The cover design is based on imagery from “Mathematical Foundations of Graph-Based Bayesian Semi-Supervised Learning,” page 1717.
Polynomial systems, homotopy continuation and applications

January 2–3, 2023 | Boston, MA

in conjunction with the Joint Mathematics Meetings

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[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society’s street address and the other bracketed information is a technical requirement of the US Postal Service.

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Who Are the Professors of Teaching?

They are called lecturers, instructors, teaching professors, professors of the practice (and more rarely in departments of mathematics, clinical professors or research professors). They are faculty holding full-time fixed terms or renewable appointments but are usually NOT eligible for tenured or tenure-track positions at their institutions. (They are NOT postdoctoral fellows). According to the Departmental Report of the Mathematical and Statistical Sciences Annual Survey of 2018, they represent roughly 33% of all the full-time faculty in mathematics and statistics throughout the country. In 2018, according to the survey, there were 5826 of them on renewable appointments (plus 1062 on fixed terms) while there were a total of 17,184 faculty with tenure or on tenure-track positions. It is to be noted that departments in large private and public research institutions have many such positions with non-tenure-track faculty representing 50% of the full-time faculty overall at mathematics or statistics departments in such institutions. While this number includes 1607 postdoctoral positions in 2018, the number of non-tenure-track faculty on renewable appointments was more than three times the number of postdoctorate faculty. 20% of them hold a master’s degree in the mathematical sciences (including statistics) as their highest degree while 80% have a PhD. There are many discussions to be had about the trends institutions of higher education are following and the perceived threat to the tenure system, but I want to focus here on those lecturers, teaching professors, and professors of the practice: who they are, their plight, and their unrealized potential. The article below reflects the author’s own experiences and his understanding of the shared experiences of some other colleagues in similar positions in a few departments of mathematics all within large private and public research institutions.

Who Are the Non-tenure-track Faculty in Mathematics?

Whatever labels their positions are given, these faculty chose to pursue an academic career focused on teaching mathematics. While teaching faculty positions are sometimes not an individual’s first choice, many chose such jobs because of their passion for teaching. This is especially the case among recent graduates. Some made the decision to seek such positions during their PhD program (and some during their postdoctorate years). While they were working on their thesis and being trained to become researchers, they realized that their love for mathematics did not have to express itself solely through active research, and that they were passionate about teaching mathematics at their very core. While some still pursue some mathematical research and publish in mathematical sciences journals, their passion is in teaching mathematics at the college level.

Students usually are not aware of the difference between the non-tenure-track faculty and their tenured or tenure-track colleagues. But if they did, they might notice that the non-tenure-track faculty who teach them look more diverse. For instance, according again to the Departmental Report of the Mathematical and Statistical Sciences Annual Survey of 2018, close to 70% of the mathematics faculty with a PhD in non-tenure-track renewable positions are women, while only 30% of tenured faculty are women.

David Crombecque is a professor of teaching at the University of Southern California. His email address is crombecq@usc.edu.
While their teaching assignments are often focused on undergraduate courses, especially lower-division courses such as Calculus, they also teach undergraduate courses for mathematics majors. They may develop new courses or modernize existing classes and they provide service to the department and the university at large, such as managing a math lab or a math center or running a math club. Many of these faculty are eager to engage with students, not only on the subject of mathematics but also on the subject of the student’s college experiences and future goals. These faculty often have much more contact with students and become de facto unofficial advisors and mentors for many undergraduate students.

Their teaching loads are two to three times those of tenured faculty, but many will still pursue scholarly work. Some continue to do research in mathematics or mathematics education, present at conferences, and publish their research. Some will engage in mathematics outreach not just on campus but in the communities around campus. While teaching math classes at their institutions is the main requirement of their position, many actually do much more.

**What Does Their Career Look Like?**

Many institutions (in particular large private and public universities) offer different paths to promote the non-tenure-track professors. Some offer a path similar to the tenure system with ranks such as assistant, associate, and full professor. Often, promotions to a higher rank lead to a new contract with longer duration (such as renewable three- or five-year contracts). In some cases, such as in the UC system, the institution provides a path to what is referred to as “security of employment” (which is not the same as tenure). In many cases, the evaluation process leading to promotion includes class observations from other (often tenured) colleagues and focuses on student evaluations and evaluation scores. As such, those faculty work hard to develop their teaching skills and generally strive for impressive teaching records. Yet, unlike the tenure-track faculty who are usually assigned a senior faculty mentor, many departments lack mentorship opportunities for the non-tenure-track faculty. The norm is that they also do not get sabbaticals and are thus often not able to participate or engage in deeper professional development opportunities to improve their teaching skills (whether at their own institution or outside). It is more difficult for them to know about the different types of scholarly work and activities they can pursue. This is a huge impediment to those in their early career trying to move up the ranks or simply trying to keep their job.

For faculty who stay in these positions at the same institution for a long time, they accumulate a lot of experience and knowledge of their institution. But unlike tenured faculty, whether or not they can seek and be awarded managerial and administrative positions (for example as vice chair for undergraduate affairs of a department, associate dean, vice dean, or dean) depends on the institution. This definitely constitutes a glass ceiling for those faculty.

Finally, as full-time professors, they usually receive benefits similar to tenured/tenure-track faculty (health care, retirement benefits), but their salaries are noticeably lower than those of their tenured/tenure-track colleagues. For example, according to the 2019–2020 Faculty Salaries Report from Mathematical and Statistical Annual Survey, for the year 2018–2019, the mean salary for new tenure-track faculty in mathematics departments at large public institutions was $95,340, while for non-tenure-track faculty (which includes postdoctorate fellows but does NOT include part-time faculty), the mean salary was $61,160. At large private institutions, the mean salary was $93,390 for new tenure-track faculty and $71,400 for non-tenure-track faculty. This can be demoralizing for this group of faculty as it can be interpreted as a signal that their institution values its education mission less than its research mission (even though universities generate substantial tuition revenue, especially private universities) and thus that their work is of lesser value than their tenure/tenure-track colleagues. More importantly, it can simply be difficult for them to make ends meet financially causing them financial hardship (though this is nothing compared to the plight of part-time faculty). They might have to rely on teaching overloads (if even available) which in turn can lead to lower teaching evaluations and thus hinder their chances of promotion. Or they simply might need to look for another job. The covid pandemic has made this worse in many places where salaries were frozen or benefits were cut (this was also the case for tenure/tenure-track faculty but the pain was greater for the non-tenure-track as their salaries were already lower).

**Time for Progress**

As many universities and departments rely more and more on faculty in these positions, we need to stop seeing such positions as some sort of failure and instead recognize the value and hard work of these faculty and more fully support them in achieving successful careers. This can start in graduate programs. While PhD advisors want to see their students obtain tenure-track positions, we have to face the reality of numbers. A significant proportion of PhD students will end
up in non-tenure-track positions, whether it is what they wanted or not. It should then be up to the PhD program to provide training opportunities to PhD students to prepare them for such positions.

Institutions and departments should provide articulated career ladders which include service, outreach, and education, but with flexibility to allow for the myriad kinds of academic endeavors that faculty may choose to engage in. Outreach and communication with society at large about the sciences and mathematics are becoming essential. Departments could encourage and support those faculty who choose to engage in activities such as creating bridges with K–12 education (math circles and math teacher circles), outreach with the local community, adult education, diversity initiatives at the university, and service outside of the math department, including faculty governance.

The evaluation of such faculty for promotion should consider everything they are doing. The teaching evaluation of these faculty should include a lot more than just student evaluations and an occasional in-class observation. For instance they may include teaching portfolios put together by the faculty with teaching statements, materials they develop, experiments they have tried, and so on. If faculty in these positions don’t have confidence that their teaching is being evaluated in a holistic way they will not be able to afford to experiment with teaching and improve. Mentoring and professional development should be provided to faculty on these ladders. Universities are in need of teaching methodologies at the undergraduate level (and graduate level for that matter) that can successfully reach an increasingly diverse student body. Teaching loads should be reasonable so that faculty have time for service or to experiment with different teaching modalities. Finally, short of offering them access to tenure eligibility (although some institutions in Canada for instance do include them in their tenure system), universities employing non-tenure-track faculty should offer similar salaries. All of this would help this group of faculty realize their full potential, from which the departments and universities will surely reap the benefit. In many places, this is a work in progress. Let’s make sure this progress benefits these faculty, their tenured colleagues, and their students.

References
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Introduction

Groups and spaces go hand in hand. For a given space, there are many groups associated to it. We can consider the group of symmetries, that is, the group of structure preserving bijections. Additionally, there is the fundamental group and also the homology and cohomology groups to name a few more. As pointed out by Hermann Weyl, these groups can give "a deep insight" into a given space. An example of this phenomenon is in the study of knots. Algebraic invariants in the form of groups show that the trefoil knot cannot be unknotted for instance. See Figure 1.

Geometric group theory takes a different perspective on this relationship between groups and spaces. Rather than using the algebraic structure and properties of groups to study spaces, the main philosophy of geometric group theory is the following.

Study groups using the topology and geometry of the spaces they act on.

That is, groups are the central objects of study and the techniques and tools used to investigate them are dynamical, geometrical, and topological in nature.
In name, geometric group theory is quite new in relation to other mathematical fields. The foundational essays by Gromov [Gro87, Gro93] introducing the notion of hyperbolic groups and initiating the study of finitely generated groups as metric spaces sparked an enormous amount of research and established lines of investigation that are still very active today. Prior to the emergence of geometric group theory, there were geometrical ideas present in group theory in the works of Dehn, Whitehead, van Kampen, and others. Additionally, Thurston’s work on 3–manifolds showed how the geometry of a manifold influences algebraic and algorithmic properties of its fundamental group. It is Gromov’s essays though that mark the beginning of the perspective where these ideas are at the forefront.

This article is intended to give an idea about how the topology and geometry of a space influences the algebraic structure of groups that act on it and how this can be used to investigate groups. As you will see, I take the approach I learned from my advisor Mladen Bestvina of favoring illustrative examples over general theory. As is true of any survey of a mathematical field, many aspects and areas of geometric group theory are not mentioned at all. The final section includes a short list of books on geometric group theory for further reading.

Groups and Spaces

As mentioned above, geometric group theory uses group actions on spaces to understand the group’s structure. What type of information could one hope to glean from an action? Are there always interesting actions to study? We will take a look at both of these questions now.

An example: $SL(2, \mathbb{Z})$. To give an illustration of how the topology of a space that a group acts on influences the group’s structure, let’s take a look at an example of a group action that appears in many areas of mathematics. We will consider the group of $2 \times 2$ matrices with integer entries and determinant equal to 1. This group is called the special linear group:

$$SL(2, \mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$ 

Is $SL(2, \mathbb{Z})$ finitely generated? That is, are there finitely many matrices $A_1, \ldots, A_n \in SL(2, \mathbb{Z})$ such that any matrix $M$ in $SL(2, \mathbb{Z})$ can be expressed as a product $M = A_1^{e_1} \cdots A_n^{e_n}$? (Note, each $A_j$ may appear multiple times.) The answer is “yes” and there is an algebraic approach to this problem, but let’s take a geometric perspective and consider an action of $SL(2, \mathbb{Z})$ on a metric space.

The space we will consider is the Farey complex which is constructed as follows. First, we start with a graph whose vertex set is the set of rational numbers $\frac{p}{q}$—always expressed in lowest terms—along with an additional point we denote $\frac{1}{0}$. Edges join two vertices $\frac{p}{q}$ and $\frac{r}{s}$ if $ps - qr = \pm 1$. Figure 2 shows a portion of this graph, known as the Farey graph.

Figure 2. The Farey graph and Farey complex.

As seen in Figure 2, the edges in the Farey graph naturally form triangles. In fact, the vertices of any such triangle always have the form $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{p+r}{q+s}$ for instance, $\frac{1}{0}$, $\frac{0}{1}$, and $\frac{1}{1}$, and also $\frac{1}{0}$, $\frac{0}{1}$, and $\frac{1}{1}$. There is an action of $SL(2, \mathbb{Z})$ on the Farey graph defined by permuting the vertices using the rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{p}{q} = \frac{ap + bq}{cp + dq}.$$ 

It is easy to check that two vertices $\frac{p}{q}$ and $\frac{r}{s}$ are connected by an edge only if their images $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{p}{q}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{r}{s}$ are. Hence, this defines an action on the Farey graph and by extension on the Farey complex, which is the space we get by filling in the triangles in the Farey graph.

You have most likely seen this space and action before but under a different guise. Indeed, the Farey complex gives a tessellation of the hyperbolic plane by ideal triangles whose vertices in the upper half plane model are either rational or $\infty$. Moreover, the action described above is none other than the usual action of $2 \times 2$ matrices with real entries and positive determinant by fractional linear transformations of the upper half plane, in particular, by
conformal maps. The conformal maps:
\[
f(z) = \frac{1 - iz}{z - i} \quad \text{and} \quad g(z) = \frac{1 + iz}{z + i}
\]
relate the two pictures. See Figure 3.

![Figure 3](image-url)

**Figure 3.** The Farey tessellation of the upper half plane by ideal triangles.

Now it is time to examine this action. Let \( \Delta \) denote the triangle in the Farey complex with vertices \( \frac{1}{0}, \frac{1}{1}, \) and \( \frac{1}{1} \). We record the key properties of the action in two claims.

**Claim 1.** For any triangle \( \Delta' \) in the Farey complex, there is matrix \( M \in SL(2, \mathbb{Z}) \) such that \( M\Delta = \Delta' \).

Indeed, suppose the vertices of \( \Delta' \) are \( \frac{p}{q}, \frac{r}{s}, \) and \( \frac{p+q}{q+s} \) where \( ps - qr = 1 \). Take \( M = [\frac{p}{q}, \frac{r}{s}] \) and observe that \( M\Delta = \Delta' \).

Let \( A = [\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}] \). Notice that \( A \cdot \frac{0}{1} = \frac{1}{1}, A \cdot \frac{1}{1} = \frac{1}{0} \), and \( A \cdot \frac{1}{1} = \frac{1}{0} \) so that \( A\Delta = \Delta \) and \( A \) acts on the triangle \( \Delta \) by a rotation.

**Claim 2.** If \( M\Delta = \Delta \), then \( M = A^k \) for some integer \( k \).

Indeed, if \( M \) fixes \( \Delta \), then it must cyclically permute the vertices \( \frac{0}{1}, \frac{1}{0}, \) and \( \frac{1}{1} \). Hence \( A^kM \) fixes the vertices \( \frac{0}{1}, \frac{1}{0}, \) and \( \frac{1}{1} \) for some \( k \). As the only conformal map that fixes three points is the identity, we see that \( A^kM = \pm I \). (Note, \(-I\) acts as the identity map.) The claim follows once we check that \( A^3 = -I \).

Let \( B = [\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}] \) and let \( \Delta_{pq} \) be the triangle that shares an edge with \( \Delta \) and has the vertex \( \frac{p}{q} \in \{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}\} \). These are labeled in Figure 2. We observe that \( B\Delta = \Delta_{1/2} \). As \( A \) rotates \( \Delta \), we also find that \( AB\Delta = \Delta_{1/2} \) and \( A^2B\Delta = \Delta_{2/1} \).

We are now in the position to show that \( SL(2, \mathbb{Z}) \) is finitely generated by the matrices \( A \) and \( B \). That is, any matrix \( M \in SL(2, \mathbb{Z}) \) can be expressed as a product of \( A \)'s and \( B \)'s:

\[
M = A^{m_1}B^{n_1} \cdots A^{m_k}B^{n_k}
\]

for some integers \( m_j, n_j \). Given \( M \in SL(2, \mathbb{Z}) \) we want to consider paths in the Farey complex from \( \Delta \) to \( M\Delta \). What do we mean by path? Specifically, we mean a sequence of triangles \( \Delta = \Delta_0, \ldots, \Delta_k = M\Delta \) where the triangles \( \Delta_{j-1} \) and \( \Delta_j \) share an edge.

Now we proceed via induction on the length of shortest path to \( M\Delta \). Claim 2 handles the case that this length is 0. Next, using a path \( \Delta = \Delta_0, \ldots, \Delta_k = M\Delta \) of minimal length we observe by Claim 1 and induction that \( \Delta_{k-1} = M_0\Delta \) where \( M_0 \) can be expressed as product of \( A \)'s and \( B \)'s.

Let's hit the whole picture with \( M_0^{-1} \): the triangle \( \Delta_{k-1} = M_0\Delta \) is sent to \( \Delta \) and the triangle \( \Delta_k = M\Delta \) is sent to an adjacent triangle, i.e., one of \( \Delta_{1/2}, \Delta_{2/3}, \) or \( \Delta_{2/1} \). Assuming for simplicity that \( M_0^{-1}M\Delta = \Delta_{1/1} \), which is equal to \( B\Delta \), we find that \( B^{-1}M_0^{-1}M\Delta = \Delta \). Claim 2 now shows that \( B^{-1}M_0^{-1}M = A^k \) and hence \( M = M_0BA^k \). Since \( M_0 \) can be expressed as a product of \( A \)'s and \( B \)'s, so can \( M \), showing that \( SL(2, \mathbb{Z}) \) is finitely generated.

**A theorem: characterizing finite generation.** What did we actually use to prove finite generation? The important topological property we used was the path-connectedness of the Farey complex so that we had a path from \( \Delta \) to \( M\Delta \) to apply induction on. The important dynamical property we used was the existence of a transitive tiling for which the stabilizer of a tile is finite and for which one tile meets only finitely many other tiles. These dynamical considerations naturally lead to the following definition.

**Definition 1.** An action of a group \( G \) on a metric space \((X,d)\) by isometries is geometric if it satisfies the following two conditions:

1. (cocompact) there exists a compact set \( K \subseteq X \) such that \( \bigcup_{g \in G} gK = X \); and
2. (properly discontinuous) for any compact set \( Y \subseteq X \), the set \( \{ g \in G \mid gY \cap Y \neq \emptyset \} \) is finite.

The requirement of a transitive tiling is captured by the cocompact condition. The properly discontinuous condition captures both requirements that the stabilizer of a tile is finite and that a tile meets only finitely many tiles.

**Technical Sidenote (i.e., feel free to ignore):** The actions of \( SL(2, \mathbb{Z}) \) on the Farey complex and on the upper half plane are not geometric. For the Farey complex the action is cocompact, but a triangle intersects infinitely many other triangles at a vertex, so the action is not properly discontinuous. We got around this problem by only considering triangles that meet along an edge—there are only finitely many such triangles. In the upper half plane the action is properly discontinuous, but the action is not cocompact. We can get around this by removing an equivariant collection of disjoint open disks tangent to the rational points. In either setting, the crucial point is that our notion of path ignores the vertices/ideal points. There is a geometric action lurking in the background here on the Farey tree that will be explored later.
Here are some examples of geometric actions.

1. The group $\mathbb{Z}^n$ acting by linearly independent translations on $\mathbb{R}^n$ equipped with the Euclidean metric.

2. More generally, any group of isometries of $\mathbb{R}^n$ equipped with the Euclidean metric that leaves a lattice $\Lambda \subset \mathbb{R}^n$ invariant and whose action on the lattice has finitely many orbits.

3. The fundamental group $\pi_1(X)$ of a compact Riemannian manifold $X$, possibly with boundary, acting by deck transformations on its universal cover $\tilde{X}$ equipped with the pull-back metric.

Arguing as we did for $\text{SL}(2, \mathbb{Z})$, we can prove the “if” direction of a geometric characterization of finite generation.

**Theorem 1.** A group is finitely generated if and only if it acts geometrically on a path-connected metric space.

For the “only if” direction, we need to introduce an important concept in geometric group theory: the Cayley graph.

A space for every group. For a finitely generated group $G$ we need to produce a path-connected metric space that admits a geometric action by $G$. This is similar to what is required to prove Cayley’s theorem from classical group theory: Every group is isomorphic to a permutation group. In the classical setting, we need to produce a set that admits a permutation action by our group. There is only one natural choice, the set is the group $G$ and the action is left multiplication.

In our current setting, the idea is similar. The metric space is built on top of the group, the extra parts of the space come from a finite generating set. The result is called a Cayley graph. Here are the details.

**Definition 2.** Let $G$ be a finitely generated group and let $S \subseteq G$ be a finite generating set. The Cayley graph, denoted $\Gamma(G,S)$, is the graph whose vertex set is $G$ and where there is an edge joining vertices $h_1, h_2 \in G$ if $h_1^{-1}h_2 \in S$, i.e., $h_2 = h_1s$ for some generator $s \in S$.

The group $G$ acts on $\Gamma(G,S)$ by permuting the vertices via left multiplication. Indeed, if the vertices $h_1, h_2 \in G$ are adjacent, then so are the vertices $gh_1, gh_2$ as $(gh_1)^{-1}(gh_2) = h_1^{-1}h_2$, and so the permutation action on the vertices extends to the entire graph.

As $S$ generates $G$, the Cayley graph $\Gamma(G,S)$ is path-connected. Figure 4 illustrates the path connecting the identity element of the group $1_G$ to the element $g = s_1s_2 \cdots s_k$ where each $s_j$ belongs to $S \cup S^{-1}$. The key point is that $s_1 \cdots s_k$ is adjacent to $s_1 \cdots s_{k-1}$.

Here are some examples of Cayley graphs.

1. $\mathbb{Z}$ and $\mathbb{Z}^2$: For $\mathbb{Z}$, we can use $S = \{1\}$ and for $\mathbb{Z}^2$ we can use $S = \{[0,1], [1,0]\}$. These graphs are pictured in Figure 5. Other generating sets are possible too, try drawing the graph $\Gamma(\mathbb{Z},\{2,3\})$. You can find this graph in the essay by Margalit and Thomas [CM17, Office Hour 7].

2. Sym(3): For the symmetric group on three elements, we can use the generating sets $S_1 = \{(1 2), (1 3)\}$ or $S_2 = \{(1 2), (1 2 3)\}$. These graphs are pictured in Figure 5 where elements in Sym(3) are listed using cycle notation and the composition is computed right to left.

3. $F_2$: For the free group of rank two, we can use a basis $S = \{a, b\}$. Recall that elements in $F_2$ are in one-to-one correspondence to words in the alphabet $\{a, a^{-1}, b, b^{-1}\}$ that are reduced in the sense that they do not contain $aa^{-1}$, $a^{-1}a$, $bb^{-1}$, or $b^{-1}b$. For example, $a^2b^{-1}a^{-1}b$ and $b^2a^{-2}b^2$ represent elements in $F_2$. The identity in $F_2$ is represented by the empty word. The group operation is concatenation followed by deletion of forbidden terms. As the reduced word representing an element is unique and as paths in the Cayley graphs read out a word representing an element as shown in Figure 4, there is a unique non-backtracking path from $1_{F_2}$ to any given element. Hence, the Cayley graph $\Gamma(F_2, \{a, b\})$ is a tree. A portion of this graph is pictured in Figure 7.
There is a metric on the vertices of \( \Gamma(G,S) \) defined as the minimum number of edges in an edge-path between a given pair of vertices. This metric can be extended to the points lying in edges by identifying (in an equivariant way) each edge with the unit interval \([0, 1] \subset \mathbb{R}\). However for most applications in geometric group theory, having a metric only on the vertices suffices. The action of \( G \) on the Cayley graph \( \Gamma(G,S) \) with this metric is by isometries.

The only item left to verify in Theorem 1 is that the action of \( G \) on \( \Gamma(G,S) \) is geometric. We can easily check these properties in turn.

1. (cocompact) Let \( K \subseteq \Gamma(G,S) \) be the union of the vertices \( \{1_G\} \cup S \) together with the edges incident on \( 1_G \) and \( s \) for each \( s \in S \). As \( S \) is finite, \( K \) is compact and clearly \( \bigcup_{g \in G} gK = \Gamma(G,S) \).

2. (properly discontinuous) Suppose that \( Y \subseteq \Gamma(G,S) \) is a finite subgraph and let \( n \) denote the number of vertices in \( Y \). If \( gY \cap Y \neq \emptyset \) then \( gh_1 = h_2 \) for a pair of vertices \( h_1, h_2 \in Y \) and hence \( g = h_2h_1^{-1} \). Thus the cardinality of \( \{g \in G \mid gY \cap Y \neq \emptyset\} \) is at most \( n^2 \).

Groups and Spaces with Negative Curvature

In the previous section, we used a path-connected space and a geometric action to derive an algebraic consequence: finite generation. Path-connectivity is a fairly weak topological property, however the notion of a geometric action is quite restrictive. For instance, by proper discontinuity the subgroup fixing a given point must be finite. What can be gained from actions on spaces with more requirements on the topology and geometry, but perhaps fewer requirements on the dynamics of the action?

One geometric property that is particularly useful is the notion of negative curvature. We will look at two instances of negative curvature in geometric group theory: trees and \( \delta \)-hyperbolic spaces.

Actions on trees. Negative curvature, say in the hyperbolic plane, influences the geometry in several ways: uniqueness of geodesics, exponential growth in the volume of balls, and a uniform bound on the diameter of an inscribed circle to a triangle to name a few. To discuss the familiar notion of curvature from differential geometry, a space requires more structure than just an ordinary metric, but several researchers have given notions of negative curvature expressed solely in terms of a distance function on an arbitrary set. Before discussing such a notion of negative curvature, let’s consider a simple example of a metric space that has the properties listed above for the hyperbolic plane: a tree.

To see an example of the usefulness of group actions on trees, let’s go back to the example of \( \text{SL}(2,\mathbb{Z}) \) and think about its finite-order elements, i.e., matrices for which some positive power is equal to the identity. We can quickly compute that \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -I = B^2 \), thus \( A^4 = I \) and \( B^4 = I \) and so \( A \) and \( B \) have finite order. Are there any others? There are obvious ones of course. Powers of \( A \) and powers of \( B \) clearly have finite order, as do their conjugates, \( CA^kC^{-1} \) and \( CB^kC^{-1} \) for any \( k \in \mathbb{Z} \) and \( C \in \text{SL}(2,\mathbb{Z}) \). But is that it? The answer to this last question is “yes” and we will see why using the action of \( \text{SL}(2,\mathbb{Z}) \) on the Farey tree, which we now describe.

Divide each triangle in the Farey complex into three quadrilaterals that meet pairwise along one leg of a tripod. Taken collectively these tripods form a tree, which is called the Farey tree. See Figure 8.

There are two types of vertices in the Farey tree: (red) degree three coming from the center of a triangle, and (green) degree two coming from an edge of a triangle. Let \( v \) denote the vertex that corresponds to the center of the triangle \( \Delta \) and let \( w \) denote the vertex that corresponds to the edge in the Farey complex between \( \frac{0}{1} \) and \( \frac{1}{2} \). These are labeled in Figure 8.

From our study of the action of \( \text{SL}(2,\mathbb{Z}) \) on the Farey complex, we conclude that every vertex in the Farey tree is a translate of \( v \) or \( w \). This follows from Claim 1 and the fact that \( A \) cyclically permutes the edges of \( \Delta \) and hence all of the vertices adjacent to \( v \). Additionally, we can conclude from Claim 2 that the stabilizer of \( v \) is the cyclic subgroup of order 6 generated by \( A \). In a similar manner, we can conclude that the stabilizer of \( w \) is the cyclic subgroup of order 4 generated by \( B \).

An important property of an action on a tree is the following claim.

**Claim 3.** Suppose that a group \( G \) acts on a tree. If \( g \in G \) has finite order, then \( g \) has a fixed point.
The center is easy to characterize. Suppose that the action we gave shows that if a group acts geometrically on a certain type of metric space, this gives an inductive way to show that all the groups in this family have some particular property or structure. In the next section, we will mention an instance where this strategy has been particularly fruitful: the mapping class group of an orientable surface.

Actions on \( \delta \)-hyperbolic spaces. Actions on trees are nice to work with, but they form a fairly restrictive class of groups. There are many interesting and natural groups in which every action on a tree has a global fixed point. For example, this is true for \( \text{SL}(n, \mathbb{Z}) \) when \( n \geq 3 \). Surely, not much can be gained in general from actions with a global fixed point.

Gromov’s influential essay [Gro87] introduced a notion of negative curvature that unifies essential properties of the hyperbolic plane, trees, and small cancellation groups—a thoroughly studied class of groups explored in the latter half of the 20th century in which geometric notions and techniques were starting to gain traction. The idea behind Gromov’s definition of a \( \delta \)-hyperbolic space is to take one of the useful consequences of negative curvature from the hyperbolic plane and use it as a definition for a metric space. Gromov gave such a definition solely using a metric \( d \) on an arbitrary set \( X \), but the most common formulation used—and one that applies to almost all the spaces one comes across in geometric group theory—requires a
geodesic metric space, which is defined as follows. A geodesic in a metric space \((X, d)\) is a function \(p : Y \to X\) where \(Y\) is a connected subset of \(\mathbb{R}\) such that \(d(p(s), p(t)) = |t - s|\) for all \(s, t \in Y\). A geodesic metric space is a metric space \((X, d)\) such that for all \(x, y \in X\), there is a geodesic \(p : [0, L] \to X\) with \(p(0) = x\) and \(p(L) = y\). A connected graph, in particular the Cayley graph of a finitely generated group, is a geodesic metric space.

There are many equivalent formulations of a \(\delta\)-hyperbolic metric space using geodesic triangles, divergence of geodesics, or nearest point projections to geodesics. We will state the most common formulation using geodesic triangles, which Gromov attributed to Rips. In the statement, \([a, b]\) represents the image of any geodesic in \(X\) from \(a\) to \(b\).

Definition 3. Let \((X, d)\) be a geodesic metric space. A geodesic triangle \(\Delta(a, b, c)\) is \(\delta\)-thin if the \(\delta\)-neighborhood of any two of the edges contains the third. That is, for all \(x \in [a, c]\) there is an \(x' \in [a, b] \cup [b, c]\) such that \(d(x, x') \leq \delta\). A \(\delta\)-hyperbolic space is a geodesic metric space where every geodesic triangle is \(\delta\)-thin.

The key point in the definition is that the same \(\delta\) works for every geodesic triangle, no matter how long the sides are. See Figure 9.

Here are some examples of \(\delta\)-hyperbolic spaces.

1. A tree is \(0\)-hyperbolic since every geodesic triangle is a tripod and so any side is contained in the union of the other two. See Figure 10. We think of thinner triangles indicating the space being more negatively curved—this is true for scalar curvature in Riemannian geometry—and so in this sense, trees are negatively curved in the extreme.

2. The hyperbolic plane is \(\log(1 + \sqrt{2})\)-hyperbolic. As every geodesic triangle is contained in an ideal triangle, we only have to compute \(\delta\) for an ideal triangle, which is a fun exercise. See Figure 11.

3. The Farey graph is \(1\)-hyperbolic. Indeed, suppose that \(\frac{1}{0}\) lies on a geodesic between the vertices \(\frac{p}{q}\) and \(\frac{n}{s}\). Let \(\frac{m}{1}\) and \(\frac{n}{1}\) be the vertices adjacent to \(\frac{1}{0}\) along this geodesic and assume that \(m < n\). As we are dealing with a geodesic, we must have \(n - m > 1\) since otherwise there is an edge between \(\frac{m}{1}\) and \(\frac{n}{1}\). Hence there is some vertex \(\frac{c}{1}\) adjacent to \(\frac{1}{0}\) such that \(m < c < n\). As the removal of the vertices \(\frac{1}{0}\) and \(\frac{c}{1}\) and also the edge connecting these two vertices disconnects the Farey graph, we see that any path from \(\frac{p}{q}\) to \(\frac{n}{s}\) must pass through either \(\frac{1}{0}\) or \(\frac{c}{1}\).

For contrast, \(\mathbb{R}^2\) with the Euclidean metric is not \(\delta\)-hyperbolic for any \(\delta\). Indeed, the geodesic triangle with vertices \((0, 0), (n, 0)\) and \((0, n)\) is \(\delta\)-thin only for \(\delta \geq n/2\). To see this, consider the point \((n/2, n/2)\).

The typical questions one may try to answer using actions on \(\delta\)-hyperbolic spaces often fit into the following categories.

1. Algorithmic: When do two words in a generating set represent the same element or conjugate elements?
2. Local-to-global: Are paths in the Cayley graph that are locally geodesics globally geodesics as well?
3. Rigidity: If two groups have geometrically similar Cayley graphs, are the groups algebraically similar? Can we characterize homomorphisms to and from the group?

We will discuss in turn geometric actions and other types of actions on \(\delta\)-hyperbolic spaces.
Geometric actions on $\delta$–hyperbolic spaces. A metric space is proper if closed balls are compact. A group $G$ is hyperbolic if it acts geometrically on a proper $\delta$–hyperbolic space\(^2\). Free groups and fundamental groups of closed hyperbolic manifolds are hyperbolic groups. It is fair to ask how common hyperbolic groups are given that we started this section noticing that useful tree actions do not always exist. Gromov introduced a model of a “random finitely presented group” that includes a parameter $0 < d < 1$ called the “density” that controls the number of relators in terms of the number of generators [Gro93, Chapter 9]. When $d < 1/2$, Gromov showed that a random group is infinite and hyperbolic. (For those curious, when $d > 1/2$ a random group has at most two elements.) Thus, it is fair to say that hyperbolic groups are quite ubiquitous.

An equivalent definition of a hyperbolic group is that $G$ is finitely generated and the Cayley graph $\Gamma(G, S)$ is $\delta$–hyperbolic for some finite generating set $S \subseteq G$. Moreover, “some” in the previous sentence can be replaced with “every.” Hyperbolic groups satisfy a long list of useful properties and besides Gromov’s original essay, there are many comprehensive works focused on these groups. See for instance the notes edited by Short [ABC+91], the chapters by Bridson and Haefliger [BH99, Chapters III.H and III.I], and the references within these works.

As hyperbolic groups are defined by a geometric condition (in several equivalent ways), from their inception researchers have wondered if there is an algebraic characterization. It is not too difficult to find algebraic obstructions. One of the first usually encountered involves the centralizer. It is not too difficult to find algebraic obstructions. Hence the coset $h(g)$ has an element whose distance from $1_G$ is at most $2L + 2\delta$. As there are only finitely many such elements and as distinct cosets are always disjoint, there are only finitely many cosets.

As a consequence, no subgroup of a hyperbolic group can be isomorphic to $\mathbb{Z}^2$. In several classes of geometrically defined groups, this turns out to be the only obstruction to hyperbolicity. For instance, this is true for the class of fundamental groups of closed 3–manifolds. In general, there are other algebraic obstructions to consider. Hyperbolic groups cannot contain a subgroup isomorphic to one of the Baumslag–Solitar groups:

$$BS(m, n) = \langle a, t \mid ta^mt^{-1} = a^n \rangle.$$  

The notation here means that $BS(m, n)$ is generated by two elements $a$ and $t$ and the only relation they satisfy is that $t$ conjugates $a^n$ to $a^m$. This is a very interesting class of groups that includes $\mathbb{Z}^2$, which is $BS(1, 1)$, and the fundamental group of the Klein bottle, which is $BS(1, -1)$.

The reason hyperbolic groups cannot contain subgroups isomorphic to a Baumslag–Solitar group relies on the two facts that (1) for $k \geq 1$ the subgroup $\langle a^k, i^k \rangle \subseteq BS(m, n)$ is never free as $t^ka^{kn}t^{-1} = a^k$, whereas (2) for infinite-order elements $g, h \in G$ in a hyperbolic group, the subgroup $\langle g^k, h^k \rangle$ is free for some large $k$. Alternatively, one can appeal to the previously mentioned fact that infinite cyclic subgroups are undistorted in hyperbolic groups.

\(^2\)In the literature, these groups are sometimes referred to as negatively curved, word hyperbolic, or Gromov hyperbolic.
It was an open question until recently if this is essentially the only obstruction. Specifically, is a group $G$ for which there exists a finite Eilenberg–MacLane space $K(G, 1)$ and that does not contain a subgroup isomorphic to $BS(m, n)$ necessarily hyperbolic? Brady gave counterexamples without the finiteness assumption [Bra99]. These examples were difficult to construct. They arise as subgroups of hyperbolic groups and hence do not have subgroups isomorphic to Baumslag–Solitar groups. The difficult part in the construction is showing these subgroups are not hyperbolic. Brady does so by showing they do not satisfy an algebraic finiteness condition, called $F_2$, known to be satisfied by hyperbolic groups. As the existence of a finite $K(G, 1)$ implies $F_2$ and also implies additional algebraic finiteness conditions satisfied by hyperbolic groups, a positive answer to the above question seemed plausible. Recently, Italiano, Martelli, and Migliorini constructed a subgroup of a hyperbolic group that is not hyperbolic but does have a finite $K(G, 1)$, answering the above question in the negative [IMM]. The hyperbolic group they construct is a quotient of the fundamental group of a finite volume cusped hyperbolic 5–manifold.

Other actions on $\delta$–hyperbolic spaces. There are many natural groups that contain subgroups isomorphic to $\mathbb{Z}^2$ and hence cannot be hyperbolic. Can we still use negative curvature to investigate these groups? Let’s relax the conditions of a geometric action and the requirement of a proper metric space, and consider an example of an important group in low-dimensional topology.

The mapping class group $\text{MCG}(\Sigma)$ of an orientable surface $\Sigma$, possibly with boundary, is the the group of orientation preserving homeomorphisms of $\Sigma$ modulo isotopy. That is, two homeomorphisms of $\Sigma$ determine the same mapping class if one can be continuously deformed to the other so that every intermediate map along the way is also a homeomorphism. When $\Sigma$ has non-empty boundary, the homeomorphisms and the isotopies need to be the identity on each boundary component. To simplify the discussion, we will assume that $\Sigma$ is compact. This group appears in the study of 3–manifolds, algebraic geometry, cryptography, symplectic geometry, dynamics, and configuration spaces. Using homeomorphisms supported on disjoint subsurfaces in $\Sigma$, it is easy to find subgroups isomorphic to $\mathbb{Z}^2$ in all mapping class groups with a few exceptions. Thus $\text{MCG}(\Sigma)$ is not hyperbolic in general.

The mapping class group acts on the curve graph $\mathcal{C}(\Sigma)$, which we now describe. A simple closed curve is an embedding of the circle $S^1 \to \Sigma$ that does not bound a disk nor an annulus in $\Sigma$ (the latter only occurs when $\Sigma$ has boundary). The curve graph is the graph whose vertex set is the set of isotopy classes of simple closed curves and two such $[c_0], [c_1]$ are joined by an edge if they have disjoint representatives. In Figure 13 some curves on $\Sigma$ are shown along with the corresponding subgraph of $\mathcal{C}(\Sigma)$. A mapping class $[f]$ acts on a vertex $[c]$ in the curve graph by sending the simple closed curve to its image: $[f] \cdot [c] = [f(c)]$. Homeomorphisms take disjoint curves to disjoint curves so this extends to an action on $\mathcal{C}(\Sigma)$ as well.

When the genus of $\Sigma$ is equal to 1, i.e., when $\Sigma$ is a torus $S^1 \times S^1$, any two non-isotopic simple closed curves necessarily intersect and so the above definition results in a graph with no edges. In this case the definition is altered slightly, $[c_0], [c_1]$ are joined by an edge if they have representatives that intersect once. Let’s take a closer look at this curve graph. Any simple closed curve on the torus is isotopic to one that winds $p$ times around the first $S^1$ factor and $q$ times around the second $S^1$ factor where $p$ and $q$ are relatively prime. As the orientation does not matter, we can assume that $q$ is positive. That is, isotopy classes of simple closed curves on the torus are parameterized by the set of rational numbers $\frac{p}{q}$ along with an additional element $\frac{1}{0}$. Moreover, the number of times the simple closed curves $\frac{p}{q}$ and $\frac{r}{s}$ intersect is $|ps - qr|$.

Sound familiar? That’s right, the curve graph of the torus is the Farey graph! In fact, the mapping class group of the torus is isomorphic to $\text{SL}(2, \mathbb{Z})$ and the two actions are the same. The action of $\text{SL}(2, \mathbb{Z})$ on the Farey graph illustrates some of the essential properties of $\mathcal{C}(\Sigma)$ and the action of $\text{MCG}(\Sigma)$ on $\mathcal{C}(\Sigma)$.

First, as we observed for the Farey graph, the curve graph is $\delta$–hyperbolic. This amazing fact was proved by Masur and Minsky [MM99] and has been reproved a number of times since. (The best estimate on $\delta$ was given by Hensel, Przytycki, and Webb who found an explicit value for $\delta$ independent of $\Sigma$ [HPW15].) It is impossible to overstate the influence of this result on the study of the mapping class group, the geometry of 3–manifolds and geometric group theory in general.
Second, the action is not properly discontinuous. Indeed, the vertex stabilizers are infinite. This is not a bug though, but a feature! Homeomorphisms that fix a simple closed curve $c$ are actually homeomorphisms of the surface obtained by cutting open along $c$. Thus the stabilizer of a vertex in $\mathcal{E}(\Sigma)$ is the mapping class group of a surface $\Sigma'$, possibly disconnected, whose components are simpler in the sense that the genus or the number of boundary components is fewer than that of $\Sigma$.

Taken together, this setting fits into the hierarchy strategy mentioned after Theorem 2 and there are numerous applications. I will mention one here that ties back to the beginning of the article: finite generation. Using the facts that the curve graphs are path-connected and that the stabilizers of vertices in the curve graph are, by induction, finitely generated, it can be shown that the mapping class group of any orientable surface is also finitely generated. That is, we promote finite generation from the stabilizers to whole group using the fact that the space is path-connected. The base case for the induction is when the surface has genus 1. The mapping class group of this surface is $SL(2, \mathbb{Z})$—the group we started our journey with! This strategy was originally employed by Dehn and he found a specific generating set for the mapping class group analogous to elementary matrices. For complete details, and much more on mapping class groups, see the text by Farb and Margalit [FM12].

Other useful properties and features of the mapping class group acting on the curve graph have been identified, isolated, and applied to the study of other groups. These include the notion of a WPD element by Bestvina and Fujiwara [BF02]; the notion of a projection complex by Bestvina, Bromberg, and Fujiwara [BBF15]; the notion of an acylindrical action by Bowditch [Bow08], which was further developed by Osin [Osi16]; and the notion of a hierarchically hyperbolic group/space by Behrstock, Hagen, and Sisto [BHS17]. The common element of each of these new tools is to exploit negative curvature in certain directions of the group. As in the case of the mapping class group acting on the curve graph, the applications have been abundant.

**Conclusion and Further Reading**

I hope that this article has given you an idea of how the topology and geometry of a space that a group acts on can influence its algebraic properties and structure. Geometric group theory is a growing field. This is in part due to the large number of questions the field generates regarding the geometry of finitely generated groups, but the field has also seen an increase in interest as a result of its applications to other areas of mathematics. A striking example of this is the recent resolution of the Virtual Haken Conjecture in 3–manifold topology. This was proved by Agol [Ago13] using tools from geometric group theory created by Scott, Sageev, Wise, and others. See the survey article by Bestvina [Bes14] for an excellent overview of this connection.

The concept of $\delta$–hyperbolicity is but one aspect of geometric group theory. There are areas of geometric group theory invoking tools from algebra (algebraic geometry, homological algebra) analysis ($L^p$–spaces, $C^*$ and von Neumann algebras), dynamics (entropy, topological Markov chains), geometry (isoperimetric functions, Lie theory) and topology (dimension, fractals). Below is a selection of books on geometric group theory for those curious to learn more, listed by publication date.

1. **Metric Spaces of Non-positive Curvature** by Martin Bridson and André Haefliger [BH99]: A comprehensive reference text focusing on various notions of non-positive curvature in metric spaces and groups.
2. **Topics in Geometric Group Theory** by Pierre de la Harpe [dlH00]: An introduction to groups as geometric objects including a multitude of examples and a broad investigation on the notion of growth in groups.
3. **A Course in Geometric Group Theory** by Brian Bowditch [Bow06]: An introductory text based on a course taught by the author with an in-depth treatment of hyperbolic groups. The audience is advanced undergraduates and beginning graduate students.
4. **Office Hours with a Geometric Group Theorist** edited by Matt Clay and Dan Margalit [CM17]: A collection of essays written by researchers on select topics in geometric group theory and central examples such as Coxeter groups and braid groups targeted to advanced undergraduates and beginning graduate students.
5. **Geometric Group Theory** by Clara Löh [Löh17]: An introductory text on geometric group theory targeted to advanced undergraduates and beginning graduate students. Fundamental topics such as quasi-isometry, boundaries and amenable groups are discussed.
6. **Geometric Group Theory** by Cornelia Drutu and Michael Kapovich [DK18]: A comprehensive text containing proofs of several fundamental results in geometric group theory including the Tits alternative and Gromov’s theorem on polynomial growth. The audience is advanced graduate students and researchers in the field.

**ACKNOWLEDGMENT.** I thank the anonymous referees as well as Dan Margalit, Lance Miller, Jean Pierre Mutanguha, and Andrew Raich for their useful comments which improved this article.
References


Credits

All figures are courtesy of Matt Clay.
Photo of Matt Clay is courtesy of Robert Green.
Bubbling Blow-Up in Critical Elliptic and Parabolic Problems

Monica Musso

Mathematical models are often expressed by nonlinear partial differential equations. Solutions of a given partial differential equation can be interpreted as attainable states for the underlying model. In steady as well as in time-dependent problems, a central issue is to determine the behaviour of solutions or the presence of blow-up.

Blow-up takes place in regions or instants where solutions, or some quantities depending on them, become unbounded or exhibit irregular behaviour. This usually means that the original model loses validity near these regions and space-time scaling is required to make an accurate description. A particular type of blow-up are the ones that are triggered by bubbling. We will briefly discuss bubbling blow-up in two classical critical elliptic and parabolic problems.

Bubbling in critical elliptic problems. Many problems of physical and geometrical interest have a variational structure. For such problems, the failure of compactness at certain energy levels reflects highly interesting phenomena related to internal symmetries of the systems under study. In several of these situations, bubbling may occur. The term
**bubbling** refers to the presence of families of solutions that at main order look like scalings of a single profile which in the limit become singular but at the same time have an approximately constant energy level. Such phenomena have been observed for the first time by Sacks-Uhlenbeck (1981) in the context of two-dimensional harmonic maps, and independently by Wente (1980) in the context of surfaces of prescribed constant mean curvature.

Classical models where bubbling occurs are semilinear boundary value problems near criticality in \( \mathbb{R}^N \). Consider the problem of finding positive solutions to

\[
\Delta u + u^q = 0 \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega, \tag{1}
\]

where \( \Omega \subset \mathbb{R}^N, N \geq 3 \), is a bounded domain with smooth boundary \( \partial \Omega \) and \( q > 1 \). This equation in (1) is sometimes called the Lane-Emden-Fowler equation. It was used first in the mid-19th century in the study of internal structure of stars, on the other hand it constitutes a basic model equation for steady states of reaction-diffusion systems, nonlinear Schrödinger equations, fast diffusion equations, and nonlinear wave equations and nonlinear wave maps [10], [16].

The critical exponent \( q = \frac{N+2}{N-2} \) sets a threshold where the structure of the solution set of (1) suffers a dramatic change. If \( q < \frac{N+2}{N-2} \) a solution may always be found by minimizing the Rayleigh quotient

\[
Q(u) \equiv \frac{\int_{\Omega} |\nabla u|^2}{\left( \int_{\Omega} |u|^{q+1} \right)^{\frac{2}{q+1}}}, \quad u \in H_0^1(\Omega) \setminus \{0\}. \tag{2}
\]

In fact, the quantity \( S_q(\Omega) \equiv \inf_{u \in H_0^1(\Omega) \setminus \{0\}} Q(u) \) is achieved thanks to compactness of Sobolev embeddings \( H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega) \) for \( q < \frac{N+2}{N-2} \). A suitable scalar multiple of a minimizer turns out to be a solution of (1). The case \( q \geq \frac{N+2}{N-2} \) is considerably more delicate: for \( q = \frac{N+2}{N-2} \), compactness of the embedding is lost while for \( q > \frac{N+2}{N-2} \) there is no such an embedding. This obstruction is not just technical for the solvability question, but essential. If \( \Omega \) is strictly star-shaped around a point \( x_0 \in \Omega \) and \( u \) solves (1) then Pohozaev’s identity (1965) yields

\[
\left( \frac{N-2}{2} - \frac{N}{q+1} \right) \int_{\Omega} u^{q+1} \, dx = - \frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 (x - x_0) \cdot \nu \, d\sigma < 0,
\]

where \( \nu \) is the unit outer normal to \( \partial \Omega \). Hence necessarily \( q < \frac{N+2}{N-2} \), and thus no solutions at all exist if \( q \geq \frac{N+2}{N-2} \).

Pohozaev’s result puts in evidence the central role of topology or geometry in the domain for solvability. Kazdan and Warner (1975) observed that Problem (1) is actually solvable for any \( q > 1 \) if \( \Omega \) is a radial annulus, as compactness in the Rayleigh quotient \( Q \) is gained within the class of radially symmetric functions. On the other hand Coron (1984) found via a variational method that (1) is solvable at the critical exponent \( q = \frac{N+2}{N-2} \) whenever \( \Omega \) is a domain exhibiting a small hole. Substantial improvement of this result was found by Bahri and Coron [1], proving that if some homology group of \( \Omega \) with coefficients in \( \mathbb{Z}_2 \) is not trivial, then (1) has at least one solution for \( q \) critical, in particular in any three-dimensional domain which is not contractible to a point. Examples showing that this condition is actually not necessary for solvability at the critical exponent were found by Dancer (1988), Ding (1989) and Passaseo (1989, 1998).

The change of structure of the solution set taking place at the critical exponent is strongly linked to the presence of unbounded sequences of solutions or **bubbling solutions**. By a **bubbling solution** for (1) near the critical exponent we mean an unbounded sequence of solutions \( u_n \) of (1) for \( q = q_n \to \frac{N+2}{N-2} \). Setting

\[
M_n \equiv \max_{\Omega} u_n = u_n(\xi_n) \to +\infty,
\]

we see then that the scaled function

\[
v_n(y) \equiv M_n^{-1} u_n(\xi_n + M_n^{-\frac{q(n-1)}{2}} y),
\]

satisfies

\[
\Delta v_n + v_n^{q_n} = 0
\]

in the expanding domain \( \Omega_n = M_n^{\frac{q(n-1)}{2}} (\Omega - \xi_n) \). Assuming for instance that \( \xi_n \) stays away from the boundary of \( \Omega \), elliptic regularity implies that locally over compacts around the origin, \( v_n \) converges up to subsequences to a positive solution of

\[
\Delta U + U^{\frac{N+2}{N-2}} = 0
\]

in entire space. Positive solutions to this equation are known from the classical works of Rodemich (1966), Aubin (1976), Obata (1972) and Talenti (1976) to be the functions

\[
U_\lambda(\xi) = \lambda^{-\frac{N-2}{2}} U \left( \frac{y - \xi}{\lambda} \right),
\]

where

\[
U(y) = (N(N-2))^{-\frac{N+2}{4}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{N-2}{2}}
\]

for any scalar \( \lambda > 0 \) and any point \( \xi \in \mathbb{R}^N \).
A natural problem is that of constructing solutions exhibiting slightly sub-critical and super-critical exponents \( q = \frac{N+2}{N-2} \pm \epsilon \). The general result can be phrased in the following terms: Given a nondegenerate critical point or a topologically nontrivial critical point of the reduced functional of \((\xi, \lambda) = (\xi_1, ..., \xi_k, \lambda_1, ..., \lambda_k) \in \Omega^k \times \mathbb{R}_+^k\),

\[
\Psi_k^{\xi, \lambda}(\xi, \lambda) = \sum_{j=1}^k H(\xi_j, \lambda_j)^{N-2} - 2 \sum_{i<j} G(\xi_i, \xi_j) \lambda_i^{N-2} \lambda_j^{N-2} - 2 \epsilon \log(\lambda_1 \cdots \lambda_k),
\]

there exists a \( k \)-bubble solution

\[
u_{\epsilon}(x) \sim \sum_{j=1}^k \lambda_{je}^{-\frac{N-2}{2}} U\left(\frac{x - \xi_j}{\lambda_{je}}\right),
\]

with \( \lambda_{je} \sim \epsilon^{-\frac{1}{N-2}} \) as \( \epsilon \to 0 \), to problem (1) with \( q = \frac{N+2}{N-2} \pm \epsilon \). Needless to say, it is a delicate task to find critical points for this reduced functional for a general domain \( \Omega \), and they may even not exist.

But what is the origin of the reduced functional \( \Psi_k^{\xi} \)? An alternative way to find a solution to (1) is as a critical point of the energy functional

\[
E_q(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{q+1} \int_\Omega u^{q+1}.
\]

The scaled bubbles give a precise description of the solution near the blow up points. Far from these points, the bubbles need to be correct to match the zero Dirichlet boundary condition. An efficient way to do this is by using a proper multiple of the regular part \( H(x, y) \) of the Green’s function. Hence a better approximate solution is given by

\[
u_{\epsilon}(x) = \sum_{j=1}^k \lambda_{je}^{-\frac{N-2}{2}} U\left(\frac{x - \xi_j}{\lambda_{je}}\right) - \lambda_{je}^{-\frac{N-2}{2}} H(x, \xi_j).
\]

The energy evaluated at \( \nu_{\epsilon} \) has the expansion, for \( \epsilon \to 0 \),

\[
E_{\frac{N+2}{N-2} \pm \epsilon}(\nu_{\epsilon}) \sim k S_N + \Psi_k^{\xi, \lambda}(\xi, \lambda).
\]

Hence finding a critical point of the reduced functional suggests the existence of a solution \( u_{\epsilon} \) close (in some topology) to the sum \( \nu_{\epsilon} \) of \( k \) corrected bubbles. As mentioned before, in the sub-critical setting \( q = \frac{N+2}{N-2} - \epsilon \), the natural topology for this problem is the energy space \( H_0^1(\Omega) \). In this regime, the reported results have been obtained by Brezis-Peletier [3], Han (1991), Rey (1990) and Bahri-Li-Rey [2]. In the super-critical regime \( q = \frac{N+2}{N-2} + \epsilon \) the embedding \( H_0^1(\Omega) \hookrightarrow L^{2q+1}(\Omega) \) is not available. New weighted \( L^\infty \) spaces were first introduced in [7] to treat the super-critical regime.

A main implication of the result in [7] states that in a domain with a small hole problem (1) with \( q = \frac{N+2}{N-2} + \epsilon \) has a two-bubble solution. More generally, if several spherical holes are drilled, a solution obtained by gluing together several two-bubbles can be found. Two-bubble solutions

These are the only positive solutions [4] and they are known as the bubbles. They corresponds precisely to an extremal of the critical Sobolev embedding

\[
S_N = \inf_{u \in C_0^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} |u|^{2N/2})^{N/2}}.
\]

Coming back to the original variable, one expects then that “near \( \xi_n \)” the behavior of \( u_n(x) \) can be approximated as

\[
u_n(x) \sim \lambda_{n}^{-\frac{N-2}{2}} U\left(\frac{x - \xi_n}{\lambda_n}\right) (1 + o(1)) \tag{4}
\]

A natural problem is that of constructing solutions exhibiting this property around one or several points of the domain when the exponent \( q \) approaches the critical value \( \frac{N+2}{N-2} \).

For \( q \) slightly sub-critical, \( q = \frac{N+2}{N-2} - \epsilon, \epsilon > 0 \), a solution \( u_{\epsilon} \) given by a minimizer of the Rayleigh quotient (2) clearly cannot remain bounded as \( \epsilon \downarrow 0 \), since otherwise Sobolev’s constant \( S_N \) in (3) would be achieved by a function supported in \( \Omega \). In this case, \( u_{\epsilon} \) has asymptotically just a single maximum point \( \xi_{\epsilon} \) and the asymptotic (4) holds globally in \( \Omega \) with \( M_{\epsilon} \sim \epsilon^{-\frac{1}{N-2}} \). Moreover, \( \xi_{\epsilon} \) approaches a critical point of Robin’s function \( H(x, x) \). Here \( H(x, y) \) is the regular part of Green’s function \( G(x, y) \) for the Laplace operator in \( \Omega \) under Dirichlet boundary conditions.

This conclusion can be refined to the case of solutions \( u_{\epsilon} \) exhibiting bubbling at multiple points, for both slightly sub-critical and super-critical exponents \( q = \frac{N+2}{N-2} \pm \epsilon \). The general result can be phrased in the following terms: Given a nondegenerate critical point or a topologically nontrivial critical point of the reduced functional of

![Figure 1. Three bubbles with different values for the concentration parameter \( \lambda \), all centered at the same point.](image-url)
are the simplest to be obtained: single-bubble solutions for Problem (1) with $q = \frac{N+2}{N-2} + \varepsilon$ do not exist, as shown by M. Ben Ayed, K. El Mehdi, M. Grossi, O. Rey (2003). Solutions with different blow-up orders, known as tower of bubbles were found in [18].

Bubbling in critical parabolic problems. The parabolic version of problem (1)
\begin{align}
\partial_t u = \Delta u + u^q & \quad \text{in } \Omega \times [0, T], \\
u = 0 & \quad \text{on } \partial \Omega \times [0, T], 
\end{align}
(7)
for $0 < T \leq \infty$, is a widely studied classical problem, usually referred to as the Fujita problem, after his work in 1969. The heat operator $\partial_t u = \Delta u$ in (7) describes the diffusion of a density-function $u = u(x, t)$, where $x$ is the space variable and $t$ denotes time, and the term $f(u) = u^q$ represents a source. This is the simplest model of semilinear parabolic equations, which are ubiquitous as they can be found in numerous applications ranging from physics and biology to materials and social sciences. We refer the reader to reference [20] for a comprehensive survey on Problem (7) and more general versions of it.

Despite its simple look, Problem (7) encodes the fundamental features of a general semilinear parabolic problem. If the initial condition $u_0 = u_0(x)$ at time $t = 0$ is smooth and has value 0 on the boundary of $\Omega$, Problem (7) has a unique (classical) solution $u = u(x, t)$ defined on some time interval $[0, T)$ with $0 < T \leq \infty$. If we call $T = T(u_0)$ the maximal possible time of existence, the solution cannot be extended beyond $T$. If $T < \infty$, then necessarily the solution blows up at $T$, in the sense that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \nearrow T$. If $T = \infty$, we say that the solution is global. In this case, two possibilities can occur: either $u$ remains bounded as $t \to \infty$, or
\[
\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]
The latter is sometimes referred to as infinite time blow-up or growth of $u$. One of the fundamental problems concerning equation (7) is whether the infinite time blow-up can actually occur for some $u_0$ or not.

When $q$ is the critical Sobolev exponent $q = \frac{N+2}{N-2}$, one expects that blow-up by bubbling for specific situations appears in the form
\[
u(x, t) \sim \sum_{j=1}^k \lambda_j(t)^{-\frac{N-2}{2}} U \left( \frac{x - \xi_j(t)}{\lambda_j(t)} \right)
\]
(8)
where now $\lambda_j(t)$ and $\xi_j(t)$ are functions of the time variable $t$, with $\lambda_j(t) \to 0$ as $t \to T$. Those solutions are usually asymptotically not self-similar and, while not generic, their presence is among the most important features of the full dynamics since they correspond to threshold solutions between different generic regimes.

Consider an initial condition of the form $u_0(x) = \alpha \varphi(x)$, where $\varphi$ is a fixed positive smooth function in $\Omega$ with zero boundary value and $\alpha$ is a positive constant, and denote by $u_\alpha(x, t)$ the unique (local) solution to (7) with this initial condition. For all sufficiently small $\alpha$, it is possible to prove that $u_\alpha(x, t)$ is globally defined and that $u_\alpha(x, t) \to 0$ uniformly for $x \in \Omega$ as $t \to \infty$. To see this, let $\lambda_1$ be the first eigenvalue of $-\Delta$ in $\Omega$ under Dirichlet boundary conditions and $\phi_1$ a positive first eigenfunction:
\[
-\Delta \phi_1(x) = \lambda_1 \phi_1(x), \quad \text{in } \Omega, \quad \phi_1(0) = 0 \quad \text{on } \partial \Omega.
\]
Let $\delta > 0$ and consider the function $\bar{u}(x, t) = \delta e^{-\gamma t} \phi_1(x)$, where $0 < \gamma < \lambda_1$. Then a direct computation gives
\[
\partial_t \bar{u} - \Delta \bar{u} = \delta \phi_1 e^{-\gamma t} \left( \lambda_1 - \gamma \right) - \delta \gamma \phi_1 e^{-\gamma t} > 0,
\]
provided $\delta > 0$ is small. By the maximum principle, $\bar{u}(x, t)$ is a supersolution of (7). Hence any solution to (7), whose initial value at time $t = 0$ is bounded above by $\bar{u}(x, 0)$, stays bounded by $\bar{u}(x, t)$ at all times. If we take $0 < \alpha$ small, then $u_\alpha(x, 0) = \alpha \varphi(x) \leq \bar{u}(x, 0)$ and hence, for some positive constant $C > 0$, and any $x \in \Omega$
\[
uu(x, t) \leq Ce^{-\gamma t}, \quad \text{as } t \to \infty.
\]
On the other hand, if we now take $\alpha$ in the initial condition $u_0(x) = \alpha \varphi(x)$ to be large, then $u_\alpha(x, t)$ blows-up in finite time. To see this, we assume that the solution $u_\alpha(x, t)$ is defined in $\Omega \times [0, T)$, we multiply the equation against $\phi_1(x)$ and integrate on $\Omega$. Using the divergence Theorem, we get
\[
\frac{\partial}{\partial t} \int_{\Omega} u_\alpha(x, t) \phi_1(x) \, dx = -\lambda_1 \int_{\Omega} u_\alpha(x, t) \phi_1(x) \, dx + \int_{\Omega} u_\alpha(x, t) \phi_1(x) \, dx.
\]
Let $g(t) = \int_{\Omega} u_\alpha(x, t) \phi_1(x) \, dx$. Then
\[
g'(t) \geq -\lambda_1 g(t) + C g^q(t)
\]
for some positive constant $C$. Besides,
\[
g(0) = \int_{\Omega} u_\alpha(x, 0) \phi_1(x) \, dx = \alpha \int_{\Omega} \varphi(x) \phi_1(x) \, dx.
\]
Then for $\alpha$ large, we have that $-\lambda_1 g(0) + C g^q(0) > 0$. Then $g'(t) \geq 0$ for all $t \in [0, T)$ and, after integration
\[
T \leq \int_{g(0)}^{\infty} \frac{1}{-\lambda_1 g + C g^q} \, dg.
\]
This gives $T < \infty$, and blow-up in finite time occurs.

A consequence of the facts we just proved is that the number
\[
\alpha_* = \sup \{ \alpha > 0 : \lim_{t \to \infty} \|u_\alpha(\cdot, t)\|_{L^\infty(\Omega)} = 0 \}
\]
is well defined and $0 < \alpha_* < \infty$. The solution $u_\alpha(x, t)$ somehow lies in the dynamic threshold between solutions globally defined in time and those that blow-up in finite time. Ni, Sacks, Tavantzis (1984) prove that this solution
is a well-defined $L^1$-weak solution of the Fujita problem, but it is not clear whether it will be smooth for all times.

When $1 < q < \frac{N+2}{N-2}$, $u_\alpha(x, t)$ is uniformly bounded and smooth, and up to subsequences it converges to a (positive) solution of the stationary problem (1). When $q > \frac{N+2}{N-2}$, $\Omega$ is a ball, and $u_\alpha$ is radially symmetric then $u_\alpha(x, t) \to 0$ as $t \to \infty$. The case $q = \frac{N+2}{N-2}$ is completely different: Galaktionov and Vázquez [13] proved that if $\Omega = B(0, 1)$ and if the threshold solution $u_\alpha$ is radially symmetric, then no finite time singularities for $u_\alpha(r, t)$ occur and it must become unbounded as $t \to +\infty$, thus exhibiting infinite-time blow up

$$\lim_{t \to \infty} \|u_\alpha(\cdot, t)\|_{L^2(\Omega)} = \infty.$$  

Galaktionov and King [12] discovered that this radial blow-up solution to (7) at $q = \frac{N+2}{N-2}$ does have a bubbling asymptotic profile as $t \to +\infty$ of the form (8) with $k = 1$ and $\lambda_1(t) \sim t^{-\frac{1}{N-2}} \to 0$ for $N \geq 5$. This critical infinite-time bubbling occurs also in the nonradial setting, as shown in [6]: there are global solutions to (7) with $q = \frac{N+2}{N-2}$ which have infinite time blow-up at any collection of points $p = (p_1, \ldots, p_k) \in \Omega^k$ if $p$ lies in the open region of $\Omega^k$ where a certain $k \times k$ matrix $\mathcal{G}(q)$ is positive definite. The matrix $\mathcal{G}$ is explicitly defined in terms of the Robin's and the Green's functions in $\Omega$, introduced in the previous section:

$$\mathcal{G}(p) = (\mathcal{G}_{ij})_{i,j=1,\ldots,k},$$  

$$\mathcal{G}_{ii} = H(p_i, p_i),$$  

$$\mathcal{G}_{ij} = -G(p_i, p_j) \quad i \neq j.$$  

In other words, if $\mathcal{G}(p)$ is positive definite, there exist an initial datum $u_0$ and smooth functions $\xi_j(t) \to p_j$ and $0 < \lambda_j(t) \to 0$, as $t \to +\infty$, $j = 1, \ldots, k$, such that the positive solution $u_\alpha$ of Problem (7) at $q = \frac{N+2}{N-2}$ has the form (8) with

$$\lambda_j(t) = O(t^{-\frac{1}{N-2}}),$$  

$$\xi_j(t) = p_j + O(t^{-\frac{2}{N-2}}) \quad as \ t \to +\infty.$$  

(9)

A consequence of the construction in [6] is that this bubbling phenomena has codimension $k$-stability in the sense that there exists a codimension $k$ manifold in $C^1(\bar{\Omega})$ that contains $u_\alpha(x, 0)$ such that if $u_\alpha$ lies in that manifold and it is sufficiently close to $u_\alpha(x, 0)$, then the solution $u(x, t)$ of problem (7) has exactly $k$ bubbling points $\bar{p}_j$, $j = 1, \ldots, k$ which lie close to the $p_j$, with the form (8).

Positive definiteness of $\mathcal{G}(q)$ trivially holds if $k = 1$. For $k = 2$ this condition holds if and only if

$$H(q_1, q_1)H(q_2, q_2) - G(q_1, q_2)^2 > 0,$$

in particular it does not hold if both points $q_1$ and $q_2$ are too close to a given point in $\Omega$. Given $k > 1$ we can always find $k$ points where $\mathcal{G}(q)$ is positive definite: it suffices to take points located at a uniformly positive distance one to each other, and then let them lie sufficiently close to the boundary.

The role of the matrix $\mathcal{G}(q)$ in elliptic bubbling phenomena of the stationary version of (1) at $q = \frac{N+2}{N-2}$ has been known for a long time. But what is its origin in the parabolic setting? The energy functional introduced in (6) is a Lyapunov functional for (7): for a solution $u(x, t)$ to (7) we compute

$$\frac{d}{dt} E_q(u(x, t)) = -\int_{\Omega} |\partial_t u|^2 \, dx.$$  

Thus along an approximate solution

$$v(x, t) = \sum_{j=1}^k \lambda_j^{-\frac{N-2}{2}} U \left( \frac{x - \xi_j}{\lambda_j} \right) - \lambda_j^{\frac{N-2}{2}} H(x, \xi_j)$$  

the energy

$$t \to E_{\frac{N+2}{N-2}}(v(x, t)) \sim k \mathcal{S}_N + \Psi_k^0(\xi(t), \lambda(t))$$

is decreasing, and we may end up at the $k$-bubble energy $k\mathcal{S}_N$ as $t \to \infty$ only if its value is greater than $k\mathcal{S}_N$. If the matrix $\mathcal{G}(p)$ is positive definite that fact is guaranteed, as a simple look at the definition of $\Psi_k^0$ in (5) suggests. A formal consideration of balancing needed for the functions $\lambda_j(t)$ and $\xi_j(t)$ yields at main order to the following system of nonlinear ODEs

$$\dot{\lambda}_j + \nabla_\lambda \Psi_k^0(\xi, \lambda) = 0, \quad \dot{\xi}_j + \nabla_{\xi_j} \Psi_k^0(\xi, \lambda) = 0.$$  

Under the positivity assumption on $\mathcal{G}$, the first equation has an admissible solution $\lambda_j(t) \to 0$, as $t \to \infty$, and the asymptotics (9) follow by a direct computation. This construction is valid on manifolds [15] but it is still open in dimensions 3 and 4, where one expects the parameters $\lambda$ and $\xi$ to satisfy a system of nonlocal nonlinear differential equations, in analogy with a known result in dimension 3 on the whole space. Solutions with different blow-up orders at infinity, generating what is now known as infinite-time tower bubbling, have been obtained in [9].

