The cover design is based on imagery from "Wave-Mean-Flow Interactions in Atmospheric Fluid Flows," page 200.
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Letter to the Editor

This is to make the mathematical community aware of the circumstances of Prof. Harold Donnelly at Purdue University. In my view, Donnelly’s treatment by his department head and his administration are outrageous. I understand that a large group of Purdue math professors objects to Donnelly’s treatment and has been trying to engage with the department head and the dean to achieve an acceptable outcome, so far to no avail.

I urge readers to learn the facts and judge for themselves. The relevant facts are summarized in the following excerpt from an email I sent to the Dean of Science at Purdue, dated 10/26/22, with a copy to Donnelly’s department head.

• After bad experiences teaching a service course (math 262) culminating in a disastrous experience during the pandemic, Donnelly was forbidden going forward to teach any course except math 262.
• Moreover, he was forbidden to teach math 262 until, among other deliverables, he produced written lectures for the whole course, “ready for delivery with no ad libbing”.
• [The math department head] threatened to reduce Donnelly’s salary to zero and to recommend to the College that his tenure be revoked for “gross negligence”.
• In connection with a hearing to consider a grievance filed by Donnelly, [the department head] asserted in writing that she had never threatened to reduce Donnelly’s salary to zero.
• Donnelly’s salary is now set at zero.

In support of the above assertions, I attach a narrative and appendices, which I asked Leonard Lipshitz to produce. If there are salient facts of which I’m unaware, justifying the actions taken against Donnelly, I would appreciate learning them from you.

The full text of my email to the dean, together with the narrative and appendices referred to above may be found at the website [https://web.math.princeton.edu/~cf/donnelly.html](https://web.math.princeton.edu/~cf/donnelly.html).

In a follow-up email to Donnelly’s department head and dean, dated 11/13/22, I told them of my intention to publicize Donnelly’s situation in the Notices of the AMS, and I asked them to point out to me any inaccuracies or misrepresentations in the above email or the accompanying narrative and appendices. As of this writing they have alleged no inaccuracies or misrepresentations.

All opinions expressed here are my own. I’m not speaking for Princeton University or its Math Department.

Sincerely,
Charles Fefferman
December 18, 2022

Response

Dear Professor Flapan,

Thank you for offering to publish a concurrent reply to the letter you received.

While the College of Science disagrees with Professor Fefferman’s characterization of events set forth in his letter, we take seriously all concerns of our community members and we value the contributions of our faculty to excellence in research and teaching.

Purdue is confident in the leadership of the Department of Mathematics, who give faculty careful and due consideration to the often-challenging and complex issues they face, while ensuring our students receive the finest instruction from engaged and expert faculty.

Without commenting on an individual case, we can assure that each faculty member in the College of Science receives regular feedback from departmental leadership about their work performance. Full professors are evaluated every three years, and leaders strive to provide clear, objective, and workable guidelines and resources for performance improvement when a faculty member’s review indicates they are not meeting expectations.

Additionally, employment decisions related to teaching expectations, performance concerns, and salary are all subject to a robust Faculty Grievance Procedure, whereby faculty can have specific employment-related issues heard.
Letters to the Editor

The College of Science regrets that Professor Fefferman chose to release incomplete confidential personnel files related to a faculty member with a long and valued career. Purdue respects the privacy of our faculty members and will not comment on any specific personnel actions.

Sincerely,
Lucy Flesch, Senior Associate Dean for Faculty Affairs
College of Science, Purdue University

Letter to the Editor

We, the undersigned math faculty, staff, and graduate students at the University of Washington, endorse the proposal by David Rohrlich in the September 2022 issue of the Notices that the AMS not hold meetings in states to which pregnant women cannot travel without risking their health and possibly their life.

Signed: Jayadev Athreya, Sara Billey, Martin Bishop, Madeleine Brown, Thomas Carr, Matthew Conroy, Natasha Crepeau, Dan Guyer, Paige Helms, Neal Koblitz, Kevin Liu, Charlie Magland, Clare Minnernath, Michael Munz, Haoming Ning, Nelson Niu, Lauren Nowak, Farbod Shokrieh, Stefan Steinerberger, Jennifer Taggart, Yirong Yang, Jonathan Zhu

Letter to the Editor

Every year for the past 70 years, Nobel laureates have come together in Lindau, Germany to meet young researchers from all over the world. What significance does this hold for mathematicians? Unfortunately none, since there has never been a Nobel Prize in mathematics. In 2013, however—inspired by the Lindau Nobel Laureate Meetings—the Klaus Tschira Foundation, the International Mathematical Union, the Norwegian Academy of Sciences and Letters, and the Association for Computing Machinery decided to create an event specifically to host the recipients of the most prestigious prizes in mathematics and computer science: the Heidelberg Laureate Forum.

Since 2013, laureates from the most distinguished prizes in mathematics and computer science—the Abel Prize, the Fields Medal, the IMU Abacus Medal (formerly Nevanlinna Prize), the ACM Prize for Computing, and the ACM A.M. Turing Award—have come to Heidelberg every year to meet and interact with 200 young researchers from all over the world. For one week, the main building at Heidelberg University becomes a nexus for interdisciplinary exchange between mathematicians and computer scientists, with not only a slew of lectures, poster sessions, and discussions, but also ample room for networking and a cross-generational exchange of ideas in an informal setting.

We would like to encourage all members of the AMS to apply for the opportunity to participate in this unique event or encourage their students, graduate students, and postdocs to apply. The 10th Heidelberg Laureate Forum will take place in Heidelberg, Germany from September 24 until September 29. Applications are currently open and can be submitted until February 11, 2023. You can go to https://www.heidelberg-laureate-forum.org to learn more.

Sincerely,
Sergei Tabachnikov
Professor, Penn State University
IMU representative, Scientific Committee Heidelberg Laureate Forum Foundation

Anna Wienhard
Director, Max Planck Institute for Mathematics in the Sciences
Scientific Chair, Heidelberg Laureate Forum Foundation

Use of Preview Editor with PDF Files

I wish to share a recent experience involving the Preview editor on a MacBook Pro using the Annotate Tools: Highlight Text, Strike Through Text, and Text. Coincidentally my coauthors and I received galley proofs in pdf format for a textbook and for a journal article within an overlapping time frame. The textbook file is approximately 500 pages and the journal article 25 pages. After many hours of markup on the textbook file, using the Save command throughout, several of the markups have vanished from the pdf file. This included markups by coauthors as well as my own. It is not a machine dependent issue.

Consulting Google revealed that it is a known problem for the Preview editor. Fortunately, there are links to recover the lost markups. The link https://github.com/julihoh/pdf_annotation_fix#readme was effective in recovering the data from the large textbook file. After this experience with the textbook file, the recovery link was applied to the journal article edits. It also recovered missing markings that would not have been known without conducting a check.

Given the pervasive use of pdf files within the mathematics community, it seems prudent to issue this alert. It would have been very helpful to us to have been aware in advance.

Sincerely,
Edward C. Waymire
Oregon State University
The February issue of the AMS Notices coincides with the nation’s celebration of Black History Month. Four interesting feature articles involving topics from fluid dynamics, diffusion problems, optics and optimal transport problems, and Riordan matrices, written by Black mathematicians, are presented here. The feature articles are: Lucy Campbell’s article “Geophysical Fluid Dynamics,” Nsoki Mavinga’s article “Steklov Spectrum and Diffusion Problems with Nonlinear Boundary Conditions,” Henok Mawi’s article “Freeform Optics: Optimal Transport, Minkowski Method and Monge Ampère Type Equations,” and Naiomi Cameron and Asamoah Nkwanta’s article “Riordan Matrices and Lattice Path Enumeration.”

The 2023 Black History Month national theme is Black Resistance, and this special issue of the Notices celebrates articles that emphasize the struggles, resistance, and triumphs of various Black mathematicians and educators. According to the Association for the Study of African American Life and History, the theme of resistance explores how “African Americans have resisted historic and ongoing oppression, in all forms, especially the racial terrorism of lynching, racial pogroms, and police killings since our arrival upon these shores. … Education, whether in elementary, secondary, or higher education institutions have been seen as a way for Black people and communities to resist the narrative that Black people are intellectually inferior.” See the link https://asalh.org/black-history-themes/ for more details of this theme and past.

This year’s BHM theme is timely here as this month’s Notices authors Ben Moynihan and collaborators remember the civil rights activist, educator, and creator of the Algebra Project, Robert P. Moses, in their memorial article for Bob Moses. In his 2001 book with co-author Charles Cobb, Jr., Radical Equations: Math Literacy and Civil Rights, Moses asserts “In today’s world, economic access and full citizenship depend crucially on math and science literacy. I believe that the absence of math literacy in urban and rural communities throughout this country is an issue as urgent as the lack of registered black voters in Mississippi was in 1961.”

Evelyn Lamb, Omayra Ortega, and Robin Wilson’s article, “The Role of Mathematics in Today’s Movement for Racial Justice” states that, “Using mathematics as a tool to critically analyze systemic racism has a long history in the United States. … W. E. B. Du Bois in 1903 predicted quite prophetically that ‘the problem of the 20th century is the problem of the color line.’” Yet in the 21st century, the color line dilemma continues to plague us.

Jesse Kass’s history article, “Joseph Carter Corbin: Arkansas’s ‘Profound Mathematician’” gives the readers a historical perspective around the theme. Corbin was born in 1833 and passed away in 1911. Kass pens, “Even though Corbin never personally experienced enslavement, his life was significantly constrained by state laws and social practice. For example, Black Ohioans were not allowed to attend public schools, and many private schools were racially segregated.” This article notes some of Corbin’s legacy. In 1882, he contributed a mathematical problem to The Mathematical Magazine, a short-term journal from 1882–1884. From 1895–1902 he regularly contributed to the problem section of the American Mathematical Monthly. Corbin was also involved in the establishment of Branch Normal College in 1875, which today is the University of Arkansas at Pine Bluff, a Historical Black College and University (HBCU). Corbin’s work demonstrates intellectual and scientific strength, plus shows determination and resistance.

Asamoah Nkwanta is a chair and professor of mathematics at Morgan State University. His email address is Asamoah.Nkwanta@morgan.edu.
Johnny Houston’s article, “Passing the Torch for NAM: A Reflection of NAM’s Development and Growth,” gives an overview of the first five decades of the National Association of Mathematicians (NAM). Readers of this article will get a glimpse of an organization that was created to resist inequities its members experienced in the math community, past and present. Houston writes, “NAM was founded under the principles of inclusivity, diversity, and equity at a time when major American mathematical sciences organizations were excluding underrepresented American mathematicians of color from their membership, editorial boards, research symposia, and other professional activities.”

