Ability and Diversity of Skills

William Geller and Michał Misiurewicz

1. Introduction

The aim of this paper is to build a simple model of problem solving, both by single agents and by teams. We realize that our model is crude and of course far from universal. Yet the results we get seem to us quite illuminating, and show the importance of both ability and diversity of skills.

The question of how to measure effectiveness of problem solving by individuals and (even more importantly) by teams, and how to choose the best individual/team, has been a subject of a lot of research. We can give as examples papers [GJIB, HP2, KI, KR], and the literature cited there. We do not address explicitly the problem of choosing a team, but our findings may serve as the basis for further research in that direction (in cases where it seems that our model may be applicable).

For a single agent, or a team of agents, we try to measure the probability of success as a function of the difficulty of a problem (or rather the easiness of the problem, measured by a variable p; the larger p, the easier the problem). In Section 3 we show that the probability of success is concave as a function of p.

In Section 4 we show that in our model for a single agent specialization is better than versatility.[1] We also show that comparing agents is difficult. In most cases, for a given agent and chosen values of p, there can be another agent, who is better at solving problems with easiness p for those chosen values, but worse at solving problems with all other values of p.

In Section 5 we consider teams of agents. We get what can be considered the main result of the paper: whenever our model can be applied, both abilities of the team members and the diversity of skills in the team matter. If any of those increases, so does the probability of success. For simplicity, we consider teams with two members, but it is clear that similar results should hold for larger teams. Interestingly, there is an example where for easier problems ability is more important, but for more difficult problems diversity is more important.

In Section 6 we show how our model can be applied to a situation where the agents are trying to defend an organization against an attack. In this application, diversity is even more important than for general problem solving.

Some of our ideas came from studying the model of L. Hong and S.E. Page [HP1]. Our model is much simpler, and can be easily investigated both by pure mathematical means and by computational means. Moreover, we avoid the main deficiency of the Hong–Page model, where high-ability teams consist basically of clones of the same agent (and as a result, ability excludes diversity).[2]

2. Preliminary Model

If an agent will be trying to solve problems that are not known in advance, her expected performance can be measured by an average over various possible problems.

Our first, preliminary model is as follows. An agent has some set of skills. This set is a subset S of $N = \{1, 2, \dots, n\}$. An immediate problem is represented by a subset P of N of cardinality p. An agent can make progress if the intersection $S \cap P$ is nonempty. Clearly, the difficulty of the problem is measured by p; problems with smaller p are more difficult.

When we want to measure the ability of an agent, we average performance of an agent over all problems of a given difficulty (that is, with a given cardinality p). The result clearly does not depend on a concrete set S of skills, but only on its cardinality s (the skillfulness of the agent).

For given n, s, p it is easy to compute the probability of making progress. If $p + s > n$, this probability is 1. If $p + s \le n$, out of all $\binom{n}{p}$ possible sets P only $\binom{n-s}{p}$ result in failure. Therefore the probability of success (that is,

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[1] *Thus a strategic agent might choose to specialize.*

[2] *A reader interested in the discussion about that model may want to look at* [T1, P, T2].

making progress) is

$$1-\frac{\binom{n-s}{p}}{\binom{n}{p}}=1-\frac{(n-s)!\,(n-p)!}{n!\,(n-p-s)!}$$
$$=1-\frac{n-p}{n}\cdot\frac{n-p-1}{n-1}\cdot\ldots\cdot\frac{n-p-s+1}{n-s+1}.$$

In Figure 1, we can see how the probability of success varies with the difficulty of the problem, for agents with various numbers of skills. If the problem is easy, the skillfulness of an agent does not matter much (provided the agent has some minimal number of skills). However, for difficult problems it matters a lot.

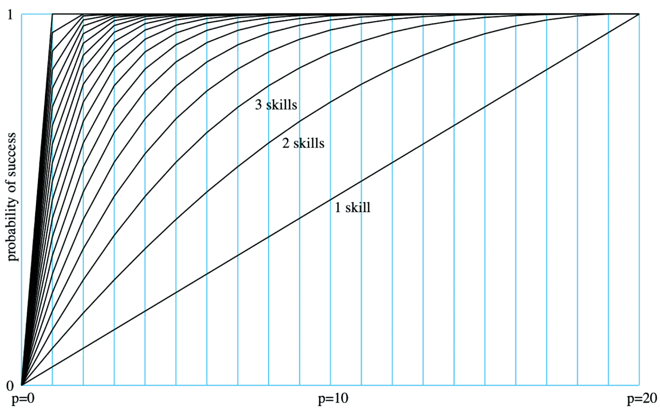

Figure 1. For $n = 20$, graphs of the probabilities of success as functions of the difficulty of the problem for various numbers of skills of the agent. As we move to the right, the difficulty of the problem decreases (that is, p increases).