When solutions blow-up in finite-time, the key issue is to understand how and where explosion can take place. The blow-up is said to be of type I if we have that

$$\limsup_{t \to T} \frac{1}{T-t} \|v(\cdot, t)\|_{L^\infty(\Omega)} < +\infty$$  

and of type II if

$$\limsup_{t \to T} \frac{1}{T-t} \|v(\cdot, t)\|_{L^\infty(\Omega)} = +\infty.$$
Type I means that the blow-up takes place like that of the ODE $v_t = u^q$, so that in the explosion mechanism the nonlinearity plays the dominant role. The second alternative is rare and far less understood. The delicate interplay of diffusion, nonlinearity and geometry of the domain is responsible for that scenario.

The role of the Sobolev critical exponent $q = \frac{N+2}{N-2}$ is well-known to be central in the possible type of blow-up. When $1 < q < \frac{N+2}{N-2}$ solutions can only have type I blow-up, as it was first established by Giga and Kohn (1984) for the case of $\Omega$ convex. This is also the case for $q = \frac{N+2}{N-2}$ and radial solutions of (7); see Filippas, Herrero, Velázquez (2000).

Type II blow-up solutions are much harder to be detected. Herrero and Velázquez (1994) found a radial solution that blows-up with type II rate, for $N \geq 11$ and $q > q_{II}(N)$ where $q_{II}(N)$ is the Joseph-Lundgren exponent defined as

$$q_{II}(d) = \begin{cases} \infty, & \text{if } 3 \leq N \leq 10, \\ \frac{4}{N-4-2\sqrt{N-1}}, & \text{if } d \geq 11. \end{cases}$$

The local profile locally resembles a time-dependent, asymptotically singular scaling of a positive radial solution of $\Delta u + u^q = 0$ in $\mathbb{R}^N$ [5]. In this range for exponents $q$, these solutions are stable. Matano and Merle [17] prove that in the radially symmetric case no Type II blow-up can take place if $\frac{N+2}{N-2} < q \leq q_{II}(N)$, a result that precisely complements that for the Herrero-Velázquez range.

A question that has remained conspicuously open for many years is whether or not type II blow-up solutions of (7) can exist in the Matano-Merle range $\frac{N+2}{N-2} < q < q_{II}(N)$.

The answer is positive [8]: in dimension $N \geq 7$ and $q = \frac{N+1}{N-3}$, the critical Sobolev exponent in dimension $N-1$ and in a class of domains with axial symmetry there exists a solution to (7) which remains uniformly bounded outside any neighborhood of a certain curve $\Gamma \subset \partial \Omega$ while

$$\lim_{t \to T} (T-t)^\gamma \|u(\cdot,t)\|_{L^\infty(\Omega)} > 0,$$

$$\gamma = \frac{(N-3)(N-4)}{2(N-5)}.$$

Notice that for $q = \frac{N+1}{N-3}$ we have $\frac{1}{q-1} = \frac{N-3}{4} < \gamma$ so that $u$ exhibits type II blow-up. This is again a blow-up by bubbling: at main order it is a bubble in dimension $N-1$ centered along a copy of the curve $\Gamma$, shifted inside $\Omega$ and at distance $d(t)$ from $\partial \Omega$, with scaled by $\lambda(t)$. The blow-up region is thus approaching the boundary, but the phenomenon still describes an interior bubbling as $\lambda(t) \sim (T-t)^{\frac{N-4}{N-3}}$, whereas $d(t) \sim (T-t)$ as $t \to T^-$.

In other words, the energy density $|\nabla u(x,t)|^2$ concentrates in the form of a Dirac mass for the curve $\Gamma$, generating bubbling blow-up along a curve. Bubbling blow-up along higher-dimensional sets, like surfaces, is still unknown.

Let me conclude this note with a brief description of the general strategy used in the proofs of the results that have been presented here. The procedure to construct solutions exhibiting the expected blow-up behaviour is to identify a first approximation with the anticipated features and then to get an actual solution, rather than an approximate one, by using a perturbation argument. Finding the remainder is a delicate and difficult step since the behaviour of a solution near the region of blow-up may depend in an intricate way on the entire dynamics, hence it is essential to have a precise control on the perturbation. The inner-outer scheme used in [6] consists in writing the solution as the approximation plus a remainder, and in expressing the remainder itself as sum of two parts, identified as the inner and the outer parts. The inner and the outer parts solve a coupled system of nonlinear partial differential equations, with the property that the main operator for the inner part catches the features of the problem near the singularity and it is expressed in the variable of the blowing-up limit profile, whilst the principal operator in the outer part sees the whole picture in the original scale. A key and delicate issue for the scheme to work is to ensure a fine control on the coupling between the inner and outer parts. Energy methods may fail to achieve this control, and sufficiently fast decay at infinity needs to be prescribed on the inner part. This general approach is quite flexible and has been successfully used in several other contexts, among which the construction of concentrated vorticities for the two-dimensional Euler’s equations for incompressible fluids, in singularity formation for the two-dimensional harmonic map flow into $S^2$ as well as in the infinite-time blow-up for the Patlak-Keller-Segel model for chemotaxis.

References


Introduction

Last year I wrote a book, aimed at a general audience, that explores how data-driven algorithms have impacted the news industry and our ability to separate fact from fiction [6]. This article zeros in on, and amplifies, some of the more mathematical aspects of that story in what I hope will be both informative and engaging to a mathematical audience. As you’ll soon see, there are many fun ingredients at play here, ranging from elementary notions (fractions, linear functions, and weighted sums) to intermediate level concepts (eigenvalues and Shannon information) to sophisticated uses of probability theory, network analysis, and deep learning.

This topic of information and misinformation is complex, multifaceted, and interdisciplinary, and in my opinion more mathematicians should try to enter the public discussions surrounding it and undertake research related to it. I believe we, the math community, can do for misinformation what we have been doing for topics like gerrymandering, where mathematicians have assisted policy makers and helped shape the discourse while also discovering marvelous math topics to explore [9]. I hope this article helps launch some readers down this path, and I would be happy for interested individuals to reach out to me on this.

Supervised Learning

Not all of the mathematical aspects of misinformation discussed here involve machine learning, but many of them do, so let’s start with a very quick review. The basic idea of supervised learning is to assume that a target variable \( y \) depends on a collection of predictor variables \( x_1, \ldots, x_p \) in some mostly deterministic way that can be deduced from the data. When the target variable is \( \mathbb{R} \)-valued this is called regression, whereas when it takes values in some finite set this is called classification. When the predictor variables are \( \mathbb{R} \)-valued, regression takes the form

\[
y = f_\omega(x_1, \ldots, x_p),
\]
where \( f_{\vec{w}} : \mathbb{R}^p \to \mathbb{R} \) is a function that depends on a potentially large number of parameters \( \vec{w} \in \mathbb{R}^N \). The underlying process may be more complex than this functional relationship—it might even be that different values of the target variable are observed for the same values of the predictor variables—but this simple setup is good enough to make useful predictions in most situations.

**Training data** refers to a set \( S \) of observed values of the predictors and target,

\[
\{(x_{1,\alpha}, \ldots, x_{p,\alpha}, y_\alpha)\}_{\alpha \in S} \subseteq \mathbb{R}^{p+1}.
\]

For regression, this is used to adjust the parameters \( \vec{w} \) to minimize the difference between the predicted value \( f_{\vec{w}}(x_{1,\alpha}, \ldots, x_{p,\alpha}) \) and the actual value \( y_\alpha \) (or some cost function applied to this difference) when \( \alpha \) ranges over a set disjoint from \( S \) called the \emph{test} data; for classification we try to minimize the misclassification rate (fraction of predictions that are incorrect) on the test data, or some other error measurement derived from the confusion matrix (the matrix whose \( ij \) entry records the number of data points of class \( i \) that were predicted to have class \( j \)).

The simplest example is linear regression, in which

\[
f_{\vec{w}}(x_1, \ldots, x_p) = w_0 + w_1 x_1 + \cdots + w_p x_p.
\]

A **neural network** is an extension of this where instead of requiring \( f \) to be linear, we let \( f \) be a composition of linear and non-linear functions of a certain type: \( f \) is a composition of any number of pairs consisting of a vector-valued linear function followed by the component-wise application of the **activation function** (typically the sigmoid function or the piecewise linear function that is 0 on negative numbers and the identity on non-negative numbers). Each such pair is called a **layer** in the network, and when the number of layers is \( \geq 2 \) this form of machine learning is called **deep learning**. In contrast to linear regression, the number of parameters \( N \) in a neural network is usually much larger than the number of predictors \( p \), and the parameters are very difficult to interpret. Much more could be said, but this is enough for what follows. For more on deep learning, I recommend the book *Deep Learning* [8].

### Text Generation

You’ve likely heard about troll farms cranking out fake news in the lead-up to the 2016 US presidential election. One concern surrounding the powerful advances in deep learning in recent years is that people could use neural networks to instantly and inexpensively generate an unlimited number of fake news articles. These fears reached a crescendo (or at least, a local maximum) in 2020 when a deep learning text generation system called GPT-3 was released.

Here’s the supervised learning task GPT-3 was trained on. A data point is a passage of \( n \) words, the predictors are the first \( n-1 \) words, and the target is the \( n^{th} \) word. For a document with \( m \geq n \) words, a window of \( n \) consecutive words is slid across, one word at a time, to produce \( m-n+1 \) training points:

**Input:** “The dog likes to play fetch with me,” \( n = 6 \)

1st prediction: The dog likes to play ???

2nd prediction: dog likes to play fetch ???

3rd prediction: likes to play fetch with ???

GPT-3 “read” in this manner a massive number of webpages and scanned books. When the prediction was correct it moved on to the next, whereas when it was incorrect the neural network parameters were adjusted based on the prediction error. To give a sense of the enormous scale, all of Wikipedia was included yet comprised only 0.5% of the training set.

When training a neural network to, say, recognize traffic lights in photos, humans need to manually indicate which training photos have traffic lights and which don’t. In contrast, no manual data labeling was necessary for GPT-3’s training task because both the predictor variable values and the target variable values come directly from the input text. This is called self-supervised learning, and it is crucial for scaling up the training process to the magnitude needed for convincing text generation.

The computational resources to train GPT-3 cost about \$10 million. That’s a lot, but the neural network need only be trained once—after that, it can be used to generate text by feeding it a passage of text and having it predict words one at a time. For example, one can feed GPT-3 the headline to a (possibly made-up) news story and GPT-3 will extend it to a full article. You don’t need to know computer programming or neural networks to use GPT-3—you just type/paste your prompt into a webpage then click a button. The price to generate 10,000 words is about one penny. Let’s give it a try:

**Headline (user prompt):** Mathematician solves famed millennium problem.

**Article (GPT-3 text):** A mathematician has solved a famed millennium problem that experts said had “defied all attempts” at a solution for more than 500 years. The problem, known as the ABC conjecture, deals with the relationship between whole numbers and prime numbers. The conjecture was first proposed by French mathematicians in the early 19th century, but it was not until 2012 that Japanese mathematician Shinichi Mochizuki claimed to have solved it. However, his proof was deemed too complex to be verified by his peers, leading some to doubt its validity. Now, after eight years of work, British mathematician Sir Andrew Wiles has published a paper in which he provides a new and simpler proof of the conjecture. If the

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2That’s why this is a common task for captch systems: your answers provide the training data for a supervised learning system, presumably one related to self-driving cars.
proof is verified, it could have major implications for the field of number theory, as well as for cryptography.

This sounds plausible enough that I Googled just now to see if this really happened and I missed the news! No, it did not—GPT-3 succeeded in creating superficially convincing mathematical misinformation. However, there are some humorous defects here—did you catch any? The ABC conjecture is not a millennium problem (although, if you’re like me, you had to consult the list to double-check this), and if it was first proposed in the 19th century (it wasn’t, by the way—it was the late 20th century, and only one of the two mathematicians who proposed it is French) then how has it defined a solution for 500 years? In general, GPT-3 output has a loose relationship with reality; nowhere in its training process is truth prioritized over falsehood. Getting GPT-3-type systems to produce more factual output is an active area of research [1, 10, 13].

GPT-3 doesn’t just predict the most likely word to follow a passage of text—it estimates a probability distribution on the possible words. (For instance, for the prompt “The dog likes to play…” it might estimate that the next word is “fetch” with 50% probability, “chase” with 25% probability, and “rough” with 25% probability.) One option is to always choose the most probable word, but then you can only get one completion for each prompt, so more common is to sample from this distribution—so that the most likely word is selected most often, but rarer words will be selected sometimes. In practice, people usually do this and produce several GPT-3 completions for a prompt then manually select the most convincing/useful of them. This is what I did with the above millennium problem prompt—some of the GPT-3 completions were not nearly as convincing or interesting as the one I included here, and many essentially just recycled old press releases from when Perelman solved the Poincaré conjecture.

One method proposed to detect text generated by systems like GPT-3 is to use such a system to score the probability of each word in the text and see whether low probability words appear at a disproportionately high rate (if they do, this suggests that the text is organic) [16]. This works reasonably well when the system used to generate the text is very similar to the system used to score the probability of the words. But there are many variants of GPT-3 available now, with many more appearing routinely, so this detection method is not too practical.

Thus far, GPT-3 has not led to the flood of fake news that some people expected—perhaps because the bottleneck is not writing fake news, it is writing fake news that will go viral, and at least so far that requires more of a human touch (additionaly, most fake news is low-quality and cheap to write manually anyway). But GPT-3 does raise the prospect of troll farms mass producing fake news and running large-scale experiments to study virality empirically with unprecedented precision and scale. Moreover, GPT-3 can be used to power more humanlike automated bot accounts on social media—and bots routinely play a large role in disinformation campaigns since they can create artificial engagement on social media posts (more on this later).

Deepfakes

One month into Russia’s invasion of Ukraine, a deepfake video of President Zelensky circulated online in which he tells Ukrainians to lay down their arms and surrender. Fortunately the video was low-quality and didn’t deceive many people, and Zelensky himself addressed it almost immediately clarifying that it was fake. To hear more about this incident and the history and possible future of deepfake videos in politics you can check out a recent Slate podcast episode I was on; here in this article I will focus instead on the mathematical question of how deepfakes are made.

There are many different types of deepfake videos (face swap, lip sync, puppeteering, speech synthesis, etc.) and many different neural network architectures involved, but here I’ll just give a flavor of the topic by discussing one particular form (see [11] for more) that relies on my favorite concept in all of deep learning: the autoencoder. I’ll also discuss a distinct use of the term “deepfake” in which a neural network doesn’t just edit faces, it creates new ones from scratch (though so far this latter technique applies only to photos, not videos).

Autoencoders. Recall that in self-supervised learning the values of the target variable are inferred directly from the data in an automatic fashion rather than entered laboriously by hand. One ingenious form of self-supervised deep learning is the autoencoder. This is a neural network that learns to compress data by passing it through a low-dimensional space. What’s so slick about this is that usually the goal isn’t the compression itself; compressing data is merely a trick to encourage the network to find meaningful patterns in the data.

Here’s how it works. An autoencoder learns to approximate the identity function $\text{id} : \mathbb{R}^p \to \mathbb{R}^p$. That is, the output is a $p$-dimensional vector $\hat{y}$ and it is trained by setting $\hat{y} = \hat{x}$, meaning the target values for each training point are by fiat equal to the predictor values. The key is that the autoencoder consists of a sequence of progressively narrowing layers followed symmetrically by a sequence of progressively widening layers—meaning it decomposes into a

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3Another amusing issue with always selecting the most probable word is that this is the least informative word, in the sense of Shannon’s information theory: the information of an event with probability $p$ is $\log(\frac{1}{p})$, so the more probable a word is, the less informative it is.

4https://slate.com/podcasts/what-next-tbd/2022/03/why-the-zelensky-deepfake-failed
where \( p > p_1 > \cdots > p_e \). The result is that the neural network must learn from the training data how to represent the original \( p \)-dimensional data in a much smaller \( p_e \)-dimensional space.

You can think of this like zipping a file—except rather than a human programmer specifying the compression algorithm, the neural network figures one out on its own. Indeed, passing the data points from the original \( \mathbb{R}^p \) down to \( \mathbb{R}^{p_e} \) is like zipping, then passing them back to the final \( \mathbb{R}^p \) is unzipping (the official parlance is encoding and decoding), and the supervised learning task means the output should resemble the input as closely as possible. By the magic of deep learning, the neural network typically uncovers meaningful multi-scale structure in the data in order to do this, and the structure it uncovers and exploits is specific to the type of data it is exposed to.

If the training data set is a collection of 100×100 images of different human faces, then typically \( p = 30,000 \) as an image is represented by the three RGB intensity values for each pixel, and if, say, \( p_e = 50 \), then the neural network must find a way of encoding each face using only 50 numbers. An overly anthropomorphic version of this would be to describe each face with numbers representing things like the person’s age, skin tone, hair color, hair style, the shape of their face, etc. In reality, the features the neural network learns are not nearly so recognizable to our human minds, but they are nonetheless a distillation of the raw pixel values into larger-scale structures.

If the training set consists of many images of one individual’s face, rather than images of many different individuals, then the neural network doesn’t need to encode things like hair color and skin tone and the shape of the face, so it might instead develop features recording things like the angle of the face and the extent to which the mouth is open, the lips are smiling, and the eyebrows are raised (this description is continuing our overly anthropomorphized version of what actually happens). That is, the neural network focuses less on distinguishing different individuals and more on distinguishing different expressions on the particular individual. This is the key behind the autoencoder’s use in the face swap deepfake.

Face swap. Suppose the goal is to swap person A’s face onto a movie of person B. The problem reduces from movies to still images by working one frame at a time, and locating faces in an image is a standard task (usually solved by deep learning), so we’re reduced to the following: we want to transform an image of face B into an image of face A—but when doing so, we need the new face A to take on the orientation and expression exhibited in the given image of face B (see Figure 1). Here’s the autoencoder approach.

We typically think of an autoencoder as comprising an encoder and a decoder that are trained in tandem. For the face swap, we’ll use two autoencoders (one trained on images of face A, one on images of face B), but we’ll force them to share the same encoder: during the training process, whenever the parameters in the encoding portion of autoencoder A are updated, the corresponding parameters in autoencoder B are forcibly updated in the exact same manner, and vice versa. This way, the low-dimensional features that are developed for encoding both faces will agree: if the first number measures how much face A is smiling, then it will also measure how much face B is smiling. Then all we do is encode face B (with either encoder, since they’re the same) and then decode it with the decoder from autoencoder A. This computes all the salient properties of face B’s expression then produces an image of face A with the same properties. If B was looking to the left with a big smile, then we’ll get A’s face looking to the left with a big smile. Doing this frame-by-frame is the cleverly elegant autoencoder approach to face swap deepfakes.

Deepfake profile photos. The term “deepfake” has two distinct meanings: (1) any form of video editing based on neural networks (this is what we have discussed so far), and (2) synthesizing lifelike photos of non-existing human faces. Unsurprisingly, the second form of deepfake has also been weaponized for disinformation. For instance, in September 2020 both Facebook and Twitter announced that they had uncovered a group of coordinated inauthentic accounts spreading anti-Biden disinformation; the accounts all used deepfake profile photos and were masquerading as American users, when in reality they were fake personas operated by the Russian government. Interestingly, what tipped Facebook and Twitter off to these accounts wasn’t their use of deepfake photos, it was the network structure of their friends/followers/actions—a topic we’ll return to later in this article.

The reason nefarious actors use deepfake photos to hide their tracks instead of just grabbing random profile pics off the internet is because the latter are easily uncovered by a
"reverse image search." If you drag and drop an image file onto Google’s search bar, then Google will scour the web for similar images. So if you use someone else’s photo for your profile, it’s easy to find the original source, thereby revealing the deception; if you use a deepfake photo, then there is no original source to find.

The math behind reverse image search is quite cool. A standard method is the following: (1) train a large autoencoder on a huge database of photos then encode all the images on the web as well as the input image; (2) the vectors for the former that are closest in some metric to that of the latter are the most similar images. Computing distances directly on the original images, viewed as matrices, does not work well because even minor modifications (cropping, rotation, translation, adjustment of the color palette, etc.) usually result in very distant image matrices. The low-dimensional representations autoencoders develop tend to capture the spirit of the image, so those kinds of minor modifications typically have little if any impact on the encoded version of the image. Morally speaking, encoding with a suitable autoencoder introduces a sort of continuity with respect to basic image manipulations that does not exist at the raw pixel level.

Deepfake photos are all created by some refinement of a general method called a generative adversarial network, or GAN for short. The idea is to pit two neural networks against each other: the generator tries to create new data points that look similar to the ones in the training set, while the discriminator tries to determine which data points are real training data and which are fake ones created by the generator. Both networks start out terrible at their respective jobs but then throughout the training process they push each other to steadily improve. It is a marvelous idea, though one should be careful about leaning too heavily on an intuitive view of this duel: the two networks improve and “learn” throughout the training process, but what and how they learn does not typically align with how human minds learn.

There are some interesting math questions surrounding GANs. For instance, this dueling neural networks setup can be viewed in game-theoretic terms, and a recent paper showed that GANs do not always have Nash equilibria but they do have a different form of zero-sum game equilibrium [3]. Since GANs can be used to generate new data points that resemble the data points in any setting, they have myriad applications, such as augmenting small training sets for supervised learning tasks; deepfake profile photo generation is when one applies GANs to a training set consisting of photos of human faces.5

Facebook’s News Feed
For years it has been believed that Facebook’s News Feed algorithm is responsible for an unsettling amount of the viral spread of misinformation online. Awareness of this issue came to the fore and was sharply clarified by the trove of internal documents released by Facebook employee-turned-whistleblower Frances Haugen in the fall of 2021. The basic problem is that Facebook algorithmically ranks the order of the posts seen by all of its 3 billion users, and it does this primarily by pushing the posts that receive the most engagement (likes, shares, comments, emoji reactions, etc.) to the top—but human psychology is such that there’s a correlation between the posts we engage with the most and posts that are divisive, offensive, and misinformative. To better understand this problem, let’s dig into the math behind Facebook’s News Feed algorithm.

The math behind the algorithm. The details of Facebook’s algorithm are kept closely guarded, but the broad strokes were outlined in a company blog post. Each user has a set of potential posts they could be shown (the posts by their friends, the pages they follow, the groups they’re in, etc.). Let’s fix a user and a moment in time. Facebook uses deep learning to predict the probability that this user will like, share, short comment on, long comment on, etc., each post in this set. Also predicted is the probability of each post violating a platform policy (Facebook prohibits hate speech, incitement to violence, and certain specific forms of misinformation). All these probabilities—call them \(q_1, \ldots, q_r\)—are aggregated into a single number, called the post’s value \(v\) (to the given user at the given moment in time), by taking a weighted sum of the probabilities: \(v = \sum w_i q_i\). The weights \(w_i\) don’t just depend on the type of engagement \(i\), they also depend on a variety of factors such as how close the poster is believed to be to the user (posts by closer friends are given more value) and the category of the post (Facebook has at times temporarily lowered the weight on political posts). After a few additional tweaks, Facebook orders your News Feed posts by these value scores \(v\).

Unsurprisingly, the weights on the engagement probabilities are positive whereas the weight on the policy violation probability is negative. Actually, the weight of the angry reaction was originally set to 5 times the weight of a like, but it was eventually lowered to 0—because it turns out angry reactions correlate with a lot of bad things, like politically polarizing content and misinformation.

In a recent Boston Globe opinion article [7], I encouraged Facebook to create a control panel where users can see and adjust all these weights. It’s unclear what impact this would have on things like misinformation in the aggregate, but at least it would allow users to customize their individual experiences—which is important since not everyone is impacted by the different forms of dangerous

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5Think you can tell deepfakes from real images? Try visiting the website https://www.whichfaceisreal.com/.
content equally. For instance, users in marginalized populations that are frequently subject to online harassment may want to weight the policy violation probability much more heavily. I also argue that congress should require all large online platforms using data-driven algorithms for ranking information to allow users to toggle on/off the various data sources the algorithms rely on. Lawyers may call this data privacy and transparency; as a mathematician, I call it letting people choose the inputs to the functions that determine what we see online.

The internal Facebook research leaked by Haugen revealed some fascinating statistics about harmful content—and how to potentially reduce it. Civic content classified as toxic receives twice as many haha reactions and 33% more angry reactions than heart reactions (this discovery factored in to Facebook’s decision to drop the angry weight to zero). Comments with heart reactions are 15 times less likely to be policy violations. Reshare depth refers to the length of chains of reshares (e.g., if A posts something then A’s friend B sees it and reshares then B’s friend C sees this reshare and reshares it, this is depth 2 since it is 2 steps away from the original post)—and it was found that strongly down-weighting posts based on reshare depth would reduce civic misinformation in the form of links by about 25% and in the form of photos by about 50%. Another interesting mathematical detail revealed in the leaked documents deserves its own subsection—so let’s have at it.

Deceptive denominators. In numerous congressional hearings and transparency reports, Mark Zuckerberg and other Facebook officials have touted the success of their machine learning approach to automatically detecting and removing hate speech, repeatedly citing a success rate around 94%. But upon closer inspection, they never asserted that 94% of hate speech is taken down—what they asserted is that among the hate speech that Facebook takes off its platform, 94% of it was detected algorithmically (the rest was flagged manually by users). So how much of the total hate speech on the platform does Facebook manage to take down?

Leaked internal documents revealed that the figure is only around 3–5%, and in some locations it is as low as half a percent. Facebook never outright lied about this, but it acted deceptively by routinely providing the impressively high percentage while keeping the shockingly low percentage secret—even when members of Congress directly asked how successful Facebook is at removing hate speech. I discussed this in an article for Wired [5] that was illustrated nicely by AMS Vice President Francis Su (see Figure 2). The punchline is that denominators really matter: when looking only at the hate speech that Facebook takes down, their machine learning algorithms work very well—but when looking at all hate speech on the platform, they are, alas, terrible.

That said, Facebook was quick to point out that their content moderation is not a binary leave up/take down decision. As you recall, each post has an estimated probability of violating a policy (such as hate speech); when this probability is above a high threshold the post is taken down automatically, but even when this threshold is not reached the post is down-ranked by the News Feed algorithm according to this probability. Facebook rightly pointed out that in recent years they have used this approach quite effectively to reduce the prevalence of hate speech—defined as the fraction of hate speech posts users see on average relative to all posts they see. Down-ranking posts reduces their prevalence without reducing the total amount of hate speech on the platform.

It has proven much harder to measure how much misinformation is on platforms like Facebook and how much of it is taken down by various forms of moderation—in large part because there is no widely accepted definition of misinformation.

Google Rankings

Days after the 2016 US presidential election, the top link in a Google search for “final election results” was a low-quality WordPress blog falsely asserting that Trump had won the popular vote. This was just one very notable instance of misinformation climbing the ranks of Google searches. Many other instances had been noted in the lead-up to the election, and people began to wonder whether this had played a role in Trump’s victory—a victory that caught most experts by surprise. Shortly after the election, Google’s CEO Sundar Pichai was asked not just whether viral misinformation (or “fake news,” as it was called then) might have played a role in the election, but whether it might have played a decisive role (meaning that it’s impact was enough to swing the electoral college victory from Clinton to Trump), and this was his response: “Sure. You know, I think fake news as a whole could be an issue.”

How did a junk blog post and other trashy news sources climb to the top of Google’s search rankings? To answer this, we need to step back to the origins of Google—and the linear algebra underlying it.
Let’s start with a slightly different question: How should one measure a user’s importance in a social media network? To make things concrete, let’s consider Twitter, which forms a directed graph in which users are vertices and an edge from $U$ to $U'$ means $U$ follows $U'$. The most obvious measure of influence of a vertex is its in-degree, which here is a user’s number of followers. But the expected number of users your tweets will reach doesn’t just depend on your number of followers—it also depends on the number of followers your followers have, and the number of followers they have, etc. So we’d like a deeper way of measuring influence in a network.

Given a graph with vertices $v_i$, one would like to assign non-negative numbers $x_i$ to the vertices such that the number on each vertex is proportional to the numbers on its neighbors:

$$x_i = c \sum_j a_{ij} x_j,$$

where $a_{ij}$ are the entries of the adjacency matrix $A$. This is the matrix equation $\vec{x} = c A \vec{x}$, so it is asking for $\vec{x}$ to be an eigenvector (with eigenvalue $\frac{1}{c}$). Since we are only interested in non-negative numbers here, by the Perron-Frobenius Theorem there is a unique (up to scaling) solution $\vec{x}$. These vertex numbers $x_i$ are called the eigenvector centrality scores for the graph. They have a beautiful random walk interpretation: when starting at random vertices in the graph and taking steps to neighboring vertices with uniform probability, the eigenvector centrality scores are proportional to the fraction of time spent at each vertex. Twitter publicly lists the follower counts on all accounts (Barack Obama currently holds the #1 spot, followed by Justin Bieber in #2 and Katy Perry in #3); I wish Twitter also listed the the eigenvector centrality scores so users could readily see which users are the most influential in this deeper sense.

The world wide web can be viewed as a directed network in which webpages are the vertices and links are represented by directed edges. Keeping in mind that people sometimes navigate the web by typing in a URL instead of clicking a link, a modified random walk process is to fix a probability $p$ and then at each step with probability $p$ walk to one of the neighboring vertices, as before, but with probability $1 - p$ jump directly to any vertex in the graph (chosen with uniform probability). The fraction of time spent at each vertex in this modified random walk process still has an eigenvector interpretation, and—more importantly for us—it was the original method Google used to rank search results; it is called PageRank. The name is a bit of wordplay: it refers both to its application in ranking webpages and to Google co-founder Larry Page.7

The problem with using a mathematical formula like PageRank for ranking search results is that over time people learn how to game the system—and even without overt attempts to game rankings, there’s nothing preventing fake news publishers and other forms of misinformation from rising to the top. This is dangerous since people typically think that if something shows up at the top of Google then it’s true.

Coordinated disinformation campaigns, sometimes organized by foreign governments, were able to land propaganda high on Google search rankings by building large networks of sites that linked to each other and that drew additional links from popular far-right “news” sites. And when a popular but low-quality site like Breitbart linked to, say, a lowly blog post about the election results, suddenly that blog post catapulted up in the search rankings.

Over time—and especially during a concerted and continuing push following 2016 to “elevate quality journalism”—Google has developed additional signals that go into search rankings. Google now uses an army of 30,000 low-paid contract workers who manually evaluate search ranking results according to a 168-page instruction manual they are provided, and their by-hand rankings form the training data for machine learning algorithms. Google has been using these manually trained machine learning signals together with the original PageRank measure together with additional factors, such as direct assessments of the quality of news sources. We don’t know too much more than that because the details are kept secret.

The punchline to this story is that Google has not solved its misinformation problem but it has made tremendous strides on it since 2016, largely by admitting that a purely mathematical ranking system is insufficient and human insight is needed to make sure fake news does not rise to the top.

**Network Dynamics**

A landmark paper on the spread of misinformation was published in *Science* in 2018 [14]. By studying the propagation of over 100,000 stories across Twitter over a 10-year span, it found that false stories traveled faster and further than true stories:

- False stories reached 1,500 people six times faster than true stories.
- Even when controlling for various differences between the original posters, such as their number of followers and whether their account was verified by Twitter, false stories were 70% more likely to get retweeted than true stories.

7Incidentally, the other Google co-founder, Sergey Brin, is the son of Michael Brin, a mathematics professor in dynamical systems. And speaking of math in the family of tech giant founders: the most popular Russian social media site, VK, was founded by the Durov brothers, one of whom is a 3-time gold medalist at the IMO who wrote a remarkable dissertation [2] under Gerd Froehling that plays an important role in the pure math research I’ve done with my brother [4]; the Durovs also founded the very popular internet messaging service Telegram.

VK, was founded by the Durov brothers, one of whom is a 3-time gold medalist at the IMO who wrote a remarkable dissertation [2] under Gerd Froehling that plays an important role in the pure math research I’ve done with my brother [4]; the Durovs also founded the very popular internet messaging service Telegram.
There are two very different ways that information can spread and reach a large number of users on Twitter: a prominent influencer could tweet a story that many followers will directly retweet, or a less prominent user could tweet a story that gets retweeted by a small number of followers who then get it retweeted by some of their followers, etc. Even if a story reaches the same number of retweets in these two scenarios, the first is considered a shallow spread and the second a deep spread since it penetrates more deeply into the social network. It was found in this study that not only did false stories ultimately reach larger audiences, but they did so with much greater depth:

- True stories seldom chained together more than 10 layers of retweets, whereas the most viral false stories reached 20 layers of retweets—and they did so 10 times as quickly as the true stories reached their 10 layers.

However, there is a crucial caveat to these results. The researchers tracked all stories that had been fact-checked by one of several reputable fact-checking organizations, so really what they were comparing was the virality of fact-checked articles that were deemed true versus fact-checked articles that were deemed false. It’s not hard to convince yourself that fact-checked articles do not form a representative sample of all news stories. Indeed, most news stories don’t receive fact-checks because they are obviously true—so it is only the suspicious stories that end up on fact-checking sites.

There have been various efforts to train supervised learning classifiers to detect fake news stories on their spread across social media. Some of these involve quite sophisticated methods, such as a fascinating “geometric” form of deep learning in which the non-Euclidean geometry of networks is heavily leveraged [12]. But a fully content-agnostic approach like this (meaning one that looks only at the network propagation patterns of posts, rather than the posts themselves) is a largely quixotic endeavor: high rates of false positives are unavoidable and propagation patterns vary tremendously across time, region, platform, and topic. That said, network propagation patterns are an important signal in the moderation process: some platforms use early indicators of virality to promote posts in the moderation queue. In other words, among the questionable posts that have been flagged for human moderators to inspect, platforms often try to assign their moderators first to the posts that are most likely to become viral—and network propagation dynamics are crucial for detecting this.

Another important application of network geometry/dynamics in the realm of curtailing misinformation is the detection of bot accounts on social media. A common strategy for fake news publishers is to use bot accounts to seed early stage virality for stories by creating artificial engagement. The engagement-based algorithms used by platforms like Facebook and Twitter pick up on this early virality and, mistaking it for authentic, broadcast the stories to a wider audience. Fake news stories are often quite controversial and tend to draw a lot of comments and reactions, so once these stories reach a wide audience, they tend to draw even more engagement—and hence, by the nature of the algorithms, even wider audiences. This is the algorithmic path to virality in which the initial bot-driven artificial engagement soon becomes authentic human user engagement.

Using bot accounts in this way is prohibited on most platforms. For instance, Facebook requires each account to correspond to a real user; Twitter does not tie accounts to individuals and it even allows some harmless bot activity, but it bans coordinated manipulation of the kind described above. But how do the platforms detect bot accounts?

In November 2020, Facebook released some details on its latest deep learning bot detection algorithm [15]. It relies on over 20,000 predictor variables that look not just at the user in question but also at all users in that user’s network of friends. Facebook didn’t disclose what these predictors are but it did say that they include demographic information, such as the distribution of ages and gender in the friend network, and connectivity properties of the friend network. The algorithm is trained in a two-tier process: first, it is trained on a large data set that has been labeled automatically, to get a coarse understanding of the task, then it is fine-tuned via training on a small data set that has been labeled manually so the algorithm can learn more nuanced distinctions. Facebook estimated in the fourth quarter of 2020 that approximately 5% of its active users were fake accounts. Throughout that year, it used this new deep learning system to remove over 5 billion accounts that were believed to be fake and actively engaging in abusive behavior—and that doesn’t include the millions of blocked attempts to create fake accounts each day.

Conclusion
The issue of widespread misinformation on the internet, and how to rein it in, has received a lot of attention in recent years from politicians, journalists, tech companies, and academic researchers. Misinformation does not belong to a single academic discipline—it has been studied by political scientists, social scientists, computer scientists, economists, psychologists, media studies scholars,
and many others. What I have found most striking is that if you draw a Venn diagram of various approaches these disciplines have taken to study misinformation, it’s not too much of a stretch to say that mathematics is what lies at the middle. Most prominently, math helps us quantify the spread of misinformation and quantify the effect of potential interventions, and it helps us understand the algorithms that create and amplify misinformation.

I encourage readers to explore the engaging math at the center of this story of misinformation. You may find a new research topic, you may find a new interdisciplinary collaboration, and you may help a congressional office better understand the complex processes it is trying to regulate. Coming at this from a math background, you’ll find the barrier to entry in this field is surprisingly low yet the potential for impact and intellectual stimulation is surprisingly high.

References


Credits

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In recent decades, science and engineering have been revolutionized by a momentous growth in the amount of available data. However, despite the unprecedented ease with which data are now collected and stored, labeling data by supplementing each feature with an informative tag remains to be challenging. Illustrative tasks where the labeling process requires expert knowledge or is tedious and
A critical aspect of machine learning, which is time-consuming, includes labeling data such as protein sequences with a protein type, texts by their topic, or videos by their genre. In these and numerous other examples, only a few features may be manually labeled due to cost and time constraints. How can we most effectively propagate label information from a small number of expensive labeled features to a vast number of unlabeled ones? This question addresses semi-supervised learning (SSL).

This article overviews recent foundational developments on graph-based Bayesian SSL, a probabilistic framework for label propagation using similarities between features. SSL is an active research area and a thorough review of the extant literature is beyond the scope of this article. Our focus will be on topics drawn from our own research that illustrate the broad range of mathematical tools and ideas that underlie the rigorous study of the statistical accuracy and computational efficiency of graph-based Bayesian SSL.

1. Semi-Supervised Learning (SSL)

Let us start by formalizing the problem setting. Suppose we are given

\[
\{(x_i, y_i)\}_{i=1}^n \quad \text{labeled data,} \\
\{x_i\}_{i=n+1}^N \quad \text{unlabeled data,}
\]

(1)

where \((x_i, y_i)\) are independent draws from a random variable with joint law \(\mathcal{L}(X, Y)\), and \(\{x_i\}_{i=n+1}^N\) are independent draws from the marginal law \(\mathcal{L}(X)\). We refer to the \(x_i\)'s as features and to the \(y_i\)'s as labels. Due to the cost associated with labeling features, in SSL applications the number \(n\) of labels is usually small relative to the number \(N\) of features. The goal is then to propagate the few given labels to the collection of all given features. Precisely, we consider the problem of using all labeled and unlabeled data to estimate the conditional mean function

\[ f_0(x) = \mathbb{E}[Y|X = x] \]

at the given features \(\{x_i\}_{i=1}^N\). We call \(f_0\) the labeling function. For ease of exposition, we will restrict our attention to regression and classification problems, where the labeled pairs are generated from:

\[
\begin{align*}
Y &= f_0(X) + \eta, \ \eta \sim \mathcal{N}(0, \delta^2) \quad \text{regression,} \\
P(Y = 1|X) &= f_0(X) \quad \text{classification,}
\end{align*}
\]

with \(\delta\) known in the regression setting. Regression and classification are prototypical examples of SSL tasks with real-valued and discrete-valued labels, respectively. To streamline the presentation, we focus on binary classification where there are only two distinct classes, labeled by 0 and 1. Several probabilistic models for binary classification are reviewed in [BLZ18]. Extensions to non-Gaussian noise or multi-class classification can be treated in a similar fashion.

As its name suggests, SSL lies between supervised and unsupervised learning. In supervised learning, the goal is to use labeled data to learn the labeling function \(f_0\), so that it may later be evaluated at new features. On the other hand, unsupervised learning is concerned with using unlabeled data to extract important geometric information from the feature space, such as its cluster structure. SSL leverages both labeled and unlabeled data in the learning procedure: all given features are used to recover the geometry of the feature space, and this geometric information is exploited to learn \(f_0\).

<table>
<thead>
<tr>
<th>Task</th>
<th>Labeled</th>
<th>Unlabeled</th>
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<tbody>
<tr>
<td>Supervised</td>
<td>✓</td>
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<tr>
<td>Unsupervised</td>
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<tr>
<td>Semi-supervised</td>
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</tr>
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Relying on unlabeled data to estimate the labeling function may seem counterintuitive at first. Indeed, the question of whether unlabeled data can enhance the learning performance in SSL has been widely debated, and different conclusions can be reached depending on the assumed relationship between the label generating mechanism and the marginal distribution of the features. We will tacitly adopt the smoothness assumption—often satisfied in applications—that similar features should receive similar labels. In other words, we assume that the labeling function varies smoothly along the feature space. Under such a model assumption, one can intuitively expect that unlabeled data may boost the learning performance: uncovering the geometry of the feature space via the unlabeled data facilitates defining a smoothness-promoting regularization procedure for the recovery of the labeling function. The main idea of the graph-based Bayesian approach to SSL is to use a graph-theoretical construction to turn pairwise similarities between features into a probabilistic regularization procedure.

2. Graph-Based Bayesian SSL

We will take a Bayesian perspective to learn the restriction of \(f_0\) to the features, denoted

\[ f_N := f_0|_{\mathcal{X}_N}. \]

We view \(f_N\) as a vector in \(\mathbb{R}^N\), with coordinates \(f_N(i) := f_0(x_i)\). In the Bayesian approach, inference is performed using a posterior distribution over \(f_N\), denoted \(\mu_N\). The posterior density \(\mu_N(f_N)\) will be large for functions \(f_N\) that are consistent with (i) the given labeled data; and (ii) our

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1. AMS Notices limits to 20 the references per article; for this reason, we refer to [VH20, GTKSSA20, SAY20b] for further pointers to the literature.

2. In machine learning, features and labels are often referred to as inputs and outputs, respectively.
belief that similar features should receive similar labels. The posterior density is defined by combining a likelihood function and a prior distribution that encode, respectively, these two requirements:

$$\mu(f_N) \propto L(f_N; y) \pi_N(f_N).$$  \hspace{1cm} (3)$$

Here and elsewhere $y \equiv \{y_1, ..., y_N\}$ is used as a shortcut for all given labels. We next describe, in turn, the definition of likelihood and prior, followed by a discussion of how the posterior distribution is used to conduct inference in the Bayesian framework.

**Likelihood function.** The likelihood function encodes the degree of probabilistic agreement of a labeling function $f_N$ with the observed labels $y$, based on the model defined by (1) and (2). The independence structure in (1) implies that the likelihood factorizes as

$$L(f_N; y) := P(y|f_N) = \prod_{i=1}^N P(y_i|f_N),$$

and the Gaussian and binomial distributional assumptions in (2) give that

$$P(y_i|f_N) = \begin{cases} 
(2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{(y_i - f_N(i))^2}{2\sigma^2}} & \text{regression,} \\
|f_N(i)|^{y_i} (1 - |f_N(i)|)^{1-y_i} & \text{classification.} 
\end{cases}$$  \hspace{1cm} (4)$$

Note that the features are not involved in the definition of the likelihood function; in particular, the likelihood function does not depend on the unlabeled data.

**Prior distribution.** The prior encodes the belief that the labeling function should take similar values at similar features. We thus seek to design the prior so that its density $\pi_N(f_N)$ is large for functions $f_N$ that vary smoothly along the feature data. We will achieve this goal by defining the prior as a transformation of a Gaussian random vector $u_N$ as follows:

$$\pi_N = \begin{cases} 
L(u_N) & \text{regression,} \\
L(\Phi(u_N)) & \text{classification.} 
\end{cases}$$

Here $\Phi : R \rightarrow (0,1)$ is a link function that ensures that in the classification setting the prior samples take coordinate-wise values in $(0,1)$. Section 3 will discuss how to use graph-based techniques to define the covariance structure of the latent vector $u_N$ so that $u_N(i)$ and $u_N(j)$ are highly correlated if the features $x_i$ and $x_j$ are similar. We term the approach “graph-based” because we will view each feature $x_i$ as a node of a graph, and use a matrix $W$ of pairwise similarities between features to define weighted edges. Then, the covariance of $u_N$ will be defined using a graph-Laplacian to penalize certain discrete derivatives of $u_N$. In applications, the pairwise similarities often take the form $W_{ij} = K(\varphi(x_i), \varphi(x_j))$, where $\varphi$ is a feature representation map that embeds the feature space in a suitable Euclidean space, and $K$ is a kernel function such as the squared exponential $K(s, t) = e^{-||s-t||^2}$, with $|| \cdot ||$ the Euclidean norm.

Note that labels are not used in the definition of the prior; instead, the prior is designed using pairwise similarities between all labeled and unlabeled features $\{x_i\}_{i=1}^N$. Bayesian inference. Bayes’s formula (3) combines likelihood and prior to obtain the posterior distribution, used to perform Bayesian inference. Before moving forward, notice again that the prior is constructed solely in terms of the features, whereas only the labels enter the likelihood function. This insight will be important in later sections.

The posterior density $\mu(f_N)$ quantifies our degree of belief that $f_N$ is the true (restricted) labeling function that generated the given data. A natural point estimator for $f_N$ is hence the posterior mode,

$$\hat{f}_N := \arg \max_{g \in \mathcal{R}^N} \mu_N(g)$$

$$= \arg \max_{g \in \mathcal{R}^N} \log L(g; y) + \log \pi_N(g).$$  \hspace{1cm} (5)$$

The right-hand side showcases that the posterior mode, also known as the maximum a posteriori estimator, can be found by optimizing an objective function comprising a data misfit and a regularization term, defined by the log-likelihood function and the log-prior density, respectively. This observation reconciles the Bayesian approach with classical optimization methods that —without a probabilistic interpretation— recover the labeling function by minimizing an objective that comprises data misfit and regularization terms.

Under the Bayesian framework, however, the posterior mean and the posterior median can also be used as meaningful point estimators that can be robust to outliers or model misspecification. Moreover, in addition to enabling point estimation, the posterior distribution also allows one to quantify the uncertainty in the reconstruction of the labeling function by computing Bayesian confidence intervals, correlations, or quantiles. All these quantities can be expressed as expectations with respect to the posterior distribution. As will be detailed later, sampling algorithms such as Markov chain Monte Carlo may be used to approximate these posterior expectations.

Illustrative example. Figure 1 contains a synthetic binary classification toy example where the features are sampled from two disjoint semi-circles, corresponding to two classes. We are given $N = 200$ features, with only $n = 10$ of them labeled (marked with red dots), and aim to classify the unlabeled ones. Using a graph-based prior defined following the ideas in Section 3, we compute the posterior mean—which estimates the probability with which each data point belongs to the lower semi-circle—and the posterior standard deviation—which represents the uncertainty in the estimation. Thresholding the posterior mean
at 0.5 for classification, the results indicate very high accuracy, with lower uncertainty in the reconstruction near labeled features. This example demonstrates a typical scenario in SSL, where the geometric information carried by the unlabeled data helps to achieve good learning performance with few labels.

Outline. Having introduced the graph-based Bayesian formulation of SSL, we are ready to outline the questions that will be explored in the rest of this article, along with their practical motivation:

1. **Prior Design:** How to define the latent field $u_N$ so that the prior $\pi_N$ promotes smoothness, facilitates computationally efficient inference, and yields a posterior that achieves optimal estimation for a large class of labeling functions?

2. **Posterior Continuum Limit:** For a fixed number $n$ of labels, do the graph-based posteriors $\mu_N$ converge to a well-defined probability measure in the limit of a large number $N$ of features?

3. **Posterior Sampling:** Can we design sampling algorithms whose rate of convergence does not deteriorate in the large $N$ limit?

4. **Posterior Contraction:** In the joint limit where both $N$ and $n$ are allowed to grow, does the posterior distribution concentrate around the true labeling function? How should $N$ scale with $n$ to achieve optimal estimation?

These questions, along with their interrelations, will be considered in the next four sections. We will focus on graph-based Bayesian SSL, but related asymptotic analyses of SSL include [NSZ09, BHL+21]. An overarching theme in our Bayesian setting will be to guarantee that the prior and the posterior distributions are well defined in the limit of a large number of features (interpreted as graph nodes). This idea is formalized through the study of continuum limits that play an essential role in understanding the statistical performance of graph-based Bayesian SSL and the scalability of sampling algorithms.

In order to set the theory on a rigorous footing, we will adopt the manifold assumption that the features lie on a hidden low-dimensional manifold [BN04] embedded in an Euclidean space; for the study of posterior contraction, $f_0$ will be assumed to be a smooth function defined in this manifold. We emphasize that the manifold setting is used only for theoretical insight, but the methodology is applicable beyond this setting. The manifold assumption is widely adopted in machine learning and high-dimensional statistics, and encapsulates the empirical observation that high-dimensional features often exhibit low-dimensional structure.

We end this section by showing, in a concrete application, the interpretation of features and labels, as well as the intuition behind manifold and smoothness assumptions. The MNIST dataset $\{x_i\}_{i=1}^N$ consists of $N = 60000$ images of hand-written digits from 0 to 9. We may want to classify images given labels $\{y_i\}_{i=1}^n$ with $y_i \in \{0,\ldots,9\}$ and $n \ll N$. Each image $x_i \in \mathbb{R}^d$ is a $d = 784$-dimensional vector, but the space of digits has been estimated [HA05] to have dimension around $m = 10$, and can be conceptualized as an $m$-dimensional manifold embedded in $\mathbb{R}^d$. The smoothness assumption reflects the idea that images that are similar are likely to correspond to the same digit, and should therefore receive the same label. As in the synthetic example of Figure 1 we need to construct a suitable prior for functions over the features $x_i$ and study posterior sampling algorithms.
3. Prior Design

In this section we discuss the definition of the Gaussian random vector \( u_N \) used to specify the prior \( \pi_N \). It will be convenient to think of \( u_N \) as a random function over \( \mathcal{M}_N \equiv \{ x_1, ..., x_N \} \), or a random process discretely indexed by \( \mathcal{M}_N \). We will denote by \( u_N(i) := u_N(x_i) \) the value of the \( i \)th coordinate of \( u_N \). Such notation will help highlight the analogies between the design of our discretely indexed random vector \( u_N \) and the design of Gaussian processes (GPs) in Euclidean domains.

In GP methodology, it is important to impose adequate smoothness assumptions. For instance, in the popular Matérn class of GPs (to be defined shortly in Section 3.2) in Euclidean space, the mean square differentiability of sample paths is tuned by a smoothness parameter. However, here we seek to define a discretely indexed random vector over abstract features—not necessarily embedded in Euclidean space—and the usual notions of smoothness are not readily applicable. To circumvent this issue, we will rely on a matrix \( W \) of pairwise similarities between features. At a high level, we would like \( u_N \) to be a random function that varies smoothly over \( \mathcal{M}_N \) with respect to the pairwise similarities, i.e., the function values \( u_N(i) \) and \( u_N(j) \) for \( i \neq j \) should be close if the similarity \( W_{ij} \) between \( x_i \) and \( x_j \) is high. In other words, if we view the features \( \mathcal{M}_N \) as a graph whose edge information is encoded in the similarity matrix \( W \), then we wish \( u_N \) to be regular with respect to the graph structure of \((\mathcal{M}_N, W)\). Techniques from spectral graph theory will allow us to construct a random vector that fulfills this smoothness requirement.

3.1. GPs over graphs. Graph-Laplacians, reviewed here succinctly, will be central to our construction. Given the similarity matrix \( W \in \mathbb{R}^{N \times N} \), let \( D \in \mathbb{R}^{N \times N} \) be the diagonal matrix with entries \( D_{ii} = \sum_{j=1}^{N} W_{ij} \). The unnormalized graph-Laplacian is then the matrix \( \Delta_N = D - W \). Several normalized graph-Laplacians can also be considered (see, e.g., [vL07]), but for our purpose we will focus on the unnormalized one. From the relation

\[
\sum_{i,j=1}^{N} W_{ij} |v(i) - v(j)|^2, \quad v \in \mathbb{R}^N
\]

we readily see that \( \Delta_N \) is positive semi-definite. Moreover, (6) implies that if we identify \( v \) with a function over \( \mathcal{M}_N \), then \( v \)'s that change slowly with respect to the similarities lead to smaller values of \( u^T \Delta_N u \). This observation suggests considering Gaussian distributions of the form \( \mathcal{N}(0, \Delta_N^{-1}) \) since its negative log density is proportional to (6) (up to an additive constant) and therefore favors those “smooth” \( v \)'s. However \( \Delta_N \) is singular, and the above Gaussian would be degenerate. To remedy this issue, and to further exploit the regularizing power of \( \Delta_N \), we will instead consider

\[
\sum_{i=1}^{N} (\tau + \lambda_{N,i})^{-s/2} \xi_i \psi_N,i \xi_i^* \sim \mathcal{N}(0, 1), \quad (8)
\]

with \( \tau, s > 0 \) and \( I_N \) the identity matrix. Here we have two additional parameters \( \tau \) and \( s \) which enhance the modeling flexibility. Roughly speaking, \( \tau \) and \( s \) control, respectively, the inverse lengthscale and smoothness of the samples (interpreted as functions over the graph). To see this, we can write the Karhunen-Loève expansion of \( u_N \) in (7)

\[
u_N = \sum_{i=1}^{N} (\tau + \lambda_{N,i})^{-s/2} \xi_i \psi_N,i \xi_i^* \sim \mathcal{N}(0, 1), \quad (8)
\]

3.2. Connection with Matérn GP. Besides the regularizing effect of the graph-Laplacian described above, the Gaussian distribution (7) is also motivated by a close connection to Matérn GPs on Euclidean spaces. To start with, recall that the Matérn covariance function takes the following form

\[
c(x, x') = \sigma^2 2^{1-v} \frac{\Gamma(v)}{\Gamma(v)} (\kappa|x - x'|)^v K_v(\kappa|x - x'|), \quad (9)
\]

for \( x, x' \in \mathbb{R}^d \). Here \( \Gamma \) is the gamma function and \( K_v \) is the modified Bessel function of the second kind. The Matérn GP is a GP with the Matérn covariance function. It is a popular modeling choice in Bayesian methodology due to the flexibility offered by the three parameters \( \sigma, v, \kappa \) that control, respectively, the marginal variance, sample path smoothness, and correlation length scale. As we will see shortly, it turns out that we can view the finite dimensional Gaussian (7) as a discrete analog of the Matérn GP where the parameters \( v, \kappa \) play similar roles as our \( \tau \) and \( s \).

The key connection is the stochastic partial differential equation (SPDE) representation of Matérn GP proved by [Whi63], which says that the Matérn GP \( u \) is the unique stationary solution to

\[
(\kappa^2 - \Delta)^{v/2 + d/4} u = \sigma \sqrt{\frac{(4\pi)^{d/2} \Gamma(v + d/2)}{\Gamma(v)}} W, \quad (10)
\]
where $\Delta$ is the usual Laplacian and $W$ is a spatial white noise with unit variance. With this in mind, we can rewrite (7) in a similar fashion as

$$(\tau I_N + \Delta_N)^{s/2} u_N = W_N,$$  \hspace{1cm} (11)$$

where $W_N \sim \mathcal{N}(0, I_N)$. Now, ignoring the marginal variance in (10), one can immediately see (11) as a discrete analog of (10) under the relation $s = \nu + d/2$ and $\tau = \chi^2$. In other words, we can interpret (7) as a Matérn GP over the graph $(\mathcal{M}_N, W)$.

3.3. Prior continuum limit. If we impose certain assumptions on the graph $(\mathcal{M}_N, W)$, it can be shown that our graph Matérn GP $u_N$ is not only a discrete analog of the usual Matérn GP, but a consistent approximation of certain continuum Matérn-type GPs. To formalize this statement, we rely on the manifold assumption that we had previously foreshadowed. Suppose now that the $x_i$'s are independently sampled from the uniform distribution in the manifold $\mathcal{M}$. We then have the following result (see [GTSA18, Theorem 4.2 (1)] and [SAY20a, Theorem 4.2] for the formal version):

**Result 3.1.** Under a manifold assumption, the graph Matérn GP (7) converges to a Matérn-type GP on $\mathcal{M}$ provided that the similarity $W$ is suitably defined and the smoothness parameter $s$ is sufficiently large.

We next provide some further context for this result. First, the limiting Matérn-type GP on $\mathcal{M}$ is defined by

$$u \sim \mathcal{N}(0, (\tau I - \Delta_M)^{-s}),$$  \hspace{1cm} (12)$$

where $I$ is the identity and $\Delta_M$ is the Laplace-Beltrami operator (the manifold analog of the usual Laplacian) on $\mathcal{M}$. By convention, $\Delta_M$ is a negative semi-definite operator, which explains the minus sign. Just as the connection between (7) and the SPDE representation of Matérn GP, we can see (12) as a manifold analog of Matérn GP defined by lifting (10). In particular, we can write a similar series representation of (12)

$$u = \sum_{i=1}^{\infty} (\tau + \lambda_i)^{-s/2} \xi_i \psi_i, \quad \xi_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1),$$  \hspace{1cm} (13)$$

in terms of the eigenpairs $\{\lambda_i, \psi_i\}_{i=1}^{\infty}$ of $-\Delta_M$. The eigenfunctions encode rich information about the geometry of $\mathcal{M}$ and form a natural basis of functions over $\mathcal{M}$. Comparing (8) and (13), it is reasonable to expect that large $N$ convergence will hold provided that we have convergence of the corresponding eigenvalues and eigenfunctions. To achieve this, we need to carefully construct the similarity matrix $W$ so that the graph-Laplacian $\Delta_N$ is a good approximation of $-\Delta_M$. If we assume that $\mathcal{M}$ is an $m$-dimensional compact submanifold of $\mathbb{R}^d$, then this is indeed the case if we set

$$W_{ij} = \frac{2(m + 2)}{N\nu_m h_N^{m+2}} 1[|x_i - x_j| < h_N],$$  \hspace{1cm} (14)$$

where $\nu_m$ is the volume of the $m$-dimensional unit ball and $h_N$ is a user-chosen graph connectivity parameter satisfying

$$\frac{(\log N)^{c_m}}{N^{1/m}} \ll h_N \ll \frac{1}{N^{1/2\gamma}}.$$  \hspace{1cm} (15)
with $c_m = 3/4$ if $m = 2$ and $c_m = 1/m$ otherwise. Small values of $h_N$ induce sparse graphs, which are easier to work and compute with; see Section 3.4 below. However, very small values of $h_N$ render graphs that are so weakly connected that they cannot induce any level of smoothness in the functions that are likely to be generated by the prior $π_N$. It is thus important to set the connectivity $h_N$ appropriately in order to take advantage of sparsity while at the same time recovering the geometric information of $ℳ$. The specific lower bound in (15) characterizes the level of resolution of the implicit discretization of the manifold induced by the $x_i$’s. We require $h_N$ to be larger than this quantity to capture the geometry of the underlying manifold. Under these conditions on $h_N$, it can be shown that $Δ_N$ converges spectrally towards $−Δ_M$. Other types of graphs such as $k$-nearest neighbors and variable bandwidth graphs can also be employed, and recent work [GT19] have shown spectral convergence in these settings.

3.4. Sparsity. So far we have discussed the construction of our prior from a modeling perspective, motivated by the regularizing power of the graph-Laplacian and the connection with usual Matérn GPs. We close this subsection by mentioning its sparseness. Notice that with our choice of weights (14), the similarity matrix $W$ —and hence the graph-Laplacian $Δ_N$— are sparse. Indeed, one can show that for

$$h_N \asymp \sqrt{\frac{(\log N)^{c_m}}{N^{1/m}}}$$

the number of nonzero entries of $Δ_N$ is $O(N^{3/2})$. Therefore, for small integer $s$ in (7), we are left with a Gaussian with sparse precision matrix, and numerical linear algebra methods for sparse matrices can be employed to speed-up computation. This is important for posterior inference algorithms that may require factorizing $Δ_N$. Similar conclusions can be reached with $k$-nearest neighbors graphs.

4. Posterior Continuum Limit

In this section we discuss the convergence of the posterior $μ_N$ for large $N$ (and fixed $n$) towards a continuum posterior $μ$ defined later. For now, it suffices to note that the continuum posterior is naturally characterized as a probability distribution over the space $L^2(ℳ)$. When formalizing a notion of convergence for posteriors, a challenge arises: the measures $μ_N$ and $μ$ are probability measures defined over different spaces, i.e., $L^2(ℳ_N)$ and $L^2(ℳ)$, respectively. In what sense should these measures be compared? In what follows, we present a possible solution to this question, which also arises in the rigorous analysis of continuum limits for prior distributions considered in the previous section.

4.1. Lifting to the space $P(TL^2)$. In order to compare the measures $μ_N$ and $μ$, we start by introducing a space where we can directly compare functions in $L^2(ℳ_N)$ with functions in $L^2(ℳ)$. We let $TL^2$ be the set:

$$TL^2 := \{(θ, g) : θ ∈ P(ℳ), g ∈ L^2(ℳ, θ)\}.$$

In words, $TL^2$ is the collection of pairs of the form $(θ, g)$, where $θ$ is a probability measure over $ℳ$ and $g$ is an element in $L^2(ℳ, θ)$. For us, the most important choices for $θ$ are the empirical measure associated to the samples $x_i$ and the data generating distribution $𝒫(X)$. We use the simplified notation $L^2(ℳ_N)$ and $L^2(ℳ)$ to denote the $L^2$ spaces for these two choices of $θ$. $TL^2$ can be formally interpreted as a fiber bundle over the manifold $P(ℳ)$: each $θ ∈ P(ℳ)$ possesses a corresponding $L^2$ fiber.

We endow $TL^2$ with the following distance:

$$d_{TL^2}((θ_1, h_1), (θ_2, h_2))^2 := \inf_{γ∈Γ(θ_1,θ_2)} \int \int_{ℳ×ℳ} \left(d_M^2(x, x') + |h_1(x) − h_2(\tilde{x})|^2\right) dγ(x, x'),$$

where $Γ(θ_1, θ_2)$ represents the set of couplings between $θ_1$ and $θ_2$ —that is, the set of probability measures on $ℳ × ℳ$ whose first and second marginals are $θ_1$ and $θ_2$, respectively— and $d_M$ denotes the geodesic distance in $ℳ$. It is possible to show that the $d_{TL^2}$ metric is, indeed, a distance function. Moreover, the topology induced by $d_{TL^2}$ in each fixed fiber $L^2(ℳ, θ)$ coincides with the topology induced by the natural topology of the Hilbert space $L^2(ℳ, θ)$, a fact that motivates the notation $TL^2$, which suggests an $L^2$-like convergence after transportation. We refer to [TPK17] for further details.

We proceed to define a notion of convergence for the posteriors $μ_N$ as $N → ∞$. As discussed above, the $TL^2$ space allows us to see $L^2(ℳ_N)$ and $L^2(ℳ)$ as subsets of the bigger common space $TL^2$. In turn, the measures $π$ and $π_N$, as well as the measures $μ$ and $μ_N$, can then be all interpreted as probability measures on the space $TL^2$. Using this “lifting” we can now interpret the statement $μ_N → μ$ as $N → ∞$, as a statement about the weak convergence of probability measures in the metric space $TL^2$. Further properties of the space $TL^2$ allow us to use a collection of theorems, such as Portmanteau’s and Prokhorov’s, to characterize convergence and compactness in the space $P(TL^2)$.

After specifying the notion of convergence of $μ_N$ towards $μ$, we can now present a result, rigorously stated in [GTSA18].

Result 4.1. Under a manifold assumption, the graph-based posterior $μ_N$ converges to a continuum limit posterior $μ$ over functions on $ℳ$, provided that the similarity $W$ is suitably defined and the smoothness parameter $s$ is sufficiently large.

Further context for this result will be given next.
describe the tools used to deduce this convergence. For ease of exposition, we focus on the regression setting.

First, we notice that the posterior distribution \( \mu_N \), introduced in Section 2 via Bayes’s formula, can be characterized variationally. Indeed, \( \mu_N \) is the solution to the optimization problem

\[
\mu_N = \arg \min_{\nu_N} J_N(\nu_N),
\]

where, for \( \nu_N \in \mathcal{P}(L^2(M_N)) \),

\[
J_N(\nu_N) = D_{KL}(\nu_N||\pi_N) + \int_{L^2(M_N)} \ell(f_N; y) \, d\nu_N(f_N).
\]

Here \( D_{KL} \) denotes the Kullback-Leibler divergence and \( \ell(f_N; y) \) denotes the negative log-likelihood. The first term in \( J_N \) will be small if \( \nu_N \) is close to the prior \( \pi_N \), while the second term will be small if \( \nu_N \) gives significant mass to \( f_N \)’s that are consistent with the labeled data. Theretore, the minimizer \( \mu_N \) of \( J_N \) represents a compromise between matching prior beliefs and matching the observed labels. Following this variational characterization, we define the continuum posterior \( \mu \) in direct analogy with the graph setting:

\[
\mu = \arg \min_{\nu} J(\nu),
\]

where, for \( \nu \in \mathcal{P}(L^2(M)) \),

\[
J(\nu) = D_{KL}(\nu||\pi) + \int_{L^2(M)} \ell(f; y) \, d\nu(f).
\]

The energies \( J_N \) and \( J \) can be extended to \( \mathcal{P}(TL^2) \) by setting them to be infinity outside the fibers \( L^2(M_N) \) and \( L^2(M) \), respectively. This extension is convenient so as to have a collection of functionals defined over a common space. The variational characterization opens the door to the use of tools in the calculus of variations, which allow to prove the convergence of minimizers of variational problems. Indeed, the following three statements together imply the convergence of the minimizer of \( J_N \) towards the minimizer of \( J \), that is, the desired convergence of posteriors.

1. For every converging sequence \( \nu_N \to \nu \) we have \( \liminf_{N \to \infty} J_N(\nu_N) \geq J(\nu) \).
2. For every \( \nu \) there exists a sequence \( \{\nu_N\}_{N=1}^{\infty} \) such that \( \limsup_{N \to \infty} J_N(\nu_N) \leq J(\nu) \).
3. Every sequence \( \{\nu_N\}_{N=1}^{\infty} \) in \( \mathcal{P}(TL^2) \) satisfying

\[
\sup_N J_N(\nu_N) < \infty
\]

is precompact.

As it turns out, it is possible to prove that, under the assumptions of Result 4.1, these three statements hold simultaneously with probability one. The structure of \( J_N \) and \( J \) —where prior and likelihood appear separately— facilitates the analysis. The most delicate part is to compare the prior distributions \( \pi_N \) and \( \pi \), that is, the first terms of \( J_N \) and \( J \). To provide some further intuition, we recall that a random variable \( u_N \) sampled from the discrete prior \( \pi_N \) takes the form:

\[
u_N = \sum_{i=1}^{N} (\tau + \lambda_{N,i})^{-s/2} \xi_i \psi_{N,i}, \quad \xi_i \sim \mathcal{N}(0,1),
\]

while a sample \( u \) from the continuum prior \( \pi \) takes the form:

\[
u = \sum_{i=1}^{\infty} (\tau + \lambda_i)^{-s/2} \xi_i \psi_i.
\]

Here we use the same random variables \( \xi_i \) in both \( u \) and \( u_N \), thereby coupling the measures \( \pi_N \) and \( \pi \). It can be shown that the \( TL^2 \) distance between \( \psi_{N,i} \) and \( \psi_i \) can be controlled with very high probability for all \( i \) up to some mode \( b \) smaller than \( N \). We can thus expect that the sum of the first \( b \) terms in \( u_N \) is close, in the \( TL^2 \) sense, to the sum of the first \( b \) terms in \( u \). For modes larger than \( b \), on the other hand, it will not be possible to obtain decaying estimates for the distance between the corresponding discrete and continuum eigenfunctions. This is to be expected as the graph cannot resolve the geometry of the manifold \( M \) at lengthscale \( \sim h \). To control the higher modes, we must use the fact that the terms \( (\tau + \lambda_i)^{-s/2} \) can be controlled by a factor of the form \( b^{-s/m} \), as it follows from the well-known Weyl’s principle describing the growth of eigenvalues of Laplace-Beltrami operators on compact manifolds. Here it is worth recalling our discussion in earlier sections regarding the level of regularity induced by higher value of \( s \): a large enough value of \( s \) can be used to control the contribution of high order modes. The above argument eventually leads to the following estimate:

\[
\min_{\gamma \in \Gamma(\pi_N, \pi)} \int_{TL^2} \int_{TL^2} (d_{TL^2}(u_N, u))^2 \, d\gamma(u_N, u)
\]

\[
\leq E \left[ d_{TL^2}(u_N, u)^2 \right] \to 0,
\]

as \( N \to \infty \). In other words, in the Wasserstein space over \( TL^2 \), the measure \( \pi_N \) converges towards \( \pi \) as \( N \to \infty \), and thus, the convergence holds also in the weak sense, implying the convergence of the prior terms. With the convergence of priors in hand, the proofs of statements 1-2-3 reduce to a careful use of lower semi-continuity properties of the Kullback-Leibler divergence. We refer to [GTSA18] for further details, and describe next why establishing continuum limits for posterior distributions is important in the design of scalable algorithms for posterior sampling.

