Topology: A Categorical Approach
Reviewed by Jose Ceniceros

This textbook is aimed at first-year graduate students who have a solid undergraduate background in linear algebra, abstract algebra, and basic set theory, as well as some knowledge of point-set topology. The authors make it clear that the main focus of *Topology: A Categorical Approach* is to present point-set topology from a categorical perspective. Therefore, the textbook does not include topics such as covering spaces, homology, and cohomology. The authors provide the reader with an extensive list of references for algebraic topology textbooks that cover the aforementioned topics.

I have several topology textbooks on my bookshelf at the office that I use to teach from or to recall some obscure theorem that I have forgotten. The point-set topology textbooks I am most familiar with are Munkres [Mun00], Armstrong [Arm83], and Kelley [Kel75]. All of these start with an introduction to set theory. *Topology: A Categorical Approach* takes a different tactic and instead the reader is first presented with some basic point-set topology definitions and basic category theory.

Chapter 0 provides the reader with the necessary preliminaries required to understand upcoming chapters. In section 0.1, the authors collect definitions and basic concepts related to topological spaces. Section 0.2 is a concise introduction to category theory. The definition of a category is presented, and the authors spend some time describing conditions satisfied by a category through commutative diagrams. This section provides a substantial list of categories most students have worked with in their undergraduate courses. Within the list of categories, the reader will find the category $\mathcal{Top}$, consisting of topological spaces as objects and continuous functions as morphisms. In addition to the list of categories, the authors explicitly show that vector fields as objects and linear
transformations as morphisms form a category. This example provides enough detail for someone encountering the definition of a category for the first time. This section ends by introducing the theme that will run throughout the rest of the textbook: an object is completely determined by its relationships with other objects. Although this section is a brief introduction to category theory, there is enough detail given to allow the reader to understand the basics. Sections 0.2.2 and 0.2.3 introduce the definition of a functor and natural transformations, respectively. In Section 0.2.3, the following result is stated and discussed in detail:

**Lemma (Yoneda Lemma).** For every object \( X \) in \( C \) and for every functor \( F : C^{\text{op}} \to \text{Set} \), the set of natural transformations from \( C(\cdot, X) \) to \( F \) is isomorphic to \( FX \),

\[
\text{Nat}(C(\cdot, X), F) \cong FX.
\]

The Yoneda Lemma roughly says that an identification can be made between an object and maps into the object. The authors use the Yoneda Lemma to support the idea that we can better understand an object by studying its relation to other objects. This chapter concludes by recalling basic definitions and results from set theory. The authors do an excellent job in Chapter 0 of reviewing the basics needed throughout the rest of the textbook and giving proper motivation for the upcoming material.

In Chapter 1, the authors shift their attention to the construction of new topological spaces from other topological spaces. Section 1.1 starts with examples of spaces as well as continuous functions. The authors introduce four basic constructions starting with subspaces, then quotients, followed by products, and lastly coproducts.

Each construction is described in three ways. The first description is an explicit construction of the topological spaces and is similar to the definition found in previous textbooks, such as [Mun00, Arm83]. The second description is given as the coarsest or the finest topology for which maps into or out of the space are continuous. The last description of each construction involves a universal property. For example, the following universal property for the subspace topology is discussed.

**Theorem.** Let \((X, \tau_X)\) be a topological space, let \(Y\) be a subspace of \(X\), and let \(i : Y \hookrightarrow X\) be the natural inclusion. The subspace topology on \(Y\) is characterized by the following property: For every topological space \((Z, \tau_Z)\) and every function \(f : Z \to Y\), \(f\) is continuous if and only if the map \(if : Z \to X\) is continuous.

Along with the three descriptions of each construction, the authors provide examples that illustrate the differences between the descriptions.