Sylvia Bozeman and collaborators’ article, “Spelman College: A Model of Success for Producing Black Women in Mathematics,” can serve as a model for other departments. Bozeman and collaborators state, “For the past century, Spelman’s mathematics department has worked deliberately to develop and support women in the mathematical sciences. We hope other institutions that want to contribute to increasing the number of women—and especially African American women—who study mathematics and pursue mathematics-related careers may find inspiration from Spelman’s proven model.”

The Early Career section contains excerpts from the book “Justice Through the Lens of Calculus: Framing New Possibilities for Diversity, Equity, and Inclusion,” edited by Matthew Voigt and collaborators. The book focuses on best practices for diversity and inclusion. Thirty case studies aim to make calculus available and accessible to previously excluded or underrepresented populations.

Houston gives the Notices readers another memorial article, “Remembering Professor Aderemi O. Kuku (1941–2022): An Internationally Recognized Mathematician and Scholar.” Kuku was a distinguished Nigerian mathematician and scholar in the field of algebra. Additionally, Lawrence Udeigwe presents an engaging opinion article titled, “Interfacing Math and Jazz.” Finally, there is a review by Sorin Gal of Gaston N’Guerekata’s book, “Almost Periodic and Almost Automorphic Functions in Abstract Spaces.”

The BHM articles in the Notices this month promote Black resistance by supporting intellectual development, inclusivity, and diversity in the mathematical sciences. In closing, I am again honored to serve as an associate editor in order to share with the Notices readers these fantastic articles.
Wave-Mean-Flow Interactions in Atmospheric Fluid Flows

Lucy J. Campbell

Geophysical fluid dynamics describes the flow of gases and liquids in the Earth’s atmosphere, oceans, and other large bodies of water such as lakes and rivers. From a mathematical perspective the evolution of the fluid flow can be described by partial differential equations derived from the physical principles of conservation of mass, momentum, and energy, and including terms that represent the effects of the rotation of the Earth and the effects of buoyancy and gravity. Analyses and solutions of these equations contribute to our understanding of the mechanisms that drive weather over the short term and climate over the long term.

Waves occur naturally in geophysical fluid flows, either in the interior of a body of fluid or at the interface between two fluids such as on the surface of the ocean. The primary mechanisms that generate waves in geophysical flows are the Coriolis force that arises from the Earth’s rotation and the buoyancy and gravitational forces. Wave interactions take place over a wide range of temporal and spatial scales and can have profound effects on the general circulation of the fluid flow. Thus, an understanding of wave properties, including their generation, propagation and interaction with the general circulation, is of great importance for accurate weather forecasting and climate modeling. Forecasting depends on frequent observations and measurements of the properties and current state of the atmosphere, analyses of the data obtained, a good understanding of the dynamics of the atmosphere, the development of mathematical models and computational algorithms, powerful computers, careful analysis and interpretation of the output of the models, and presentation of the output in the form of forecasts.

Mathematical analyses of the fluid equations provide insights that inform the development of the models. Waves may be represented in the equations as perturbations to a suitably averaged background fluid flow. This perspective leads to nonlinear perturbation equations for the wave quantities, which can be linearized and solved exactly in certain special configurations, and can be analyzed in more general situations using the methods of perturbation theory, hydrodynamic stability analysis, and asymptotic analysis. In general, the goal is to understand the wave-mean-flow interactions, i.e., the interchange of energy and momentum between the perturbations and the large-scale background mean flow.
This feature article discusses a mechanism by which certain types of waves interact with a background mean flow in a geophysical fluid context: the critical layer interaction. This type of interaction occurs when a wave propagating with a given phase speed reaches a location called a critical line or critical level, where the speed of the background mean flow is equal to the wave phase speed. In the critical layer surrounding this line, the wave transfers momentum and energy to the background fluid flow, a process called wave absorption, and modifies the flow as a result. This can cause reversal of wind direction, changes in the mean vorticity and temperature, wave breaking, and turbulence. Mathematically, the critical layer corresponds to a singularity in the linearized inviscid equations for steady-amplitude waves.

This article is not intended to be a comprehensive review of atmospheric waves and critical layer phenomena. The focus of the discussion shall be on three types of waves that are ubiquitous in the atmosphere, namely internal gravity waves, planetary Rossby waves, and vortex Rossby waves, and their corresponding critical layer theories. In spite of the differences between the physical characteristics of these waves, there are some similarities in the underlying mathematics and in the approaches used for analyses and for obtaining solutions. The article aims to highlight some of these similarities by referring to critical layer studies of the author and other researchers.

Gravity waves exist in geophysical fluid flows as a result of the competing effects of the downward force of gravity and the upward buoyancy force that causes fluid particles to rise. They include surface gravity waves, such as the familiar water waves that form at the interface between a body of water and the air above.

Internal gravity waves occur in the interior of a fluid, e.g., within the atmosphere or within the ocean, and can propagate vertically as well as horizontally. In the atmosphere, the most common examples are orographic waves that are generated by airflow over mountains and other large obstacles and convective waves that form above cloud layers and arise from vertical motion in the lower atmosphere. The spatial scale of these waves is small relative to the planetary scale; their horizontal wavelengths are typically no more than a few hundred kilometers.

Figure 1 shows an image of gravity waves over stratocumulus cloud layers above the Indian Ocean; they appear as multiple cloud bands corresponding to the troughs and crests of the waves. A similar structure is seen in the clouds in the photograph in Figure 2.

Internal gravity waves have frequent and profound effects on the general circulation of the atmosphere over different time scales and they thus affect both weather and climate. The amplitude of an internal gravity wave tends to increase as the wave propagates upwards. This is because the wave kinetic energy is proportional to the product of the atmospheric density and the square of the amplitude of the fluctuation in the velocity. Conservation of the wave energy in an atmosphere with density decreasing with altitude implies that the wave amplitude increases with altitude. In practice, the rate of increase could be modified by dissipation; this is seen, for example, in the measurements reported by Tsuda [Tsu14]. The amplitude increase can lead to an instability and possible wave breaking at high altitudes. Moreover, the presence of a critical level acts to filter out the waves, as observed by Whiteway and Duck [WD96], and may also lead to breaking. Breaking waves generate the turbulence that is often experienced by air travelers especially when flying over mountain ranges or over convective clouds.

On a longer timescale, internal gravity waves propagating into the stratosphere transfer momentum and energy to the background flow and are known to contribute to the
development of the quasi-biennial oscillation (QBO). This is a large-scale oscillation in which the wind direction alternates between eastward and westward phases with a period of 26–30 months. The QBO has profound effects on the stratospheric flow globally, beyond the tropics; it affects the transport and distribution of chemical constituents in the atmosphere and may even affect the strength of hurricanes [B+01]. Given the important role that atmospheric gravity waves are known to play in such weather and climate events, it is important to represent and simulate them accurately in the general circulation models that are used for weather prediction and climate modeling. Historically, this presented a challenge due to their relatively small wavelengths and short time periods. The development of the mathematical theory of gravity waves, including critical layer interactions, has contributed to our understanding of their effects on the general circulation and helped to improve the accuracy of their representation in general circulation models in recent decades [FA03].

Rossby waves are oscillations that result from the variation of the rotational or Coriolis force due to the curvature of the Earth. Gravitational and buoyancy effects also play an albeit less significant role in their generation and propagation. These waves play a significant role in weather in the midlatitude regions and are associated with oscillations in the jet stream. Rossby waves are named after the Swedish-born meteorologist Carl-Gustaf Arvid Rossby, who first identified them in 1939 as large-scale wavy patterns in the atmospheric flow and subsequently described their characteristic features. The mathematical theory of Rossby wave critical layers dates back to the 1950s and 60s [Lin55]. The seminal mathematical studies on the nonlinear inviscid time-dependent problem, presented simultaneously by Stewartson [Ste78] and Warn and Warn [WW78] in 1978, describe asymptotic solutions that are commonly called the “SWW” solutions. The SWW theory provides insight into the observed formation of the so-called stratospheric “surf-zone,” a large region of planetary wave breaking that extends from northern latitudes towards the tropics [MP84].

Rossby waves also exist in the atmosphere on a smaller scale as vortex waves, which take the form of outward-propagating disturbances within cyclonic vortices such as hurricanes [Mac68], and result primarily from the force due to the radial gradient of the cyclone vorticity. A hurricane is a cyclone with maximum wind over 32 ms⁻¹ that develops in the atmosphere over the tropical ocean. Typically, a hurricane has a calm area with low atmospheric pressure at its centre. This is called the eye and it is surrounded by a ring of extremely high angular velocity called the primary eyewall. In high-intensity hurricanes, a secondary eyewall sometimes develops as a larger outer ring, increases in intensity and moves inward to replace the primary eyewall. This process repeats as an eyewall replacement cycle. The structure of the vortex at two stages in this process is shown in Figure 3.

Vortex Rossby waves interact with the background vortex flow if there is a critical radius where the vortex flow angular velocity is equal to the wave phase speed. The presence of a critical radius affects the structure and intensity of the hurricane and it has been suggested that vortex Rossby waves may contribute to the secondary eyewall development and the eyewall replacement cycle, possibly through a critical radius interaction mechanism [MK97].

All these examples underscore the importance of characterizing and understanding atmospheric waves and their behavior at critical layers. This article discusses the critical layer theory for the three types of waves mentioned above. In each case, the discussion is on situations where the wave dynamics is represented in a two-dimensional spatial domain by perturbation equations which are analyzed using...
asymptotic methods or solved numerically. The focus is on problems involving forced waves, i.e., where an oscillatory boundary condition is applied at one end of the two-dimensional domain and generates a wave with a specified wavelength which propagates into the interior of the region where it encounters a critical layer.

In Section 1 the basic equations of fluid dynamics are presented and perturbation equations are obtained from these. Sections 2–4 give respective overviews of planetary Rossby waves, vortex Rossby waves, and internal gravity waves, and the aspects of their critical layer theories that are similar. Some results of recent numerical studies of the author are presented as illustrations.

1. Geophysical Fluid Flows and Waves

The dynamics of geophysical fluid flows can be described mathematically by equations based on the laws of conservation of mass, momentum, and energy. In a geophysical fluid configuration the flow exists in a three-dimensional domain defined by the spherical Earth, but depending on the spatial scale of the phenomena of interest, we may be able to justify using a two- or three-dimensional representation with Cartesian coordinates as an approximation for spherical geometry.