5. Posterior Sampling

As noted in Section 2, the construction of point estimates and confidence intervals in Bayesian inference rests upon computing expectations with respect to the posterior distribution. For instance, finding the posterior mean, marginal variances, and quantiles requires one to compute \( E_{\mu_N}[h] \)
for various test functions \( h : \mathbb{R}^N \to \mathbb{R} \). When the posterior is not tractable —such as in SSL classification— expectations can be approximated using sampling algorithms. The goal of this section is to show how the continuum limit of posteriors described in Section 4 can be exploited to design Markov chain Monte Carlo (MCMC) sampling algorithms with a rate of convergence that is independent of the number \( N \) of features. Subsection 5.1 contains the necessary background on the Metropolis-Hastings MCMC algorithm. In Subsection 5.2 we introduce the graph pre-conditioned Crank-Nicolson (pCN) algorithm, a Metropolis-Hastings scheme that exploits the continuum limit to ensure scalability to large datasets. Finally, in Section 5.3 we discuss how the large \( N \) scalability of the graph pCN algorithm can be formalized through the notion of uniform spectral gaps.

5.1. Metropolis-Hastings sampler. Metropolis-Hastings MCMC is one of the most widely used algorithms in science and engineering, and is a cornerstone of computational Bayesian statistics. The basic idea is simple: for a given sample size \( K \), the Metropolis-Hastings sampler approximates

\[
E_{\mu_N}[h] \approx \frac{1}{K} \sum_{k=0}^{K} h(f_N^{(k)}), \quad (16)
\]

where \( \{f_N^{(k)}\}_{k=0}^{K} \) are samples from a Markov chain whose kernel \( p_{MH} \) satisfies detailed balance with respect to \( \mu_N \), that is,

\[
\mu_N(f) p_{MH}(f,g) = \mu_N(g) p_{MH}(g,f). \quad \forall f,g. \quad (17)
\]

The detailed balance condition (17) guarantees that \( \mu_N \) is the stationary distribution of the Markov chain, and consequently, \( f_N^{(k)} \) will be approximately distributed as \( \mu_N \) for large \( k \), under mild assumptions.

The Metropolis-Hastings algorithm is built upon an accept/reject mechanism that turns a given proposal kernel into a Metropolis-Hastings Markov kernel \( p_{MH} \) that satisfies the desired detailed balance condition.

Given the \( k \)-th sample \( f_N^{(k)} \), the \((k+1)\)-th sample is obtained following a two-step process. First, a proposed move is sampled \( g_N^{(k)} \sim q(f_N^{(k)}, \cdot) \) from the given proposal kernel \( q \). Second, the proposed move is accepted with probability \( a(f_N^{(k+1)}, g_N^{(k+1)}) \) and rejected with probability \( 1 - a(f_N^{(k+1)}, g_N^{(k+1)}) \). If the move is accepted, one sets \( f_N^{(k+1)} = g_N^{(k+1)} \), if rejected, \( f_N^{(k+1)} = f_N^{(k)} \). The Metropolis-Hastings acceptance probability

\[
a(f,g) := \min \left\{ 1, \frac{\mu_N(g) q(g,f)}{\mu_N(f) q(f,g)} \right\}
\]

is defined in such a way that the procedure renders a Markov chain whose kernel \( p_{MH} \) satisfies (17). Moreover, under mild assumptions the distribution \( \mu_N^{(k)} \) of the \( k \)-th sample \( f_N^{(k)} \) converges to \( \mu_N \) as \( k \to \infty \). How fast this convergence occurs —and, as a consequence, how accurate the approximation (16) is for a given sample size \( K \) — depends crucially on the choice of proposal kernel \( q \). In the following subsection we introduce the graph pCN algorithm: a Metropolis-Hastings MCMC algorithm that uses a specific proposal kernel to ensure that the rate of convergence of the chain \( \mu_N^{(k)} \) to the posterior \( \mu_N \) does not deteriorate in the large \( N \) limit.

5.2. The graph pCN algorithm. The proposal kernel \( q_{pCN} \) of the graph pCN algorithm [BLSZ18] is chosen so that it satisfies detailed balance with respect to the prior distribution \( \pi_N \). For ease of exposition, we present the algorithm in the regression setting. Let \( \vartheta \in (0,1) \) be a tuning parameter, and set

\[
g_N^{(k)} = (1 − \vartheta^2)^{1/2} f_N^{(k)} + \vartheta \xi_N^{(k)}, \quad \xi_N^{(k)} \sim \pi_N, \quad (18)
\]

where \( \pi_N \) is the prior on \( f_N \) introduced in Section 3, with covariance \( C_N \). A direct calculation shows that the Markov kernel

\[
q_{pCN}(f, \cdot) = N((1 − \vartheta^2)^{1/2} f, \vartheta^2 C_N)
\]

implicitly defined by the proposal mechanism (18) satisfies detailed balance with respect to \( \pi_N \). Therefore, for the graph-pCN algorithm, the Metropolis-Hastings acceptance probability is given by

\[
a_{pCN}(f,g) = \min \left\{ 1, \frac{\mu_N(g) q_{pCN}(g,f)}{\mu_N(f) q_{pCN}(f,g)} \right\}
\]

\[
= \min \left\{ 1, \frac{L(g,y) \pi_N(g) q_{pCN}(g,f)}{L(f,y) \pi_N(f) q_{pCN}(f,g)} \right\}
\]

\[
= \min \left\{ 1, \frac{L(g,y)}{L(f,y)} \right\},
\]

where we used detailed balance of \( q_{pCN} \) with respect to \( \pi_N \) in the last equation. Note that the probability of accepting a move is hence completely determined by the value of the likelihood at the proposed move relative to its value at the current state of the chain. In particular, moves that lead to a higher likelihood are always accepted. Putting everything together, the graph pCN algorithm [BLSZ18, GTKSSA20] is outlined in Algorithm 1.

Notice that the prior distribution —and hence the unlabeled features— are only used in the proposal step, while the likelihood function —and hence the labels— are only used in the accept/reject step. Therefore, one would expect that the acceptance rate should not fundamentally depend on the number of unlabeled features, provided that the prior approaches a continuum limit and the number of labels is kept fixed. This insight can be formalized into
Algorithm 1 Graph pCN.

Input: Prior \( \pi_N \), likelihood \( L(\cdot; y) \), \( \vartheta \in (0, 1) \).

Initialize: Pick \( f_N^{(0)} \).

For \( k = 0, 1, 2, \ldots \) do:
1. Proposal step: Set
   \[
   g_N^{(k)} = (1 - \vartheta^2)^{1/2} f_N^{(k)} + \vartheta g_N^{(k)} + \tau N \sim \pi_N.
   \]
2. Accept/reject step: Compute
   \[
   a(g_N^{(k)}, f_N^{(k)}) := \min \left\{ 1, \frac{L(g_N^{(k)}; y)}{L(f_N^{(k)}; y)} \right\}
   \]
   and set
   \[
   f_N^{(k+1)} := \begin{cases} g_N^{(k)} & \text{w.p. } a(g_N^{(k)}, f_N^{(k)}), \\ f_N^{(k)} & \text{w.p. } 1 - a(g_N^{(k)}, f_N^{(k)}). \end{cases}
   \]
3. \( k \to k + 1 \).

Output: \( f_N^{(k)} \), \( k = 0, 1, \ldots \)

a rigorous guarantee of algorithmic scalability, discussed next.

5.3. Uniform spectral gap. As noted above, under mild assumptions on the likelihood function, it is possible to show that the distribution \( \mu_N^{(k)} \) of the \( k \)-th sample \( f_N^{(k)} \) of the pCN algorithm converges to \( \mu_N \) in the large \( k \) limit. More precisely, for a suitable distance \( d \) between probability measures, one can show that there are constants \( c > 0 \) and \( \varepsilon_N \in (0, 1) \) such that

\[
\text{d}(\mu_N^{(k)}, \mu_N) \leq c (1 - \varepsilon_N)^k, \quad k = 0, 1, \ldots
\]

The largest \( \varepsilon_N \) satisfying this requirement is called the spectral gap of the chain. A large spectral gap implies fast convergence of the chain. In particular, a positive spectral gap is sufficient to ensure the consistency and asymptotic normality of the estimator (16) for suitable test functions. It is therefore important to understand if the spectral gaps \( \varepsilon_N \) deteriorate (i.e. decay to zero) as \( N \) grows. The resulting formalization, formalized in [GTKSSA20], indicates that the spectral gaps for the graph pCN algorithm are uniform, meaning that they are bounded from below by a positive constant independent of \( N \).

**Result 5.1.** Under the conditions that ensure the existence of a continuum limit for the posteriors \( \mu_N \), the graph pCN algorithm has a uniform spectral gap in Wasserstein distance.

The result hinges on the continuum limit of posteriors discussed in Section 4 and on the use of the graph pCN algorithm, which exploits it. Standard MCMC algorithms based on random walks or Langevin dynamics fail to satisfy a uniform spectral gap. The proof is based on a weak Harris theorem [HMS11] that provides necessary conditions for the existence of a Wasserstein spectral gap. An \( L^2 \) spectral gap can be obtained as a corollary.

6. Posterior Contraction

In Section 4 we studied convergence of posteriors \( \mu_N \) towards their continuum limit as \( N \to \infty \) with \( n \) fixed. The previous section showed that exploiting this continuum limit is essential in order to design sampling algorithms that scale to large number \( N \) of features. In this section, we study the performance of graph-based Bayesian SSL when both \( N \) and \( n \) tend to infinity. The analysis of this double limit discerns if, and how, unlabeled data enhances the learning performance. We provide an affirmative answer for regression and classification, with a quantitative analysis of the scaling of \( N \) with \( n \) required to achieve optimal learning performance.

Subsection 6.1 formalizes the problem setting and our criterion used to quantify the learning performance. We then show in Subsection 6.2 how the performance of graph-based Bayesian SSL can be analyzed by bringing together the continuum limit of graph-based priors in Subsection 3.3 with the theory of Bayesian nonparametrics.

6.1. Background. To formalize our setting, recall that we are given labeled data \( \{(x_i, y_i)|i=1\}^n \) sampled independently from the model (2) and unlabeled data \( x_{N_n} \) sampled independently from \( \mathcal{L}(X) \). Notice that we have introduced a subscript to the total number \( N_n \) of features since we are interested in studying its scaling with respect to \( n \). We assume that the labels are generated from a fixed truth \( f_0 \) and aim to study the performance of learning \( f_0 \) with the graph-based Bayesian approach. For our theory, we will view the truth \( f_0 \) as a function defined on the manifold from where the features are assumed to be sampled.

We will use the notion of posterior contraction rates [GGvdV00] to quantify the learning performance. This concept, which we will overview in what follows, provides a rigorous footing for the analysis of Bayesian techniques from a frequentist perspective. We will say that the posteriors \( \mu_{N_n} \) contract around \( f_0 \) with rate \( \delta_n \) if, for all sufficiently large \( M > 0 \),

\[
\mu_{N_n}(f \in \mathbb{R}^{N_n} : \|f - f_0\|_n \leq M\delta_n) \xrightarrow{n \to \infty} 1
\]

in probability, where

\[
\|f - f_0\|_n^2 := \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - f_0(x_i))^2.
\]

Here again we identify a vector in \( \mathbb{R}^{N_n} \) with a function over \( \{x_i| i=1\}^{N_n} \). The convergence (19) implies that, asymptotically, the sequence of posteriors \( \mu_{N_n} \) will be nearly supported on a ball of radius \( O(\delta_n) \) around \( f_0 \). Therefore, \( \delta_n \) characterizes the rate at which the posterior “contracts” around \( f_0 \), and can be intuitively interpreted as the convergence rate of the posterior distribution towards the truth. As a consequence of the convergence (19), the point

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estimator
\[ \hat{f}_n := \arg \max_{g \in \mathbb{R}^{N_n}} \left( \mu_{N_n} (f \in \mathbb{R}^{N_n} : \| f - g \|_n \leq M \delta_n) \right) \]
converges (in probability) to \( f_0 \) with the same rate \( \delta_n \). This observation provides convergence rates for Bayesian point estimators that can be compared with the optimal rates from the minimax theory of statistical inference.

6.2. Performance and unlabeled data. We have the following result, formalized in [SAY20b, Theorem 2.1].

Result 6.1. Under a manifold assumption, optimal posterior contraction rates can be achieved if \( N_n \geq n^{2m} \).

The result suggests that unlabeled data helps and gives a quantitative required scaling of unlabeled and labeled data. We will now illustrate the main ideas behind it. First of all, in order for the unlabeled data to help, there should be some correlation between the truth \( f_0 \) and the marginal distribution of \( X \). This then relates back to the manifold assumption that we have used throughout. The intuition is then that if the truth \( f_0 \) is a smooth function over the manifold, a better understanding of the underlying geometry through the unlabeled data may improve the learning of \( f_0 \). This, in terms of our graph-based prior, is reflected by the fact that it is constructed using all of the features. Since the graph-based prior approximates an underlying continuum prior on the manifold, incorporating the unlabeled data allows one to get a better approximation at the level of the prior, which leads to better learning performance at the level of the posterior.

Another important ingredient in our analysis is that the continuum prior gives optimal learning performance. Recall that the continuum prior is the Matérn type GP as in (12) or (13) and \( s \) characterizes the smoothness properties of the sample paths. It turns out that if the truth \( f_0 \) is \( \beta \)-regular (belonging to a Besov-type space \( \mathcal{H}^{\beta,\infty} \)), then the posterior with respect to the continuum prior with parameter \( s = \beta + m/2 \) contract around \( f_0 \) with rate \( n^{-\beta/(2\beta + m)} \) (up to logarithmic factors), which is the minimax optimal rate of estimating a \( \beta \)-regular function. The key is that, for optimal performance, the prior smoothness parameter \( s \) needs to match (up to an additive constant that depends only on the intrinsic dimension) the smoothness \( \beta \) of the truth \( f_0 \). This agreement is also needed when working with Matérn GPs on Euclidean spaces.

Now the final step is to combine the above two main observations:
1. The graph-based prior approximates the continuum prior.
2. The continuum prior gives optimal posterior contraction rates.

A result from the Bayesian nonparametrics literature then implies that if the graph-based prior approximates the continuum prior sufficiently well (satisfying an error rate of \( n^{-1} \) in an \( L^\infty \) version of the \( d_{TL2} \) metric introduced in Section 4), then the graph-based prior gives the same posterior contraction rates as the continuum prior, which is again optimal. Therefore the remaining piece is to quantify the approximation error of the continuum prior by the graph-based prior, which is shown to be on the order of \( N_n^{-1/2m} \). Therefore the scaling in Result 6.1 is obtained by matching \( n^{-1} \) and \( N_n^{-1/2m} \). The message is that the convergence rate of the graph-based prior suffers from the curse of dimensionality, which is not surprising since the resolution of the \( x_i \)'s scales like \( N_n^{-1/m} \). But the abundance of the unlabeled data alleviates such an issue and leads to an accurate approximation of the underlying continuum prior, based on which optimal performance can be achieved.

7. Summary and Open Directions
In this article we have overviewed the graph-based Bayesian approach to SSL. We have emphasized how the study of continuum limits provides a rigorous foundation for the design of prior distributions and sampling algorithms with large number of features, and is also a key ingredient in the statistical analysis of posterior contraction. The foundations of graph-based Bayesian learning are still emerging, and we expect that future contributions will require the development and the synergistic use of a broad range of mathematical tools, including topology, calculus of variations, spectral graph theory, ergodicity of Markov chains, optimal transport, numerical analysis, and Riemannian geometry. We conclude this article with some theoretical, methodological, and applied open directions.

7.1. Theory. The uniform spectral gap of the graph pCN algorithm ensures its independent rate of convergence in the limit \( N \to \infty \) with fixed number \( n \) of labels. However, the rate of convergence of this algorithm would deteriorate in the joint limit \( N, n \to \infty \). The exploration of MCMC algorithms that scale in this joint limit is an interesting open direction. The contraction of the posterior distribution in this regime has been discussed in Section 6. Existing results assume an a priori known smoothness of the labeling function in order to achieve optimal contraction rates. We believe these results can be extended to achieve statistical adaptivity: the smoothness could, in principle, be inferred without hindering the contraction rate. This is an interesting theoretical question, which may also lead to the design of more flexible prior models. Finally, the manifold assumption that our continuum limits rely on is an idealization of the intuitive idea that features often contain some low-dimensional structure while living in a high-dimensional ambient space. In applications, however, data are noisy and it is important to ensure that algorithms designed under a manifold assumption are not sensitive to small perturbations in the data. In this regard, the
We envision new opportunities to develop active learning strategies for the adaptive labeling of features. However, how best to utilize the Bayesian probabilistic perspective is its ability to provide uncertainty quantification. Finally, an important asset of the Bayesian perspective is its ability to provide uncertainty quantification.

6. A related topic that deserves further research is the modeling of flexible nonstationary graph-based GPs by appropriate choice of graph-Laplacian and similarities between features. Finally, an important asset of the Bayesian perspective is its ability to provide uncertainty quantification. However, how best to utilize the Bayesian probabilistic framework in the SSL context also requires further research. We envision new opportunities to develop active learning strategies for the adaptive labeling of features.

7.2. Methodology. The design of graph-based prior GPs in SSL takes inspiration from, and shares ideas with, the design of GPs in spatial statistics, where numerous techniques have been developed to enhance the scalability of GP methodology to large datasets. Some of these connections are investigated in [SAY20a], but there are still numerous opportunities for cross-pollination of ideas. For instance, [SAY21] analyzes the finite element approach from spatial statistics using the techniques outlined in Section 6. A related topic that deserves further research is the modeling of flexible nonstationary graph-based GPs by appropriate choice of graph-Laplacian and similarities between features. Finally, an important asset of the Bayesian perspective is its ability to provide uncertainty quantification. However, how best to utilize the Bayesian probabilistic framework in the SSL context also requires further research. We envision new opportunities to develop active learning strategies for the adaptive labeling of features.

7.3. Applications. The ideas and techniques that underpin the foundations and algorithms outlined in this article are bound to be useful beyond the SSL regression and classification problems that have been our focus. Graph-based Bayesian techniques can find application, for instance, in nonlinear inverse problems. In this direction, [HJKSA22, HSAY20] investigate PDE-constrained inverse problems on manifolds, where both the prior distribution and the likelihood function involve differential operators supplemented with appropriate boundary conditions. Graphical approximations of these operators call for new continuum limit analyses.

References


N. García Trillos

D. Sanz-Alonso

R. Yang

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1. Introduction

Calculus of variation is a branch of mathematics at the intersection between analysis and geometry that focuses on techniques to find minima (and more generally critical points) of functionals, that usually represent physical or geometrical energies, and to study their regularity. Some famous examples are the Dirichlet energy, used to model the shape of an elastic membrane, and the Plateau problem, named after the physicist Joseph Plateau and introduced by Lagrange to model the shape of soap films.

A basic question in partial differential equations (PDEs) is the existence of solutions given certain initial/boundary conditions. Calculus of variations is a collection of techniques that allows to answer this type of questions. The starting point is a basic fact in real analysis: the first derivative of a smooth function at an interior minimum is zero. Thus, by analogy, in order to find solutions to certain elliptic PDEs, one can look for minima of suitably defined energies, whose first derivative, often called first variation or Euler-Lagrange equation, is precisely the desired PDE. Again in analogy with the classical theory of differentiable functions, in order to guarantee the existence of such minima, compactness of the domain of the functional is often
needed, and this leads to the introduction of new spaces of functions (or geometrical objects), such as Sobolev spaces, whose elements are not in general differentiable functions, and so will not solve any PDE in the classical sense. Thus, the problem of regularity of minimizers arises. In the study of such regularity questions, a very useful tool is given by the so-called monotonicity formulas. They allow to study the infinitesimal behavior of minimizers at a given point by reducing it to the classification of global homogeneous minimizers. Once again the underlying idea is based on a simple fact that I usually ask my undergraduate analysis students as part of their midterm problems:

**Theorem 1.1.** Let \( f : (0, 1] \to \mathbb{R} \) be a differentiable function and suppose that \( f'(x) \geq 0 \) for every \( x \in (0, 1] \), then there exists \( \lim_{x \to 0} f(x) \in \mathbb{R} \).

The goal of this note is to describe a set of techniques, inspired by basic concepts in analysis, such as Theorem 1.1, that have been of central importance to prove existence and regularity of minimizers in a variety of problems in the calculus of variation, such as harmonic maps, minimal surfaces and free-boundary problems, to name a few. Our toy model will be the Dirichlet energy. In particular we will see in the next sections how this energy can be used to prove the existence of classical harmonic functions, which are in fact analytic. This will be achieved using tools that are general enough to be applied to a variety of variational problems, such as the one mentioned above and many others.\(^1\) In order to describe the full depth of these tools and how they are used in more complicated situations involving certain non-linear elliptic PDEs, we will also briefly address two of these problems: harmonic maps and minimal surfaces.

This note should be used as a roadmap for students interested in calculus of variations, to understand some fundamental ideas which lay at the foundation of many major results in the field. In what follows, I will only use symbols and formulas that any student who has taken a graduate analysis sequence should be able to understand. However, I often find certain formulas clearer than their descriptions, and for this reason I will insert them when needed.

2. **Existence of Harmonic Functions:**

   **The Dirichlet’s Principle**

   Given an open, bounded, connected domain \( \Omega \subset \mathbb{R}^n \), an harmonic function \( u : \Omega \to \mathbb{R} \) in \( \Omega \) is a \( C^2 \) function which solves the so-called *Laplace equation*, that is

   \[
   \Delta u(x) \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x) = 0 \quad \text{for all } x \in \Omega,
   \]

   where \( \partial^2 u/\partial x_i^2 \) denotes the second partial derivative in direction \( e_i \in \mathbb{R}^n \), for every \( i = 1, \ldots, n \). The Laplace equation was introduce in 1799 by Pierre-Simon Laplace to prove that the solar system is stable over astronomical timescales, and since then it has been used to construct models for surface tension (soap bubbles and minimal surfaces), heat, electromagnetism, fluid mechanics, and many other physical phenomena.

   Roughly speaking, the Laplace equation requires that the sum of the eigenvalues of the Hessian matrix of \( u \) at every point of \( \Omega \) is equal to zero, that is from any point of \( \Omega \) you can go up as much as you can go down. This property is known as *maximum principle*, and, stated formally, it says that the maximum and the minimum of \( u \) are achieved only on \( \partial \Omega \), unless \( u \) is constant. Another key feature of this equation is its linearity, that is

   \[
   \Delta (c_1 u_1 + c_2 u_2) = c_1 \Delta u_1 + c_2 \Delta u_2,
   \]

   for every \( c_1, c_2 \in \mathbb{R} \), \( u_1, u_2 \in C^2(\Omega) \). As a simple exercise, one can combine these two facts to obtain uniqueness of solutions for the so-called Dirichlet problem, that is given a smooth function \( g \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) any solution of the following problem is unique

   \[
   \begin{cases}
   \Delta u = 0 & \text{in } \Omega \\
   u = g & \text{on } \partial \Omega.
   \end{cases}
   \tag{2.1}
   \]

   There are several methods to prove existence of solutions of (2.1) (see for instance [10]). One of them is the Dirichlet principle, introduced by Riemann, and then formally justified by Weierstrass and finally Hilbert, who in 1900 developed the so-called *direct method in the calculus of variations*. The basic observation is that if \( u \in C^2(\Omega) \) minimizes the energy

   \[
   D(v, \Omega) := \int_{\Omega} |\nabla u|^2 \, dx
   \]

   in the space of functions \( C^2_0(\Omega) := \{ v \in C^2(\Omega) : v|_{\partial \Omega} = g \} \), then \( u \) solves the Dirichlet problem (2.1), and vice versa.
This follows from a standard fact from real analysis, that is for any variation \( v \in C^2_0(\Omega) \) with zero boundary condition on \( \Omega \), the differentiable function
\[
R \ni t \to g(t) := \mathcal{D}(u + tv, \Omega)
\]
has a minimum at \( t = 0 \), and so \( g'(0) = \int_{\Omega} \nabla v \cdot \nabla u \, dx = 0 \). An integration by parts and the convexity of the energy then give the desired correspondence.

After observing this equivalence, Riemann took for granted the existence of minimizers of \( \mathcal{D} \), and it was only 50 years later that Hilbert, following an observation of Weierstrass, gave a rigorous proof of this fact by turning once again to real analysis. Indeed, a sufficient condition for a function \( f : K \subset \mathbb{R}^n \to \mathbb{R} \) to have a minimum is that \( K \) is compact and \( f \) is lower semicontinuous in the same topology in which \( f \) is compact. In order to use this observation, one introduces the so-called Sobolev space \( W^{1,2}_g(\Omega) \) of, roughly speaking, \( L^2 \) functions with gradient in \( L^2 \) (see [10]) and whose suitably defined trace on \( \partial \Omega \) is \( g \). Notice that, since \( L^2 \) functions are only defined almost everywhere, this is non-trivial and requires some assumptions on the regularity of \( g \) and \( \partial \Omega \), which we will not discuss here. It is then possible to equip \( W^{1,2}_g(\Omega) \) with a suitable topology that makes it compact and with respect to which \( \mathcal{D}(u, \Omega) \) is lower semicontinuous, thus obtaining existence of a minimizer in \( W^{1,2}_g(\Omega) \). This is also one of the starting points in the study of functional analysis, that is the study of spaces of functions, of which \( W^{1,2}_g(\Omega) \) is but one of many examples.

Since minimizers a priori only belong to the space \( W^{1,2}_g(\Omega) \), the remaining question to prove the existence of solutions to (2.1) is the regularity of such minimizers, that is if they are \( C^2(\Omega) \) and achieve their boundary datum \( g \) smoothly, so to solve (2.1) in the classical sense. An affirmative answer to this question has been known for a long time in the case of harmonic functions. For more general minimization problems, such as harmonic maps or the Plateau problem, however, answering this regularity question has been one of the major focuses of research in analysis in the last 100 years. We will explain why in the next sections.

3. Regularity of Sobolev Functions: Decay and Growth

We want to discuss the interior regularity of minimizers \( u \in W^{1,2}_g(\Omega) \) of the energy \( \mathcal{D} \), that is their differentiability at interior points of \( \Omega \). It is easy to see that, if \( u \) is a minimizer of \( \mathcal{D} \) in \( \Omega \) then it is a local minimizer of \( \mathcal{D} \), that is
\[
\mathcal{D}(u, B_1) \leq \mathcal{D}(u + v, B_1)
\]
for every \( v \in W^{1,2}_0(B_1) \) where we assume without loss of generality that \( B_1 \subset \Omega \). The advantage of this formulation is that it highlights how neither the boundary of \( \Omega \) nor the Dirichlet datum \( g \) play any role in the study of interior regularity questions.

For local minimizers we want to understand the following questions:

1. Is a local minimizer of \( \mathcal{D} \) differentiable? Is it \( C^2 \), so that we can conclude existence of harmonic functions? Is it \( C^\infty \)?
2. Is a local minimizer an analytic function?

The reason why we state the two questions separately is that, while the first is essentially a question about the decay of some suitably chosen norm of \( u \), the second requires also some form of non-degeneracy, that is polynomial growth, of \( u \). To get an understanding of this, one can consider the classical example of \( C^\infty \) non-analytic function \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases} \tag{3.2}
\]
This function decays at 0 faster than any polynomial, and in fact it is \( C^\infty \) there, but in doing so it fails to have a polynomial growth, and so to be analytic.

Let us discuss question 1 first. As hinted above, in order to prove regularity of a Sobolev function (or even just \( L^2 \)), one tries to prove some form of decay of its Sobolev norm at any given point. A celebrated such criterion for regularity is Campanato’s lemma, which states:

**Lemma 3.1** (Campanato’s lemma (see [15])). Suppose \( u \in L^2(B_2) \), \( \alpha \in (0, 1] \), \( \beta > 0 \), with \( B_2 \subset \mathbb{R}^n \), and
\[
\inf_{\lambda \in \mathbb{R}} \frac{1}{\rho^n} \int_{B_\rho(y)} |u - \lambda|^2 \, dx \leq \beta^2 \rho^{2\alpha} \tag{3.3}
\]
for every ball \( B_\rho(y) \subset B_2 \). Then there is a Hölder continuous representative \( \tilde{u} \) for the \( L^2 \)-class of \( u \) with
\[
|\tilde{u}(x) - \tilde{u}(y)| \leq C_{n,\alpha} \beta |x - y|^{\alpha}, \tag{3.4}
\]
for every \( x, y \in B_1 \), where \( C_{n,\alpha} \geq 0 \) depends only on \( n, \alpha \).

The intuition is the following. If \( u \) is Hölder continuous in a ball \( B_\rho(y) \), that is if (3.4) holds in \( B_\rho(y) \), then we can write it as a constant \( \lambda \) (possibly zero), plus a remainder that does not exceed \( \rho^\alpha \) times a constant (say \( \beta \)). Vice versa, if this holds in every ball, then \( u \) is \( C^{0,\alpha} \). Inequality (3.3) is an integral version of this property, that holds for functions that are merely in \( L^2(B_2) \). The reason we divide the integral by \( \rho^\alpha \) is to make the quantity on the left-hand side of the inequality independent of the measure of the ball over which we are integrating, that is to consider a scale invariant quantity. It is not difficult to check that (3.4) holds if \( \lambda = 0 \) (the value of \( f \) at 0), and any \( \alpha \in (0, 1] \).

Campanato’s lemma holds also for the gradients of \( W^{1,2} \) functions, since they are \( L^2 \) functions themselves, and
in this case gives $C^{1,\alpha}$ regularity. For the gradient statement, (3.3) needs to be replaced by

$$\inf_{\lambda \in \mathbb{R}^n} \frac{1}{\rho^{n-2}} \int_{B_r(y)} |Du - \lambda|^2 \, dx \leq \beta^2 \rho^{2\alpha},$$

(3.5)

where one should think of $\lambda \in \mathbb{R}^n$ as the linear function that is the gradient of $u$ at the point $y$. Notice also that the power of $\rho$ in the left-hand side of the inequality changed to keep the quantity scale invariant. Combining this with the Poincaré inequality, it is possible to prove the following $\varepsilon$-regularity theorem.

**Theorem 3.2** ($\varepsilon$-regularity [15]). There exists $\varepsilon > 0$, depending only on the dimension $n$, such that if $u$ is a local minimizer of $\mathcal{D}$ in $B_1$ and

$$\int_{B_1} |Du|^2 \, dx < \varepsilon,$$

(3.6)

then $u \in C^{1,\alpha}(B_{1/2})$, for every $\alpha \in (0, 1)$.

Roughly speaking, the smallness of the Sobolev norm in (3.6), combined with the minimality of $u$ in $B_1$, is enough to guarantee the decay assumptions of Lemma 3.1 in $B_{1/2}$, and thus regularity. Since derivatives of harmonic functions are still harmonic, it is possible to bootstrap the above result to obtain $C^{\infty}$ regularity at points where (3.6) is satisfied. This is true for more general elliptic PDEs, under suitable regularity assumptions on their coefficients, and is known as Schauder theory.

Let us now discuss briefly a property strictly related to question 2., that is unique continuation. Let us consider a function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open bounded connected domain. Then we say that $f$ satisfies

- the unique continuation property if $f \equiv 0$ in a non-empty open subset $U \subset \Omega$ implies that $f \equiv 0$ in $\Omega$;
- the strong unique continuation property if the condition that $f$ vanishes of infinite order at a point $x_0 \in \Omega$, i.e.,

$$\lim_{r \to 0} \frac{1}{r^m} \int_{B_r(x_0)} |u|^2 \, dx = 0 \quad \forall m \in \mathbb{N},$$

implies that $f \equiv 0$ in $\Omega$.

It is easy to see that an analytic function in $\Omega$ satisfies both of the above properties. On the other hand the function $f$ in (3.2) does not satisfy either of them. In particular, strong uniqueness continuation makes precise the statement that analytic functions cannot decay too fast at a point, in an $L^2$ sense.

Of course, there are much simpler techniques to prove the $C^{\infty}$ regularity and analyticity of local minimizers of $\mathcal{D}$ than Theorem 3.2 (see [10]). However, differently from the ones described in this note, many of them are not flexible enough to be applied to more complicated problems, such as harmonic maps or minimal surfaces.

4. Monotonicity Formulas: Energy Density and Frequency Function

We will see two monotone quantities associated to local minimizers of $\mathcal{D}$: the energy density and the frequency function. The name energy density requires no explanation, while the name frequency function is probably due to the fact that its value at 0 is related to the first non-trivial term in the analytic expansion of an harmonic function, or analogously to the first non-trivial term in its Fourier expansion. The starting point is that, by choosing the variation $\nu$ properly in the definition of local minimizers (3.1), it is possible to prove the following two integral identities, valid for a local minimizer $u \in W^{1,2}(B_1)$ and $B_r(x) \subset B_1$:

$$\int_{B_r(x)} |\nabla u|^2 \, dx = \int_{\partial B_r(x)} u \partial_r u \, d\sigma,$$

$$\int_{\partial B_r(x)} |\nabla u|^2 \, d\sigma = \frac{(n-2)}{r} \int_{\partial B_r(x)} |Du|^2 \, d\sigma + 2 \int_{\partial B_r(x)} |\partial_r u|^2 \, d\sigma$$

(4.1)

where $\partial_r u = \frac{x}{|x|} \cdot \nabla u$ is the radial derivative of $u$ and $d\sigma$ is the spherical measure. It is then a simple exercise in calculus to show that the energy density, defined by

$$\Theta(r, x, u) := \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2 \, dx$$

and the frequency function, defined by

$$N(r, x, u) := \frac{r}{\mathcal{S}_{B_r(x)}} \int_{\partial B_r(x)} |\nabla u|^2 \, d\sigma$$

for $B_r(x) \subset B_1$, are monotone non-decreasing functions in $r$. For example, using (4.1), we have

$$\Theta(r, u) - \Theta(s, u) = 2 \int_{B_r \setminus B_s} \frac{|\partial_r u|^2}{|x|^{n-2}} \, dx,$$

(4.2)

where we drop the dependence on the point $x = 0$. We can then use these two monotone quantities to answer questions 1 and 2 from the previous section.

To answer question 1 we need to show that $\lim_{r \to 0} \Theta(r, u) = 0$, so that Theorem 3.2 can be applied. By Example 1.1, we know that there exists

$$\Theta(0, u) := \lim_{r \to 0} \Theta(r, u) \in \mathbb{R}.$$

A standard technique to compute it, common to many variational problems where monotonicity formulas are available, is the so-called blow-up procedure, that, as hinted in the introduction, together with Theorem 3.2, allows to reduce the study of regularity to the understanding of global homogeneous minimizers.

**Theorem 4.1** (Blow-up procedure). Let $u$ be a local minimizer and let $(u_r)_r$ be the sequence of function $u_r(x) := u(rx)$.
There exist a subsequence \((u_{r_k})_k\), with \(r_k \to 0\) as \(k \to \infty\), and a function \(u_0 \in W_{\text{loc}}^{1,2}(\mathbb{R}^n)\) such that

1. \(u_0\) is a 0-homogeneous local minimizer of \(D\) in \(\mathbb{R}^n\), that is \(u_0(x) = u_0(x/|x|)\) for every \(x \in \mathbb{R}^n\);
2. \(\Theta(u, 0) = \Theta(u_0, \sigma)\) for every \(\sigma > 0\).

The key ideas in the proof of this theorem are a compactness result for sequences of minimizers, that is such sequences converge strongly, up to subsequence, to a minimizer, and the use of (4.1) to prove homogeneity of the limit of the \(u_r\).

The last step is the classification of 0-homogeneous solutions:

**Theorem 4.2** (Liouville’s theorem). Every 0-homogeneous minimizer of \(D\) in \(\mathbb{R}^n\) is a constant function.

It follows that \(\Theta(0, u) = \Theta(\sigma, \phi) = 0\), as desired, thus giving regularity of minimizers at any interior point.

To answer question 2, one can proceed in a similar way, that is

- Let \(N(0, u) := \lim_{r \to 0} N(r, u) \in \mathbb{R}\).
- Consider the sequence \(u_r(x) := u(rx)/r^{N(0,u)}\) and prove a blow-up theorem to obtain a global minimizer \(u_0\) which is \(N(0,u)\)-homogeneous.
- Classify \(N(0,u)\)-homogeneous minimizers, to show that they are harmonic polynomials of degree \(N(0,u) \in \mathbb{N}\).

Intuitively, \(N(0,u)\) is the order of the first non-trivial term in the analytic expansion of \(u\) at 0 and \(u_0\) is the first non-trivial term. It is then possible to apply the same procedure with \(u - u_0\), instead of \(u\), and reasoning inductively in this way one can write the analytic expansion of \(u\) at 0. Of course there are several technical steps to check, such as the uniqueness of \(u_0\) and the convergence of the series. As an example, consider the harmonic function given in polar coordinates by \(f(r, \theta) = r^3 \sin(3\theta) + r^6 \sin(6\theta)\), then \(N(0,u) = 3\) and \(u_0(r, \theta) = r^3 \sin(3\theta)\), as in the following picture.

At the next step \(N(0,u) = 6\) and \(u_0(r, \theta) = r^6 \sin(6\theta)\), as in the following picture.

Notice that the number of oscillations increases at every step, thus the name frequency function.

Using the monotonicity of the frequency function it is possible to prove unique and strong unique continuation in a straightforward manner, and in fact this is the reason why Almgren introduced it in his celebrated Big regularity paper ([11]). The idea is that

\[
\frac{d}{dr} \ln \left( \frac{1}{r^{n-1}} \int_{\partial B_r} u^2 d\sigma \right) = 2N(r),
\]

and integrating this simple ODE in \(r\) and using the monotonicity of the frequency one obtains a so-called doubling condition, that is

\[
\int_{B_{2r}} u^2 dx \leq e^{C_r N(1,u)} \int_{B_r} u^2 dx \quad \forall r \in (0,1),
\]

from which both unique continuation properties follow easily (see for instance [13]).

5. Harmonic Maps and Energy Density
We fix a smooth manifold \(M \subset \mathbb{R}^N\) and we consider locally minimizing harmonic maps, that is functions \(u \in W^{1,2}(B_1; M)\) such that

\[
D(u, B_1) \leq D(u + v, B_1),
\]

for every \(v \in W_0^{1,2}(B_1; \mathbb{R}^N)\) such that \(u(x) + v(x) \in M\) for every \(x \in B_1\). That is, \(u\) is a vector-valued function that minimizes the Dirichlet energy among all functions taking values in the manifold \(M\). The Euler-Lagrange equation associated to this problem is a system of non-linear equations, where the non-linearity is given by a suitable expression involving the second fundamental form of \(M\).

It is possible to show that if \(u\) is a locally minimizing harmonic map, then analogous versions of Theorem 3.2 and 4.1 hold, and the energy density \(\Theta(u, r)\) is monotone. However, Theorem 4.2 fails. As an example one can take \(M = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}\) and \(u \in W_0^{1,2}(\mathbb{R}^3, S^2)\) defined by \(u_0(x) = x/|x|\). This function is a 0-homogeneous, local minimizer in \(\mathbb{R}^3\), is not constant, and in fact it is not \(C^{1,\alpha}\) at the origin, where Theorem 3.2 cannot be applied, since the energy density is not 0.
For this reason, one defines the regular and singular sets, $\text{Reg}(u)$ and $\text{Sing}(u)$, of a local minimizer $u$ in $B_1$ to be respectively the collection of points $x$ in $B_1$ such that $u$ is smooth in a neighborhood of $x$ and its complement. At difference from the case of harmonic functions, $\text{Sing}(u)$ might be non-empty, as shown by the previous example. In fact, singular points can be characterized precisely as those points at which the energy density does not converge to 0, that is

$$\text{Sing}(u) := \{x \in B_1 : \Theta(0, x, u) \neq 0\}.$$ 

A typical question to ask at this point is how big is the singular set? Usually one looks at how big the singular set of blow-up functions is, that is of global homogeneous minimizers, and then tries to extend the same bound to general local minimizers by using once again the monotonicity formulas. In our example above, we have a point of singularity in $\mathbb{R}^3$, and in fact it is possible to prove that

**Theorem 5.1** (Dimension of singular set of blow-ups [15]). Let $u$ be a $0$-homogeneous, locally minimizing harmonic map in $\mathbb{R}^n$, then the Hausdorff dimension of $\text{Sing}(u)$ is at most $n - 3$.

We remark that, using Theorem 3.2, the scaling of the energy density and a simple covering argument, it is easy to show that the $(n - 2)$-volume of the singular set is 0, where $(n - 2)$ is the same power of $r$ as in the definition of energy density. However, in order to get the optimal bound $(n - 3)$ one needs to use a more refined method, called dimension reduction, introduced by Federer, which formalizes the idea that the maximal dimension of the singular set of possible blow-up functions should bound the dimension of the singular set of any local minimizer. A very recent result of Naber-Valtorta (see [14]) actually improves this regularity to rectifiability of the singular set and finite $(n - 3)$-volume:

**Theorem 5.2** (Dimension of singular set of harmonic maps). Let $u$ be a locally minimizing harmonic map in $B_2$, then $\text{Sing}(u)$ has locally finite $(n - 3)$ Hausdorff measure and it is rectifiable.

### 6. An Application of Frequency Function Monotonicity: Area Minimizing Currents

In the previous section we suggested that in those problems where a monotonicity formula is available and the blow-up procedure can be carried out, understanding the maximal dimension of the singular set of admissible blow-ups is enough to understand that of general local minimizers. This is not always the case. As an example we can consider the case of 2-dimensional area minimizing surfaces in $\mathbb{R}^4$. It is well known that every holomorphic curve in $\mathbb{C}^2$ is locally area minimizing (see [11]), so an example of area minimizing surface is given by

$$\Gamma := \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}.$$ 

Notice that this surface is not regular at the origin, where it is not embedded but only immersed. In this setting it is also possible to define an energy density, prove an $\varepsilon$-regularity theorem, and run the blow-up argument (by one-homogeneous rescalings). However, running this procedure for $\Gamma$ at the origin would yield as unique blow-up the plane $\{z = 0\}$ counted with multiplicity 2, which is regular, even though the minimal surface itself is singular: that is the singular structure of a blow-up might give no information on that of the minimizer itself, the problem of course being the presence of multiplicity.

Without entering too much into details, in order to prove the existence of area minimizing surfaces, Federer and Fleming introduced a notion of generalized surfaces, called integral currents (see [12]), which play the same role as Sobolev spaces in the previous sections. This generalized surfaces are not smooth in general, and even minimizers might not be smooth in light of the previous example. However, a major achievement of Almgren-Chang, later completed by De Lellis-Spadaro and myself, is the following theorem:

**Theorem 6.1** (Almgren-Chang’ regularity theorem [2, 6–9]). Let $\Gamma$ be a 2-dimensional integral current, locally area minimizing in $B_1 \subset \mathbb{R}^{2+k}$, $k \geq 2$. Then the singular set of $\Gamma$ is locally finite and at every singular point $\Gamma$ is locally the union of holomorphic curves intersecting only at the singularity.

Among the many ideas needed to prove this result, one of the most successful is the Almgren’s frequency function. Almgren’s insight was to compare the expansion of area minimizing surfaces in holomorphic pieces to the analytic expansion of harmonic functions, and then observing that this last expansion can be done using the frequency function, as explained in the previous sections. In fact, he used this idea first to prove a more general bound on the dimension of the singular set of area minimizing currents in the celebrated big regularity paper (see [1, 3–5]).

**References**


The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section with assistance from Early Career Intern Katie Storey. Next month’s theme will be Good Ideas. All Early Career articles organized by topic are available at https://www.angelagibney.org/the-ec-by-topic.

Planting Seeds for Community

Ellen Eischen and Catherine Hsu

It can feel disorienting to find yourself in a new setting without adequate sources of support and connection, in other words without sufficient community. Given the repeated moves an academic career typically entails, as well as some math-specific cultural norms, it is important for young mathematicians to develop robust skills for building and rebuilding their sense of community. Even though you can sign up for official community groups offered by institutions, building a rich sense of community for yourself frequently necessitates taking an active role in connecting with people with whom you can relate about work, hobbies, and other aspects of life. You will likely need to go beyond simply showing up to institutional community-building meetings.

Like seeds planted in a garden, it is impossible to predict which attempts at finding communities will be the most fruitful in the long run. For both of us (EE and CH), the most fortuitous connections have arisen serendipitously from our involvement in a wide range of groups. Making new connections both in and out of work can lead to unexpected sources of joy, learning, and growth. Moreover, building a foundation for your community that can move with you helps ease transitions between stages in your career.

In practice, how can you build a sense of community for yourself that gives you an authentic sense of belonging, social and academic support, connections to future opportunities, and so on? Here are specific examples of community-building that have influenced our own lives, professionally and beyond.

1. **Get out of your comfort zone.** Some opportunities to build community appear to be far from ideal because of an apparent mismatch in interests. As new postdoctoral fellows (EE in Chicago, IL, and CH in Bristol, UK),

Ellen Eischen is an associate professor of mathematics at the University of Oregon. Her email address is eeischen@uoregon.edu. Her work is partly supported by NSF CAREER Grant DMS-1751281.

Catherine Hsu is an assistant professor of mathematics at Swarthmore College. Her email address is chsu2@swarthmore.edu.

DOI: https://dx.doi.org/10.1090/noti2563
our initial sense of camaraderie with postdocs in our respective departments came from get-togethers that our peers held at bars and pubs, even though we dislike not only beer but also some of the activities around which these events were ostensibly organized. As we experienced, the stated theme of a social event, such as a Super Bowl party, often turns out to be secondary to connecting with others.

2. **Reflect on what you want.** You might seek outreach programs, people with whom to explore a new field, a group focused on professional development, or something entirely different. If you have a clear sense of what you are looking for, it can be easier to notice when key opportunities arise. For several years, EE wrote in her journal about her desire to create a "Women in Math Reading Room," a dedicated space that would include books and articles to help women and members of other underrepresented groups thrive in math and that would also serve as a meeting space for formal or informal discussions sparked by these resources. She felt this could foster community but seemed prohibitively costly to create. Years later, someone asked her if she had ideas for how to spend new funds set aside for women in math, and soon after, EE and CH brought the original idea to fruition. Thanks to support from the department, it has been part of the University of Oregon Mathematics Library for four years now.

3. **Take advantage of built-in support.** A natural place to seek community could be amongst your "academic family," starting with your advisor's students. Although CH was EE's sole graduate student when she began attending conferences, several students of EE's PhD advisor (and of her advisor's advisor) welcomed and encouraged CH. During the pandemic, we started holding virtual "Eischen Group Gatherings," open to everyone mentored by EE, ranging from an undergraduate summer researcher to CH, who is now in a tenure-track position. Rather than focus on mathematical topics, the gatherings provide an opportunity to build and strengthen relationships between participants, who have a wide range of mathematical backgrounds. Depending on attendees' preferences, the meetings have included ice-breakers, career guidance from more senior members, and informal chatting.

4. **Follow up on invitations you receive.** Some of CH's most productive collaborations have arisen from her sending a follow-up email to someone who proposed a potential project idea in passing at a conference. An invitation to speak in a seminar (even if initially posed informally), a chance to discuss a grant application, or an offer to talk about your research over a meal at a conference is a signal that someone is interested in your work. It can be easy to overlook or dismiss informal invitations, but if they interest you, taking the initiative to follow up on them can help you develop in new directions.

5. **Look for opportunities to mentor.** Although early on it can be easy to focus solely on finding mentors for yourself, at every stage of your academic career, you can usually find someone eager for your mentoring. Becoming part of a vertically integrated community can provide a strong foundation for long-term connections. CH is still passing down application tips, cover letter templates, and teaching advice that she received from older graduate students many years ago. Sharing your own knowledge with younger academics can be rewarding in itself, and it strengthens your department to have continuity between different years of graduate students.

6. **Team up with people from adjacent fields.** You might be able to build a robust community by including people who work in related fields, or even outside mathematics, and have goals in common with members of your field. For instance, writing is a nearly universal requirement in academia, so you could seek out a writing accountability group, either informally with people you know from a variety of disciplines or more formally through a program such as the National Center for Faculty Diversity and Development's 14-Day Writing Challenges.1

7. **Find an online community.** A vital source of community for EE comes from the four (or more) hours each month she spends meeting—over Zoom—with the Advisory Board and Exhibits Committee of the newly forming Seattle Universal Math Museum (SUMM). Virtual tools vastly increase the possibilities for meeting with groups who share a common interest, especially if that interest is rare. To avoid Zoom fatigue, you might limit the number of online communities you try simultaneously and just consider ones with no in-person analogue available to you.

8. **Join affinity groups.** When you are the only one or one of only a few in a particular demographic group in your department or school, affinity groups can provide a much-needed sense of community and belonging. We participated in Women in Numbers (WIN) workshops, which facilitate new research collaborations among women in number theory. In addition to their mathematical value, such communities can combat isolation. Attending for the first time as an assistant professor, EE and several other women discovered they all had been grappling with similar challenges alone. Attending as a graduate student, CH was inspired by the role models she met who have dedicated time and energy to promoting women in mathematics research.

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1 Information about NCFDD writing programs is provided at https://www.facultydiversity.org/14-day-challenge.
9. **Participate in activities purely for fun.** The workload and pressure to excel in mathematics research, among other difficulties, can quickly become overwhelming. Being proactive about finding hobbies that you do for your own enjoyment can help you build a fulfilling community. CH has enjoyed organizing intramural sports teams; her math department’s inner tube water polo team, the Floating Tori, even won back-to-back championships towards the end of her graduate career. Similarly, EE’s experience during her postdoctoral years was enriched by participating in (theatrical) improvisation classes, a hobby that was new to her at the time but later grew to intersect with her professional life.

10. **Don’t let rejection hold you back.** We have both enjoyed the culture of potlucks and informal gatherings of friends and neighbors in Eugene, Oregon (home to the University of Oregon). Before moving to Oregon, we found that invitations to similar events were sometimes met with less enthusiasm. It can feel deflating when your efforts at community-building are met with resistance or rejection, but you can respond productively by reaching out to broader groups of people or asking for suggestions from someone familiar with local customs.

Building community is a continuing process. As you grow and circumstances change, your approach to building community will evolve to serve your needs. For years, CH’s mathematical community has sprouted from mathematicians linked to EE. More recently, EE has started to meet mathematicians who introduce themselves as conference friends of CH. You never know how a new connection might expand your community.

**Women in Mathematical Biology**

**Rebecca Segal**

Nine years ago, newly tenured and pregnant with my second child, I was feeling a bit unfocused. I heard about a collaborative workshop for women in mathematical biology hosted at the Institute for Mathematics and its Applications (IMA). I was accepted, reaped the benefits of attending, and have now spent the years since organizing similar workshops. Since graduate school, I have been part of the AWM community, but this research workshop at the IMA was a new dimension for me. Working on a nascent research project for a week with a group of smart, interesting, and enthusiastic women brought me full circle back to my undergraduate days at Bryn Mawr College. The intense shared experience, the mentoring both formal and informal, the networking, and the development of a group bond has been a source of continued joy in my mathematical career and one that I hope carries others forward as well.

Kristin Lauter started a Women In Numbers (WIN) workshop in order to grow and support more women researchers in Number Theory. The success of that workshop has inspired 25 different research networks to be established, Women in Mathematical Biology (WIMB) being one. The structure is generally established so that senior women will mentor and collaborate with bright young women in their field on a part of their research agenda of their choosing, while the junior participants will develop a network of colleagues and supporters and encounter important new research areas to work in, thereby improving their chances for successful research careers.

A primary goal of the workshops in the research networks is to create and foster high-quality research collaborations between women. In the WIMB workshops, we also aim to provide our groups an opportunity for positive professional growth and development for all participants. Group collaboration is exciting in this context because it provides a tangible and supportive way to learn about a new problem, develop new skills, and connect with new colleagues. New and fresh eyes bring unexpected observations that can lead the project forward. However, we also acknowledge that group collaboration can be challenging, especially when members do not have prior history together or common training. Some time is required to learn each other’s strengths. As an organizer, I work with the groups to make sure that each member of the group feels able to contribute and grow as part of the research effort.

**Ellen Eischen**  
**Catherine Hsu**

**Credits**

Photo of Ellen Eischen is courtesy of Miles Truesdell. Photo of Catherine Hsu is courtesy of Kevin Brinker.

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Rebecca Segal is a professor of mathematics at the Virginia Commonwealth University. Her email address is rasegal@vcu.edu.

DOI: https://dx.doi.org/10.1090/noti2564
The impact of these workshops has been felt by all the participants. The post workshop survey responses have been overwhelmingly positive—100% of the respondents feel that the workshop was worth their time. This includes the group leaders as well as the junior participants. A few responses from the post workshop survey sampled here indicate the positive response: “My group was productive and enthusiastic the whole week and it was a great work environment. Doing math in an all-female group was a really positive experience and I left feeling super motivated.” Another participant said her favorite part of the workshop was “Establishing a new group of collaborators. I’ve honestly never developed this skill and I’m glad to have had this opportunity.” The workshop also provides a time to have in-depth conversations about work-life balance and issues of starting a family or two-body problems (i.e., job searching with a partner who is also an academic) as well as dealing with the promotion and tenure process. For many women participants, they were able to establish a research project despite being in a high-teaching-load institution, or post-pregnancy leave. The workshop has enabled participants to jump start stalled research programs and gain tenure and/or promotion when the prospects had previously seemed unlikely.

Many of our past participants have now returned as group leaders or organizers. As the community continues to grow and flourish, bringing new generations of women into the fold, we remain thankful to the mathematical, government, and industry institutes who facilitate the funding and logistics. It’s so fun to go to conferences and see that mini-symposia related to the WIMB work have been organized by workshop participants. I can catch up with past participants and hear about their ongoing successes and to see how the research projects have developed over the years. Personally, I gained a new sense of direction and excitement about my research portfolio and met many colleagues I know I can count on for career advice and encouragement. I’m lucky to have been able to participate in the first Women in Mathematical Biology workshop and feel equally privileged to continue organizing these workshops and to meet the up-and-coming researchers in my field.

Collaborating Across Disciplines

Benjamin Braun and Pooja Sidney

As a mathematician whose research is in geometric and algebraic combinatorics (Braun) and a psychologist who studies how children understand math concepts (Sidney), it might not be immediately clear how we benefit from collaborating. However, we have found many shared interests and goals in the realm of mathematics teaching and learning, and our collaboration has developed in positive and unexpected ways. Our goal in this article is to share some of the things we have learned about cultivating and maintaining a productive collaboration across disciplines and to explain why this has been valuable for each of us.

Collaboration has many forms. Many of us collaborate within our own disciplines. Perhaps someone else within our field has disciplinary expertise or access to tools that we do not. Often, these collaborations are transactional, enduring only to serve a given project and ending when that project is completed. Sometimes, collaborations have more depth, are more personal, and reflect relationships in which we continue to invest over time. Many of these long-term collaborations are focused on generating new questions and ideas or making connections between different fields. In this setting, our collaborators often become our friends and help drive our work forward in new and more interesting ways. In our experience, interdisciplinary collaboration can also take both transactional and relational forms, with relational collaboration both requiring more work and being ultimately more fruitful and gratifying. So, what does it take to build a relational, interdisciplinary collaboration? We have found that there are five key ingredients: build a network, identify shared interests and goals, work to understand each other’s disciplines, value multiple types of outcomes, and be patient.

Build a network. One of the critical ingredients in starting a good collaboration is (and yes we know this sounds lame but it is incredibly important) to invest time in building your network. If you don’t know people outside your discipline, then it is difficult to make connections with people outside your discipline! For example, we met because Ben had served on committees outside the math department with a senior psychology faculty member, Christia Spears Brown. When Pooja was hired, Christia told her to talk to Ben to make a connection with the math department. We

Benjamin Braun is a professor of mathematics at the University of Kentucky. His email address is benjamin.braun@uky.edu.

Pooja Sidney is an assistant professor of psychology at the University of Kentucky. Her email address is pooja.sidney@uky.edu.

DOI: https://dx.doi.org/10.1090/noti2562

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Photo of Rebecca Segal is courtesy of Rebecca Segal.
met for coffee, found out that we have several common interests related to how students learn math, and this led to an invitation for Pooja to speak at an event in the Math Department Teaching and Learning Seminar. Note that this collaboration was the result of multiple actions: Ben agreed to serve the university beyond his department and made a point of engaging in those activities. Pooja sought out connections with others in a new place, and Christia made a point of mentoring Pooja as a new faculty member by helping expand her network. Then, Pooja took Christia’s advice and sent Ben the following email:

I am a new faculty member in the Psychology Department, and my primary area of research is children’s mathematical thinking and learning. Christia Brown suggested to me that you might be a good person to know here at UK! Do you have any interest in a brief meeting, perhaps over coffee, sometime in the next couple of weeks? I’d love to know more about your interests in mathematics teaching & learning and your work with pre-service teachers here.

I look forward to meeting you!

So this leads to a key ingredient of collaboration: quality conversations do not happen on a whim, they require continuous engagement so that everyone is in the right place at the right time for a connection to form.

Identify shared interests. Once you have made a connection with someone, our view is that the best way to identify shared interests is to have conversations. While there are other ways, for example, checking out someone’s professional webpage and looking at their identified areas of interest, recent projects, and current products, a static document never tells the whole story. In a conversation, you can get a better sense of your prospective collaborator’s thought process. What problems do they seek to solve? What approaches do they take? What informs their thinking about potential solutions? Sometimes connections are immediately clear; in our case, we immediately identified that we were both interested in improving mathematics education for preservice teachers. Some connections are more distant, yet the potential for developing shared interests is there. An example of this for us is that we are both interested in the role of identity in mathematical learning, but the ways in which our interests overlap only became clear after many conversations across several years. What is important is to be attentive to potential connections, even if they are not immediately obvious; we have both had experiences with attempted collaboration where others were not actually committed to learning something new.

Not every conversation with a potential collaborator will necessarily result in future collaboration. If there are not common intellectual or professional interests or themes, then this might not be the best collaboration, at least at that moment. Since all of our interests evolve over time, building a network means having the opportunity to revisit a possible connection later. Also, just as with any other relationship, sometimes a potential collaborator is not a good fit.

Work to understand each other’s disciplines. One major challenge of identifying shared interests and engaging in intellectual conversation across disciplines is grappling with differences in disciplinary culture, history, and language. Unfortunately, it is common for faculty in different academic disciplines to be siloed in their respective departments and colleges. One consequence of this is that in our doctoral training, especially in mathematics, it is uncommon for students to take graduate courses outside our primary discipline. Another consequence is that faculty often need to put in effort to generate and maintain disciplinary knowledge outside their field.

This is already a challenge within disciplines, given the highly specialized nature of modern research. For example, there are typically many bridges that researchers in PDE, combinatorics, and machine learning must cross in order to develop a deep understanding of techniques and methods in the other fields. However, meaningful connections between these areas exist. Similarly, within psychology, there is often little discussion across experimental and clinical areas, despite clearly overlapping interests (for example, in addressing mathematics anxiety). Across disciplinary boundaries, the obstacles are even greater, especially when crossing between the humanities, social sciences, and physical/natural sciences. In many cases, methods, goals, professional standards, and even epistemological foundations are distinct and sometimes contradictory.

Because of this, interdisciplinary collaboration requires that everyone involved be genuinely curious to learn more about each other’s fields, and be willing to put in substantial effort to learn new modes of thought and new models for what it means to know. What does this mean for us? For Ben, it means reading papers in psychology, learning more about survey design from Pooja. For Pooja, it means learning about advanced mathematics, goals and expectations for junior mathematicians, different formulations of success in mathematical work, aspects of the history of mathematics and math education, what mathematical teaching and learning look like in practice, and ongoing debates about all of these topics.

Value multiple types of outcomes. Given the amount of effort needed to build our interdisciplinary knowledge, a natural question to ask is: “what is the purpose of investing time to understand other disciplines?” In academia, our reward systems are set up to focus our energy on a narrow range of outcomes, including high-impact publications. While these are certainly possible outcomes from a collaboration, there are many other possible outcomes that are
equally valuable, even if they are not rewarded institutionally in the same way. Ironically, many of these valuable outcomes have a significant positive impact institutionally, professionally, and personally. In our own collaboration, we maintained open-ended goals focused on learning about each other’s work instead of aiming to produce specific products. We acknowledged from the beginning that our collaborative efforts might end up producing traditional academic products (and this is turning out to be true), but this was not something that either of us required. Having this philosophy as a foundation for our work was a critical ingredient for us, and allowed us to be flexible in exploring all possible ways forward.

So, in addition to learning about shared interests, an initial conversation with someone can also include a discussion about how a potential collaboration might support your general intellectual and professional goals. Concrete goals might include designing a workshop, developing an intervention, conducting a research project, or writing a paper. More open-ended goals might include understanding another person’s disciplinary perspective, identifying connections between disciplines, facilitating connections for our students, or maintaining an updated understanding of research developments in a related field.

For us, it was helpful to identify a specific intersection point. Initially, we focused our conversations on the topic of how math majors conceptualize arithmetic involving rational numbers. Since one of Pooja’s research areas is on understanding how children learn division by fractions, and Ben has a lot of experience teaching math majors (including pre-service high school teachers), this turned out to set the stage for rich conversations. Following many conversations, we decided to design an experiment involving math majors and rational numbers conducted in Pooja’s lab, and we started collecting data during the 2019–2020 academic year. Unfortunately, this project was halted by the COVID-19 pandemic. While reflecting on the partial data we had managed to collect, we realized that our study design was not going to achieve what we had wanted it to. Here, the open-ended nature of our goals allowed us to shift to new ideas. Moreover, for Ben, this was a valuable experience in learning how intellectual work in the social sciences progresses, with pilot studies informing later studies leading to published work. There have been many more outcomes from our collaboration (more on these below), but this is an example of how collaborative efforts often take time to fully develop and mature.

Be patient. One theme that has come up repeatedly so far is worth pointing out explicitly: collaboration across disciplines requires patience, as each of the other four ingredients to successful collaboration take time. Patience is required to build a network and allow that network to generate authentic connections. Once we have formed a connection, we shouldn’t expect to identify all of our common interests in a first conversation. Sometimes there will be obstacles to communication arising from differences in disciplinary culture or individuals’ approaches to brainstorming and problem solving. In our collaboration, having patience has given us time to build trust and mutual respect and to cultivate our focus on cooperation, understanding, and curiosity rather than competition, judgment, and evaluation.

Patience is the key ingredient needed to allow the process of building a relationship to unfold and develop and to support the other ingredients for successful collaboration.

Maintaining the collaboration. Once a collaboration is established with shared interests and goals, how do we keep it going? One effective approach is to build time for your collaboration into your normal routines. This can be done in ways ranging from as simple as scheduling a regular coffee break or walk to ambitious activities such as applying for a joint grant. For us, two things that have helped maintain our collaboration have been to develop research projects to work on, for example the project about math majors and rational numbers that we mentioned previously, and to broaden our shared network and welcome other people into our collaboration.

Collaboration is fundamentally about bringing together multiple perspectives and drawing on the diverse strengths of a collective. As our ideas expanded, our group did as well. In our first project, running from 2019–2020, we invited a mathematics graduate student (Julianne Vega) and a psychology undergraduate thesis student (Gabrielle Eismann) to share in its development and execution. While this project has not yet resulted in any published products, it did produce a draft of a survey to measure student self-evaluation of mathematical proficiency and some pilot data regarding that survey. During the pandemic and following the racial justice protests of 2020, we were part of a team of six co-organizers who started a University of Kentucky working group on Ethics, Equity, Inclusion, and Justice in the Mathematical Sciences (EEIJMS), and this group attracted faculty and graduate students from mathematics, psychology, STEM education, and engineering. The EEIJMS working group meets six times per semester, so this builds regular opportunities for conversation into our schedules which helps to maintain our collaboration. It has also increased the scope of our collaborative efforts; for example, as part of EEIJMS, a subgroup of ten faculty and students updated the survey from our previous work and we have administered it to hundreds of first- and second-year undergraduates in mathematics courses as part of a new study of identity and belongingness in undergraduate mathematics education.

It is possible that our collaborative work will lead to concrete outcomes, such as peer-reviewed articles, presentations, and grant submissions. However, even without these, our collaboration has had many positive impacts. Through EEIJMS, we generate new ways of thinking about shared problems that we take back to our disciplinary work and into our classrooms. Pooja thinks differently now about the “math” part of math learning, based on a
richer understanding of mathematics as a discipline and practice, in turn influencing her other research collaborations within psychology. She has also been inspired to dramatically change her own teaching practice with respect to group work, based on conversation and experiences in the EEIJMS meetings. Ben interprets research papers in math education and psychology with a different perspective now that he’s been through the data collection process and has learned more about study design. His perspective on social sciences research was particularly influenced after spending one session in an online discussion group with Pooja’s research network, learning about the various factors (funding, disciplinary norms, disciplinary politics, etc.) that play a role in determining research directions in the social sciences. This changed how he thinks about the ways that social scientists develop their research programs. Also, in his own classes, Ben thinks about question design in new ways, taking into account what he has learned regarding the factors that impact how students answer questions. This has also influenced his approach to mentoring mathematics PhD students.

These are not outcomes that either of us would have predicted during our first conversation, and this is one of the wonderful qualities of collaboration, especially across disciplinary boundaries. While we can’t predict what the next few years will bring, we know that our collaboration will continue to enrich our intellectual, professional, and personal lives.

Credit
Photo of Benjamin Braun is courtesy of Benjamin Braun. Photo of Pooja Sidney is courtesy of Pooja Sidney.

Erica Winterer
At twenty-two years old, I started my teaching career in a New Orleans high school. My oldest student was twenty-one and on my fifth day, a sophomore named Malaysia asked me if I was trying to be Hilary Swank (referring to her Freedom Writers portrayal of Erin Gruwell, the American teacher known for her unique teaching method used to inspire at-risk youth to further their education). She wasn’t wrong. Even though I knew next to nothing about my students’ community, I thought I would make an excellent teacher because I understood the content. Misguided and over-confident, I muddled through my first two years of teaching. I wasn’t the worst, but I definitely wasn’t the best. My students undoubtedly deserved better.

After two years of teaching, I could finally manage the paperwork, mostly manage upwards of thirty teenagers, and keep the rats from eating the corners off the reward Starbursts; however, I was still blindly flailing through attempts to build “classroom culture.” Always a pillar of professional development, but never clearly defined, classroom culture was a slippery, murky creature I would rather avoid than chase after. Every example offered to me seemed like some wizard teacher with a unicorn personality was able to effortlessly inspire groups of teenagers with their nebulous teacher moves. All the magic seemed to hinge on these teachers’ personalities which were far removed from mine. I am not a particularly funny or gregarious person. I prefer small groups of people, have very few close friends, and avoid attention. The best choice I made was to stop imitating these examples.

Students valued my authenticity more than my ability to entertain them. We built community by building trust. I demonstrated care for students through high expectations, highly organized lesson plans, and praise for their individual progress. My investment in students’ success and our shared responsibility for the course combined to foster and grow a community. I will never be Jaime Escalante (the Bolivian-American educator portrayed in the 1988 film Stand and Deliver) and that is fine. We can still build community without magic.

I had to repeat this mantra the first time I saw my graduate advisor, Uri Treisman, teach his freshman calculus class. He is one of those wizard instructors, seamlessly

Erica Winterer earned a degree in biomedical engineering from Tulane University and is currently a doctoral student in STEM education at The University of Texas. Her email address is ewinterer@utexas.edu.

This article is reprinted from Count Me In: Community and Belonging in Mathematics, https://bookstore.ams.org/clrm-68/. DOI: https://dx.doi.org/10.1090/noti2565
integrating expertise from mathematics and psychology to forge a transformative experience for students. A highly respected figure in the community of mathematics, he has long been committed to equity-minded teaching, starting with his work at UC Berkeley and the creation of the Emerging Scholars Program [6]. I spent the first two years as his TA just trying to catalogue (and organize) all the moving pieces of the course. As I read more of the literature, I realized the magic of the course was a combination of expert knowledge, intentionality, and many iterations. Every decision that is made, no matter how small, is considered and informed by research and our collective teaching experience. How should we frame the exam results? If we say XYZ, what will students hear? What do they need in this specific moment? Which students need more challenge? Which ones need more support? Do they all need a win to lift morale? Everything matters. I am now in my fifth year of my doctoral program and have graduated to co-teacher of this freshman calculus course. The course has extra resources (e.g., undergraduate TAs, me as a co-teacher) but we consider it more of a laboratory than a replicable model.

We work with leading psychologists, including David Yeager, to design classroom environments that promote positive learning mindsets. Specifically, we carefully conceive and employ structures, routines, and rituals we think will positively influence students’ growth mindset, purpose, and belonging. These structures, routines, and rituals in combination with personal connection and shared responsibility help to create a sense of community in the course. In this chapter, I will share some specific examples of how we try to build connections with and between students, in the hope that they might be helpful in building community within your own course. These examples were chosen based on student interviews I conducted two years after a section of the course ended. They were the course components which most students remembered and which seemed to have a lasting impact. While each course component is based on a combination of mathematics, education, and psychological literature, the intention of this chapter is to simply share strategies we employ and student reflections. For that reason, I do not include a detailed discussion of the academic literature [2, 3, 7, 8, 10].

