flutters into the world.

(C) The Great Doodle Page features doodles by seven women mathematicians.

Figure 13. We inspire and are inspired by mathematics.

convex pentagonal tiling discovered by amateur mathematician Marjorie Rice in her kitchen. Of the now-proven complete list of fifteen pentagonal tiling types, Rice discovered four, working in her spare time [7]. Her story serves as a reminder that mathematicians can be anyone, anywhere. Our hope is that *Mathemalchemy* fulfills a similar purpose, inviting its audience to see mathematics as an enterprise open to all intrepid explorers.

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Credits

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