In terms of Cartesian coordinates $x$, $y$, and $z$, the conservation of mass is written in terms of the fluid density (mass per unit volume) $\rho$, the components $u$, $v$, and $w$ of the fluid velocity in the respective directions, and the time $t$, as the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$ 

The special case of an incompressible flow, where the rate of change of the density along fluid parcel trajectories is considered to be zero, gives a simplified form of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$ 

The conservation of momentum is given by Newton's Second Law, which says that the rate of change of the fluid momentum (mass multiplied by the velocity) in some control volume of fluid is balanced by the sum of any applied forces. Considering a fluid parcel of unit volume moving with velocity components $u$, $v$, and $w$, this can be written as

$$\rho \frac{D u}{D t} = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + F_x,$n

$$\rho \frac{D v}{D t} = \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + F_y,$n

$$\rho \frac{D w}{D t} = \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + F_z - g\rho.$$ 

The differential operator $\frac{D}{D t}$, defined above, gives the rate of change of a fluid property following a fluid parcel moving with velocity $(u, v, w)$; this is known as the Lagrangian derivative.

The terms on the right-hand side are the forces acting on the fluid: $p$ is the pressure (the force per unit area), the spatial derivatives of $p$ are the components of the pressure gradient force, $F_x, F_y, F_z$ are the components of the Coriolis force and $g$ is the acceleration due to gravity, which gives a gravitational force that acts downward and thus only appears in the vertical momentum equation. Other forces that could be present in a geophysical flow situation have been omitted here, for example the viscous or frictional force and any forces due to electromagnetic effects.

Given the current state of the fluid flow (at $t = 0$) in some specified region in space, and some information about the flow at the boundaries, the goal is to solve the equations to find out its state at a later time (for $t > 0$) within that region. This can be written as an initial-boundary-value problem for the system of partial differential equations. To facilitate mathematical analyses, and in particular, to assess the relative magnitude and importance of the various terms, the equations may be considered in nondimensional form.

In a weakly-nonlinear analysis, waves in a fluid are considered as perturbations to a background mean flow with each fluid flow quantity being written as the sum of its mean and a perturbation which is taken to be of small amplitude relative to the mean flow. In a nondimensional framework, we write a quantity $\varphi_{\text{total}}$ as

$$\varphi_{\text{total}}(x, y, z, t) = \varphi + \varepsilon \varphi(x, y, z, t),$$

where $\varphi$ is a temporal and spatial average of the total quantity, usually the mean in the horizontal west-east ($x$) direction, and $\varepsilon$ is a small nondimensional constant which gives a measure of the magnitude of the perturbation relative to the background flow quantity. This leads to nonlinear equations for the perturbation quantities, which may be linearized as a first approximation. In the linear equations each perturbation quantity $\varphi$ is considered to be an oscillatory and periodic function in one or more spatial directions or in time, and can be represented using Fourier modes, i.e., combinations of sine and cosine functions written in terms of complex exponential functions.

A wave may reach a location in the fluid flow where the background flow speed is equal to the wave phase speed. In an actual three-dimensional flow, this would, in general, be a surface oriented in three-dimensional space. In a two-dimensional representation, it is a critical line. For planetary Rossby waves propagating on a horizontal plane, it is a critical latitude; for vortex Rossby waves traveling radially outward from the center of a vortex, it is a critical radius; and for gravity waves in a vertical plane, it is a critical level.
In the linearized theory for steady-amplitude waves in a flow with no viscosity, the critical line corresponds to a singularity in the ordinary differential equation for the wave amplitude. However, the singular behavior is a mathematical artifact resulting from the omission of the nonlinear terms and viscous terms and the assumption that the wave amplitude is steady. The methods of asymptotic analysis are useful for obtaining an approximate “inner” solution in the critical layer, which is then matched with the solution in the “outer” region away from the singularity. The inner solution is obtained by restoring the viscous terms to the equations in the critical layer, by considering waves with time-dependent amplitude, or by carrying out a weakly-nonlinear analysis in which the previously neglected nonlinear terms are included but they are considered to be “small” relative to the linear terms. In the nonlinear time-dependent formulation, the methods of multiple-scale analysis are used as well; we introduce a suitably scaled “slow” time variable and obtain a slowly varying solution to complete the description of the solution at late time.

The focus of the discussion here will be on weakly-nonlinear time-dependent solutions of the inviscid equations. The development of nonlinear time-dependent critical layer theory for Rossby waves and internal gravity waves has an extensive history developed over the past 60 years. A brief overview of critical layer theory is given here, along with some illustrative numerical simulations for Rossby waves (Sections 2 and 3) and internal gravity waves (Section 4). Each of the problems discussed is based on a two-dimensional simplification of the full three-dimensional system of conservation laws. In each case, the geometry of the problem is defined as either a horizontal or a vertical plane, to give an approximate representation of the direction of propagation of the type of wave studied. Such two-dimensional studies provide us with insights into the dynamics of the respective types of waves in more realistic three-dimensional situations.

2. Planetary Rossby Waves

In this section, the two-dimensional configuration involving Rossby waves in a barotropic shear flow is discussed. A barotropic fluid flow is one in which the density is a function of the pressure only, so that the surfaces of constant density coincide with the surfaces of constant pressure. In situations where the wave propagation is primarily horizontal, a barotropic flow on a two-dimensional horizontal plane may be a reasonable first approximation for the full three-dimensional representation. On the horizontal plane the density is taken to be constant; thus, mass conservation is described by the incompressible form of the continuity equation.

We consider an inviscid barotropic fluid flow on a horizontal plane defined by Cartesian coordinates $x$, in the west-to-east direction, and $y$, in the south-to-north direction, and described by the momentum and mass conservation equations

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{-1}{\rho} \frac{\partial p}{\partial x} + fu, \tag{2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{-1}{\rho} \frac{\partial p}{\partial y} - fu, \tag{3}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4}
\]

The terms $fv$ and $-fu$ represent the horizontal components of the Coriolis acceleration in the eastward and northward directions respectively, and $f$ is Coriolis parameter, which is defined as $f = 2 \Omega \sin \theta$, where $\theta$ is latitude and $\Omega$ is the Earth’s angular velocity. In the $\beta$-plane approximation, the Coriolis parameter is linearized about some reference latitude, say $\theta_0$. The difference in latitude $\theta - \theta_0$ is written in terms of the rectangular coordinate $y$, using the arc length formula, and the Coriolis parameter is approximated by a linear function of $y$. Writing $f \approx f_0 + \beta y$, where $f_0$ is the value of $f$ at $\theta_0$, gives the beta-plane approximation. This approximation is valid for a situation where the latitudinal extent of the domain under consideration is narrow, relative to the scale of the Earth, as shown in Figure 4. Under the beta-plane approximation, a narrow strip like that shown in the diagram, centered at latitude $\theta_0$ is represented by a rectangular domain centered along the line $y = 0$ and with periodic boundary conditions in the longitudinal or zonal direction.

The form of the incompressible continuity equation in two variables (4) allows us to define a streamfunction $\Psi(x, y, t)$ by $\frac{\partial \Psi}{\partial y} = -u$, $\frac{\partial \Psi}{\partial x} = v$. This is a function whose level curves in the $xy$-plane define the streamlines, curves which are instantaneously tangent to the velocity vector. The Laplacian of the streamfunction $\nabla^2 \Psi$ is the vertical component of the vorticity, which measures the extent of local rotation of a fluid parcel at a given location in the flow. Differentiating (2) with respect to $y$ and (3) with respect to $x$ and taking the difference of the resulting equations gives a description of the flow in terms of the streamfunction and vorticity called the barotropic vorticity equation,

\[
\nabla^2 \Psi_t + \Psi_x \nabla^2 \Psi_x - \Psi_y \nabla^2 \Psi_x + \beta \Psi_x = 0. \tag{5}
\]
Here the subscripts x, y, and t denote differentiation with respect to the respective variables.

The total streamfunction is written in the form suggested by (1). We write
\[ \Psi(x, y, t) = \hat{\psi}(y) + \varepsilon \psi(x, y, t), \]
where \( \hat{\psi}(y) \) is the basic flow streamfunction, taken as the initial mean in the zonal (x) direction, and \( \psi(x, y, t) \) is the perturbation streamfunction, and \( \varepsilon \) is a “small” parameter that may be considered as \( \varepsilon \ll 1 \) in an asymptotic analysis. A simple representation of an atmospheric flow is one in which the basic flow has a zonal (x) velocity component \( \bar{u}(y) \) (where the prime denotes the y-derivative) and its meridional or latitudinal (y) velocity component is zero. This is an example of a shear flow, where the x-averaged basic flow speed varies with y. This gives a nonlinear equation from which the perturbation quantities can be determined in terms of the specified background flow,
\[
\nabla^2 \psi_t + \bar{u}(y) \nabla^2 \psi_x + [\beta - \bar{u}''(y)] \psi_x \nabla^2 \psi_x + \varepsilon(\psi_x \nabla^2 \phi_y - \phi_y \nabla^2 \psi_x) = 0. \tag{6}
\]
If we consider \( \varepsilon \ll 1 \), we can justify neglecting the nonlinear terms to obtain a linear equation,
\[
\nabla^2 \psi_t + \bar{u}(y) \nabla^2 \psi_x + [\beta - \bar{u}''(y)] \phi_x = 0. \tag{7}
\]

Different approaches may be taken to analyze this linear equation, depending on the specification of the background flow and domain and on the information that we wish to obtain about the solutions. In the case where the mean flow speed \( \bar{u} \) is constant, equation (7) has constant-amplitude plane wave solutions of the form
\[ \psi(x, y, t) = Ae^{i(kx + ly - \omega t)} + \text{c.c.} \tag{8} \]
Here \( A \) is the complex constant amplitude and “c.c.” denotes the complex conjugate of the function given, so that the wave solution is a real function. The constants \( k \) and \( l \) are the wavenumbers in the \( x \) and \( y \) directions, respectively, and define the number of oscillations within an interval of length \( 2\pi \) and the constant \( \omega \) is the wave frequency, which defines the number of oscillations in a time interval of \( 2\pi \).