Community is sustained through the connections between its members. Without these connections, intended messages can be lost, misinterpreted, or ignored. We try to build an environment that fosters connection not only between students and the instructional team but between students themselves. Courses, especially large lectures, can often feel impersonal and students wear their anonymity like camouflage. Instructors are often happy to play along. It is difficult to feel a sense of responsibility to a student you know nothing about, and teaching is one of many demands on our time. We must work purposefully over the course of the semester to disrupt these social norms and practices that isolate students from their professors and each other.

**Welcome Students**

This starts on day one by welcoming students to the course. The purpose of day one is to signal to students that we want to know them, induct them into the mathematics community, and set the expectation that the course is difficult by design. We start by trying to learn our students’ names before the first day of class. With class sizes of about 130, this means we quiz ourselves with flashcards made from their ID photos of questionable quality. Clearly, we never get all their names correct that day or even within the first two weeks; we do get enough names right to shock students and let them know we see them. Many students have reported that just using their names made them “feel more comfortable and like the class was less daunting.” They have also said that just “calling someone by their name draws you to that person and lets them know you see them as a person—not just a student.” Learning students’ names is time consuming on the front end, but the return on that investment runs the course of the semester.

We also use the first day to establish common ground in the content. We display their math genealogy, introduce them to their ancestors, and tell them we are now and forever their teachers. We frame the content of the course as their legacy, i.e. something that they are entitled to and do not have to earn. We get mixed reactions—

I guess he would list a bunch of old mathematicians and say they were our ancestors? At first, I was like what is this guy talking about? Afterwards, I was like OK, he’s just saying if they were able to do it, we should be able to do it.

I do remember feeling like it made me, like a part of a group. Like a part of a very smart group who knew very smart things, and I could do it. Ya know? I’m up there with the, uh, triangle guys.
He made us feel like math was a part of us. Which was really nice because I know for a lot of people if they find a topic really difficult, they tend to give up on it and say this is not for me, like I’m not meant to be doing this. But, even though it was a simple thing of him being like, look at how all these people are related, it made us feel like—oh, we can actually do this. Or like, we actually have the ability to do this, like, inside of us. It’s not just something that I just wasn’t born to do.

I kind of questioned the credibility, if it was true or not. But I liked the idea of it. Um, it was just showing that we’re somebody in this long lineage of people—at least instilling in us that you’re somebody. You’re not just here to take a course credit and get out of my class. Like, you’re going to be here, and you’re going to be my student forever kind of thing rather than you’re just going to be here and you’re going to move on.

Conducting these interviews two years later, I realized that I underestimated the impact of the genealogy ritual. This simple ritual resonated with students in a way I did not expect—and in some cases reinforced their sense of capability and membership in the mathematics community. Over the course of the semester, Uri references these ancestors to communicate that while we expect students to learn the material quickly, these ideas are complicated and famous mathematicians struggled with them before us. He will explicitly message that there are specific tricks that his advisor passed on to him and he will pass to students. These tricks are not things that they should know in advance or be able to derive through their knowledge or common sense. We want to prevent students from questioning their intelligence if they do not immediately understand an idea. We cannot expect students to feel like valued members of a community if they doubt their own capability and contributions.

Toward this end we do our best to encourage students to attribute any difficulty or struggle in the course to our intended design and not personal deficiencies. We hope that these first impressions help to sow seeds of trust, but it is tentative; they still don’t know us.

Make the First Move

As instructors we need to demonstrate that we care about students and that this is not just some formal, impersonal classroom. Social norms establish distance between professors and students. Often there is an unwritten agreement that they don’t bother us and we don’t bother them. We both show up and play our respective roles while maintaining a comfortable distance. Intentional work is required to disrupt these social norms that isolate students from their instructors and each other. These are norms in our society which make real teaching difficult. Unless we disequilibrate them, and do so with integrity, real teaching cannot occur. We need to explicitly tell students that we want to know them, and we expect them to know us, while creating conditions for new, more productive norms to evolve.

We know relationships with faculty are important to students and research shows that these connections positively influence students’ academic outcomes. However, students intimidated by professors are likely to avoid faculty and instead seek help from their peers or struggle in isolation. For these students, professors’ expert knowledge paired with a serious or hyper-professional demeanor makes faculty seem unapproachable. These students miss out on opportunities when they don’t make connections with professors, and we miss the chance to know our students, to see first-hand what sense students made of a lesson or lesson-slice, or at the macro-level, how the course is shaping students’ interests and aspirations.

This disequilibration starts with us making the first move. There are simple ways we can initiate conversations and connect with students to avoid these missed opportunities. For example, students describe instructors who just ask them, “how are you,” and genuinely listen to their response as “caring.” Taking the time to ask students about their degree plans and to encourage them can make students feel validated in their academic pursuits, like their potential is recognized. As little as one conversation with a professor can dampen students’ doubts and reaffirm their belief that they can be successful in college.

To quote a former student, “It’s one thing to say you want to get to know your students, it’s another thing to actually do something about it.” We do a few things in class that signal to students that we want to know them and care about their progress:

- We make small talk before and after class with individual students. Ask them simple questions like “how are your other classes going?”, “how is your day going?”, etc.
- We schedule test reviews outside of regular lectures, to interact with students in a more informal environment.
- We learn and use students’ names in lectures and discussion sessions (not always perfectly, but they appreciate that we try!).
- We announce to students that everyone is expected to attend at least one office hour.

Every professor is required to schedule office hours; not every professor actually wants students to show up. If we want students to come to office hours or to reach out for help, we need to signal that we actually want them there, that we are invested in their progress. To signal that we actually want students to come to office hours, we notice who hasn’t shown up and email those students. Now, I don’t think it’s necessary to send office hour invites to every student who hasn’t shown up, but I do track who has attended. The idea is just to let students know you want to see them. If we email a few individual students, word will get out that we meant it when we said we want to see
everyone in office hours. It is our responsibility to make the first move because students will always follow our lead.

Even two years later, students remembered the efforts to engage with them and open lines of communication:

He made the effort to come up to me, ya know, a few times … and say, ‘Sophia!’, because obviously he knew everyone’s name, and like kind of just catch up … He a lot of the time would make the first move. Which is not really a thing you see with professors happen at all … In a lot of other classes, professors just are not going to speak until they are spoken to. Which is, like, a really common thing. But I think Professor Treisman, like, he didn’t care. He was like, I’m going to introduce myself to everybody. …

I remember now, even in lecture, he would make a really big deal about like, you coming to his office hours. He would say, ‘I still haven’t seen so and so people’ or … he would really emphasize wanting every single person to go to his office hours or make an appointment so he could get to know them.

At the end of class he would randomly stand in front of the classroom and just like ya know, that’s where he was before and as people were leaving he would just maybe just wave or smile or be like hey how are you doing, how’s this and that or something we previously talked about.

I think Professor Treisman also made a big deal in wanting to know about the student also. He would, ya know, like, ask you ‘How are your classes going? What classes do you want to take?’ Or he would say, ‘What are you interested in?’ … ‘Oh you’re physics? Are you going to do FRI [freshman research initiative]?’ Ya know, he kind of had an idea of how the system works here and kind of would like, ask us questions seeing where we are in that system, like what are we doing that is right. And we would just talk about what’s going on, like, how was your morning going, or he’s going to a flight directly after class and he has a suitcase with him. Yeah, it didn’t have to be complicated with him I think.

Let Students Know You

Even though students appreciated these efforts, it is unfair for them to be the only ones sharing about themselves. We try to share personal anecdotes, stories about our work, interests, and hobbies to disrupt the power dynamic and make students feel more comfortable. As instructors, we may not perceive ourselves as intimidating or unapproachable. However, knowing that students are often intimidated, we must make a conscious effort to build relationships. An important part of relationship-building is letting students know you. Having informal, balanced, personal interactions with students can have a significant influence on the classroom dynamic and student performance. Although it takes work, research has shown that increased rapport with students impacts participation as well as effective and cognitive learning. It’s easier to approach someone you feel like you know personally, and students rate approachable professors as more effective overall.

As a teacher of undergraduate students, I recognize that many students walk into class already intimidated by professors, determined to never show confusion. Students look up at professors and see someone who has climbed above them, separate and fundamentally different from themselves. This dynamic simply exists within the current system, and we anticipate it when the semester begins. To make students more comfortable, we try to chip away at this “Us vs. Them” power structure. We humanize ourselves by sharing information about our lives and work. It could be anything that is honest and authentic to ourselves. We might tell students about: a conference that we went to, a paper we read, some of our research, a hobby, a short anecdote, our recent travels.

Sharing personal stories or interests helps to close the distance between us and our students. We recognize that students need to know a little bit about us before they can be vulnerable with their confusion. We try to communicate to students that it is not innate intelligence that separates us, but education, training, and practice—experiences that they can also collect over time. Also, who doesn’t like to just know a little something about the people they work with?

In interviews, students emphasized how professors sharing information about themselves humanized them and made the class feel more connected:

He also shared his own personal story a lot. Which, um, was pretty good, because that really humanized him ‘cause he wasn’t just some big authority on campus in calculus, he was also like, he’s from the Bronx right? [laughter] Yeah, also a guy from the Bronx, so that was pretty cool.

There was, like, one story that I always remember he told … that he was a fan of the baseball team from Brooklyn and they moved from Brooklyn to Los Angeles. Um, and he told the story and it was very funny but, then he just went right onto teaching. But, I think that kind of thing makes a professor a lot less intimidating. You can just tell a story like that, and it’s like, you’re not forcing it, it’s just like … somehow it came up. I don’t know how. And you distill it, and then you know, you’re just seeing that human aspect.

A lot of the times you don’t really know a lot about a professor besides, like, Rate My Professor, ya know?
But it’s nice to get, like, the story behind the person ‘cause nowadays it’s more like … in the students at UT it’s kinda like us versus them … where students are tryin’ to pass this class, get through it somehow, but more than that it’s like … I feel like there’s a lot behind professors … there’s so many weird amazing things I find out about my professors, even the ones I don’t like, and it’s like I never learned to appreciate it because I was so stressed out from the class … I guess it’s nice to know, like, I dunno, there’s much more to a professor.

Help Students Know Each Other

It’s not only important for us to forge connections with our students, but we should also foster connections between students themselves. My own greatest fear in undergrad was asking a stranger a question. I was sure the other students in engineering knew everything and asking any questions would just confirm that I was the only one confused. If I had a question, I always ran it by a friend I trusted. I needed to know that my question was reasonable and would not make me look ridiculous. If I asked a silly question, I knew they wouldn’t assume I was a silly person.

I conveniently forgot this experience in my first few years of teaching. Despite explaining group work procedures with precision, offering explicit directions and exemplar groups, students rarely took risks or interacted in the ways I envisioned. In fact, without intervention, students often spend an entire semester sitting next to the same people without knowing their names. When working with others, students usually subscribe to the norms of interaction, are nervous to ask for help, and are intimidated by the fear of looking unintelligent. Research shows group work and collaboration benefit learning and future careers; however, a quick Google search reveals people are not naturally good at working together. Putting people in the same space or telling them to talk does not result in higher productivity; the collaboration must be strategic. So, how do we connect students and effectively coach them to work together?

Watching Uri teach, I realized students only fully engage with the content when they feel safe presenting their work and ideas. Since it is easier to feel comfortable working with people you know, Uri starts class by encouraging students to know each other’s names. It seems overly simplistic. However, students have told us that just knowing another student’s name makes it easier to ask them a question or to collaborate on an answer. I was chatting with a student about this and he said, “Yeah, what am I going to do if I don’t know their name? Say yeah, hey you, can you help me with this?”

Beyond learning names, we encourage students to work together throughout the semester, not just on the first day of class. For example, we:

• Make time for students to meet each other/exchange contact information on the first day of class.
• Offer extra credit for the first homework assignment if it is completed in a study group. (Students email us the members of their group, where they met, length of the session, brief description of how it went, and a hook ‘em horns selfie of the group.)
• Assign group problems during lecture and instruct groups to spend the first minute on introductions.
• Set the expectation that students should know each other and cold-call students at the beginning of class, asking them to name three people around them.
• Remind students to study in groups throughout the semester and give a quick mid-semester Google survey to make sure everyone has a study group. We might try to connect isolated students to a few different groups to give them options.

This intentional connecting not only builds community but supports students’ academic success. It is something I wish my courses had emphasized when I was a student. Listening to our students reflect on how they remembered routines designed to get them to know each other, I realized how easy it would be to underestimate the impact of this simple act. They really valued the formal time during lecture dedicated to helping students connect and noted how it was not something they experienced in most undergraduate courses.

We go to panels and the students are always like, ‘Yeah like make sure you’re like talking to the people around you in class. Ya know it’s super important throughout your college experience. Ya know study groups …’ I feel like even before I went to college people always said that. But I think that Dr. Treisman recognized that that’s not something people are just going to do. Ya know, like no matter how many times you hear it—oh you should really talk to the people around you and get to know them— that students won’t do it.

Starting the first day of class he would call on you and be like, ‘hey, who are the people that are around

Figure 2. Extra credit selfie of a study group.
you?’ And that kind of fostered more of a community environment than any other classes that I’ve had before. Calculus is really difficult and if you don’t have that kind of support system, you’re not used to meeting your classmates and forming study groups, it’s really hard to succeed, especially as a freshman.

We had that thing where we would tell each other our names, or he’d ask us who was sitting next to us. That, I loved that . . . it was just the best . . . I felt comfortable. And I think when you feel comfortable, you feel like you can ask for help.

In the very beginning it was like, why do you want me to know the person’s name next to me? Like, he’s just going to sit next to me every day in class, and that’s how it is in other classes. It’s like those people sit next to you and you never really acknowledge them. And so, in the beginning it was like why are you making me acknowledge this person next to me? And then it became evident, why he was having us work in groups. I think the homework sessions made me realize the importance of that.

I could see how some people would get stressed out and be like, man, why do I have to learn these people’s names? But, I think it’s kind of an important thing, ‘cause especially so in college you get really disconnected. And especially at a school like UT, there’s so many students here, like thousands, but you can feel like isolated, um, really easily. So, I think even just knowing someone’s name can open up . . . like, it’s more important than just inside the class, maybe outside the class even?

Honestly, I loved that you were forced to talk, not forced, but it was like, he encouraged us to talk to the people around us and getting to know others’ names and getting to know people we wouldn’t normally talk to. Um, it really helped because in a class like that one, where the content is really hard and it’s like you have to spend a lot of time studying and stuff, knowing that there was other people around you and putting a name to their face, and understanding that they were going through the same thing, made it a lot easier for you to actually reach out to people being like ‘hey, I don’t understand this can you help me?’ And then as well, like, it being . . . was my first semester, being encouraged to talk to people around me really helped me make friends and I still talk to a lot of people from the class. And, although none of us are in the same class [now] it’s just . . . since we were given the opportunity to get to know each other we kind of bonded over the class itself and then it grew into a better friendship which was really nice and I really liked that a lot.

Stereotype Threat, Belonging, and the Importance of Community

Students are usually aware of negative stereotypes related to their identity and their awareness often amplifies as they age. Stereotype threat refers to the psychological impact of stereotypes that allege inferiority of marginalized groups in a certain domain [1, 5]. In a situation where a student assesses their group’s stereotype as relevant, a student may feel an extra psychological burden relative to their peers who are not in the same group. This activation of stereotype threat is dangerous as it can lead to disidentification with academic subjects and undermine emotions that intrinsically motivate students to learn. For example, Catherine Good demonstrated that stereotype threat can suppress the test performance of even the most qualified women in college-level mathematics [4].

Evidence points to the potential of stereotype threat to interfere with a student’s problem-solving capacity in our classrooms. We should acknowledge this and work to create a classroom environment that mitigates the effects of stereotype threat by promoting feelings of belonging. Gregory Walton, Geoffrey Cohen, and David Yeager, among others, have produced extensive work around this topic [7, 9]. The previously-described strategies to facilitate connections with students should help with this but we should also encourage students to interpret threats to their belonging as a common experience that is shared, normalized, and transient among the undergraduate population. To create this effect, we explicitly message to students that the course is difficult by design and that students struggle through it every year. We also try to share our own stories of academic struggle and we bring in a panel of past students to discuss how they struggled, persisted, and then excelled on the final exam. Again, our goal is that students attribute experiences of struggle to the design of the course and consider it a normal experience rather than any perceived deficiency on their end.

Our students really appreciate these shared stories of struggle, both from me and Uri, and our former students. In the interviews I conducted, they expressed how these examples gave them perspective and normalized the struggle they were experiencing. Here are a few comments from those interviews:

I remember my freshman year, you were saying, back when you were doing engineering, you had people tell you at one point that you shouldn’t do engineering anymore. And, I, recently because of health issues had my GPA drop and last semester my advisors were like you should just withdraw from the semester and do it over again. And I was like, no, I really think that I can still do this—it doesn’t matter.
my GPA dropped a bit because of health things … it’s still possible. So, thank you for telling me that.

I remember one day we had a review session and he invited three people and one of them told her story about she herself was coming from a high school with less than enough calculus experience and she was scared and wasn’t doing as well as she wanted. And then she’s up here, success story, like, she’s doing great. It provided a sort of encouragement … he’s not just telling us that we can be good at these things … he’s bringing an example for us … the effort just made me feel like it was worth working hard for.

When you’re struggling it feels like you’re just going to be there forever. Whenever he had his students come and talk to us, I was like, OK, there’s an end to this madness. Everything’s going to be all right. They did fine. Somehow, one way or another, we’re all going to be fine. I guess, it made me not lose hope because I feel like … struggling in math class was like losing hope and gaining hope, losing hope and gaining hope.

Make It Your Own

While the strategies outlined in this chapter have been effective for us, they make up only a small slice of our course. These examples were offered as potential ways to begin building community, not to imply this is all that is needed or the only way to do it. There are essential elements woven into the fabric of our course: established trust, shared responsibility, expert content, and pedagogical knowledge, etc. I suspect the described strategies would not have the same effect if they were implemented in isolation or without a certain level of established trust between instructors and their students. That being said, I would remind readers who are interested in building community, not to imply this is all that we can do our best to make sure all of them feel like they belong in our classrooms, mathematics, and our institutions. That starts with making connections and building community.

References


Credits

Figure 1 is courtesy of Lana Fukisawa/Department of Mathematics, UC Berkeley.
Early Career

Todxs Cuentan en ECCO
Building a Mathematical Community

Federico Ardila-Mantilla and Carolina Benedetti-Velásquez

Aprendí que para uno encontrarse tiene que buscar en la raíz. …
Aprendí que no soy sólo yo, y que somos muchos más.
—Hugo Candelario González, Grupo Bahía

What is ECCO?
The Encuentro Colombiano de Combinatoria is a biannual gathering of students and researchers from Colombia, Latin America, California, and many other places. It is a two-week long summer school, featuring mini-courses by experts, collaborative problem workshops, research talks and posters, open problem sessions, a discussion panel, a hike, and visits to some of Colombia’s legendary salsa clubs. It is also much more than a summer school, and we hope to capture a bit of its spirit in these pages.

ECCO is designed to give every participant opportunities to interact closely with people at all stages of the mathematical career. We do our best to build a very professional and very warm atmosphere. We are collaborators and we are also a community.

The Encuentro started as a small gathering for combinatorics students in Colombia and the San Francisco Bay Area. They had taken classes together, as part of the SFSU—Colombia Combinatorics Initiative described in [1], and it had become clear that they wanted to meet in person, build closer ties, and find ways to collaborate.

Since then, ECCO has broadened and gained a strong reputation. Students from many different countries now attend, and combinatorics experts also ask to participate. We communicate our goals clearly. This is not a regular conference; it is a school and an encuentro: a coming together. We ask experienced researchers to do problem sets with the students, to present research questions that they would like help with, to offer advice, and to join the dance floor at some point. They have been wonderfully helpful and inspiring mentors, they have recruited students, and—perhaps most meaningfully to us—several have mentioned that their experiences at ECCO have influenced their work at their home institutions.

As one becomes more experienced organizing events, one becomes more conscious of their shortcomings. ECCO is certainly an imperfect event. After seventeen years, it is still an event under construction, and we hope it continues to be. But ECCO has been tremendously inspiring and energizing to us, and has taught us a lot about what it might mean to truly find community and belonging in a mathematical space. The goal of this article is to share a few of the lessons that we have learned from helping to build it.

Community Agreement, Part 1

When prospective participants are applying to ECCO, they encounter our Community Agreement. The first part reads:

A rewarding experience for all. The Encuentro Colombiano de Combinatoria aims to offer a rewarding, challenging, supportive, and fun experience to every participant. We will build that rich experience together by devoting our strongest available effort to all ECCO activities. You will be challenged and supported. Please be prepared to take an active, critical, patient, and generous role in your own learning and that of the other participants.

When we meet in person, we start ECCO by reminding everyone about this agreement. We ask people to get in pairs, read it out loud to each other, and spend a few minutes discussing it: What stands out to you about this agreement? What can it look like to put it in practice?

We’re not gonna lie. While some participants jump right in, many look confused, and if we are reading their body language correctly, a few seem to think: I can’t believe you are asking me to do this: what am I, a kindergartner? But we insist. Everyone participates.

To initiate a dialogue, we ask each group to underline a few words in the agreement that resonate with them, and share them with everyone. Some are excited that they will be challenged; some that they will be supported; some point out that the combination is crucial. We discuss how to be productively critical of each other’s work and what generosity might mean in a mathematical setting. We talk about how sometimes we are very good at being patient with others, but not so good at being patient with ourselves.

We wrote this agreement to communicate, from day one, the kind of space we are trying to build collectively. Johan, one of the participants of Días de Combinatoria,2 shared with us an experience that became an unforeseen consequence of the agreement. He told us that reading it on the webpage of Días was the push he needed to apply, and to attend; for the first time, he felt he was welcome at an event like this.

2The Días summer school is one of the offsprings of ECCO.
Community Agreement, Part 2

The second part of the community agreement reads:³

A welcoming experience for all. ECCO is committed to creating a professional and welcoming environment that benefits from the diversity of experiences of all its participants. We will not tolerate any form of discrimination or harassment. We aim to offer equal opportunity and treatment to every participant regardless of their mathematical experience, gender identity, nationality, race or ethnicity, religion, age, marital status, sexual orientation, disability, or any other factor.

Behavior or language that is welcome or acceptable to one person may be unwelcome or offensive to another. Consequently, we ask you to use extra care to ensure that your words and actions communicate respect for others. This is especially important for those in positions of authority or power, since individuals with less power have many reasons to fear expressing their objections regarding unwelcome behavior.

If a participant engages in discriminatory or harassing behavior, ECCO organizers may take any action they deem appropriate, from warning the offender to immediately expelling them from the event.

If you are being harassed, you feel uncomfortable with the way you are being treated, you notice that someone else is being harassed, or you have any other concerns, please contact Carolina Benedetti or Federico Ardila immediately. If you prefer not to speak in person, you may e-mail us (anonymously, if you wish) at the account ___________@__________._____, which only Federico and Carolina access.

Again, we make sure everyone actively engages with this text, reading it out loud in pairs and discussing it, awkward as they might find that. We are direct: social events are an essential part of ECCO, and we explicitly ask participants not to use them as excuses for romantic advances. We bring together more than 100 strangers from many different cultural and mathematical backgrounds for an intense shared experience; it is essential to have an agreement that clarifies expectations, and gives the organizers the power to react to potential incidents. We regularly revisit our protocol to respond to incident reports; we have found the Geek Feminism Wiki protocol⁴ to be a useful starting point.

³This part of the agreement was based on a code of conduct written by Ashe Dryden, a former programmer turned diversity advocate and consultant; see ashedryden.com.

⁴https://geekfeminism.fandom.com/wiki/Conference_anti-harassment/Responding_to_reports

We have co-created these community agreements with our students; our hope is to reach a collective understanding that is actually ours, that everyone is committed to as a whole. The critical feedback of participants has helped us strengthen our prevention and intervention protocols. We plan to add a short training on bystander intervention to the schedule of future ECCOs; this training helps participants recognize potentially harmful interactions and intervene to prevent them from escalating. We need to understand harassment and discrimination as community issues, and not individual issues, if we want to truly transform the harmful practices that our societies have normalized.

In the ‘Any additional comments?’ question on the exit survey of ECCO 2018, almost all participants who identified as women and/or LGBTQ+ praised the community agreement, and several said they would like to have one in all math events. Two participants wrote:

I thought the community agreement was an excellent idea. The openness allowed us … to make a giant community out of everyone, which made the conference very special. I felt I could finally be myself after years of feeling caged in.

We made an agreement to acknowledge each other’s differences and try our best to create a positive experience for everyone and it worked! We came together and did math without fear or judgment. It was so much fun! I think that the community agreement and the leadership of the organizers, TAs, and Colombians were driving forces behind making that possible. We all played a part by putting our hearts into creating the environment we were longing for. I left feeling fired up about bringing ECCO home with me. I would love it if all of my classes started off with a community agreement at the beginning of the semester.

A senior participant later told us: I was very surprised at first, and looked at [the agreement] as an oddity. Then I remembered what it was like being a grad student at conferences and all the weird guys I had to avoid. So I figured, yeah, why not? Another participant, who had been assaulted in a mathematical space before, told us that she simply does not attend conferences that do not have a plan to ensure her safety.

Mathematics has lost too many people—primarily women and people of color—to harassment and discrimination, and silence has never protected the victims. Perhaps by sharing with you how we are confronting these problems in our context, we may help you confront them in yours.

Breaking Power Structures

In any group of people there is a hidden power structure that influences who leads the discussion, who participates, whose voices are listened to, and whose ideas are seen as important.
We must return ourselves to a state of embodiment in order to deconstruct the way power has been traditionally orchestrated in the classroom.

—bell hooks [4]

The dance floor is one of the most democratic spaces of the tremendously unequal societies we live in. At ECCO it is a place of joy, and also a place of pedagogy, for professors and students alike.

Problem Workshops: Thinking Simply About Deep Things

Mathematically, ECCO aims for a low-floor, high-ceiling approach. We want the courses and activities to be designed so that everyone is able to engage with them at some level, and no one runs out of questions to explore. Every participant should find interesting things to learn. This is perhaps best exemplified in the way that problem workshops are structured.

Each mini-course meets four times, and each 60-minute class meeting is followed by a 90-minute problem workshop. People self-identify their level of expertise, and we split them into groups as heterogeneously as we can. A typical group will include a professor or postdoc, a graduate student in combinatorics, and two undergraduates with scarce combinatorial experience. Many participants speak very little English, and many speak very little Spanish, so everyone has something to learn and something to teach. We offer materials in Spanish or English and the unofficial language of mathematical discussions is Spanglish. People are welcome to use the language they wish. Interestingly, many choose to communicate in a foreign language for the first time since this is their opportunity to try it.

The first problems on each list ask people to carry out a small example, to ensure that everyone understands the key constructions or results in the class. The last few problems on the list can be very challenging and may take days or weeks to solve.

We ask each group to keep in mind our community agreement: how can they make the problem session rewarding, challenging, supportive, and fun for every participant? The result has always exceeded our expectations.

We realize this approach is unusual. Occasionally, it faces some resistance. A few of the more-experienced participants have asked: “Why don’t you let the beginners work on the easy problems together, and we can focus on the hardest problems?” But this is how these experts have been operating for most of their career. Why not learn something new? It is very rare for an undergraduate to collaborate with an expert of one field on questions about a different field and see: Experts struggle too! How do they productively struggle? These are very valuable lessons for the undergraduates. It is also very rare for an expert to collaborate with a relative newcomer to mathematics as equals. When they find a way

Our activities are most successful—for teachers and for students—when we are able to disrupt those power structures as much as possible, when every participant feels that their presence is important and their thoughts are valuable. We try to do this constantly, in several ways; a particularly successful one occurs outside of the classroom.

On Saturday nights, ECCO moves to the dance floor of the best salsa club we can find. The truth is that many of our international experts look a bit intimidated when they first walk in. For most of them, this is not the kind of place they visit often, if ever. Few people at the discoteca look like them; they might feel like they don’t really belong there.

Very soon, the students approach them and invite them to dance. They don’t accept “I don’t know how to dance” for an answer; they teach them, patiently, kindly, from the beginning, or just persuade them to dance as they will.

We won’t pretend our guests become expert dancers overnight; that really does not matter. But they always seem really grateful to the students who make sure they are comfortable, who guide them through a few steps, and who probably help them find a bit of freedom inside their body. Some of us have known these professors for years and we get to see a smile that they have never shown us before.

We like to ask our course instructors to keep in mind the feeling of discomfort they might have had entering the discoteca and the feeling of growth and joy they hopefully had walking out. Many ECCO students—who have never met so many accomplished mathematicians, who may have never attended a math conference before—are probably feeling a similar discomfort when they walk into the classroom. We want them to have that sense of belonging, growth, and empowerment when they leave. Since the professor was vulnerable in front of the student, the student can more comfortably say “I don’t understand, can you explain this to me?” when needed. Since the student showed generosity and patience on the dance floor, the professor naturally shows a similar generosity and patience in the classroom.

![Figure 1. Lecturers and students on the dance floor.](image)
to do it, they inevitably deepen their understanding of the subject.

The last few minutes of the problem workshop are spent sharing solutions. We ask the people who are usually very comfortable speaking up to make space for others. We invite the least-experienced or the least-vocal participants to present their work—they are the ones who can grow the most from doing so, and with the right atmosphere and maybe a bit of extra encouragement, they are usually happy to speak.

Andrés Vindas-Meléndez, who was a master’s student at the time, described his experience:

The exercises were mathematically meaningful, but what is noteworthy is that all group members played an active role in reaching a solution and understanding of the concepts. I observed that the more experienced mathematicians went directly to thinking about the abstraction of the problems, whereas the younger students emphasized a more concrete approach to exemplify the theory occurring in the problem. Of course both ways of thinking are valuable.

This reminds us of Gelfand’s request when encountering a new mathematical topic:

*Explain this to me in a simple example; the difficult example I will be able to do on my own.*

—Israel M. Gelfand

Satisfying this request can be very challenging for beginners and experts alike, and it can also be surprisingly rewarding and enlightening. The celebrated Ross Mathematics Program extols the value of thinking deeply about simple things: We agree wholeheartedly and propose the counterpart as well: there is a tremendous amount to be learned from thinking simply about deep things.

### Universal Design

Mathematicians from overrepresented groups in mathematics often ask “Why do you need these math conferences for minorities? Don’t we all do the same mathematics?” To try to answer, allow us to digress for a moment.

Let us share an embarrassing confession: The first time we rode a public bus in North America, and someone in a wheelchair got on, we could not believe our eyes. Are all 50 of us really going to wait all this time for one person to get on? Did the city really spend all this money putting all this equipment on every bus for such a small percentage of the population? We both grew up riding the ramshackle buses of Bogotá, jumping in and out of them while they were still in motion, collecting frequent minor bruises along the way. We should have known better.

The term “universal design,” coined by architect Ron Mace, describes the concept of designing all products and built environments to be aesthetic and usable to the greatest extent possible by everyone, regardless of their age, ability, or status in life. What may be unintuitive about universal design is that, what may seem like designing for a small minority, ends up being a better design for the majority. In fact, once it becomes widely used, it is no longer seen as serving special needs.

We often forget that sidewalk ramps were installed in every US city thanks to the Americans with Disabilities Act of 1990, after decades of activism. They were originally designed for people who use wheelchairs to go on and off sidewalks easily. Today, everyone uses them: a kid on a tricycle, a parent with a stroller, a traveler with a suitcase, a skate boader, or the two of us when we were dealing with injuries. Everyone benefits from them.

With this in mind, let us propose an analogy:

*Mathematics education cannot truly improve until it adequately addresses the very students who the system has most failed. … We need a central focus on students who are Latinx, Black, and Indigenous …, developing practices and measures that feel humane to those specific communities as a means to guide the field.*

—Rochelle Gutiérrez [3]

This does not come naturally, or without some opposition. Sexism, racism, classism, and centralism often lead to a small, homogeneous group of students being valued more than the rest, tacitly or explicitly. Academic elitism centers the voices and interests of the “top” students from the “top” schools, whatever “top” might mean. Furthermore, in Colombia, we always seem to put the needs of our foreign guests above our own.

At ECCO we are intentional and unapologetic about focusing on the needs and interests of the local students, the less-experienced students, and the students from regional universities that have less access to activities like this. It is our belief, and our experience, that when we find practices and structures that truly serve these students, we are doing much more than serve these students. We find practices and structures that benefit the wider mathematical community.

For example, we must confess that the organizing committee had not explicitly thought about the experience of the LGBTQ+ community at ECCO. But our struggles are connected, sometimes in ways that we do not foresee. Postdoc Aram Dermenjian wrote:

*The single biggest reason I loved this conference was the diversity and inclusiveness. In recent years I felt like the only gay person doing mathematics. I’ve started to feel more and more lonely in my math community. All my friends are amazing, and they* [3]

*We would like to thank May-Li Khoe for teaching us about universal design and its wide applicability.*
always try to make me feel welcome, but it’s just not the same.

Having a community agreement allowed everyone to be open about themselves allowing queer people to be out.

It gave me the confidence to do something I had never done before. I invited my math friends at ECCO to a gay club. Sure, more than half of us weren’t gay, and sure, the gay club wasn’t that amazing, but just being out, in a gay club, with mathematicians … was amazing. I felt I belonged.

Professor Viviane Pons wrote:

One question arose from the students: why do professors come and teach at ECCO? They saw clearly what was the gain for them, but the reason we would spend time and energy there was not clear to all of them. So let me tell you what I (and the scientific community as a whole) gain from that investment. I can help shape the academic world to something better and to something I like. Being part of ECCO is one step in this direction, because that is the kind of math community I want.

The Colombian Way

Worldwide, people have (accurate or inaccurate) ideas about the “French style,” “Hungarian style,” or “Japanese style” of doing mathematics. In a country that is relatively new to research in mathematics, perhaps we still have the opportunity to shape what “Colombian mathematics” might look and feel like.

Many would argue that mathematics does not distinguish a person’s culture or nationality. Unfortunately, this widespread belief has led many of us to feel forced to leave our humanity at the door and struggle to fit in with the dominant mathematical cultures and practices. But our cultures are too rich to be dismissed when we enter mathematics and dismissing them is a loss to mathematics itself.

Research has shown that creating learning environments that value and incorporate students’, families’, and community members’ cultural and linguistic strengths into instruction creates a nexus to mathematics cognition. … Culture and mathematics learning are intertwined in that they are both transformed through everyday lived experiences and are shaped by those experiences.

—Michael Orosco and Naheed Abdulrahim [5]

We are not interested in patriotism, but we are interested in culture and values. How might we use the cultural practices of Colombian communities to positively influence the cultural practices of Colombian mathematical communities—or at least the cultural practices of ECCO?

Colombians pride themselves in being good hosts, and making every effort to help our guests feel welcome and comfortable. We are proud of our food, our music, our rebusque, and our stories. At ECCO, these all end up playing a central role.

Colombia has a unique salsa culture, where each salsoteca has a wall full of hard-to-find vinyl records from the 60s and 70s—mostly salsa, cumbia, and West African music—and this is what the DJ plays all night long. People of all ages dance with their family, their friends, their coworkers, and with any stranger who asks to dance with them. No one does those fancy turns and slick moves they teach in American salsa lessons. Everyone sings along as they dance. At ECCO, we organize a visit to the salsoteca with the deepest music collection we can find. César gives everyone (foreigners and bogotanos alike) dance lessons, the venue gives us maracas and cowbells to play along (and they take them away if our rhythm is not on point), and everyone dances together. And once the dancing starts, it does not stop. Imagine a conference where participants work hard during the day, dance during the night, and get up fresh and early next morning all over again, for two weeks. Well, dancing drains some energy out of us, but it also fuels us to return to the conference the next day and give it our all. In the final survey, undergraduate student Eliana said:

I think that dancing is an important part of ECCO and it changes the whole dynamic in a very positive way. In Colombia, mathematics is danced.

Colombians are used to working with a shortage of resources, and we have a strong culture of rebusque: this means that in the face of difficulty, there is always an ingenious solution to be found within our means. The first time we organized a summer course in geometric combinatorics for undergraduates, we were advised by foreigners that Colombian students would not have the preparation necessary to understand these topics, that we should teach a basic course in abstract algebra instead. But, for better or for worse, a lack of preparation has never stopped a Colombian from trying to accomplish something. We don’t really believe in deficit mindsets. This culture of rebusque shapes our conviction that even without a lot of experience with mathematics, if you are hard-working and resourceful, you can take a class
What Does ECCO Wish to Be?

For us, it is crucial to be mindful of how we engage with and produce high quality mathematics. Valuing and promoting respect and difference has been essential to the development of ECCO. We seek to create an environment where each participant is empowered to take the space that belongs to them and share their voice, ideas, experiences, and world views, inside and outside the classroom. More than a conference, ECCO has become a space where learning mathematics is as important as recognizing each other as mathematicians and as individuals. If we are aware of what makes us different mathematically and personally, we can take advantage of these differences to complement each other.

The difficult, but also essential part, is to value respect and difference positively; not as minor, inevitable nuisances, but as the elements that enrich life and encourage creation and thought.—Estanislao Zuleta [6]

ECCO brings together a close-knit community of mathematicians who are spread out all over the world. This community has led to the founding of the Seminario Sabanero de Combinatoria in Bogotá, and it has strengthened our mathematical connections with ALTENUA in Colombia, and with CIMPA worldwide.

To achieve a more lasting effect, we also challenge ECCO participants to continue to mold their mathematical knowledge so that it can be used beyond the creation and understanding of science as a tool of positive societal and human impact. This has led to the construction of other mathematical communities that celebrate different ways of learning and help dehomogenize the concept of academia in Colombia. Examples include Días de Combinatoria and Círculos Matemáticos.

ECCO wishes to help build a strong and dynamic research network that collaborates regionally and internationally and produces very interesting mathematical work. We also wish to help create a culture of sharing mathematical knowledge with the public and using this knowledge to have a positive impact in all sectors of society. Finally, we wish to be very mindful of how we do this work, putting our humanity, our values, and the diversity of our cultures about current research directions from the world experts in the field, learn from it, and contribute to it.

The final activity of ECCO is a panel discussion where we talk about personal issues that most of us struggle with at every stage of our careers, but we rarely or never talk about. Discussion ranges from everyday topics such as “What does a typical day in your life look like?” to more transcendental ones such as “What tools have worked for you to deal with stress, anxiety, or a sense of not belonging in academia?” We choose a broad range of panelists, from professors to undergrads, from all parts of Colombia and the world.

To be truthful, we are not big fans of math panels in general. Why does this one feel different to us? Perhaps it’s the strong sense of community and trust that has been built by the end of ECCO that leads to a very honest conversation that does not shy away from strong emotions. Perhaps it’s simply that people, and their stories, are really important to us. Daniel wrote:

(The panel) is one of the most relevant things at ECCO. Breaking with the idea of math as a selfish and lonely task should be a priority. Talking like human beings, with our emotions and conflicts, is fundamental. Unfortunately this is rarely done in academic events. Congratulations to the organizers for recognizing the need to humanize math and mathematicians. This was a cathartic experience.

This encuentro is an intense mental, physical, and emotional experience. We seem to have a tacit agreement to store lots of energy prior to ECCO and budget a few days of recovery afterwards. We start as a bunch of strangers and end, with tears of joy, promising to keep in touch, planning our next encuentro.

Figure 3. ECCO 2018 participants from six countries.

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6SeSaCo is a weekly seminar that rotates locations among five universities in Bogotá.
7ALTENUA is a research group in algebra and number theory with a very strong presence in many regions of Colombia.
8CIMPA is a nonprofit organization founded in France that promotes research in mathematics in developing countries.
9Días is a summer school in basic combinatorics, geared towards undergraduates who haven’t had access to classes in this area. More than half of Días alumni, from 16 different universities in 9 different cities, went on to attend ECCO.
10Círculos is now a national program that helps high school students from public schools fall in love with mathematics in an inclusive, non-competitive setting.
at the center of everything that we do. These are the goals that guide our work. In our minds, they are inseparable.

ACKNOWLEDGMENTS. We would like to extend our warm gratitude to the ECCO family, who have become our mathematical home. They have allowed us to experience true community and belonging in a mathematical space.

References

Credits
Figure 1 is courtesy of Andrés R. Vindas Meléndez.
Figure 3 is courtesy of Natalia Saavedra.
Kenneth Kunen (1943–2020) made deep and wide-ranging contributions to mathematics: to topology, analysis, theoretical computer science, and algebra, but first and foremost, to set theory. Ken supervised over thirty doctoral students, several of whom have mathematical descendants of their own. The topologist Mary Ellen Rudin, in her note for the 2011 Topology and its Applications tribute to Ken, observed that “he didn’t make waves, but he made both mathematics and mathematicians.” Another close colleague, the set-theorist Arnie Miller, wrote that Ken was “not just a brilliant and productive mathematician but what we really admire most about him is his generosity with his mathematical ideas, conjectures, and problems.” And he perfectly captured Ken’s personality, by describing him as “always affable and always unflappable.” Ken did what he enjoyed: even after retiring, he still thought about interesting problems and discussed them with colleagues, and continued to publish results.

We thank the AMS for the opportunity to edit the following compilation of personal recollections and technical accounts of Ken’s work. We are also grateful to all who participated in this project for their enthusiastic responses and insightful contributions: Alan Dow, Michael Hrušák, Stephen Jackson, István Juhász, H. Jerome Keisler, Jan van Mill, Arnold W. Miller, Justin Moore, Dilip Raghavan, John Steel, Frank Tall, and Hugh Woodin. In particular, István was the first to propose this idea, and Steffen made it possible for us, as Ken’s students and colleagues, to volunteer for this job. As our long (but still not exhaustive) list of contributors suggests, thanks to Ken and the scope of his work, keeping our tribute within the prescribed limits was yet another nontrivial problem.

István Juhász

I was deeply shocked by the death of Ken Kunen, because I was unaware that my old friend had any health problems. In fact, a relatively short time before learning this, for me, completely unexpected news, we corresponded via email concerning a math problem that I had turned to him about, as I had done so often. Also, he was younger than

1The contribution of Mathias is included in the section by Steel and Woodin. István Juhász is a research professor emeritus of mathematics at the Alfréd Rényi Institute of Mathematics. His email address is juhasz.istvan@renyi.hu.
I, by exactly one month, and in my eyes he had “a professor’s head on an athlete’s body.” His devotion to physical fitness was well-known. I remember how, sometime in the 70s, Paul Erdős greeted Ken not by asking about mathematics, as he usually did, but with “How is your cycling?”

Our acquaintance and friendship goes back a long time. I first heard his name mentioned in 1969, at a memorable ASL summer meeting in Manchester, England. That was just after he proved that there is no non-trivial elementary embedding of $V$, the set-theoretic universe, into itself. Though technically relatively simple, this result was completely unexpected, so it was perhaps the biggest sensation there. (I remember how excitedly Jan Mycielski tried to convince us at the previous ASL meeting in Italy that assuming the existence of such an elementary embedding is intuitively obvious.)

Another sensation, at least for me, at this meeting was the appearance of Martin’s Axiom. After the meeting, Dana Scott, who had been Ken’s PhD advisor and, like me, was visiting Amsterdam that year, proudly told me that the axiom could just as well be called Kunen’s Axiom because Ken invented it independently of Tony Martin. This initiated my correspondence with Ken about topological applications of Martin’s Axiom and his advice helped me a lot in writing the chapter on Martin’s Axiom in my tract on cardinal functions in topology.

We first met in person at the Cambridge Set Theory Summer School in 1971, and that was when we worked on our first joint paper [JK73]. I’ll omit the mathematical details of this as well as our later joint work because in my article for the 2011 special issue of Topology and its Applications, I wrote in detail about Ken’s decisive and plentiful contributions to set-theoretic topology.

We really got close in the academic year 1974/75 that I spent at the UW–Madison. Although Mary Ellen Rudin arranged my invitation to Madison, Ken was also a gracious host. For instance, he took care of finding a place for us to stay in the same building where he then lived.

My professional benefits from this visit were enormous, mainly due to my connection with Ken. Our numerous exchanges about our common interests resulted, for example, in the triple paper [JKR76], one of the most cited works in our field. More importantly, these exchanges taught me an awful lot. In addition, I had the good fortune to attend a recursion theory course that Ken taught and that opened up a completely new chapter of logic for me.

This visit of mine to Madison and Ken was followed by uncountably many others. I even visited him in Austin in 1980, when he temporarily left Madison. My last visit to Ken in Madison took place in 2009, at his retirement meeting. After that we only kept in frequent touch by email.

In contrast, Ken visited Hungary just four times. The first time was in the summer of 1978 at a topology conference in Budapest. In his talk he presented his very deep result on the existence of weak $P$-points in $\omega^\ast$, [Kun80a]. Just recently, this result played an essential role in a result of ours that is to appear in a volume of Topology and its Applications commemorating Ken. His second visit to us was for the August 1998 summer topology conference, where he presented results on Bohr topologies. His many papers on Bohr topologies include one with Walter Rudin. So, given the earlier triple paper [JKR76] that he, Mary Ellen, and I published, the question Ken and Walter answered makes Ken the only person to have published results jointly with each of the Rudins. Following TOPOSYM2001 in Prague, where he was an invited speaker, Ken visited Budapest again, when 9/11 had just occurred. His last visit was in 2003, to a meeting that was organized on the occasion of my 60th birthday.

I hoped to convince Ken to take advantage of the Hungarian Academy of Sciences distinguished visitor program, but it turned out that before I could lure Ken back to Budapest he ran out of time.

H. Jerome Keisler

I first met Ken in 1968, when he was in his last year as a graduate student at Stanford, and he gave a lecture at UCLA, where I was visiting for the logic year. I had just been promoted to Professor at Wisconsin. The logic group at Wisconsin then consisted of Stephen Kleene and Barkley Rosser, who were late in their illustrious careers, and myself. In 1968, Ken was universally recognized as a budding star, and was probably the most sought after new PhD in the world of logic. It was a major coup for Wisconsin to add Ken to our department. Although Ken worked mainly in set theory and I worked mainly in model theory, our research was closely related. Throughout our careers we had a common interest in properties of ultrafilters. For the next 34 years, I had the incredible luxury of having Ken in the office next door to mine. He was always available to exchange ideas and answer any questions I had in set theory. There was no one better to ask. Ken would get to the heart of a question and explain things in the clearest possible way. He was a superb teacher whose courses were highly popular. His graduate textbook on set theory is widely regarded as the best. During our careers, I worked closely with several of Ken’s PhD students, and Ken worked closely with several of mine, where model theory and set theory overlapped.

H. Jerome Keisler is a professor emeritus of mathematics at the University of Wisconsin–Madison. His email address is keisler@math.wisc.edu.
Ken had an immediate and profound influence on the logic program at Wisconsin. His arrival made Wisconsin one of the few programs that covered all of computability theory, model theory, and set theory. That gave graduate students the option to learn all of them and then choose a research area. Ken had a great influence on the research of others. Ken's work as a graduate student and his seminar in set theory in his first year at Wisconsin inspired Rosser to write his book “Simplified Independence Proofs.” When Ken arrived, Mary Ellen Rudin joined the Wisconsin logic seminar, and from then on much of her research was on the borderline between topology and set theory. Early in his career, in 1971–72, after spending the academic year at Berkeley, Ken returned with three Berkeley graduate students, William Fleissner, Judith Roitman, and Aki Kanamori. Each of them added much to the Wisconsin logic group as visiting students and then went on to have distinguished careers. For the next forty years, Ken attracted many of the top people in the field as visitors to Wisconsin, and some visited several times.

During the 1970s, Ken, Jon Barwise, and I worked together on the logic program at Wisconsin and on many other things, such as the Handbook of Mathematical Logic, and the Kleene Symposium in honor of Kleene's 70th birthday. I was extremely fortunate to have had the opportunity to work with Ken for so many years.

Steffen Lempp

When I arrived in Madison in 1988, twenty years after Ken, I felt like a very junior colleague among senior giants (including my current coauthors Jerry Keisler and Arnie Miller as well as Terry Millar, who unfortunately passed away two years ago). Each of them had his own unique style. The three things I remember the most about Ken are his boundless energy, his uncanny ability to attract very good students with a fairly hands-off approach, and his truly amazing ability to generate interesting qualifying exam problems.

During my first 25 years at UW, we almost always had our semesterly logic picnic at Devil’s Lake State Park, and we always scaled East Bluff from the south shore. It was hard to keep up with Ken running up the steep path; in fact, most of the time, Ken didn’t even bother with the path but just climbed up the boulder field in between, with the rest of us panting behind him (or taking the path instead).

He was immensely popular with students, many of whom went on to highly distinguished careers of their own, and almost all of whom finished their degree and found their niche; in fact, Ken's last student graduated the summer Ken passed away! He was very conscientious about meeting with his students, but especially after his retirement, they had to come in early, since Ken arrived in the department at 7 AM and was gone by 10 AM!

Finally, each semester when we had to make up the qualifying exam, and even long after Ken retired, he contributed a lot of problems. My problem with this was that I had a hard time even solving the ones he labeled as for the “elementary” section! Fortunately, Ken gave the solutions for later publication along with the problems, so I could check if the problems were doable for our students without spending hours trying to solve them!

Arnold W. Miller

I want to say something about Ken’s teaching. Some time ago one of our graduate students made a very astute observation about Ken’s style of lecture. He said “Professor Kunen never gives the proof of anything; he just makes remarks.” I attended a great many of Ken’s lectures and can explain what the student meant. Here is how a typical Kunen lecture went.

First he states the Theorem and its relevant definitions. Then he explains what the Theorem says, maybe illustrates a few consequences, special cases, Corollaries, and problems it doesn’t settle. Then he begins remarking about how the proof might go. “You might think” he says “that we could prove it this way—but that wouldn’t work.” But then he says “or you might prove it this way—but that doesn’t work. But maybe something similar does.” After discussing several false starts and why they don’t work, he discusses strategies and how to handle various problems and details. He says “let’s consider the following strategy. Prove Lemma 1, 2, and 3 would give us the Theorem. Then he says “Now to prove 1 you might think we could do this—but that wouldn’t work.” And then he says “Well a possible proof of Lemma 1 would be a, b, then c.” After a while he starts to discuss Lemma 2. “Now” he says “Lemma 2 is false, and here’s why...but maybe we could prove Lemma 2”... So by the end of the hour the whole class sees the particular path the proof must take and how to take care of the problems and details along the way. And then finally Ken says “OK ah ah that’s enough discussion, let’s give the proof of the Theorem.” So he writes on the board:

“Proof—see above remarks—.”

Arnold W. Miller is a professor emeritus of mathematics at the University of Wisconsin–Madison. His email address is miller@math.wisc.edu.

Steffen Lempp is a professor of mathematics at the University of Wisconsin–Madison. His email address is lempp@math.wisc.edu.
Justin Tatch Moore

Unfortunately, I arrived too late to know Ken well. I remember meeting him on only two occasions: once as the external examiner of my 2000 PhD thesis at the University of Toronto and later when I visited Madison in 2006. Instead, I was part of a generation of set-theorists educated through his book [Kun80b].

In 1995 while an undergraduate at Miami University, I worked on a summer research project with Dennis Burke funded through the college. There was a budget for “materials” and Dennis picked three books for me: Open Problems in Topology, The Handbook of Set-theoretic Topology, and Ken’s Set Theory. All are still on my bookshelf; the last is by far the most worn. I still recommend “Kunen” to any student who expresses an interest in studying set theory. From transfinite recursion to combinatorial set theory to constructibility to iterated forcing, it gives a complete foundation for further reading in the subject. The treatment is timeless. The exercises are legendary. The tone is both precise and conversational.

Above all, it is a study in discipline. At the time the book was written, set theory was rapidly developing in many different directions. It must have been extremely tempting to include more, but Ken somehow knew the correct boundaries and kept within them. As a result, the book has stood the test of time extremely well. It is still the place I send students to learn about forcing or how to construct a Souslin tree. Even Chapter V on defining definability—which seemed somewhat obscure back in the 1990s—was prescient in light of the role that HOD plays in modern inner model theory.

Of course Ken’s research itself was transformative and inspirational: the nonexistence of a nontrivial elementary $j : V_{κ+2} → V_{κ+2}$, the inner model for a measurable cardinal, his work on random reals and RVMs, saturated ideals on small cardinals, and his S and L spaces. Most of these have had a significant influence on my own work. Still, his book stands out as an extremely important contribution to set theory and surely a lasting part of his legacy.

Frank Tall

I’ve written my reminiscences of Ken in my contribution to the forthcoming memorial issue of Topology and its Applications. I’ll just repeat one item here. Ken had a genius for providing clear answers to murky questions. You could go to his office with a vaguely formulated question and he would divine its essence and answer it. Furthermore, he knew “everything” known about set theory.

I only really learned set theory when I started teaching from Ken’s book [Kun80b]. Many of my students who seriously worked through the exercises became accomplished set theorists. His innovation of teaching Martin’s Axiom before forcing was revolutionary at the time. For a discussion of Ken’s expository talents, see Kanamori’s article in the upcoming special issue of Annals of Pure and Applied Logic.

My most-cited article with Ken, and Ken’s third most cited joint paper is Between Martin’s axiom and Souslin’s hypothesis [KT79]. I had noticed that most consequences of Martin’s Axiom fell into two natural categories: combinatorial propositions about the real line or sets of natural numbers and propositions implying Souslin’s Hypothesis. I asked Ken if these two could be distinguished. He had already done so! Ken had a habit of jotting down 4-page handwritten notes containing proofs of interesting facts that were too small to be an actual paper, but that he could pull out of his filing cabinet when the occasion called for them. One of these was that property $K$ forcing preserved Souslin trees. It immediately followed that Martin’s Axiom restricted to property $K$ partial orders plus the negation of the Continuum Hypothesis did not imply Souslin’s Hypothesis, although it did imply all the usual combinatorial consequences. This paper spawned many others; its terminology has become so commonplace that its origin is often not cited. A survey on this topic by Bagaria, entitled The relative strength of fragments of Martin’s Axiom, will appear in the issue of Annals of Pure and Applied Logic mentioned above.

My second-most cited article with Ken, [KST86], provided an unexpected strong connection between saturated ideals and a problem Russian topologists were interested in—the existence of Baire irresolvable spaces.

Ken had high standards concerning what was publishable. Several topologists, e.g. Wis Comfort, had asked me about whether our Baire irresolvable results applied at singular cardinals, so I thought we should publish the fact that they did as a short note. Ken didn’t want to, because it was essentially the same proof as in the regular case, so we posted it on Topology Atlas with the title On the consistency of the non-existence of Baire irresolvable spaces.

I had two other papers with Ken: [BGKT78] and [KT00]. The first was the usual story—the other authors couldn’t solve a central problem in their work, so asked Ken. The second was a true collaboration, answering the question of whether an uncountable elementary submodel containing the real line as a member actually includes all of it. Not surprisingly, the answer depends on your set theory.
Ken and $\beta\mathbb{N}$

**Alan Dow, Michael Hrušák, and Jan van Mill**

Ken’s enormous contribution to the study of the *space of ultrafilters* $\beta\mathbb{N}$, and dually the Boolean algebras $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\text{fin}$, cannot be overstated. Dealing with the difficult task of deciding which of his many fundamental results we should highlight, we have decided to focus almost exclusively on his $\mathsf{HJR}$ theorems leaving aside his many important consistency proofs such as (1) the non-existence of selective ultrafilters, (2) the early results on cardinal invariants of the continuum ($t < c$, $u < c$ and $a < c$), (3) his joint work with Bell that every point in $\beta\mathbb{N}$ has $\pi$-character $\aleph_1 < c$ and there is a point with $\pi$-character at least $cf(\omega)$, (4) the non-existence of gaps other than Hausdorff and Rothberger together with $\mathsf{MA} + \neg \mathsf{CH}$ while inventing the technique of “freezing” a gap, and (5) along with some more “topological sounding” statements, e.g., the fact that $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ cannot be covered by nowhere dense closed $\mathcal{P}$-sets under $\mathsf{CH}$, the consistency of the statement that $\mathbb{N}^*$ does not map onto all compact spaces of weight $\omega$, and the consistency of $\mathbb{N}^* \setminus \{p\}$ is $C^*$-embedded in $\mathbb{N}^*$ for every $p \in \mathbb{N}^*$, to mention but a few.

In the presence of $\mathsf{CH}$, information about $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\text{fin}$ can often be found by a transfinite procedure dealing essentially with countable structures. This is no longer true in the absence of $\mathsf{CH}$ if one’s aim is to get results in $\mathsf{ZFC}$, since regardless of additional set theoretic assumptions one can, for example, run into a Hausdorff gap. It was Kunen [Kun72] who found in 1972 a remedy for this seemingly insurmountable barrier by creating beforehand enough “space” to make sure that a recursive process does not prematurely terminate. This is what we now call a “guided transfinite recursion.” We shall briefly outline this brilliant innovative idea that turned out to be extremely powerful in solving open problems dealing with ultrafilters in $\mathcal{P}(\mathbb{N})$. One first identifies an “independent” matrix of subsets of $\mathbb{N}$ with $\omega$-many rows. The cofinite filter is independent modulo the matrix in the sense that every member of it intersects an arbitrary finite collection of sets, each chosen from a different row of the matrix, in an infinite set. Suppose that the tranfinite process allows for steps in which only a finite number of rows of the matrix is used. Then at every intermediate step of the process, only fewer
than \(\mathcal{P}\)-many rows of the matrix are used and there are still \(\mathcal{P}\)-many left for future use, hence the transfinite process does not terminate until it reaches its successful end.

Kunen applied it in two papers, solving major open problems. In [Kun72], motivated by the work of Frolik and M. E. Rudin, he used it to prove the existence of incomparable ultrafilters in the Rudin-Keisler order. Kunen’s result which was generalized by Shelah and Rudin, had a very high impact. In the same paper [Kun72], motivated by problems from Model Theory, Kunen used the new method to prove the existence of so-called good ultrafilters in ZFC, which improved a well-known result of Keisler who proved it earlier using CH. The Rudin-Keisler incomparability of ultrafilters also had definitive topological applications. In [Kun90], Kunen used it to show that no infinite compact \(F\)-space (and more generally, no product of compact \(F\)-spaces at least one of which is infinite) is (topologically) homogeneous, which greatly improved an earlier result by Frolik for extremally disconnected compacta.

Kunen’s second main, and still more striking, application of his method [Kun80a] that there are weak \(P\)-points in \(N^*\) gave the ultimate proof of non-homogeneity of \(N^*\). W. Rudin showed that \(P\)-points exist in \(N^*\) under CH, thereby demonstrating that \(N^*\) is not homogeneous by producing two points with obviously distinct topological behavior (recall that a \(P\)-point is a point with the property that the intersection of any countable family of its neighborhoods is again a neighborhood). That some set theoretical hypothesis is essential in Rudin’s result was shown by Shelah, while a ZFC proof of non-homogeneity of \(N^*\) is due to Frolik. However, Frolik’s proof is based on a cardinality argument, and does not yield two points with obvious different topological behavior. A point is a weak \(P\)-point if it is not in the closure of any countable subset contained in its complement. Every \(P\)-point is a weak \(P\)-point (but not vice-versa). Kunen’s result about weak \(P\)-points in \(N^*\) not only gives an “honest” proof of the nonhomogeneity of \(N^*\), but also shows that the Shelah \(P\)-point independence theorem is in a certain sense sharp. The matrix of sets needed for the proof is much more complicated than the one from [Kun72]. Indeed the matrix discussed above had only two elements in each row and was shown to exist by Hausdorff; however for the results on weak \(P\)-points each row had \(\mathcal{P}\)-many elements and proving such a structure exists in ZFC is in itself a major result. The ideas in [Kun80a] had again enormous impact. Several other major open problems on \(N^*\) were solved by applying Kunen’s method: Simon’s result that there is a separable closed subspace of \(N^*\) that is not a retract of \(N^*\), van Mill’s results on weak \(P\)-points in general Cech-Stone remainders and construction of many other special points of \(N^*\), and Dow’s solution to van Douwen’s problem that there is a nontrivial copy of \(N^*\) in \(N^*\) (the latter result was recently generalized by Dow and van Mill that there even exists a nowhere dense weak \(P\)-set copy of \(N^*\) in \(N^*\)).

**Kunen’s Work on Determinacy and Descriptive Set Theory**

**Stephen Jackson and Donald A. Martin**

Kunen’s work in the early 70s in descriptive set theory, and particularly in the development of determinacy theory, deserves to be considered a fundamental and groundbreaking achievement.

**Background for Kunen’s work in determinacy.** The axiom of determinacy, AD, which states that every two-player game on \(\omega\) (or equivalently on \(2 = \{0, 1\}\)) has a winning strategy, was introduced by Mycielski and Steinhaus in 1962. They proposed using this axiom to develop a theory of the sets of reals, as this axiom avoids pathological sets constructed from AC. Although AD contradicts AC, it was understood that this axiom was meant to apply to a more restricted universe, such as \(L(\mathbb{R})\), in which the sets of reals have a more explicitly definable structure. However, it wasn’t until much later through the work of Martin, Steel, and Woodin in the late 80s that it was shown that large cardinals imply AD\(^*(\mathbb{R})\). The early researchers realized,

**Stephen Jackson is a Regents professor of mathematics at the University of North Texas. His email address is stephen.jackson@unt.edu.**

**Donald A. Martin is a professor emeritus of mathematics at the University of California, Los Angeles. His email address is dam@math.ucla.edu.**
though, that AD, and even weaker forms such as projective determinacy, PD, might be enough to develop a theory of the projective sets and beyond similar to the theory of the Borel and analytic sets developed by the classical descriptive set theorists of the early 20th century (see Kechris’s textbook).

In the late 60s, just prior to Kunen’s main work in this area, several important developments occurred. Martin and Moschovakis (1968) independently proved the first periodicity theorem for propagating the prewellordering property under $\forall \omega^\omega$ assuming a determinacy hypothesis. This was followed by Moschovakis’s second periodicity theorem for propagating scales and the scale property under the same determinacy hypothesis (see Moschovakis’s textbook for more history). The periodicity theorems give a structural representation for the projective sets, but the theory that emerges is largely in terms of the so-called projective ordinals. By definition,

$$\delta_n^1 = \sup\{\langle \leq \rangle : \leq \in \Delta^\omega_\delta \text{ is a prewellordering of } \omega^\omega\},$$

that is, it is the supremum of the lengths of the $\Delta^\omega_\delta$ prewellorderings of $\omega^\omega$. As we will discuss shortly, the initiation of a program for computing the $\delta_n^1$ would be an important contribution of Kunen’s.

A second development in the late 60s which helps to set the stage for Kunen’s work was Martin’s proof of $\Pi^1_1$ determinacy from a measurable cardinal. Aside from forging a connection between determinacy hypotheses and large cardinal axioms, implicit in this proof was the notion of a homogeneous tree. This notion also implicitly appears in the 1969 paper of Martin and Solovay [MS69]. What we now call the Martin-Solovay construction shows how to construct a scale/Suslin representation for $\omega^\omega \setminus A$ from a scale on $A$ given that the tree $T$ giving the Suslin representation for $A$ is weakly homogeneous. The modern notion of a homogeneous tree and weakly homogeneous tree was formulated independently by Martin and Kechris in 1981, but arose implicitly earlier as just mentioned.

Kunen’s work. With this as background, in 1971 Kunen began producing a series of remarkable results which would shape the future of descriptive set theory and determinacy theory. During this time, Kunen (and others) communicated their results through handwritten notes which were circulated to the other researchers in the area. While Kunen’s work in determinacy became well-known, he did not publish this work.

In the same time period, Kunen and Martin independently proved what we now call the Kunen-Martin theorem. This theorem states that a $\kappa$-Suslin well-founded relation $\prec$ has length $|\prec| < \kappa^+$. This result has immediate ramifications for computing the $\delta_n^1$. It shows that $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^\omega$ and that $\delta_{2n+1}^1 = \lambda_{2n+1}$, where $\lambda_{2n+1}$ is the cardinal where $\Sigma^1_{2n+1}$ sets and $\Pi^1_{2n}$ sets admit scales. Kunen then developed a plan for computing the projective ordinals. Roughly speaking, if we assume $\delta_{2n-1}^1$ is known, and this is where the $\Sigma^1_{2n}$ sets admit weakly homogeneous scales, and if we can transfer the scales on the $\Sigma^1_{2n}$ sets to the $\Pi^1_{2n}$ sets (something akin to the Martin-Solovay construction) then we would compute $\lambda_{2n+1}$ and thus $\delta_{2n+1}^1$, and thus complete the induction.

A central part of Kunen’s plan for carrying out the above program is to establish partition properties of the projective ordinals. If one assumes the strong partition property at $\delta_{2n-1}^1$, and the tree $T_{2n-1}$ of a scale on a complete $\Pi^1_{2n-1}$ set is homogeneous then the Martin-Solovay construction will propagate the tree $T_{2n-1}$ to a homogeneous Suslin representation for a complete $\Pi^1_{2n-1}$ set which will be on the ordinal $\sup_{\mu} j_\mu(\delta_{2n-1}^1)$ where $\mu$ ranges over the measures in the homogeneous tree for $T_{2n-1}$, and $j_\mu$ is the ultrapower embedding. Thus we would compute $\lambda_{2n+1} = \sup_{\mu} j_\mu(\delta_{2n-1}^1)$ and thereby compute $\delta_{2n+1}^1$.

The above program of Kunen’s depends on two key points. First, we must be able to prove the partition properties of the $\delta_{2n+1}^1$, and second we must be able to analyze the homogeneity measures $\mu$ well enough to be able to compute the ultrapowers $j_\mu(\delta_{2n-1}^1)$. Concerning the partition properties, Martin showed that $\delta_1^1 = \omega_1$ has the strong partition property, and established a general framework for proving partition results from determinacy. In a remarkable result which contained several important ideas, Kunen showed that $\delta_3^1$ had the weak partition property $\delta_3^1 \rightarrow (\delta_1^1)^\lambda$ for all $\lambda < \delta_3^1$. Kunen’s proof went by showing that there is a $\Delta_4^1$ coding of the subsets of $\omega^\omega$, and then quoting the Martin framework (see Jackson’s chapter in the Handbook of Set Theory for an exact statement). In particular, Kunen came up with a short but very clever argument that to analyze the subsets of $\lambda$, it suffices to analyze the measures on $\lambda$. Using the theory of indiscernibles, Kunen analyzed the measures on the $\omega_n$, which then allowed him to show the weak partition relation on $\delta_3^1$ (see [Sol78] for a presentation of Kunen’s argument). A similar elegant result proved by Kunen is that under AD the set of measures on any ordinal $\alpha < \Theta$ is wellordered.

An important technical ingredient in the analysis of measures on the $\omega_n$ under AD is the notion of the Kunen tree. This concept, introduced by Kunen, plays a central role in almost all arguments involving the $\omega_n$ under AD. The Kunen tree $T$ is a tree $T \subseteq (\omega \times \omega_1)^{\omega\omega}$ on $\omega \times \omega_1$ such that for all $f : \omega_1 \rightarrow \omega_1$ there is an $x \in \omega^\omega$ with the section $T_x = \{s \in \omega^{\omega\omega} : (x \upharpoonright |s|, s) \in T\}$ of $T$ wellfounded, such that for all $\alpha < \omega$ in a c.u.b. set we have

$$f(\alpha) < |T_x \upharpoonright \alpha|$$

(the rank of the $T_x$ restricted to ordinals less that $\alpha$). An
The immediate consequence of the Kunen tree, for example, is a bound for the ultrapower $j^{\omega_1}_{\omega_1}(\omega_1) \leq \omega_{n+1}$ where $W_n$ denotes the $n$-fold product of the normal measure on $\omega_1$. From the perspective of Kunen’s program, this can be viewed as computing an upper bound for $\delta_3$ as we have $\delta_3^1 = (\lambda_3)^+ \leq \sup_n j^{\omega_1}_{\omega_1}(\omega_1) \leq (\omega_\omega)^+ = \omega_{\omega+1}$.

The program of computing the projective ordinals via Kunen’s program stalled after the results mentioned above. It turns out that Kunen’s overall plan was still sound, but several important ingredients were missing for extending the program to higher levels. One of the missing ingredients was a generalization of the Kunen tree to higher cofinalities above $\omega$. For functions on the higher $\delta_{2n+1}^1$, one needs the Martin tree, an appropriate generalization of the Kunen tree introduced by Martin in the early 80s. A second ingredient necessary for extending the program is the notion of a description. These are hereditarily finite objects which “describe” how to generate functions via certain iterated ultrapowers (in Jackson’s chapter in the Handbook of Set Theory the earlier theory is redone from this perspective). Although the description analysis does not rely on the theory of indiscernibles, it is important to note that Kunen’s idea of analyzing the sets of ordinals by analyzing the measures is still a key component in the analysis. Using the Martin tree and the description analysis, Jackson computed $\delta_3^1$ and proved the strong partition relation on $\delta_3^1$ in the early to mid 80s [Jac99], and then extended this analysis to complete Kunen’s program of computing all of the $\delta_k^1$ and establishing the strong partition relation at all of the odd projective ordinals. Extending the results of the projective analysis throughout the cardinal structure of a determinacy model below $\Theta$ remains an elusive goal.

