Isadore M. Singer (1924–2021) In Memoriam Part 1: Scientific Works

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Introduction

This Memorial Collection, which will appear in two consecutive issues of the Notices of the AMS, celebrates the life and work of Isadore Singer, one of the most influential mathematicians of the past 75 years. Singer was born on May 3, 1924 and died on February 11, 2021. Various easily accessible sources recount in depth the rich story of his life.[1] We have decided not to duplicate them here. The corresponding Memorial Collection for Atiyah covers (among other things) the most famous work of both Atiyah and Singer, the Index Theorem.[2] In this first issue of the present collection, we have sought to emphasize those aspects of Singer's work which were not joint with Atiyah. The second issue consists of reminisces of colleagues and family members.

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[1] *The series Science Lives on the Simons Foundation website,* https://www.simonsfoundation.org/2009/08/25/isadore-singer/; *the interview on the occasion of the 150th anniversary of MIT,* https://www.youtube.com/watch?v=5FoaMcCJnmQ; *the shelf on* https://celebratio.org/Singer_IM/article/652/; *the Abel Prize Interview with Atiyah* https://www.youtube.com/watch?v=UOv9wJyPGUQ.

[2] https://www.ams.org/journals/notices/201910/rnoti-p1660.pdf; https://www.ams.org/journals/notices/201911/rnoti-p1834.pdf.

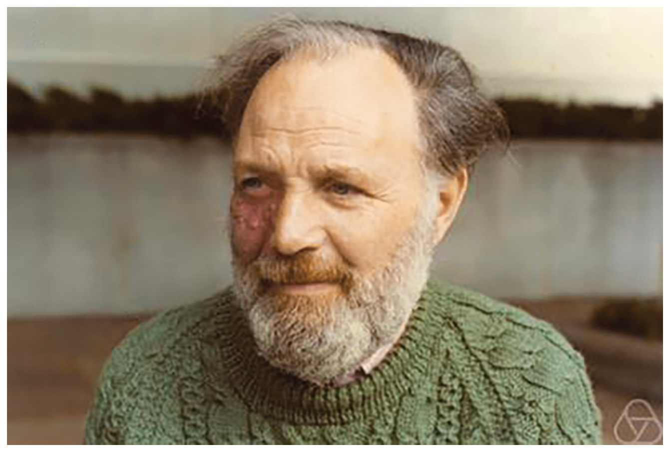

Figure 1. Singer in Oberwolfach, 1977.

Jean-Michel Bismut

In 1963, in the *Bulletin of the AMS*, Atiyah and Singer announced the proof of their index theorem. In 1967, in a landmark paper, McKean and Singer [14] suggested that the heat equation could be used to prove the index theorem, at least for the de Rham complex and the associated Euler characteristic. They discuss the Minakshisundaram small time expansion of the heat kernel of the Laplacian on the diagonal, whose coefficients depend only locally on the metric. They apply the method to the Hodge Laplacian acting on differential forms. They discover the simple fact that the alternating sum of the traces of the time dependent heat kernel on the forms of various degrees is just

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the time independent Euler characteristic. Therefore the singular terms in the small time expansion of the alternating sum of the traces of the heat kernel on the diagonal (which can be computed locally in terms of the metric) integrate out to 0, and the integral of the constant term is just the Euler characteristic. And here is a prophetic statement: "It is natural to hope that some fantastic cancellation will also take place *in the small*," i.e., locally, and that the local constant term in the expansion will coincide with Chern's explicit formula for Chern–Gauss–Bonnet in terms of the curvature of the Levi-Civita connection. Here *cancellation* refers to the fact that even though the local traces in various degrees are known to be singular when time tends to zero, it is expected that the alternating sum will cancel the singularities. And they continue: "The even dimensional proof eludes us," except in dimension 2, and they leave the question as an open problem.

At the same time [3], Atiyah and Singer had discovered the Dirac operator in the context of index theory. This is a first-order differential elliptic operator that exists on any compact Riemannian oriented spin manifold,[3] of which the de Rham–Hodge operator for the de Rham complex, the Dolbeault–Hodge operator for complex manifolds, and the signature operator were already known special cases. The Levi-Civita connection appears explicitly in the construction of the Dirac operator. The square of the Dirac operator is a generalized Laplacian given by Lichnerowicz's formula, and McKean–Singer heat equation formula extends to a formula for the index of the Dirac operator.

Several major steps were made to establish that *fantastic cancellations* also occur for Dirac operators on even-dimensional oriented spin manifolds, in which the local constant term in the asymptotic expansion in the difference of the traces of the heat kernel on even and odd spinors is obtained using the explicit local Chern–Weil natural representative in the corresponding index formula. First came the proofs by Patodi that such cancellations occur for the de Rham–Hodge operator, and for the Dolbeault–Hodge operator for Kähler manifolds, then the proof by Gilkey for the signature operator. Subsequently, Atiyah, Bott, and Patodi established this result for an arbitrary Dirac operator, and showed that proving the index theorem for the Dirac operator is enough to establish the index theorem for arbitrary elliptic operators. The proofs of the local cancellations are based on the fact that the coefficients of the local expansion of the difference of the traces of the heat kernel are universal polynomials of the curvatures and their covariant derivatives for the considered vector bundles.

In 1975, Atiyah–Patodi–Singer [2] established an index theorem for even-dimensional oriented spin manifolds with boundary.[4] The Dirac operator and the above local cancellations play a key role in the proofs of this result. Of necessity, global boundary conditions appear, so that the domain of the considered Dirac operator consists of sections whose restrictions to the boundary lie in the direct sum of eigenspaces associated with the nonnegative (resp. negative) eigenvalues of the Dirac operator on the boundary. Progress made by Atiyah–Patodi–Singer in their choice of global boundary conditions was crucial in their treatment of the signature formula for manifolds with boundary. When the metric is a product near the boundary, there are two distinct contributions to the index. One is local in the interior, and is just the usual Chern–Weil differential form that appears in the case of closed manifolds. The other is a global spectral invariant of the Dirac operator of the boundary, the η-invariant. The η-invariant is the value at 0 of a function $\eta(s)$ that is well-defined for $\operatorname{Re} s >> 0$. The fact that it extends by analytic continuation to $s = 0$ is one main result proved by Atiyah–Patodi–Singer. The η-invariant should be thought of as a renormalized version of the difference between the numbers of positive eigenvalues and negative eigenvalues of the Dirac operator on the boundary, and is in general not an integer. The variation of $\eta(0)$ in $\mathbf{R}/\mathbf{Z}$ with respect to metrics and connections can be computed by an explicit local formula.

The above result was used by Atiyah, Donnelly, and Singer to give a proof of the Hirzebuch conjecture expressing the signature defect of Hilbert modular varieties in terms of the value at 0 of Shimizu's L-function. Müller gave an independent proof based on an L_2 index theorem and on Selberg's trace formula.

In his work on L_2 cohomology [10], Cheeger discussed the signature on manifolds with isolated conical singularities and also obtained a formula for the signature of such manifolds, in which the contribution of the cone tip is given by the η-invariant of the cross section. While implicitly, Atiyah–Patodi–Singer's analysis takes place on the infinite cylinder attached to the boundary, Cheeger's analysis is made instead on an attached cone. It is interesting to note that the cone structure encodes the global boundary conditions of Atiyah–Patodi–Singer in implicit form.

While Singer was working with Atiyah on all aspects of index theory, Ray and Singer made two striking contributions that, at the time, seemed to have no relation to index theory and were concerned with the whole spectrum of the Hodge Laplacians.

[3] *The spin condition is a global topological condition.*

[4] *Their index theorem is valid for more general operators. Here, we only review the case of Dirac operators.*

In 1971, Ray and Singer [17] consider the Reidemeister torsion of a compact odd-dimensional manifold equipped with a unitarily flat vector bundle, a combinatorial invariant. On the other hand, given a Riemannian metric, they introduce the analytic torsion, a spectral invariant of the Hodge Laplacian, given by a suitable linear combination of the derivative at 0 of the zeta function of the Hodge Laplacian in various degrees, which they prove does not depend on the choice of metrics. The proof uses the fact that the small time asymptotic expansion of the trace of the heat kernel does not contain a constant term. They verify that these two invariants behave in the same way under natural operations on the manifolds. Because of this similarity of behavior, they made the stunning conjecture that the two torsions should be equal.

When properly rephrased, the Ray–Singer conjecture asserts that two natural metrics on the determinant of the cohomology of the flat vector bundle, one combinatorial, and the other analytic, should be equal.

The conjecture was first proved independently by Cheeger [9] and Müller [15]. The history of the proof is discussed further in this collection.

In 1973, in a subsequent paper [18], Ray and Singer considered the case of compact complex Kähler manifolds, replacing the de Rham complex by the Dolbeault complex twisted by unitarily flat vector bundles, and they define the corresponding holomorphic torsion. Because the real dimension is even, holomorphic torsion now depends locally on the metric. Still, they verify that, given two unitary representations, the ratios of the holomorphic torsions remain constant. They specialize the construction to Riemann surfaces. In the case of one-dimensional complex tori with a flat metric, using Poisson's summation formula, they express the holomorphic torsion in terms of ϑ-functions and of the Dedekind η-function. For Riemann surfaces of higher genus, they use instead Selberg's trace formula to evaluate the ratio of the torsions in terms of the value at 1 of the Selberg zeta function.

It is impossible to overestimate the value of these two papers for future developments. In the case of de Rham torsion, Ray and Singer formulated a simple and powerful conjecture. In the case of holomorphic torsion, they uncovered the tip of the iceberg, its relation to complex algebraic geometry.

But even more was to come: the fact that these two theories have deep relations to the index theorem.

The index theorem was proved just before the revolution introduced in mathematics by modern quantum physics. The fact that index theory could be relevant to questions of modern physics was deeply tantalizing to both Atiyah and Singer. They both embraced the new ideas coming from physics enthusiastically.