This formulation, when substituted into (7) gives the dispersion relation which expresses the frequency as a function of the wavenumbers
\[ \omega = k \bar{u} - \frac{\beta k}{k^2 + l^2}. \]
The ratio \( c = \omega/k \) is the wave phase speed in the \( x \) direction; it gives the speed at which a given phase of the wave, e.g., a crest or trough of the wave, travels in the \( x \) direction. Thus, dividing the dispersion relation by \( k \) tells us that the phase speed \( c \) is always westward (negative) relative to the background flow speed \( \bar{u} \). This is a characteristic feature of Rossby waves. Differentiating the dispersion relation partially with respect to \( k \) or \( l \) gives the \( x \) and \( y \) components of the group velocity vector
\[ (c_{gx}, c_{gy}) = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right), \]
which defines the direction of propagation of the wave energy. To illustrate the difference between the phase velocity and the group velocity, we may consider a wave packet like that shown in the diagram in Figure 5 propagating in the direction of a horizontal or zonal variable \( x \). The \( x \)-component of the group velocity \( c_{gx} \) gives the speed at which the “envelope” of the wave packet, indicated by the dotted curve, travels in the \( x \) direction, while \( c \) gives the speed of propagation of the crests and troughs of the oscillation. If \( c \neq c_{gx} \), as shown in this diagram, then the wave packet is said to be dispersive; this is the case for Rossby waves.

In the general case with mean flow speed \( \bar{u}(y) \), we seek solutions of equation (7) that are periodic and monochromatic in \( x \) and \( t \) (i.e., with a fixed wavenumber and frequency) and have an amplitude that depends on \( y \). This is called the normal mode form, written as
\[ \psi(x, y, t) = \phi(y)e^{ik(x-c\cdot t)} + \text{c.c.}, \tag{9} \]
where \( k \) is the wavenumber, \( c \) is the zonal phase speed, \( \phi(y) \) is the complex amplitude and as before, “c.c.” denotes the complex conjugate of the function given. The amplitude function \( \phi(y) \) satisfies the Rayleigh–Kuo equation,
\[
(\bar{u}(y) - \varepsilon)[\phi''(y) - k^2\phi(y)] + [\beta - \bar{u}''(y)]\phi(y) = 0. \tag{10}
\]
This is a generalization of the Rayleigh equation, which was first introduced by Lord Rayleigh in the 1880s for the situation with \( \beta = 0 \).

In the approach of hydrodynamic stability theory, equation (10) is considered in the context of an eigenvalue
problem, where \( c = c_R + ic_I \) is a complex constant and the solution (9) is thus proportional to \( e^{kc_I t} \). The term stability refers to the situation where \( c_I \leq 0 \) and so the amplitude of the perturbation either decays to zero with \( t \) or remains constant. We seek to determine the conditions for stability.

The main result obtained from the stability analysis is that if a wave perturbation is unstable, with an amplitude that grows exponentially with \( t \), then we must have \( \beta - \bar{u}'(y) \) somewhere in the flow. This type of instability is called a barotropic instability. The converse of this result is that if \( \beta - \bar{u}'(y) \) is nonzero everywhere in the flow, then the flow is stable to perturbations of the form (9). This is useful information that must be taken into account when we carry out numerical simulations involving barotropic Rossby waves; it is important to ensure that the gradient of the basic flow vorticity \( \bar{u}'(y) \neq \beta \) for all \( y \) in the domain under consideration in order to avoid barotropic instability. The case with \( \beta = 0 \) gives Rayleigh’s inflection point theorem which says that the existence of an inflection point in the background flow speed, \( \bar{u}'(y) = 0 \), is a necessary condition for instability.

The Rayleigh–Kuo equation also describes the behavior of forced Rossby waves near a critical line. In the forced wave problem, an oscillatory boundary condition with a specified wave speed \( c \) and \( x \)-wavenumber \( k \) is imposed at the one boundary of the spatial domain generating a steady-amplitude travelling wave solution of the form (9). The Rayleigh–Kuo equation describes the variation of the complex wave amplitude \( \phi \) with \( y \). The equation has a regular singular point \( y = y_c \) where \( \bar{u}(y_c) = c \); this corresponds to the critical line. The method of Frobenius is used to obtain a series solution in powers of \( (y - y_c) \) and a second solution that includes a logarithmic term of the form \( (y - y_c) \log(y - y_c) \). For \( y < y_c \), the logarithm is defined as

\[
\log (y - y_c) = \log |y - y_c| + i \theta,
\]

where the argument \( \theta = \pm \pi \). Lin [Lin55] showed that the correct branch of the log is given by \( \theta = -\text{sgn}(\bar{u}'(y_c))\pi \). This is called a logarithmic phase shift or phase change. Taking this into account in the solutions, we see that there is a discontinuity in the wave amplitude across the critical layer. A wave incident on the critical line from the “north” \( (y > y_c) \) has its amplitude greatly reduced across the critical layer. The physical interpretation of this is that the wave is absorbed by the background mean flow. In order to represent the wave absorption correctly, however, we need a more realistic model in which the wave amplitude is allowed to vary with \( t \) in which case the equation for the wave perturbation is nonsingular.

We consider the forced wave problem in which the solutions of (7) are in the form of waves with time-dependent amplitude

\[
\psi(x, y, t) = \hat{\psi}(y, t)e^{ikx} + c.c. \tag{11}
\]

This leads to an initial-boundary-value problem for a partial differential equation involving \( t \) and \( y \) derivatives of the amplitude function \( \hat{\psi} \). Dickinson [Dic71] and Warn and Warn [WW76] carried out time-dependent analyses using a Laplace transform to find an outer solution, valid for large \( t \), and then asymptotic matching to find an inner critical layer solution. They showed that for large \( t \) the solution approaches a state with a quasi-steady amplitude with similar features to the steady-amplitude state analyzed earlier by Lin [Lin55], namely the same logarithmic phase shift and the corresponding reduction of the wave amplitude across the critical layer. In the inner region, however, the solution grows with time and it becomes apparent that consideration of the neglected nonlinear terms is needed in order to avoid this secular behavior [WW76].

Stewartson [Ste78] and Warn and Warn [WW78] continued the time-dependent investigations by considering the nonlinear equation (6). The nonlinear “SWW” solution, as it is commonly called, behaves like the linear solution at early time in the sense that there is a reduction in the wave amplitude across the critical layer. This reduction is balanced by a transfer of momentum to the mean flow indicating that the wave is indeed absorbed by the mean flow, as the linear solutions suggest. In the nonlinear formulation, the wave absorption results in a wave-induced mean flow acceleration in the vicinity of the critical line. An equation that describes how the mean flow velocity changes in time as a result of the wave interaction is obtained by taking the \( x \)-average of the \( x \)-momentum equation (2) over one \( x \)-wavelength. In (2) \( u \) represents the total \( x \)-component of the velocity, which is equal to the background flow \( \bar{u}(y) \) plus the order \( \varepsilon \) perturbation, and \( v \) represents the \( y \)-component of the velocity, which is of order \( \varepsilon \). Thus, the nonlinear terms involve products of the perturbations which are of order \( \varepsilon^2 \). Defining the change in the mean flow velocity as \( u_0(y, t) \), we obtain

\[
\frac{\partial u_0}{\partial t} = -\varepsilon^2 \frac{\partial \bar{u}v}{\partial y}. \tag{12}
\]

The “over-bar” on the right-hand side denotes the \( x \)-average taken over one \( x \)-wavelength and \( u \) and \( v \) are the \( x \)- and \( y \)-components of the wave velocity, \( u = -\psi_y, v = \psi_x \). The \( x \)-averaged quantity \( \bar{u}v \) is the mean meridional \( (y) \) momentum flux. In the absence of a critical line, this quantity would be approximately constant, so its \( y \) derivative would be small and there would be no significant change in the mean velocity over time. In a configuration with a critical line, there is a large change in the momentum flux in the \( y \)-direction across the critical layer. As a result the \( y \)-derivative of the momentum flux on the right-hand side of (12) is large in the vicinity of the critical layer and

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there is a corresponding change \( u_0 \) in the mean flow velocity. There is a corresponding change in the wave energy across the critical layer as energy is transferred from the wave modes to the mean flow.

Over time, additional nonlinear features develop in the critical layer. At late time momentum and energy from the mean flow are transferred to the wave modes as the wave is “reflected” by the mean flow. The timescale on which these nonlinear effects develop depends on the parameter \( \varepsilon \) which defines the amplitude of the perturbation relative to the mean flow. Larger amplitude waves evolve into a nonlinear reflecting state sooner. For sufficiently large \( \varepsilon \), the system may evolve into a state of over-reflection where the reflection of momentum and energy is greater than the absorption [WW78] and then subsequently continue over the long term to alternate between states of net-absorption and over-reflection [Bel76] and ultimately to a limiting state of perfect reflection [KM85].

Contour plots of the total vorticity, calculated from the asymptotic solutions [Ste78, WW78] and from numerical solutions [Bel76, Cam04], show closed contours known as “cat’s eyes” in the critical layer, that eventually overturn and break. Figure 6 shows a contour plot of overturning vorticity contours in a nonlinear Rossby wave critical layer. This result was obtained from a time-dependent numerical solution of (6) in a rectangular domain given by nondimensional variables \( 0 \leq x < 2\pi, \quad -5 \leq y \leq 5 \).

A Fourier spectral representation was used in the \( x \) direction and a finite difference approximation in the \( y \) direction [Cam04]. In this nondimensional formulation, all the functions and variables in the problem are defined by scaling the dimensional quantities using reference length and timescales that could reasonably describe atmospheric Rossby wave development. The background zonal-mean velocity is chosen as \( \overline{u}(y) = \tanh y \) and the nondimensional parameter \( \beta \) is set to 1; these choices guarantee that \( \beta - \overline{u}'(y) \neq 0 \) over the interval of \( y \) and prevent the development of a barotropic instability. The waves are forced with a wavenumber of \( k = 2 \) and a phase speed of \( c = 0 \) at the upper boundary of the domain, \( y = 5 \), and propagate towards a critical line located at \( y = 0 \) where \( \overline{u} = 0 = c \). This two-dimensional configuration gives a representation of planetary Rossby waves forced at northern latitudes, propagating horizontally southwards and encountering a critical latitude.

The change in the mean flow velocity is computed numerically as \( u_0(y, t) = \varepsilon \hat{u}(0, y, t) \), where \( \hat{u}(x, y, t) \) is the Fourier transform of the \( x \)-component of the wave velocity, and \( \hat{u}(0, y, t) \) is the Fourier component corresponding to the zero wavenumber \( \kappa = 0 \). Figure 7 shows \( u_0(y, t) \) as a function of \( y \) at fixed \( t \). The peak in the magnitude of \( u_0 \) close to the critical value of \( z = 0 \) is due to the wave-mean-flow interaction which is seen in Figure 6. These results bear a resemblance to the characteristic features of waves that break and result in an acceleration of the background mean flow in the northern stratospheric surf zone [MP84].