This was his generalization of a 1971 theorem of Solovay [Sol71] which, by sacrificing translation-invariance, extended Lebesgue measure to a countably additive measure defined on all subsets of $[0,1]$. For this, Solovay had to assume the existence of a measurable cardinal. This is an especially bold large cardinal axiom: measurable cardinals are so large that there were serious suspicions back then that their existence is inconsistent. Their consistency has stood the test of time long enough for these suspicions to have quieted down, but there is still good reason to avoid relying on their consistency when possible.

Kunen needed an even larger cardinal, known as “strongly compact,” to extend Solovay’s theorem to a whole class of countably additive measures. He did this by way of the Product Measure Extension Axiom (PMEA), which can be succinctly described as extending, for all cardinals $\kappa$, the Haar measure on the product of $\kappa$-many copies of the 2-element group to a measure on all subsets of these groups.

This axiom turned out to be the key to the first of two breakthroughs that settled the Normal Moore Space Problem, a problem of great interest to set-theoretic topologists since 1937, when F. Burton Jones published a proof, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that every normal Moore space with a countable dense subset is metrizable. In her booklet, Rudin devotes a whole chapter “strictly to the normal Moore space conjecture.” The PMEA was exactly what I needed to get a complete generalization of Jones’s topological theorem under a different axiom: every normal Moore space is metrizable.

Kunen’s proof was finally published in 1984 by Bill Fleissner, in his chapter in the Handbook of Set-theoretic Topology, where he also presented the general result in the opposite direction. This was his construction of a non-metrizable normal Moore space using an axiom that is so weak, that to negate it entails the consistency of there being a proper class of measurable cardinals!

Kunen made several significant contributions to another topic to which Mary Ellen devoted a chapter (and hefty parts of two others) in her booklet: the theme of S- and L-spaces. An S-space is a regular hereditarily separable2 space that is not hereditarily Lindelöf; to define L-space, switch “separable” and “Lindelöf.”

Early on, the conventional wisdom was that the existence of an S-space is equivalent to that of an L-space. This was refuted by J. T. Moore’s sensational construction of an L-space from ZFC combined with an earlier result by S. Todorcevic showing the consistency of there being no S-spaces.

\[ \text{Peter Nyikos is a professor of mathematics at the University of South Carolina. His email address is nyikos@math.sc.edu.} \]

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[1] The word “hereditarily” refers to all subspaces having the stated property. A separable space is one with a countable dense subset, and a Lindelöf space is one for which every open cover has a countable subcover.
One of Kunen’s best papers on this topic [Kun77] contributed to the conventional wisdom long before Moore’s breakthrough. It has to do with spaces whose finite powers are all S-spaces or all L-spaces, known as strong S-spaces, resp. strong L-spaces. Zenor had shown that if there is a strong S-space, there is a strong L-space, and vice versa. Under CH, there are many constructions of strong S-spaces (including some by Kunen) and, by Zenor’s theorem, strong L-spaces. Kunen’s big breakthrough in [Kun77] was to show that MA + ¬CH implies that there are no strong S-spaces, and the proof itself dualized easily to imply that there are no strong L-spaces, without invoking Zenor’s theorem.

The “Kunen line” [JKR76] is probably the best known of Kunen’s contributions in this area. It is a refinement of the topology of the real line under CH to a locally countable and locally compact, perfectly normal S-space. Kunen also constructed various examples of compact L-spaces under various axioms. Some, like his Corson compact L-space using CH, are relevant to functional analysis.

A very different construction was in a joint article with Eric van Douwen. They showed that the following are equivalent: (1) a combinatorial principle that they designated ↓; (2) $\mathcal{P}(\omega)$ with the Vietoris topology has an L-subspace; (3) $\mathcal{P}(\omega)$ with the Vietoris topology has an S-subspace. The equivalence of (2) and (3) is obtained by taking complements of the respective subsets of $\omega$, as are the proofs that each statement is equivalent to ↓, again in line with the illusion of S- and L-duality. They also showed that these statements follow from CH, via a stronger principle designated ↑ [mnemonic: What goes up must come down.] and are negated by MA + ¬CH.

Kunen and Forcing

Dilip Raghavan

Kunen made seminal contributions to all of the principal threads in the study of forcing, forcing axioms, and their applications. Here are some of the greatest hits.

Iterated forcing was invented by Solovay and Tennenbaum to prove the consistency of Suslin’s Hypothesis. Several people realized that the details of Solovay and Tennenbaum’s argument could be carried out with all c.c.c. partial orders, leading to the formulation of Martin’s Axiom (MA), the first example of a forcing axiom. Kunen, who had independently formulated MA, began exploring its consequences in his thesis of 1968. In Section 14 of his thesis, Kunen showed that MA implies that every set of reals of cardinality less than $2^{\aleph_0}$ is of strong measure zero. He went on to observe that if $\kappa \leq 2^{\aleph_0}$ is real-valued measurable, then because of $\Theta_1$ indescribability, there is a set of reals of cardinality less than $\kappa$ which is not Lebesgue measurable, thereby proving that MA is incompatible with the assertion that the continuum is real-valued measurable. Section 14 also contains the first construction of a generalized Luzin set under MA. In Section 12 of his thesis, Kunen showed that MA implies that every subset of $\mathbb{R} \times \mathbb{R}$ belongs to the $\sigma$-algebra generated by arbitrary rectangles, while showing that this is not the case in the Cohen model. Section 13 contains a proof that the cardinal characteristic $\mathfrak{b}$ equals $2^{\aleph_0}$ under MA.

Kunen was the first to investigate the gap structure of $\mathcal{P}(\omega)/\text{fin}$ under forcing axioms. Introducing the pivotal technique of freezing a gap in his handwritten notes from August 1975, Kunen was able to show that $\mathfrak{M}_\alpha$, alone is not sufficient to determine the gap structure of $\mathcal{P}(\omega)/\text{fin}$. Although they were never published, the mimeographed notes of 1975 were widely circulated and attracted a great deal of attention, featuring in Woodin’s solution to Kaplan’s conjecture. The idea of freezing gaps went on to play a crucial role in the eventual complete characterization of the gap structure of $\mathcal{P}(\omega)/\text{fin}$ under PFA, which in turn, was vital to the proof that $2^{\aleph_0} = \aleph_2$ under PFA. The reader may consult Todorcević’s 1989 monograph for more details of this. Kunen went on to publish most of the key ideas from his 1975 notes in [Kun88], where using the technique of freezing gaps once again, he was able to show that the cardinal characteristic $\mathfrak{m}$, which marks the place where MA first fails, could consistently be equal to $\aleph_1$. In [Kun88], Kunen asked if it is possible for $\mathfrak{m}$ to be a singular cardinal of cofinality other than $\omega_1$. Kunen’s question is yet to be fully resolved.

A major application of MA to general topology is presented in [Kun77] where Kunen proved that strong S and L spaces do not exist under MA. More recently, Kunen and his collaborators have investigated the effect of forcing axioms on continua and differentiable functions in $\mathbb{R}^n$. In 2011, Hart and Kunen proved that PFA implies that every uncountable subset of $\mathbb{R}^n$ meets some $C^1$ arc in an uncountable set and that this is not provable from $\mathfrak{M}_\kappa$. In [Kun12], Kunen showed that PFA implies that if $E$ is any subset of $\mathbb{R}$ of size at most $\kappa$, then $E^2$ can be covered by countably many graphs of $C^1$ functions and their inverses.

Kunen pioneered techniques for obtaining consistency results from large cardinals through iterated forcing. What is now known as the Kunen-Paris forcing was introduced by Kunen and Paris [KP70] in 1970 to show that it is consistent for a measurable cardinal $\kappa$ to have $2^{2\kappa}$ normal measures on it. At the end of their paper Kunen and Paris asked...
whether it is possible for the number of normal measures on a measurable cardinal \( \kappa \) to be strictly between 1 and \( 2^{2^{\kappa}} \), a question which was fully resolved only in 2009 by Friedman and Magidor. Kunen’s 1972 result, published in 1978 in [Kun78], showing that it is consistent relative to a huge cardinal to have an \( \aleph_2 \)-saturated ideal on \( \omega_1 \), has come to be regarded as a landmark in set theory. Kunen’s method has been adapted and improved numerous times to show the consistency of various saturation type properties from large cardinals. While it is now known how an \( \aleph_2 \)-saturated ideal on \( \omega_1 \) can be obtained from more optimal large cardinal hypotheses, Kunen’s original technique and its variations remain an indispensable item in every set theorist’s toolkit. We refer the reader to Section 7 of Kunen’s [Kun78].

Kunen systematically studied the combinatorial properties of the Cohen and Random models. In his thesis, Kunen pointed out that there are no towers of length \( \omega_2 \) in \( \mathcal{P}(\omega)/\text{fin} \) in the Cohen model. The fact that Ramsey ultrafilters could not be constructed in \( \text{ZFC} \) was established by Kunen’s observation in [Kun72] that they do not exist in the Random model. In the expository article [Kun84], Kunen introduced the crucial notion of an invariant c.c.c. ideal. This notion allowed him to abstract away the shared features of the Cohen and Random models, and to point out the reasons for the differences between them. Towards the end of his article Kunen asked for a classification of all invariant c.c.c. ideals. Kunen’s notion of an invariant c.c.c. ideal and his call to classify them turned out to be very influential, inspiring a sequence of important works by Kechris and Solecki, Farah and Zapletal, and Roslanowski and Shelah which provided partial answers to Kunen’s problem. Jointly with Juhász in 2001, Kunen sought axioms that capture the combinatorics of \( \mathcal{P}(\omega) \) and \( \omega^\omega \) in the Cohen model. The idea behind this interesting line of research is to identify a handful of principles that hold in the Cohen model from which most of the known properties of the Cohen model could be axiomatically derived. Kunen was one of the first to obtain consistency results about combinatorial cardinal characteristics of the continuum by forcing. Kunen observed that \( \mathfrak{u} = \aleph_1 \) in the Cohen model, a result he published in his textbook [Kun80b]. He was the first to prove the consistency of \( \mathfrak{u} < 2^{\aleph_0} \). There were, in fact, two models of this. Bell and Kunen [BK81] produced a model with \( \mathfrak{u} = \aleph_1 \) and \( 2^{\aleph_0} = \aleph_1 \) (more is true in their model – every ultrafilter on \( \omega \) has \( \pi \)-character \( \aleph_1 \)). Exercise (A10) in Chapter VIII of [Kun80b] asks the student to build Kunen’s simpler model where \( \mathfrak{u} = \aleph_1 \) and \( 2^{\aleph_0} \) can be arbitrary. Kunen asked whether it is possible to have a uniform ultrafilter on \( \omega_1 \) that is generated by fewer than \( 2^{\aleph_1} \) sets, that is, whether \( \mathfrak{u}(\aleph_1) < 2^{\aleph_1} \) is consistent. His question remains wide open.

I will end with some personal recollections. Kunen was an outstanding advisor, a perfect match for my personality. He was able to inspire his students by the sheer example of his deeply original work. I had complete intellectual freedom in choosing what I wanted to work on, receiving guidance only if I was hopelessly stuck and decided to ask him. And in that case he always had some valuable remarks. I was able to acquire what is arguably the single most important skill in research: discerning the right problems to tackle. He was the ideal advisor for me. Kunen was also well-known for providing generous financial support to his students through his research grants. I was the beneficiary of this generosity twice.

I saw Kunen for the last time in May 2018 in Madison, WI. At that time, I was working on some problems about the order dimension of uncountable partial orders. I explained to him the results I had obtained with my collaborators and told him some of the problems I couldn’t solve at the time. He thought briefly about what I had said, told me enthusiastically that the results were really interesting, and remarked casually that the model obtained by adding \( \aleph_3 \) Cohen reals is likely to be the right place to look for a counterexample to one of the open problems. He was absolutely right. Lemma 5.1 of [KR21] proves Kunen’s remark from our final meeting that day.

Kulen and Inner Model Theory

**John Steel and Hugh Woodin**

Kulen began his graduate studies at Stanford in 1965, and finished in 1968 with a PhD thesis that is truly remarkable for its depth and breadth, and for the importance of its ideas in later work. The first half of the thesis deals with \( L[U] \), the canonical minimal inner model with one measurable cardinal. An augmented version of it was published in [Kun70], a classic paper that has become standard material in graduate set theory texts.

Kulen’s thesis adviser Dana Scott had proved in 1961 that the existence of measurable cardinals implies that there are nonconstructible sets, that is, the full universe \( V \) of sets is larger than Gödel’s canonical inner model \( L \). Rowbottom and Gaifman had shown in 1964 that if there are...
measurable cardinals, then in fact there are only countably many constructible real numbers, and Silver had identified a canonical “least” nonconstructible real, now known as $0^\sharp$. $L[0^\sharp]$ is thus a canonical inner model slightly larger than $L$, but if there are measurable cardinals, then it too has only countably many reals. Solovay had observed that if $U$ is an ultrafilter witnessing the measurability of a cardinal $\kappa$, then in $L[U]$, $\kappa$ is measurable via $U \cap L[U]$, and the GCH holds above $\kappa$. So $L[U]$ is an inner model with a measurable cardinal, but we cannot say yet that it is canonical, because it seems to depend on the arbitrary parameter $U$. Silver showed in [Sil71] that the full GCH holds in $L[U]$, and its set of reals is independent of $U$.

With this as a foundation, Kunen established the basic canonicity theorems for $L[U]$. Silver’s work on $L[U]$ had used Rowbottom’s method of indiscernibles. Kunen’s key first idea was to build on Gaifman’s method of iterated ultrapowers instead. With that as his starting point, he showed that the model $L[U]$ depends only on the measurable cardinal $\kappa$, and not on $U$, and that the first order theory of $L[U]$ is independent of $\kappa$ as well. Behind these results was a Comparison Lemma, which in Kunen’s work took the form: if $\lambda > \kappa$ is any regular cardinal and if $W$ is the filter generated by the closed unbounded subsets of $\lambda$, then in $L[W]$, $W \cap L[W]$ is an ultrafilter on $\lambda$, and the inner model $L[W]$ is an iterated ultrapower of $L[U]$.

With $L[U]$ established as a canonical object, it is natural to ask whether there are other ways to construct it, and whether there are canonical inner models for stronger large cardinal hypotheses. Kunen’s [Kun70] took some important first steps in these directions. He showed that if there is a strongly compact cardinal, then there is a canonical inner model $L[\hat{U}]$ with a proper class of measurable cardinals. By later work of Magidor, the existence of strongly compact cardinals does not imply the existence of two measurable cardinals, so this construction of $L[\hat{U}]$ cannot be the simple one that Solovay identified. Building on work of Solovay ([Sol71]), Kunen showed that if there is a $\kappa^+$-saturated (uniform and $\kappa$-complete) ideal on $\kappa$, then the canonical inner model with a measurable cardinal exists. (This leads to an equiconsistency.) The hypothesis here does not imply the existence of a measurable cardinal. Finally, Kunen showed that if the GCH fails at a measurable cardinal, then there are indiscernibles for $L[U]$. Again, the indiscernibles cannot come from a second measurable cardinal. Each of these theorems stands near the beginning of some substantial line of development in set theory.

Shortly after leaving Stanford for Madison, Kunen proved his well-known, very useful theorem that if there is a nontrivial elementary embedding $j$ from $L$ to itself, then $0^\sharp$ exists. (The converse is easy.) The key concept of an $\mathcal{M}$-ultrafilter is studied in his thesis, and the equivalence of $0^\sharp$ with an iterable $L$-ultrafilter is at least implicit there. The heart of the new work is that the $L$-ultrafilter derived from $j$ is iterable. Here Kunen’s technique of considering the hull of $\{\alpha \mid j(\alpha) = \alpha\}$, much used by him and by others later, plays the key role.

Kunen left inner model theory not long after moving to Madison, but it is truly remarkable how far he went, near the beginning, and in such a short time. Since then, the theory has been extended so as to produce canonical inner models for much stronger large cardinal hypotheses, under a wide variety of assumptions. Kunen’s Comparison Lemma has been greatly extended, both as to the models being compared, and as to the iteration methods used to compare them. Nevertheless, we still have very little information concerning canonical inner models at the level of strongly compact cardinals, and there are significant open problems well below that level.

Adrian Mathias was a Research Associate at Stanford in 1967–68 and a Visiting Lecturer at Madison in 1968–69. He writes of those times:

“...He had two excellent qualities not, alas, shared by every distinguished academic: if you asked to try a proof on him, he would listen carefully and give valuable feedback; and he scrupulously gave credit to others for their work. I remember receiving a letter once from him answering a question I’d posed, in which one paragraph began “An argument due to you shows that...” One knows mathematicians who hate to admit that someone else has had a good idea; but Ken was free of that fault.

We had speculative conversations, and listened to each other’s proposed proofs, but there was no intense collaboration such as I have had with some people; nor was there that intense competitiveness that has made me wary of some others. We were simply colleagues who helped each other. I learnt a lot from him: I remember his coming to my Stanford office to test a proof on me: the result was that if $U$ is a normal ultrafilter on $\kappa$ and you iterate $L[U]$ up to a larger regular cardinal $\lambda$, what you get is $L[F]$ where $F$ is the club filter on $\lambda$; from which many things follow.”
In Madison, as spring approached, Kleene organised a picnic for the logicians, and at it Ken came up to me and said “I can prove that 0^2 exists.” I asked him if he was feeling quite well, and then he admitted that his proof would require an assumption.

The assumption was that there is a nontrivial elementary embedding from L to itself.

References


Credits

Figure 1 and Figure 2 are courtesy of Eva Coplakova. Figure 3 is courtesy of George Bergman. Figure 4 is courtesy of Anne Kunen.
Isadore M. Singer (1924–2021)
In Memoriam
Part 2: Personal Recollections
Robert Bryant, Jeff Cheeger, and Phillip Griffiths

Figure 1. Singer giving the Killian lecture at MIT, 2006.

Lenore Blum

At Is’ 90th birthday celebration, Mike Sipser, then head of the MIT Math Department, read a letter Is wrote in the spring of 1962 suggesting how he might contribute to the department on returning from sabbatical in the fall. “Am willing to serve as Faculty Counselor or Freshman advisor, whichever you need most. …Happy to lecture in 18.01–18.02 or any of the special calculus sections. In fact, am willing to teach any elementary course so long as it is not advanced calculus for engineers.” He also added, “would be happy to teach some form of Modern Algebra…whatever the department desires.”

I was stunned. That letter, I realized, had changed my life.

Since I was 10 years old, math was my favorite subject, though I also loved art. When I told my high school math teacher that I wanted to major in math in college, he said, “Why would you want to do that? The best math was done 2000 years ago.” Not so strange in retrospect. After all, this was a missionary high school, and my math teacher was a missionary who taught Euclidean geometry. Not wanting to go into a dead field, I decided that architecture would combine my love for art and math, that is, until I took advanced calculus for architects. During my second year in college (Carnegie Tech), I was able to switch to math by taking an experimental course in computer science (the first such academic course ever given on the planet). Although fascinating, this was not the math I was looking for. I married that summer, moved to Boston, and transferred

Lenore Blum is a career professor of computer science at Carnegie Mellon University. Her email address islblum@cs.edu.
to Simmons College. The head of the Math Department, Marion Walter, a wonderful teacher said, “we don’t have anything more for you here” and arranged for me to be a special student at MIT. So, in the fall of 1962, I enrolled in Modern Algebra at MIT. The teacher was Is Singer. His course was just so beautiful. It was what I had always been looking for. The first semester was abstract algebra, the second semester linear algebra, but really terrific abstract linear algebra with Grassmannians. It was beautiful underlying theory and pretty advanced. I loved it and did really well. So that spring, I got up my courage and applied to grad school at MIT.

I remember going for an interview with the head of the Math Department in his office. As I walked in, he handed me a list of schools and said, “if I had a daughter who was going to graduate school, these are the places I’d tell her to go. MIT is not a place for women.” I was devastated. But then, I got a ‘lucky’ break. The next Saturday the Math Department had a faculty party. Somehow, at the party they were joking about this girl who was applying to the PhD program in mathematics. Is was there and asked, “who are you talking about?” They gave my name and he said, “Oh! She’s the best student in my class.” I got accepted the next day.

Robert Bryant

My first meeting with the legendary Is Singer was when I was a graduate student at the University of North Carolina at Chapel Hill. He, Atiyah, and Hitchin had just announced their beautiful work classifying instantons on \( S^4 \) using an artful combination of index theory and ideas of Penrose, and he came to our department to deliver one of those inspiring colloquium talks that live in my memory as highlights of my introduction to the world of research mathematics. I remember the contagious delight that he radiated as he told his story. He managed to convey the importance and beauty of the techniques and results to us graduate students without hauling out all the machinery that underlay the proofs. For example, when he needed to pass from the case of instantons with finite action on \( \mathbb{R}^4 \) to instantons defined on \( S^4 \), he referred to Karen Uhlenbeck’s “beautiful removable singularities theorem” and urged us to read about her work. (I believe that this was the first that I heard of Karen’s work.) I understand now, as I did not then, how important inspiring colloquium talks are to young mathematicians. That Is took the task seriously and didn’t just give a brilliant technical seminar talk to the experts in the audience is an indication of how much he cared about the life of our mathematical community.

I got to experience Is’ generosity in another way that evening. Back then, it was customary for a faculty member to host a reception/party in their home for the colloquium visitor, and graduate students were invited. After such a great talk, I couldn’t pass up the opportunity. I had an ulterior motive, though. In my study of Élie Cartan’s works on Lie transformation groups, which was just beginning, I had been referred to a paper by Singer and Sternberg [4, Part I], for a modern treatment. I wanted to ask Professor Singer, “What happened to Part II?” In Part I, they had thoroughly explained Cartan’s method of classifying the primitive, transitive Lie transformation groups and even pointed out places where Cartan’s arguments were incomplete (and how to complete them). Moreover, whereas Cartan worked in what was essentially a holomorphic category, they were able, with some extra hypotheses, to extend his results to the smooth category. However, this was just in the transitive case, and it was clear that there were many interesting intransitive simple Lie transformation groups; for example, the gauge group of a principal \( G \)-bundle where \( G \) is a simple (finite dimensional) Lie group. When I got up the courage to ask my question at the party, Is gave me his full attention, found a quiet corner where we could sit and talk, and told me the story of how he and Sternberg had become interested in Cartan’s work, what had motivated them, and what still intrigued him about Cartan’s old papers on the subject. The intransitive case had turned out to have several new and unexpected features, and they had intended to continue to work on it, but, after Part I was finished, Singer had gone on leave to the UK to visit Atiyah, and Atiyah immediately engaged him in the work that ultimately led to the their famous Index Theorem. Singer had never had time since then to return to work on Part II and, as far as I can tell, he never did, but he encouraged me to keep working on developing Cartan’s ideas. The enormous amount of attention that Professor Singer paid to a random graduate student far from the

Figure 2. Singer in the Army Signals Corps during WWII, ca. 1945.
MEMORIAL TRIBUTE

major centers in geometry is just one example of the way that he fostered and inspired a generation of mathematicians.

Of course, over the years, as I became aware of the enormous service that he did for the mathematics and physics communities and the National Academies, his example has remained an inspiration to me in my own career.

Perhaps the most direct way that he influenced my career, though, was that he, Shiing-shen Chern, and Calvin Moore joined forces in the early 1980s to propose a new NSF-sponsored research institute, the Mathematical Sciences Research Institute, in Berkeley, CA. They conceived of it as a permanent, independent entity from UC-Berkeley, supported by a consortium of academic institutions and with national reach.

Dan Burns

“Very dashing, Dr. Singer,” Ronald Reagan said in reaction to Is’ brightly colored cravat as the President rushed into his meeting with the White House Science Council. Is relished recounting this moment which had gone the way he felt all one’s life should go: vivaciously full-throttle. Earlier, however, during the Vietnam War era, he marched in protest at MIT in a suit, cravat, and sunglasses to increase the ‘credibility’ of the largely hippie student throng. He remarked, shocked, that his old friend Dirk Struik had not recognized him in his more ‘establishment’ outfit!

Many know that Is grew up poor during the Depression in Detroit. He was wistful recalling the letter from the University of Michigan awarding him a tuition scholarship of $50 per semester, making it possible for him to attend. Many years later he returned with his three youngest children to show them the campus and recount his days in the student coops living on less than $2.50 per week, taking turns preparing the meals for the residents and other duties the members shared. He had a deep sense of the challenges overcome in that part of his life and wanted to pass on some of that resilience to his daughters.

Is is famous for his endless series of very influential seminar courses on a broad array of the latest active areas in mathematics, and he would push his stable of students to pick up expertise in these areas to bring back to the group. The topics would range from algebraic geometry to PDE’s and later to mathematical physics. His advising of grad students could be similarly ‘bird’s eye view,’ students just being assigned a very general area to explore. I remember his oracular pronouncement long ago that the Poincaré Conjecture would be solved by a PDE method, his confidence based on the pattern of the uniformization theorem in dimension two. He was right, of course, but it was unreachable until much later, by methods he certainly did not foresee in any degree of detail.

In the time I worked with him, he was often away on leave and would return about once every month or two and hold a marathon session all in one day to catch up on theses, projects of post-docs, and so on. I found it was always a good idea to be last in line since his energy even late at night seemed boundless, and when it was getting late, with no dinner in the meantime, the last student could get dragged along to a continuation of the scientific discussion over a midnight sandwich or more at Ken’s in Kenmore Square. Discussions often drifted over to Boston’s need for more jazz and Chicago blues clubs, some of his musical loves.

Is went to college to study either literature or physics. He said much later that he had thought of trying to have another go at writing, perhaps buoyed by being a relative of Isaac Bashevis Singer, though my memory is not confident I have that claim correct now. But even with his enthusiastic embrace of intellectual breadth, Is admitted that he thought that people really didn’t excel at such cross-sectional careers, that it was already a rare enough gift to be good at even one pursuit, even if taken in the broad sense he showed in mathematics and physics. Is continued actively to a very ripe age, retiring in 2010 at the age of 86. He was always keen to be active, on the move. He had very little use for dwelling on past accomplishments. In 1984, Jeff Cheeger and I and others discussed having a 60th Birthday conference for him, an idea he shot down unceremoniously. Five years later he was awarded an honorary degree from the University of Michigan, and we had a small ‘birthday-like’ cake in the department common room for him, which he admitted he was grateful for, but we shouldn’t think he didn’t notice the unmentioned coincidence with his 65th birthday. He said to come back in twenty years, maybe then he would feel like 65 and be ready for a party. That was exactly what he got with the famous Gang of Four Conference for him and his friends Atiyah, Bott, and Hirzebruch right on time for his 85th.

I think of Abraham Pais’ exquisite biography of Einstein, Subtle Is the Lord, where he says that for such a committed scientist, his science was of the essence in any biography since that was an integral part of his view of life. This is certainly true of Is, and hopefully some day somebody will write such a history. These few recollections are admittedly personal and barely scientific, but they are a part of our placeholders until this scientific biography is written.

Dan Burns is a professor of mathematics. His email address is dburns@umich.edu.
**Alain Connes**

Is Singer was a great analyst who discovered and developed, completely independently of his work on index theory, many new fundamental notions of analysis with great impact in geometry such as analytic torsion. He was also a rare example of a mathematician with true influence in physics with, for instance, the use of zeta function regularization in renormalization (he told me about his lecture on that topic in front of Richard Feynman). Of course, the same independence holds for his contributions to operator algebras, such as the problem he formulated with Dick Kadison, whose resolution took more than fifty years.

**Harold Donnelly**

Although I wrote only one joint paper with Atiyah and Singer [1], these two mathematicians were crucial in my early career. In 1973, S.S. Chern, my thesis advisor at Berkeley, suggested that I visit MIT for a semester, to meet some of the mathematicians in Cambridge. Professor Singer kindly agreed to be my host. Patodi was also visiting MIT at that time, and we had many joint discussions about about spectral geometry. A few years later, Patodi and I wrote a joint paper. Patodi was then at the Tata Institute, and I was a Moore instructor at MIT. This long-distance collaboration might never have taken place without our previous acquaintance. Jeff Cheeger was visiting Harvard, and we had a discussion concerning Chern–Simons invariants. These two important contacts were only possible because of Singer’s generosity in acting as host.

I completed my doctoral program at Berkeley in 1974 and accepted a two-year Moore instructorship at MIT. Singer was the likely sponsor, because he was the faculty member most cognizant of my thesis work and some other results. During the years 1974 to 1976, there were some discussions with Singer about mathematics. For example, he liked my paper about the heat equation and volume of tubes, which solved a conjecture by Seeley. It was somewhat disappointing that Singer was on leave for much of the years 1974–76. However, I talked extensively with Bott at Harvard and Ray at MIT. Bott informed me of an old conjecture of his, which I solved. My interaction with other faculty at MIT was also very enlightening. So, again Professor Singer was indirectly supportive, although the personal contact was quite limited.

After my time at MIT, I spent two years at Johns Hopkins, 1976–78. For the years 1978–79, a Sloan Fellowship supported a return to Berkeley. Professor Singer was one of the letter writers in my application for the Sloan. Professor Singer had moved from MIT to Berkeley by that time. During the year, I was working on spectral theory for complete Riemannian manifolds, partly with Peter Li. Again, Singer attended my lectures and made favorable comments. Purdue was looking for a mathematician

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**Note:**

Alain Connes is a professor emeritus at the Collège de France. His email address is alain@connes.org.

Harold Donnelly is a professor of mathematics at Purdue University. His email address is hdg@purdue.edu.
specializing in partial differential equations and differential geometry. The department head contacted Singer who suggested my name. I have been a tenured faculty member at Purdue since 1979.

My collaboration with Atiyah and Singer began in 1980. Several years earlier, they had provided an outline for the solution of a conjecture by Hirzebruch. The outline suggested that only one crucial step was missing. I worked on that step and proposed a joint paper to Singer. However, the problem was more difficult than anticipated. The work was only completed and published in 1983. An independent proof was given by Werner Muller; [2].

David Ebin

My contact with Singer was limited to being his student—I don't recall talking to him except about mathematics. Thus, I am limited to describing him in his role as an advisor. He was everything a student would want. He would both ask and answer questions. Then he would ask me to write out pieces of my work and then make comments and suggestions.

Singer was my thesis advisor from 1965 to 1967. This was shortly after the Atiyah–Singer index theorem came out, so Singer was very much in demand. However he always made time for students. For a thesis problem, Singer first suggested that I work on the conjugate locus of a Riemannian manifold. Two previous students, Frank Warner and Nathan dos Santos, had written theses in this area previously, and John Mather had recently achieved new results on singularities of differentiable mappings. Singer thought I should try to apply Mather’s results to the conjugate locus. Unfortunately I was unable to make any headway in this area. As far as I know, this still has not been done.

Instead, I started working on the space of all Riemannian metrics on a given manifold. The group of diffeomorphisms naturally acts on this space, and my thesis came out as a construction of a subspace transversal to its orbits. It involved a lot of technicalities using Sobolev spaces and finally encompassed about a hundred and fifty pages. This is where I believe Singer showed his value as a fine advisor. He exhibited infinite patience while I worked it out, again sharing the asking and answering of questions. I say much patience because the project went well beyond the end of the semester and into the summer. This became a bit complicated because Singer was in California for the summer. However, he accepted my hand-written work and gave it to two other readers who were willing to sign off on the project.

Once during this time I went to the Singer house and met Mrs. Singer and their daughter Natasha, who was then about six years old. I saw Natasha again recently at a social function in New York, and she confirmed that she is the Natasha Singer who writes for the New York Times. I found that rather curious because Singer had several times told me that he was the world’s worst writer. For many of us in mathematics, the strenuous task of writing out the details does not compare to the satisfaction of working out the ideas.

One might ask how a graduate student in mathematics manages to write a thesis? It is usually the first time that one faces the daunting task of solving an original problem. The task is double for there are two a priori unknowns: ‘Can this problem be solved?’ and ‘Can I solve it?’ Singer’s advice and patience provided the help necessary to surmount the difficulties.

Phillip Griffiths

Although over the years Is and I were together at conferences and other gatherings, my main contacts with him were through his mathematical works. With his special and, in many ways, unique perspectives from physics, Is was a central pioneer in the confluence in mathematics between topology and analysis that has taken place.

Among the interactions, direct and indirect, that I had with Is, three stand out. In the mid-1960s, we were together at a small conference in Mexico City. I had recently finished my degree at Princeton under Don Spencer and, not surprisingly, was interested in deformations of complex structures, a subject that had then been recently originated by Kodaira and Spencer. Is was also interested in

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David Ebin is a professor of mathematics at Stonybrook University. His email address is ebin@math.stonybrook.edu.
this, especially how the Maurer–Cartan equation

\[ D\phi + [\phi, \phi] = 0 \]  

(1)

enters and the subsequent properties of its solutions. Here \( \phi \) is in a graded Lie algebra and \( D \) is a derivation. In application to geometry, \( \phi \) is usually a section of a bundle over a manifold and \( D \) is in a first-order linear differential operator. In deformations of the complex structure on a compact manifold \( X \), in a tour de force Kuranishi had just completed his work using properties of (1) to show the existence of a unique versal deformation space, the Kuranishi space, \( \text{Def}(X) \). Although by no means an expert, Is asked me to explain as much as I could of the technical aspects of Kuranishi’s argument. It was later that I came to understand his interest as a variant of the inhomogeneous version of (1) appears in deforming connections in gauge theory as, e.g., in his work with Atiyah and Hitchin on instantons.

A second instance was many years later during the period when I was Director of the Institute for Advance Study. The Institute had organized a special event commemorating the establishment of the School of Mathematics. For one part of the program, Is, Raoul Bott, and Michael Atiyah were invited to come and give a general discussion of the year in the mid 1950s they had been together at the IAS. This was the beginning of one of those golden periods in an era of mathematics where there is a confluence of areas, in this case topology and analysis, that together solve major outstanding problems and create a whole new area in the subject. The three of them were among the principal founders of this new area, and their reminiscences and reflections, at once mathematical and personal, were the highlight of the event.

The third is indirect and reflects Is’ great breadth of interests. In the 1970s, I was working in Nevanlinna theory, which is the general subject of the geometric properties of holomorphic mappings between complex manifolds. Two milestones in the classical theory were due to Lars Ahlfors. The classical defect relations of R. Nevanlinna concerned the solutions to the equation

\[ f(z) = w \]  

(2)

where \( f \) is an entire meromorphic function and \( w \) is a point on the Riemann sphere \( \mathbb{P}^1 \). One may view (2) as a holomorphic mapping \( f : \mathbb{C} \rightarrow \mathbb{P}^1 \), and Ahlfors extended the theory to the case of a holomorphic mapping

\[ f : \mathbb{C} \rightarrow \mathbb{P}^n. \]  

(3)

The second was Ahlfors’ interpretation of the classical Schwarz lemma, formulated as saying that a holomorphic mapping \( f : \Delta \rightarrow \Delta \), (\( \Delta \) = unit disc) is distance decreasing in the hyperbolic metric. Ahlfors showed that the same result is true if one uses any metric on the image whose Gauss curvature is \( \leq -1 \). This ‘method of negative curvatures’ can be used to give a differential-geometric proof of the defect relation. Reflecting his broad interests in almost anything that combines geometry and analysis, Is was interested in Nevanlinna theory and asked one of his students, Michael Cowen, if the method of negative curvature could be used to prove Ahlfors’ defect relation for (3). Michael talked with me and together we were able to do this. Although far removed from Is’ main interests involving PDEs, operator theory, and topology, it was the combination of differential geometry and analysis that he appreciated. And this example is only one of the many instances that, through his own work and through his personal interactions with his students and colleagues, Is had such a central and unique role in our field.

Victor Guillemin

As a devotee of Is Singer, I sat in on many of the graduate-level courses that he taught at MIT during the years that we were colleagues in the math department. His lucidity and insights into the mathematics of spectral theory, index theory, and the geometry and topology of differential manifolds made me an ardent Singer fan. Moreover, I had an opportunity several years back to teach a course dealing with the Atiyah–Singer index theorem, and, as a result, an opportunity to appreciate the depth and beauty of one of his greatest achievements. I am extremely pleased that Is is being remembered in this volume of the Notices.

Richard Palais

The way I am most closely connected with Singer is through volume 57 of the Annals of Math Studies book series, published in 1967 with the title Seminar on The Atiyah–Singer Index Theorem (SASIT). This lists me as its ‘Author,’ though I was in fact the sole writer of only nine of its twenty-one chapters, and there were six other participants (Atiyah, Borel, Floyd, Seeley, Shih, and Solovay) who wrote the remaining chapters or otherwise made major contributions to the volume, so I should better be referred to as its ‘Editor.’

This volume has an interesting history. In Bonn, in the Spring of 1962, Michael Atiyah gave a fascinating Arbeitstagung talk outlining his recent joint work with Singer: a remarkable new connection they had discovered between analysis and topology. This was what has come to be

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Victor Guillemin is a professor of mathematics at MIT. His email address is vwg @math.mit.edu.

Richard Palais is an adjunct professor of mathematics at the University of California, Irvine, and a professor emeritus of mathematics at Brandeis University. His email address is palais@uci.edu.
known as their Index Theorem. Armand Borel and I were in the audience and, like everyone else present, we both were excited by what we heard. Borel knew that I would be at IAS the next year and asked if I would be willing to cooperate with him on a seminar working out the details. (Borel, Deane Montgomery, and I had run a successful Seminar on Transformation Groups when I had been at IAS in 1959). I of course readily agreed, but then in mid-summer, after our seminar had been announced, Borel wrote to me that something had come up and he would have to spend the better part of the academic year in Paris, so would I be willing to take over the direction of the Seminar. I felt I had to agree, and then spent the next two and a half years, first at IAS, with other Institute members, working through the details that had been roughly outlined in the Atiyah–Singer announcement, and then back at Brandeis, carefully writing up those details for publication in SASIT. It was a very rewarding but difficult experience, and it would have been a great help if Singer had been at MIT where I could have easily consulted with him. But he was in England, working with Atiyah on the details of an even more powerful result, using a different approach, and since this was long before the period of instant email communication, I had to get along without that resource. SASIT was published a couple of years prior to when Atiyah and I wrote to me that something had come up and he would have to spend the better part of the academic year in Paris, so would I be willing to take over the direction of the Seminar. I felt I had to agree, and then spent the next two and a half years, first at IAS, with other Institute members, working through the details that had been roughly outlined in the Atiyah–Singer announcement, and then back at Brandeis, carefully writing up those details for publication in SASIT. It was a very rewarding but difficult experience, and it would have been a great help if Singer had been at MIT where I could have easily consulted with him. But he was in England, working with Atiyah on the details of an even more powerful result, using a different approach, and since this was long before the period of instant email communication, I had to get along without that resource. SASIT was published a couple of years prior to when Atiyah and Singer felt satisfied enough with their revised version to publish it, so for a while SASIT was the ‘go-to’ source for the Index Theorem, and even now it is sometimes still suggested as a source for less-experienced mathematicians to get started learning the prerequisites for its understanding.

Hugo Rossi

Once I had passed my orals, I was officially qualified to do research with my advisor, Professor Singer. My job became that of formulating problems, looking at examples to understand them, and ultimately solving them. Throughout this period I had weekly meetings with my advisor. Sometimes these were group meetings with all of his advisees, but mostly they were one-on-one. Rarely did I have some progress to report—almost all of the meetings were spent trying to understand why the latest method of attack failed.

At one of those meetings in early March of my third year, my advisor stopped me in the middle of a tortuous computation of what we thought was a signal example, and said, "Good. Write it up." I asked, "Write what up?" and he said, "Your thesis. What you’ve been telling me for the past months. That’s your thesis. Congratulations. And, by the way, I am nominating you for a postdoc at Princeton, and an abstract of your thesis is due there by the end of the week. Get something to me by Thursday and I’ll add it to my letter."

I was shocked. What did I do? Where was the fabulous new result? The remarkable insight? The surprising twist? What was there in anything I’d reported over the past year that could impress anyone at Princeton? He could see what I was thinking. "Just the facts. No embellishment. Just ‘here’s the problem I’ve been working on, here’s where I’ve gotten, here’s how, and here’s what I want to do next.’ Just as if you were talking to me, here in this room."

So, I went home that evening and started work on an abstract just as he had advised. Indeed, I had solved some problems, but not those originally proposed. I realized that I had to invert history: start by stating that the problems I had solved was the object of the study and end with the original problems as prospects for future work. Over the next few days I followed that course and went to Singer’s office with a draft of my abstract. To me, it was a catalogue of failures—a confession of incompetence.

When my advisor looked at it, he took a different view. "This will work just fine." He noticed the look of despondency in my eyes. "OK, it’s a work in progress, but there’s no doubt you’re on the right track, and there’s lots of promise in this approach. They’ll appreciate that in Princeton. Don’t worry, you’ll do just fine." I didn’t feel just fine. I knew in my bones that—on the day I arrived in Princeton—they’d uncover the hoax we were perpetrating on them, and expose me as a fraud.

"But, Dr. Singer… I’ve applied for a lectureship at Dartmouth College, and I’ve been told that I’ll probably get it. I love to teach. I think I prefer…" "That’s fine. It’s good to have options. If they don’t take you at Princeton, we can talk about options. Maybe Dartmouth is one. But I think you’ll get the Princeton postdoc, so let’s proceed on that premise."

I must have looked like I was being led to the gallows. "Look, Hugo—you’ve learned a lot of things here at MIT. You have the tools to learn what Spencer is doing and to work with him. While we’ve been scratching around in the desert looking for gold, he’s been out in the galaxy finding diamonds. You go there for a year and next summer come..."
back here and tell me all about it. I really want to learn that stuff. So, you better get busy writing your thesis in detail, because it’s due in the office by the second week of April—about a month. And it has to be read by your committee before you can submit it.”

**James Simons**

While I was an undergraduate at MIT I first saw Is around midnight at Jack and Marion’s restaurant in Brookline where he was sitting with Warren Ambrose (whom I already knew) doing mathematics. I thought that was very cool—being a mathematician was certainly a great job! Thereafter I saw them frequently there but didn’t talk with either of them.

I graduated MIT after three years but stayed on as a graduate student working under Is. He taught me about Lie groups and Lie algebras as well as some differential geometry, which I very much enjoyed. Later in the year he told me that Chern was about to leave U. Chicago and go to Berkeley. He urged me to transfer to Berkeley so I could work under him. That sounded good. I got a very nice fellowship and hightailed it to the West Coast. Much to my disappointment, Chern was spending his first year at Berkeley on sabbatical leave!

Soon I met Bert Kostant, whom I liked, and he took me on as his student. I proved a few things that seemed interesting and said they might be used to solve an open problem: Why are all holonomy groups on undecomposable Riemannian manifolds transitive on the unit sphere? I said I would like to work on that, but he said don’t even try. He said Borel and Singer had each tried and failed. That just got me excited!

I made some progress with the problem and by mail shared my progress with Singer. He was encouraging. I kept making progress but then got stuck. It was then that my newly married wife and I were going to Boston over Christmas, and I made an appointment to meet Is at his office. That day there was a huge blizzard, but somehow he and I reached his office. I showed Is where I got stuck, and he immediately pointed out that I hadn’t used the undecomposable assumption, and, in a second, I became unstuck! After a few months back in Berkeley, I finished the proof.

In the next several years, we socialized occasionally but did not work together. Then I became Chair of Math at Stony Brook and became friendly with C.N. Yang, the famous physicist. One day Yang told me about the Bohm–Aharonov effect. It showed that if one constructed a magnetic field completely confined to the z-axis in 3-space and sent electrons in a circle around that axis but very far away, then as they met, they made a phase change that would vary with the strength of the magnetic field. It is quite amazing! Mathematically, it could be described as a flat vector bundle with varying holonomy.

That year Singer was in New York. We met for dinner, and I told him about Bohm–Aharonov. He got very excited, and I believe that inspired him to start doing physics with Atiyah.

In subsequent years, we met socially from time to time. He and his wife Rosemarie accompanied us on our new boat, Archimedes, thirteen years ago. We occasionally visited them at their home, and once or twice they visited us at our home. When we started the Center for Geometry and Physics at Stony Brook, Is became co-chair of the Board. He gradually began to fail. The last time I visited him, he recognized me and was glad to see me, but it seemed clear that the end was nigh.

I wish I had seen more of him in these past sixty-plus years, but each occasion was a great pleasure. I miss him very much.

**Elliot Singer**

Is Singer, my father, was the quintessential Boston driver. He was the only person who could get to Logan Airport without taking the bridge or the tunnel. He never demeaned himself with paid parking for Red Sox games—free side-street spots miraculously opened on yet another go-around. He would find circuitous routes to avoid traffic jams on Memorial Drive. When, in my middle-age, I pointed out this saved no time, he acknowledged the fact, but had no patience for the inelegant solution.

As a child, there were always mathematicians at the house speaking a foreign language, which fascinated me—words like ‘fiber-bundle,’ ‘isomorphic,’ and ‘manifold.’ Before Dad’s 50-plus years as a professor at MIT, he did the typical pre-tenure itinerant faculty gigs—UCLA, Columbia, Institute for Advanced Study in Princeton. The year in New York, when I was 4, Dad and Dick Kadison would visit jazz clubs, late into the night, and Dad got to sip coffee with Billie Holiday, one of his most precious
memories. (Stale black coffee and two packs a day of Chesterfields were his staples.)

The next year, in Princeton, I knocked over J. Robert Oppenheimer while learning to ride a bike, with Dad running alongside, holding the seat. Arnold Shapiro was another of the mathematicians there—Arnold’s wife, Janine, was a second mother to me—and Dad always remained close to Janine and their son Gregor. The Shapiros had a copy of the newly released Harry Belafonte calypso album, and “Jamaica Farewell” became our family song. I still know it by heart.

We used to go to Celtics games with Ambrose. Dad had been in the Signal Corps with Johnny Most, the famous Boston sports announcer, and we would visit the broadcast booth, where I was allowed to announce. I never knew Ambrose had a first name, and I never called anyone ‘Mr.’ or ‘Mrs.’ Dad referred to some mathematicians, like Dick and Arnold, only by their first names. Others like (Warren) Ambrose, (Raoul) Bott, and (Michael) Atiyah, had only last names.

I was 11 when we spent the year at Oxford, where the Atiyah–Singer Index Theorem was born. I remember putting shillings into the meter for heat, how bad the food was, and learning algebra and Latin, and scuffing my dress shoes playing football (the English kids brought boot-black to retouch) at Magdalen College School for Boys. Before retuning to Boston, we visited Paris, where Dad taught me quadratic equations on the banks of the Seine.

Memories are funny, and I don’t really have many of Dad from my interminable years in junior and senior high. We moved to a more suburban-style house in Newton, and he hated the need to keep up appearances, like weeding and mowing (often assigned to me, contributing to my own lifelong contempt for suburbia). We went to see Goldfinger together, after which I devoured, in a week, his collection of Bond novels, before graduating to Hammett and Chandler (I still have his copies, in tatters). Then Kafka, The Brothers K., and browsing together at the Paperback Booksmith in Harvard Square—Dad, absolutely not school, instilled in me a love of literature. Of course, there was Casablanca, and the Bogart festival at the Brattle Theatre. We had no television.

By the late 1960s, when I was an undergraduate at MIT, I was a budding anti-war and civil rights protestor, which Dad never discouraged, though himself not an activist. October 1969 was the first huge Boston anti-war march, and Dad and his colleagues dressed in suits, with the hope this would make a difference. A few months later, when we were occupying Jerry Wiesner’s office to protest the war, Dad was among those outside the door (along with Chomsky). In later years, Dad took great pride in the photo of me on the front page of the Boston Globe from my graduation, handing a red armband to Wiesner, though I don’t think he was so sanguine at the time.

Dad always said, “science is just one of the humanities.” I sometimes try to reconcile my own research, as a folklorist, with his, as an abstract mathematician who considered String Theory, ‘applied.’ We both hated the term, ‘social science,’ and he loved my dismissal, stolen from Tom Lehrer, of anti-logocentric philosophers who incessantly bemoan the failings of analytical scholarship: “the very least you can do is shut up!” Somehow his search for simple, elegant, proofs and explanations must be isomorphic with my exhaustive (and exhausting) search for interconnectedness of versions and variants.

I was fortunate enough to attend the 1979 Joys of Research Einstein centennial conference at the Smithsonian Institution in Washington, where Dad was one of the presenters—another was Linus Pauling, whom we were both thrilled to meet. This was probably the occasion on which he best shared his views, in light of his personal experiences, and those interested in his intellectual biography should be familiar with this easily overlooked source (Walter Shropshire, Jr., ed., The Joys of Research, Washington, D. C., Smithsonian Institution Press, 1981), from which I quote:

My own motivations for doing research are: first, the private joy in the exercise of one’s talents. When I was a youngster I envied many around me. They had much talent. There were those who could play musical instruments well and those who were good at sports. (A great tragedy of my youth was I couldn’t hit a curve ball!) I found, however, that I seemed to be able to think more abstractly than most. When I learned about science and mathematics as a teenager, I discovered that the manipulation of abstract objects, their construction, and their rearrangement, were things I could do very well. Exercising this talent has always been a joy...

I find great satisfaction in creativity, for I then feel a kinship with artists and scientists the world over. A Matisse exhibit thrills and inspires me. I rush home and attack my own research problems with zest, feeling I am part of the world of Matisse. A good ballet affects me the same way. I love it and am inspired to go home and try to do my little bit—like a juggler before the gates of heaven.

Isadore Singer (On His Early Years)

(As told to Hugo Rossi)

I was in Luzon when the war ended [in 1945], and nine months later I was shipped home by boat and train—that is to Chicago, where I was discharged. Before returning
home, I thought it would be a good idea to stop by the University of Chicago to see what it would take for me to become a graduate student in mathematics or physics. After talking with some people in the Math department, I was told, “You are now a graduate student in Mathematics—show up here in September.”

Three years later I received my PhD—my advisor was Irving Segal, and the subject was mathematical physics. I was offered a visiting assistant professorship at MIT. Paul Halmos suggested that I get in to see Warren Ambrose when I got there late that summer. I didn’t know it at the time, but he had written to Ambrose to tell him, “look out for this kid—you’ll like him.”

When I arrived at MIT late in early September, I looked for Ambrose. When I walked in his office, he looked at me and said, “You’re Singer. I’m Ambrose. Sit down and let’s talk.” And we started talking, but about 15 minutes into that I told him that I had better go to the Math office to meet the chair and associate chair. “Nonsense,” Ambrose said, “There’s plenty of time for that. What’s going on here is what is important.” So we talked in his office for an hour and a half, and then Ambrose said, “I’ve heard that Chern is doing some fantastic stuff—completely changing geometry. Tell me about it.” I said that I had little knowledge of what Chern was doing; I was a student of Segal—a mathematical physicist. Ambrose retorted, “You’re from Chicago, that’s good enough for me. You’ll teach me Chern’s differential geometry this Fall.”

Ambrose, Halmos, and I were abstract analysts of one stripe or another, that’s what connected us. But here was Ambrose wanting to be in on what was going on that was really new, and I was his link to that. So, Ambrose and I were abstract analysts, but he saw, and I accepted, that the action at the time was in geometry. I told him that I’d prepare weekly meetings on Chern’s geometry. He said, “Where have you been since you got here?” I answered, “North Station and the train here.” He said, “Let me show you around.” I protested that I didn’t want to take up his time, and he said, “My time and your time are not at stake—when you get up tomorrow morning, it will all be there. But Boston is here now and you have to get to know Boston, if you want to do mathematics at MIT.” I had no choice.

We drove around and talked. Ambrose was infatuated with what was new and profound. His gut feeling was that Chern’s geometry was right at the forefront and will finally make geometry understandable to mathematicians. As it turns out, we had a couple of great years and came to understand and further develop this new technique in geometry. Some time later, Ambrose was asked: “how would you define geometry?” His response was: “Geometry is the study of things that are invariant under a change of notation.” Chern and a few others knew what he meant. I was one of them.

Ambrose was a taciturn fellow, uninterested in conventions. If you came up to him and said, “Hello, Warren, I’m…,” He would just walk away. He was Ambrose, not Warren Ambrose, not Dr. Ambrose, and not even ‘hey, you.’ Just Ambrose. He once wrote a review of a paper, saying that “it filled a much needed gap in the literature.” That was his whole review.

In any event, we drove all around Boston, up and down the coast and stopped to see something that would have been just a tilted post, until Ambrose explained what it was. We ended up late in the evening, at a coffee shop—Jack and Marion’s. Ambrose explained that this was a hangout for the wives of mathematicians, whose husbands were in their studies proving theorems. He said that we could do our mathematics there, benefiting from the energy propelled by the women.

Indeed, when we got there, he took me over to a table and introduced me to Mrs. X, Y, Z, and so forth. I was in awe to meet the wives of these famous men—who I had not yet met! Was Ambrose trying to tell me something?

After two years at MIT as a visiting Assistant Professor, I accepted a position at UCLA as a tenured Associate Professor. I went back to abstract algebra and happily and excitedly worked with [a colleague] at UCLA. We did great stuff, but it didn’t have significant impact on mathematics. The work of “Ambrose–Singer” did. In 1957, I returned to MIT—not just to work with Ambrose, but to develop my own approach to geometry—especially as a tool to understand physics, both macro and micro.

In 1974, the Soviet Union opened up enough to hold a meeting of differential geometers in Novosibirsk. The main theme of the conference was the impact of Atiyah–Singer and Chern–Simons. I had not been invited, so I contacted the organizers (friends I had developed over the years) to remind them that I was the Singer of Atiyah–Singer. The response I received was that there were political issues, but they’d get me an invitation.

As they did. It was wonderful—I gave a half-hour talk, and the following discussion was another half-hour (that conference was advertised as taking place from 2pm to 6pm, but, every day, went on until midnight). It turned out that the issue was that I am a cousin of Isaac Bashevis Singer—and therefore a threat in their eyes.
Natasha Singer

My father Isadore, known to his colleagues and friends as ‘Is,’ passed away in February, and with him went the first person who taught me how to tell a story.

During his 50-year career as a mathematician at M.I.T., he produced and collaborated on major research discoveries that helped catalyze developments in both math and physics.

But, to his family and students, my dad was first and foremost a teacher.

For years, professors and students in the Boston area would gather for his weekly seminars at M.I.T. where he hosted mathematicians and physicists presenting their latest ideas on string theory and other fast-evolving science topics. The seminars always ran late, largely on purpose, so that students and faculty could go out together afterward for Chinese food and continue the discussions.

Even after my father became an Institute Professor at M.I.T., a status that freed him from formal teaching duties, he continued working with undergraduates. For several years toward the end of his career, he even volunteered to work as a teaching assistant for a freshman course: first-semester calculus. The gig appealed to him, he said, partly because T.A.s in the course were able to freely mentor students—without needing to test or grade them.

“The students understood that I was there to help them and not to judge,” he said in an interview in 2010, adding that he thought empathy was the key to being a good teacher.

My father often said that math was “just one of the humanities” and he had a gift for explaining the intricacies of science in simple terms to non-scientists. One time when I was in sixth grade, I remember having dinner with him at a restaurant in Harvard Square and asking him to explain how the telephone worked. Delighted, he quickly launched into a lesson that started off with an ode to Marconi’s discoveries on radio waves, covered Alexander Graham Bell’s work on electrical currents and ended with a caution on government wire-tapping.

Along the way, he grew so animated, using his arms to illustrate the undulating electrical currents, that people at neighboring tables began to stare at us. I was 11 years old at the time and recall feeling embarrassed by the attention.

When the science lesson ended, however, a stranger at the table next to us immediately leaned over to my dad. “That was amazing!” she said in a loud whisper. “What are you going to explain next?”

It wasn’t until much later that I absorbed the real lessons of that evening: that understanding how science and technology work can be a powerful tool—and that explaining the inner workings of tech power to others can be a public service.

Nancy Stanton

I met Is Singer the first day of his Complex Manifolds course in the spring of 1970. He came in the first day and said, “For those of you enrolled in the course, I require no work. I would like volunteers to write up notes.” I volunteered—and worked very hard. By the middle of the semester, it was clear that I would be one of Is’ ‘complex manifolds’ generation of students.

Several things stand out about his course. He introduced numerous important examples in the first few classes, so we would not just learn theory without knowing what it applied to. When possible, he introduced multiple ways of thinking about concepts, usually with a theorem starting “The following are equivalent,” and communicated that which way was best depended on the question. The course continued for several semesters, bringing a group of students to the point that we could do research in the area.

One of my nonmathematical memories from the first semester of the course is chatting with Is on the march from MIT to Boston Common for an anti-Vietnam War rally in the spring of 1970. Is wore his banker suit to help make sure the protesters wouldn’t be described as just some radical students. Is’ son Elliot, who was an MIT undergrad at the time, refused to march next to Is because he would be embarrassed to be seen with a ‘banker,’ but he did give Is a red armband to wear on his suit.

Nancy Stanton is a professor emerita of mathematics at Purdue University. Her email address is stanton@nd.edu.

Natasha Singer is a tech accountability reporter at The New York Times and a Knight Science Journalism Fellow at MIT.
From Is I learned the importance and beauty of bringing together different areas of mathematics. The interplay and relations between different areas of mathematics and also physics was a theme throughout his research. As an advisor, he made sure his students worked on problems which involved several areas of mathematics so we would not become narrowly focused, but also so we would appreciate the unity of mathematics. The interplay between complex geometry, several complex variables, and partial differential equations became a theme through much of my work.

In the years after I received my PhD, Is inspired me to work on serious, interesting hard problems. When I struggled with a difficult problem he always encouraged me to continue working on it and made me feel that it was tractable. His strong encouragement was crucial to me at many points of my career.

I have very fond memories of Is and his wife Rosemarie welcoming Is’ former students to their house for dinner during the 1999 conference in honor of Atiyah, Bott, Hirzebruch, and Is. I last saw Is when he and Rosemarie again warmly welcomed us, along with other attendees and colleagues, to a garden party at their house at the end of the 2009 conference in honor of Is’ 85th birthday.

I am very happy to have been Is’ student, thankful for his teaching me to bring together different areas of mathematics and to look at the big picture, and grateful for his inspiration throughout my career.

Shlomo Sternberg

At the time of our collaboration, Singer was an established mathematician. See for example, the Ambrose–Singer theorem and others. I had proved then one big theorem. The Linearization Theorem. We started working together on Cartan’s theory of Infinite Lie Groups [4]. We worked every evening in the living room of our first rented apartment in Brookline, Mass. These sessions lasted very late into the night.

Is was devoted to his son and brought him to our home for regular visits. I particularly recall our evening together with this young son at our Passover Seder. Is was a joyous man. This joyousness that he so clearly brought to his mathematics was always there. It was part of who he was.

References

Credits
Figure 1 is courtesy of MIT.
Figures 2–7 are courtesy of Natasha Singer.

Shlomo Sternberg is a professor emeritus of mathematics at Harvard University. His email address is shlomo@math.harvard.edu.
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Apply on MathPrograms.org
“In every job that must be done
There is an element of fun
You find the fun and snap!
The job’s a game” \[1\].

In the iconic “Spoonful of Sugar,” by the Sherman brothers, Mary Poppins invites the Banks children to overcome the daunting task of cleaning their room by finding the fun within. Robert Sherman got the idea for the song when informed by his son that he had swallowed the newly developed polio vaccine on top of a cube of sugar. We might wish that such a tactic could help overcome the vaccine hesitancy of our times. Regardless, the song continues to appeal to our human desire for play as one means of addressing the difficulty and drudgery of our days.

Francis Su has identified play as one of the fundamental human desires that reveal our mathematical natures and that help build virtues that lead to human flourishing \[2\]. Virtues that Su identifies as awakened by play include hopefulness, confidence in struggle, patience, and perseverance. Each of these has been made more vital for human flourishing due to the challenges of facing a deadly global pandemic. Su asks his readers how we would teach mathematics differently, if we thought of it as a playful sport rather than a performance sport.

A rich resource for answering Su’s question is provided by *Teaching Mathematics Through Games*. The audience for this book is college-level math instructors who desire to add a bit of fun to their curricula. The goal is to elicit more enjoyment and engagement in courses ranging from general education through upper-level electives, including independent research projects.

There is a long history of connections between mathematics and games. Games have been more traditionally used in teaching K–12 mathematics, and this book is an effort to expand that into college-level mathematics. The book is part of the MAA Classroom Resource Materials series. As an MAA publication, its pedagogical sensibilities are informed by the MAA’s *Instructional Practices Guide* \[3\]. The emphasis is on collaborative learning, classroom community, persistence in problem solving, and inquiry-based learning.

The editor, Mindy Capaldi, is a Professor of Mathematics and Statistics at Valparaiso University whose research interests are in mathematics education and the scholarship of teaching and learning. Capaldi, along with her undergraduate coauthor, Timothy DeRolf, contributed the final chapter of the collection on using the game Carcassonne to explore mathematics. For those, like me, unfamiliar with the game, Carcassonne is a tile-based game, named after a medieval French fortified town famed for its city walls, in which players randomly draw tiles and lay them out to earn points by building cities, roads, monasteries, and fields. Capaldi describes how she has used Carcassonne for many years to reinforce ideas from probability, Bayes’ Theorem, and expected value in a Finite
Mathematics course. DeRolf follows up with a description of an undergraduate research project in which he explored ideas from abstract algebra and topology in the context of the game. This chapter exemplifies the diversity of curricular levels that the book addresses.

The other sixteen chapters are collectively written by twenty-nine authors who represent a variety of institutional sizes and missions, including community colleges, liberal arts colleges, and regional state universities. The authors are mostly mathematics faculty members with a few mathematics education faculty in the mix. Many, although not all, are self-professed “gamers.” I certainly would not describe myself as such, although like most people, I enjoy playing games. Moreover, like most mathematicians, and if this book is successful, many math students, I enjoy trying to understand the strategy and structural underpinnings of games that I play.

According to the preface, “The primary goal of Teaching Mathematics Through Games is to inspire instructors of mathematics as well as present the tools necessary for them to incorporate games into their classrooms.” The chapters are self-contained, and as such can be used independently by the reader for either inspirational or practical purposes. However, the editor encourages the reader to read all of the chapters, “as you may be surprised at what inspires you.” I would recommend this as well, as in doing so I learned a variety of enjoyable and useful tidbits about games, mathematics, and pedagogy.

With a focus on the practical as well as the inspirational, the “chapters are intended to allow for easy adoption of materials with as little work as possible required by an instructor.” A key feature of the material is that all of the resources are classroom-tested. The authors thus credibly describe how to implement their ideas and are able to point out potential pitfalls along the way.

Another attribute of the collection in service of usability is its variety. The chapters are organized in order to proceed from topics with broader appeal to more specific, upper-level mathematics courses. In addition to varying curricular levels, the chapters are also varied in their representation of type of game, mathematical topic, and length of activity.

Games discussed include commercial favorites (at least in our household), such as Risk®, Apples to Apples®, and SET®. Other familiar games that are represented range from the childhood favorite of Tic-Tac-Toe, to several casino games such as Roulette and Keno, to the ubiquitous Sudoku. There are even games developed by the authors for exploring abstract mathematical ideas (Bandits on a Wall by Michael P. Allocca and Jennifer Franko Vasquez) and for developing quantitative literacy while exploring structural inequities (Fantasy Forest by Vivian Y. Lim and colleagues).

You may expect any collection exploring mathematics and games to include ideas from probability, Markov chains, game theory, combinatorics and graph theory, and this book is no exception. Other mathematical topics featured in the book include calculus, number theory, eigenvalues/eigenvectors, group actions, symmetries, data analysis, and hypothesis testing. The intended duration of the activities ranges from one class period (representing about half of the chapters), to semester-long projects, with a few in between. The section “How to Use This Book” includes a helpful roadmap so that you find what interests you based on the course, length of activity, and mathematical topic.

Another aspect of the book that aids its usability is the common structure employed throughout the chapters. Each chapter begins with an introduction and overview introducing the game and the context in which it has been used by the author(s). This is followed by a section in which the game is more fully described and an outline of how the game can be used in the relevant course(s) is provided. Each chapter ends with a conclusion and suggestions for additional directions for exploration. The hope is that the reader will be able to use the materials as ending points or starting points, by “expand[ing] upon an idea by using a different game or applying the given game in a new way.”

Some chapters include further aids to classroom implementation in the form of appendices and/or supplemental materials available via a link from the book’s page on the AMS bookstore. The supplemental materials range in scope from a one-page handout (for the chapter on analyzing Tic-Tac-Toe) to a zip file containing several files such as assignments and implementation notes (for using and building upon Fantasy Forest to develop quantitative literacy).

The book’s greatest strength is its breadth. For a slim (159 pages) volume, the amount of material included is considerable. Moreover, its inherent variety gives it broad appeal. It ranges in complexity from the simple familiarity of Tic-Tac-Toe to open research questions in mathematics that would be of interest to many AMS members.

Some of my favorite chapters were those addressed to “gamifying” pedagogy by mimicking well-known and popular games in order to develop mathematical communication skills. The chapter by Jacob Heidenreich uses the game Battleship as a model for helping students correctly use mathematical language and build conceptual understanding when studying functions. The chapter by Kayla Blyman and Marie Meyer uses the game Telestrations® as a model for helping students represent and communicate mathematics in multiple ways, with examples of implementation in a wide variety of courses. The commercial version of Telestrations® is a favorite of my family, because it is almost always guaranteed to produce shared fits of
side-splitting hysteria. I doubt that the in-class version would have quite the same result, but it gives me confidence when the authors attest that it has the effect of almost “tricking their students into learning because they are having so much fun playing the game.”

Much of the material offered here meets the stated goal of ease of use. However, the one critique I would offer is that the book is somewhat uneven in that regard. It seems to me that some of the chapters with more sophisticated and less familiar games (e.g., Arkham Horror), deeper mathematical ideas (e.g., topological linear algebra), or more extended and intricate in-class implementation (e.g., a semester-long project exploring structural inequity through a simulation game) are not quite “plug and play.” Nevertheless, interested readers with the virtues of patience and perseverance will find rich rewards throughout.

The book description on the back cover asserts that “Active engagement is the key to learning,” which is a well-attested claim [4]. The book’s cover art nicely evokes the connections between mathematics and games, with its call-out to Chutes and Ladders® using images of game implementations and mathematical objects in place of the childhood activities shown in the commercial version. In addition, it demonstrates the authors’ shared philosophy that teaching mathematics through games will promote curiosity, stimulate student questions, and support active and engaged learning, with the three characters shown in turn considering the outcome of a dice roll, wondering about a hand of cards, and reaching towards the next higher level of mathematical play.

In fact, look closely! There are no chutes, only ladders. Thus, the book provides a resource for inviting our students higher up and deeper into the “game” of mathematics. I look forward to using many of the ideas and resources presented here to bring elements of fun to my courses in hopes that my students will, “snap!,” find learning mathematics to be an unexpectedly sweet game.

References
Anachronisms in the History of Mathematics: Essays on the Historical Interpretations of Mathematical Texts
Edited by Niccolò Guicciardini

We’re likely all familiar with the word “anachronism” referring to something that is not in its correct historical period. From a historical standpoint, “anachronism” can also refer to instances in which work has been mistranslated or incorrectly evaluated.

One such instance occurred in 1906 when Roberto Bonola wrote a book which characterized Nikolai Lobachevskii’s and János Bolyai’s work as traditional elementary geometry. Bonola failed to recognize that their work was revolutionary in its time and overlooked Lobachevskii’s discovery of formulae associated with hyperbolic trigonometry. Bonola’s book devalued the work of Lobachevskii and Bolyai, along with that of other early contributors to non-Euclidean geometry, such as Gauss. Consequently, from Bonola’s perspective, Riemann’s contributions to the field appeared even more revolutionary than they were. Anachronisms in the History of Mathematics is a collection of case studies like this one, each of which discusses an instance where historical mathematics has been misrepresented.

Another type of anachronism can occur when concepts in modern mathematics are incorrectly attributed to historical figures. This idea is explored with respect to Euler’s work on infinite series, particularly the series $1 - 1 + 1 - 1 + \ldots$ which Euler thought of as $\frac{1}{1-1} = 1 + x + x^2 + x^3 + \ldots$ where $x = -1$. This representation led him to conclude that $1 - 1 + 1 - 1 + \ldots$ was equal to $\frac{1}{2}$. Using the rigorous methods developed by Cauchy, a Calculus II student would be able to explain why this series is divergent. However, in the late nineteenth century, as Cesàro (and later Hardy) developed summability theory to provide methods for finding the sum of divergent series, Euler’s work on divergent series would be recalled. According to Hardy, Euler was a mathematician ahead of his time and his work was the inspiration behind some techniques in summability theory. The author of this chapter cautions this line of thought, arguing that it is an anachronism to say that Euler laid the groundwork for summability theory, since this ascribes insights and beliefs to Euler that he did not actually hold.