The importance of connections to physics already appeared at the very birth of Chern–Weil theory, part of which already emerged inside Yang–Mills theory. That physicists were working with explicit models insured that secondary objects in mathematical index theory, which did not have a name in mathematics, appeared naturally inside the physical model, as in questions related to anomalies. What follows is a brief review of some developments in mathematical index theory with connections to physics, but with limited reference to the important physics literature.

In 1982, in a paper in the *Journal of Differential Geometry*, Witten gave an analytic proof of the Morse inequalities that used a deformation of the de Rham operator associated with a Morse function, and he conjectured that instanton effects could ultimately be responsible for the computation of the Betti numbers from the critical points. This was later proved by Helffer and Sjöstrand. Witten's arguments were further elaborated by Zhang and myself to give another proof of the Ray–Singer conjecture.

Álvarez-Gaumé introduced a new approach to the index theorem of Dirac operators that was based on supersymmetry. This is best understood in the Lagrangian formalism, in which the trace of the heat kernel is evaluated in terms of a path integral on the loop space of the manifold. The path integral for the heat supertrace of the Dirac operator involves a supersymmetric Lagrangian that, beyond the Bosonic classical energy, also contains Fermionic supersymmetric variables. This idea was reformulated in the case of Dirac operators acting on spinors by Atiyah and Witten, who viewed the Lagrangian as a differential form on the loop space that is closed with respect to an equivariant de Rham operator. They obtain the index formula as a consequence of a formal equivariant cohomology localization of the integral on the manifold itself. This is best explained by referring to the localization formulas of Duistermaat–Heckman and Berline–Vergne. These are formulas expressing the integral of certain differential forms on manifolds equipped with the action of a torus as another integral evaluated on the zero set of the vector field generating the torus action. The argument of Atiyah and Witten consists in formally extending the argument to the loop space of a manifold, using the natural action of the circle on the loop space, while noting that the original manifold is exactly the zero-set of the associated vector field and, finally, in observing that the index formula appears to be a formal consequence of equivariant localization.

The first most significant progress in rigorous mathematics was Getzler's approach which incorporates the Clifford variables in a supersymmetric rescaling method, so that the $\widehat{A}$-genus appears as the value on the diagonal of the heat kernel associated with a harmonic oscillator. Further progress was made when we gave a probabilistic proof of the index theorem and where contact was made between the formal arguments of equivariant localization, and the *fantastic cancellations* being viewed as a "consequence" of equivariant localization for all the twisted Dirac operators.

In a 1985 paper published in *Topology*, Quillen invented the concept of superconnection, an object that combines connections and operators, that led eventually to the possibility of putting together Chern–Weil theory and index theory.

Given a proper submersion of smooth manifolds, with corresponding elliptic operators acting along the fiber, this family defines a virtual vector bundle on the base of the fibration. When the kernel and cokernel of the fiberwise elliptic operators are smooth, the virtual vector bundle is just the formal difference of the kernel and the cokernel. The families index theorem of Atiyah–Singer consists in the computation of the Chern character of this virtual vector bundle.

Using superconnections, I gave a heat equation proof [5] of a version of the families index theorem for families of Dirac operators. The proof is local on the base and uses the formalism of superconnections of Quillen in an infinite-dimensional context, with an associated Levi-Civita superconnection. The proof can be viewed as an *adiabatic limit* of the proof of the local index theorem for a global Dirac operator, when the metric of the base is blown up to infinity. Chern–Simons transgression is naturally part of the superconnection formalism.

Other advances were made on holomorphic torsion. Motivated by the work of Faltings on Mordell's conjecture, Quillen [16] introduced the Quillen metric on the determinant of the cohomology of a family of Riemann surfaces, which is constructed using the Ray–Singer holomorphic torsion of the fiber. In the case of a trivial family of Riemann surfaces equipped with the family of complex structure on a given vector bundle, he proved a curvature theorem for the first Chern form of the determinant line bundle equipped with the Quillen metric.

This was the beginning of new developments that incorporated the η-invariant, holomorphic and real torsion as part of more general globalinvariants naturally obtained by transgression in the superconnection formalism. While the index theorem expresses the index (associated with the zero eigenvalue of the Dirac operator) as a local quantity, these secondary invariants involve the full spectrum of the Dirac operator or of its square and provide the connecting machine between the index and its local expression.

Motivated by the work of Quillen on holomorphic torsion, Freed and I proved a curvature theorem for a canonical unitary connection on the determinant line bundle of a smooth family of Dirac operators in even dimension. A conjecture of Witten expresses the holonomy of the determinant line bundle of a family of Dirac operators along a closed curve as the adiabatic limit of the η-invariant of the associated cylinder when the metric of the base is blown up, and was solved independently by Freed and me in a 1986 paper in *Communications in Mathematical Physics* and also by Cheeger in a 1987 paper in the *Journal of Differential Geometry*. For families of odd-dimensional, oriented spin manifolds, the η-invariant of the fibers appears as the piece of degree 0 of a family of even forms, the $\tilde{\eta}$-forms, obtained by transgression by Cheeger and me [6]. For families of even-dimensional manifolds, the $\tilde{\eta}$-forms are odd, the component of degree one being a connection form on the corresponding determinant line bundle. With Cheeger [6], we showed that the adiabatic limit[5] of the η-invariant of a fibered manifold can be expressed in terms of such $\tilde{\eta}$-forms. With Cheeger, we also proved an index theorem for a family of manifolds with isolated conical singularities [7] that extends the earlier index theorem of Cheeger, where the $\tilde{\eta}$-forms now appear as the contribution of the singular strata. This can also be viewed as a families version of the APS theorem for manifolds with boundary.

Quillen's theory on holomorphic torsion was extended in two directions. With Gillet and Soulé [8], we considered the Knudsen–Mumford holomorphic determinant line bundle[6] associated with a proper holomorphic projection of complex manifolds. The determinant line bundle was equipped with the Quillen metric coming from the Ray–Singer holomorphic torsion. We proved that the Quillen metric is smooth. In the case of locally Kähler fibrations,[7] a curvature theorem was established, stating that the curvature of the determinant line bundle equipped with the Quillen metric is given by the integral along the fiber of the Riemann–Roch–Grothendieck natural characteristic forms in Chern–Weil theory. This result is related to my previous result with Freed, but does not follow from it. It puts the families index theorem of Atiyah–Singer and the holomorphic torsion of Ray–Singer in the same basket: Holomorphic torsion is part of a secondary theory that refines the families index theorem of Atiyah–Singer at the level of differential forms. In particular, the Polyakov anomaly formulas for the determinant of the Laplacian on Riemann surfaces were shown to be part of the Riemann–Roch–Grothendieck formula for the curvature of an

[5] *Given a metric on the manifold, we add a large multiple of the pull-back of a metric on the base.*

[6] *This is the line bundle constructed by Knudsen–Mumford that is associated with a coherent sheaf.*

[7] *A fibration is said to be locally Kähler if for any small open set in the base, its pull-back can be equipped with a Kähler metric.*

associated determinant bundle and extended to arbitrary dimension. Higher versions of the holomorphic torsion were also defined. The functorial behavior of Quillen metrics and their higher analogues was studied by various authors including Berthomieu, Kähler, Lebeau, Ma, and myself in the case of embeddings and projections. Related work on forms and currents was done by Gillet–Soulé and me. These joint efforts culminated in the Riemann–Roch–Grothendieck theorem in Arakelov geometry of Gillet and Soulé that reassembles the distinct strands coming from algebraic geometry, number theory, and index theory in a single machine.

A significant extension on Singer's ideas on analytic torsion also came from the work of Kontsevich and Vishik, in which determinants of elliptic pseudo-differential operators appeared, and they studied the defect to multiplicativity of such determinants.

In the spirit of the explicit computations of holomorphic torsion of flat tori by Ray and Singer, Yoshikawa considered the equivariant holomorphic torsion of a family of $K3$-surfaces with involution, and he showed that it coincides with an associated Borcherds modular form on the corresponding moduli space. Holomorphic torsion has found many other applications in the context of mirror symmetry, as in Bershadsky, Cecotti, Ooguri, and Vafa. Analytic and real torsion also appears repeatedly in a considerable body of physics-oriented literature, to which Singer himself contributed in a major way. One reason is that when localizing infinite-dimensional functional integrals at critical points of the action, infinite-dimensional determinants appear, which can be interpreted as being related to analytic torsion. We refer to Witten, where the volumes of the moduli spaces of flat connections on a Riemann surface are computed in that spirit.

Progress was also made on real torsion. With Lott, we established a version of the families index theorem for the direct images of flat vector bundles, with values in the real odd cohomology of the base. Real analytic torsion appears as a natural transgression of a corresponding equality of cohomology classes. Higher-dimensional analytic real torsion forms were also defined. The proof of their relation to Igusa's higher torsion is in progress.

New developments also happened in local index theory. Dirac operators associated with metric connections with nonzero torsion T appear naturally, for instance on non-Kähler manifolds. If $\langle T \wedge \theta \rangle$ denotes the 3-form that is obtained by antisymmetrization, I showed that if this form is closed, a version of the local index theorem still holds. This sufficient condition is also essentially necessary. For complex manifolds, this just means that if ω denotes the canonical 2-form form, then $\bar{\partial}\partial\omega = 0$.

Jeff Cheeger

I was fortunate to have met Isadore Singer (Is) at the end of my last year as a graduate student. He had just given a lecture on analytic torsion. Though it was very inspiring, I hadn't understood much. We were introduced by Jim Simons, who was my teacher and to whom Is had been a mentor. Though we were a generation apart, through common mathematical interests and our mutual connection with Jim, we became friends. Is was one of the most influential mathematicians of the second half of the 20th century. Here, I want to describe how so much of my own work was rooted in his.

The first instance was indirect. About year after I had gotten my degree, Jim mentioned to me the problem of finding a lower bound for the smallest eigenvalue of the Laplacian. It was Singer who had called it to his attention. I was led to the inequality, $\lambda_1 \geq \frac{1}{4}h^2$, where h is a certain isoperimetric constant. Eventually, this inequality acquired an extensive and varied collection of descendants.