3. Main Model

The preliminary model is very crude, because for each skill an agent either has it or does not. However, one should allow an agent to have partial skills. Then S becomes a *strength function* $S : N \to [0, 1]$. If $k \in P$, then the probability that the agent can make progress using skill number k is $S(k)$. We assume that those probabilities for different k are independent. This means that it is easier to use in computations the *weakness function* $R = \mathbb{1} - S$, where $\mathbb{1}$ is the constant function 1. Then the probability $F(R, P)$ of failure for a given agent and given problem is equal to the product of the numbers $R(k)$ over $k \in P$.

Often instead of speaking of the strength and weakness functions we will speak of the *strength and weakness vectors* $(S(1), S(2), \ldots, S(n))$ and $(R(1), R(2), \ldots, R(n))$.

For a given R, the sum of $F(R, P)$ over all sets $P \subset N$ of cardinality p is equal to

$$\sum_{|P|=p}\prod_{i\in P}R(i),$$

where $|P|$ denotes the cardinality of P. Observe that this number is equal to the coefficient for the polynomial

$$Q_R(x)=\prod_{i=1}^{n}(x+R(i))$$

of x^{n-p}. Thus, the average probability $F_R(p)$ of failure over all sets P of cardinality p is equal to this coefficient divided by $\binom{n}{p}$. Note that $\binom{n}{p}$ is the coefficient of x^{n-p} for the polynomial $Q_{\mathbb{1}}$. This in particular means that if an agent has no skills (so $R = \mathbb{1}$), her probability of failure is 1 no matter what.

Clearly, if σ is a permutation of the set N then $F_R(p) = F_{R\circ\sigma}(p)$. Therefore we may assume that $R(1) \le R(2) \le \ldots \le R(n)$. Sometimes, if we do not want to make this assumption, we will say that $S \circ \sigma$ is a permutation of S.

Of course, if R takes only values 0 and 1, we get the previous model.

Let us investigate some basic properties of the function F_R.

Proposition 1. *We have $F_R(p + 1) \le F_R(p)$, and equality holds only if either both numbers are equal to 0 or $R = \mathbb{1}$.*

Proof. Replace each subset $P \subset N$ of cardinality $p+1$ by $p+1$ pairs (P, j), where $j \in P$. Then the average of $\prod_{i\in P} R(i)$ over all such pairs will be equal to $F_R(p + 1)$. Similarly, when we replace each subset $P \subset N$ of cardinality p by $n-p$ pairs (P, j), where $j \in N\setminus P$, the average of $\prod_{i\in P} R(i)$ over all such pairs will be equal to $F_R(p)$. However, there is a natural one-to-one correspondence between the pairs of the first and of the second type. Namely, if $|P| = p$ and $j \in N\setminus P$, then $|P\cup\{j\}| = p+1$ and $j \in P\cup\{j\}$. Since always $\prod_{i\in P\cup\{j\}} R(i) \le \prod_{i\in P} R(i)$, we get $F_R(p + 1) \le F_R(p)$.

Suppose that we have the equality. Then for every $P \subset N$ of cardinality p and every $j \in N \setminus P$ we have either $\prod_{i\in P} R(i) = 0$ or $R(j) = 1$. If for every $P \subset N$ of cardinality p we have $\prod_{i\in P} R(i) = 0$, then $F_R(p+1) = F_R(p) = 0$. Otherwise, there exists $P \subset N$ of cardinality p with $\prod_{i\in P} R(i) > 0$, so we have $R(j) = 1$ for every $j \in N \setminus P$. Unless $R = \mathbb{1}$, there is $k \in P$ for which $R(k) < 1$. Choose one $j \in N \setminus P$ and consider the set $V = P \cup \{j\} \setminus \{k\}$. Then

$$\prod_{i\in V\cup\{k\}}R(i)=R(k)\prod_{i\in V}R(i)<\prod_{i\in V}R(i),$$

a contradiction. This proves the second part of the proposition. □

Proposition 2. *We have*

$$\frac{F_R(p+2)+F_R(p)}{2}\ge F_R(p+1), \qquad (1)$$

so the function F_R is convex (and the success function, $\mathbb{1} - F_R$, is concave). Equality holds if and only if either $F_R(p) = 0$ or there is at most one $i \in N$ such that $R(i) \ne 1$.