3. Vortex Rossby Waves in a Tropical Cyclone

Vortex Rossby waves also exhibit critical layer interactions which can modify the background flow. Although they differ greatly from planetary Rossby waves with respect to their scale, their structure, and the mechanisms that generate them, there are nevertheless strong qualitative similarities between their critical layer behavior and that of planetary Rossby waves. The development of vortex wave critical layer theory is more recent; in particular, in the past 25 years there have been a number of investigations into the linear initial value problem on a horizontal plane, with constant Coriolis parameter, e.g., [BM02, MK97]. More recently, we [NC15a, NC15b] examined a forced wave
problem in a configuration analogous to the SWW problem, where the waves are forced by a oscillatory boundary condition at one end of the domain and propagate towards a critical layer.

In our configuration, we consider a background vortex flow with steady uniform rotation in an annular domain given by \( r_1 < r < \infty \), \( 0 \leq \lambda < 2\pi \). The radius \( r_1 \) is taken to be the inner eyelaw of the vortex. The vortex flow velocity has an azimuthal (\( \lambda \)) component of \( \bar{v}(r) \) and a zero radial (\( r \)) component, and its angular velocity is \( \bar{\Omega}(r) = \bar{v}(r)/r \). This is a simple barotropic approximation for a cyclone like what is shown in Figure 3. A radially symmetric oscillatory boundary condition with a specified frequency \( \omega \) and wavenumber \( k \) is imposed at \( r = r_1 \) and generates a wave perturbation of the form

\[
\psi(r, \lambda, t) = \phi(r)e^{i(k\lambda - \omega t)} + c.c. \tag{13}
\]

The wave propagates radially outwards until it reaches a critical radius, \( r = r_2 \), where \( \bar{\Omega}(r_2) = \omega/k \). We considered a special choice of mean flow angular velocity profile in the form of quadratic function of \( r \), which was suggested by [BM02]. The equations include the “beta-effect,” i.e., the terms involving the parameter \( \beta \) which model the effects of the Coriolis force, as well as the nonlinear terms, which are multiplied by the parameter \( \varepsilon \). Thus, the problem is governed by two parameters \( \beta \) and \( \varepsilon \), and the behavior of the solutions depends on their relative magnitude.

The quadratic profile considered for the mean flow angular velocity gives steady-amplitude solutions of the linearized perturbation equation in terms of hypergeometric functions. These solutions show behavior that is similar to that of planetary Rossby waves in the analogous steady-amplitude configuration [Lin55]: the amplitude of the perturbation is attenuated at the critical radius and there is a logarithmic phase change of \(-\pi\). The configuration in which the wave amplitude varies with time \( t \) gives linear and nonlinear problems analogous to the planetary Rossby wave critical layer problems studied by Warn and others [Dic71, Ste78, WW76, WW78]. Following a similar procedure, using a Laplace transform and asymptotic matching, we are able to obtain linear asymptotic solutions with time-dependent amplitude, valid for large \( t \), in the outer region and in the inner region near the critical radius, first for the case with constant Coriolis parameter (\( \beta = 0 \)) [NC15a]. In this case, the linear outer solution approaches a steady state in the limit of infinite time, but the inner solution has secular terms that grow with time. The is an indication that the nonlinear terms and the beta-effect terms need to be considered as well.

A weakly-nonlinear analysis of the nonlinear time-dependent problem, with the beta-effect included, reveals certain aspects of the solutions that bear resemblance to the observed features of the secondary eyewall replacement cycle in tropical cyclones [NC15b]. In particular, there is an inward displacement of the location of the critical radius with time and additional rings of high wave activity develop corresponding to the critical radii of the components of the solution that arise from the beta-effect.

Some numerical solutions for this configuration are shown in Figure 8: the wave streamfunction \( \psi(x, z, t) \) in Figure 8(a) and the wave vorticity \( \nabla^2 \psi(x, z, t) \) in Figure 8(b). The contour plot of the streamfunction shows that the wave forced at the inner boundary of the annular domain with a wavenumber of \( k = 1 \) propagates radially outwards and its amplitude is reduced at the critical radius, which is indicated by the dashed circle. At this time, a very thin ring of high vorticity has developed in the vicinity of the critical radius resembling a secondary eyewall. This is analogous to the situation of overturning vorticity contours for the related planetary Rossby wave configuration, shown in Figure 6, which was first seen in the SWW asymptotic and numerical solutions [Bél76, Ste78, WW78]. In this vortex Rossby waves configuration as well, we found that higher wavenumber components develop with time; however, the low wavenumber components continue to dominate even at late time. This conclusion is consistent with the observations of Hurricane Gloria (1998) which are analyzed and discussed by Shapiro and Montgomery [SM93].

4. Internal Gravity Wave Packet Generated by an Isolated Mountain

In this section we give an overview of linear and nonlinear critical layer problems for internal gravity waves in a two-dimensional configuration. Internal gravity waves affect the general circulation of the atmosphere through critical level interactions and through wave breaking and saturation. Gravity wave saturation occurs when a wave propagating upwards in the atmosphere reaches a level where its amplitude becomes too large to be sustained and the wave breaks partially to reduce its amplitude and maintain stability. Given the relatively short wavelengths of gravity waves, historically, general circulation models have had difficulty in resolving them adequately and correctly representing the drag force that results from gravity wave critical layer and saturation effects. Consequently, the need to develop and understand the mathematical theory of atmospheric gravity waves has been recognized for many decades.

Gravity wave critical layer theory, in particular, has a rich history that dates back to the 1950s and which developed in parallel to that of planetary Rossby waves. Because they propagate vertically as well as horizontally, the simplest two-dimensional model for gravity waves would be in a vertical plane with one horizontal coordinate tangent to the curved surface of the Earth and one vertical coordinate, normal to the tangent plane.
Under the Boussinesq approximation, there is a linear relationship between the temperature and the density and this allows us to write the energy conservation equation in a simplified form in terms of the density. Moreover, the continuity equation can be written in its simplified incompressible form
\[
\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z} = 0. \tag{14}
\]
This is analogous to (4), but in terms of one horizontal variable and one vertical variable, rather than two horizontal variables. Again, this two-dimensional continuity equation allows us to define a streamfunction, but in this case it is given by \( \frac{\partial \psi}{\partial z} = -u \), \( \frac{\partial \psi}{\partial x} = \omega \).

Differentiating the x-momentum equation by z and the z-momentum equation by x and taking the difference of the resulting equations gives an equation in streamfunction-vorticity form, which includes the effect of the gravitational force:
\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x} = 0.
\]

Here the Laplacian \( \nabla^2 \) is the component of the vorticity in horizontal direction, perpendicular to the vertical plane. The total streamfunction is written in the form (1) as
\[
\psi(x, z, t) = \bar{\psi}(z) + \epsilon \psi(x, z, t),
\]
where \( \bar{\psi}(z) \) is the basic flow streamfunction, taken as the initial mean in the horizontal \( (x) \) direction, and \( \psi(x, z, t) \) is the perturbation streamfunction, with \( \epsilon \) again being a small parameter. The basic flow is a shear flow with a horizontal \( (x) \) velocity component \( \bar{u}(z) = -\bar{\psi}'(z) \) (where the prime now denotes the z-derivative), and a vertical \( (z) \) velocity component of zero. This gives a nonlinear equation for the streamfunction and vorticity perturbations and a similar substitution for the total density gives a nonlinear equation for the density perturbation,
\[
\frac{\partial}{\partial t} \nabla^2 \psi_t + \bar{u}(z) \nabla^2 \psi_t - \bar{u}''(z) \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \psi_t + N^2 \psi_{xx} = 0.
\]

The constant \( N \), known as the Brunt–Väisälä frequency or buoyancy frequency, is defined by
\[
N^2 = -\frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial z}.
\]
and gives a measure of the extent of stratification of the density.

The problems defined by equations (15)–(17) share some similar features with the corresponding Rossby wave problems described in Section 2. All the approaches for analysis described in Section 2 can be applied here as well. If we consider both the mean flow speed \( \bar{u} \) and the Brunt–Väisälä frequency to be constant, then equation (17) has constant-amplitude plane wave solutions of the same form as (8). In this case, we write

\[
\psi(x, z, t) = A e^{i(kx + mz - \omega t)} + \text{c.c.} \tag{18}
\]

where the horizontal and vertical wavenumbers are denoted, respectively, by \( k \) and \( m \). When substituted into (17), this gives the dispersion relation for the wave frequency \( \omega \) as a function of the wavenumbers,

\[
\omega = k \bar{u} \pm \frac{Nk}{(k^2 + m^2)^{1/2}}.
\]

Also, in analogy with the Rossby wave problem, the derivatives of the dispersion relation with respect to \( k \) and \( m \) give the horizontal and vertical components of the group velocity vector which defines the direction of propagation of the wave energy and the waves are dispersive.

In the general case with mean flow speed \( \bar{u}(z) \), we seek steady-amplitude normal mode solutions of equation (17), written as

\[
\psi(x, z, t) = \phi(z) e^{i(kx - \omega t)} + \text{c.c.,} \tag{19}
\]

where \( k \) is the wavenumber, \( \omega \) is the zonal phase speed, \( \phi(y) \) is the complex amplitude, and as before, “c.c.” denotes the complex conjugate of the function given.

The amplitude function \( \phi(z) \) satisfies the Taylor–Goldstein equation,

\[
\phi''(z) + \frac{N^2}{(\bar{u}(z) - c)^2} \left( \frac{\bar{u}''(z)}{\bar{u} - c} - k^2 \right) \phi = 0, \tag{20}
\]

which is the analog of the Rossby wave Rayleigh–Kuo equation (10) for gravity waves.

As noted in Section 2, the Rayleigh–Kuo theorem gives us a necessary condition for stability of normal mode solutions (9) of (10). The analogous stability theorem for the gravity wave configuration gives a condition in terms of the Richardson number of the background flow. The Richardson number is a function of \( z \) which measures the relative effects of the stratification, given by the buoyancy frequency \( N \), and the shear, given by the vertical derivative \( \bar{u}''(z) \) of the mean flow velocity. It is defined as

\[ Ri(z) = \frac{N^2}{(\bar{u}''(z))^2}. \]

The Miles–Howard theorem states that a necessary condition for instability is that the local Richardson number \( Ri(z) \) be somewhere less than 1/4.