In a visionary paper, "Curvature and the Eigenvalues of the Laplacian," [14], Singer and Henry McKean proposed the possibility of a heat equation proof of the Atiyah–Singer index theorem. Specifically, they discussed in detail the case of the Euler characteristic. Let $e^{-t\Delta_i}$ denote the fundamental solution of the heat equation for the Hodge Laplacian on i-forms. It was known that as $t \to 0$, there is an asymptotic expansion for the *pointwise trace*, $e^{-t\Delta_i}(x,x) \sim \sum_j a_{i,j}(x,x)t^{-(n/2)+j}$, where the $a_{i,j}(x,x)$ are given by polynomial expressions in the curvature tensor and its higher covariant derivatives. The orders of the covariant derivatives increase with j. A simple cancellation argument shows that the alternating sum of *global traces* $\sum_i(-1)^i\mathrm{trace}(e^{-t\Delta_i})$ is actually *independent of* t. As $t \to \infty$, it converges to the alternating sum of the dimensions of the spaces of harmonic i-forms and hence, to the Euler characteristic $\chi(M^n)$. As $t \to 0$, it follows that the alternating sum of the coefficients of the negative powers of t must cancel, leading to the McKean–Singer formula $\chi(M^{2n}) = \sum_i(-1)^i a_{i,n/2}$. They conjectured that the alternating sum of *pointwise* traces $\sum_i(-1)^i a_{i,n/2}(x,x)$ is in fact equal to the Chern–Gauss–Bonnet form $P_\chi(\Omega)$, a certain invariant polynomial in the curvature tensor Ω (specifically a multiple of the Pfaffian). If true, this would give a new proof and a new understanding of the Chern–Gauss–Bonnet theorem $\chi(M^{2n}) = \int_{M^{2n}} P_\chi(\Omega)$. In dimension 2, they were able to verify it by direct computation. In higher dimensions, though their computations rapidly became unmanageable, they noted that were the conjecture to hold, "fantastic cancellations" of the local terms involving higher covariant derivatives of curvature would have to take place. By the early 1970s, primarily through independent work of Patodi and Gilkey, the program was realized for all operators of Dirac type.

The next logical step was to extend the heat equation proof to manifolds with boundary. However, for the signature operator on oriented $4k$-manifolds (which depends on a choice of *orientation*) it was known that no *local* elliptic boundary condition has the right index. After some time, Atiyah, Patodi and Singer (APS) found a *global* boundary condition that would work. The APS condition is akin to requiring the vanishing of the negative Fourier coefficients of $f \mid \partial D^2$, where D^2 is the unit disc and $f : D^2 \to \mathbf{C}$. (This holds for holomorphic functions on the disc.) The crowning achievement of their revolutionary series of three papers was the Atiyah–Patodi–Singer formula; see in particular [2].

$$\mathrm{Sig}(M^{4k}) = \int_{M^{4k}} P_L(\Omega) + \eta(\partial M^{4k})\,.$$

The differential form, $P_L(\Omega)$, is the invariant polynomial in curvature (Chern–Weil form) corresponding to the Hirzebruch L-class. The invariant $\eta(\partial M^{4k})$ is a certain *global spectral invariant* of the boundary with its induced metric. There can be no *canonical local formula* for the η-invariant since examples show that it can fail to behave multiplicatively when one passes from a space to a nontrivial covering space. By using the fact that the *variation* of the η-invariant under change of metric *is locally computable*, APS defined topological invariants called ρ-invariants which measure the extent of the failure of the η-invariant to behave multiplicatively.

As was understood at the time, the term $\int_{M^{4k}} P_L(\Omega)$ could be viewed as an instance of a Chern–Simons invariant, or more precisely, a differential character. Chern–Simons invariants had been defined and their theory developed at the end of the 1960s. What is not so well known is that the initial motivation for these now famous invariants came from a failed attempt to find a local combinatorial formula for the signature of a compact oriented 4-manifold, a problem which was first posed in the 1940s.

Yet another revolutionary contribution to spectral theory was made by Singer with his MIT colleague Dan Ray; [17]. This concerned analytic torsion, a global spectral invariant whose definition involves a choice of Riemannian metric. No choice of orientation is required. The definition imitates in a sophisticated manner, a reformulation of that of Reidemeister torsion, a combinatorially defined topological invariant, though not a homotopy invariant. Among other fundamental properties, they showed that the analytic torsion is indeed is independent of the choice of Riemannian metric used to define it. The Ray–Singer conjecture posited the equality of the two torsions.

In the early 1970s, I was trying to learn enough about the heat equation to have a chance of participating in these exciting developments. During that time, my work with Simons on differential characters got stuck on a problem which is still open. Namely, the conjecture that there exist simplices in S^3 with totally geodesic faces and rational dihedral angles, whose volumes are irrational multiples of that of the sphere. Finally, I decided to give up and try my luck on the Ray–Singer conjecture. After a few years, I succeeded in proving it; [9]. An independent proof was given at the same time by Werner Muller; [15].

One step in my proof was the following: First, as in doing a surgery, remove from a closed Riemannian manifold M^n, a tubular neighborhood $T_r(S^k)$ of radius r, of an embedded sphere S^k whose normal bundle is trivial. Then understand the relation as $r \to 0$ of the heat kernels on differential forms of $M^n \setminus T_r(S^k)$ with absolute and absolute boundary conditions, and the corresponding heat kernels on differential forms of M^n. In particular, describe the eigenforms of the Laplacian for which the corresponding eigenvalues either are zero, or approach zero as $r \to 0$. Note in this connection that when $r = 0$, the topology changes. Note also the close analogy with the problem which led to the lower bound $\lambda_1 \geq \frac{1}{4}h^2$. A key technical issue was the need to control the relevant Green's functions as $r \to 0$. I had learned from Jackson's well known text "Classical Electrodynamics" that in $\mathbb{R}^n$, the formula in polar coordinates for the Green's function, $G(r_1, y_1, r_2, y_2)$, can be obtained by using separation of variables. The use of polar coordinates amounts to viewing $\mathbb{R}^n$ as the *metric cone* with cross-section the *unit sphere* S^{n-1}. However, near points at which $r_1 = r_2$, the resulting series doesn't converge uniformly and for $r_1 = r_2$ it only converges in a weak sense. I noticed that almost by definition, the formula in Jackson's book could be interpreted as a parameterized family of Poisson like kernels associated to the relevant Laplacian on the cross-section S^{n-1}. For say $r_1 \leq r_2$, the ratio r_1/r_2 of the radial variables, plays the role of e^{-t} in the Poisson like kernels. As a consequence, these kernels, and hence the Green's functions, could be understood by means of the *known functional calculus* for the relevant Laplacians on S^{n-1}.

I realized that the same method, "the strong form of the method of separation of variables," could be used to construct a functional calculus for Laplacians on differential forms, on metric cones with an arbitrary smooth closed Riemannian manifold as cross-section. With this, one could do spectral theory on manifolds with isolated conical singularities; [10]. In particular, it gave a new interpretation and a new proof of the APS theorem in which the η-invariant of the cross-section of the cone arises intrinsically as a needed correction term to the asymptotic expansion of the heat kernel trace coming from the singularity. Another consequence was my 1982 work with Michael Taylor on diffraction of waves by cones of arbitrary cross-section, which appeared in *Communications in Pure and Applied Mathematics*. A different continuation concerned Poincaré duality for singular spaces (pseudomanifolds) via L_2-cohomology, [11], and index theory/spectral theory on these spaces in a 1983 paper in the *Journal of Differential Geometry*. The latter included a solution to the original problem of finding a local combinatorial formula for the signature. Recall that triangulated $4k$-manifold has a *canonical geometric realization*. It is built from the combinatorial triangulation by using n-simplicies, all of whose edge lengths are equal to 1. For this piecewise flat metric, which is invariant under simplicial automorphisms, the heat equation method can be used to express the signature as the sum of the η-invariants of links of vertices, with their canonical induced piecewise constant curvature 1 metrics. This provides a (canonical) local combinatorial formula for the signature. However, since η-invariants are global spectral invariants, in most cases they are not explicitly computable. On the other hand, by a simple cancelation argument, given any local combinatorial formula for the signature formula, one gets a new formula by subtracting the boundary of any combinatorially defined 1-chain. In recent work (in preparation) I use this freedom to replace the *global spectral* aspect of the η-invariant by a *global combinatorial construction*. Finally, I should mention that my friendship with Jean-Michel Bismut, and our work on adiabatic limits of η-invariants, $\tilde{\eta}$-forms and the local families index theorem for manifolds with boundary, arose from our common interest in the work of Is and his collaborators; [6], [7].

Offshoots of Is' work have consumed much of my mathematical thought processes over the past 50 years. He was also a much admired friend. Not so long ago, I asked him how, despite his advancing years, he was able to maintain his obvious zest for math, tennis, and life. His answer: "Every day I have a ball!"

Simon Donaldson

I only had a few opportunities to meet Is Singer—the first in 1984 during his spell in Berkeley and the last at a celebration held for his birthday in Harvard in 2017—but I learnt a lot from his work.

Singer was a pivotal figure in the mathematical developments around gauge theories, starting in the late 1970s. He and others have written about lectures he gave in Oxford and elsewhere at the start of this development, introducing the 1975 "Wu and Yang dictionary" to mathematicians [19]. This is a dictionary between physics and mathematics terminology, starting with *gauge type = principal fibre bundle* and moving on through *electromagnetism = connection in a U(1) bundle* and and *isotopic spin gauge field = connection in an SU(2) bundle*. Singer wrote in [19] "It would be inaccurate to say that after 30 years studying mathematics I felt ready to return to physics. Instead, elementary particle physics turned to modern mathematics and some of us found the interplay full of promise."