These and other anachronisms, both within and outside of Europe, are considered in this collection. Mathematicians without historical training may find this a challenging read that is well worth the effort! Historians (and certainly historian of mathematics) will find this an interesting and thought-provoking book.

Coming Home to Math: Become Comfortable with the Numbers that Rule Your Life
by Irving P. Herman

A concern (and, if I’m being honest, frustration) of mine is how many people openly admit to hating or being terrible at math and how socially acceptable it is to proclaim those feelings. My guess is that many of us share that concern and, like me, are trying to find ways to combat these feelings and the culture they promote. Coming Home to Math is written for just this purpose and would make a great gift for any math-phobic people in your life who want to overcome their fears of math.

The book starts with a description of arithmetic operations and builds up from there. It has chapters dedicated to exponential growth and decay, probability, statistics, and optimization. Within these chapters topics such as compound interest, mortgages and annuities, the normal distribution, ranking and voting systems, and financial investments are discussed. An interested reader can skip around the book and focus on the topics that interest them most or read it from cover-to-cover. This is a friendly and unintimidating book that works to empowers individuals to feel comfortable working with numbers, thinking about what they mean, and hopefully, remove the phrase “I’m not good at math” from their lives.
The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world’s leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit bookstore.ams.org to explore the entire collection of AMS titles.

A Conversation on Professional Norms in Mathematics
Edited by Mathilde Gerbelli-Gauthier, Pamela E. Harris, Michael A. Hill, Dagan Karp, and Emily Riehl

This collection of essays paints a broad but nuanced picture of the current state of the mathematics profession. Its authors address today’s compelling issues, highlight areas where improvements are needed, provide individual observations, and describe creative and effective initiatives that have been put into practice. Topics include access to education and job opportunities, effective mentoring and communication, equitable work conditions, safety from harassment, and the relevance of mathematics in dealing with larger challenges facing the world.

The tone and perspectives of the essays found in this book form a rich and diverse tapestry. One essay describes the tremendous luck that seemed necessary to build a mathematical career, another brings attention to the trials of toxic interactions with advisers, from an unwelcoming environment to outright harassment. Several essays bring new language into the discourse, examining the value of both “congressive” and “ingressive” personalities in mathematics, understanding the meaning of intersectional feminism through a mathematical analogy, or discussing practical techniques for better teaching and effective communication. Two essays address complacency in academia, with one questioning the level of fairness through which privilege, compensation, and security are granted to faculty of different ranks, and the other arguing that universities are not adequately recognizing and dealing with the imminent dangers brought by climate change and pandemics.

This book provides an opening for conversations about the norms that are often, wrongly, taken for granted. To this end, the book presents reflections, snapshots, questions, and ideas, some of which may resonate and others challenge or even provoke. In either case, the book provides fodder and inspiration for thoughtful discussions and is worth having on the shelf as a resource for students, faculty, and administrators to read and periodically revisit as they transition through the different stages of their mathematical careers.

A Festival of Mathematics
A Sourcebook
By Alice Peters and Mark Saul

The Julia Robinson Math Festival (JRMF) is a non-profit organization founded by Nancy Blachman that aims to spread the joy of mathematics to middle and high school students. The hands-on non-competitive mathematical activities provide a welcoming and inclusive alternative for curious and talented students who may be put off by the more prevalent math competitions at many schools. JRMF events typically take place in a school cafeteria or gymnasium, where students can freely wander around and choose from amongst a variety of activities set up at tables.

The activities are designed to draw students in with something simple yet enticing, like arranging colored disks according to a set of rules or designing your own magic squares from an incomplete template. The philosophy of JRMF is for facilitators to interfere as little as possible as students work through the carefully crafted problems. For students who persist in one activity, the sequence of problems leads them to increasingly deep and sophisticated developments.

This Sourcebook, assembled by two longtime organizers and proponents of the JRMF philosophy, is an accessible and easy-to-implement guide for individuals who would like to host their own JRMF events. Each chapter lays out explicit step-by-step sequences of problems that advance gradually and compellingly to increasingly high levels. Historical background, mathematical context, and pedagogical motivation are inserted as needed in concise and timely snippets. Interested students of all ages may also enjoy reading the book on their own.

The AMS Bookshelf was contributed by AMS Book Acquisitions Consultant Eriko Hironaka. For questions or comments, please send email to acquisitions@ams.org.
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For more information and to apply, please go to www.ams.org/ams-congressional-fellowship.

Deadline for receipt of applications: February 1, 2023

“...I wanted to find ways to utilize my mathematical skills to affect real world change, but I doubted whether any of my mathematical skills could be applied outside of academia. The AMS Congressional Fellowship has removed all of my doubts. I have learned in the past year that there is overwhelming need for mathematicians in all areas of government. The skills I have as a mathematician are highly valued in Congress and I became an integral part of a Senate office assisting in housing, economic development, and consumer protection. The Fellowship has allowed me to use my skills to help make the world a better place and I feel immensely privileged to have had that opportunity.”

— A.J. Stewart, AMS Congressional Fellow 2021–2022
Intriguing patterns emerge when you create a plot in polar coordinates of the points \((r, \theta) = (p, p)\) for prime numbers \(p\). At an intermediate scale, the points seem to fall in 20 spiral arms. Zoom out, and the spirals give way to 280 nearly straight rays arranged mostly in clusters of four. What causes the patterns? That was the hook of a 2019 video by Grant Sanderson, one of the most successful math communicators on the internet.

On his YouTube channel 3Blue1Brown, Sanderson brings non-experts into the world of mathematics through compelling visualizations and clear explanations. The prime spirals video, which amassed 3.7 million views, explored rational approximations of \(\pi\) and Dirichlet’s theorem, all with a friendly tone and minimal jargon. Inviting viewers to play with numbers, Sanderson emphasized that “frolicking around in the playground of data visualization” and asking seemingly superficial questions can easily lead to deep mathematical concepts.

With more than 4.5 million subscribers, 3Blue1Brown shows that online videos about math can attract an eager audience beyond the typical boundaries of the mathematics community. And Sanderson is not alone. The past decade has seen the rise of many successful YouTube video creators—or “YouTubers,” as they are commonly called—who share the joys of both recreational and research mathematics through imagery, stories, humor, and interviews.

In a sense, math YouTubers are the new Martin Gardner, inspiring the public just as Gardner’s “Mathematical Games” column in Scientific American once did. In 2021, 81% of Americans, including 95% of those ages 18–29, used YouTube [AG19], the most popular social media site in the country. The creators featured in this article hail from the US, Australia, and the United Kingdom, and their reach extends farther: The majority of their viewers reside outside the US, with substantial contingents in India and various European countries.

These YouTubers aren’t reaching everyone. Their viewership skews heavily male, a gender disparity that extends to STEM YouTube videos in general [Lan21]. People of all ages watch math videos, but the largest audience segment comprises 18- to 24-year-olds. Even so, “it’s probably true to say that more people get attracted to mathematics through YouTube than through anything else at the moment,” says Burkard Polster, known on the platform as the Mathologer. “Forget about schools, forget about uni, […] it just pales into insignificance when you compare it to the impact that [YouTube has].”

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Scott Hershberger is a master’s student in science communication at the University of Wisconsin–Madison. When he wrote this piece, he was the communications and outreach content specialist at the AMS. His email address is scotthersh42@gmail.com.

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DOI: https://doi.org/10.1090/noti2559
Elaborate Animations or Whiteboard Drawings

Many YouTubers begin their channels as side projects. Sanderson started 3Blue1Brown for fun while studying mathematics and computer science at Stanford University. After completing his bachelor’s degree, he spent a year creating online instructional content for Khan Academy before diving into 3Blue1Brown full-time in 2016.

3Blue1Brown video topics range from puzzles like computing \(\pi\) using colliding blocks and solving Wordle using information theory to major areas of mathematics like Fourier series and linear algebra. In all cases, Sanderson sees his work as complementary and supplementary to traditional math education. “[The channel is] not meant to substitute for any actual class that someone’s taking,” he says. “The usual aim is that it helps more people self-identify as loving math, whether […] they already kind of do and it fans those flames, or if it can spark that interest in some way.”

As he writes a script for a video, Sanderson makes sure to lead with concrete examples before moving into abstract concepts—a structure that entices viewers’ curiosity and guides them toward true understanding. He typically recruits a friend or two for a sample lesson to gauge the effectiveness of explanations and calibrate the right level of detail. In his delivery, Sanderson avoids the rapid-fire, almost breathless pace of many viral videos. He instead speaks slowly and deliberately, giving viewers plenty of opportunities to “pause and ponder.”

3Blue1Brown videos feature intricate animations of diagrams and equations, often interspersed with the reactions of anthropomorphized \(\pi\) creatures. Sanderson uses his own custom Python library to generate the animations programmatically. While this unique approach yields an eye-catching and polished visual style, he points out that visuals don’t need to be fancy to be effective.

Mithuna Yoganathan is one YouTuber who embraces a simpler style, filling the screen with colorful diagrams and equations hand-drawn on a whiteboard. Her channel, Looking Glass Universe, originated as a place for her to solidify her learning of quantum mechanics concepts by explaining them. She cites fellow video creator Vi Hart, a winner of the 2018 Joint Policy Board for Mathematics Communications Award, as a major influence on her early videos.

Yoganathan posted occasional videos throughout her PhD in quantum computing, finding an audience of hobbyists eager to learn alongside her. “They don’t necessarily already know quantum mechanics, but they’re pretty comfortable with doing maths and comfortable with watching longer videos,” she says. Yoganathan keeps her explanations less technical than a typical lecture—for example, she might disregard the normalization constants in wave functions or omit the entries of matrices. Still, she finds that her viewers want more details than are included in hand-wavy popularizations elsewhere.

In fact, one of the most popular Looking Glass Universe videos is a guide to self-studying quantum mechanics, which inspired viewers ranging from high school students to a retired biochemist. In other videos, Yoganathan speaks candidly about the nature of mathematics research and the winding path of her PhD work at Cambridge University. She shifted to making YouTube videos full-time after earning her doctorate in early 2021. Now, Yoganathan is producing a series of videos demonstrating how to do “Quantum Experiments at Home.”

“I find it a very enriching experience to be on YouTube,” she says. “I’m making very niche videos about quite technical topics in quantum mechanics, and I can still get a decent audience for that, which is only possible because of the scale of YouTube.”

Inspiring Students to Pursue Math

Sanderson and Yoganathan each write, record, and edit most of their videos single-handedly. By contrast, Numberphile, one of the longest-running YouTube channels...
about math, has found success with a rotating and ever-expanding cast of guest mathematicians. According to video journalist Brady Haran, who films and produces Numberphile, well over 100 mathematicians have starred in his videos since the channel’s 2011 debut.

Since 2013, the Mathematical Sciences Research Institute (MSRI) has sponsored Numberphile, and Haran frequently travels to Berkeley to interview the mathematicians who visit MSRI. When preparing to film with a new presenter, “We just sit down over a cup of tea and talk about it. The key is that the person feels relaxed,” he says. “Most mathematicians are better at explaining their work than people give them credit for—the bigger challenge is easing the nerves that a camera can generate. That’s often about creating trust and a rapport.”

Haran has worked on more than a dozen YouTube channels throughout his career, including still-active channels on chemistry, physics, and astronomy. Compared to those other projects, the biggest challenge for Numberphile is the lack of something “spectacular” to display on camera, he says. “Astronomers can show me pretty galaxies, chemists can perform explosive experiments, but mathematicians are often limited to a pen and paper.”

Yet a thick pen and a large roll of brown paper, the visual hallmark of Numberphile videos, turn out to be plenty to capture the imaginations of viewers. Numberphile is by far the most successful of Haran’s projects, with a cumulative 650 million views as of April 2022. Teachers around the world show Numberphile videos in their classrooms. When students in one class met frequent Numberphile presenter James Grimes at school, they said it was “like meeting a rock star.”

YouTube was launched in 2005, making it the same age as today’s high school seniors—so the newest wave of math YouTubers can testify firsthand that the platform galvanizes students to pursue STEM. “Watching the very early YouTubers […] really did inspire me in high school to […] look into science in the first place,” says Toby Hendy. Hendy went on to major in physics and mathematics at the University of Canterbury in New Zealand, then began a PhD program in biophysics. All the while, she made YouTube videos for a growing audience. Less than a year into her PhD, Hendy realized that her passion lay in science communication rather than science research. She left academia to create videos full-time on her channel Tibees.

With a calm, soothing demeanor, Hendy analyzes math exams from around the globe and shows documents from the lives of famous mathematicians and physicists. When some commenters described her as “the Bob Ross of math,” she leaned into the comparison to the host of *The Joy of Painting*. On a windy day, she lugged a chalkboard, easel, camera, and tripod to the top of a scenic hill. There, she explained logarithms “Bob Ross style,” complete with drawings of trees and logs. Other videos in the series—filmed at an overlook, in a forest, or by a lake—explore Euler’s identity, parabolas, and sine waves.

Hendy continued bending genres with her most ambitious project to date, “Finding X.” Narrated as a poem and featuring animations made from felt, the allegorical short film tells the story of x’s search for their mathematical—and human—value. Like Hendy’s Bob Ross–style videos, “Finding X” struck a chord with viewers who previously associated math with stressful experiences.

“If this had been mandatory viewing when I was around 10 years old - I wouldn’t have dreaded mathematics existentially, fearing continuous failure and wrong solutions,” wrote one commenter. “Thank you so very much for this beautiful new narrative [of] what mathematics is actually about: creative exploring of all the patterns and relations we can think of!”
Figure 6. In one of the most ambitious Mathologer videos to date, Burkard Polster presented the proofs that $\pi$ and $e$ are transcendental.

Funding the Creative Process and Building Community

Writing, filming, editing, promoting—releasing high-quality videos on a regular basis is a time-consuming endeavor. To turn their passion into a profession, creators typically depend on funding sources beyond ad revenue. Many math YouTubers accept sponsorships from educational organizations or crowdfund their work on a platform such as Patreon. Both Yoganathan and Hendy won grants from Screen Australia and Google Australia to carry out their latest projects.

But not all successful math YouTubers create videos full-time. In addition to running his YouTube channel Stand-up Maths, Matt Parker gives public talks, performs stand-up comedy, and writes popular books about math. Burkard Polster creates Mathologer videos between research and teaching as an associate professor at Monash University—a juxtaposition of pursuits made possible by the support of his colleagues and institution.

Mathologer videos do not shy away from mathematical detail, so typical viewers hold at least a bachelor’s degree in mathematics or engineering. Polster frequently engages with his audience by proposing challenges related to his videos. For example, an invitation for viewers to code interactive apps for modular multiplication circle diagrams yielded at least three dozen submissions. “I think the world is a richer place now that people can just go there and play with these apps,” Polster says.

Some viewers’ contributions may shine, but other remarks dampen the glow. Like other social media sites, YouTube struggles with abusive behavior, especially toward female creators [AA21]. Negative comments “can be very personal in nature, attacking my appearance or personality,” Hendy says. “My job has required me [to form] a ‘thick skin’ with regards to hateful comments and [do] what I can to protect my privacy.” In the comments of a Mathologer video about why $0.9999\ldots = 1$, Polster even received death threats.

For Hendy, positive interactions far outweigh the negative ones. Yoganathan similarly prioritizes community-building on her channel, often posting videos that directly respond to viewers’ conceptual questions. She says that this give-and-take fosters collaborative learning and even enhances her own understanding of quantum mechanics.

Math Communicators Rise and Unsolved Problems Fall

Following the path pioneered by Numberphile, 3Blue1Brown, and others, creative and skilled communicators continue stepping up to fill the public’s appetite for engaging video content about mathematics. In 2021, Sanderson of 3Blue1Brown held a “Summer of Math Exposition” to encourage people to explain math online. He received a staggering 1,200 submissions, most of them videos. Undergraduates, graduate students, educators, researchers, engineers, and even video game developers took part.

Sanderson coordinated a peer review process to evaluate submissions, and then he assessed the top 100. “One of the coolest outcomes of it was that even before I announced the ones that I […] chose as winners, many of them had actually picked up a lot of traction on YouTube,” he says. The winning videos exposited on Bézier curves, lock patterns on smartphones, Pick’s theorem in geometry, and the envelopes of light in coffee mugs. At press time for this article, a second Summer of Math Exposition was underway.

In at least one instance, a series of YouTube videos for a broad audience directly led to new mathematics research. A 2015 Numberphile video explored the conjecture that all positive integers (except those with remainder 4 or 5 when divided by 9) can be expressed as the sum of three cubes of positive or negative integers. At the time, such expressions were known for all numbers below 100 except for 33, 42, and 74. Viewer Sander Huisman, a physicist, was inspired to write a computer program to search for solutions—and he succeeded for 74 [Hui16].

The follow-up Numberphile video, “74 is cracked,” spurred another researcher to develop new algorithms that identified a solution for 33 [Boo19]. Finally, the expanding group of collaborators found a way to write 42 as the sum of three cubes in 2019 [Mil19]. “We started with this one video and just released it out into the wild and said, ‘Look at this interesting thing’,” Haran says. “As a result of it, […] all these amateur [and] professional mathematicians—people who may never have met—have been working on papers together.”

2https://www.3blue1brown.com/blog/some1-results.
The original conjecture remains unproven. But if and when a proof is found, Haran will record a video about it. “I would love it if someone who watched the Numberphile videos is the person who proves it.”

As with any craft, there is no single best approach to math communication. Polster used to improvise his explanations based on an outline, but he now follows scripts that he writes; meanwhile, Sanderson is moving away from full scripts in favor of outlines. Similarly, a YouTuber must decide with each project whether they hope to create “the perfect video” for a small audience or “a pretty good video” for a larger audience, Sanderson says.

Research mathematicians who want to join the ecosystem of math communication on YouTube don’t have to start a new channel from scratch. An appearance on Numberphile is one possible point of entry. Sanderson invites researchers to connect with him as well. “More active collaboration with mathematicians is something I would greatly welcome,” he says.

Hendy concurs: “Having partnerships between research mathematicians and us YouTube science communicators, I think, makes the platform better for everyone.”

A list of links to the YouTube channels and videos mentioned in this article is available at https://www.ams.org/about-us/scotthershberger.

References
[Mil19] Sandi Miller, The answer to life, the universe, and everything, Massachusetts Institute of Technology, September 2019.

Credits
Figures 1 and 2 are courtesy of Grant Sanderson.
Figure 3 is courtesy of Mithuna Yoganathan.
Figure 4 is courtesy of Brady Haran.
Figure 5 is courtesy of Toby Hendy.
Figure 6 is courtesy of Burkard Polster.
Author photo is courtesy of Scott Hershberger.
The tenure system is a ubiquitous feature of American higher education. Of the roughly 1,400 four-year institutions that the Carnegie Classification system categorizes as bachelor’s, master’s, or doctoral institutions, 87 percent reported in the 2020 Integrated Postsecondary Education Data System (IPEDS) survey, conducted by the US Department of Education’s National Center for Education Statistics (NCES), that they have a tenure system. From the perspective of AAUP policy, having a tenure system is essentially just a matter of granting tenure, which the AAUP defines as an indefinite appointment terminable only for cause or under extraordinary circumstances, such as financial exigency. However, the tenure system is now generally identified with having a system of reviews that lead to a tenure evaluation.

While tenure practices vary among institutions, systematic studies of these variations are rare. The institutional survey component of the National Study of Postsecondary Faculty (NSOPF), also conducted by NCES, included questions about tenure practices the four times it was administered between 1988 and 2004, and Cathy A. Trower’s 2000 study of faculty handbooks and collective-bargaining agreements, Policies on Faculty Appointment, analyzed the prevalence of several additional features of the tenure system at that time. However, not only are these existing studies dated, but they do not address features of the tenure system that have come into focus more recently, in particular those related to diversity, equity, and inclusion.

The present study provides information about the prevalence of general tenure practices and policies, including accommodations related to family obligations, as well as considerations of diversity, equity, and inclusion as they relate to the tenure process. Some of the survey questions were taken from NSOPF in order to allow comparisons of how these features have developed over time. Other questions were included to capture recent developments in the tenure system. Yet another set of results provides information about the extent to which tenure continues to be threatened by institutional policies and practices, such as post-tenure review programs and replacements of tenure lines by faculty on contingent appointments. The results of this study can inform discussions on individual campuses about potential changes to prevailing tenure policies, practices, and standards.

This study is intended to be generalizable to the roughly 1,200 four-year institutions that have a tenure system and a Basic Carnegie Classification of doctoral, master’s, or bachelor’s institution. The study thus excludes two-year institutions and four-year specialized institutions, such as freestanding law schools, art institutes, and seminaries. In the following, we generally refer to the group from which the sample was drawn as “four-year institutions with a tenure system.” The questionnaire was administered to a random sample of 515 chief academic officers and had a response rate of 52.8 percent. For additional methodological information, please see the appendix.

The Probationary Period and Stopping the Tenure Clock

The central innovation of the 1940 Statement of Principles on Academic Freedom and Tenure was a probationary period of fixed length, after which any full-time faculty member who is reappointed has tenure, regardless of the faculty...
The present survey finds that 96.8 percent of four-year institutions with a tenure system have a probationary period of fixed length. In 2004, according to NSOPF, 90.5 percent of the same type of institutions had fixed-length probationary periods—a relatively small change. The mean length of the probationary period was 6.3 years in 2004, compared to 5.7 years now. The tenure review thus generally occurs right around the six-year point, as the 1940 Statement recommends. These two findings, like those in a 2020 AAUP survey on the prevalence of standards related to academic freedom and due process, which reported that three-quarters of four-year institutions with a tenure system base their academic freedom policy on the 1940 Statement, demonstrate the lasting impact of this eighty-year-old statement on US higher education.²

Over the last fifty years, the inflexible, fixed nature of the 1940 Statement’s probationary period has raised concerns about how faculty members with family obligations pursue tenure within a reasonable amount of time given that childbearing age for many faculty members tends to coincide with the probationary period. In 2001, the AAUP issued a statement recommending that institutions allow probationary faculty members to stop the tenure clock “for up to one year for each child,” with a maximum of two times. “Stopping the tenure clock” is a bit of a misnomer, because in cases where faculty members do not take leaves of absence, the practice doesn’t so much stop as extend the probationary period by an additional year or two. Immediately before the AAUP issued that recommendation, Cathy Trower’s 2000 survey of faculty handbooks and collective bargaining agreements found that 17 percent of four-year institutions with a tenure system permitted the stopping of the tenure clock for family reasons. Today, 82.3 percent of institutions provide this opportunity, which represents a large shift in institutional policies over the past twenty years. Of those that offer policies to stop the tenure clock, 92.5 percent make the option available to faculty members regardless of gender, in recognition that partners can be coequal caretakers of newborn or newly adopted children. Only 50.5 percent of institutions explicitly permit stopping the tenure clock for elder care; however, many respondents stressed in their comments that their colleges and universities provided opportunities to negotiate stopping the tenure clock on an individual basis for many or unspecified reasons.

Some of the results in this report will be broken down by Basic Carnegie Classification (bachelor’s, master’s, doctoral) and institutional size, which is based on Carnegie categories as well: small (fewer than two thousand students), medium-sized (between two thousand and five thousand students), and large institutions (more than five thousand students). The availability of options to stop the tenure clock varies among institutions of different Carnegie categories and sizes, with about three-quarters of small and bachelor’s institutions providing the opportunity, compared to all large and essentially all doctoral institutions doing so, with medium-sized and master’s institutions falling in between (see Figure 1).

The ubiquity of policies to stop the tenure clock notwithstanding, the long-term, disparate, and gendered impact of such policies on primary caregivers has been noted for some time, as women are known to be more likely to stop the tenure clock for childbearing and childrearing and thereby delay promotions and concomitant raises in base salary, which has a compounding effect over time.³ There certainly is a need to look for alternative arrangements that avoid this result.


Tenure Quotas and Standards

At some institutions, the percentage of faculty members who can be tenured at any given point is limited by a tenure quota, a mechanism originally proposed in the 1970s to address a perceived problem resulting from the spread of the tenure system: institutions considered to be “overtenured” were thought to face financial challenges and limited flexibility to respond to the changing circumstances of that decade. One result of tenure quotas is that probationary faculty members at institutions that have them would not be granted tenure, regardless of whether they met the substantive requirements, if doing so would result in the number of tenured faculty members exceeding the tenure quota. At the time, the AAUP issued a policy statement opposing tenure quotas, in which it characterized this outcome as “nullifying probation.”

In 1988, when the question about tenure quotas was last included in the NSOPF institutional survey, 17.8 percent of four-year institutions with a tenure system reported having either formal or informal tenure quotas. Now formal or informal tenure quotas can be found at 8.6 percent of institutions. The prevalence of tenure quotas has thus been reduced by about 50 percent over the past three and a half decades. Moreover, only small and medium-sized institutions reported tenure quotas in the present survey, and such quotas were more prevalent at bachelor’s and master’s than at doctoral institutions (see Figure 2).

In opposing tenure quotas, the AAUP’s statement suggested that, rather than introducing tenure quotas, “stricter standards for the awarding of tenure can be developed over the years, with a consequent decrease in the probability of achieving tenure.” The rationale for this option was that similar limitations on the number of tenured faculty members could result without nullifying probation. Since that time, questions about the “ratcheting up” of expectations for tenure have periodically surfaced, and in 1993 the AAUP itself criticized the increasing expectations for tenure as “cruel to members of the faculty, as individuals, and…counterproductive for our students’ education.” It added, “Institutions should define their missions clearly and articulate appropriate and reasonable expectations against which faculty will be judged.”

The NSOPF institutional survey regularly asked whether tenure standards had been made “more stringent” during the past five years. In 2004, 13.3 percent of four-year institutions with a tenure system had made their tenure standards more stringent, which is similar to the finding today of 17.6 percent. The current survey shows some differences by institutional type. These are particularly pronounced when we consider size, with 9.4 percent of small institutions, 16.2 percent of medium-sized institutions, and 38.7 percent of large institutions reporting that tenure standards had been made more stringent (see Figure 3). That difference is smaller when we compare institutions by Carnegie Classification or by institutional control.

Among institutions that made standards more stringent, 78.9 percent reported that this occurred with respect to research standards, 41.1 percent about teaching standards, 24.2 percent about service standards, and 14.0 percent about other standards. Those who reported that “other” standards had been made more stringent included in their comments examples like community engagement, student success, collegial relations with administration, and mentoring and advising.

Tenure Practices Related to Diversity, Equity, and Inclusion

Efforts to improve the institutional climate for diversity, equity, and inclusion (DEI) can focus on tenure practices and standards. Certainly, it is reasonable to consider, for instance, whether the underrepresentation of women faculty or faculty of color among tenured faculty at an institution may be due to tenure practices, or whether an institution’s stated mission to advance DEI should be reflected in standards for promotion and tenure. Examples of possible types of implicit bias in tenure criteria when it comes to the work of faculty of color or women faculty include requiring research publications in journals that do not focus on research areas in which such scholars may be more often represented, not accounting for systematic differences in how students rate faculty of color or women faculty relative to white male instructors on student course evaluations, and not accounting for mentorship and service obligations that fall disproportionately on faculty of color.
The survey focused on three policy responses regarding tenure and DEI: whether standards for tenure include DEI criteria, whether existing standards for tenure had been reviewed with respect to potential implicit bias during the past five years, and whether faculty serving on promotion and tenure committees had been trained regarding implicit bias during the past five years. While these are certainly not the only ways institutions can address DEI concerns in tenure practices, we identified these three responses as central through interviews with subject matter experts and administrators responsible for faculty professional development. To the best of our knowledge, no previous systematic studies of the prevalence of these responses exist, and thus there are no bases to compare our results with previous findings.

DEI criteria were found in tenure standards at 21.5 percent of institutions. While there were differences among institutions based on Carnegie Classification, with 29.2 percent of doctoral institutions reporting the practice, compared to 18.5 percent and 17.9 percent at master’s and bachelor’s institutions, respectively, the largest difference was by size, with 45.6 percent of large institutions reporting having DEI criteria in tenure standards, compared to 15.5 percent and 14.5 percent at medium-sized and small institutions, respectively.

At 39.4 percent of institutions criteria for tenure had been reviewed for implicit bias. Again, there were differences among institutions by Carnegie Classification and size, with a larger difference by size: 63.5 percent of large institutions reported having reviewed standards for implicit bias, compared to 38.6 percent and 29.0 percent of medium-sized and small institutions, respectively.

Respondents who indicated that tenure criteria had been reviewed for implicit bias were asked what actions had been taken as a result of that review. By far the largest number of responses indicated revisions to student course evaluation forms or the elimination of student course evaluations from the tenure process. In addition, respondents reported the broadening of tenure standards, including more explicitly recognizing service obligations that fall more heavily on faculty of color and being more inclusive in scholarly expectations. Regarding student course evaluations, one respondent explained that the institution had added “a mentoring questionnaire so students who may or may not be in [a] faculty member’s classes can be invited to address the value of mentoring.” Two respondents pointed out that they had received NSF ADVANCE grants to support their review of tenure practices from a DEI perspective.

Forty percent of institutions had provided training on implicit bias to members of promotion and tenure committees in the last five years. The difference by size, which was again the most pronounced, was similar to the previous area, with 61.4 percent of large institutions reporting having offered such training, compared to 38.6 percent and 31.1 percent of medium-sized and small institutions, respectively.

Some respondents who sought to advance DEI in the tenure process pointed to challenges that originated from outside of the institution. One respondent observed, “While the university might be able to do a holistic review, there still remains bias in scholarly fields controlling publication.” Another noted, “We are in a state where proposed legislation would greatly restrict our ability to provide training regarding implicit bias.”

For each of the three practices, respondents who indicated that they had not been undertaken at their institution were asked whether the institution was considering undertaking them in the future. Of those who indicated that their institution did not have DEI criteria for tenure, 49.9 percent indicated that it was considering adding them in the future. Similarly, around half (54.8 percent) of respondents who had indicated that no review of tenure criteria for implicit bias had occurred in the last five years indicated that their institution was considering including such reviews in the future. A somewhat higher percentage (70.5 percent) of those who had responded that no training on implicit bias had been provided to promotion and tenure committee members indicated that they were considering it for the future. Figures 4, 5, and 6 combine the responses to the three questions about the presence of DEI-related practices with the questions about whether the institution intends to implement them.
## FIGURE 4
Institutions including DEI criteria in tenure standards

<table>
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<th>Category</th>
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<th>Private</th>
<th>Doctoral</th>
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<th>Bachelor’s</th>
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<td>DEI criteria included</td>
<td>21.5%</td>
<td>28.4%</td>
<td>17.8%</td>
<td>29.2%</td>
<td>18.5%</td>
<td>17.9%</td>
<td>49.5%</td>
<td>15.5%</td>
<td>14.5%</td>
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<td>41.4%</td>
<td>38.4%</td>
<td>38.9%</td>
<td>35.5%</td>
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<tr>
<td>Review is not being considered</td>
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<td>35.8%</td>
<td>43.4%</td>
<td>29.3%</td>
<td>43.1%</td>
<td>45.2%</td>
<td>18.8%</td>
<td>43.4%</td>
<td>46.8%</td>
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</table>

Source: 2022 AAUP Tenure Survey.
Note: Findings are from four-year institutions with a tenure system.

## FIGURE 5
Institutions where tenure standards have been reviewed for implicit bias

<table>
<thead>
<tr>
<th>Category</th>
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<th>Private</th>
<th>Doctoral</th>
<th>Master’s</th>
<th>Bachelor’s</th>
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<td>Standards have reviewed</td>
<td>39.4%</td>
<td>44.4%</td>
<td>35.5%</td>
<td>52.9%</td>
<td>32.4%</td>
<td>35.3%</td>
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<td>Review is under consideration</td>
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<td>26.8%</td>
<td>44.7%</td>
<td>23.3%</td>
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<td>39.9%</td>
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<td>40.8%</td>
<td>20.0%</td>
<td>13.2%</td>
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</tbody>
</table>

Source: 2022 AAUP Tenure Survey.
Note: Findings are from four-year institutions with a tenure system.

## FIGURE 6
Institutions where promotion and tenure committee members have been trained on implicit bias

<table>
<thead>
<tr>
<th>Category</th>
<th>All</th>
<th>Public</th>
<th>Private</th>
<th>Doctoral</th>
<th>Master’s</th>
<th>Bachelor’s</th>
<th>Large</th>
<th>Medium</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Training has provided</td>
<td>40.0%</td>
<td>43.0%</td>
<td>37.5%</td>
<td>56.6%</td>
<td>30.5%</td>
<td>38.5%</td>
<td>61.4%</td>
<td>38.6%</td>
<td>31.1%</td>
</tr>
<tr>
<td>Is under consideration</td>
<td>42.3%</td>
<td>39.8%</td>
<td>44.4%</td>
<td>32.0%</td>
<td>46.7%</td>
<td>46.7%</td>
<td>26.8%</td>
<td>41.2%</td>
<td>58.1%</td>
</tr>
<tr>
<td>Is not considered</td>
<td>17.7%</td>
<td>17.2%</td>
<td>18.1%</td>
<td>12.6%</td>
<td>22.7%</td>
<td>18.8%</td>
<td>11.8%</td>
<td>20.2%</td>
<td>18.8%</td>
</tr>
</tbody>
</table>

Source: 2022 AAUP Tenure Survey.
Note: Findings are from four-year institutions with a tenure system.
Continuing Threats to Tenure

Replacement of tenure lines. An assessment of the prevalence of tenure practices should also account for ways that institutional actions undermine tenure. For several decades, the AAUP has noted the increase in contingent appointments as a proportion of the academic workforce. According to results prepared by the AAUP research department based on IPEDS data, in 2019, 10.5 percent of appointments were tenure track, 26.5 percent tenured, 20 percent full-time contingent, and 43 percent part-time contingent.

The institutional component of NSOPF reported in 2004 that 17.2 percent of four-year institutions with tenure had replaced tenured with fixed-term positions in the previous five years. As some respondents to the question on this year’s survey noted, the question as formulated describes a specific scenario: a replacement of tenured positions that are vacated by retirement or resignation with fixed-term positions. It does not capture scenarios in which fixed-term positions are added, for example. It also isn’t a measure of how many tenure lines have been replaced but rather only of whether any tenure line has been replaced at the institution. That said, in the present survey 53.5 percent of institutions reported having replaced tenured with fixed-term positions in the last five years, a threefold increase in institutions reporting the practice. Larger proportions of master’s institutions, medium-sized institutions, and public institutions reported replacing tenure lines with contingent positions (see Figure 7). Some respondents noted that they had at the same time converted fixed-term positions to tenure lines; others noted that the conversion had occurred at the request of the department or unit, or as a result of states’ disinvestment in higher education.

Post-tenure review. For some time, an additional threat to tenure has been the spread of post-tenure review policies. Post-tenure review typically is a comprehensive periodic review of tenured faculty members that is separate from annual reviews. Although the AAUP does not view all post-tenure review policies as at odds with its standards, it considers policies that place the burden of proof on faculty members to demonstrate that they should be retained as inimical to academic freedom and tenure. The present survey asked whether post-tenure review was practiced at the institution and, for those institutions that practiced it, whether the process can result in the termination of appointments. Although the Association’s particular concern in the latter case is whether essential elements of academic due process are provided, the survey could not really explore such details. Nevertheless, since the AAUP’s policies on post-tenure review recommend that such reviews be developmental rather than summative, the possibility that reviews can result in termination raises concerns about their conformance with AAUP standards.

In Cathy Trower’s 2000 study, 46 percent of institutional regulations contained a post-tenure review policy. In the current survey, 58.2 percent of institutions reported having a post-tenure review policy. Thus, while common, post-tenure review policies are hardly universal. Our study found a difference by institutional control, with 67.6 percent of public institutions reporting a post-tenure review program, compared to 50.7 percent of private institutions. This difference may be due to state legislative requirements for post-tenure review or a perceived need by public institutions to conduct such a review in the face of legislative hostility toward tenure. There was no noticeable difference by Carnegie Classification. Large institutions more frequently reported post-tenure review policies than did small and medium-sized ones (see Figure 8).

Of institutions that reported having a post-tenure review program, 47.2 percent indicated that the outcome of a review could lead to termination, and thus only about 27 percent of all four-year institutions with a tenure system have post-tenure review programs that can result in termination. Some respondents added that such terminations would be the result of failing to adhere to an improvement plan rather than a direct result of the post-tenure review.

FIGURE 7
Institutions that have replaced tenure lines with contingent appointments in the last five years

<table>
<thead>
<tr>
<th></th>
<th>All</th>
<th>Public</th>
<th>Private</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>53.5%</td>
<td>62.1%</td>
<td>46.8%</td>
</tr>
<tr>
<td>Doctoral</td>
<td>45.9%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Master’s</td>
<td>41.7%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bachelor’s</td>
<td>52.7%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Large</td>
<td>61.0%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>49.2%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Small</td>
<td>42.3%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source: 2022 AAUP Tenure Survey.
Note: Findings are from four-year institutions with a tenure system.

\[4\] See, for example, the recent investigation of the University of Georgia system for an extended discussion of the threat posed by the systemwide introduction of changes to a post-tenure program.
Institutions that have a post-tenure review program

<table>
<thead>
<tr>
<th>Category</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>All</td>
<td>58.2%</td>
</tr>
<tr>
<td>Public</td>
<td>50.7%</td>
</tr>
<tr>
<td>Private</td>
<td>58.0%</td>
</tr>
<tr>
<td>Doctoral</td>
<td>57.9%</td>
</tr>
<tr>
<td>Master’s</td>
<td>57.0%</td>
</tr>
<tr>
<td>Bachelor’s</td>
<td>67.6%</td>
</tr>
<tr>
<td>Large</td>
<td>48.2%</td>
</tr>
<tr>
<td>Medium</td>
<td>22.4%</td>
</tr>
<tr>
<td>Small</td>
<td>58.0%</td>
</tr>
</tbody>
</table>

Source: 2022 AAUP Tenure Survey.
Note: Findings are from four-year institutions with a tenure system.

Conclusion

Although it was not the original intent of the 1940 Statement, which proposed that tenure be obtained by virtue of reappointment after the conclusion of the probationary period rather than as a result of a promotion-like review, the tenure system has today become identified with an extensive review process. While the AAUP continues to hold that tenure is acquired due to length of service, which is at times referred to as de facto tenure, it has issued a number of policy statements related to the tenure review process. The present study provides information about the prevalence of tenure practices, some of which relate to the Association’s policy statements.

Among highly prevalent practices that conform with the Association’s standards, the presence of a probationary period of fixed length that generally results in a tenure decision in the sixth year dates to the 1940 Statement. Permitting probationary faculty members to stop the tenure clock for reasons of childbearing or child rearing, which the AAUP adopted as a policy recommendation more recently, is also a common practice at this point. Conversely, tenure quotas and post-tenure review programs that can result in termination of appointment, both of which are at odds with AAUP policies, are relatively rare. In contrast, the practice of replacing tenure lines with contingent appointments, one entirely at odds with the Association’s standards, has markedly increased in the past two decades.

The present study also provides information about tenure practices related to diversity, equity, and inclusion. Press reports of such efforts at individual institutions have at times focused on opposition to them by faculty members, outside organizations, or state legislatures. It should be noted that AAUP policy generally does not provide substantive grounds to oppose tenure practices and standards that promote DEI. For instance, AAUP policy does not support claims that DEI criteria for promotion and tenure are political litmus tests or somehow akin to loyalty oaths. Nevertheless, the focus on opposition to such activities may have caused them to be viewed as “controversial,” which may help explain the relatively high percentage of institutions that are not considering undertaking them. The fact that several respondents reported changes to the format or use of student evaluations points to a need to consider in more detail how institutional practices in this area are changing.

While tenure is regularly under attack, by both institutional practice and legislation, it continues to serve as the bulwark in the defense of academic freedom. As such, it is essential to study practices related to tenure systematically and on a regular basis, as this study has done.

Appendix: Methodology

As mentioned above, this study is intended to be generalizable to a population of some 1,200 four-year institutions that have a tenure system and a Basic Carnegie Classification of being a doctoral, master’s, or bachelor’s institution. Within that population, 30.3 percent are bachelor’s institutions, 39.8 percent master’s institutions, and 29.9 percent doctoral institutions; and 47.5 percent are small institutions, 31.1 percent are medium-sized, and 21.4 percent are large. The above table contains information about the distribution of four-year institutions with a tenure system by size and Carnegie Classification.

The questionnaire was administered to a stratified random sample of 515 chief academic officers and had a response rate of 52.8 percent, using the American Association for Public Opinion Research definition of response rate (RR2). In a few instances, the survey was completed by another academic affairs administrator (such as an associate provost), who was designated because of their responsibility for the promotion and tenure process.

We stratified the population by Carnegie Classification and size into five strata and drew disproportionate, random samples from each stratum, oversampling small strata and undersampling large ones. The five strata were bachelor’s institutions, small master’s institutions, medium-sized and large master’s institutions, small and medium-sized doctoral institutions, and large doctoral institutions. The purpose of these sampling choices was to ensure adequate numbers of institutions within each stratum for further analysis.

In order to increase participation, the AAUP sought the endorsement of national associations that are members of the Washington Higher Education Secretariat. The endorsements were listed on correspondence inviting
Distribution of Institutions by Size and Carnegie Classification

<table>
<thead>
<tr>
<th></th>
<th>Small</th>
<th>Medium</th>
<th>Large</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bachelor’s</td>
<td>28.3%</td>
<td>2.0%</td>
<td>0.0%</td>
<td>30.3%</td>
</tr>
<tr>
<td>Master’s</td>
<td>16.9%</td>
<td>19.1%</td>
<td>3.8%</td>
<td>39.8%</td>
</tr>
<tr>
<td>Doctoral</td>
<td>2.3%</td>
<td>10.0%</td>
<td>17.6%</td>
<td>29.9%</td>
</tr>
<tr>
<td>Total</td>
<td>47.5%</td>
<td>31.1%</td>
<td>21.4%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

participation. The following associations endorsed the survey: the American Council on Education, the American Association of Colleges and Universities, the Association of Research Libraries, the College and University Professional Association for Human Resources, the Council of Independent Colleges, the Council on Social Work Education, the Educational Testing Service, and the Phi Beta Kappa Society. We would like to thank these organizations for their support.

Two responses were excluded from the final analysis because they indicated that the institution does not have a tenure system. Nineteen responses were breakoffs that had extensive item nonresponse and were excluded as well. To improve the accuracy of the estimates, we weighted the results of the study with design weights to account for the disproportionate selection across the different strata and weights to account for unit nonresponse.

Estimates of proportions in the population made on the basis of a sample have a margin of sampling error. That margin depends on the sampling design (the “design effect”), the size of the sample, the finite population correction, and the estimated proportion itself. With 271 respondents in this stratified sample, the margin of error is about +/-5.4 points when the proportion reported is 50 percent, which is the proportion at which the margin of error is largest for a given sample size. Thus, for example, the estimate that 50.5 percent of institutions allow faculty members to stop the tenure clock for elder care has a 95 percent confidence interval of 45.1 percent to 55.9 percent. The margin of error is larger when proportions are reported for smaller subpopulations (such as by Carnegie Classification, size, and so forth).

Hans-Joerg Tiede

Credits

Figures 1–8 are courtesy of the AAUP. Photo of Hans-Joerg Tiede is courtesy of the AAUP/Michael Ferguson.

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What does it mean to indigenise mathematics education and research, and how and why should we do it?

Addressing these questions in a university setting especially was the subject of the first Indigenising University Mathematics Symposium, which took place over September 20–21, 2021. It was jointly hosted by the University of Newcastle’s Priority Research Centre for Computer-Assisted Research Mathematics and its Applications (CARMA), and its Wollotuka Institute of Indigenous Engagement and Advancement. Judy-anne Osborn (University of Newcastle), cochair of the organising committee and a major driving force behind the symposium, described it as having been sparked in part by a challenge issued to academic staff at the university from across the disciplines: “We have to indigenise the curriculum. The government says we’ve got to do it, Universities Australia says we’ve to do it, we’ll lose funding if we don’t do it, and it’s morally the right thing to do.” The symposium grew out of her grappling with what this meant and how to do it in the context of mathematics.

The symposium took place entirely online, with an opportunity for the participants to gather virtually between sessions. The online format may have been forced on the organisers by the ongoing covid19 pandemic, but it almost certainly enabled more people to attend: well over 100 people participated, connecting in from locations including Australia, Canada, Hawai’i, New Zealand, the United Kingdom, and the continental United States.

The symposium was designed within a framework of mutual sharing and learning, where we would all benefit from the experiences and knowledge of all present. To this end each session was followed by a yarning session (yarning in the sense of sharing a story), where we could reflect on and share our ideas about what had been said so far. These drew upon the principles of the Yarning or Talking Circles seen in Australian Aboriginal and North American Indigenous cultures, adapting them as best possible to the online space we were meeting in and the time constraints of the symposium. The meeting chat offered another channel in which people could respond, and was lively with the exchange of ideas throughout.

The symposium was organised around the six interconnected themes of Re-imagining the Living Present; Country and Mathematics; Language and Oral Traditions; Indigenous Mathematics; Love and Pedagogy; and Traditional Knowledge. Each theme was chaired by a partnership of indigenous and non-indigenous speakers, who gave a presentation and then led a yarning session. A seventh keynote session discussed decolonisation, and there were five shorter community contributions. The presented portions of the symposium were all recorded but the yarning sessions were not, so that participants could freely exchange thoughts and ideas without fear of any half-formed ideas being recorded for posterity. The videos of the sessions can all be viewed on the symposium webpage [IUM1].
Some Key Ideas

The symposium was fascinating, thought-provoking, and challenging. I learnt a lot and was very glad I’d taken part. Here are some of the key ideas I took from it. Space limitations preclude me from mentioning everything that I would like to.

In the process of writing this article I reviewed the videos of the presentations to refresh my memory of what had been said. In doing so I was struck again by the depth of knowledge and thinking of the presenters, and the impossibility of doing them justice in a short article. I encourage you to visit the symposium webpage [IUM1] and view the videos yourself, so that you can hear their thoughts in full and in their own words.

Indigenous knowledges are grounded in evidence and observation, are still living and relevant today, and should be respected and valued as such. In Theme 1 Reimagining the Living Present Mark Mac Lean (University of British Columbia) addressed a fallacy he sees prevalent in Western education of presenting indigenous peoples as historical, and as either an object of study, or as needing to be brought into the world of “modern mathematics and modern mathematical thinking.” He spoke of the need to “replace that fallacy with the truth, that indigenous peoples have knowledge and pedagogies, including mathematical knowledge and pedagogies, rooted deep in these long histories, in some cases tens of thousands of years, and that are still vibrant and still relevant in the present, and still a living thing in these peoples around the world.”

Kathleen Butler (Director, Wollotuka Institute, University of Newcastle; Aboriginal, of the Bundjalung and Worimi people) then spoke of the ways in which indigenous knowledge and oral traditions are all too often seen as secondary to Western knowledge, and regarded as metaphor, or accepted as true only when evidence is found to support them. She sees this as problematic. It is debated how the Australian megafauna went extinct, with some arguing that they were hunted to extinction shortly after humans arrived in Australia, while others argue that all evidence points to Aboriginal people having coexisted with the megafauna for over 20,000 years. What no one seems to be doing, Butler says, “is going back to the oral traditions, which tell us very clearly that we coexisted with megafauna.” Butler says “I’m really interested in how we could think about indigenous knowledges as something which have their own inherent value, rather than only having a value when evidence supports their existence. Because I think that indigenous knowledges do come from a really clear evidence base.”

Butler spoke of the way in which our use of numbers can create something artificial that separates us from the natural world. She gave the Gregorian calendar as an example of this, with its twelve months that don’t align with the phases of the moon, in contrast to the lunar calendars used by many indigenous peoples. “I think that what indigenous knowledges bring to our understanding of mathematics is very much embedding us in what is happening in the world around us. That the numbers that we [use] to artificially create something, don’t change our natural world, and in fact, in doing that artificiality, we remove ourselves from a really important element of connection.” Butler says many indigenous academics and thinkers argue that this removal of ourselves from the natural world greatly contributes to wicked problems such as climate change.

Doing mathematics is a universal human endeavour. The idea that doing mathematics is simply part of being human is one that came through several times over the course of the symposium, in both the keynote presentations and the yarning sessions. Rowena Ball (Australian National University; Aboriginal and Irish descent) put it as follows in the Decolonisation Discussion:

Doing maths and mathematical thinking are universal human imperatives across all societies and cultures, as natural and intrinsic to our humanity as doing art, and probably as ancient.

Michael Assis (CARMA and the University of Melbourne) put a case for this idea in Theme 3 Language and Oral Traditions. Assis presented a series of mathematical texts from ancient civilisations, ranging from Babylonian and ancient Chinese texts on the Pythagorean Theorem, to a Mayan work on predicting lunar eclipses. Some of the cultures were geographically close enough to influence each other, but others were geographically well removed. This leads him to the conclusion that mathematics is a human endeavour that all cultures have engaged in — that wherever you go there is mathematical thought — and moreover that the development of mathematics doesn’t have a single linear history, with the same ideas arising multiple times in multiple places.

What is mathematics, and what is indigenous mathematics? In Theme 4 Indigenous Mathematics, Edward Doolittle (First Nations University of Canada; Indigenous North American, of the Kanyen’kehaka people, commonly known in English by the exonym Mohawk) addressed the three questions What is indigenous? What is mathematics? and What is indigenous mathematics?

Doolittle sees the first as very simple: indigenous means connected to place, and importantly connected to the culture associated with the place. The second he sees as much more problematic, and in fact the wrong question to ask: rather than try to define the noun “mathematics” he thinks the better question is to define the adjective
“mathematical”—just as we can more easily define the adjective “infinite” than we can the noun “infinity.” In answer to this better question Doolittle referred to a list of six fundamental activities that Bishop [Bis88] argues are universal to all cultural groups, and necessary and sufficient for the development of mathematical knowledge: counting, locating, measuring, designing, playing, and explaining.

To answer the third question Doolittle puts these two answers together: indigenous mathematics is mathematical knowledge that is local, and acquired using the methodologies of the place.

Ball gave an example of indigenous mathematics in the Decolonisation Discussion. She described songlines transforms, an Aboriginal solution to the wayfaring problem of navigating a route of hundreds or thousands of kilometres which no one in the party has travelled before, in the absence of portable maps and satellite navigational aids.

The songline for a route encodes it in the winter stars, so that elders who know the route can teach it over winter to the party who are to travel it the coming summer. Ball argues that the songline is a mathematical transform, akin to the Fourier transform: “The songline transform is an approximation to the route, just as a truncated Fourier series is an approximation to the function. You remove distracting noise in the songline, just as you remove high frequency components of a Fourier transform signal.”

Ball believes that much Aboriginal and Torres Strait Islander mathematical knowledge is alive and can still be studied by those who are willing to do so; and that the discipline of mathematics stands to be broadened and enriched by including the mathematics of indigenous cultures.

Mathematics and colonisation. In the Decolonisation discussion, Ball argued that mathematics played a role in colonisation, and colonisation played a role in shaping mathematics. To support this claim Ball discussed the two major strands in the development of her field of applied dynamical systems and stability theory: the three-body problem and the stability of the centrifugal fly-ball governor.

Ball argues that these problems were crucial to the colonial enterprise, and this provided powerful motivation for solving them. The centrifugal fly-ball governor was used to control the steam engines of the Industrial Revolution, but the instability of the governor meant that they could become hopelessly inefficient, requiring more than the labour they replaced to control them. This, Ball says, posed a risk to the wealth of the industrialists seeking to profit from the raw materials such as cotton and whale oil obtained through colonialism.

The success of sea voyages of colonial trade and exploitation depended critically on being able to calculate longitude at sea, in order to navigate safely there and back. Major European sea powers including Spain, the Netherlands, and Great Britain all offered substantial rewards for simple and practical methods of computing longitude to sufficient accuracy. Ball says this motivated efforts to solve the three-body problem, because longitude could be calculated by the lunar distance method if the position of the moon were known sufficiently accurately, and led to the development of a great deal of sophisticated mathematics.

Data sovereignty and self determination. Several speakers argued that each community should be able to determine its wants and needs and priorities, and the means by which to achieve them. This is closely connected to issues of data sovereignty. Here for example is Butler speaking in Theme 1 Reimagining the living present:

There are so many ways in which statistics have been used to present us [Aboriginal peoples] in deficit. When we talk about things like “close the gap” — a term used in Australia to try and have Aboriginal and Torres Strait Islander data for things such as health, education, housing and employment, to have those meet that of the mainstream Australian population. And again what that does for us in terms of language, is it says that the mainstream or the West is our benchmark. But that may not be what our communities see as our particular benchmark. When we look at things like NAPLAN, which is our national testing in schools, that may not be the measure that Aboriginal and Torres Strait Islander communities see as success in education. [...] If we have different priorities, and our community priorities are different to what’s recognised in government reports, then government funding is not going to be shared equitably to meet our particular outcomes. So addressing issues around who's gathering the data, for what purpose, and for what question, is really important.

Good pedagogy is good for everyone. In Theme 5 Love and Pedagogy Michael Donovan (Macquarie University; Aboriginal, of the Gumbaynggir people) told us about the findings of his 2016 PhD thesis [Don16] What form(s) of pedagogy are necessary for increasing the engagement of Aboriginal school students? He asked 15–17-year-old Aboriginal students what made a good teacher, a good school, and a good curriculum; their responses mirrored what pedagogical theorists have argued for the last 50 years, showing that the best methods of engaging indigenous students are the same methods that work for all students, and are the things that students do want.

So how do we indigenise mathematics? The symposium was never intended as the last word on this question, but rather as the first step of an ongoing process. At the close of Theme
1 Reimagining the living present Mac Lean offers a view of what this will involve.

The thing to remember, Mac Lean says, is to place indigenous peoples, and their knowledge and their thinking in the very centre of the conversation—and indeed, to let them be the ones to speak. For those of us who are part of colonising cultures—“that disrupted other cultures, and basically prevented them from continuing and creating their own future spaces that would be universities”—he says it’s a time for backing off, giving space for the conversation to happen, listening carefully, contributing our expertise where it can be of help—and importantly, working to ensure the necessary funding is made available.

Next Steps

An edited book organised around the conference themes is planned, and special sessions on indigenising university mathematics were held at the December 2021 meetings of the Australian and Canadian Mathematical Societies. A second symposium IUM2: Indigenizing University Mathematics 2 will be held at the end of November 2022, taking place at three venues: online, First Nations University of Canada, and the University of Newcastle’s Wollotuka Institute. For details see the symposium webpage [IUM2].

More information can be found on CARMA’s Indigenising University Mathematics Project webpage [CAR].

References


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— Randy LeVeque, SIAM Member, Fellow, and Board Member

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SHORT STORIES

Proofs Without Words
Danny Calegari

2 × 3 = 3 × 2. Ecce!

Figure 1. ***** ******* *****

In this ‘proof,’ terms in the equation correspond to objects in the figure, and the identity is ‘proved’ by appealing to geometric symmetries of the figure. Symmetries mean a group $G$, acting on suitable objects in some space $X$.

Many proofs without words take the following form: an identity $A = B$ is ‘proved’ by exhibiting an object $\alpha$ in $X$ ‘representing’ $A$, which may be cut into pieces which can be rearranged (i.e., translated by elements of the group $G$) to form a new object $\beta$ ‘representing’ $B$. Such proofs often work best when $A$ and $B$ take values in an abelian group $\mathcal{P}$ (which need bear no direct relation to $G$), and the pieces into which $\alpha$ and $\beta$ are decomposed also represent elements in $\mathcal{P}$. The group $\mathcal{P}$ should be abelian because in a (planar) figure composed of several pieces there is no obvious order in which to ‘multiply’ the pieces. Pictorial identities in such groups $\mathcal{P}$ are called scissors congruences.

An example is the Pythagorean identity $A^2 + B^2 = C^2$ for the sides of a right-angled triangle; see Figure 2.

In this example $\mathcal{P}$ is $\mathbb{R}$ (where the areas $A^2$, $B^2$ and $C^2$ of the three squares take values) and $G$ is the group of isometries of the (Euclidean) plane.

Everyone knows the ‘proof without words’ for the commutative law for multiplication. There is another commutative law which is a bit trickier to see, one involving information. Let’s play twenty questions, and to save time let’s reduce the play to two questions. Let’s further assume that the questions allow only yes-no answers. You think of an object $X$ drawn from some class $\Omega$ of possible objects and I ask two yes-no questions $\alpha$ and $\beta$. Whatever we may mean by ‘information,’ it stands to reason that the amount of information I obtain from asking my questions does not depend on the order I ask them: if $H$ denotes the information I gain from asking a series of questions, and $\alpha \land \beta$ means I first ask $\alpha$ and then $\beta$, then I should have

$$H(\alpha \land \beta) = H(\beta \land \alpha).$$

Danny Calegari is a professor of mathematics at the University of Chicago. His email address is dannyc@math.uchicago.edu.

Communicated by Notices Associate Editor Steven Sam.

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DOI: https://doi.org/10.1090/noti2557
On the other hand, I can compute \( H(\alpha \land \beta) \) another way. Evidently \( H(\alpha \land \beta) \) is equal to \( H(\alpha) \) (the information I get by asking \( \alpha \)) plus the information I get from \( \beta \) conditioned on having already obtained an answer to \( \alpha \) (this is the chain rule for information). This second quantity is denoted \( H(\beta|\alpha) \), and now my commutative law takes the form

\[
H(\alpha) + H(\beta|\alpha) = H(\beta) + H(\alpha|\beta).
\]

Our goal is to give a ‘proof without words’ for this equation.

To translate this into more traditional mathematics, \( \Omega \) is a space with a probability measure on it and \( X \) is a random variable. For concreteness let’s take \( \Omega \) to be the unit interval \([0, 1]\) and \( X \) to be a random number drawn from this interval. A question about \( X \) amounts to choosing some (measurable) subset of \( \Omega \) and asking whether \( X \) is in this subset or not. Again, for concreteness, let’s take \( \alpha \) to be the question ‘is \( X < p \)?’ and \( \beta \) to be the question ‘is \( X < q \)?’ for some fixed \( 0 < p < q < 1 \).

Entropy is a numerical measure of information; it is given by a (rather opaque) formula

\[
H(\alpha) := -p \log(p) - (1 - p) \log(1 - p)
\]

which we write as a function \( h(p) \) of \( p \) since it depends only on \( p \). One heuristic way to think about this formula is to observe that probabilities of independent events multiply, whereas we want to think of information as additive. Thus we define the entropy of an event as the log of its probability, and then the entropy of a question is the expectation of the entropy of an answer to that question. The conditional information \( H(\beta|\alpha) \) is either zero if \( X < p \) (because in that case we already know \( X < q \)) and otherwise is equal to \( h(\frac{1-q}{1-p}) \) because if we already know that \( X > p \), the question ‘is \( X < q \)?’ has a ‘no’ answer with probability \((1-q)/(1-p)\). Thus our entropy equation becomes

\[
h(p) + (1 - p) h\left(\frac{1-q}{1-p}\right) = h(q) + q h\left(\frac{p}{q}\right)
\]

and this is the form in which we shall give a proof without words.

OK. To present our proof without words we need three things: a class of geometric objects which will represent allowable terms in our equation, a group \( G \) of symmetries so that geometric objects that differ by an element of \( G \) will be considered to represent ‘the same’ term, and a list of allowable dissections of objects that will represent addition of terms in the equation.

The objects in our figure will be certain plane polygons assembled from pieces called \( H \)-trapezoids. A \( H \)-trapezoid is a quadrilateral with two parallel sides. A trapezoid is horizontal if the two parallel sides are horizontal. Unless it is a parallelogram, the other two sides, when extended, intersect at a point. An \( H \)-trapezoid is a horizontal trapezoid whose non-horizontal sides intersect at a point on the \( x \)-axis (for simplicity we’ll only consider \( H \)-trapezoids which lie on the positive side of the \( x \)-axis).

What about the group \( G \)? It is generated by three kinds of transformations which permute \( H \)-trapezoids:

1. horizontal translations \((a, b) \to (a + \lambda, b) \) for \( \lambda \in \mathbb{R} \);
2. horizontal shears \((a, b) \to (a + \lambda b, b) \) for \( \lambda \in \mathbb{R} \);
3. dilations \((a, b) \to (\lambda a, \lambda b) \) for \( \lambda \in \mathbb{R}^+ \).

Translations and dilations together generate the affine group of the line \( \mathbb{R} \rtimes \mathbb{R}^+ \). Shears commute with both translations and dilations; thus \( G \cong \mathbb{R} \oplus (\mathbb{R} \rtimes \mathbb{R}^+) \).

Finally we must say what sorts of ‘dissections’ are permitted; this is the geometric avatar of ‘addition’ in the abelian group where information will take its values. We shall define an abelian group \( \mathcal{P} \) generated by \( G \)-equivalence classes of \( H \)-trapezoids modulo the relation \( A + B = C \) whenever \( C \) is an \( H \)-trapezoid decomposed into two \( H \)-trapezoids \( A \) and \( B \) by a straight line \( \ell \) of one of the following two sorts:

1. (horizontal cut): \( \ell \) is any horizontal line; or
2. (vertical cut): \( \ell \) is any non-horizontal line concurrent with the non-horizontal edges of \( C \).

These two decompositions are illustrated in Figure 4.

Figure 3. \( G \)-equivalent \( H \)-trapezoids.

Figure 4. Addition of \( H \)-trapezoids.
we define

will follow from an analogous identity in

Then we may compute

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equivalence class with vertices at

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In other words, for any

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\(\phi \colon (x, y) \to (x', \log y)\) is a diffeomorphism from the upper

half space to \(\mathbb{R}^2\); if we think of \(\mathbb{R}^2\) as \(T^*\mathbb{R}\) with its tauto-

logical symplectic form \(\omega\), then \(\phi^*\omega\) is the hyperbolic area

form and the group \(G\) acts by symplectomorphisms.

In fact, one may reinterpret Figure 5 as a scissors congruence in a somewhat different group. In place of the group \(G\) one can consider the group \(G'\) generated by horizontal translations, dilations, and reflection in the \(y\)-axis \((a, b) \to (-a, b)\); and then we may define \(\mathcal{P}'\) to be the abelian group generated by \(G'\)-equivalence classes of \(H\)-trapezoids modulo horizontal or vertical cuts. Now \(G'\) really is a group of hyperbolic isometries (it is the ‘parabolic’ subgroup of hyperbolic isometries fixing a point at infinity) and \(\mathcal{P}'\) is a rather interesting refinement of \(\mathcal{P}\). The decomposition in Figure 5 proves a valid identity in \(\mathcal{P}'\) which is now harder to interpret because of the more complicated structure of \(\mathcal{P}'\) as an abelian group.

The \(\mathbb{R}\)-vector space spanned by formal symbols \([x]\) for \(0 < x < 1\) modulo the entropy equation is denoted \(\beta_2\), and is a kind of ‘additive linearization’ of the Bloch group, which arises in the geometry of 3-dimensional hyperbolic polyhedra. There is a short exact sequence called the Cathelineau complex

\[0 \to \beta_2 \to \mathbb{R} \otimes \mathbb{R}^+ \to \Omega^1_{\mathbb{R}/\mathbb{Q}} \to 0\]

where \(\Omega^1_{\mathbb{R}/\mathbb{Q}}\) is the (additive) group of Kähler differentials of \(\mathbb{R}\) over \(\mathbb{Q}\). After our identification of \(\mathcal{P}\) with \(\mathbb{R} \otimes \mathbb{R}^+\), the first map is \([a] \to T(a, a) + T(1 - a, 1 - a)\) and the second map is \(T(a, b) \to (a/b)db\).

This sequence is closely related to the famous ‘Dehn–Sydler–Jessen’ sequence which encodes (among other things) the fact that volume and the Dehn invariant together are a complete invariant of scissors congruence for 3-dimensional Euclidean polyhedra. What is the deeper connection between 3-dimensional polyhedra, hyperbolic and symplectic geometry, Kähler differentials, \(H\)-trapezoids, and the chain rule for conditional entropy? I am at a loss for words.

Figure 5. The entropy equation as a scissors congruence in \(\mathcal{P}\).

\[T(nw, h) = nT(w, h) = T(w, h^n)\] for any (positive) integer \(n\), as can be seen by repeated shears, dilations, and the addition laws.

Identifying \(\mathcal{P}\) with \(\mathbb{R} \otimes \mathbb{R}^+\) shows how to make \(\mathcal{P}\) into an \(\mathbb{R}^+\)-module: \(\mathbb{R}^+\) acts by (scalar) multiplication on the left factor. In other words, for any \(t \in \mathbb{R}^+\) and any \(T(w, h)\) we define \(t \cdot T(w, h) = T(tw, h)\). Geometrically this is achieved by stretching an \(H\)-trapezoid in the horizontal direction by a factor of \(t\) (or, what is equivalent as classes in \(\mathcal{P}\), stretching it in the vertical direction by a factor of \(1/t\)).

Where does entropy come in? To see this we must take an unexpected detour into hyperbolic geometry. Poincaré’s upper half-space model of the hyperbolic plane is the subset \(H \subset \mathbb{R}^2\) consisting of points \((x, y)\) where \(y > 0\), except that at points of height \(y\) we scale the Euclidean metric infinitesimally by the factor \(1/y\).

The action of \(G\) on the upper half-space does not preserve the hyperbolic metric; however it does preserve the hyperbolic area form. It follows that hyperbolic area defines a (surjective) homomorphism from \(\mathcal{P}\) to \(\mathbb{R}\).

The connection to entropy becomes clearer once we compute that the hyperbolic area of \(T(a, a)\) is \(-a \log a\). To see this, observe that \(T(a, a)\) has a representative of its \(G\)-equivalence class with vertices at \((0, 1), (a, 1), (0, a), (a^2, a)\). Then we may compute

\[
\text{area}(T(a, a)) = \int_{y=a}^{y=1} \int_{x=0}^{x=ay} \frac{1}{y^2} \, dx \, dy = -a \log a.
\]

In particular, \(h(p)\) is the area of the expression \([p] := T(p, p) + T(1 - p, 1 - p) \in \mathcal{P}\), and the entropy equation will follow from an analogous identity in \(\mathcal{P}\):

\[
[p] + (1 - p) \left[ \frac{1 - q}{1 - p} \right] = [q] + q \left[ \frac{p}{q} \right].
\]

At last we may give the proof of this identity without words; this is Figure 5. On the left \([p]\) is in blue and \((1 - p)\left[ \frac{1 - q}{1 - p} \right]\) is in yellow while on the right \([q]\) is in blue and \(q\left[ \frac{p}{q} \right]\) is in yellow. These are evidently equidecomposable in \(\mathcal{P}\).

One may think of this picture in several ways. The map \(\phi : (x, y) \to (x', \log y)\) is a diffeomorphism from the upper half space to \(\mathbb{R}^2\); if we think of \(\mathbb{R}^2\) as \(T^*\mathbb{R}\) with its tautological symplectic form \(\omega\), then \(\phi^*\omega\) is the hyperbolic area form and the group \(G\) acts by symplectomorphisms.

Figure 5. The entropy equation as a scissors congruence in \(\mathcal{P}\).

AUTHOR’S NOTE. Cathelineau’s paper (which contains much more than is discussed here) is “Remarques sur les différentielles des polylogarithmes uniformes” and appeared in Ann. Inst. Fourier (Grenoble) 46 (1996), no. 5, 1327–1347. I learned about this paper, and its connection to the Dehn–Sydler–Jessen theorem, from Daniil Rudenko. I’d like to thank Rich Schwarz, Amie Wilkinson and the anonymous referee for comments on earlier versions of this note.

Credits

Figure 1. Figures 3–5, and author photo are courtesy of Danny Calegari. Figure 2 is courtesy of Scott Sheffield.

Danny Calegari
Backlog of Mathematics Research Journals

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Once a publisher’s journals are accepted for inclusion, the publisher must designate a contact person or persons to supply data about the journals to the AMS. While the AMS makes every effort to obtain the data from the designated contacts, if data about a journal is not supplied, then that journal will not appear in the backlog report.
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## Research Journals Backlog

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AMS Prizes & Awards

AMS Claytor-Gilmer Fellowship

The AMS established the Claytor-Gilmer Fellowship to further excellence in mathematics research and to help generate wider and sustained participation by Black mathematicians. One fellowship in the amount of $50,000 will be awarded for the 2023–2024 academic year.

About this Fellowship
Awardees may use the fellowship in any way that most effectively enables their research --for instance, for release time, participation in special research programs, travel support, childcare, etc. The award is issued through the recipient's institution, and no part of it may be utilized for indirect costs. Given the aims of the fellowship, the most likely awardee will be a mid-career Black mathematician based at a US institution whose achievements demonstrate significant potential for further contributions to mathematics.