Proof. In a similar way as in the proof of Proposition 1, we get the following four equalities (in the first one, we have to look at the set $N \setminus \{j\}$ instead of N):

$$\underset{j\in N}{\forall} \quad (p+1) \sum_{\substack{|V|=p+1\\ j\notin V}} \prod_{i\in V} R(i) = \sum_{\substack{k=1\\ k\neq j}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} R(j) \prod_{i\in W} R(i), \tag{2}$$

$$(n-p-1) \sum_{|V|=p+1} \prod_{i\in V} R(i) = \sum_{j=1}^{n} \sum_{\substack{|V|=p+1\\ j\notin V}} \prod_{i\in V} R(i), \tag{3}$$

$$(n-p)(n-p-1) \sum_{|W|=p} \prod_{i\in W} R(i) = \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} \prod_{i\in W} R(i), \tag{4}$$

$$(p+2)(p+1) \sum_{|P|=p+2} \prod_{i\in P} R(i) = \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} R(j)R(k) \prod_{i\in W} R(i). \tag{5}$$

Since $(1-R(j))(1-R(k)) \geq 0$, we have

$$1 + R(j)R(k) \geq R(j) + R(k). \tag{6}$$

Now, from (5), (4), and (6), we get

$$\begin{aligned} &(p+2)(p+1) \sum_{|P|=p+2} \prod_{i\in P} R(i) \\ &\quad + (n-p)(n-p-1) \sum_{|W|=p} \prod_{i\in W} R(i) \\ &= \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} (1+R(j)R(k)) \prod_{i\in W} R(i) \qquad (7) \\ &\geq \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} (R(j)+R(k)) \prod_{i\in W} R(i). \end{aligned}$$

From (2) and (3), we get

$$\begin{aligned} \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{\substack{|W|=p\\ j,k\notin W}} R(j) \prod_{i\in W} R(i) &= \sum_{k=1}^{n} (p+1) \sum_{\substack{|V|=p+1\\ k\notin V}} \prod_{i\in V} R(i) \\ &= (p+1)(n-p-1) \sum_{|V|=p+1} \prod_{i\in V} R(i). \end{aligned} \tag{8}$$

From (7) and (8) (note that (8) holds also with j and k switched) we get

$$\begin{aligned} (p+2)(p+1) \sum_{|P|=p+2} \prod_{i\in P} R(i) + (n-p)(n-p-1) \sum_{|W|=p} \prod_{i\in W} R(i) \\ \geq 2(p+1)(n-p-1) \sum_{|V|=p+1} \prod_{i\in V} R(i), \end{aligned}$$

that is,

$$\begin{aligned} (p+2)(p+1)\binom{n}{p+2}F_R(p+2) + (n-p)(n-p-1)\binom{n}{p}F_R(p) \\ \geq 2(p+1)(n-p-1)\binom{n}{p+1}F_R(p+1). \end{aligned}$$

Since

$$\begin{aligned} (p+2)(p+1)\binom{n}{p+2} &= (n-p)(n-p-1)\binom{n}{p} \\ &= (p+1)(n-p-1)\binom{n}{p+1}, \end{aligned}$$

we get

$$F_R(p+2) + F_R(p) \geq 2F_R(p+1).$$

This proves the first part of the proposition.

To prove the second part, notice that by (7), equality in (1) holds if and only if for every $W \subset N$ of cardinality p and every $j,k \in N \setminus W$ such that $j \neq k$, either $R(j) = 1$, or $R(k) = 1$, or $\prod_{i\in W} R(i) = 0$.

If $F_R(p) = 0$, then for every $W \subset N$ of cardinality p we have $\prod_{i\in W} R(i) = 0$. If there is at most one $i \in N$ such that $R(i) \neq 1$, then $j \neq k$ implies $R(j) = 1$ or $R(k) = 1$. In all those cases we get equality in (1).