One of the first fruits of this interplay was the calculation by Atiyah, Hitchin, and Singer of the dimensions of the moduli spaces of Yang–Mills instantons over the 4-sphere. The instanton equation is a nonlinear PDE for a connection on a bundle and the relevant $SU(2)$ bundles are classified by a Pontryagin number $k \geq 1$. Physicists had found explicit constructions of solutions depending on $5k+4$ parameters (for $k \geq 3$). Atiyah, Hitchin, and Singer proved that there were more solutions: the full moduli space has dimension $8k-3$. The basis for their analysis was the Atiyah, Hitchin, Singer deformation complex

$$\Omega^0(\text{ad } P) \xrightarrow{d_A} \Omega^1(\text{ad } P) \xrightarrow{d_A^-} \Omega^2_-(\text{ad } P). \tag{1}$$

Here A is an instanton connection and ad$P$ is the bundle of Lie algebras associated to the adjoint representation. The middle term represents deformations $A + a$ of the connection. The equation $d_A^- a = 0$ is the linearization of the nonlinear instanton equation, for small a. The first term $\Omega^0(\text{ad } P)$ represents infinitesimal gauge transformations, or automorphisms of the bundle P. An infinitesimal deformation $a = d_A u$ does not change the solution, up to geometric equivalence. The tangent space of the moduli space of solutions modulo equivalence is given by the cohomology in the middle term of the complex (1). More precisely, this depends on the fact that the other cohomology groups vanish; the desired dimension is minus the Euler characteristic of the complex and this Euler characteristic is computed by the Atiyah–Singer index theory. This kind of analysis had precedents in the Kodaira–Spencer–Kuranishi theory of deformations of complex structures but was novel in this differential geometric context, and the techniques have been applied to a host of other moduli problems over the years since, becoming part of the differential geometers tool kit.

The solution of the general deformation problem for instantons appeared in [1] but that paper covered much more. It is the foundational paper for contemporary four-dimensional differential geometry, and the special features in this dimension associated with "self-duality."

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In the 1981 paper in *Physics Scripta*, Singer wrote "We wonder whether the techniques and insights of mathematics can contribute directly to the solution of quantum field theory problems. We believe that a careful study of the space of orbits ... of the group of gauge transformations will be useful." The infinite-dimensional geometry point of view and its connection to quantum field theory runs through much of Singer's work and represents some of his most distinctive contributions. The set-up is that we have the space A of connections on a principal G-bundle P over a Riemannian manifold M, where G is a compact Lie group. This is an affine and hence contractible space (corresponding to the deformations a above). The interest comes from the group $\mathcal{G} = \mathrm{Aut}\, P$ of gauge transformations which acts on $\mathcal{A}$ and the geometrically meaningful infinite-dimensional space is the quotient $\mathcal{A}/\mathcal{G}$. There is an analogous discussion in Riemannian geometry, also prominent in Singer's work, with the diffeomorphism group of a manifold acting on the space of Riemannian metrics. While the gauge theory case was less familiar to differential geometers at the time we are discussing, it is in fact much simpler—"less nonlinear"—and that was an important feature of its role in developments in modern differential geometry.

To work on the quotient space $\mathcal{A}/\mathcal{G}$, one can try to "fix a gauge," in other words find a slice in $\mathcal{A}$ transverse to the $\mathcal{G}$-orbits and containing exactly one representative of each orbit. This can be done locally, for small variations of the connection, but as Singer showed in a paper in *Communications in Mathematical Physics* in 1978, one of his first papers in the area, it is not possible globally. Ignoring some important technicalities, the quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{G}$ is the universal bundle $E\mathcal{G} \to B\mathcal{G}$ for the gauge group $\mathcal{G}$. Global gauge fixing corresponds to a section of this bundle, and this can exist if and only if $\mathcal{G}$ is contractible (which would imply the same for $\mathcal{A}/\mathcal{G}$). But in fact, for base manifolds M of interest, $\mathcal{G}$ and $\mathcal{A}/\mathcal{G}$ have complicated topology. For example when the base manifold M is the 2-sphere the space $\mathcal{A}/\mathcal{G}$ has the homotopy type of the loop space of the group G.

The topology of the space $\mathcal{A}/\mathcal{G}$ is bound up with index theory, but now for families of operators. If M is a spin manifold, a connection A and a representation of the group G define a coupled Dirac operator D_A, and the index construction for families yields an element $\mathrm{ind} D_A$ of the K-theory of $\mathcal{A}/\mathcal{G}$. These ideas are important in connection with anomalies in quantum field theory. For example, the determinant of the family of Dirac operators is intrinsically a section of a line bundle over $\det \mathrm{ind} D_A$ over $\mathcal{A}/\mathcal{G}$ and the first Chern class of this line bundle is a topological obstruction to defining the determinant as a function. The paper [4] of Atiyah and Singer was one of the first mathematical treatments of these ideas and the foundation for many further developments through the work of Quillen and others.

In another direction, Singer studied the differential geometry of the infinite-dimensional space $\mathcal{A}/\mathcal{G}$. This has a natural Riemannian metric, induced by the L^2 metric on the bundle-valued 1-forms a, which represent tangent vectors in $\mathcal{A}/\mathcal{G}$. Singer obtained a formula for the sectional curvature corresponding to a pair a, b of tangent vectors: $K(a,b) = 3\langle \Delta_A^{-1}(a * b), a * b\rangle$. Here $a * b$ denotes the pointwise bilinear algebraic operation combining the inner product on 1-forms and Lie bracket on the bundle $\mathrm{ad}\, P$ and Δ_A is the coupled Laplace operator. Going further, Singer discussed the Ricci curvature of $\mathcal{A}/\mathcal{G}$. This is given formally by summing an infinite collection of sectional curvatures which corresponds to the trace of a certain operator T. The problem is that T is not of trace class, and, to get around this, Singer introduces the regularization $\mathrm{Tr}\ T\Delta_A^{-s}$ which is defined for large enough s. The meromorphic continuation has in general a pole at $s = 0$, but Singer found that the residue there has an expression given by an integral over M involving the scalar curvature. Thus he obtained the striking result that the Ricci tensor of $\mathcal{A}/\mathcal{G}$ can be defined when working over a base manifold M of zero scalar curvature.

Nigel Hitchin

As a student, postdoc and collaborator of Michael Atiyah, I inevitably encountered Isadore Singer many times. Since they first met at the Institute in Princeton in 1955, Atiyah and Singer were part of the postwar group of mathematicians who were redefining the interactions between algebraic geometry, topology, differential geometry, and analysis. A sabbatical in Oxford in 1962 started their most notable collaboration, which of course resulted in the Atiyah–Singer index theorem, but it was another visit in 1977 that led to an important development in which I was fortunate to take part.

Atiyah often crossed the Atlantic to see Singer, or Raoul Bott, bringing home news about current directions in geometry. When he invited them back to Oxford, his Monday seminar was the forum for us all to learn about these issues. In early 1977, Singer visited and spoke about the self-dual Yang–Mills equations. As I recall, he also gave lectures about the geometry behind Bäcklund transformations, but it was the Yang–Mills problem which gained traction.

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Singer had a long-held interest in physics and the problem he presented to us was a differential geometric interpretation of the question of "instantons," in the language of physicists. The scenario consists of Euclidean 4-space $\mathbb{R}^4$ and a connection. This is a covariant derivative operator defined by $\nabla_i = \partial/\partial x_i + A_i$ where A_i is a function with values in (for this problem) 2×2 skew-Hermitian matrices of trace zero, the Lie algebra of SU(2). The curvature $[\nabla_i, \nabla_j] = F_{ij}$ is a differential 2-form with values in the Lie algebra and the Yang–Mills functional is the integral of $|F|^2$. The problem is to characterize all absolute minima of this integral.

The functional is conformally invariant, and, in the course of the weekly seminar, Singer showed how, with suitable decay assumptions on the curvature (later verified by Karen Uhlenbeck) $\mathbb{R}^4$ could be replaced by the sphere S^4 by adding a point at infinity. In this setting the physicist's instanton charge k becomes the second Chern class of a vector bundle E. Moreover the integral expression for the Chern class provides a lower bound for $\|F\|^2$, achieved when the connection is self-dual.

Self-duality is a special property of four-dimensional Riemannian geometry: the exterior 2-forms decompose into ± 1 eigenspaces of the Hodge star operator and F is self-dual if it lies in the $+1$ eigenspace at each point. In the context of the curvature of a 4-manifold, this feature was implicit in a earlier paper of Singer's [20], which I knew as a student. (In fact, that paper implies that the spinor bundle on S^4 with its constant curvature metric is an instanton of charge 1.) In Oxford, Singer introduced a whole family of solutions of different charges produced by the physicists. The obvious question was: Are there any more? Conjugation of ∇ by an automorphism of the bundle (a gauge transformation) clearly gives an infinite-dimensional space of such but one needed to know all solutions modulo this equivalence.

The self-duality equations are nonlinear in the connection coefficients A_i and one can study the linearization. For self-duality, a first order deformation $\dot{A}$ satisfies a linear equation $D_{-}\dot{A} = 0$ as a section of the vector bundle $\operatorname{End} E \otimes \Lambda^2_{-}$. A first order gauge transformation gives $\dot{A} = \nabla\psi$ for ψ a section of $\operatorname{End} E$, so the linearization, as far as equivalence is concerned, is the kernel of a differential operator D_{-} modulo the image of ∇. This is the first cohomology group of an elliptic complex

$$\operatorname{End} E \xrightarrow{\nabla} \operatorname{End} E \otimes \Lambda^1 \xrightarrow{D_{-}} \operatorname{End} E \otimes \Lambda^2_{-}\,.$$

Elliptic complexes are precisely the context of the Atiyah–Singer index theorem, and, together with vanishing theorems given the positive curvature of the sphere the theorem gives the number $(8k-3)$ for the solution of the linearization problem. This led to a short announcement followed by an expanded paper [1], which incorporated the other input from the Oxford seminars, from Penrose's group, and in particular his student Ward. It placed the instanton problem in a wider context, where the sphere is replaced by a 4-manifold with a self-dual conformal structure—a condition on the Weyl tensor of a metric. Penrose's twistor theory added a new dimension to the original problem, linking it directly with algebraic geometry and quite quickly it yielded a concrete construction for the whole $(8k-3)$-dimensional family.