The focus of Chapter 2 is the interaction between the constructions introduced in Chapter 1 and the topological properties of connectedness, Hausdorff, and compactness. In Section 2.1, the basic definitions, results, and examples of connectedness are introduced. The formal proofs for connectedness are left out, but the authors include several references for the interested reader. In Sections 2.2 and 2.3, the properties of Hausdorff and compactness are introduced. Section 2.3 also contains classic results such as the Bolzano-Weierstrass theorem, the Heine-Borel theorem, and Tychonoff’s theorem. In addition to providing a list of definitions and theorems the authors provide insight into some of the results. For example:

**Theorem.** Let \(X\) be a space and \(f : X \to Y\) be a surjective map. If \(Y\) is connected in the quotient topology and if each fiber \(f^{-1}y\) is connected, then \(X\) is connected.

The authors emphasize that this result fits into a much more general strategy in mathematics of extending knowledge of parts to knowledge of the whole. As someone who teaches at a liberal arts college, I appreciate that the authors note the significance of the result and how it fits into the larger picture of mathematics rather than just presenting the result and proof.

Chapter 3 takes a slightly different approach than Chapter 2. In Chapter 2, topological properties were introduced, and the interaction with category theory was discussed. In Chapter 3, the authors explore topological properties familiar to analysis students and take a break from discussing topology from a category theory perspective. The basic definitions and results regarding the closure, limit points, and sequences are introduced in Sections 3.1 and 3.2. The chapter continues with the introduction of filters along with some essential results. In Section 3.3, filters are used to characterize the Hausdorff property, closure, and continuity. The chapter ends with a proof of Tychonoff’s theorem that makes use of filters.

Chapter 4 returns to category theory. The material in this chapter is not limited to just the category Top, but is instead defined in general for any category. Section 4.1 starts with an intuitive discussion of limits and colimits. The formal definitions of a limit and a colimit are presented in Section 4.2. The chapter ends with examples along with a useful table that summarizes common categorical limits and colimits.

Chapter 5 starts with the following quote by Freeman Dyson [Dys09]:

*Birds fly high in the air and survey broad vistas of mathematics out of the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time.*

This quote sets the tone for the remaining two chapters. In Chapter 5, the authors introduce adjoint functors. This
chapter follows a similar structure to that of the previous chapters. In Sections 5.1 and 5.4, the authors present basic definitions and results about the adjoint functor, respectively. In Sections 5.2, 5.3, and 5.5, the authors provide the reader with examples that include free constructions in algebra, forgetful functors from Top, and the Stone-Čech compactification. The remaining portion of the chapter is devoted to carefully defining a topology on the set of continuous functions from space $X$ to a space $Y$. This chapter provides a balance of both big picture ideas and enough detail for topology students of all levels to find useful.

In the last chapter of the textbook, the authors return to the central theme introduced in Chapter 0, that information can be obtained about an object by analyzing its relationship to other objects. In particular, the authors define the fundamental group of a topological space and examine it from a category theory perspective. In Sections 6.1 and 6.2, the authors introduce homotopies and homotopy classes of loops with a basepoint, respectively. Then Section 6.3 introduces the fundamental group of a topological space with a basepoint. In Sections 6.4 and 6.5, the authors introduce the smash-hom adjunction and the suspension-loop adjunction, respectively. These adjunctions are introduced in order to prove $\pi_1(S^1) \cong \mathbb{Z}$ in Section 6.6. The textbook ends with Section 6.7, which introduces the Seifert-Van Kampen theorem and presents several examples.

In summary, the authors have written a textbook that should be accessible to most first-year graduate students. Each chapter ends with a set of exercises that range from elementary to challenging. Many of the proofs of key results are omitted, with several proofs left as exercises for the reader. This textbook is an excellent guide for students transitioning from undergraduate mathematics to graduate-level mathematics. It would be fantastic for an advanced undergraduate in an independent study course or for graduate students to read on their own.

This book is an excellent resource for students trying to deepen their understanding of point-set topology. Additionally, the textbook serves as a concise introduction to category theory. The authors include a substantial bibliography for readers interested in continuing their study of point-set topology and category theory. Students that are willing to read with pencil and paper in hand will gain a great deal of knowledge from this textbook.

References


[Dys09] Freeman Dyson, Birds and frogs, Notices Amer. Math. Soc. 56 (2009), no. 2, 212–223. \[MR2483565\]


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