Now assume that $F_R(p) \neq 0$ and $R(i) \neq 1$ for at least two indices $i \in N$. Then there are two possible cases. Either there are two or more zeros among $R(i)$, $i \in N$, or there is at most one zero there. In the first case, we can choose $j,k \in N$ such that $j \neq k$ and $R(j) = R(k) = 0$. Then, since $F_R(p) \neq 0$, there is $W \subset N \setminus \{j,k\}$ of cardinality p, such that $\prod_{i\in W} R(i) > 0$. In the second case, we choose $j,k \in N$ with $j \neq k$ and $R(j), R(k)$ as small as possible, and again there is $W \subset N \setminus \{j,k\}$ of cardinality p, such that $\prod_{i\in W} R(i) > 0$. In both cases, there is no equality in (1). □

4. Specialization and Versatility

We would like to be able to measure the skillfulness of an agent in our model. There may be various ways of doing this, and as we will see in Theorem 5, we cannot expect to find a perfect one. We will settle on what seems the most natural way of doing it, by defining it to be $S(1) + S(2) + \cdots + S(n)$, where S is the strength function of the agent.

Within our model, one of the first questions that comes to mind is what is the best distribution of strengths given the skillfulness of an agent. The agent can be more specialized or more versatile. We will show that in our model specialization is better than versatility.

The simplest case is when we have two agents, one with $S_1(1) = S_1(2) = 1/2$ and $S_1(k) = 0$ for $k > 2$, and the other one with $S_2(1) = 1$ and $S_2(k) = 0$ for $k > 1$. The first agent is more versatile and the second one more specialized. We have

$$Q_{R_1} = (x+1/2)^2(x+1)^{n-2} = (x^2+x+1/4)(x+1)^{n-2}$$
$$= Q_{R_2} + \frac{1}{4}(x+1)^{n-2}$$

and

$$Q_{R_2} = x(x+1)^{n-1}.$$

Thus, $F_{R_1}(p) = F_{R_2}(p)$ for $p < 2$, and $F_{R_1}(p) > F_{R_2}(p)$ for $p \geq 2$. Let us make exact computations.

The coefficient of x^{n-p} for the polynomial $x(x+1)^{n-1}$ is the same as the coefficient of x^{n-p-1} for the polynomial $(x+1)^{n-1}$, that is, $\binom{n-1}{n-p-1}$. Thus,

$$F_{R_2}(p) = \frac{(n-1)!}{(n-p-1)!\,p!} \cdot \frac{(n-p)!\,p!}{n!} = \frac{n-p}{n} = 1 - \frac{p}{n}.$$

To get the coefficient of x^{n-p} for the polynomial $(x+1/2)^2(x+1)^{n-2}$, we have additionally to add the coefficient of x^{n-p} for the polynomial $\frac{1}{4}(x+1)^{n-2}$, so

$$F_{R_1}(p) = 1 - \frac{p}{n} + \frac{1}{4} \cdot \frac{(n-2)!}{(n-p)!\,(p-2)!} \cdot \frac{(n-p)!\,p!}{n!}$$
$$= 1 - \frac{p}{n} + \frac{1}{4} \cdot \frac{p(p-1)}{n(n-1)}.$$

This means that while the graph of the probability of success as a function of p lies on the straight line from $(0,0)$ to $(n,1)$ for R_2, it lies on a parabola from $(0,0)$ to $(n,3/4)$ for R_1 (see Figure 2).

In this example specialization is better than versatility (see Figures 3 and 4 for other examples).

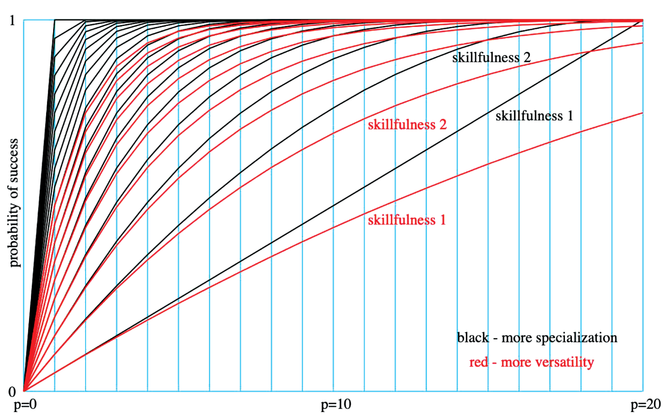

Figure 2. The same picture as in Figure 1, with additional red graphs showing probabilities of success with skillfulness an integer, from 1 to 10, but S taking only values 0 and 1/2. Black graphs represent more specialization and red ones more versatility.

We considered only a simple example, but it turns out that in more complicated situations the result is the same.