Perhaps the most important aspect in all of this was Singer's observation, a postscript in the announcement [20], that one could use the methodology of Kuranishi (for the moduli space of complex structures on a compact complex manifold) to construct from the linearization of the self-duality equations an actual moduli space of instantons. This was probably the first time that the notion of moduli space escaped from algebraic geometry into the wider mathematical world. It provided the framework for Simon Donaldson's famous later work replacing the sphere by a more general 4-manifold with no differential geometric assumptions.

At a personal level, Is Singer always took an interest in what I was doing. I remember in particular when I was a postdoc in New York in 1973 and Michael Atiyah was visiting MIT. I went up one cold December day to give a seminar. Maybe it was the semester break or the bad weather but the audience was Singer, Atiyah, Kostant, Irving Segal, and no more! I spoke about the Dirac operator on bounded domains in $\mathbb{R}^n$ and conformal invariance (conformally invariant operators were the centre of my interest at the time, including the complex above, but without the connection). Afterwards I was unsure of the reception, but a week later I received from Is several pages of carefully written notes pointing out a more sophisticated way of doing things. He is greatly missed.

H. Blaine Lawson and Mikhail Gromov

We have been asked to present our perspective on the index theorem, which puts us in the position of a blind person describing what he/she perceives of an elephant upon touching the elephant's tail. Index theory is a world of its own, as the reader can begin to see from the article of Connes and Kouneiher in the November, 2019 issue of the AMS *Notices*. As geometers we are well aware only of what

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is directly related to our field. With that understood, we can begin.

Over his long career Is Singer made many revolutionary contributions to mathematics. Of all his work, perhaps the most far-reaching were his results with Atiyah on the index theorem. It was not just that they established a formula which generalized the Riemann–Roch–Hirzebruch theorem. What was even more important was their discovery of a fundamental elliptic operator, which certainly paved the way to the index theorem, but which also engendered profound applications, which to date can be proved in no other way.

For background, recall that a manifold of dimension n is *orientable*, if its structure group (where the jacobians of coordinate changes live) can be reduced from GL_n to GL_n^+, the connected component of the identity. The manifold is *spin* (with $n \geq 3$) if the structure group GL_n^+ can be lifted to the universal covering group $\widetilde{\mathrm{GL}}_n^+$. At first this seems unimportant since $\widetilde{\mathrm{GL}}_n^+$ has no finite-dimensional representations that are not pulled back from GL_n^+. However, if one restricts to maximal compact subgroups $\mathrm{SO}_n \subset \mathrm{GL}_n^+$ and $\mathrm{Spin}_n \subset \widetilde{\mathrm{GL}}_n^+$ (these inclusions are homotopy equivalences), then Spin_n *does* have representations that are not pulled back from SO_n.

In the 1960s, many decades had passed since the fundamental work of É. Cartan and P.A.M. Dirac, but the fundamental operator on a spin manifold with a Riemannian metric was still unknown. Borel and Hirzebruch had proved that on a spin manifold M of dimension $4k$ a certain topological invariant $\widehat{A}(M)$ is always an integer, and this is not always true for M that are not spin. Atiyah had conjectured that perhaps this was because $\widehat{A}(M)$ was the index of an operator that only existed on spin manifolds. It was Singer who found that operator, and they were on their way. They showed in fact that, in the spin case, every elliptic operator was equivalent to this fundamental operator $\not{D}$, twisted by a coefficient bundle, and the index formula is a natural generalization of Hirzebruch's. With this insight they were able, with more work, to also treat the non-spin case.

It is important to understand that to define the operator $\not{D}$ it is necessary to have a metric on the spin manifold. Given this metric, the operator satisfies a Bochner-type identity, found by E. Schrödinger and A. Lichnerowicz:

$$\not{D}^2 = \nabla^*\nabla + \frac{1}{4}\kappa,$$

where $\nabla^*\nabla \geq 0$ with kernel consisting of parallel sections, and κ is the scalar curvature (the average of all the sectional curvatures), which is a very weak invariant. However, from this formula one sees that on a spin $4k$-manifold with $\widehat{A} \neq 0$, there does not exist a Riemannian metric with $\kappa > 0$.

If one considers the general case of $\not{D}_E$ (i.e., $\not{D}$ tensored with a vector bundle with connection E), there is the formula

$$\not{D}_E^2 = \nabla^*\nabla + \frac{1}{4}\kappa + \mathbf{R}_E,$$

where $\mathbf{R}_E$ is explicitly defined in terms of the curvatures of E. This allows for interesting generalizations of the scalar curvature result above.

Atiyah and Singer went on to give a different proof of the index theorem, which was based on topological K-theory (= the Grothendieck group of isomorphism classes of complex vector bundles on a given space—work of Atiyah and Hirzebruch). They also found many important generalizations.When the operator is equivariant with respect to a compact Lie group G, the index lies in the representation ring of G, and this result has many applications, including new results in number theory. In a different direction, suppose we have a family of elliptic operators over a parameter space X, then there is a theorem where the index of the family is an element of the K-theory of X. Perhaps one of the most profound of these generalizations was the index theorem for $\mathrm{Cl}_k(\mathbb{R})$-linear operators, where $\mathrm{Cl}_k(\mathbb{R})$ is the Clifford algebra on $\mathbb{R}^k$ with its standard positive definite metric. Here the index is an element in the quotient of the Grothendieck group of $\mathrm{Cl}_k(\mathbb{R})$-modules which can be identified with $\mathrm{KO}_{-k}(\mathrm{pt})$, the real K-theory group for a point. Interestingly, these groups are torsion in certain dimensions. Now every spin n-manifold has a fundamental $\mathrm{Cl}_n(\mathbb{R})$-linear Atiyah–Singer operator with index in the group $\mathrm{KO}_{-n}(\mathrm{pt})$. This gives a ring homomorphism $\widehat{A} : \Omega_*^{\mathrm{Spin}} \to \mathrm{KO}_{-*}(\mathrm{pt})$ from the spin cobordism ring, that must vanish on every cobordism class that contains a manifold that admits a metric with $\kappa > 0$. In particular, by a result of N. Hitchin, for all $n \equiv 1$ or $2 \pmod 8$, $(n \geq 8)$, *half of the exotic spheres do not carry a metric of positive scalar curvature!* From the classical geometer's point of view this was pure magic.

Somewhat later, Stephan Stolz showed that, in the world of simply-connected spin manifolds of dimension > 4, the index $\widehat{A}$ is the only invariant obstructing positive scalar curvature. For manifolds with nontrivial fundamental group, there is an analogous, yet more wide-ranging and still unfinished, story.

We want to say that Is Singer brought more than great insights to mathematics. His talks were magical. His enthusiasm was totally contagious. He spent many years bringing together the worlds of mathematics and physics, and this transformed both fields. Is Singer not only opened people to wonders in mathematics, but he made them excited to be part of the wonderment. He was always inclusive, and he and Rosemarie were very warm and outreaching. Is Singer made mathematics a better place.

Werner Müller

Another milestone in the work of Singer is the invention of the analytic torsion [17] which he, together with Ray, introduced in 1971 as an analytic counterpart of the Reidemeister torsion. The Reidemeister torsion (or the Reidemeister–Franz torsion) is a topological invariant of a compact manifold M and a representation of its fundamental group $\pi_1(M)$ that was introduced by Reidemeister in 1935 for 3-manifolds and generalized to higher dimensions by Franz and de Rham. In fact, the Reidemeister torsion is defined for every finite CW complex K and a unitary representation ρ of $\pi_1 \coloneqq \pi_1(K)$ on a finite-dimensional vector space V_ρ. Let $\tilde{K}$ be the simply connected covering space of K with π_1 acting on $\tilde{K}$ as group of deck transformations. Consider K as being embedded as a fundamental domain in $\tilde{K}$, so that $\tilde{K}$ is the set of translates of K under π_1. In this way, the real co-chain complex becomes an $\mathbb{R}(\pi_1)$-module. Let $C^*(K;\rho) \coloneqq C^*(\tilde{K}) \otimes_{\mathbb{R}(\pi_1)} V_\rho$ be the twisted co-chain complex. In the real vector space $C^q(K;\rho)$ one can choose a preferred base $(x_i \otimes v_j)$, where x_i runs through the preferred base of the $\mathbb{R}(\pi_1)$-module $C^q(\tilde{K})$ given by the cells of K and v_j through an orthonormal base of V_ρ. A preferred base gives rise to an inner product in $C^*(K;\rho)$. Let $\delta\colon C^q(K;\rho) \to C^{q+1}(K;\rho)$ denote the co-boundary operator and δ^* the adjoint operator with respect to the inner product in $C^*(K;\rho)$. Define the combinatorial Laplacian $\Delta^{(c)}$ by $\Delta^{(c)} = \delta\delta^* + \delta^*\delta$. Assume that the cohomology $H^*(K;\rho)$ of the complex $C^*(K;\rho)$ vanishes. Then $\Delta^{(c)}$ is invertible, and, as shown by Ray and Singer, the Reidemeister torsion $\tau(K,\rho) \in \mathbb{R}^+$ of K and ρ is given by

$$\log \tau(K,\rho) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1} q \log\det(\Delta_q^{(c)}), \qquad (2)$$

where n is the dimension of the top-dimensional cells of K. This is not quite the original definition of the Reidemeister torsion, but, as shown by Ray and Singer, it is equivalent to the original one. If $H^*(K;\rho) \neq 0$, one has to choose a volume form $\mu \in \det H^*(K;\rho)$. For any choice of μ, one can define the Reidemeister torsion $\tau_M(\rho;\mu) \in \mathbb{R}^+$, which depends on μ.