The Taylor–Goldstein equation (20) is singular at a critical level \( z_c \) where \( \bar{u}(z_c) = c \). Two linearly-independent solutions valid near the singular point can be obtained using the method of Frobenius. The leading-order terms are

\[
\phi(z) = (z - z_c)^{1/2} e^{\pm i N(z - z_c)/2} (1 + O((z - z_c))),
\]

where \( \gamma = \left( \frac{N^2}{(\bar{u}''(z_c))^2} - \frac{1}{4} \right)^{1/2} = \left( Ri_c - \frac{1}{4} \right)^{1/2} \) and the ratio

\[
\frac{N^2}{(\bar{u}''(z_c))^2} = R_i_c
\]

is the Richardson number at the critical level. Noting that that the solutions can be written as

\[
\phi_\pm(z) \sim (z - z_c)^{1/2} e^{\pm i \gamma \log \left| z - z_c \right|},
\]

we observe that we have again arrived at a situation where there is a logarithmic phase shift \( \theta = \pm \pi \). The correct choice of branch of the log depends on the direction of propagation of the waves, with either upward or downward group velocity, and on the wind direction relative to the phase speed of the waves, i.e., whether \( \bar{u} - c \) is positive or negative as the waves approach the critical level. In any case, we find that the wave amplitude is reduced across the critical layer [Mil61] by a factor of \( e^{-\gamma} \). As a result, there is a discontinuity in the vertical momentum flux of the wave. The Eliassen–Palm theorem states that the horizontal \( x \) average of the vertical momentum flux constant in the absence of a critical layer, but when there is a critical layer, the phase shift results in a decrease in the wave amplitude and in the momentum flux across the critical layer, analogous to the situation that occurs for the Rossby wave critical layer.

Booher and Bretherton [BB67] examined the linear time-dependent problem for waves forced by an oscillatory periodic lower boundary. They found a solution including transients that approach zero in the limit of infinite time, as well as a steady singular solution with the same \( -\pi \) phase shift as in the normal mode analysis. This indicates that there is wave absorption by the background flow in the critical layer. In the nonlinear time-dependent problem, the reduction in wave amplitude across the critical layer results in a change in the mean flow near the critical level, given by

\[
\frac{\partial \bar{u}_0}{\partial t} = -\varepsilon^2 \frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho} \bar{u} \omega)}{\partial z}
\]

Brown and Stewartson [BS82] carried out a series of investigations using weakly-nonlinear analysis to examine the late-time behavior of the nonlinear solution and found terms indicating wave reflection from the critical layer, in addition to the wave absorption of earlier time. Our numerical solutions [CM03] for forced waves with a single horizontal wavenumber show agreement with [BS82] with regard to the absorption and reflection of the wave at the critical layer. The long-term behavior is qualitatively similar to that of the nonlinear Rossby wave critical layer, where under certain circumstances with sufficiently large \( \varepsilon \), a state of reflection or over-reflection may be attained. In addition, at late time, there may be overturning vorticity and
density contours and secondary instabilities in the critical layer, occurring in regions where the local Richardson number, which was initially everywhere greater than 1/4, becomes less than 1/4 or even negative, due to large density gradients.

In [CM03] the purpose was to investigate a configuration where the wave forcing is in the form of a horizontally localized wave packet comprising a continuous spectrum of horizontal wavenumbers, such as $\psi = e^{-\mu^2 x^2} e^{i k x}$, on an infinite horizontal domain. The graph of this function takes the form of the solid curves in Figure 5 with an envelope of the form of the dashed curves. This is representative of a situation where a wave packet is generated by a lower boundary condition such as a mountain range with multiple peaks. The analytical and numerical problem is then governed by two small parameters $\mu$ measuring the horizontal extent of the “mountain range” wave packet and $\varepsilon$ measuring the vertical height of the peaks. There is a “fast” scale which is defined by the oscillations within the wave packet and a “slow scale” which is defined by the horizontal extent of the packet. We found that the absorption of the wave packet continues to later time, there is a lesser extent of wave reflection and secondary instabilities compared with the case with monochromatic forcing [CM03]. This is because while the wave packet momentum and energy are absorbed by the mean flow in the critical layer, there is an outward flux of momentum and energy in the horizontal direction.

In [CN14] we considered the case where the wave forcing is of the form $\psi = e^{-\mu^2 x^2}$ with $\mu \ll 1$, as shown in Figure 9, or more generally, it is any function of the form $\psi = A(\mu x)$, with $\mu \ll 1$, and $A \rightarrow 0$ as $x \rightarrow \pm \infty$. In that case, the wave packet comprises a continuous spectrum of horizontal wavenumbers centered at the zero wavenumber. This is representative of a situation where a wave packet is generated by an isolated mountain. It was found that in this case the onset of the time regime in which the nonlinear effects become significant occurs on a later time frame and that the wave absorption was prolonged compared with the case with multiple peaks. Some numerical solutions are shown in Figures 10 and 11 illustrating the time evolution of the critical layer interaction for the configuration studied by [CN14].

The total density $\rho(z) + \varphi(x,z,t)$ in the vicinity of the critical layer is shown in Figure 10 as a contour plot at a fixed late-time value of $t = 200$. Even at this point in time, the system remains stable; there are no density overturning regions, where the local Richardson number is negative. Indeed, calculating the local Richardson number we find that it is larger than 1/4 everywhere. The plots of the perturbation streamfunction in Figure 11 show that the basic structure of the packet is maintained even up to $t = 400$, but the shape of the closed contours is modified with time and the wave packet becomes more elongated in the horizontal direction.

5. Concluding Remarks

This article gives an overview of a type of wave-mean-flow interaction, the critical layer interaction, that is observed in atmospheric fluid flows and is described robustly by mathematical theory. Three types of waves are discussed, each in a simplified two-dimensional configuration derived from the three-dimensional conservation laws, with an emphasis on the similarities between them. Each of these mathematical problems gives analytical and numerical solutions that are consistent with the situations described in the literature (e.g., [MP84], [SM93], [WD96], [FA03], and references therein). The wave amplitude is greatly reduced when the waves reach the critical layer, the waves are “absorbed” by the background flow, there is a wave-induced mean flow acceleration, and sharp vorticity gradients in the critical layer. At later time, other phenomena such as wave reflection and over-reflection may occur.

There are some obvious shortcomings to the type of studies discussed here, including the fact that they are based on two-dimensional geometry and
weakly-nonlinear equations. A more realistic three-dimensional configuration, either in rectangular geometry over a localized region or in spherical geometry on the global scale, would allow investigations of other types of waves, for example, Rossby waves with a vertical component of propagation, vortex Rossby waves, and vortex gravity waves in a more realistic vertically varying cyclone; the inclusion of processes such as water vapor condensation and evaporation in a tropical cyclone simulation would allow investigations of other types of gravity waves forced by isolated topography, Geophys. Astrophys. Fluid Dyn. 108 (2014), no. 5, 503–535, DOI 10.1080/03091929.2014.903944. MR3250930

The critical layer interaction is just one of the many mechanisms by which atmospheric waves influence the background general circulation and ultimately affect weather and climate. Other related wave-mean-flow interaction mechanisms that have well-developed mathematical theories include barotropic and baroclinic (three-dimensional) instabilities and the process of internal gravity wave saturation. Analyses and numerical simulations of these phenomena provide information and insight to advance our understanding of the dynamics of the atmosphere.

References


Figure 11. Wave streamfunction \( \psi(x, z, t) \) at nondimensional time (a) \( t = 200 \) and (b) \( t = 400 \) from a numerical solution of equations (15)–(16) with a lower boundary condition of the form shown in Figure 9. The contour levels are evenly spaced over the interval of values of each function with blue indicating negative values and red indicating positive values.


Lucy J. Campbell

Credits

Figure 1 is courtesy of NASA/GSFC/LaRC/JPL, MISR Team. Figure 3 was produced by Hal Pierce (SSAI/NASA GSFC). All other figures and photos are courtesy of the author.
Steklov Spectrum and Elliptic Problems with Nonlinear Boundary Conditions

Nsoki Mamie Mavinga

Problems with nonlinear boundary conditions arise naturally in many applications. For instance, in population dynamics where an impact of habitat-edges (boundary) on the dispersal pattern of species as they reach the boundary takes place in spatial ecology [CC06]. They occur when the biochemical reactions take place at or near the boundary, for example, in the limb bud development of a chick in which a chemical reaction produces outgrowth due to cell growth and division, and interactions between morphogens produced in several zones of the limb bud [DO99]. They also appear in noninvasive testing methods to locate defects in a medium by using boundary data measurements (see, e.g., [CCMM16]). In cryosurgery (a minimally invasive treatment used to treat some types of cancers and some conditions that may become cancer), a highly exothermic reaction takes place in a thin layer around the boundary in order to destroy abnormal tissue [LOS98]. These examples are not exhaustive.

Diffusion-type equations play a crucial role in these problems, and associated steady state problem and eigenvalue problems are critical in understanding the dynamics of diffusion-type equations. Hence, the qualitative (analytical) study of such equations is essential for better understanding and modeling nonlinear processes. The investigation of problems with nonlinear boundary conditions has therefore attracted a lot of attention in recent years; see for instance [Ama76, MN10, Mav12, MP17, LS18, CL15] and references therein.

In this article, we first introduce the spectral problem for elliptic equations with spectral-parameter dependent boundary conditions. We then discuss some recent results on the solvability of nonlinear diffusion problems when the nonlinearity on the boundary interacts in some sense with the spectrum, especially the effect of the first eigenvalue. We will present some of the results without proofs. References will be mentioned as appropriate.

In the following sections, we will first consider the linear Steklov problem in which the spectral parameter is in the boundary condition. Then, we discuss in-depth the properties of the first eigenvalue as well as briefly consider the one-dimensional case. In the last section, we take up the case of nonlinear perturbations of the linear Steklov problem, and set up the problem as a nonlinear first-order boundary-flux equation with a second-order elliptic partial differential equation “constraint” inside the domain. Considering asymptotic conditions on the boundary nonlinearity, we present existence, bifurcation and multiplicity results. A sketch of the bifurcation diagram is also provided.
Steklov Spectrum and its Properties

In the seminal paper entitled “Sur les problèmes fondamentaux de la physique mathématique (suite et fin),” published in 1902 in the Annales Scientifiques de l’École Normale Supérieure [Ste02], W. Steklov considered the (spectral) problem of finding a harmonic function \( v \) inside a convex bounded region \( \Omega \) in the plane \( \mathbb{R}^2 \) with smooth boundary surface \( S = \partial \Omega \) which satisfies the boundary condition

\[
\frac{\partial v}{\partial n} = \lambda \phi v \quad \text{on } S,
\]

where \( \partial/\partial n \) denotes the directional derivative in the direction of the (unit) outward normal vector to the boundary \( S \), \( \lambda \in \mathbb{R} \) is a spectral parameter, and \( \phi \) is a given smooth positive (weight) function. The convexity condition on the region \( \Omega \) and the smoothness of the surface \( S \) were relaxed by H. Poincaré, and the positive weight function \( \phi \) was introduced by E. Le Roy. Soon after that S. Zaremba considered the more general (spectral) problem with a lower-order term

\[
\Delta v + \xi v = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial v}{\partial n} = \lambda \phi v \quad \text{on } S,
\]

where \( \Delta \) is the Laplace operator and \( \xi \) is a (fixed) constant.