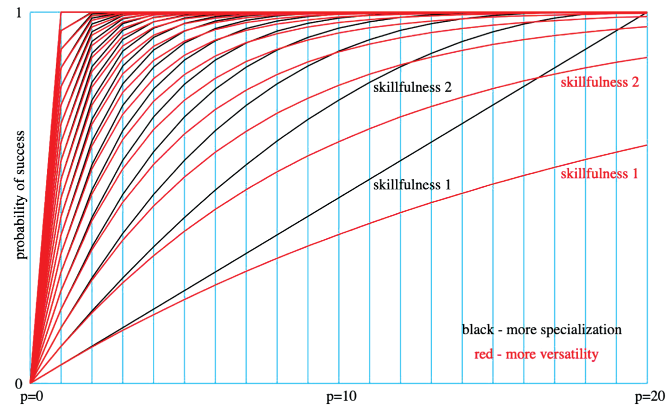

Figure 3. The same picture as in Figure 2, but the red graphs showing probabilities of success with skillfulness an integer, from 1 to 20, spread evenly (that is, $S(i)$ is the same for all i). This represents even more versatility than in the preceding figure.

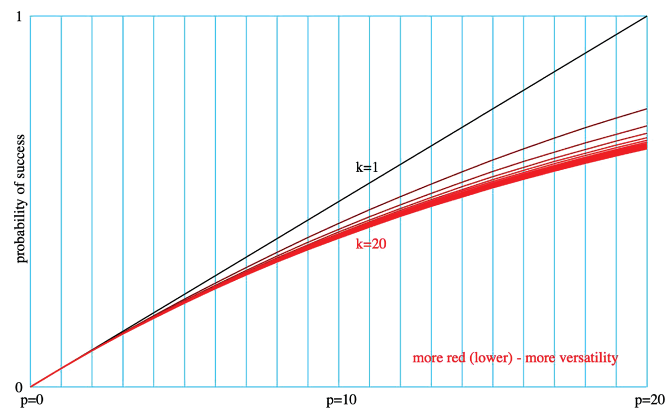

Figure 4. Here skillfulness is 1, but it is spread equally into k skills, and k varies from 1 (the highest graph) to 20 (the lowest graph).

Lemma 3. *Let $a, b \in (0,1)$, $n \geq 2$. Let a strength function S_1 be such that $S_1(1) = a$ and $S_1(2) = b$. If $a+b \leq 1$, set $S_2(1) = a+b$, $S_2(2) = 0$ and $S_2(k) = S_1(k)$ for $k > 2$. If $a+b > 1$, set $S_2(1) = 1$, $S_2(2) = a+b-1$, and $S_2(k) = S_1(k)$ for $k > 2$. Then, in both cases, $F_{R_1}(p) \geq F_{R_2}(p)$ for all p and $F_{R_1}(p) > F_{R_2}(p)$ for at least one p.*

Proof. Assume first that $a+b \leq 1$. Then for some polynomial T of degree $n-2$ with nonnegative coefficients we have

$$Q_{R_1} = (x+1-a)(x+1-b)T(x)$$
$$= \big(x^2 + (2-a-b)x + (1-a-b+ab)\big)T(x)$$

and

$$Q_{R_2} = (x+1-a-b)(x+1)T(x)$$
$$= \big(x^2 + (2-a-b)x + (1-a-b)\big)T(x).$$

Thus, $F_{R_1}(p) \geq F_{R_2}(p)$ for all p, and $F_{R_1}(p) > F_{R_2}(p)$ for at least one p.

Assume now that $a + b > 1$. Then the formula for Q_{R_1} stays the same, and we have

$$Q_{R_2} = x(x + 2 - a - b)T(x) = \big(x^2 + (2 - a - b)x\big)T(x).$$

We have $0 < 1 - a - b + ab$, so the result is the same as in the first case. □

The way we can restate this lemma is that if an agent has at least two strengths other than 0 and 1, then we can change her strength function, keeping the same skillfulness, in such a way that none of the probabilities of failure $F(p)$ increases, and at least one of them strictly decreases. This change of the strength function is in the direction of specialization.

Observe that given a skillfulness ξ, there is a unique (up to permutations) strength function with at most one value in $(0, 1)$ giving this skillfulness. Let us denote this function by S_ξ. This is the strength function of the most specialized possible agent of skillfulness ξ.

Theorem 4. *Given a strength function S with skillfulness ξ and probabilities of failure $F(p)$, and the function S_ξ with probabilities of failure $F_\xi(p)$, we have $F_\xi(p) \le F(p)$ for every p, and unless S is a permutation of S_ξ, there is at least one p for which $F_\xi(p) < F(p)$.*

Proof. Use Lemma 3 inductively. □

We can interpret this result as saying that for an individual problem solver in our model, specialization is better than versatility.