Let M be a closed Riemannian manifold and ρ an orthogonal representation of $\pi_1(M)$. Let $E_\rho \to M$ be the flat orthogonal vector bundle associated to ρ. Then by the Hodge–de Rham theorem, $H^*(K;\rho)$ is identified with the space of harmonic forms with values in E_ρ. Using the global inner product on harmonic forms, we get a volume form μ. It turns out that $\tau_K(\rho;\mu)$ is invariant under subdivisions. Furthermore, any two smooth triangulations of M admit a common subdivision. Thus $\tau_K(\rho;\mu)$ is independent of the choice of K and we write $\tau_M(\rho) \coloneqq \tau_K(\rho;\mu)$. This is the Reidemeister torsion of M with respect to ρ and $\mu \in \det H^*(M;E_\rho)$.

The original interest in Reidemeister torsion came from the fact that it is not a homotopy invariant and so can distinguish spaces which are homotopy invariant but are not homeomorphic. Especially, Reidemeister used Reidemeister torsion to classify three-dimensional lens spaces up to homeomorphism and this was generalized by Franz to higher dimensions. The classification includes examples of homotopy equivalent three-dimensional manifolds which are not homeomorphic. Reidemeister torsion is closely related to Whitehead torsion, which is a more sophisticated invariant of chain complexes. It is related to the concept of simple homotopy equivalence.

Following a suggestion of Arnold Shapiro, Ray and Singer were looking for an analytic description of the Reidemeister torsion of a closed Riemannian manifold. The inspiration for the definition came from formula (2). Let $(\Lambda^*(M;E_\rho), d)$ be the de Rham complex of E_ρ-valued differential forms on M and let $\Delta \coloneqq dd^* + d^*d$ be the Laplace operator acting in $\Lambda^*(M;E_\rho)$, where d^* is the formal adjoint of d with respect to the inner product induced by the Riemmanian metric g and the fibre metric in E_ρ. Then the idea is to replace $C^*(K;\rho)$ by $\Lambda^*(M;E_\rho)$ and the combinatorial Laplacian $\Delta_q^{(c)}$ by the Hodge Laplacian Δ_q. The problem is that Δ_q is acting in an infinite-dimensional space and the determinant is not well defined. To overcome this problem, one uses the zeta function regularization. Regarded as an unbounded operator in the Hilbert space of L^2-forms of degree q with values in E_ρ, Δ_q is essentially self-adjoint and nonnegative. Since M is compact, it follows that Δ_q has a pure point spectrum consisting of eigenvalues $\lambda_j \geq 0$, $j \in \mathbb{N}$, of finite multiplicity $m(\lambda_j)$. Let $\zeta_q(s;\rho) \coloneqq \sum_{\lambda_j>0} m(\lambda_j)\lambda_j^{-s}$, $s \in \mathbb{C}$, be the spectral zeta function. As shown by Seeley, the series converges absolutely and uniformly on compact subsets of the half plane $\Re(s) > n/2$, admits a meromorphic extension to $\mathbb{C}$ and is holomorphic at $s = 0$. Then the regularized determinant of Δ_q is defined by $\det(\Delta_q) \coloneqq \exp(-\frac{d}{ds}\zeta_q(s;\rho)|_{s=0})$. Replacing formally $\det(\Delta_q^{(c)})$ in (2) by the regularized determinant of Δ_q has led Ray and Singer to the following definition of the analytic torsion $T_M(\rho) \in \mathbb{R}^+$

$$\log T_M(\rho) \coloneqq \frac{1}{2}\sum_{q=0}^{n}(-1)^q q \frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}. \qquad (3)$$

By its definition, $T_M(\rho)$ depends on the whole spectrum of the Laplace operators Δ_q, $q = 0,\dots,n$, and is therefore

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a more sophisticated spectral invariant. Ray and Singer proved that the analytic torsion satisfies the same formal properties as the Reidemeister torsion, which supported their conjecture that

$$T_M(\rho) = \tau_M(\rho) \tag{4}$$

for any orthogonal representation ρ. The Ray–Singer conjecture was eventually proved independently by Cheeger [9] and Müller [15] (and is now often referred to as the Cheeger–Müller theorem). The proofs of Cheeger and Müller are different, but similar in spirit. The strategy of both proofs is to show that $T_M(\rho) - \tau_M(\rho)$ remains invariant under surgery which reduces the problem to the case of the sphere for which the equality can be verified explicitly. The proof of Cheeger is based on analytic surgery methods. Müller uses the Whitney approximation of the de Rham complex by the co-chain complex and the finite element approximation of eigenvalues, which goes back to the work of Dodziuk and Patodi.

The equality of analytic torsion and Reidemeister torsion has been extended in various ways. Müller has shown that (4) holds for unimodular representations ρ of $\pi_1(M)$ ($|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$). Recently, this result has found some interesting applications to the study of the cohomology of arithmetic groups (see below).

Finally, Bismut and Zhang treated the general case of an arbitrary finite-dimensional representation ρ. The framework is slightly different. They work with the metrics on $\det H^*(M, E_\rho)$ induced by the analytic torsion and the Thom–Smale complex associated to a Morse function on M. In general, an equality does not hold anymore. There appears a defect term, which, however, can be described explicitly. The defect is a kind of obstruction for ρ being unimodular. Bismut and Zhang gave a completely new proof, which uses the Witten deformation of the de Rham complex associated to the Morse function.

The equivariant case was first studied by Lott and Rothenberg and then by Lück. Also Bismut and Zhang extended their result to the equivariant case, using again the Witten deformation of the de Rham complex.

It is natural to try to generalize (4) to other classes of manifolds. The first obvious case is compact manifolds with a nonempty boundary. A corresponding result was announced by Cheeger in his paper proving the Ray–Singer conjecture. Then Lück has derived a formula using the double of a compact Riemannian manifold M with boundary which reduces the problem to the equivariant case. This approach requires that the metric of M is a product near the boundary. On the analytic side one has to impose absolute or relative boundary conditions for the Laplacians. The resulting formula comparing analytic and Reidemeister torsion involves a correction term which is given by the Euler characteristic $\chi(\partial M)$ of the boundary. The general case, i.e., without any assumption on the behavior of the metric near the boundary, was treated by Ma and Brüning. The boundary contribution, which is called anomaly, is in general more complicated.

There were various attempts to generalize (4) to singular spaces. The first case is manifolds with conical singularities. The study of analytic and topological torsion on singular spaces with conical singularities started with work of A. Dar. She proved that on singular spaces with isolated conical singularities, the analytic torsion is well-defined. On the combinatorial side she used the middle intersection complex to define the intersection Reidemeister torsion, which one expected to be equal to the analytic torsion. This turned out not to be true. There are recent results by Albin–Rochon–Sher, Hartman–Spreafico, and Ludwig that establish a formula relating analytic torsion and intersection torsion. This is not an equality, but the defect term can be described explicitly.

Guided by the definition of the real analytic torsion, Ray and Singer introduced an analog of the analytic torsion for complex manifolds. The role of the flat vector bundle is played by a holomorphic vector bundle $E \to X$ over a compact complex manifold and the de Rham complex is replaced by the $\bar{\partial}$-complex of $(0, q)$-forms with values in E. The complex analytic torsion has found important applications in arithmetic-algebraic geometry and theoretical physics. This will be discussed in a separate section.

The equality of analytic torsion and Reidemeister torsion has recently found interesting application in the study of the growth of torsion in the cohomology of arithmetic groups. This idea goes back to Bergeron and Venkatesh. The origin of this kind of application is the following observation. Let X be a compact Riemannian manifold and let ρ be a representation of $\pi_1(X)$ on a finite-dimensional real vector space V. Suppose that there exists a lattice $M \subset V$ which is invariant under $\pi_1(X)$. Let $\mathcal{M}$ be the associated local system of finite rank free $\mathbb{Z}$-modules over X. Note that $\mathcal{M} \otimes_{\mathbb{Z}} \mathbb{R} = E_\rho$. The cohomology $H^*(X; \mathcal{M})$ of X with coefficients in $\mathcal{M}$ is a finitely generated abelian group. Suppose that ρ is acyclic, i.e., $H^*(X, E_\rho) = 0$. Then $H^*(X; \mathcal{M})$ is a finite abelian group. Denote by $|H^q(X; \mathcal{M})|$ the order of $H^q(X; \mathcal{M})$. Then, as observed by Cheeger, the Reidemeister torsion satisfies

$$\tau_X(\rho) = \prod_{q=0}^{n} |H^q(X; \mathcal{M})|^{(-1)^{q+1}}. \tag{5}$$

Using the equality of $T_X(\rho) = \tau_X(\rho)$, (5) provides an analytic tool to study the torsion subgroups in the cohomology. This has been used by Bergeron and Venkatesh in the following way: Let $\mathbf{G}$ be a connected semi-simple algebraic group over $\mathbb{Q}$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. Let $G = \mathbf{G}(\mathbb{R})$. Then Γ is a lattice in G. An example is $G = \mathrm{SL}(n,\mathbb{R})$ and $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. We assume that Γ is co-compact and torsion free. Let $K \subset G$ be a maximal compact subgroup and $\widetilde{X} \coloneqq G/K$ the associated Riemannian symmetric space of nonpositive curvature. Then Γ acts freely on $\widetilde{X}$ and $X \coloneqq \Gamma\backslash\widetilde{X}$ is a compact locally symmetric manifold. Consider a decreasing sequence of congruence subgroups $\cdots \subset \Gamma_{j+1} \subset \Gamma_j \subset \cdots \subset \Gamma$ with $\cap_j \Gamma_j = \{1\}$. Put $X_j \coloneqq \Gamma\backslash\widetilde{X}$, $j \in \mathbb{N}$. Then $X_j \to X$ is a finite covering. Let ϱ be a rational irreducible representation of $\mathbf{G}$ on a finite-dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. Then there is a lattice $M \subset V_{\mathbb{Q}}$ that is invariant under $\varrho(\Gamma)$. Let $V = V_{\mathbb{Q}} \otimes \mathbb{R}$ and $\rho \coloneqq \varrho|_{\Gamma}$. Then $M \subset V$ is a Γ-invariant lattice. Let $T_{X_j}(\rho)$ be the analytic torsion of the covering X_j of X with respect to $\rho|_{\Gamma_j}$. By an appropriate assumption on the highest weight of ϱ, there is a uniform spectral gap at the origin for all Laplacians on X_j, uniformly in $j \in \mathbb{N}$. In this case, it follows that the limit $\log T_{X_j}(\rho)/\operatorname{vol}(X_j)$ as $j \to \infty$ exists and equals a constant $t^{(2)}_{\widetilde{X}}(\rho)$ that depends only on $\widetilde{X}$ and ρ. In fact, $\operatorname{vol}(X) \cdot t^{(2)}_{\widetilde{X}}(\rho)$ is the L^2-torsion of X. Let $H^q(\Gamma_j; M)$ be the cohomology of Γ_j with coefficients in the Γ_j-module M. Note that $H^q(\Gamma_j; M) \cong H^q(X_j; \mathcal{M})$, where $\mathcal{M}$ is the local system associated to M. Then, as shown by Bergeron and Venkatesh, it follows from (5) combined with (4)