Later on in [Pay67], Payne presented a physical problem that describes the vibration of an elastic membrane with its whole mass uniformly distributed on the boundary with density \( \phi \) leading to the problem

\[
\Delta v = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial v}{\partial n} = \lambda \phi v \quad \text{on } S,
\]

where \( \Omega \) denotes an \( n \)-dimensional body with boundary \( S = \partial \Omega \).

There have been many results and generalizations of the Steklov problem. We mention the book by Bandle [Ban80] and the papers by Auchmuty [Auc04] and Mavinga [Mav12] for higher dimensions with mild regularity conditions on the data. We refer also to Amann [Ama76], who discussed the existence of the first eigenvalue of the spectrum for this problem under somewhat strong regularity conditions on the data. Although a more general linear operator with lower order terms was considered in [Ama76], the techniques there used the theory of positive operators (Krein–Rutman theorem); which of course does not apply when trying to obtain higher eigenvalues. The arguments in [Auc04], which yielded higher eigenvalues as well, involved maximization of the boundary functional on bounded closed convex subsets of the Sobolev space \( H^1(\Omega) \).

In this article, we present an approach used in [Mav12] where the minimization of the (differential) functional on an appropriate subspace of \( H^1(\Omega) \) is used.

Steklov eigenproblem. Let \( \Omega \subset \mathbb{R}^N, N \geq 2 \), be a bounded domain with smooth boundary \( \partial \Omega \). Consider the second-order elliptic equation

\[
-\Delta v + c(x) v = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial v}{\partial n} = \mu \sigma v \quad \text{on } \partial \Omega,
\]

where the (given) functions \( c : \Omega \to \mathbb{R} \) and the weight \( \sigma : \partial \Omega \to \mathbb{R} \) satisfy the following conditions.

(C) \( c \in L^\infty(\Omega) \) and \( \sigma \in L^\infty(\partial \Omega) \) are nonnegative functions such that \( \int_\Omega c(x) \, dx > 0 \) and \( \int_{\partial \Omega} \sigma \, dS > 0 \). Here, \( L^\infty \) denotes the real Lebesgue space of bounded functions.

Throughout this article, \( H^1(\Omega) \) denotes the usual real Sobolev space of functions on \( \Omega \); which is a Hilbert space endowed with the \( c \)-inner product defined by

\[
(u,v)_c = \int_\Omega \nabla u \nabla v + \int_\Omega c(x) uv,
\]

with the associated norm denoted by \( \|v\|_c \). This norm is equivalent to the standard \( H^1(\Omega) \)-norm.

Besides the Sobolev spaces, we make use, in what follows, of the real Lebesgue space \( L^2(\partial \Omega) \) of square integrable functions, and the compactness of the trace operator \( \Gamma : H^1(\Omega) \to L^2(\partial \Omega) \) (see, e.g., [Bre11] and references therein). Sometimes we will just use \( u \) in place of \( \Gamma u \) when considering the trace of a function on \( \partial \Omega \). We will denote the \( L^2(\partial \Omega) \)-inner product by \( (u,v)_\partial \) and the associated norm by \( \|u\|_\partial \). We also set

\[
(u,v)_\sigma = \int_{\partial \Omega} \sigma(x) uv \quad \text{and} \quad \|u\|_\sigma^2 = \int_{\partial \Omega} \sigma(x) u^2,
\]

for \( u,v \in H^1(\Omega) \). Observe that because of condition (C), it can happen that \( \sigma = 0 \) in a subset of positive measure (on the boundary \( \partial \Omega \)) where \( u \neq 0 \). In this case \( \|u\|_\sigma \) is only a semi-norm. Set \( V_\sigma(\Omega) := \{u \in H^1(\Omega) : \|u\|_\sigma = 0\} \). It is readily seen that \( V_\sigma(\Omega) \) is a closed linear subspace of \( H^1(\Omega) \). Observe that if \( \sigma(x) > 0 \) on \( \partial \Omega \), then the subspace \( V_\sigma(\Omega) \) reduces simply to \( H^1_0(\Omega) \). Let us denote the \( c \)-orthogonal complement of \( V_\sigma(\Omega) \) by \( H_\sigma(\Omega) = [V_\sigma(\Omega)]^\perp \). Therefore, one can split

\[
H^1(\Omega) = V_\sigma(\Omega) \oplus_c H_\sigma(\Omega)
\]

as a direct orthogonal sum (in the sense of \( H^1,c \)-inner product); that is, every \( u \in H^1(\Omega) \) can be written in a unique way in the form \( u = u_1 + u_2 \), where \( u_1 \in V_\sigma(\Omega) \) and \( u_2 \in H_\sigma(\Omega) \) with \( (u_1,u_2)_\sigma = 0 \).

Definition. The Steklov eigenproblem (in its variational form) is to find a pair \( (\mu, \varphi) \in \mathbb{R} \times H^1(\Omega) \) with \( \varphi \neq 0 \) such that

\[
\int_\Omega \nabla \varphi \nabla \psi + \int_\Omega c(x) \varphi \psi = \mu \int_{\partial \Omega} \sigma(x) \varphi \psi
\]
for every $\psi \in H^1(\Omega)$. The real number $\mu$ is called an eigenvalue of (2) and the function $\varphi$ is said to be an eigenfunction associated to the eigenvalue $\mu$.

Now, choosing $\psi = \varphi$, one sees immediately that if there is such an eigenpair, then $\mu > 0$ and $\int_{\partial \Omega} \sigma \varphi^2 > 0$. Otherwise, $\varphi$ would be a constant function; which would contradict the assumption (C) imposed on $c(x)$. Therefore $\varphi \perp V_\varphi(\Omega)$ in the $H^1$-$c$-inner product defined in (3) above. Notice also that, if $u \in V_\varphi$ with $u \neq 0$, then $\|u\|_c > 0$ and the quotient $\|u\|/\|u\|_c \varepsilon \infty$.

The set $\Sigma$ of $\mu \in \mathbb{R}$ such that (2) has a nontrivial solution, is called the Steklov spectrum.

In what follows, we present some results on the properties of the Steklov spectrum. Namely that $\Sigma$ forms a countably infinite set $\{\mu_k : k \in \mathbb{N}\} \subset \mathbb{R}^+$ without finite accumulation point. Thus, its elements can be arranged in an increasing sequence. We omit the details and refer to [Mav12] for the proof.

Variational characterization of the Steklov spectrum. Assume that the condition (C) holds, then

(i) The Steklov eigenproblem (2) has a sequence of real eigenvalues

$$0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_j \leq \ldots \rightarrow \infty, \text{ as } j \rightarrow \infty,$$

and these eigenvalues satisfy the variational characterizations

$$\mu_1 = \inf_{u \in H^1_\Omega \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 + \int_{\Omega} c(x)u^2}{\int_{\partial \Omega} \sigma(x)u^2} \quad (6)$$

and for $j = 1, 2, \ldots$

$$\mu_{j+1} = \inf_{u \in W_j \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 + \int_{\Omega} c(x)u^2}{\int_{\partial \Omega} \sigma(x)u^2}, \quad (7)$$

where $W_j = \{u \in H^1_\Omega : (u, \varphi_i)_\sigma = 0 \text{ for } i = 1, \ldots, j\}$ and $\varphi_i$ are the eigenfunctions corresponding to $\mu_j$. (Hence, each eigenvalue has a finite-dimensional eigenspace.)

(ii) The normalized eigenfunctions provide a complete $c$-orthonormal basis of $H^1_\sigma(\Omega)$. Moreover, each function $u \in H^1_\sigma(\Omega)$ has a unique representation of the form

$$u = \sum_{j=1}^{\infty} c_j \varphi_j \text{ with } c_j = \frac{1}{\mu_j} (u, \varphi_j)_\sigma = (u, \varphi_j)_\sigma$$

and $\|u\|_c^2 = \sum_{j=1}^{\infty} \mu_j |c_j|^2$.

In addition

$$\|u\|_c^2 = \sum_{j=1}^{\infty} |c_j|^2.$$

Observe that the variational characterization (6) gives the trace inequality

$$\mu_1 \int_{\partial \Omega} \sigma(x)u^2 \leq \int_{\Omega} |
abla u|^2 + \int_{\Omega} c(x)u^2 \quad (8)$$

for all $u \in H^1(\Omega)$. Moreover if equality holds in (8), then $u$ is a multiple of an eigenfunction of Eq. (2) corresponding to $\mu_1$.

On the other hand, for every $u \in \oplus_{i \leq j} E(\mu_i)$ and $w \in \oplus_{i \geq j+1} E(\mu_i)$ we have that

$$\|u\|_c^2 \leq \mu_j \|u\|_c^2 \quad \text{and} \quad \|u\|_c^2 \leq \mu_{j+1} \|w\|_c^2, \quad (9)$$

where $E(\mu_i)$ denotes the $\mu_i$-eigenspace and $\oplus_{i \leq j} E(\mu_i)$ is the span of eigenfunctions associated to eigenvalues up to $\mu_j$.

Hence, this gives a splitting of the space $H^1_\sigma(\Omega)$ (and hence of $H^1(\Omega)$).

Remark. Note that if $c \equiv 0$ (i.e., the harmonic equation case) then $\mu = 0$ is the first eigenvalue of (2) with eigenfunction $\varphi \equiv 1$ on $\Omega$. Let us mention here that the eigenvalues and eigenfunctions of the harmonic operator are used in fluid mechanics, heat transmission, electromagnetism, and material design (see, e.g., [Lip98, CCMM16]). They play an important role in the study of isoperimetric inequalities (see, e.g., [Pay67]).

Properties of the first eigenvalue. The first eigenvalue $\mu_1$ is principal; that is, it is simple (i.e., its associated eigenfunctions are each a constant multiple of one another) and the associated eigenfunction $\varphi_1$ doesn’t change sign in $\Omega$ (i.e., it is either strictly positive or strictly negative).