In Figures 2 and 3 we see pairs of graphs of the probability of success (extended piecewise linearly to functions on $[0, n]$) that have intersections not only at 0 (more such graphs can be seen in Figures 6 and 8). For each such pair there is only one intersection apart from 0. Thus, a question arises whether this is a general phenomenon. We now show that this is very far from being true.

Theorem 5. *Let $R : N \to (0, 1)$ be an injection, and let τ be a function from N to $\{-1, +1\}$. Then there exists a function $\widetilde{R} : N \to (0, 1)$ such that $F_R(p) > F_{\widetilde{R}}(p)$ if $\tau(p) = -1$ and $F_R(p) < F_{\widetilde{R}}(p)$ if $\tau(p) = +1$.*

Proof. We may assume that R is a strictly increasing function. We have $F_R(p) = a_p / \binom{n}{p}$, where

$$x^n + \sum_{p=1}^{n} a_p x^{n-p} = Q_R(x) = \prod_{i=1}^{n} (x + R(i)).$$

If $\varepsilon > 0$ is sufficiently small, then the zeros of the polynomial $x^n + \sum_{p=1}^{n} (a_p + \varepsilon\tau(p)) x^{n-p}$ are all real and contained in the interval $(-1, 0)$. Thus, there is a function $\widetilde{R} : N \to (0, 1)$ such that this polynomial is equal to $Q_{\widetilde{R}}(x)$. We have $F_{\widetilde{R}}(p) = (a_p + \varepsilon\tau(p)) / \binom{n}{p}$, so $F_R(p) > F_{\widetilde{R}}(p)$ if $\tau(p) = -1$ and $F_R(p) < F_{\widetilde{R}}(p)$ if $\tau(p) = +1$. □

This theorem illustrates the difficulty of measuring the ability of agents (see also, for example, [HP2, KR]). Theorem 5 shows that given a typical agent and a specified set of problem difficulties, there can be another agent who is better at solving problems with those difficulties but worse at solving problems with all other difficulties.

One can ask whether we can remove the assumptions that R maps N to the open interval $(0, 1)$ and that it is an injection. The answer is "no," as the following simple examples show.

Let $n = 2$ and consider two weakness vectors, $R = (1/2, 1)$ and $\widetilde{R} = (a, b)$. We have $Q_R(x) = x^2 + (3/2)x + 1/2$ and $Q_{\widetilde{R}}(x) = x^2 + (a + b)x + ab$. If $a + b \ge 3/2$ then $1/2 \le a \le 1$, so

$$\begin{aligned} ab &\ge -a^2 + \frac{3}{2}a \ge \min\left(-\left(\frac{1}{2}\right)^2 + \frac{3}{2}\cdot\frac{1}{2}, -1^2 + \frac{3}{2}\cdot 1\right) \\ &= \min\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

Thus, we cannot have $F_R(1) < F_{\widetilde{R}}(1)$ and $F_R(2) > F_{\widetilde{R}}(2)$.

Similarly, if $R = (1/2, 1/2)$ and $\widetilde{R} = (a, b)$, then $Q_R(x) = x^2 + x + 1/4$ and $Q_{\widetilde{R}}(x) = x^2 + (a + b)x + ab$. If $a + b \le 1$ then by the inequality between the geometric and arithmetic means, $\sqrt{ab} \le (a + b)/2 \le 1/2$, so $ab \le 1/4$. Thus, we cannot have $F_R(1) > F_{\widetilde{R}}(1)$ and $F_R(2) < F_{\widetilde{R}}(2)$.

5. Teams

Let us consider now what our model tells us about teams of agents. Suppose we have a team of two agents,[3] and for skill i, their weakness is $R_1(i)$ and $R_2(i)$, respectively. We assume independence (in the sense of probability theory), so the probability of not making progress on a problem using skill i is the product $R_1(i) \cdot R_2(i)$. As for a single agent, we will speak of the strength and weakness vectors of a team.

We can take the diversity of a team to be the lack of overlap of their strengths. While this is not a formal definition, we can often say which of two teams has larger diversity. Similarly, we can speak of the ability of the team. Here we can use the skillfulness as the measure, although Theorem 5 suggests that it is not an ideal measure. However, again we can often say which of two teams (or members of the team) has larger ability.