$$\liminf_{j\to\infty} \sum_q \frac{\log |H^q(\Gamma_j; M)|}{[\Gamma : \Gamma_j]} \geq C_{G,M}, \tag{6}$$

where the sum runs over the integers q such that $q + \frac{\dim(X)-1}{2}$ is odd and $C_{G,L} \geq 0$. Moreover, if the fundamental rank $\operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K)$ is 1, then $C_{G,M} > 0$. The latter condition is satisfied, for example, for hyperbolic manifolds and $X = \Gamma\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with $n = 3, 4$. For these cases it follows from (6) that the order of the torsion of the cohomology grows exponentially. There is a conjecture with a more precise statement saying that the exponential growth happens exactly in the middle degree. Müller and Rochon extended this result to the case of finite volume hyperbolic manifolds. This includes Bianchi subgroups $\Gamma_D = \mathrm{SL}(2, \mathcal{O}_D)$ of $\mathrm{SL}(2,\mathbb{C})$, where $\mathcal{O}_D$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$, $D > 0$, square free. The complementary case is if the lattice is fixed and the rank of the module M increases. This case has been studied by Müller, Pfaff, and Rochon with analogous results on the growth of torsion.

There are other interesting developments related to real analytic torsion which, due to the limited space, could not be discussed here.

Adam Marcus, Daniel Spielman, and Nikhil Srivastava

When Dan Spielman and Is Singer were colleagues at MIT, they discussed Singer's tennis game and his observations on the winding of the stairways in Building 2, but their mathematical interests seemed completely unrelated. It is a testament to the breadth of Singer's impact that, many years later, Spielman was led by Graph Theory and Computer Science to the Kadison–Singer Problem. We now explain the path from the original statement of the Kadison–Singer Problem in terms of operator algebras to the problem in Graph Theory that motivated its solution.

Operator algebras were originally introduced by von Neumann as a rigorous mathematical framework for quantum mechanics, in which bounded self-adjoint operators in the C^*-algebra $B(H)$ play the role of physical observables, for H a complex separable Hilbert space. Abelian subalgebras of $B(H)$ play a special role in that they are generated by observables that commute, implying that they can be measured simultaneously without being constrained by an uncertainty principle. Kadison and Singer were motivated by the following question of Dirac (which he believed had an affirmative answer):

> Given a quantum system (such as an electron in a hydrogen atom) does knowing the outcomes of all measurements with respect to a maximal set of commuting observables (such as the quantum numbers n, ℓ, m, s) *uniquely* determine the outcomes of all possible measurements of all possible observables?

To turn this into a mathematical question, recall that a *state* on a C^* algebra $\mathcal{A}$ is a normalized positive continuous linear functional $\phi : \mathcal{A} \to \mathbb{C}$. States correspond to physical states of a quantum system, and a state which is not a convex combination of other states is called a *pure state*. Dirac's question asks whether every pure state on a maximal abelian subalgebra $\mathcal{A}$ of $B(H)$ has a unique extension to a state on $B(H)$. By a classification theorem, it is sufficient to study $\mathcal{A}$ isomorphic to either the "continuous" algebra $\mathcal{A}_c$, the algebra of essentially bounded multiplication operators on $L^2(0,1)$, or the "discrete" agebra $\mathcal{A}_d = D(\ell^2(\mathbb{N})))$, the algebra of bounded infinite diagonal matrices. In their 1959 paper, Kadison and Singer

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[12] showed that the unique extension property *fails* for $\mathcal{A}_c$. They were unable to settle the case of $\mathcal{A}_d$, which became known as the *Kadison–Singer Problem*:

> Does every pure state on $D(\ell^2(\mathbb{N}))$ have a unique extension to a state on $B(\ell^2(\mathbb{N}))$?

It is easy to see that this problem is equivalent to showing that $\rho(M) = 0$ for every pure state ρ on $D(\ell^2(\mathbb{N}))$ and every $M \in B(\ell^2(\mathbb{N}))$ with zero diagonal. The difficulty is that the set of pure states on $D(\ell^2(\mathbb{N}))$ is quite complicated, containing points corresponding to nonprincipal ultrafilters, and thus rather inaccessible in concrete terms. Kadison and Singer outlined an elegant approach to solving their problem without having to say too much about these states directly. The key observation is that if $P \in D(\ell^2(\mathbb{N}))$ is a diagonal projection and ρ is a pure state on $D(\ell^2(\mathbb{N}))$ then $\rho(P) = 0$ or $\rho(P) = 1$. This leads to the associated notion of a *paving* of an operator.

Definition 0.1. An ϵ-paving of an operator $M \in B(\ell^2(\mathbb{N}))$ is a finite collection of diagonal projections $P_1, \dots, P_k$ satisfying $P_1 + \dots + P_k = I$ and

$$\|P_i M P_i\| \leq \epsilon \|M\|,$$

for every $i = 1, \dots, k$.

Kadison and Singer showed via a simple Cauchy–Schwartz type argument that if $M \in B(\ell^2(\mathbb{N}))$ has an ϵ-paving, then $|\rho(M)| \leq \epsilon$, reducing the KSP to the *infinite paving conjecture*: For every $\epsilon > 0$, every zero diagonal $M \in B(\ell^2(\mathbb{N}))$ has an ϵ−paving. The pleasing feature of this conjecture is that it makes no mention of pure states.

The next simplification was achieved by Anderson in 1979, who showed via a compactness argument that the *infinite paving conjecture* is implied by a simply stated combinatorial conjecture about *finite* matrices.

Conjecture 0.2 (Finite Paving Conjecture). *For every $\epsilon > 0$ there is a $k = k(\epsilon)$ such that for every n, every zero diagonal complex $n \times n$ matrix M can be ϵ−paved with k projections.*

The most important feature of this conjecture is that the number of projections k is allowed to depend only on ϵ and *not on the dimension* n, and this is because any dependence on n precludes a limit.

In the decades since Anderson's result, the paving conjecture was shown (using various finite-dimensional linear algebra arguments) to be equivalent to several other statements about partitioning matrices or sets of vectors into submatrices or subsets which are "smaller" in some appropriate sense. In particular, the work of Casazza et al. shows that it is equivalent to a number of other conjectures in various fields of pure and applied mathematics. A very tangible, combinatorial such statement is the following conjecture of Weaver, which is actually a family of statements indexed by $r \in \mathbb{N}$. The validity of the conjecture for any r implies the paving conjecture.

Conjecture 0.3 (Weaver KS_r). *There are universal constants $\epsilon, \delta > 0$ such that the following holds. Suppose $v_1, \dots, v_m \in \mathbb{C}^n$ are vectors satisfying $\sum_{i=1}^m v_i v_i^* = I$ and $\|v_i\| \leq \delta$. Then there is a partition of $\{1, \dots, m\}$ into $T_1 \cup T_2 \dots \cup T_r$ such that for every $j \in \{1, \dots, r\}$:*

$$\left\| \sum_{i \in T_j} v_i v_i^* \right\| \leq 1 - \epsilon. \tag{7}$$

We now describe how a problem in Graph Theory leads to the same core problem KS_2, and which is in fact how the present authors were introduced to this problem. The question we consider is:

> Can every finite undirected graph be approximated by a graph with very few edges?

The answer to this question depends on the notion of approximation, and this is where the Laplacian operator comes in. Recall that the discrete Laplacian of a weighted graph $G = (V, E, w)$ may be defined as the following sum of rank one matrices over the edges:

$$L_G = \sum_{(a,b) \in E} w_{(a,b)} (e_a - e_b)(e_a - e_b)^T,$$

where e_a is the elementary unit vector in direction a.

Two graphs G and H on the same vertex set V are *spectral approximations* of each other if their Laplacian quadratic forms multiplicatively approximate each other:

$$\kappa_1 \cdot L_H \preccurlyeq L_G \preccurlyeq \kappa_2 \cdot L_H,$$

for some approximation factors $\kappa_1, \kappa_2 > 0$, where $A \preccurlyeq B$ means that $B - A$ is positive semidefinite. An important example is given by a d−regular Ramanujan graph on n vertices, G_n, and the complete graph on n vertices, K_n:

$$(1 - 2\sqrt{d-1}/d) L_{K_n} \preccurlyeq (n/d) L_{G_n} \preccurlyeq (1 + 2\sqrt{d-1}/d) L_{K_n}.$$

So, $(n/d)G$ (i.e., G with edge weights scaled up by n/d) is a good sparse approximation of K_n.