Application Period
Applications will be collected via MathPrograms.org September 1, 2022–December 1, 2022. All applicants will be notified in February 2023 whether or not they have been chosen to receive the fellowship.

Find more application information at https://www.ams.org/claytor-gilmer. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; prof-serv@ams.org; 401-455-4189.

AMS Centennial Research Fellowship Program

The AMS Centennial Research Fellowship Program makes awards annually to outstanding mathematicians to help further their careers in research. One fellowship in the amount of $50,000 will be awarded for the 2023–2024 academic year.

About this Fellowship
The eligibility rules are as follows: The primary selection criterion for the Centennial Fellowship is the excellence of the candidate's research. Preference will be given to candidates who have not had extensive fellowship support in the past. Recipients may not hold the Centennial Fellowship concurrently with another research fellowship such as a Sloan or NSF Postdoctoral fellowship. Under normal circumstances, the fellowship cannot be deferred. A recipient of the fellowship shall have held his or her doctoral degree for at least three years and not more than twelve years at the inception of the award (that is, must be received between September 1, 2011, and September 1, 2020). Applications will be accepted from those currently holding a tenured, tenure track, postdoctoral, or comparable (at the discretion of the selection committee) position at an institution in North America. Applications should include a cogent plan indicating how the fellowship will be used. The plan should include travel to at least one other institution and should demonstrate that the fellowship will be used for more than reduction of teaching at the candidate's home institution. The selection committee will consider the plan, in addition to the quality of the candidate's research, and will try to award the fellowship to those for whom the award would make a real difference in the development of their research careers. Work in all areas of mathematics, including interdisciplinary work, is eligible.

Application Period
Applications will be collected via MathPrograms.org September 1, 2022–December 1, 2022. All applicants will be
Calls for Nominations & Applications
FROM THE AMS SECRETARY

Joan and Joseph Birman Fellowship for Women Scholars

The Joan and Joseph Birman Fellowship for Women Scholars is a mid-career research fellowship specially designed to fit the unique needs of women. This fellowship program is made possible by a generous gift from Joan and Joseph Birman. One fellowship in the amount of $50,000 will be awarded for the 2023–2024 academic year.

About this Fellowship
The fellowship seeks to address the paucity of women at the highest levels of research in mathematics by giving exceptionally talented women extra research support during their mid-career years. The most likely awardee is a mid-career woman, based at a US academic institution, with a well-established research record in a core area of mathematics. The fellowship will be directed toward those for whom the award will make a real difference in the development of their research career. Candidates must have a carefully thought-through research plan for the fellowship period. Special circumstances (such as time taken off for care of children or other family members) may be taken into consideration in making the award. The fellowship can be used to provide additional time for research of the awardee, or opportunities to work with collaborators. This may include, but is not limited to, course buy-outs, travel money, childcare support, or support to attend special research programs.

Application Period
Applications will be collected via MathPrograms.org September 1, 2022–December 1, 2022. All applicants will be notified in February 2023 whether or not they have been chosen to receive the fellowship.

Find more application information at https://www.ams.org/Birman-fellow. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; prof-serv@ams.org; 401-455-4189.

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New to the AMS: www.ams.org/join

Current eligible members who have not yet paid 2023 dues: www.ams.org/account

*Annual statement of unemployed status is required.
The American Mathematical Society (AMS) has reciprocity agreements with a number of mathematical organizations around the world. A current list of the reciprocating societies appears here; for full details of the agreements, see www.ams.org/membership/individual/mem-reciprocity.

Allahabad Mathematical Society
Argentina Mathematical Society
Australian Mathematical Society
Austrian Mathematical Society
Azerbaijan Mathematical Society
Balkan Society of Geometers
Bangladesh Mathematical Society
Belgian Mathematical Society
Berliner Mathematische Gesellschaft
Bharata Ganita Parishad
Brazilian Mathematical Society
Brazilian Society of Computational and Applied Mathematics
Calcutta Mathematical Society
Canadian Mathematical Society
Catalan Society of Mathematicians
Chilean Mathematical Society
Colombian Mathematical Society
Croatian Mathematical Society
Cyprus Mathematical Society
Danish Mathematical Society
Dutch Mathematical Society
Edinburgh Mathematical Society
Egyptian Mathematical Society
European Mathematical Society
Finnish Mathematical Society
German Mathematical Society
German Society for Applied Maths & Mechanics
Glasgow Mathematical Association
Hellenic Mathematical Society
Icelandic Mathematical Society
Indian Mathematical Society
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Iranian Mathematical Society
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Korean Mathematical Society
London Mathematical Society
Luxembourg Mathematical Society
Malaysian Mathematical Society
Mathematical Society of France
Mathematical Society of Japan
Mathematical Society of the Philippines
Mathematical Society of the Republic of China
Mathematical Society of Serbia
Mexican Mathematical Society
Mongolian Mathematical Society
Nepal Mathematical Society
New Zealand Mathematical Society
Nigerian Mathematical Society
Norwegian Mathematical Society
Palestine Society for Mathematical Sciences
Parana’s Mathematical Society
Polish Mathematical Society
Portuguese Mathematical Society
Punjab Mathematical Society
Ramanujan Mathematical Society
Romanian Mathematical Society of Mathematicians
Royal Spanish Mathematical Society
Saudi Association for Mathematical Sciences
Singapore Mathematical Society
Sociedad Matemática de la República Dominicana
Sociedad Uruguaya de Matemática y Estadística
Société Mathématiques Appliquées et Industrielles
Society of Associations of Mathematicians and Computer Scientists of Macedonia
Society of Mathematicians, Physicists, and Astronomers of Slovenia
South African Mathematical Society
Southeast Asian Mathematical Society
Spanish Mathematical Society
Swedish Mathematical Society
Swiss Mathematical Society
Tunisian Mathematical Society
Turkish Mathematical Society
Ukrainian Mathematical Society
Union of Bulgarian Mathematicians
Union of Czech Mathematicians and Physicists
Union of Slovak Mathematicians and Physicists
Vietnam Mathematical Society
Vijñāna Parishad of India
Semail Ülgen juggles much on a daily basis. During the day, she focuses on her work as chair of industrial engineering at Antalya Bilim University in Turkey. In the evening, she cooks, cleans, and takes care of her children, ages 14, 12, and 7. After the kids go to bed, she stays up late to tackle her pure mathematics research on Finsler geometry.

When Ülgen came to the US in 1998 to attend graduate school, other mathematicians told her that “if you have kids, you cannot be a mathematician.” Ülgen proved the naysayers wrong, giving birth to her first child while she was an assistant professor at Northwestern University. But it has been a stressful journey.

Raising children while pursuing an academic career in mathematics has always been difficult, especially for mothers, who still tend to shoulder the majority of child care and housework. Mathematicians who become parents depend on the support of family, colleagues, and institutions. On top of the challenges of everyday life, parents face logistical hurdles each time they want to attend a conference, whether in person or virtually.

To help parents participate more fully in conferences, the AMS offers child care grants for the Joint Mathematics Meetings (JMM) as well as AMS Sectional Meetings. Sixty mathematicians received these grants in spring 2022. “I can’t really explain how grateful I am to the AMS,” said Ülgen, who has received grants four times for virtual meetings.

“You have to spend some time for your kid, and you have to spend time for your profession. But you have to do it in a really wise way.” With enough planning and support, balancing mathematics and family is doable, she emphasized.

Facilitating Conference Attendance in Person and Virtually

Theresa Anderson gave birth to her first child, Lucian, in August 2018. Less than 18 months later came JMM 2020 in Denver. Lucian was still breastfeeding until shortly before the meeting, so traveling without him was not an
AAMS COMMUNICATION

option. An AMS child care grant enabled Anderson, an assistant professor at Purdue University at the time, to bring her husband along as a caregiver while she attended the conference. "Without that, I would have chosen not to go," she said.

At the JMM, Anderson gave a talk on her research in harmonic analysis and went to a session on automorphic forms and $L$-functions, where she met number theorist Amita Malik. Their discussion after Malik’s talk sparked an ongoing research collaboration that might never have formed if Anderson had not traveled to Denver.

Soon afterward, the pandemic closed daycares and schools and sent conferences online, presenting a new predicament for mathematicians trying to keep up with their field while taking care of children.

"Theoretically, it’s easier to attend" online conferences, said Anton Dochtermann, the father of 7-year-old Arlo and an associate professor at Texas State University. Yet for him, "it’s almost somehow flipped, because at least when I’m physically away, it’s clear that I’m not the caregiver. [...] But when you’re at home, and you’re trying to be on Zoom, it’s this in-between space where you’re not exactly 100% there, and also not 100% at the conference."

Child care grants enable attendees of virtual meetings to focus their undivided attention on mathematics. Dochtermann received grants for JMM 2022 and the 2022 Spring Western Sectional. During the weekdays of JMM, the grant went toward after-school care since Arlo’s kindergarten ended at 3 pm. During the weekend sectional, Dochtermann was able to hire a babysitter to come to the house while he attended talks either in another room or at his wife’s office. “[Paying a babysitter] $20 an hour adds up pretty fast, but it definitely helps to get a few hours” covered by a grant, he said.

In Turkey, the bulk of each AMS conference corresponded to Ulgen’s evening. Thanks to grants for JMM 2021, JMM 2022, and two AMS Sectional Meetings, she hired a babysitter to take care of her kids away from home from 4 pm until midnight.

At JMM 2021, Ulgen served as a judge for the online student poster presentations, which motivated her to organize a similar event for her department later in the year. This year, Ulgen attended talks related to non-commutative geometry, algebra, and topology, as well as sessions on applied mathematics that she found useful for her engineering research.

Anderson gave birth to her second child in spring 2021. Both boys spent the weekend with her parents during the 2022 Spring Eastern Sectional, with the grant going toward food and activities. “That money was just so helpful to make this a worthwhile experience instead of having the stress of watching children [and] attending the meeting at the same time,” she said.

Balancing Mathematics and Family

For Anderson, now at Carnegie Mellon University, the importance of child care grants extends beyond finances. The program “acknowledges what parents go through and makes it more of a norm in the mathematical community.”

Despite some (but limited) progress in creating a more inclusive environment for individuals from underrepresented communities, discussions of how raising children fits into a career in mathematics seem to lag behind, Dochtermann said. He talks frankly with collaborators and colleagues about child care duties but has found little space in the academic world for formal discussions about that time commitment. When he went up for tenure at Texas State, Dochtermann was encouraged to talk about how his work was affected by the pandemic. Yet he understood implicitly that “the fact that I had been raising a kid during that whole time, I felt, was not really something I was supposed to even mention”—even though taking care of Arlo was a central part of the pandemic’s impact on his life. Dochtermann adds that these burdens typically fall on the shoulders of women and people of color (his partner, also an academic, does a large share of

Figure 2. Theresa Anderson says that Lucian is sweet and imaginative. Her younger son, Jasper, loves to climb and cuddle.

Figure 3. Anton Dochtermann describes his son Arlo as “inquisitive, confident, and playful.”
the child care at home) and hence should be part of broader discussions of equity.

Even before the pandemic, traveling to conferences as a parent of a young child was a logistical headache for Dochtermann. His wife, a professor of geography, also traveled for work. Sometimes the pair would combine trips so that one parent could watch Arlo while the other worked. Still, each had to miss conferences they would have liked to attend.

Universities in the US often do not reimburse faculty for child care costs during conferences [1]. The National Science Foundation allows money from NSF conference awards to be used for child care [2] (though not the children’s travel expenses [3])—but only if the grant recipient’s institution also allows it. As awareness grows of the financial burden on academic parents, some universities are changing their policies to make traveling with children more affordable.

When institutions support parents pursuing mathematical careers, those efforts can also contribute to their children developing an interest in mathematics. Anderson’s oldest son, for instance, looks up to her as someone in an important profession. Now almost four years old, Lucian already knows that an 8 on its side is infinity. On one occasion, completely unprompted, he brought his mother a paper triangle and correctly identified it as isosceles.

“However difficult that you think [raising children] is, it is more difficult than that,” Anderson said. “Now on the flip side, it is also more rewarding than [you think].”

Learn more about how to apply for AMS child care grants at https://www.ams.org/meetings-child-care-grants.

References
[1] Tien Nguyen, For academic parents, work travel can be costly—but some universities are stepping up their support, Science, Sept. 4, 2019. DOI: 10.1126/science.caredit.aaz3871.
Professor Thomas J.R. Hughes won William Benter Prize in Applied Mathematics 2022

Professor Thomas J.R. Hughes, Peter O’Donnell Jr. Chair in Computational and Applied Mathematics and Professor of Aerospace Engineering and Engineering Mechanics, University of Texas at Austin, US, won the William Benter Prize in Applied Mathematics 2022.

Professor Hughes, a phenomenal leader in the field of computational science, engineering and mathematics, created isogeometric analysis, a novel approach to integrating computer-aided design (CAD) and computer-aided engineering (CAE). The ingenious idea is to derive finite element basis functions from CAD, which has revolutionised numerous engineering applications. His fundamental paper on this topic has been cited over 6,000 times on Google Scholar and is widely referred to throughout industry, national labs and academia.

Variational multiscale methods (VMS) and stabilised methods are the groundbreaking results of Professor Hughes’ research in computational fluid dynamics (CFD). They overcome a fundamental mathematical obstacle, the Babuska-Brezzi condition, and have enabled the construction of widely used methods for simulating viscous compressible flows. VMS is leading to new turbulence models and multiphase flow algorithms.

Additionally, Professor Hughes applied his expertise in computational mechanics to a different field: blood flow modelling. This work has evolved into the concept of “predictive medicine”. The underlying methodology has been commercialised to provide non-invasive, patient-specific, coronary disease diagnosis based on CFD.

Among his many prestigious awards, Professor Hughes has received the ASME Medal from the American Society of Mechanical Engineers (ASME), the John von Neumann Medal from the U.S. Association for Computational Mechanics (USACM), and the Gauss-Newton Medal from the International Association for Computational Mechanics (IACM).

There are two awards under the name of Professor Hughes: the Thomas J. R. Hughes Medal of USACM, and the Thomas J.R. Hughes Young Investigator Award of ASME.

Professor Hughes is a member of the US National Academy of Sciences, the US National Academy of Engineering, and the American Academy of Arts and Sciences, and is a foreign member of the Royal Society of London. He has been a plenary lecturer at the International Congress of Mathematicians.

The William Benter Prize will be presented during the opening ceremony for the International Conference on Applied Mathematics (ICAM 2023), which is co-organised by CityU’s Liu Bie Ju Centre for Mathematical Sciences (LBJ) and the Department of Mathematics.

The William Benter Prize in Applied Mathematics was set up by LBJ in honour of Mr William Benter for his dedication and generous support to the enhancement of the University’s strengths in mathematics. The prize recognises outstanding mathematical contributions that have had a direct and fundamental impact on scientific, business, finance and engineering applications. The cash prize of US$100,000 is given once every two years.

– City University of Hong Kong
2022 IMU Awards Announced

The International Mathematical Union (IMU) awarded several prestigious prizes at the International Congress of Mathematicians (ICM), held virtually in July 2022. The prize ceremony and prize lectures took place in person in Helsinki, Finland.

**Abacus Medal: Mark Braverman**

Mark Braverman of Princeton University was honored with the Abacus Medal “for his path-breaking research developing the theory of information complexity, a framework for using information theory to reason about communication protocols. His work has led to direct-sum theorems giving lower bounds on amortized communication, ingenious protocol compression methods, and new interactive communication protocols resilient to noise.” The citation reads: “Mark Braverman led the development of the theory of information complexity, the interactive analog of Shannon’s information theory. This theory has been successfully used to obtain tight lower bounds on the communication cost of interactive communication protocols for the joint computation of some function when the input is distributed across two or more parties. Braverman’s early work (with Rao) shows that in any interactive communication protocol between two players, the information revealed by the protocol to the players about the inputs hidden from them exactly equals the amortized communication complexity of computing independent copies of the same function. This extended his earlier work (with Barak, Chen, and Rao) that bounded amortized communication cost from below in terms of communication cost and internal information cost. These works led to a more nuanced understanding of different information cost measures, and along with later independent advances, establish that for randomized protocols, amortized communication costs can be significantly less than the non-amortized costs. It also spurred the design of new protocols achieving better compression, and new coding schemes that exhibit improved resilience to transmission errors. Braverman’s work has led to the use of information complexity in other interactive communication regimes as well, including distributed statistical estimation, memory requirements in streaming algorithms, and distributed error-correction. It has also found beautiful application in a notable non-communication setting: direct-product theorems for parallel repetition.

“In addition to his work on information complexity, Braverman has made contributions to diverse areas at the interface of theoretical computer science and mathematical sciences. These contributions include the settling of the long-standing conjecture concerning pseudorandomness and constant depth Boolean circuits—polylog-wise independence fools $\mathsf{AC}^0$, understanding the decidability frontier for geometric objects such as quadratic Julia sets (joint with Yampolsky), and developing tools for algorithm mechanism design.”

Mark Braverman received his PhD in 2008 from the Department of Computer Science at the University of Toronto under the supervision of Stephen Cook. He was a postdoctoral researcher at Microsoft Research New England from 2008 to 2010. In 2010 he was appointed assistant professor at the Departments of Mathematics and Computer Science at Toronto. He joined Princeton in 2011, where he has been professor of computer science since 2015. Braverman’s honors include the Stephen Smale Prize (2014), the EMS Prize (2016), the Presburger Award (2016) of the European Association for Theoretical Computer Science, and the NSF Waterman Award (2019). He has also been the recipient of a Sloan Research Fellowship, a Turing Fellowship from the John Templeton Foundation, an NSF CAREER Award, and a Packard Fellowship.

The IMU Abacus Medal is awarded for outstanding contributions in Mathematical Aspects of Information Sciences. It is a continuation of the Rolf Nevanlinna Prize.
Barry Mazur of Harvard University has been awarded the 2022 Chern Medal “for his profound discoveries in topology, arithmetic geometry and number theory, and his leadership and generosity in forming the next generation of mathematicians.” The prize citation reads in full: “Barry Mazur has shaped the modern landscape in arithmetic, by way of tackling the most difficult problems in the area, pioneering exciting new directions, and guiding generations of mathematicians to fertile new terrain. His numerous fundamental contributions place him squarely within the ranks of the greatest mathematicians of the 20th century.

“His proof of the torsion conjecture for elliptic curves gives an explicit list of possibilities for torsion subgroups of elliptic curves over the rational numbers. Aside from being a result of supreme elegance and finality, it is remarkable for the method of proof, which applies sophisticated methods of arithmetic geometry, notably the theory of group schemes in the flat topology and the geometry of arithmetic moduli spaces, to the study of a concrete diophantine question.

“The deep study of the ring theoretic properties of completed Hecke algebras which this work initiated, along with the deformation theory of Galois representations which Mazur pioneered several decades later, laid the groundwork for Andrew Wiles’ proof of the modularity of elliptic curves and Fermat’s last theorem. Indeed, it has enabled an avalanche of results in the Langlands program and its applications, as well as advances in the study of special values of L-functions following the conjecture of Bloch and Kato.

“Mazur’s proof of the Iwasawa main conjecture with Andrew Wiles in the early 1980s is another monumental achievement, which realizes the zeroes of the Kubota-Leopoldt p-adic zeta function as the eigenvalues of a natural operator acting on the ideal class groups of cyclotomic towers, and can thus be viewed as the analogue of the Riemann hypothesis for the p-adic zeta functions attached to Dirichlet characters.

“The extension of the methods of Iwasawa theory to the setting of elliptic curves and more general motives was largely spearheaded by Mazur, through his construction with Peter Swinnerton-Dyer of the p-adic L-function of a modular elliptic curve, his formulation of the p-adic Birch and Swinnerton-Dyer conjecture with John Tate and Jeremy Teitelbaum, and his seminal program for the study of abelian varieties over towers of number fields.

“These great achievements themselves are but a sampling of highlights within a broad, deep, and sustained range of influential perspectives that have enriched mathematics over the last fifty years, touching on such topics as the generalized Schoenflies problem, p-adic cohomology theories, the Fontaine–Mazur conjecture characterizing the p-adic Galois representations that arise in geometry, the theory of ‘Euler systems,’ and the study of rational points on curves and higher dimensional varieties, to name just a few.

“Mazur’s influence is cemented by his role as a mentor and a teacher, with close to sixty PhD students, of whom a great many have actively and fruitfully pursued his capacious intellectual legacy.

“Barry Mazur’s view of the subject is a supremely pluralistic one, in which pure curiosity interacts constantly with visionary theory-building, and the pursuit of deep structures for their own sake is just as important as computational experiments and the resolution of difficult problems. The impact he has had on mathematics dramatically illustrates the revolutionary potential of such a versatile perspective.”

Barry Mazur received his PhD from Princeton University in 1959. He followed this with a research fellowship at the Institute for Advanced Study (1958–1959) and a junior fellowship at Harvard University from 1959 to 1962, when he was named assistant professor at Harvard. He has been professor at Harvard since 1969. His honors include the AMS Veblen Prize in Geometry (1965), the Cole Prize in Number Theory (1982), and the Steele Prize for Seminal Contribution to Research (1999). He is also a recipient of the Chauvenet Prize of the Mathematical Association of America (1994) and the 2011 National Medal of Science. He is a Fellow of the AMS and an elected member of the American Academy of Arts and Sciences, the National Academy of Sciences, and the American Philosophical Society.

The IMU Chern Medal is awarded to an individual whose accomplishments warrant the highest level of recognition for outstanding achievements in the field of mathematics. It carries a prize of US$500,000, of which half is a cash prize and half is designated by the recipient to an organization of his or her choice to support research, education, or outreach programs in mathematics.
Lieb has proved several celebrated results in the area of matrix analysis including what became known as Lieb’s concavity theorem. As a consequence of the latter, he and Ruskai were able to prove the strong subadditivity of the von Neumann entropy, a result which decades later has become the foundational result in the very active modern area of quantum information theory.

“Lieb has made fundamental contributions to the areas of functional analysis and functional inequalities. He has both proved a long list of new functional inequalities, some of which now bear his name, as for instance the Brascamp–Lieb inequality, and brought existing ones into their sharp form, as for instance the Young and the Hardy–Littlewood–Sobolev inequalities. He has developed and masterly applied symmetrization and compactness methods. These works of Lieb are widely used by analysts, probabilistic and mathematical physicists and continue to have a tremendous impact in mathematics.

“Elliott H. Lieb’s work has a truly outstanding combination: as mathematical work, his contributions have a hard-to-rival impact on other sciences, and as applied work, his contributions have a hard-to-rival mathematical depth. It continues the tradition of a dialogue at the highest level between mathematics and physics, and beautifully demonstrates the power of mathematics as a theoretical and practical tool to understand nature.”

Lieb earned his PhD in 1956 from the University of Birmingham, England. He was a Fulbright Fellow at Kyoto University from 1946 to 1957, followed by research associate positions at the University of Illinois (1957–1958) and Cornell University (1958–1960). From 1960 to 1963 he served as staff theoretical physicist at International Business Machines Corporation Research Center. He spent the year 1961–1962 as visiting senior lecturer at Fourah Bay College in Sierra Leone, then became associate professor of physics at Yeshiva University. He has been professor of physics at Northeastern University (1966–1968) and professor of applied mathematics (1968–1973) and of mathematics and physics (1973–1974) at the Massachusetts Institute of Technology. He has been at Princeton University since 1974 and is now professor emeritus. Among his many honors are two Guggenheim Foundation Fellowships (1972, 1978); the Heineman Prize in Mathematical Physics (1978); the AMS–SIAM Birkhoff Prize in Mathematical Physics (1978) and the AMS Levi Conant Prize (2002); the Rolf Schock Prize (2001); the Poincare Prize (2003); a Simons Foundation Fellowship (2012); and the Erwin Schrödinger Institute Medal and Prize (2021). He is a Fellow of the AMS, of the American Physical Society, and of the American Association for the Advancement of Science. He is a member of the US National Academy of Sciences, the Austrian Academy of Sciences, the Royal Danish Academy of Sciences and Letters, and the Norwegian Academy of Science and Letters.

### Gauss Prize: Elliott H. Lieb

Elliott H. Lieb of Princeton University was awarded the 2022 Gauss Prize “for deep mathematical contributions of exceptional breadth which have shaped the fields of quantum mechanics, statistical mechanics, computational chemistry, and quantum information theory.” The citation reads in full: “Elliott H. Lieb is a mathematical physicist who has made outstanding contributions to physics, chemistry, and pure mathematics. Reminiscent of Gauss and other 18th- and 19th-century giants, Elliott H. Lieb, driven by problems in and applications to physics, has unraveled elegant and fundamental mathematical structures, vastly transcending the original motivations.

“In doing so, Lieb has introduced concepts which have shaped whole fields of research in mathematics even beyond his original area, while having a transformative impact on physics and chemistry. His work from the 60s discovering the exact solvability of some fundamental models (square ice, Lieb–Liniger, Lieb–Mattis, Temperley–Lieb) contributed to the definition of much of modern statistical mechanics, and to the foundations of the modern field of integrable probability. His exact solutions of percolation and coloring problems, introducing what is now called the Temperley–Lieb algebra along the way, also had long-lasting algebraic and combinatorics implications, for knot invariants, quantum groups, braid groups and braid statistics, and also conformal field theory.

“Lieb developed a whole program of the analysis of ‘stability of matter’ and ‘existence of the thermodynamic limit’ putting on firm mathematical grounds the study of quantum particles interacting through physically realistic potentials. An essential ingredient in this analysis is the celebrated Lieb–Thirring inequality estimating the sum of negative eigenvalues of Schrödinger operators. It has had an enormous impact over the years on spectral estimates, in particular on semi-classical analysis.

“One of the most important computational tools of modern chemistry is density functional theory (DFT). Lieb provided a mathematically sound formulation of it, the now famous Lieb or Levy–Lieb functional most widely used in theoretical chemistry. A very important concept in DFT is to understand what is known as the indirect Coulomb energy. Lieb, partly in collaboration with Oxford, gave rigorous bounds on it, providing an ultimate benchmark for all suggested forms of the indirect energy. These very important bounds have been abundantly used and cited by chemists.

Lieb earned his PhD in 1956 from the University of Birmingham, England. He was a Fulbright Fellow at Kyoto University from 1946 to 1957, followed by research associate positions at the University of Illinois (1957–1958) and Cornell University (1958–1960). From 1960 to 1963 he served as staff theoretical physicist at International Business Machines Corporation Research Center. He spent the year 1961–1962 as visiting senior lecturer at Fourah Bay College in Sierra Leone, then became associate professor of physics at Yeshiva University. He has been professor of physics at Northeastern University (1966–1968) and professor of applied mathematics (1968–1973) and of mathematics and physics (1973–1974) at the Massachusetts Institute of Technology. He has been at Princeton University since 1974 and is now professor emeritus. Among his many honors are two Guggenheim Foundation Fellowships (1972, 1978); the Heineman Prize in Mathematical Physics (1978); the AMS–SIAM Birkhoff Prize in Mathematical Physics (1978) and the AMS Levi Conant Prize (2002); the Rolf Schock Prize (2001); the Poincare Prize (2003); a Simons Foundation Fellowship (2012); and the Erwin Schrödinger Institute Medal and Prize (2021). He is a Fellow of the AMS, of the American Physical Society, and of the American Association for the Advancement of Science. He is a member of the US National Academy of Sciences, the Austrian Academy of Sciences, the Royal Danish Academy of Sciences and Letters, and the Norwegian Academy of Science and Letters.”
Academy, and the American Academy of Arts and Sciences, a foreign member of the Royal Society, and a member of the Academia Europaea. The Gauss Prize is awarded to individuals whose mathematical research has had an impact outside of mathematics. It is awarded jointly by the German Mathematical Union and the IMU.

**Leelavati Prize: Nikolai Andreev**

Nikolai Andreev of the Steklov Mathematical Institute of the Russian Academy of Sciences (Moscow) has been named the recipient of the IMU Leelavati Prize “for his contribution to the art of mathematical animation and of mathematical model-building, in a style which inspires the young and the old alike, and which mathematicians around the world can adapt to their varied uses—as well as for his indefatigable efforts to popularize genuine mathematics among the public via videos, lectures, and a prize-winning book.” The full citation reads: “Nikolai Nikolayevich Andreev is being awarded the Leelavati Prize 2022, for his wondrous art of mathematical animation and of mathematical model building. Animation differs from simulation: it is a computer-generated video of a mathematical story unfolding in front of our eyes. Models differ from 3D printing: they are made of wood or paper, playfully simple yet executed with consummate craftsmanship. Andreev’s signature style of this art captures the imagination of both the young and the old, and offers potential for a variety of uses in the popularization of mathematics. In parallel, he is recognized for tremendous resilience in often overcoming hardship to continue kindling enthusiasm for mathematics among a large number of people of all life circumstances, via web resources, lectures, and a book.

“We should recall that Eastern Europe has a rich tradition, harking back at least to the era of Tchebyshev, of organizing nationwide ladders, so to speak, of mathematical activities from small children to college students; some of it has been exported abroad, as witnessed by ‘math circles’ that flourish in hubs of initiative around the globe today. Out of this tradition came *Kvant*, arguably the highest-quality magazine of popularization in mathematics and theoretical physics that the world has seen, commanding in its heyday a circulation of $2 \times 10^3$. It is to this tradition that Andreev, or Kolya to his many friends, was born in 1975 in Saratov, and to this tradition that he has claim to be a leading successor in our 21st century.

“His early training was as a researcher. Andreev graduated from the Faculty of Mechanics and Mathematics of Moscow State University, completing a candidate’s degree (PhD) in 2000 in the area of extremal problems and approximation theory, codes and designs. In the same year he began working at Steklov Mathematical Institute, where he has been based ever since.

“2002 marked a watershed: Andreev gathered a team of R. A. Koksharov (senior developer, web design), M. A. Kalinichenko (graphics, video producer), and N. M. Panyunin (mathematics) and created the project ‘Mathematical Études.’ The project is a treasure trove of animation videos, available to everybody free of charge. Each video gives a brief but genuine mathematical experience of an interesting point that is elementary but little known; as such, it is orthogonal to the common style of presenting journalistically some fashionable topic. He also recruited A. D. Leshinskii, an artist of stunning skill who realizes mathematical phenomena in beautiful wooden models. In 2010 Andreev was made head of the Laboratory of Popularization of Steklov Institute. The productions of his lab include, alongside the ongoing growth of Études plus models, the collection ‘mechanisms by Tchebyshev,’ moveable gadgets which make intriguing uses of mathematics, and the book Математическая составляющая (which we may translate freely as *Mathematical Take on Things*), an anthology of about thirty prominent mathematicians on a luxuriant diversity of material reminiscent of *Kvant*. The book, first published 2015 by [Andreev], S. P. Konovalov, Panyunin, and Koksharov, earned a gold medal for scientific writing [in] 2017; the second edition, more than double in content, followed in 2019.

“Andreev travels the length and breadth of Russia to deliver lectures, well over 1,000 in twenty years, reaching out to by now countless members of the public, especially adolescents.

“Time and again extraordinary dedication and perseverance saw his cause through chronic administrative and financial trammels, endless negotiations and setbacks. Whenever his team’s funding dried up, he divided his own salary in equal parts among himself and the other staff of the team, in order to keep the work alive.

“For all his accomplishments, much of Andreev’s career is still ahead of him. We salute Kolya, as one representative of the community of mathematicians through the centuries who gave of themselves selflessly to doing mathematics with each rising generation. ‘We look forward to being raised by his future work for decades to come.’”

Andreev’s honors include the Prize of the President of the Russian Federation in the Area of Sciences and Innovations for Young Scientists (2010) and the Gold Medal of the Russian Academy of Sciences (2017). His specialties
are function approximations, trigonometric polynomials, orthogonal polynomial systems, cubic formulas, and discrete geometry.

The prize, sponsored by Infosys, recognizes outstanding contributions to increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors. It carries a cash prize of 1,000,000 Indian rupees (approximately US$12,600).

—Elaine Kehoe from IMU announcements

Credits

Photo of Mark Braverman is courtesy of Lance Murphey.
Photo of Barry Mazur is courtesy of Lance Murphey.
Photo of Elliott H. Lieb is courtesy of Lance Murphey.
Photo of Nikolai Andreev is courtesy of Sergei Amirdzhanov.
Deaths of AMS Members

Hassoon S. Al-Amiri, of Bowling Green, Ohio, died on October 14, 2017. Born on September 25, 1927, he was a member of the Society for 50 years.

John S. Alin, of McMinnville, Oregon, died on June 8, 2019. Born on August 14, 1940, he was a member of the Society for 52 years.

Daniel D. Anderson, of Coralville, Iowa, died on April 24, 2022. Born on December 20, 1948, he was a member of the Society for 48 years.

Richard B. Barrar, of Happy Valley, Oregon, died on August 5, 2019. Born on October 12, 1923, he was a member of the Society for 68 years.

Manjit S. Bhatia, of Potomac, Maryland, died on November 22, 2020. Born on October 12, 1936, he was a member of the Society for 37 years.

Sherwood F. Ebey, of Macon, Georgia, died on December 23, 2019. Born on March 7, 1932, he was a member of the Society for 63 years.

R. C. Entringer, of Scottsdale, Arizona, died on June 18, 2020. Born on May 17, 1931, he was a member of the Society for 59 years.

James E. Hall, of Willoughby, Ohio, died on October 23, 2019. Born on October 3, 1936, he was a member of the Society for 57 years.

Gordon B. Hare, of Walla Walla, Washington, died on September 16, 2019. Born on August 14, 1930, he was a member of the Society for 58 years.

Eleanor Green Jones, of Virginia Beach, Virginia, died on March 1, 2021. Born on August 10, 1929, she was a member of the Society for 50 years.

Wilfred M. Kincaid, of Ann Arbor, Michigan, died on December 12, 2015. Born on September 13, 1918, he was a member of the Society for 72 years.

Phyllis M. Kittel, of Nathrop, Colorado, died on December 14, 2019. Born on March 13, 1938, she was a member of the Society for 49 years.

James W. Lea, Jr., of Murfreesboro, Tennessee, died on May 21, 2022. Born on March 17, 1941, he was a member of the Society for 52 years.

Reinhard Mennicken, of Germany, died on June 13, 2019. Born on March 16, 1935, he was a member of the Society for 34 years.

Randal P. Miller, of Fairlawn, Ohio, died on March 30, 2022. Born on July 7, 1938, he was a member of the Society for 45 years.

Mary M. Neff, of Atlanta, Georgia, died on February 3, 2021. Born on January 20, 1930, she was a member of the Society for 66 years.

Don L. Pigozzi, of Oakland, California, died on November 17, 2021. Born on June 29, 1935, he was a member of the Society for 54 years.

William M. Priestley, of Sewanee, Tennessee, died on June 22, 2022. Born on August 4, 1940, he was a member of the Society for 60 years.

Larry Dennis Smith, of Huntington Beach, California, died on March 7, 2022. Born on November 15, 1937, he was a member of the Society for 46 years.

Andrew Sterrett, Jr., of Granville, Ohio, died on May 22, 2021. Born on April 3, 1924, he was a member of the Society for 62 years.

R. Stuik, of Boulder, Colorado, died on February 9, 2022. Born on December 2, 1930, he was a member of the Society for 64 years.

Jack P. Tull, of San Antonio, Texas, died on March 5, 2022. Born on December 2, 1930, he was a member of the Society for 64 years.

Dale E. Varberg, of Minneapolis, Minnesota, died on April 18, 2019. Born on September 9, 1930, he was a member of the Society for 62 years.

Richard S. Varga, of Avon, Ohio, died on February 25, 2022. Born on October 9, 1928, he was a member of the Society for 70 years.

W. V. Vasconcelos, of Princeton, New Jersey, died on June 17, 2021. Born on May 17, 1937, he was a member of the Society for 58 years.

Evelyn K. Wantland, of Urbana, Illinois, died on February 4, 2022. Born on June 22, 1917, she was a member of the Society for 73 years.

Robert H. Wasserman, of East Lansing, Michigan, died on March 2, 2022. Born on January 3, 1923, he was a member of the Society for 73 years.

James V. White, of North Hampton, Massachusetts, died on May 5, 2021. Born on May 20, 1941, he was a member of the Society for 45 years.
Jitomirskaya Awarded Ladyzhenskaya Prize

Svetlana Jitomirskaya of the Georgia Institute of Technology has been awarded the inaugural Ladyzhenskaya Prize in Mathematical Physics (OAL Prize) by the World Meeting for Women in Mathematics (WM²) “for her seminal and deep contributions to the spectral theory of almost periodic Schrödinger operators.” The prize citation reads as follows:

“Svetlana Jitomirskaya has made profound, transformative contributions to the spectral theory of almost periodic Schrödinger operators and Jacobi matrices. The prototypical example is the almost Mathieu operator, a three-parameter family of discretized Schrödinger operator on a one-dimensional lattice with an incommensurate periodic potential, of great interest both for mathematicians and physicists. Physically it describes an electron on a two-dimensional lattice in a perpendicular magnetic field. For this model and related almost periodic Schrödinger operators and Jacobi matrices, Jitomirskaya developed new techniques to obtain landmark results on the spectrum, the Lyapunov exponents, and the delicate arithmetic effects at non-Diophantine frequency parameter. She and her work have a big impact on the community, both by drawing senior mathematicians to the field and by raising a new generation of young researchers.

“In particular, she established the existence of the metal-insulator transition for the almost Mathieu operator: she proved that as one varies the coupling parameter the spectrum goes from absolutely continuous to pure point, with singular continuous spectrum at the transition point. This was the first non-perturbative result on a problem with small denominators, for which previous results, based on Kolmogorov–Arnold–Moser perturbation theory, applied only for very small or (by duality) very large coupling parameter. With Jean Bourgain, Jitomirskaya proved the first general result on the continuity of the Lyapunov exponents. With Artur Avila, she established a key component in the solution of the celebrated Ten Martini Problem, by showing that the pure point spectrum is a Cantor set. With Igor Krasovsky, she settled the Thouless 1/2-conjecture, by proving that the Hausdorff dimension of the spectrum at critical coupling is at most 1/2 for all irrational values of the frequency parameter.”

Jitomirskaya was born in Kharkiv, Ukraine, and received her PhD from Moscow State University in 1991 under the supervision of Yakov Sinai. She took a position at the University of California, Irvine, rising to the rank of distinguished professor. She joined Georgia Tech in July 2022 as the inaugural Elaine M. Hubbard Chair. Her awards and honors include a Sloan Fellowship (1996), the AMS Ruth Lyttle Satter Prize in 2005, and the Dannie Heineman Prize for Mathematical Physics in 2020. She is a member of the American Academy of Arts and Sciences and of the U.S. National Academy of Sciences.

The World Meeting for Women in Mathematics is organized every four years by the International Mathematical Union (IMU) as a satellite event of the International Congress of Mathematicians. The OAL Prize is awarded at a joint session of WM² and the Probability and Mathematical Physics Conference. The prize is funded by the Simons Foundation and carries a cash award of 10,000 euros (approximately US$10,200).

—From an IMU announcement

2022 Dirac Medals Awarded

Joel Lebowitz of Rutgers University, Elliott Lieb of Princeton University, and David Ruelle of the Institut des Hautes Études Scientifiques (IHES) have been awarded the 2022 Dirac Medals for Mathematical Physics by the International Centre for Theoretical Physics (ICTP) “for groundbreaking and mathematically rigorous contributions to the understanding of the statistical mechanics of classical and quantum physical systems.” According to the prize citation, their work “has very significantly deepened and expanded our mathematical understanding of physical systems in many new directions, sometimes different from the traditional ones. Their major contributions include, among others, the study of non-equilibrium physics and large deviations; the proof of the stability of matter; the analytic solution...
of two-dimensional models; seminal results in quantum information theory; the definition of Gibbs states for infinite systems; and the analysis of chaos and turbulence.”

Lebowitz earned his PhD from Syracuse University in 1956 under the supervision of Peter Bergmann. He held positions at Yale University, the Stevens Institute of Technology, and Yeshiva University before joining the faculty at Rutgers. With Elliott Lieb, he proved that the Coulomb interactions obey the thermodynamic limit, and he established the Lebowitz inequalities for the ferromagnetic Ising model. He was editor in chief of the Journal of Statistical Physics from 1975 to 2018. Among his awards and honors are the Boltzmann Medal (1992), the Nicholson Medal of the American Physical Society (1994), the Poincaré Prize (2000), the Max Planck Medal (2007), the Grande Médaille of the French Academy of Sciences (2014), and the Heineman Prize for Mathematical Physics (2021). He is a Fellow of the AMS and of the American Physical Society and a member of the National Academy of Sciences.

Lieb received his PhD from the University of Birmingham, England, in 1956. After receiving his doctorate, he held positions at Kyoto University, the University of Illinois, and Cornell University. From 1960 to 1963 he was staff theoretical physicist at IBM Corporation Research Center. He returned to academia with positions at Yeshiva University, Northeastern University, and the Massachusetts Institute of Technology before joining the faculty at Princeton in 1974. His honors include the Heineman Prize in Mathematical Physics (1978); the AMS–SIAM Birkhoff Prize in Mathematical Physics (1978) and the AMS Levi Conant Prize (2002); the Rolf Schock Prize (2001); the Poincaré Prize (2003); a Simons Foundation Fellowship (2012); the Erwin Schrödinger Institute Medal and Prize (2021); and the Gauss Prize of the International Mathematical Union (2022). He is a Fellow of the AMS, of the American Physical Society, and of the American Association for the Advancement of Science. He is a member of the US National Academy of Sciences, the Austrian Academy of Sciences, the Royal Danish Academy, and the American Academy of Arts and Sciences, a foreign member of the Royal Society, and a member of the Academia Europaea.

Ruelle received his PhD from the Université Libre de Bruxelles in 1959 under the direction of Res Jost. He held positions at ETH Zurich and the Institute for Advanced Study, Princeton, before becoming professor at IHES in 1964; he is currently emeritus professor at IHES and distinguished visiting professor at Rutgers University. His honors include the Heineman Prize for Mathematical Physics (1985), the Boltzmann Medal (1986), the Holweck Prize in physics (1993), the Matteucci Medal (2004), the Poincaré Prize (2006), and the Max Planck Medal (2014). He is a Fellow of the AMS and a member of the French Academy of Sciences and the Academia Europaea. He is an honorary international member of the American Academy of Arts and Sciences, an international member of the National Academy of Sciences, and a foreign member of the Accademia Nazionale dei Lincei. He tells the Notices: “Earlier in life I did quite a bit of hiking and camping, mostly alone, in remote parts of Mexico and other distant places. This has formed an important part of my view of the world.”

—From ICTP announcements

AWM Presidential Recognition Award

The 2022 Association for Women in Mathematics (AWM) Presidential Recognition Award was presented to the founders of Mathematically Gifted and Black (MGB): Erica Graham (Bryn Mawr College), Raegan Higgins (Texas Tech University), Candice Price (Smith College), and Shelby Wilson (Johns Hopkins University). The citation reads: “The website Mathematically Gifted and Black (MGB), founded in 2016, highlights twenty-eight living Black mathematicians each year during Black History month. The honorees are featured on the website with a photo and their...”

responses to questions about their lives and mathematical interests. The first-person storytelling and the frequent contextualization of math in terms of a full life experience make each of these contributions a small treasure trove of insight and inspiration. Honorees’ expertise ranges from K–12 education to government and industry to highly technical research mathematics, reflecting a broad and inclusive definition of who is a mathematician that the rest of the math community would benefit from emulating. As of today, MGB offers almost 200 mathematical role models for the Black community and provides abundant stories of Black excellence in math and in life. MGB also offers a tremendous resource to the broader math community, both for historical documentation and as a useful reference.” The AWM Presidential Recognition Award was established in 2014 to recognize those individuals who or programs that have significantly increased and/or supported women in mathematics.

—From an AWM announcement

AWM Service Awards

The Association for Women in Mathematics has announced the recipients of its 2022 Service Awards, which recognize individuals “for helping to promote and support women in mathematics through exceptional voluntary service.” The recipients are Katherine Dowd of the University of Minnesota, Robin Marek, Chair of the AWM Fund Development Committee, and Tracy Weyand of the Rose-Hulman Institute of Technology.

Dowd is administrative director of the School of Mathematics and assistant director of the Institute of Mathematics and Its Applications (IMA) at the University of Minnesota. She was recognized for the “extraordinary professionalism, wisdom, and care she bestowed upon the AWM as host of the 2022 AWM Research Symposium.” The citation reads in part: “In the face of pandemic-related setbacks, last-minute schedule changes, unusual requests from the AWM, and various technical and logistic difficulties, Katherine Dowd remained calm, flexible, organized, responsive, patient, thoughtful, and thorough. …She communicated with the staff at the conference venue to ensure that everything AWM suggested would work and she consistently communicated with the AWM to make sure that the event was carefully orchestrated.” As assistant director of IMA, she “provided support for the Roots of Unity Workshop—designed to support women, particularly women of color, who are in years 1–3 of graduate school and are considering research in algebra, combinatorics, geometry, topology, or number theory—and she arranged for collaboration space for the WinCompTop Research Network.” Dowd received her PhD in chemistry from the University of Illinois at Urbana-Champaign. She served as a lecturer in chemistry and as the outreach coordinator for the College of Science and Engineering at the University of Minnesota.

Marek was recognized for her “exceptionally generous contribution of time and expertise in helping the AWM establish a more professional and trustworthy fund development program.” Her experience as a fund development professional includes serving as development director for the AMS, as well as holding fund development positions for major universities and large health care organizations. The prize citation states: “Marek was instrumental in securing two major gifts and generally offers advice and guidance in every aspect of fundraising and fund development. …She continues to guide the development committee as AWM identifies and sets fundraising priorities, establishes an AWM case for funding, and develops a culture of philanthropy.”

Weyand was recognized “for building communities in which women in mathematics can thrive and feel welcome.” The citation reads in part: “Tracy Weyand has been involved in the founding of two Student Chapters of the AWM. As a postdoc at Baylor University in 2015, she was the co-founder of the Baylor chapter of the AWM, where she served as the co-advisor for the remainder of her postdoc. In 2018, as a tenure-track assistant professor, Weyand founded and has served as the faculty advisor for the Rose-Hulman Student Chapter of the AWM.” She has led the chapter in creating and holding a Sonia Kovalevsky Day for high school girls on the Rose-Hulman campus, as well as building community through other mathematical activities. The conclusion reads: “At an institution where 75% of the student body is male, Tracy Weyand’s outreach activities have been a vital step towards building a more welcoming and inclusive community.” Weyand received her PhD from Texas A&M University in 2014. She was a postdoctoral associate at Baylor University before joining the Rose-Hulman faculty. Her areas of interest are analysis and mathematical physics, specializing in spectral graph theory.

—From an AWM announcement
Hurtubise and Deng Awarded CMS Prizes

The Canadian Mathematical Society (CMS) has awarded prizes for 2022. Jacques Hurtubise of McGill University received the David Borwein Distinguished Career Award, and Qin Deng of the Massachusetts Institute of Technology was awarded the Blair Spearman Doctoral Prize. Hurtubise was honored “for his exceptional, continued, and broad contributions to mathematics.” According to the prize citation, he has gained “worldwide recognition for his seminal work on the topology of the moduli spaces of instantons, as well as on integrable systems.” Hurtubise earned his doctorate from Oxford University in 1982 as a Rhodes Scholar. He held a teaching position at the University of Quebec at Montreal before joining McGill in 1988. He served as Chair of the Department of Mathematics and Statistics at McGill from 2009 to 2015 and from 2019 to 2022. He held posts of deputy director and director of the Centre de Recherches Mathématiques (CRM) between 1996 and 2003 and of president of the CMS from 2010 to 2012. He was a Fellow of the Institute for Advanced Study (1987–1988) and was awarded an AMS Centennial Fellowship for 1993–1994. In 1993 he received the Coxeter–James Prize of the CMS. He is a Fellow of the Royal Society of Canada and was an inaugural Fellow of the AMS, as well as of the CMS. Deng was honored for a thesis that, according to the prize citation, “demonstrates [his] ability to assimilate a wide variety of deep, technical results and originality in combining them to make substantial advances.” The citation states: “Qin Deng is an outstanding researcher working in metric and Riemannian geometry as well as geometric analysis. Deng’s thesis contains a solution to a long-standing open problem in the theory of manifolds with lower Ricci curvature bounds and RCD spaces.” Deng received his PhD from the University of Toronto under the supervision of Vitali Kapovitch. He has been the recipient of the Ida Bulat Teaching Award, the Malcolm Slingsby Robertson Prize, a George F. D. Duff Graduate Fellowship, and an NSERC Alexander Graham Bell Canada Graduate Scholarship. He is currently a C. L. E. Moore Instructor at MIT.

—From CMS announcements

NCTM Lifetime Achievement Award

Carolyn Maher of Rutgers University has been awarded the 2022 Lifetime Achievement Award of the National Council of Teachers of Mathematics (NCTM). She is the founder and director of the Robert B. Davis Institute for Learning at Rutgers and editor of the Journal of Mathematical Behavior. She received her EdD in mathematics education and statistics in 1972 from Rutgers. She was a secondary mathematics teacher and a professor and assistant dean at Middlesex Community College in Edison, New Jersey, before joining the faculty at Rutgers, where she is Distinguished Professor. Her research focuses on longitudinal and cross-sectional studies of students’ mathematical reasoning and argumentation and on the importance of collaboration in research on learning. She has given invited talks throughout the world and has organized workshops and courses in South Africa, Brazil, and Mozambique. She helps teachers to develop effective teaching methods through attention to deep knowledge of mathematics and to student differences and strengths and to be aware of student diversity and ways of building knowledge. She has been president of the North American Group of the Psychology of Mathematics Education, chair of the American Education Research Association, and an elected member to the Holmdel Public Schools Board of Education. She has also served on the editorial boards of the British Journal of Educational Studies and the Journal for Research in Mathematics Education. She tells the Notices: “I love classical music, ballet, long walks, travel, and special time with family (especially my three grandchildren).”

—From an NCTM announcement

Dimitrov and Gao Awarded 2022 David Goss Prize

Vesselin Dimitrov of the University of Toronto and Ziyang Gao of Leibniz Universität Hannover were named recipients of the 2022 David Goss Prize in Number Theory. According to the prize citation, Dimitrov “has solved several classical problems in number theory and arithmetic geometry with very original and imaginative methods. His work touches on many topics regarding the heights of algebraic points, G-series, and modular forms.” Gao “has made a remarkable contribution to arithmetic geometry
Marrakchi Awarded 2022 Zemánek Prize

Amine Marrakchi (Ecole Normale Supérieure de Lyon, France) has been awarded the 2022 Jaroslav and Barbara Zemánek Prize in functional analysis with emphasis on operator theory “for his groundbreaking achievements in the theory of operator algebras, ergodic theory and geometric group theory, and especially for his contributions to the study of the Connes’s bicentralizer problem for type III von Neumann algebras.” Marrakchi received his PhD in 2018 from the Université Paris-Sud under the supervision of Cyril Houdayer. He held a postdoctoral position at Kyoto University from 2018 to 2019 and is a CNRS Junior Researcher at ENS Lyon. The Zemánek Prize was founded by the Institute of Mathematics of the Polish Academy of Sciences to encourage research in functional analysis, operator theory and related topics. The Prize is awarded to mathematicians under thirty-five years of age who have made important contributions to the field.

—From a Polish Academy of Sciences announcement

Credits
Photo of Svetlana Jitomirskaya is courtesy of UCI Physical Sciences Communications.
Photo of Joel Lebowitz is courtesy of Joel Lebowitz.
Photo of Elliott Lieb is courtesy of Lance Murphey.
Photo of the founders of Mathematically Gifted and Black is courtesy of Erica Graham, Raegan Higgins, Candice Price, and Shelby Wilson.
Photo of Tracy Weyand is courtesy of Bryan Cantwell.
Photo of Jacques Hurtubise is courtesy of Owen Egan.
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*Benefit rates subject to change based on yearly renewals and budget allocations.

The American Mathematical Society is committed to creating a diverse environment and is proud to be an equal opportunity employer. The AMS supports equality of opportunity and treatment for all individuals, regardless of sex, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disability status, economic background, veteran or immigration status, or any other social or physical component of their identity.
Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices may be submitted to notices@ams.org.

NSF Launching Early-Career Academic Pathways in the Mathematical and Physical Sciences Program

The National Science Foundation (NSF) Launching Early-Career Academic Pathways in the Mathematical and Physical Sciences (LEAPS-MPS) call has an emphasis to help launch the careers of pretenure faculty in Mathematical and Physical Sciences (MPS) fields at institutions that do not traditionally receive significant amounts of NSF-MPS funding. LEAPS-MPS has the additional goal of achieving excellence through diversity and aims to broaden participation to include members from groups historically excluded and currently underrepresented in the mathematical and physical sciences. These grants are intended to support MPS principal investigators, particularly at the aforementioned institutions, for whom LEAPS funding would enable the PI to submit a subsequent successful proposal to a traditional, already-existing NSF funding opportunity. Proposals from institutions of higher education are due by January 26, 2023. For more information, see https://www.nsf.gov/pubs/2022/nsf22604/nsf22604.pdf.

—From an NSF announcement

NSF Mathematical Sciences Infrastructure Program

The primary aim of the Mathematical Sciences Infrastructure Program is to foster the continuing health of the mathematical sciences research community as a whole. In addition, the program complements the Workforce Program in the Mathematical Sciences in its goal to increase the number of well-prepared US-based individuals who successfully pursue careers in the mathematical sciences and in other professions in which expertise in the mathematical sciences plays an increasingly important role. The DMS Infrastructure program invites projects that support core research in the mathematical sciences, including: (1) novel projects supporting research infrastructure across the mathematical sciences community, (2) training projects complementing the Workforce Program, and (3) conference, workshop, and travel support requests that include cross-disciplinary activities or have an impact at the national scale. The due date for full proposals is February 7, 2023. See https://beta.nsf.gov/funding/opportunities/mathematical-sciences-infrastructure-program.

—From an NSF announcement

AAUW Educational Foundation Fellowships and Grants

The American Association of University Women (AAUW) has programs supporting women students and scholars at various stages of their careers. American Fellowships support women in full-time study to complete dissertations, conduct postdoctoral research, or prepare research for publication. International Fellowships support women pursuing graduate or postdoctoral study in the United States who are not US citizens or permanent residents. Selected Professions Fellowships support women students in areas in which women’s participation has traditionally been low, including computer/information sciences and mathematics/statistics. For deadline dates and more information, see the website https://www.aauw.org/resources/programs/fellowships-grants/.

—From AAUW website
AWM Essay Contest

The Association for Women in Mathematics (AWM) and Math for America cosponsor an annual essay contest for biographies of contemporary women mathematicians and statisticians in academic, industrial, and government careers. This contest is open to students in Grades 6–8, Grades 9–12, and college undergraduates. The deadline for entries is February 1, 2023. AWM is also currently seeking women mathematicians to volunteer as the subjects of these essays. For more information, see https://awm-math.org/awards/student-essay-contest/.

—AWM announcement

News from MSRI/SLMath


Call for Program Proposals

MSRI/SLMath invites proposals for full- or half-year programs to be held at MSRI. A scientific program at SLMath generally consists of up to nine months of concentrated activity in a specific area of current research interest in the mathematical sciences. SLMath usually runs two programs simultaneously, each with about forty mathematicians in residence at any given time. The most common program length is four months (typically in the form of a Fall or Spring semester program). Each program begins with a Connections workshop and an Introductory workshop, the purpose of which are to introduce the subject to the broader mathematical community and connect early-career researchers, especially women, gender-expansive individuals, and members of minority groups, to senior mentors in the field. The programs receive administrative and financial support from the Institute, allowing organizers to focus on the scientific aspects of the activities. Deadlines for proposals are March 1, October 1, and December 1. For more information, see www.msri.org/proposal. Proposals are also invited for Hot Topics Workshops and Summer Graduate Schools.

Spring 2023 Scientific Workshops

- **January 23–27, 2023. Introductory Workshop: Algebraic Cycles, L-Values, and Euler Systems.** Organizers: Henri Darmon (McGill University), Ellen Eischen (University of Oregon; lead organizer), Benjamin Howard (Boston College), Elena Mantovan (California Institute of Technology), Andrew Snowden (University of Michigan), Linquan Ma (Purdue University), Karen Smith (University of Michigan), Andrew Snowden (University of Michigan), Irena Swanson (Purdue University).

2023 Summer Research in Mathematics (SRiM) Program

MSRI/SLMath invites applications for the 2023 Summer Research in Mathematics (SRiM) program. This program provides space, funding, and the opportunity for in-person collaboration to small groups of mathematicians, especially women and gender-expansive individuals, whose ongoing research may have been disproportionately affected by various obstacles, including family obligations, professional isolation, or access to funding. Groups of two to six mathematicians with partial results on an established program may apply. Visits to the program must take place between June 5, 2023, and July 14, 2023. For full details, see www.msri.org/summer. The deadline for applications is December 1, 2022.
2023 ADJOINT Workshop
MSRI/SLMath invites applications for its 2023 African Diaspora Joint Mathematics Workshop, to be held June 19–30, 2023, in Berkeley, California. The main objective of ADJOINT is to provide opportunities for in-person research collaboration to US mathematicians, especially those from the African Diaspora, who will work in small groups with research leaders on various research projects. For more information and to apply, see www.msri.org/adjoint. The deadline date for applications is December 1, 2022.

Call for Membership for 2023–2024 Scientific Research Programs
MSRI/SLMath invites applications for its 2023–2024 scientific research programs and for research memberships and postdoctoral fellowships. The research programs are:

- **August 21–December 20, 2023:** Algorithms, Fairness, and Equity
- **August 21–December 20, 2023:** Mathematics and Computer Science of Market and Mechanism Design
- **January 16–May 24, 2024:** Commutative Algebra
- **January 16–May 24, 2024:** Noncommutative-Algebraic Geometry

Research Memberships are intended for researchers who will be making contributions to a program and who will be in residence for one or more months. The deadline for applications for Research Memberships is December 1, 2022. Postdoctoral Fellowships are intended for those who have received their PhDs no longer than five years before the start of the program. The deadline for applications for Postdoctoral Fellowships is December 1, 2022. For more information and to apply, see www.msri.org/application.

—From MSRI/SLMath announcements

News from IMSI

The Institute for Mathematical and Statistical Innovation (IMSI) invites applications for Research Memberships for two long programs in 2023–24:

- **Algebraic Statistics and Our Changing World: New Methods for New Challenges, September 18–December 15, 2023**
- **Data-Driven Materials Informatics: Statistical Methods and Mathematical Analysis, March 4–May 24, 2024**

Research Memberships are intended for researchers who will be in residence at IMSI for two weeks or longer to participate in one of the programs. Funding is available to support travel and lodging expenses. Further information and applications are available at www.imsi.institute/programs. Applications which are received by January 1, 2023 will have first priority for consideration. IMSI encourages applications from researchers at all career stages, including PhD students, and aims to make its programs broadly accessible to interested participants.

IMSI also welcomes proposals for research activity involving applications of statistics and mathematics to problems of significant scientific and societal interest. Areas of specific interest are climate and sustainability, data and information, health care and medicine, materials science, quantum computing and information, and uncertainty quantification. There are two proposal cycles each year, with deadlines on March 15 and September 15. Typical frameworks for activity include:

- Long programs
- Workshops
- Interdisciplinary Research Clusters
- Research Collaboration Workshops

For more information, see www.imsi.institute/proposals. To discuss ideas before submitting a proposal, please contact the Director at director@imsi.institute.

—Kevin Corlette
Classified Advertising
Employment Opportunities

ILLINOIS

University of Chicago
Assistant Professor

Position Description
The University of Chicago Department of Mathematics invites applications for the position of Assistant Professor. Successful candidates are typically two to four years past the Ph.D. These positions are intended for mathematicians whose work has been of outstandingly high caliber. Appointees are expected to have the potential to become leading figures in their fields. The appointment is generally for four years, with the possibility for renewal. The teaching obligation is up to three one-quarter courses per year.

Applicants are strongly encouraged to include additional information related to their teaching experience, such as evaluations from courses previously taught, as well as an AMS cover sheet. If you have applied for an NSF Mathematical Sciences Postdoctoral Fellowship, please include that information in your application, and let us know how you plan to use it if awarded. Questions may be directed to judygarza@uchicago.edu. We will begin screening applications on November 1, 2022. Screening will continue until all available positions are filled.

Qualifications
Completion of a Ph.D. in mathematics or a closely related field is required at the time of appointment.

Application Instructions
Required materials are: (a) a cover letter, (b) a curriculum vitae, (c) three or more letters of reference, at least one of which addresses teaching ability, (d) a description of previous research and plans for future mathematical research and (e) a teaching statement.

Applications must be submitted online through https://www.mathjobs.org/jobs/list/20321.

We seek a diverse pool of applicants who wish to join an academic community that places the highest value on rigorous inquiry and encourages diverse perspectives, experiences, groups of individuals, and ideas to inform and stimulate intellectual challenge, engagement, and exchange. The University’s Statements on Diversity are at https://provost.uchicago.edu/statements-diversity.

Job seekers in need of a reasonable accommodation to complete the application process should call 773-834-3988 or email equalopportunity@uchicago.edu with their request.

The Notices Classified Advertising section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society’s standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

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US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to clasads@ams.org.
Position Description
The University of Chicago Department of Mathematics invites applications for the position of L.E. Dickson Instructor. This is open to mathematicians who have recently completed or will soon complete a doctorate in mathematics or a closely related field, and whose work shows remarkable promise in mathematical research. The initial appointment is for a term of up to three years. The teaching obligation is generally four one-quarter courses per year. If you are two or more years post Ph.D., we encourage you to apply for our Assistant Professor position as well at [https://www.mathjobs.org/jobs/list/20322](https://www.mathjobs.org/jobs/list/20322).

Applicants are strongly encouraged to include additional information related to their teaching experience, such as evaluations from courses previously taught, as well as an AMS cover sheet. If you have applied for an NSF Mathematical Sciences Postdoctoral Fellowship, please include that information in your application, and let us know how you plan to use it if awarded. Questions may be directed to judygarza@uchicago.edu. We will begin screening applications on November 1, 2022. Screening will continue until all available positions are filled.

Qualifications
Completion of all requirements for a Ph.D. in mathematics or a closely related field is required at the time of appointment.

Application Instructions
Required materials are: (a) a cover letter, (b) a curriculum vitae, (c) three or more letters of reference, at least one of which addresses teaching ability, (d) a description of previous research and plans for future mathematical research and (e) a teaching statement.

Applications must be submitted online through [https://www.mathjobs.org/jobs/list/20322](https://www.mathjobs.org/jobs/list/20322).

We seek a diverse pool of applicants who wish to join an academic community that places the highest value on rigorous inquiry and encourages diverse perspectives, experiences, groups of individuals, and ideas to inform and stimulate intellectual challenge, engagement, and exchange. The University’s Statements on Diversity are at [https://provost.uchicago.edu/statements-diversity](https://provost.uchicago.edu/statements-diversity).

The University of Chicago is an Affirmative Action/Equal Opportunity/Disabled/Veterans Employer and does not discriminate on the basis of race, color, religion, sex, sexual orientation, gender identity, national or ethnic origin, age, status as an individual with a disability, protected veteran status, genetic information, or other protected classes under the law. For additional information please see the University’s Notice of Nondiscrimination—[https://www.uchicago.edu/non-discrimination](https://www.uchicago.edu/non-discrimination).

Job seekers in need of a reasonable accommodation to complete the application process should call 773-834-3988 or email equalopportunity@uchicago.edu with their request.

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**ILLINOIS**

**University of Chicago**

L.E. Dickson Instructor

**Position Description**

The Mathematics Department at MIT is seeking to fill positions in **Pure and Applied Mathematics**, and Statistics at the level of Instructor beginning September 1, 2023 (for the 2023–2024 academic year). Appointments are based primarily on exceptional research qualifications. Appointees will be expected to fulfill teaching duties and pursue their own research program. PhD in Mathematics or related field required by employment start date.

The Department of Mathematics offers supportive mentorship to junior faculty and instructors, an exceptional environment for mathematical inquiry, and a strong commitment to an inclusive, welcoming culture. MIT is an equal employment opportunity employer. All qualified applicants will receive consideration for employment and will not be discriminated against on the basis of race, color, sex, sexual orientation, gender identity, religion, disability, age, genetic information, veteran status, ancestry, or national or ethnic origin.

For more information and to apply, please visit [www.mathjobs.org](https://www.mathjobs.org). To receive full consideration, submit applications by December 1, 2022.

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**MASSACHUSETTS**

**Massachusetts Institute of Technology (MIT)**

**Cambridge, MA**

**Department of Mathematics**

The Mathematics Department at MIT is seeking to fill positions in **Pure and Applied Mathematics** at the level of tenure-track Assistant Professor or higher beginning July 1, 2023 (for the 2023–2024 academic year, or as soon thereafter as possible). Appointments are based primarily on exceptional research qualifications. Appointees will be expected to fulfill teaching duties and pursue their own research program. PhD in Mathematics or related field required by employment start date.

The Department of Mathematics offers supportive mentorship to junior faculty and instructors, an exceptional environment for mathematical inquiry, and a strong
commitment to an inclusive, welcoming culture. MIT is an equal employment opportunity employer. All qualified applicants will receive consideration for employment and will not be discriminated against on the basis of race, color, sex, sexual orientation, gender identity, religion, disability, age, genetic information, veteran status, ancestry, or national or ethnic origin.

For more information and to apply, please visit [mathjobs.org](http://www.mathjobs.org). To receive full consideration, submit applications by December 1, 2022.

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**MASSACHUSETTS**

**Boston University**

**Tenure-track Assistant Professor: Number Theory**

The Department of Mathematics and Statistics at Boston University invites applications for a tenure-track Assistant Professor position in number theory. A Ph.D. is required. All areas of number theory will be considered, with some preference given to automorphic representation theory and arithmetic geometry. Our Department is committed to building a diverse community of scholars and invites applicants not only to share their thoughts on teaching and research but also to indicate ways in which they may be able to contribute to the creation of an equitable and inclusive community in the Department.

Submit cover letter, CV, research statement, and teaching statement online to mathjobs.org, and arrange for the online submission to the same address of four recommendation letters, one of which addresses teaching. Application deadline: December 1, 2022. Pending final administrative and budgetary approval, the appointment will start on July 1, 2023. We welcome applications from all eligible candidates without regard to race, color, religion, sex, sexual orientation, gender identity, national origin, disability status, protected veteran status, or any other characteristic protected by law. Boston University is an equal opportunity employer and a VEVRAA Federal Contractor.

To Apply: [https://www.mathjobs.org/jobs/list/20425](https://www.mathjobs.org/jobs/list/20425)

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**UTAH**

**University of Utah, Department of Mathematics**

**Current Positions (2022–2023)**

The Department of Mathematics at the University of Utah invites applications for the following positions:

- Full-time tenure-track or tenured appointment at the level of Assistant Professor, Associate Professor, or Professor in applied mathematics, specifically in the Mathematics of Materials and related fields.
- Non-tenure-track appointments at the level of Wiley Assistant Professor in pure and applied mathematics. These are 3-year, termed [postdoctoral] positions which involve research and teaching, including Wiley Assistant Professorships funded by the Department of Mathematics and grants obtained by our faculty. These additional opportunities and federal funding sources include (1) an NSF Research Training Grant (RTG) on “Optimization and Inversion for the 21st Century Workforce” ([www.math.utah.edu/agtrtg2019/](http://www.math.utah.edu/agtrtg2019/)). Outstanding candidates in all areas of pure mathematics will be considered. Applicants to either RTG program must have a Ph.D. in Mathematics or a closely related field and must be US citizens, nationals, or permanent residents.
- Non-tenure-track appointment at the level of Assistant Professor to work on partial differential equation models for sea ice and its role in the climate system, with Distinguished Professor Ken Golden and colleagues. This is a 3-year, termed [postdoctoral] position which involves research and teaching, funded by an ONR grant on sea ice and the Department of Mathematics. Applicants must have a Ph.D. in Mathematics or a closely related field. Please see our website at [http://www.math.utah.edu/about/faculty-hiring.php](http://www.math.utah.edu/about/faculty-hiring.php) for more information regarding available positions and application requirements. Applications must be completed through [https://www.mathjobs.org/jobs/list/20425](https://www.mathjobs.org/jobs/list/20425)

Review of complete applications for the tenure-track position will begin on November 1, 2022. Completed applications for postdoctoral positions received before January 1, 2023 will receive full consideration. Review of applications can continue beyond these deadlines until the positions are filled.