Let us consider the simple example where there are two agents in the team, and each of them has two skills of strength 1/2. There are three possibilities: two, one, or none of the skills coincide. Then we get for the team three possible strength vectors: $(3/4, 3/4, 0, \dots, 0)$, $(3/4, 1/2, 1/2, 0, \dots, 0)$, and $(1/2, 1/2, 1/2, 1/2, 0, \dots, 0)$. Figure 5 illustrates the results. We see that with the same levels of abilities of the members of the team, more diversity in

[3] *The situation should be similar for larger teams.*

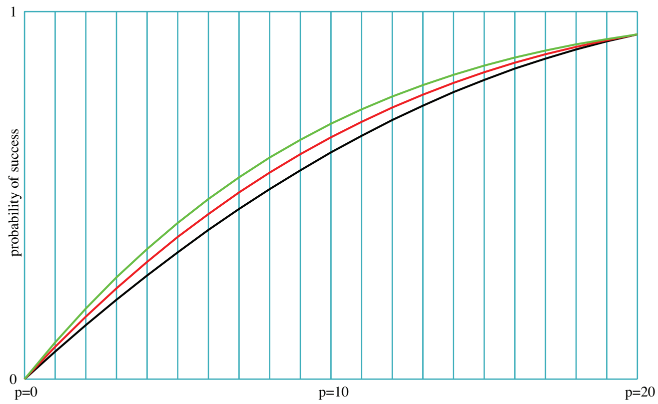

Figure 5. Team of two agents, each with two skills of strength 1/2. The black graph corresponds to a team whose skills coincide, the red graph to a team sharing one skill, and the green one to a team with no skills in common.

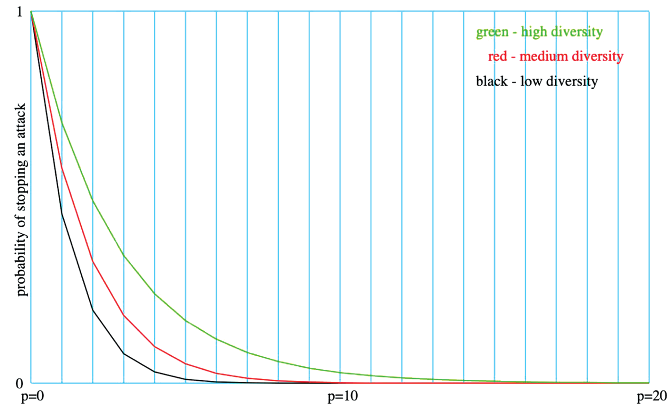

Figure 7. Team of two agents for the security problem. The black graph corresponds to the strength vector consisting of ten strengths 0.91 and ten 0; the red graph to the vector consisting of five strengths 0.91, ten 0.7, and five 0; and the green one to the vector consisting of twenty strengths 0.7.

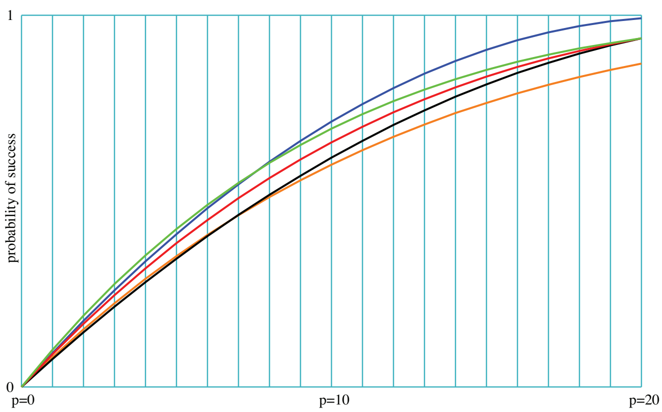

Figure 6. Like Figure 5, but with two additional graphs. The dark blue graph corresponds to the team strength vector $(0.91, 0.91, 0, \dots, 0)$ (that is, strength of skills 0.7, but no diversity), the orange graph to $(0.4, 0.4, 0.4, 0.4, 0, \dots, 0)$ (that is, strength of skills 0.4 and maximal diversity).

their skills gives better results, except for the easiest problems.