In 2012, Batson, Spielman, and Srivastava proved that *every* weighted graph G has an approximation that is almost this good, namely a weighted graph H with average degree at most $2d$ which approximates G with $\kappa_1 = \left(1 - \frac{1}{\sqrt{d}}\right)^2$ and $\kappa_2 = \left(1 + \frac{1}{\sqrt{d}}\right)^2$. Their proof had very little to do with graphs. In fact, they derived their result from the following theorem about sparse weighted approximations of sums of rank one matrices.

Theorem 0.4. *Let $v_1, v_2, \dots, v_m$ be vectors in $\mathbb{R}^n$ with $\sum_i v_i v_i^T = V$. For every $\epsilon \in (0,1)$, there exist nonnegative real numbers s_i with $|\{i : s_i \neq 0\}| \leq \lceil n/\epsilon^2 \rceil$ so that*

$$(1-\epsilon)^2 V \preccurlyeq \sum_i s_i v_i v_i^T \preccurlyeq (1+\epsilon)^2 V. \tag{8}$$

Taking V to be a Laplacian matrix written as a sum of outer products and setting $\epsilon = 1/\sqrt{d}$ immediately yields the theorem about approximating graphs.

Theorem 0.4 is very general and turned out to be useful in a variety of areas including graph theory, numerical linear algebra, and metric geometry (see for instance Naor's 2011 Bourbaki talk). One of its limitations is that it provides no guarantees on the weights s_i that it produces, which can vary wildly. So it is natural to ask:

> Is there a version of Theorem 0.4 in which all the nonzero weights s_i are the same?

In the case $V = I$, it is easy to see that equal weights are impossible whenever there is a vector v_i with large norm. The punch line is that Weaver's conjecture KS_2 is *equivalent* to the refined assertion that for $V = I$, equal weights can be attained in Theorem 0.4 whenever all vectors have small norm—the desired partition is obtained by taking $T_1 = \{i \leq m : s_i \neq 0\}$ and $T_2 = [m] \setminus T_1$. This similarity was originally pointed out to us by Gil Kalai.

The main result of [13] is that KS_2 is true, settling the Kadison–Singer problem as well as the speculation about unweighted graph sparisfication. The key idea of the proof is to study the *expected characteristic polynomial* of the random matrix

$$\bigoplus_{j=1,2}\left(\sum_{i\in T_j} v_i v_i^*\right)$$

in the setting of Conjecture 0.3, where the partition is chosen uniformly at random. Miraculously, it turns out that the polynomials corresponding to bad partitions of large operator norm "cancel each other out" in this average, and the expectation has real roots which are bounded away from 1 by an absolute constant. The proof of the real-rootedness utilizes tools from the *geometry of polynomials*, a classical subject studying among other things the dynamics of complex roots of univariate and multivariate polynomials under differential operators. Originally motivated by considerations in PDEs and mathematical physics (specifically, the Lee–Yang theorem) in the 1950s, this area saw several breakthroughs in the 2000s, in particular the work of Gurvits on the van der Waerden conjecture and of Brandën and Borcea, whose ideas played a crucial role in our proof. The remaining ingredient, necessary for proving the quantitative bound on the largest root, was to adapt the argument used to prove Theorem 0.4 (especially the inductive use of the Stieltjes transform to control eigenvalues) from the setting of matrices to the new setting of expected characteristic polynomials.

Edward Witten

I first met Is Singer in 1979, while visiting the University of California at Berkeley, where he was a professor at the time. Singer was then running a very active seminar on quantum field theory and differential geometry. He had been one of the first mathematicians or physicists to appreciate the new opportunities for interaction of math and physics that were opened up by the emergence of the modern Standard Model of particle physics, which is based on nonabelian gauge theory.

Through his vision and his contributions, Is Singer played a major role in the emerging interaction of mathematics and quantum field theory. One of the early landmarks was the role in physics of the Atiyah–Singer index theorem. A puzzle in strong interaction physics was a missing symmetry—a symmetry of the classical action of the Standard Model that was not visible in experiment. It turned out that the solution to this puzzle revolves around "instantons" of four-dimensional gauge theory, and the Atiyah–Singer index theorem. This had become clear in 1976, through the work of Gerard 't Hooft as interpreted by Albert Schwarz. As a result, instantons and the index theorem were prominent at the interface of physics and differential geometry in the late 1970s, when I first met Singer. By this time, Singer, with Michael Atiyah and Nigel Hitchin, had developed the mathematical foundations of the study of instantons on a general four-manifold. All this was one of the topics in his seminar.

Within a few years, the many sides of the index theorem became familiar to physicists. For example, in 1982 it turned out that the mod 2 index theorem of Atiyah and Singer governs an obstruction to a certain generalization of the Standard Model. In 1984, Atiyah and Singer—in what proved to be their last joint paper—interpreted the "anomalies" of Steve Adler, John Bell, and Roman Jackiw in terms of the families index theorem (the index theorem for a family of elliptic differential operators). These anomalies are an important constraint on the consistency of the Standard Model and its possible extensions. With Henry McKean, Singer had initiated the heat kernel approach to the index theorem. This approach became familiar to physicists as it is natural in quantum field theory and is closely related to the supersymmetric proof of the index theorem. Over time, the influence of the index theorem in physics spread beyond elementary particle physics and relativistic quantum field theory, where it started. By now, the index theorem is also a familiar and important tool in condensed matter physics as well.

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Another aspect of Singer's work that has been important in physics involves the η-invariant that was introduced by Atiyah and Singer with Vijay Patodi. The Gauss–Bonnet formula for the Euler characteristic of a manifold with boundary has a boundary contribution that involves the integral of a local invariant involving the extrinsic curvature of the boundary. Atiyah, Patodi, and Singer showed that for a Dirac-like operator on a boundary, in general there is no local elliptic boundary and no local invariant analogous to the extrinsic curvature. Instead they introduced a "global" boundary condition and proved an index theorem in which a spectral invariant that they called η appears as a boundary contribution. All this is relevant in quantum field theory at multiple levels. The most basic is that the global APS boundary condition encodes the ground state of a quantum field theory; in fermion quantum field theory, the APS index theorem has a direct physical interpretation. The APS η-invariant has several other manifestations in quantum field theory that are loosely related to this. It controls "charge fractionalization," an important phenomenon, first explored in the early 1970s by Jackiw and Claudio Rebbi (without knowing the relation to index theory and the η-invariant), in which the ground state of a quantum field carries fractional quantum numbers. The η-invariant is also important in a refined understanding of the Adler–Bell–Jackiw anomaly and its generalizations. All these facets of the η-invariant have manifestations in condensed matter physics as well as relativistic field theory.

Finally, I should mention the role in physics of Ray–Singer analytic torsion. In 1971, Daniel Ray and Singer discovered an invariant of a flat bundle on a compact manifold M that is expressed in terms of a regularized determinant of the Laplace operator acting on differential forms on M. Ray and Singer conjectured that their "analytic torsion" is equivalent to the "combinatorial torsion" of Reidemeister, and this was later proved by Jeff Cheeger and Werner Müller. Two years later, Ray and Singer formulated a version of analytic torsion for holomorphic vector bundles over over a complex manifold. Here, there is no combinatorial counterpart of the analytic torsion. Analytic torsion for complex manifolds is important in string theory in multiple ways. The complex manifold is usually either the worldsheet of a string, or a factor in spacetime. Analytic torsion for real manifolds appears in physics primarily in topological field theory and in simple models of quantum gravity. When one expands the three-dimensional Chern–Simons topological field theory in perturbation theory, the leading approximation can be expressed in terms of the analytic torsion and the η-invariant. In higher orders, one runs into more subtle invariants, which were studied by Singer and Scott Axelrod in two relatively well-known papers in the early 1990s. The most interesting application of analytic torsion to gravity involves the relation of a simple model of gravity in two dimensions that is known as Jackiw–Teitelboim or JT gravity to the volumes of moduli spaces of Riemann surfaces; the celebrated results of Maryam Mirzakhani on those volumes are also part of this story. The relation of JT gravity to the volumes is a special case of a more general statement about analytic torsion.

Apart from specific applications of analytic torsion in physics, there was a technical step in the work of Ray and Singer that has also been important. To define a regularized determinant of a nonnegative self-adjoint differential operator, they first defined the ζ function $\zeta(s) = \sum_i \lambda_i^{-s}$, where λ_i are the eigenvalues of the operator, and then, assuming $\zeta(s)$ has an analytic continuation that is holomorphic at $s = 0$, they define the determinant as $\exp(-\zeta'(0))$. The analytic continuation to $s = 0$ can be made in fairly wide circumstances, using the ideas of McKean and Singer concerning the heat kernel. Determinants of differential operators were known to be important in physics, but the methods that physicists were using to define these determinant were less incisive than the ζ-function regularization of Ray and Singer. ζ-function regularization was taken into physics in the 1970s by Stuart Dowker and Raymond Critchley, followed by Stephen Hawking. I know from multiple discussions that Singer was particularly proud of this application of his work to physics.

Is developed many active collaborations with physicists—his collaborators included Orlando Alvarez, Laurent Baulieu, and Axelrod, among others. Is' interest in high energy physics was not limited to theory. He became very interested in what was going on in experiments where fundamental ideas of physics are tested. My wife and I were on sabbatical in the spring of 2009 at the European particle physics laboratory CERN when Is visited. We toured the ATLAS detector at CERN, one of the two detectors that discovered the Higgs particle three years later.

One thing that Is and I had in common, apart from an interest in math and physics, is that we both took up tennis relatively late in life (in our forties) and became passionate about the game. It ended up that we never got to play; the one time that this almost happened, Is ended up bowing out and leaving me on the court with Cumrun Vafa and his sons. But I am pretty sure that Is' tennis was on a different level from mine. He had been an athlete in his youth—a minor league baseball player for a time, in fact. I can remember many discussions about tennis. Perhaps 15 years ago, he told me that he had finally developed a strong serve. On another occasion, he expressed the ambition of eventually becoming the United States Tennis Association champion in the oldest age group. Yet another time, he explained that he had assured the dean at MIT that he would retire once the dean could beat him in tennis.

It was a pleasure to know Is and to learn from him over the years.

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