The University of Utah is an Affirmative Action/Equal Opportunity employer and does not discriminate based upon race, national origin, color, religion, sex, age, sexual orientation, gender identity/expression, status as a person with a disability, genetic information, or Protected Veteran status. Individuals from historically underrepresented groups, such as minorities, women, qualified persons with disabilities and protected veterans are encouraged to apply. Veterans’ preference is extended to qualified applicants, upon request and consistent with University policy and Utah state law. Upon request, reasonable accommodations in the application process will be provided to individuals with disabilities. To inquire about the University’s
nondiscrimination or affirmative action policies or to request disability accommodation, please contact: Director, Office of Equal Opportunity and Affirmative Action, 201 S. Presidents Circle, Rm 135, (801) 581-8365. Additional information can be found at [http://www.utah.edu/nondiscrimination/](http://www.utah.edu/nondiscrimination/)

The University of Utah values candidates who have experience working in settings with students from diverse backgrounds and possess a strong commitment to improving access to higher education for historically underrepresented students.

A diverse scholarly community stimulates innovation and educational excellence. The Department of Mathematics at the University of Utah works to maintain a respectful, inclusive, and supportive environment where everyone can flourish. We are actively working to increase our diversity and to promote belonging and community for all. We value constructive input and welcome feedback from our community.

**CHINA**

Tianjin University, China

Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn.
For more information, please visit [cam.tju.edu.cn](http://cam.tju.edu.cn) or contact Mr. Albert Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.
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New Books Offered by the AMS

Algebras, Lattices, Varieties
Volume II
Ralph S. Freese, University of Hawaii, Honolulu, HI, Ralph N. McKenzie, Vanderbilt University, Nashville, TN, George F. McNulty, University of South Carolina, Columbia, SC, and Walter F. Taylor, University of Colorado, Boulder, CO

This book is the second of a three-volume set of books on the theory of algebras, a study that provides a consistent framework for understanding algebraic systems, including groups, rings, modules, semigroups and lattices. Volume I, first published in the 1980s, built the foundations of the theory and is considered to be a classic in this field.

The long-awaited volumes II and III are now available. Taken together, the three volumes provide a comprehensive picture of the state of art in general algebra today, and serve as a valuable resource for anyone working in the general theory of algebraic systems or in related fields.

The two new volumes are arranged around six themes first introduced in Volume I. Volume II covers the Classification of Varieties, Equational Logic, and Rudiments of Model Theory, and Volume III covers Finite Algebras and their Clones, Abstract Clone Theory, and the Commutator. These topics are presented in six chapters with independent expositions, but are linked by themes and motifs that run through all three volumes.

This item will also be of interest to those working in logic and foundations and discrete mathematics and combinatorics.

Algebras, Lattices, Varieties
Volume II
Ralph S. Freese, Ralph N. McKenzie, George F. McNulty, Walter F. Taylor

Mathematical Surveys and Monographs, Volume 268

Algebras, Lattices, Varieties
Volume II
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Mathematical Surveys and Monographs, Volume 271

Categories and Representation Theory
With A Focus on 2-Categorical Covering Theory
Hideto Asashiba, Shizuoka University, Suruga-ku, Shizuoka, Japan, Kyoto University, Sakyo-ku, Kyoto, Japan, and Osaka Central Advanced Mathematical Institute, Sumiyoshi-ku, Osaka, Japan

This book gives a self-contained account of applications of category theory to the theory of representations of algebras. Its main focus is on 2-categorical techniques, including 2-categorical covering theory. The book has few prerequisites beyond linear algebra and elementary ring theory, but familiarity with the basics of representations of quivers and of category theory will be helpful. In addition to providing an introduction to category theory, the book develops useful tools such as quivers, adjoints, string diagrams, and tensor products over a small category; gives an exposition of new advances such as a 2-categorical generalization of Cohen-Montgomery duality in pseudo-actions of a group; and develops the moderation level of categories, first proposed by Levy, to avoid the set theoretic paradox in category theory.

The book is accessible to advanced undergraduate and graduate students who would like to study the representation theory of algebras, and it contains many exercises. It can be used as the textbook for an introductory course on the category theoretic approach with an emphasis on 2-categories, and as a reference for researchers in algebra interested in derived equivalences and covering theory.

Mathematical Surveys and Monographs, Volume 271

Categories and Representation Theory
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**NEW BOOKS**

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**Discrete Analogues in Harmonic Analysis**

**Bourgain, Stein, and Beyond**

**Ben Krause**, King's College, London, UK

This timely book explores certain modern topics and connections at the interface of harmonic analysis, ergodic theory, number theory, and additive combinatorics. The main ideas were pioneered by Bourgain and Stein, motivated by questions involving averages over polynomial sequences, but the subject has grown significantly over the last 30 years, through the work of many researchers, and has steadily become one of the most dynamic areas of modern harmonic analysis.

The author has succeeded admirably in choosing and presenting a large number of ideas in a mostly self-contained and exciting monograph that reflects his interesting personal perspective and expertise into these topics.

—Alexandru Ionescu, Princeton University

Discrete harmonic analysis is a rapidly developing field of mathematics that fuses together classical Fourier analysis, probability theory, ergodic theory, analytic number theory, and additive combinatorics in new and interesting ways. While one can find good treatments of each of these individual ingredients from other sources, to my knowledge this is the first text that treats the subject of discrete harmonic analysis holistically. The presentation is highly accessible and suitable for students with an introductory graduate knowledge of analysis, with many of the basic techniques explained first in simple contexts and with informal intuitions before being applied to more complicated problems; it will be a useful resource for practitioners in this field of all levels.

—Terence Tao, University of California, Los Angeles

**Algebras, Lattices, Varieties**

**Volume III**

**Ralph S. Freese**, University of Hawaii, Honolulu, HI, **Ralph N. McKenzie**, Vanderbilt University, Nashville, TN, **George F. McNulty**, University of South Carolina, Columbia, SC, and **Walter F. Taylor**, University of Colorado, Boulder, CO

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**This item will also be of interest to those working in logic and foundations and discrete mathematics and combinatorics.**

Mathematical Surveys and Monographs, Volume 269

December 2022, Softcover, ISBN: 978-1-4704-6798-2, LC 2017046893, 2010 Mathematics Subject Classification: 08–02, 06Bxx, 03C05, 06Exx, List US$125, AMS members US$100, MAA members US$112.50, Order code SURV/269


**Graduate Studies in Mathematics, Volume 224**


**Graduate Studies in Mathematics Volume 224 Analysis**

**Ben Krause**, King’s College, London, UK

This timely book explores certain modern topics and connections at the interface of harmonic analysis, ergodic theory, number theory, and additive combinatorics. The main ideas were pioneered by Bourgain and Stein, motivated by questions involving averages over polynomial sequences, but the subject has grown significantly over the last 30 years, through the work of many researchers, and has steadily become one of the most dynamic areas of modern harmonic analysis.

The author has succeeded admirably in choosing and presenting a large number of ideas in a mostly self-contained and exciting monograph that reflects his interesting personal perspective and expertise into these topics.

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—Terence Tao, University of California, Los Angeles
Differential Equations

The Mathematical Analysis of the Incompressible Euler and Navier-Stokes Equations
An Introduction

Jacob Bedrossian, University of Maryland, College Park, MD, and
Vlad Vicol, Courant Institute of Mathematical Sciences, New York University, NY

The aim of this book is to provide beginning graduate students who completed the first two semesters of graduate-level analysis and PDE courses with a first exposure to the mathematical analysis of the incompressible Euler and Navier-Stokes equations. The book gives a concise introduction to the fundamental results in the well-posedness theory of these PDEs, leaving aside some of the technical challenges presented by bounded domains or by intricate functional spaces.

Chapters 1 and 2 cover the fundamentals of the Euler theory: derivation, Eulerian and Lagrangian perspectives, vorticity, special solutions, existence theory for smooth solutions, and blowup criteria. Chapters 3, 4, and 5 cover the fundamentals of the Navier-Stokes theory: derivation, special solutions, existence theory for strong solutions, Leray theory of weak solutions, weak-strong uniqueness, existence theory of mild solutions, and Prodi-Serrin regularity criteria. Chapter 6 provides a short guide to the must-read topics, including active research directions, for an advanced graduate student working in incompressible fluids. It may be used as a roadmap for a topics course in a subsequent semester. The appendix recalls basic results from real, harmonic, and functional analysis. Each chapter concludes with exercises, making the text suitable for a one-semester graduate course.

Prerequisites to this book are the first two semesters of graduate-level analysis and PDE courses.

This item will also be of interest to those working in mathematical physics.

Number Theory

Primes of the Form $x^2 + ny^2$
Fermat, Class Field Theory, and Complex Multiplication. Third Edition with Solutions

David A. Cox, Amherst College, MA
With contributions by Roger Lipsett

This book studies when a prime $p$ can be written in the form $x^2 + ny^2$. It begins at an elementary level with results of Fermat and Euler and then discusses the work of Lagrange, Legendre and Gauss on quadratic reciprocity and the genus theory of quadratic forms. After exploring cubic and biquadratic reciprocity, the pace quickens with the introduction of algebraic number fields and class field theory. This leads to the concept of ring class field and a complete but abstract solution of $p = x^2 + ny^2$. To make things more concrete, the book introduces complex multiplication and modular functions to give a constructive solution. The book ends with a discussion of elliptic curves and Shimura reciprocity. Along the way the reader will encounter some compelling history and marvelous formulas, together with a complete solution of the class number one problem for imaginary quadratic fields.

The book is accessible to readers with modest backgrounds in number theory. In the third edition, the numerous exercises have been thoroughly checked and revised, and as a special feature, complete solutions are included. This makes the book especially attractive to readers who want to get an active knowledge of this wonderful part of mathematics.

AMS Chelsea Publishing, Volume 387

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Graduate Studies in Mathematics, Volume 225

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New in Memoirs of the AMS

Differential Equations

Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups
Cai-Heng Li, Southern University of Science and Technology, Guangdong, China, and Binzhou Xia, The University of Melbourne, Parkville, Australia

Memoirs of the American Mathematical Society, Volume 279, Number 1375

Coefficient Systems on the Bruhat-Tits Building and Pro-p Iwahori-Hecke Modules
Jan Kohlhaase, Universität Duisburg-Essen, Germany

Memoirs of the American Mathematical Society, Volume 279, Number 1374

Analysis

Maximal Functions, Littlewood–Paley Theory, Riesz Transforms and Atomic Decomposition in the Multi-Parameter Flag Setting
Yongsheng Han, Auburn University, GA, Ming-Yi Lee, National Central University, Chung-Li, Taiwan, and National Center for Theoretical Sciences, Taipei, Taiwan, Ji Li, Macquarie University, Sydney, Australia, and Brett Wick, Washington University, St. Louis, Missouri

Memoirs of the American Mathematical Society, Volume 279, Number 1373

Algebra and Algebraic Geometry

Intrinsic Approach to Galois Theory of $q$-Difference Equations
Lucia Di Vizio, Université de Versailles-St Quentin, France and Charlotte Hardouin
With a Preface to Part 4 by Anne Granier

Memoirs of the American Mathematical Society, Volume 279, Number 1376
Geometry and Topology

Floer Cohomology and Flips
François Charette, Barnard College, Columbia University, New York City, NY, and Chris T. Woodward, Rutgers University, Piscataway, NJ

Memoirs of the American Mathematical Society, Volume 279, Number 1372

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Stochastic Games and Related Concepts
T. Parthasarathy, Chennai Mathematical Institute and Indian Statistical Institute, Chennai, and Sujatha Babu, Chennai Mathematical Institute, India

This set of lecture notes is based on a series of ten lectures given by T. Parthasarathy at the Chennai Mathematical Institute. Topics in matrix and bimatrix games, stochastic games (finite, infinite, and undiscounted), and cooperative games are covered.

Hindustan Book Agency; 2020; 156 pages; Softcover; ISBN: 978-93-86279-79-8; List US$56; AMS members US$44.80; Order code HIN/78

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The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Meetings in this Issue

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IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

**New: Sectional Meetings Require Registration to Submit Abstracts.** In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

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**Chattanooga, Tennessee**

*University of Tennessee at Chattanooga*

**October 15–16, 2022**  
Saturday – Sunday

**Meeting #1181**  
Southeastern Section  
Associate Secretary for the AMS: Brian D. Boe

**Deadlines**  
For organizers: Expired  
For abstracts: To be announced

Program first available on AMS website: Not applicable  
Issue of Abstracts: Volume 43, Issue 4

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

**Invited Addresses**

- Giulia Saccà, Columbia University, *Lagrangian fibrations in Hyper-Kähler geometry.*  
- Chad Topaz, Williams College and the Institute for the Quantitative Study of Inclusion, Diversity, and Equity, *Mathematical and Computational Approaches to Social Justice.*  
- Xingxing Yu, Georgia Institute of Technology, *Graph structure and graph coloring.*

**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

- **CANCELED:** Active Learning Methods and Pedagogical Approaches in Teaching College Level Mathematics, Hashim Saber, University of North Georgia.  
- Applied Knot Theory, Eleni Panagiotou, Arizona State University, Jason Cantarella, University of Georgia, and Eric J Rawdon, University of St. Thomas.
Boundary Value Problems for Differential, Difference, and Fractional Equations, John R. Graef and Lingju Kong, University of Tennessee at Chattanooga, and Min Wang, Kennesaw State University.

Combinatorial Commutative Algebra, Keri K. Sather-Wagstaff and Hugh Geller, Clemson University.

Deterministic and Stochastic PDEs: Theoretical and Numerical Analyses, Pelin G. Geredeli, Iowa State University, and Hakima Bessaih, Florida International University.

CANCELED: Ends and Boundaries of Groups: On the Occasion of Mike Mihalik’s 70th Birthday, Craig R Guilbault, University of Wisconsin-Milwaukee, and Kim E Ruane, Tufts University.

Enumerative Combinatorics, Vince Vatter and Miklós Bóna, University of Florida.

CANCELED: Geometric and Topological Generalization of Groups, Bikash Das, University of North Georgia.

Geometry and Arithmetic of Hyperkähler Manifolds, Giulia Saccà, Columbia University, and Laure Flapan, Massachusetts Institute of Technology.

Interactions Between 3-Manifolds and 4-Manifolds, Jonathan Simone, Georgia Tech, Bulent Tosun, University of Alabama, and Hannah Turner, Georgia Institute of Technology.

Modern Applied Mathematics and Spectral Analysis, Boris Belinskiy and Roger Nichols, University of Tennessee at Chattanooga.

Multiplicative Ideal Theory and Arithmetical Properties of Monoids and Domains, Jim Coykendall, Clemson University, Scott Chapman, Sam Houston State University, and Richard Erwin Hasenauer, Northeastern State University.

New Developments in Operations Research and Management Sciences, Aniekan Ebiefung and Lakmali Weerasena, University of Tennessee at Chattanooga.

Nonstandard Elliptic and Parabolic Regularity Theory with Applications, Tuoc Phan, University of Tennessee, and Hongjie Dong, Brown University.

CANCELED: Probability and Statistical Models with Applications, Sher Chhetri, University of South Carolina, Sumter, and Cory Ball, Independent.

Qualitative Analysis and Control Theory of Evolutionary Partial Differential Equations, George Avalos, University of Nebraska-Lincoln, and Weiwei Hu, University of Georgia.

Recent Advances in Mathematical Biology, Xiunan Wang, University of Tennessee at Chattanooga, and Jin Wang, UTC.

Reliable and Efficient Machine Learning for Scientific Forward and Inverse Problems, Feng Bao, Florida State University, Anuj Abhishek, University of North Carolina at Charlotte, Lan Gao, University of Tennessee at Chattanooga, Taufiquar Khan, UNC Charlotte, and Jin Wang, UTC.

Structural and Extremal Graph Theory, Zhiyu Zhang, Georgia Institute of Technology, Xingxing Yu, Georgia Tech, and Hao Huang, Emory University.

Topological, Measurable, and Symbolic Dynamics, and Interactions with Geometry, Chris Johnson, Western Carolina University, and Martin J. Schmoll, Clemson University.

Salt Lake City, Utah
University of Utah

October 22–23, 2022
Saturday – Sunday

Meeting #1182
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of Abstracts: Volume 43, Issue 4

Deadlines
For organizers: Expired
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Bhargav Bhatt, University of Michigan, Algebraic geometry in mixed characteristic.
Jonathan Brundan, University of Oregon, Derived equivalences and odd symmetric functions.
Mariel Vazquez, University of California Davis, The topology of DNA and RNA.
Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

- 4-dimensional topology, Mark Hughes, Brigham Young University, Maggie Miller, Stanford University, and Patrick Naylor, Princeton University.
- Algebraic Combinatorics and Applications in Harmonic Analysis, Sung-Yell Song and Joseph W. Iverson, Iowa State University, and Bangteng Xu, Eastern Kentucky University.
- Approximation Theory and Numerical Analysis, Vira Babenko, Drake University, and Akil C. Narayan, University of Utah.
- Arithmetic Dynamics, Bianca Thompson, Westminster College, Jamie Juul, Colorado State University, and Bella Tobin, Oklahoma State University.
- Building Bridges Between Commutative Algebra and Nearby Areas, James Cameron, University of Utah, Sarasij Maitra, University of Virginia, and Tim Tribone, Syracuse University.
- Commutative Algebra, Adam Boocher, University of San Diego, Eloísa Grifo, University of California, Riverside, and Jennifer Kenkel, University of Kentucky.
- Extremal Graph Theory, József Balogh, University of Illinois, and Bernard Lidicky, Iowa State University.
- Geometry and Representation Theory of Quantum Algebras and Related Topics, Bach Nguyen, Xavier University of Louisiana, Mee Seong Im, United States Naval Academy, Annapolis, and Arik Wilbert, University of Georgia.
- Graphs and Matrices, Mark Kempton, Emily J. Evans, and Ben Webb, Brigham Young University.
- Heegaard Floer Homology in Topology, Algebra, and Physics, Andrew Manion, NC State University, and Aaron Lauda, University of Southern California.
- Higher Topological and Algebraic K-Theories, John Lind, California State University, Chico, Agnès Beaudry, University of Colorado Boulder, and Jonathan Campbell, Duke University.
- Hypergeometric functions and q-series, Howard Cohl, National Institute of Standards and Technology, Robert Maier, University of Arizona, and Roberto Costas-Santos, Universidad de Alcalá.
- Inverse Problems, Robert M. Owczarek, University of New Mexico, and Hanna E. Makaruk, Los Alamos National Laboratory, Los Alamos, NM.
- Mathematical Modeling of Biological and Social Systems, Daniel Brendan Cooney, Daniel Gomez, and Hyunjoong Kim, University of Pennsylvania.
- Mathematics of Collective Behavior, Daniel Lear and Roman Shvydkoy, University of Illinois at Chicago.
- Partitioning and Redistricting, Wesley Hamilton, University of Utah, and Tyler Jarvis, Brigham Young University.
- PDEs, Data, and Inverse Problems, Jared P. Whitehead, Brigham Young University, Adam Larios, University of Nebraska-Lincoln, and Vincent Martinez, CUNY Hunter College & Graduate Center.
- Quantum groups, Hopf Algebras and Applications: In honor of Professor Earl J. Taft, Siu-Hung Ng, Louisiana State University, and Susan Montgomery, University of Southern California.
- Recent Advances in Algebraic Geometry and Commutative Algebra in or Near Characteristic p (associated with the Invited Address by Bhargav Bhatt), Bhargav Bhatt, University of Michigan, and Karl Schwede, University of Utah.
- Recent Advances in the Theory of Fluid Dynamics, Elaine Cozzi, Oregon State University, and Magdalena Czubak, University of Colorado Boulder.
- Recent Developments in Inverse Problems for PDEs and Applications, Loc Nguyen, UNC Charlotte, Dinh-Liem Nguyen, Kansas State University, and Fernando Guevara Vasquez, University of Utah.
- Topics in Graphs, Hypergraphs and Set Systems, David J. Galvin, University of Notre Dame, John A. Engbers, Marquette University, and Clifford D. Smyth, University of North Carolina, Greensboro.
- Topology and geometry of multi-stranded nucleic acids (associated with the Invited Address by Mariel Vazquez), Mariel Vazquez, University of California Davis, Christine E. Soteros, University of Saskatchewan, Margherita Maria Ferrari, University of Manitoba, and Javier Arsuaga, University of California, Davis.
Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023
Wednesday – Saturday

Meeting #1183
Associate Secretary for the AMS: Steve Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: Volume 44, Issue 1

Deadlines
For organizers: Expired
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/national.html.

Joint Invited Addresses

Laura G. DeMarco, Harvard University, Rigidity and uniformity in algebraic dynamics (AWM-AMS Noether Lecture).
Omayra Ortega, Sonoma State University, Who are we Serving with our Scholarship: A Covid Model Case Study (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).
Grant Sanderson, 3blue1brown, Title to be announced (JPBM Communications Award Lecture).
Bernd Sturmfels, Max Planck Institute for Mathematics in the Sciences, The Quadratic Formula Revisited (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).
Talithia Williams, Harvey Mudd College, The Power of Talk: Engaging the Public in Mathematics (JPBM Communications Award Lecture).

AMS Invited Addresses

Rodrigo Banuelos, Purdue University, Sharp inequalities in probability and harmonic analysis.
Richard G. Baraniuk, Rice University, Title to be announced (AMS Josiah Willard Gibbs Lecture).
Eugenia Cheng, School of the Art Institute of Chicago, Associativity, Commutativity and Units: a Higher-dimensional ballet (AMS Erdős Lecture for Students).
Wilfrid Gangbo, UCLA, Recent Progress on Master equations in Mean Field Games.
Ling Long, Louisiana State University, A Stroll in the Garden of Hypergeometric Functions (AMS Maryam Mirzakhani Lecture).
Chris Rasmussen, Center for Research in Math and Science Education, Three Models of Successful Department Change Approaches for Infusing Active Learning in Introductory Mathematics Courses (AMS Lecture on Education).
Nikhil Srivastava, University of California, Berkeley, Four Ways to Diagonalize a Matrix (AMS von Neumann Lecture).
Rekha Rachel Thomas, University of Washington, Ideals and Varieties of the Pinhole Camera.

Invited Addresses of Other JMM Partners

Jeremy David Avigad, Carnegie Mellon University, The promise of formal mathematics (ASL Invited Address).
Peter Cholak, Notre Dame, Ramsey like theorems on the rationals (ASL Invited Address).
Edray Goins, Pomona College, Distance Makes the Math Grow Deeper: Rational Distance Sets, Nate Dean, and Me (PME J. Sutherland Frame Lecture).
Franziska Jahnke, University of Münster, Model theory of perfectoid fields (ASL Invited Address).
Apoorva Khare, Indian Institute of Science, Analysis applications of Schur polynomials (ILAS Invited Address).
Stephen S. Kudla, University of Toronto, Modularity of generating series of divisors on unitary Shimura varieties (AIM Alexanderand Award Lecture).
Luis Antonio Leyva, Vanderbilt University-Peabody College of Education & Human Development, Undergraduate Mathematics Education as a White, Cisheteropatriarchal Space and Opportunities for Structural Disruption to Advance Queer of Color Justice (Spectra Lavender Lecture).
Sandra Müller, Technical University of Vienna, Universally Baire sets, determinacy and inner models (ASL Invited Address).
MEETINGS & CONFERENCES

Robert Santos, US Census Bureau, *Title to be announced* (ASA Committee of Presidents of Statistical Societies Lecture).
Lynn Scow, California State University, San Bernardino, *Semi-retractions and the Ramsey Property* (ASL Invited Address).
Assaf Shani, Harvard University, *Classifying invariants for Borel equivalence relations* (ASL Invited Address).
Mariel Vazquez, University of California Davis, *Topology and Evolution of DNA and RNA* (SIAM Invited Address).

Invited Addresses of Other Organizations


AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

Advances in Markov Models: Gambler’s Ruin, Duality and Queueing Applications I (Code: SS 60A), Alan Krinik, California State Polytechnic University, Pomona, and Randall James Swift, California State Polytechnic University.
Advances in Mathematical Modeling Mosquito-borne Disease Dynamics and Control Methods I (Code: SS 28A), Zhuolin Qu, University of Texas at San Antonio, and Michael A. Robert, Virginia Tech.
Advances in Nonlinear Boundary Value Problems I (Code: SS 81A), Nsoki Mavinga, Swarthmore College, Maya Chhetri, UNC Greensboro, and R. Pardo, Complutense University of Madrid.
Advances in Operator Algebras I (Code: SS 20A), Sarah Browne, University of Kansas, Priyanga Ganesan, Texas A&M University, and David Jekel, University of California, San Diego.
Advances in Partial Differential Equations, Numerical Analysis, and their Applications I (Code: SS 2A), Andrew Miller, Bridgewater State University, and Joshua L. Flynn, McGill University.
Advances in Qualitative Theory and Applications to Life Sciences of Differential, Difference, and Dynamic Equations I (Code: SS 22A), Elvan Akin, Missouri University S&T, and Naveen K. Vaidya, San Diego State University.
Analysis and Differential Equations at Undergraduate Institutions I (Code: SS 4A), Ryan Alvarado, Amherst College, and Lyudmila Korobenko, Reed College.
Applications of Riemann Surfaces I (Code: SS 71A), Mark Syd Bennett and Bernard Deconinck, University of Washington, Charles Wang, Harvard University, and Turku Ozlum Celik, Bogazici University.
Applications of Tensors in Computer Science I (Code: SS 36A), Harm Derksen, Northeastern University, Neriman Tokcan, Broad Institute, and Benjamin Lovitz, University of Waterloo.
Applied Category Theory (a Mathematics Research Communities session) I (Code: SS 96A), Charlotte Aten, University of Denver, Layla H.M. Sorkatti, Southern Illinois University, and Abigail Hickok, University of California, Los Angeles.
Applied Enumerative Geometry I (Code: SS 83A), Frank Sottile, Texas A&M University, and Taylor Brysiewicz, Max Planck Institute For Mathematics In the Sciences.
Arithmetic Geometry Informed by Computation I (Code: SS 25A), Jennifer Balakrishnan, Boston University, and Bjorn Poonen and Andrew V. Sutherland, Massachusetts Institute of Technology.
Automorphic Forms and Representation Theory I (Code: SS 8A), Spencer Leslie, Duke University, and Solomon Friedberg, Boston College.
Coding Theory for Modern Applications I (Code: SS 40A), Allison Beemer, University of Wisconsin-Eau Claire, Hiram H. Lopez, Cleveland State University, and Rafael D’Oliveira, Clemson University.
Complex and Arithmetic Dynamical Systems I (Code: SS 18A), Laura G. DeMarco and Niki Myrto Mavraki, Harvard University, and Max Weinreich, Brown University.
Complexity and Topology in Computational Algebraic Geometry I (Code: SS 77A), Ali Mohammad Nezhad and Saugata Basu, Purdue University.
Complex Systems in the Life Sciences I (Code: SS 61A), Xiang-Sheng Wang, University of Louisiana at Lafayette, Zhisheng Shuai, University of Central Florida, and Gail S. Wolkowicz, McMaster University.
Current Directions in the Philosophy of Mathematics I (Code: SS 45A), Bonnie Gold, Monmouth University, and Kevin Iga, Pepperdine University.

Current Progress in Computational Biomedicine (Code: SS 7A), Nektarios Valous, National Center for Tumor Diseases Heidelberg, German Cancer Research Center, Heidelberg, Germany, Anna Konstorum, Yale School of Medicine, Heiko Enderling, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, FL, USA, and Dirk Jäger, National Center for Tumor Diseases, German Cancer Research Center, Heidelberg, Germany.

Data Science at the Crossroads of Analysis, Geometry, and Topology (a Mathematics Research Communities session) (Code: SS 97A), Hitesh Gakhar, University of Oklahoma, Harlin Lee, University of California, Los Angeles, and Josué Tonelli-Cueto, Inria Paris & IMJ-PRG.

Definability, Computability, and Model Theory: A Special Session dedicated to Gerald E. Sacks I (Code: SS 51A), Nathanael Leedom Ackerman, Harvard University, Ted Slaman, University of California, Berkeley, and Cameron E. Freer, Massachusetts Institute of Technology.

Discrete and Hybrid Dynamical Systems: Time Scales and Fractional Approaches I (Code: SS 26A), Billy Jackson, University of Wisconsin Madison.

Distance Problems in Continuous, Discrete and Finite Field Settings I (Code: SS 76A), Abdul Basit, Johns Hopkins University, Eyvindur Ari Palsson, Virginia Tech University, and Steven Joel Miller, Williams College.

Dynamics, Geometry & Group Actions I (Code: SS 57A), Kathryn Lindsey, Boston College, and Boris Hasselblatt, Tufts University.

Dynamics of PDEs on Homogeneous Domains: Theory & Applications I (Code: SS 55A), Denis Daniel Patterson, Princeton University, Ryan Nolan Goh, Boston University, and Jonathan Touboul, Brandeis University.

Ecological and Evolutionary Dynamics in Life and Social Sciences I (Code: SS 74A), Sabrina H. Streipert, McMaster University, and Yun Kang and Lucero Rodriguez Rodriguez, Arizona State University.

Excursions in Arithmetic Geometry I (Code: SS 9A), Tony Shaska, Research Institute of Science and Technology.


Geometric PDEs I (Code: SS 33A), Theodora Bourni, University of Tennessee, Knoxville, and Brett Kotschwar, Arizona State University.

Geometry and Dynamics in Moduli Spaces of Abelian Differentials I (Code: SS 43A), Chris Johnson, Western Carolina University, Martin J. Schmoll, Clemson University, Chris Martin Judge, Indiana University, and Jane Wang, Indiana University Bloomington.

Homotopy Theory: Connections and Applications I (Code: SS 63A), Elden Elmanto, Harvard University, and Daniel C. Isaksen, Wayne State University.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences I (Code: SS 91A), Rolando de Santiago, Purdue University, and David Goldberg, Math Alliance/Purdue University.

Integrable Systems and Symplectic Group Actions I (Code: SS 64A), Joseph Palmer and Susan Tolman, University of Illinois Urbana-Champaign.

Integral Equations and Applications I (Code: SS 52A), Irina Mitrea, Temple University, and Shari Moskow, Drexel University.


Langlands Program I (Code: SS 53A), Shanna Dobson, University of California, Riverside.

Lessons Learned from Successful Departmental Efforts to Transform Precalculus and Calculus I (Code: SS 62A), Chris Rasmussen, Center for Research in Math and Science Education.

Math Circle Activities as a Gateway into Mathematics I (Code: SS 78A), Lauren L. Rose, Bard College, Brandy S. Wiegens, Central Washington University, Gabriella A. Pinter, and Nick Rauh, Julia Robinson Math Festivals.


Mathematics and Fiber Arts I (Code: SS 1A), sarah-marie belcastro, MathILy and Smith College, and Carolyn Ann Yackel, Mercer University.
MEETINGS & CONFERENCES

Mathematics and the Arts I (Code: SS 66A), Karl M. Kattchee, University of Wisconsin-La Crosse, Doug Norton, Villanova University, and Anil Venkatesh, Adelphi University.

Mathematics Standards, Equity, Policy, and Politics I (Code: SS 49A), Yvonne Lai, University of Nebraska-Lincoln, Tyler Kloefkorn, American Mathematical Society, Dave Kung, Charles A. Dana Center, The University of Texas at Austin, and Blain Patterson, Virginia Military Institute.

Modeling Collective Behavior in Biology I (Code: SS 46A), Alexandro Volkening, Purdue University, and Philip Maini, University of Oxford.

Models and Methods for Sparse (Hyper) Network Science (a Mathematics Research Communities session) (Code: SS 98A), Sarah Tymochko, Michigan State University, Jessalyn Bolkema, California State University, Dominguez Hills, Himanshu Gupta, University of Delaware, Fangfei Lan, University of Utah, and Nicholas W. Landry, University of Colorado Boulder.

Modular Forms, Hypergeometric Functions, Character Sums and Galois Representations I (Code: SS 23A), Ling Long, Louisiana State University, Wen-Ching Winnie Li, Pennsylvania State University, William Yun Chen, Institute for Advanced Study, and Holly Swisher, Oregon State University.

New Developments in Differential Geometry and Topology I (Code: SS 56A), Megan M. Kerr, Wellesley College, and Catherine Searle, Wichita State University.

Nonlinear Evolution Equations and Their Applications I (Code: SS 67A), Guoping Zhang, Gaston Mandata N’Guekakata, Xuming Xie, Mingchao Cai, and Jemal S. Mohammed-Awel, Morgan State University.

Nonlocal Frameworks in Analysis and Mathematical Modeling I (Code: SS 34A), Nicole Buczkowski, University of Nebraska, Lincoln, and Petronela Radu and Anh Vo, University of Nebraska-Lincoln.

Number Theory at Non-PhD Granting Institutions I (Code: SS 85A), Steven Joel Miller, Williams College, Naomi Tanabe, Bowdoin College, Harris Daniels, Amherst College, Enrique Treviño, Lake Forest College, and Alia Hamieh, University of Northern British Columbia.

Orthogonal Polynomials and their Applications I (Code: SS 47A), Ahmad Barhoumi, University of Michigan, Roozebeh Gharakhloo, Colorado State University, and Andrei Martinez-Finkelshtein, Baylor University.

Partial Differential Equations and Complex Variables I (Code: SS 89A), Hyun-Kyoung Kwon, University At Albany, SUNY, Bingyuan Liu, The University of Texas Rio Grande Valley, and Qi Han, Texas A & M University San Antonio.

Perspectives on Eigenvalue Computation I (Code: SS 94A), Nikhil Srivastava, University of California, Berkeley, Peter Buergisser, Technische Universität Berlin Institut Für Mathematik, and James Demmel, University of California, Berkeley.

Polynath Jr: Mentoring and Learning I (Code: SS 84A), Steven Joel Miller, Williams College, Johanna Franklin, Hofstra University, Adam Sheffer, Baruch College, CUNY, and Yunus E. Zeytuncu, University of Michigan - Dearborn.

Polynomial Systems, Homotopy Continuation and Applications (Code: SS 95A), Margaret Regan, Duke University, and Timothy Duff, University of Washington.

Promoting Equity Through Active Learning in Undergraduate Mathematics: Precalculus I (Code: SS 73A), Jose Maria Menendez, Pima Community College, Ksenija Simic-Muller, Pacific Lutheran University, and Anthony Fernandes, University of North Carolina - Charlotte.

Quadratic Forms, Modular Forms, and Applications I (Code: SS 59A), Fang-Ting Tu, Louisiana State University, Gene S. Kopp, Purdue University, and Jingbo Liu, Texas A&M University-San Antonio.

Quaternions I (Code: SS 58A), Johannes Hamilton, Borough of Manhattan Community College, CUNY, Chris McCarthy, BMCC, City University of New York, and Terrence Richard Blackman, Medgar Evers Community College, CUNY.

Recent Advances in Arithmetic Dynamics I (Code: SS 16A), Joseph H. Silverman, Brown University, Jacqueline Anderson, Bridgewater State University, and John R. Doyle, Oklahoma State University.

Recent Advances in Nonlinear Partial Differential Equations and the Applications I (Code: SS 79A), Qi Han, Texas A&M University-San Antonio, and Jing Tian, Towson University.

Recent Development in Partial Differential Equations Related to Geometric and Harmonic Analysis I (Code: SS 15A), Meijun Zhu, University of Oklahoma, and Xiaodong Wang, Michigan State University.

Recent Developments in Geometric Measure Theory I (Code: SS 88A), Camillo De Lellis, Institute For Advanced Study, Princeton, Antonio De Rosa, University of Maryland, and Luca Spolaor, University of California, San Diego.

Recent Developments in Numerical Methods for PDEs I (Code: SS 21A), Leo G. Rebholz, Clemson University, and Michael Neilan, University of Pittsburgh.

Recent Trends in Discrete-Time Ecological and Epidemiological Models I (Code: SS 42A), Mustafa R. Kulenovic, University of Rhode Island, and Abdul-Aziz Yakubu, Howard University.

Research Community in Algebraic Combinatorics I (Code: SS 87A), Rosa C. Orellana and Nadia Lafrenière, Dartmouth College.
Research from the Graduate Research Workshop in Combinatorics (GRWC) I (Code: SS 32A), Steve Butler, Iowa State University, Xavier Perez-Gimenez, University of Nebraska-Lincoln, and Puck Rombach, University of Vermont.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs I (Code: SS 17A), Darren A. Narayan, Rochester Institute of Technology, Khanh Tran, California State University, Fresno, Mark Daniel Ward, Purdue University, John C. Wierman, Johns Hopkins University, and Christopher O’Neil, San Diego State University.

Resolutions of Singularities and Cohomology in Geometry and Representation Theory I (Code: SS 44A), Iva Halacheva, Northeastern University, Roman Bezrukavnikov, Massachusetts Institute of Technology, Peter Crooks, Utah State University, and Valerio Toledano Laredo, Northeastern University.

Rethinking Number Theory I (Code: SS 35A), Allechar Serrano Lopez, Harvard University, Lea Beneish, University of California, Berkeley, and Soumya Sankar, Ohio State University.

Riemannian Manifolds with Lower Scalar Curvature Bounds I (Code: SS 38A), Brian Allen, University of Hartford, and Demetre Kazaras, Duke University.

Scholarship on Teaching and Learning Introductory Statistics I (Code: SS 3A), Jennifer McNally, Laura Kyser Callis, and Steven LeMay, Curry College.


Statistics and Data Science Curriculum in a Mathematics Department I (Code: SS 50A), Qing Wang and Anny-Claude Joseph, Wellesley College.


Stochastic Analysis and Applications I (Code: SS 5A), Parisa Fatheddin, Ohio State University, Marion, and Michael A. Salins, Boston University.

Tensor Representation, Completion, Modeling and Analytics of Complex Data I (Code: SS 19A), Ivo D. Dinov and Joshua Welch, University of Michigan.

The Combinatorics and Geometry of Jordan Type and Lefschetz Properties I (Code: SS 31A), A. Iarrobino, Northeastern University, and Leila Khatami, Union College.

The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors I (Code: SS 10A), Laurel Ohm, Princeton University, Shanise Walker, University of Wisconsin Eau Claire, and Ziva Myer, Duke University.

The History of Mathematics I (Code: SS 30A), Jemma Lorenat, Pitzer College, Adrian Rice, Randolph-Macon College, Deborah Kent, University of St. Andrews, and Daniel E. Otero, Xavier University.

The Math and Art of Mathemalchemy I (Code: SS 27A), Samantha Pezzimenti, Penn State Brandywine, Carolyn Ann Yackel, Mercer University, and Edmund O. Harriss, University of Arkansas.

The Mathematics of RNA and DNA I (Code: SS 75A), Chris McCarthy, BMCC, City University of New York, and Johannes Hamilton, Borough of Manhattan Community College, CUNY.

The Scholarship of Teaching and Learning: Past, Present, and Future I (Code: SS 13A), Jacqueline M. Dewar, Loyola Marymount University, Thomas F. Banchoff, Brown University, Curtis D. Bennett and Brian P. Katz, California State University, Long Beach, Lewis D. Ludwig, Denison University, and Larissa Schroeder, University of Nebraska Omaha.

The Teaching and Learning of Undergraduate Ordinary Differential Equations: An interdisciplinary approach I (Code: SS 69A), Viktoria Savatorova, Central Connecticut State University, Itai Seggev, Wolfram Research, Iordanka Panayotova, Christopher Newport University, and Beverly H. West, Cornell University.


Topology, Structure and Symmetry in Graph Theory I (Code: SS 37A), Mark Ellingham, Vanderbilt University, and Lowell Abrams, George Washington University.

Trees in Many Contexts (a Mathematics Research Communities session) I (Code: SS 99A), Ann Wells Clifton, Louisiana Tech University, Fadekemi Janet Osaye, Alabama State University, Lora Bailey, Grand Valley State University, Alex Wiedemann, Randolph-Macon College, and Reem Mahmoud, Virginia Commonwealth University.

Undergraduate Research Activities in Mathematical and Computational Biology I (Code: SS 41A), Timothy D. Comar, Benedictine University, Hannah Callender Highlander, University of Portland, and Anne E. Yust, University of Pittsburgh.
Understanding COVID-19: Three Years of Mathematical Models to Address the Global Pandemic (Code: SS 12A), Lauren M. Childs, Virginia Tech, Hwayeon Ryu, Elon University, and Kamila Larripa, Humboldt State University.

Variational Methods, Optimal Control and Hamilton-Jacobi Equations (Code: SS 11A), Wilfrid Gangbo, UCLA, Andrzej Swiech, Georgia Tech, Alpar Meszaros, University of Durham, and Chenchen Mou, City University of Hong Kong.

Women in Automorphic Forms (Code: SS 70A), Mathilde Gerbelli-Gauthier, Institute for Advanced Study, Maria Fox, University of Oregon, and Manami Roy, Fordham University.

AIM Special Sessions
Automorphic Forms and Special Cycles (Code: AIMSS1A), Tonghai Yang, University of Wisconsin, Madison, Stephen S. Kudla, University of Toronto, and Jan Hendrik Bruinier, Technical University of Darmstadt.

Little School Dynamics: Cool Dynamics Research by Researchers at PUIs (Code: AIMSS2A), Kimberly Ayers, California State University, San Marcos, Ami Radunskaya, Pomona College, Han Li, Wesleyan University, David M. McClendon, Ferris State University, and Andrew Parrish, Eastern Illinois University.

ASL Special Sessions
Model-theoretic and “Higher Infinite” Methods in Descriptive Set Theory and Related Areas (Code: ASLSS1A), Rehana Patel, AIMS-Senegal, Alexander Kechris, California Institute of Technology, Alejandro Poveda, Hebrew University of Jerusalem, and Assaf Shani, Harvard University.

Tame Geometry and Applications to Analysis (Code: ASLSS2A), Alexi Block Gorman and Elliot Kaplan, McMaster University, and Daniel Miller, Emporia State University.

AWM Special Sessions
AWM Workshop: Women in Commutative Algebra (WiCA), I (Code: AWMWK1A), Claudia Miller, Syracuse University, and Jan Striuli, Fairfield University.

Celebrating the Mathematical Contributions of the AWM (Code: AWMSS4A), Michelle Ann Manes, University of Hawaii, Kathryn E Leonard, Occidental College, Donatella Danielli, Arizona State University, and Ami Radunskaya, Pomona College.

Recent Developments in the Analysis of Local and Nonlocal PDEs (Code: AWMSS3A), Alaa Haj Ali and Donatella Danielli, Arizona State University.

Women, Art, and Mathematics: Mathematics in the Literary Arts and Pedagogy in Creative Settings (Code: AWMSS1A), Shanna Dobson, University of California, Riverside, Stephanie Lewkiewicz, Temple University, and Elizabeth Donovan, Murray State University.

Women in Graph Theory (Code: AWMSS2A), Karen L. Collins, Wesleyan University, Sandra R. Kingan, Brooklyn College and the Graduate Center, CUNY, Brigitte Servatius, WPI, and Ann N. Trenk, Wellesley College.

COMAP Special Sessions
COMAP’s Modeling Contests: Engaging Students and Faculty in Mathematical Modeling (Code: COMAPSS1A), Amanda I Beecher, Ramapo College of New Jersey, Steve Horton, US Military Academy (Emeritus), and Kayla Blyman, Saint Martin’s University.

ILAS Special Sessions
Innovative and Effective Ways to Teach Linear Algebra (Code: ILASS2A), David M. Strong, Pepperdine University, Gil Strang, MIT, Sepideh Stewart, University of Oklahoma, and Megan Wawro, Virginia Tech.

Matrices and Operators (Code: ILASS3A), Mohsen Aliabadi, Iowa State University, and Tin-Yau Tam and Pan-Shun Lau, University of Nevada, Reno.

Matrix Analysis and Applications (Code: ILASS1A), Hugo Woerdeman, Drexel University, and Edward Poon, Embry-Riddle Aeronautical University.

The Inverse Eigenvalue Problem for a Graph and Zero Forcing (ILAS-AIM) (Code: ILASS4A), Mary Flagg, University of St. Thomas, and Bryan A Curtis, Iowa State University.

MSRI Special Sessions
Summer Research in Mathematics (SRiM): Analytic Number Theory (Code: MSRI9A), Ayla Gafni, University of Mississippi, Amita Malik, Max Planck Institute, Bonn, and Sneha Chaubey, Indian Institute of Information Technology Delhi.

Summer Research in Mathematics (SRiM): Applied and Computational Mathematics (Code: MSRI3A), Yunan Yang, Cornell University, Jingwei Hu, University of Washington, and Yifei Lou, The University of Texas at Dallas.

Summer Research in Mathematics (SRiM): Cluster Algebras and Related Topics (Code: MSRI7A), Sunita Chepuri, University of Michigan, Elizabeth Kelley, University of Illinois at Urbana-Champaign, and Esther Banaian, University of Minnesota.

Summer Research in Mathematics (SRiM): Differential and Metric Geometry (Code: MSRI6A), Catherine Searle, Wichita State University, Lee T. Kennard, Syracuse University, and Elahe Khalili Samani, University of Notre Dame.

Summer Research in Mathematics (SRiM): Dynamics and Operator Algebras (Code: msri11A), Sarah Reznikoff, Kansas State University, Sarah Browne, University of Kansas, Elizabeth Anne Gillaspy, University of Montana, and Lauren Chase Ruth, Mercy College.

Summer Research in Mathematics (SRiM): Geometric and Topological Combinatorics (Code: MSRI5A), Margaret M. Bayer, University of Kansas, Marija Jelic Milutinovic, University of Belgrade, and Julianne Vega, Kennesaw State University.

Summer Research in Mathematics (SRiM): Mathematical Modeling and Analysis in Eye Research (Code: MSRI8A), Atanaska Dobreva and Erika Camacho, Arizona State University.

Summer Research in Mathematics (SRiM): Unknotting Operations (Code: MSRI10A), Hannah Turner, Georgia Institute of Technology, and Samantha Allen, Duquesne University.

The MSRI Undergraduate Program (MSRI-UP) (Code: MSRI2A), Federico Ardila, San Francisco State University.

NSF Special Sessions
Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences, I (Code: NSFOUT1A), Michael Ferrara and Mindy Capaldi, Division of Undergraduate Education, National Science Foundation, John R. Hadlock, National Science Foundation, Elise Nicole Lockwood, Division of Undergraduate Education, National Science Foundation, and Lee L. Zia, National Science Foundation.

PMA Special Sessions
Mathematical Research in Budapest for Students and Faculty (Code: PMASSA), Kristina Cole Garrett, Budapest Semesters in Mathematics.

SIAM Minisymposium
Applications of the Maslov Index (Code: SIAM6A), Christopher K. R. T. Jones and Emmanuel Fleurantin, University of North Carolina.

Combinatorial Optimization (Code: SIAM1A), Annie Raymond, University of Massachusetts.

Fractional Dynamics (Code: SIAM4A), Lukasz Plociniczak, Wroclaw University of Science and Technology, and Krzysztof Burnecki, Wrocław University of Science and Technology.

Imaging and Inverse Problems (Code: SIAM5A), Andrea Arnold, Worcester Polytechnic Institute.


Quantitative Justice (a NAM-SIAM Joint Session), I (Code: SIAM3A), Ron Buckmire, Occidental College, Omayra Ortega, Sonoma State University, and Carrie Diaz Eaton, Bates College.

Quantum Algorithms (Code: SIAM7A), Dong An, University of Maryland, and Di Fang and Lin Lin, University of California, Berkeley.

SIAM ED Session on Education as Research and Research as Education (Code: SIAM8A), Benjamin Galluzzo and Kathleen Kavanagh, Clarkson University.

SPECTRA Special Sessions
Research by LGBTQ+ Mathematicians (Code: SPECTSS1A), Juliette Emmy Bruce, University of California, Berkeley, Christopher Goff, University of the Pacific, and Rebecca R.G., George Mason University.
Atlanta, Georgia
Georgia Institute of Technology

**Meeting #1184**
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

**March 18–19, 2023**
Saturday – Sunday

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**
- **Betsy Stovall**, University of Wisconsin-Madison, *Title to be announced*.
- **Blair Sullivan**, University of Utah, *Title to be announced*.
- **Yusu Wang**, University of California San Diego, *Title to be announced*.
- **Amie Wilkinson**, University of Chicago, *Title to be announced* (Erdős Memorial Lecture).

**Special Sessions**
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

- **Advanced Topics in Graph Theory and Combinatorics** (Code: SS 1A), **Songling Shan**, Illinois State University, and **Guangming Jing**, Augusta University.
- **Advances in Applied Dynamical Systems and Mathematical Biology** (Code: SS 25A), **Chunhua Shan**, The University of Toledo, and **Guihong Fan**, Columbus State University.
- **Advances in Mathematical Finance and Optimization** (Code: SS 24A), **Ibrahim Ekren**, **Arash Fahim**, and **Lingjiong Zhu**, Florida State University.
- **Algebraic Methods in Algorithms** (Code: SS 16A), **Kevin Shu** and **Mehrdad Ghadiri**, Georgia Institute of Technology.
- **Combinatorial Matrix Theory** (Code: SS 22A), **Zhongshan Li**, **Marina Arav**, and **Hein Van der Holst**, Georgia State University.
- **Combinatorics, probability and computation in molecular biology** (Code: SS 38A), **Christine Heitsch** and **Brandon Legried**, Georgia Institute of Technology.
- **Commutative Algebra and its Interactions with Algebraic Geometry** (Code: SS 10A), **Michael Brown** and **Henry K. Schenck**, Auburn University.
- **Contact and Symplectic Topology in Dimensions 3 and 4** (Code: SS 34A), **Akram Alishahi**, **Peter Lambert-Cole**, and ** Gordana Matic**, University of Georgia.
- **Diversity in Mathematical Biology** (Code: SS 36A), **Daniel Alejandro Cruz**, University of Florida, and **Margherita Maria Ferrari**, University of Manitoba.
- **Dynamics of partial differential equations** (Code: SS 18A), **Gong Chen**, Georgia Institute of Technology, **Hao Jia**, University of Minnesota, and **Dallas Albritton**, Princeton University.
- **Fractal Geometry and Dynamical Systems** (Code: SS 11A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and **Scott Kaschner**, Butler University.
- **Geometric and Combinatorial Aspects of Lie Theory** (Code: SS 40A), **William Graham**, University of Georgia, **Amber Russell**, Butler University, and **Scott Larson**, University of Georgia.
- **Geometric Group Theory** (Code: SS 4A), **Ryan Dickmann**, Georgia Institute of Technology, **Sahana H Balasubramanya**, University of Münster, and **Abdoul Karim Sane** and **Dan Margalit**, Georgia Institute of Technology.
- **Harmonic Analysis** (Code: SS 14A), **Betsy Stovall**, University of Wisconsin-Madison, **Benjamin Jaye**, Georgia Tech, and **Manasa Vempati**.

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 44, Issue 2

**Deadlines**
- For organizers: Expired
- For abstracts: To be announced
High-dimensional Convexity and Probability (Code: SS 13A), Galyna Livshyts and Orli Herscovici, Georgia Institute of Technology, and Dan Mikulincer, MIT.


Logic, Combinatorics, and Their Interactions (Code: SS 27A), Anton Bernshteyn, Georgia Institute of Technology, and Robin Tucker-Drob, University of Florida.

Macdonald Theory at the Intersection of Combinatorics, Algebra, and Geometry (Code: SS 37A), Olya Mandelshtam, University of Waterloo, Sean Griffin, UC Davis, and Andy Wilson, Kennesaw State University.

Mathematical modeling and simulation techniques in fluid structure interaction problems (Code: SS 17A), Pejman Sanaei, Georgia State University.

Mathematical Modeling of Populations and Diseases Transmissions (Code: SS 33A), Yang Li, Georgia State University, Jia Li, University of Alabama in Huntsville, and Xiang-Sheng Wang, University of Louisiana at Lafayette.

Multiscale Approaches to Modeling Ecological and Evolutionary Dynamics (Code: SS 28A), Daniel Brendan Cooney, University of Pennsylvania, Denis Daniel Patterson, Princeton University, Olivia Chu, Dartmouth College, and Chadi M Saad-Roy, University of California, Berkeley.

Qualitative aspects of nonlinear PDEs: Well-posedness and Asymptotics (Code: SS 23A), Atanas G. Stefanov, University of Alabama Birmingham, Fazel Hadaifard, University of California - Riverside, and Jiahong Wu, Oklahoma State University.

Quasi-periodic Schrödinger operators and quantum graphs (Code: SS 35A), Fan Yang, Louisiana State University, Matthew Powell, UCI, and Burak Hatinoglu, UC Santa Cruz.

Recent Advances and Applications in Imaging Sciences (Code: SS 39A), Carmeliza Luna Navasca, University of Alabama at Birmingham, Fatou Sanogo, Bates College, and Elizabeth Newman, Emory University.

Recent Development in Advanced Numerical Methods for Partial Differential Equations (Code: SS 21A), Seulip Lee and Lin Mu, University of Georgia.

Recent Developments in Commutative Algebra (Code: SS 5A), Thomas Polstra, University of Alabama, and Florian Enescu, Georgia State University.

Recent Developments in Graph Theory (Code: SS 32A), Guantao Chen, Georgia State University, Zhiyu Wang, Georgia Institute of Technology, and Xingxing Yu, Georgia Tech.

Recent Developments in Mathematical Aspects of Inverse Problems and Imaging (Code: SS 20A), Yimin Zhong and Junshan Lin, Auburn University.


Recent Trends in Structural and Extremal Graph Theory (Code: SS 29A), Joseph Guy Briggs and Jessica McDonald, Auburn University.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 6A), Daniel K Nakano, University of Georgia, Chun-Ju Lai, Institute of Mathematics, Academia Sinica, Taipei 10617 Taiwan, and Weiqiang Wang, University of Virginia.


Spectral Theory (Code: SS 19A), Rudi Weikard, University of Alabama at Birmingham, and Stephen P Shipman, Louisiana State University.

Stochastic Analysis and its Applications (Code: SS 7A), Parisa Fatheeddin, Ohio State University, Marion, and Kazuo Yamazaki, Texas Tech University.

Stochastic Processes and Related Topics (Code: SS 15A), Ngartelbaye Guerngar, University of North Alabama, and Le Chen, Erkan Nane, and Jerzy Szulga, Auburn University.

Topological Persistence: Theory, Algorithms, and Applications (Code: SS 12A), Luis Scoccola, Northeastern University, Hitesh Gakhar, University of Oklahoma, and Ling Zhou, The Ohio State University.

Topology and Geometry of 3- and 4-Manifolds (Code: SS 3A), Beibei Liu, Georgia Institute of Technology, Siddhi Krishna, Georgia Institute of Technology and Columbia University, and Miriam Kuzbari, Georgia Institute of Technology.

Undergraduate Mathematics and Statistics Research (Code: SS 30A), Leslie Julianna Meadows, Georgia State University, Tsz Ho Chan and Asma Azizi, Kennesaw State University, and Mark Grinshpon, Georgia State University.
Spring Eastern Virtual Sectional Meeting

meeting virtually, EDT (hosted by the American Mathematical Society)

April 1–2, 2023
Saturday – Sunday

Meeting #1185
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: February 9, 2023
Issue of Abstracts: To be announced

Deadlines
For organizers: Expired
For abstracts: January 31, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Kirstin Eisentraeger, Pennsylvania State University, Title to be announced.
Jason Manning, Cornell University, Title to be announced.
Jennifer Mueller, Colorado State University, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

- Analysis of Markov, Gaussian and Stationary Stochastic Processes, Randall J. Swift, Cal Poly Pomona, and Alan C. Krinik, California State Polytechnic University, Pomona.
- Cybersecurity and cryptography, Lubjana Beshaj, Army Cyber Institute, Shekeba Monshref, IBM, and Angela Robinson, NIST.
- Fractal Geometry and Dynamical Systems, Mrinal Kanti Roychowdhury, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Sangita Jha, National Institute of Technology Rourkela, India, and Saurabh Verma, Indian Institute of Information Technology Allahabad.
- Hypergeometric functions, q-series and generalizations, Howard Saul Cohl, National Institute of Standards and Technology, Robert Maier, University of Arizona, and Roberto Costas-Santos, Universidad Loyola de Andalucía.
- Modeling, Analysis, and Control of Populations Impacted by Disease and Invasion, Rachel Natalie Leander, Middle Tennessee State University, and Wandi Ding, Middle Tennessee State University.
- Quasiconformal analysis and geometry on metric spaces, Dimitrios Ntalampakis, Stony Brook University, and Hrant Hakobyan, Kansas State University.
- Recent Advances in Differential Geometry, Bogdan D. Suciu, California State University Fullerton, Adara M. Blaga, West University of Timișoara, Romania, Cezar Oniciuc, “Al.I.Cuza” University of Iași, Romania, Marian Ioan Munteanu, “Al.I.Cuza” University of Iași, Iași, Romania, Shoo Seto, California State University, Fullerton, and Lihan Wang, California State University, Long Beach.
- Recent advances in infinite-dimensional stochastic analysis, Vincent R Martinez, Hunter College (CUNY), Hung Nguyen, UCLA, and Nathan E Glatt-Holtz, Tulane University.
- Recent advances in ion channel models and Poisson-Nernst-Planck systems, Zilong Song, Utah State University, and Xiang-Sheng Wang, University of Louisiana at Lafayette.
- Recent progress in Chromatic Graph Theory, Hemanshu Kaul, Illinois Institute of Technology, and Samantha Dahlberg, Illinois Institute of Technology.
Cincinnati, Ohio
University of Cincinnati

April 15–16, 2023
Saturday – Sunday

Meeting #1186
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: February 23, 2023
Issue of Abstracts: To be announced

Deadlines
For organizers: Expired
For abstracts: February 14, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Johnny Guzman, Brown University, Title to be announced.
Lisa Piccirillo, MIT, Title to be announced.
Krystal Taylor, The Ohio State University, Title to be announced.
Nathaniel Whitaker, University of Massachusetts-Amherst, Einstein Public Lecture.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Dispersive Partial Differential Equations (Code: 13A), William R. Green, Rose Hulman Institute of Technology, William R. Green, University of Illinois at Urbana Champaign, and Mehmet Burak Erdogan, University of Illinois.


Algorithms, Number Theory, and Cryptography (Code: 8A), Jonathan P. Sorenson, Butler University, and Jonathan Webster, Butler University.

Arithmetic Statistics (Code: 19A), Brandon Alberts, Eastern Michigan University, and Soumya Sankar, Ohio State University.

Brauer Groups in Algebraic Geometry and Arithmetic (Code: 2A), Craig R. Guilbault, Tufts University, and Kim E. Ruane, Tufts University.

Combinatorial and Geometric Knot Theory (Code: 11A), Micah Chrisman, The Ohio State University, Robert G. Todd, Mount Mercy University, and Sujoy Mukherjee.


Ends and Boundaries of Groups: On the Occasion of Mike Mihalik's 70th Birthday (Code: 2A), Craig R. Guilbault, Tufts University, and Kim E. Ruane, Tufts University.

Extremal Graph Theory (Code: 18A), Neal Bushaw, Virginia Commonwealth University, Vic Bednar, Virginia Commonwealth University, Calum Buchanan, University of Vermont, and Puck Rombach, University of Vermont.

Geometric and Analytic Methods in PDE (Code: 20A), Dennis Kriventsov, Rutgers University, Mariana Smit Vega Garcia, Western Washington University, and Mark Allen, Brigham Young University.

Growth Models, Random Media, and Limit Theorems (Code: 6A), Magda Peligrad, University of Cincinnati, Xiaoqin Guo, University of Cincinnati, and Wlodek Bryc, University of Cincinnati.

Harmonic Analysis and its Applications to Signals and Information (Code: 12A), Dustin G. Mixon, The Ohio State University, and Matthew Fickus, Air Force Institute of Technology.

Inequalities in Harmonic Analysis (Code: 26A), Ryan Gibara, University of Cincinnati, Kabe Moen, University of Alabama, and Leonid Slavin, University of Cincinnati.

Interactions Between Analysis, PDE, and Probability in Non-Smooth Spaces (Code: 1A), Nageswara Shanmugalingam, University of Cincinnati, Luca Capogna, Smith College, and Jeremy T. Tyson, University of Illinois.
Interactions Between Noncommutative Ring Theory and Algebraic Geometry (Code: 3A), Jason Gaddis, Miami University, and Robert Won, George Washington University.


Nonlinear Partial Differential Equations From Variational Problems and Fluid Dynamics (Code: 16A), Changyou Wang, Purdue University, and Hengrong Du, Vanderbilt University.


Probabilistic and Extremal Combinatorics (Code: 14A), Jozsef Balogh, University of Illinois at Urbana Champaign, and Tao Jiang, Miami University.

Quantitative Aspects of Symplectic Topology (Code: 15A), Jun Li, University of Michigan, Richard Keith Hind, University of Notre Dame, and Olguta Buse, IUPUI.

Recent Advances in Finite Element Methods: theory and applications (Code: 21A), Tamas L. Horvath, Oakland University, and Giselle Sosa Jones, University of Houston.

Recent Trends in Graph Theory (Code: 23A), Adam Blumenthal, Westminster College, and Katherine Perry, University of SOKA.

Recent Trends in Integrable Systems and Applications (Code: 5A), Deniz Bilman, University of Cincinnati, and Robert J. Buckingham, University of Cincinnati.

Representation Theory, Geometry and Mathematical Physics (Code: 22A), Daniele Rosso, Indiana University Northwest, and Jonas T. Hartwig, Ohio State University.

Stochastic Analysis and its Applications (Code: 9A), PO-HAN Hsu, University of Cincinnati, Ju-Yi Yen, University of Cincinnati, and Tai-Ho Wang, Baruch College, CUNY.

Structures on Complexes of Projective Modules (Code: 27A), Michael Debelleveu, Syracuse University, and Josh Pollitz, University of Utah.


Fresno, California
California State University, Fresno

May 6–7, 2023
Saturday – Sunday

Program first available on AMS website: March 16, 2023
Issue of Abstracts: Not applicable

Deadlines
For organizers: Expired
For abstracts: March 7, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Sami Assaf, University of Southern California, Title to be announced.
Natalia Komarova, University of California, Irvine, Title to be announced.
Joseph Teran, University of California, Los Angeles, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances by Scholars in the Pacific Math Alliance, Andrea Arauza Rivera, California State University, East Bay, Mario Banuelos, California State University, Fresno, and Jessica De Silva, California State University, Stanislaus.

Advances in Functional Analysis and Operator Theory, Michel L. Lapidus, University of California, Riverside, Marat V. Markin, California State University, Fresno, and Igor Nikolaev, St. John’s University.
Algebraic Structures in Knot Theory, Carmen Caprau, California State University, Fresno, Sam Nelson, Claremont McKenna College, and Neslihan Gügümcu, Izmir Institute of Technology in Turkey.

Algorithms in the Study of Hyperbolic 3-manifolds, Robert Haraway, III and Maria Trnkova, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application, Bhuvaneswari Sambandham, Dixie State University, and Aghalaya S. Vatsala, University of Louisiana at Lafayette.

Artin-Schelter Regular Algebras and Related Topics, Ellen Kirkman, Wake Forest University, and James Zhang, University of Washington.

Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf), Sami Assaf, University of Southern California, Nicole Gonzalez, University of California, Los Angeles, and Brendan Pawloski, University of Southern California.

Complexity in Low-Dimensional Topology, Jennifer Schultens, University of California, Davis, and Eric Sedgwick, DePaul University.

Data Analysis and Predictive Modeling, Earvin Balderama, California State University, Fresno, and Adriano Zambom, California State University, Northridge.

Inverse Problems, Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Math Circle Games and Puzzles that Teach Deep Mathematics, Maria Nogin and Agnes Tuska, California State University, Fresno.

Mathematical Biology: Confronting Models with Data, Erica Rutter, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova), Natalia Komarova and Jesse Kreger, University of California, Irvine.

Methods in Non-Semisimple Representation Categories, Eric Friedlander, University of Southern California, Los Angeles.

Julia Pevtsova, University of Washington, Seattle, and Paul Sobaje, Georgia Southern University, Statesboro.

Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution, Lale Asik, Texas Tech University, Khanh Phuong Nguyen, University of Houston, and Angela Peace, Texas Tech University.

Research in Mathematics by Early Career Graduate Students, Doreen De Leon, Marat Markin, and Khang Tran, California State University, Fresno.

Scientific Computing, Changho Kim, University of California, Merced, and Roummel Marcia.

The Use of Computational Tools and New Augmented Methods in Networked Collective Problem Solving, Mario Banuelos, California State University, Fresno, Andrew G. Benedek, Research Centre for the Humanities, Hungary, and Agnes Tuska, California State University, Fresno.

Women in Mathematics, Doreen De Leon, Katherine Kelm, and Oscar Vega, California State University, Fresno.

Zero Distribution of Entire Functions, Tamás Forgács and Khang Tran, California State University, Fresno.

Buffalo, New York

University at Buffalo (SUNY)

September 9–10, 2023

Saturday – Sunday

Meeting #1188

Eastern Section

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: July 27, 2023

Issue of Abstracts: To be announced

Deadlines

For organizers: February 9, 2023

For abstracts: July 18, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Jennifer Balakrishnan, Boston University, Title to be announced.

Sigal Gottlieb, University of Massachusetts, Dartmouth, Title to be announced.

Samuel Payne, University of Texas, Title to be announced.
Omaha, Nebraska
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Creighton University

**October 7–8, 2023**
*Saturday – Sunday*

**Meeting #1190**
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: August 17, 2023
Issue of Abstracts: To be announced

**Deadlines**
For organizers: March 7, 2023
For abstracts: August 8, 2023

Mobile, Alabama
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University of South Alabama

**October 13–15, 2023**
*Friday – Sunday*

**Meeting #1189**
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: August 24, 2023
Issue of Abstracts: To be announced

**Deadlines**
For organizers: March 13, 2023
For abstracts: August 15, 2023

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**

- Theresa Anderson, Carnegie Mellon University, *Title to be announced*
- Laura Miller, University of Arizona, *Title to be announced*
- Cornelius Pillen, University of South Alabama, *Title to be announced*

**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

**Mathematical Modeling of Problems in Biological Fluid Dynamics** (Code: SS 1A), Laura Miller, University of Arizona, and Nick Battista, The College of New Jersey.

Albuquerque, New Mexico
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University of New Mexico

**October 21–22, 2023**
*Saturday – Sunday*

**Meeting #1191**
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: August 31, 2023
Issue of Abstracts: Not applicable

**Deadlines**
For organizers: March 21, 2023
For abstracts: August 22, 2023
Auckland, New Zealand

December 4–8, 2023

Monday – Friday

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

San Francisco, California

Moscone West Convention Center

January 3–6, 2024

Wednesday – Saturday

Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Tallahassee, Florida

Florida State University in Tallahassee

March 23–24, 2024

Saturday – Sunday

Southeastern Section

Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Eastern Section

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Western Section

Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable

Issue of Abstracts: Not applicable

Deadlines

For organizers: To be announced

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.
Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Recent Advances in Differential Geometry, Zhiqin Lu, University of California, Shoo Seto and Bogdan Suceava, California State University, Fullerton, and Lihan Wang, California State University, Long Beach.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of Abstracts: Not applicable

Deadlines
For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday

Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday

Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
Keeping the Lights On

When you flip a switch to turn on a light, where does that energy come from? In a traditional power grid, electricity is generated at large power plants and then transmitted long distances. But now, individual homes and businesses with solar panels can generate some or all of their own power and even send energy into the rest of the grid. Modifying the grid so that power can flow in both directions depends on mathematics. With linear programming and operations research, engineers design efficient and reliable systems that account for constraints like the electricity demand at each location, the costs of solar installation and distribution, and the energy produced under different weather conditions. Similar mathematics helps create “microgrids”—small local systems that can operate independent of the main grid.

A microgrid can keep the power flowing when a natural disaster knocks the main grid offline. For example, after Japan’s devastating earthquake and tsunami in 2011, a microgrid powered in part by solar panels provided energy to a university and hospital in Sendai. A recently completed microgrid in Chicago’s historically Black Bronzeville area—the US’s first neighborhood-scale microgrid—demonstrates that such technology can increase the resilience of the communities most impacted by natural disasters. Researchers continue to develop their methods, often turning equations into computer simulations to visualize new microgrid designs. A related area of study is solar forecasting. Mathematical models that predict levels of sunlight days in advance will allow grid operators to better plan the flow of power from the Sun to your home, school, and community.

NEW RELEASE from the AMS

Lost in the Math Museum
A Survival Story

Colin Adams, Williams College, Williamstown, MA

From the twisted imagination of best-selling author Colin Adams (Zombies & Calculus, The Knot Book) comes this tale of sixteen-year-old Kallie trying to escape death at the hands of the exhibits in a mathematics museum. Kallie crosses paths with Carl Gauss, Bertrand Russell, Sophie Germain, G. H. Hardy, and John von Neumann as she tries to save herself, her dad, and his colleague Maria from the deadly Hairy Ball theorem, the harrowing Hilbert Hotel, the bisecting Ham Sandwich machine, and a variety of other mathematical menaces. It’s a wild romp through a mathematical bestiary featuring the bizarre, the exotic, and the counterintuitive. You’ll never think of math the same way again.

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