This phenomenon is easy to explain. Consider two vectors R_1 and R_2, the same except the values at some i, j. They come from a team of agents with prescribed skills, slightly differently placed. For R_1 we have more diversity, so $R_1(i) = a \in (0,1)$ comes from one agent, and $R_1(j) = b \in (0,1)$ from another agent. For R_2, diversity is smaller, so $R_2(i) = ab$ and $R_2(j) = 1$. Then there is a polynomial T, with nonnegative coefficients, such that $Q_{R_1}(x) = T(x)(x+a)(x+b)$ and $Q_{R_2}(x) = T(x)(x+ab)(x+1)$. We have $(x+a)(x+b) = x^2+(a+b)x+ab$ and $(x+ab)(x+1) = x^2+(ab+1)x+ab$ and $(ab+1)-(a+b) = (1-a)(1-b) > 0$, the coefficients of the polynomial Q_{R_2} are strictly larger than the coefficients of the polynomial Q_{R_1} (except the first and last coefficients, for which we have the equality). This means that if ability is kept constant, by increasing diversity of skills we get better chances for success.

This result differs from what we saw about specialization and versatility. This is because the skillfulness of a team is usually smaller than the sum of each member's skillfulness.

We can ask what happens if we change the abilities of the members of the team. In Figure 6 we added two graphs. One of them corresponds to larger abilities but no diversity; the other one corresponds to smaller abilities but larger diversity. By comparing the two lowest graphs with each other, and two highest graphs with each other, we see that to some degree ability and diversity of skills are exchangeable. However, in this example, for easier problems ability is more important, while for more difficult ones diversity is more important. Of course, we do not know how this applies to real life situations, since our model may or may not fit them (cf. Theorem 5).

However, if the ability of one or more team members increases (and no other changes are made), the coefficients of the polynomial Q for the team decrease, so we get better chances for success for problems of all difficulties.

6. Security

Let us look at a possible adaptation of our model to a security problem. Here the agents are trying to defend an organization against an attack (for instance, by hackers).

The attacker has p possible lines of attack (out of n possible), and for each of them the skillfulness of the agent gives us the probability of stopping this line of attack. Thus, the average probability of stopping an attack of strength p is $F_S(p)$. Note that for problem solving large p meant an easy problem; here large p means a strong attack.

In earlier sections we wanted to minimize our probability of failure $F_R(p)$. In contrast, we want now to *maximize*

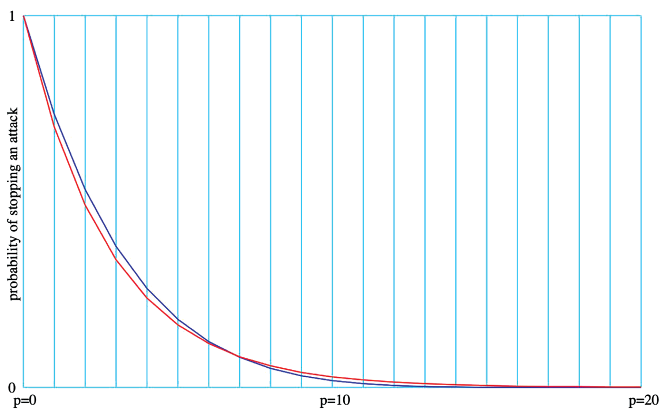

Figure 8. Team of two agents for the security problem. The red graph corresponds to the vector consisting of twenty strengths 0.7; and the blue one to the vector consisting of four strengths 0.97, twelve strengths 0.9, and four strengths 0.

the attacker's chance of failure $F_S(p)$. Therefore, by the results for problem solving, for security purposes versatility is better than specialization.

This also means that in this application diversity of skills in a team plays an even larger role than for problem solving. Diversity corresponds to more uniform spread of strengths, which for the security problem is useful even for one agent. An example similar to the one from Figure 5 is illustrated in Figure 7. We consider a team consisting of two agents, each of them having 10 strengths 0.7. Then we compare three possibilities: all, half, or none of the strengths coincide. We get for the team three strength vectors. The first one consists of ten strengths 0.91 and ten 0; the second one of five strengths 0.91, ten 0.7, and five 0; and the third one of twenty strengths 0.7.

Here also diversity and ability are to some degree interchangeable. For example, if we have two agents, one with sixteen strengths 0.7, and the other one with four strengths 0.7, then in the most diverse case we get for the team twenty strengths 0.7. If the first agent's strengths are 0.9 instead of 0.7 and we consider the least diverse case, we get for the team four strengths 0.97, twelve strengths 0.9, and four strengths 0. The first team will be better for strong attacks, but the second one will be better for weak attacks (see Figure 8).

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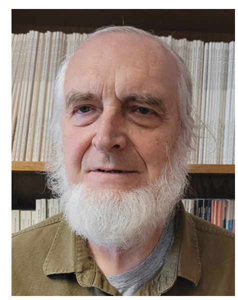

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Credits

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