We first show that $\varphi_1$ does not change sign in $\Omega$. Indeed, suppose it does, and let $\varphi_1^+ = \max\{\varphi_1, 0\}$ and $\varphi_1^- = \min\{\varphi_1, 0\}$, we know that $\varphi_1^+$ and $\varphi_1^-$ are in $H^1(\Omega)$.

By the characterization of $\mu_1$ it follows that $(\varphi_1, \varphi_1)_c = \mu_1(\varphi_1, \varphi_1)_\sigma$. Therefore,

$$\begin{align*}
0 &\leq (\varphi_1^+, \varphi_1^+)_c + (\varphi_1^-, \varphi_1^-)_c - \mu_1(\varphi_1^+, \varphi_1^-)_\sigma - \mu_1(\varphi_1^-, \varphi_1^+)_\sigma \\
&= (\varphi_1, \varphi_1)_c - \mu_1(\varphi_1, \varphi_1)_\sigma = 0.
\end{align*}$$

It follows immediately that $\varphi_1^+$ and $\varphi_1^-$ are also eigenfunctions corresponding to $\mu_1$. From [Bre11], we get that $\varphi_1^+ > 0 \text{ a.e } \Omega$ and $\varphi_1^- < 0 \text{ a.e } \Omega$, which is impossible. Thus $\varphi_1$ does not change sign in $\Omega$.

Next, we claim that $\mu_1$ is simple if and only if $\varphi_1$ does not change sign. Indeed, if $\varphi_1$ changes sign then $\varphi_1^+$ and $\varphi_1^-$ are also eigenfunctions corresponding to $\mu_1$; they are linearly independent. Hence, $\mu_1$ is not simple. On the other hand, suppose that $\mu_1$ is not simple and let $\varphi$ and $\psi$ be two eigenfunctions corresponding to $\mu_1$; they are linearly independent. If $\varphi$ or $\psi$ changes sign then the claim is proved. Otherwise suppose without loss of generality that $\varphi$ and $\psi$ are positive then we will prove that there exists $a \in \mathbb{R}$ such that the eigenfunction (corresponding to $\mu_1$)
We refer to \((\varphi + \alpha \psi)\) changes sign. Indeed, suppose that for all \(\alpha \in \mathbb{R}\), 
\(\varphi + \alpha \psi\) does not change. Let the function \(h : \mathbb{R} \to \mathbb{R}\) be defined by \(h(\alpha) = \int \varphi + \alpha \int \psi\). Since \(h\) is continuous there exists \(a \in \mathbb{R}\) such that \(h(a) = \int \varphi + a \int \psi = 0\). Hence, 
\(\varphi = -a \psi\) which contradicts the fact that \(\varphi\) and \(\psi\) are linearly independent. Thus, \(\varphi + \alpha \psi\) changes sign.

**Remark.** If the boundary \(\partial \Omega\) is smooth and the functions \(c\) and \(\sigma\) are Hölder continuous, then by the regularity arguments for elliptic equations (see, e.g., [Bre11, Theorem 9.26 and Theorem 9.34]) it follows that \(\varphi_1 \in C^{2,\gamma}(\Omega) \cap C^0(\partial \Omega)\) where \(0 < \gamma < 1\). The Hopf Lemma and the subsequent strong maximum principle (or Boundary Point Lemma) shows that the outer normal derivative \(\frac{\partial \varphi_1}{\partial n}(x) < 0\) whenever \(\varphi_1(x) = 0\) with \(x \in \partial \Omega\). Hence, one has that \(\varphi_1 > 0\) on \(\Omega\).

**Remark.** As shown above, we have completely described the spectrum of the Laplace operator. We would like to mention that in the case of the p-Laplacian operator, the Steklov spectrum is not completely known, although one may still obtain an infinite sequence of eigenvalues. Moreover, if \(\sigma\) changes sign appropriately on \(\partial \Omega\), then problem (2) possesses an infinite sequence of positive eigenvalues and an infinite sequence of negative eigenvalues

\[-\infty \leftarrow \cdots \leq \mu_{-2} < \mu_{-1} < 0 < \mu_1 < \mu_2 \leq \cdots \to \infty,\]

as \(j \to \infty\). In addition, \(\mu_{-1}\) and \(\mu_1\) are both principal eigenvalues. We refer to [CL15] for details.

So far, we have presented the Steklov spectrum for the \(N\)-dimensional equation with \(N \geq 2\). For sake of completeness, let us make some comments on the one-dimensional case.

The one-dimensional case. Consider the one-dimensional domain \(\Omega = (0, 1)\) with \(c \equiv 1\) and \(\sigma \equiv 1\). The spectral problem (2) can be rewritten as a second order ordinary differential equation

\[
\begin{align*}
-\nu'' + v &= 0 \quad \text{in} \ (0, 1), \\
-\nu'(0) &= \mu \nu(0) \\
\nu'(1) &= \mu \nu(1)
\end{align*}
\]

(10)

In this case, the differential equation can be solved explicitly by using the characteristic polynomial technique, and the general solution of (10) is of the form \(u(x) = c_1 e^x + c_2 e^{-x}\), where \(c_1\) and \(c_2\) are constants. Taking into account the boundary conditions, we obtain only two (simple) eigenvalues

\[
\mu_1 = \frac{e - 1}{e + 1} < \mu_2 = \frac{1}{\mu_1} = \frac{e + 1}{e - 1}.
\]

The eigenfunctions associated to \(\mu_1\) and \(\mu_2\) are given by

\(\varphi_1(x) = e^x + e^{1-x}\) and \(\varphi_2(x) = e^x - e^{1-x}\), respectively. Observe that \(\varphi_1(0) = \varphi_1(1) = 1 + e\) and \(\varphi_1(1) > 0\) for all \(x \in [0, 1]\), and \(\varphi_2(0) = 1 - e = -\varphi_2(1)\). We see that \(\varphi_2\) changes sign and it is orthogonal to \(\varphi_1\) with respect to the \(L^2((0, 1))\)-inner product as well as the \(H^1((0, 1))\)-inner product.

More generally, when \(c \in L^1((0, 1))\) is nonnegative and \(\int_0^1 c > 0\), and the weight \(\sigma : [0, 1] \to \mathbb{R}\) is a nonnegative function with \(\sigma(0) + \sigma(1) \neq 0\), then by using variational characterizations discussed above, we also obtain exactly two positive eigenvalues

\[
\begin{align*}
\mu_1 &= \inf_{u \in H^1((0, 1))} \int_0^1 \left( |u'|^2 + c(x)u^2 \right) \frac{\sigma(0)u^2(0) + \sigma(1)u^2(1)}{\sigma(0)u(0) + \sigma(1)u(1)} \\
\mu_2 &= \inf_{u \in W} \int_0^1 \left( |u'|^2 + c(x)u^2 \right) \frac{\sigma(0)u^2(0) + \sigma(1)u^2(1)}{\sigma(0)u(0) + \sigma(1)u(1)},
\end{align*}
\]

where \(W = \{ u \in H^2((0, 1)) : (u, \varphi_1)_\sigma = (\sigma(1)\varphi_1(1)u(1) + \sigma(0)\varphi_1(0)u(0) = 0 \text{ with } \varphi_1 \text{ an eigenfunction corresponding to } \mu_1, \text{ and } ||u||_\sigma = [\sigma(0)u^2(0) + \sigma(1)u^2(1)]^{1/2} \text{. Moreover, } \mu_1 \text{ is principal and } \mu_2 \text{ is simple with eigenfunction changing sign.}

We notice that in the one-dimensional case the Steklov spectrum has only two elements whereas in \(N\)-dimensional case with \(N \geq 2\), the Steklov spectrum is unbounded, infinite and discrete.

**Linear nonhomogeneous Steklov problem.** Consider the linear nonhomogeneous problem

\[
-\Delta u + c(x)u = 0 \quad \text{a.e. in } \Omega, \\
\frac{\partial u}{\partial \eta} = (\mu_k + \lambda)\sigma(x)u + h(x) \quad \text{on } \partial \Omega,
\]

(11)

where \(h \in L^2(\partial \Omega), \mu_k\) is a Steklov eigenvalue of (2), and \(\lambda \in \mathbb{R}\). From the Fredholm alternative theorem [Bre11], we have that

(i) If \(\mu_k + \lambda\) is not an eigenvalue of (2), then Equation (11) has a unique solution for every \(h\).

(ii) If \(\mu_k + \lambda\) is an eigenvalue of (2), then Equation (11) has a solution if and only if \(h\) is orthogonal to the eigenspace associated to \((\mu_k + \lambda)\).

Because of the properties of the first eigenvalue \(\mu_1\), our analysis will only be focused on the case where \(\mu_k = \mu_1\). From the above discussion, the Fredholm alternative-type arguments describe completely the structure of the solution set of (11).

In what follows, we will be concerned with nonlinear perturbations of equation (2). We will also analyze the structure of the solution set in the framework of bifurcation from infinity.
Problems with Nonlinear Boundary Conditions

Consider a nonlinear perturbation of equation (2) given by

\[-\Delta u + c(x)u = 0 \quad \text{in } \Omega,\]
\[\frac{\partial u}{\partial \eta} = (\mu_1 + \lambda)\sigma(x)u + g(x,u) + h(x) \quad \text{on } \partial \Omega. \tag{12}\]

Here \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\), where \(N \geq 2\). Although much weaker regularity conditions may be considered on the data as seen in the previous sections, we assume for the sake of simplicity and clarity of the presentation that the coefficient-functions \(c\) and \(\sigma\), as well as the nonhomogenous term \(h\) and the nonlinearity \(g\), are smooth on their domains of definition, and that \(g(x,u)\) is asymptotically sublinear at infinity in \(u\) uniformly in \(x\) (see below). Moreover, we assume that \(c\) and \(\sigma\) are nonnegative with \(\int_{\partial \Omega} \sigma dS > 0\). In addition, \(\mu_1\) is the first Steklov eigenvalue of equation (2) and the (real) parameter \(\lambda\) varies in a neighborhood of zero.

When \(g \neq 0\) is a (genuine) nonlinearity, the structure of the solution set may be quite different from that of the nonhomogeneous linear equation (11). Therefore, we will present some results on the solution set structure; namely, the location and behavior of the solution set for the nonlinear problem (12) for \(\lambda\) in a neighborhood of zero (i.e., \(\mu_1 + \lambda\) in a neighborhood of \(\mu_1\)), and the nonlinearity \(g\) satisfies some asymptotic conditions. In particular, the existence of multiple solutions with (potentially) large norms.

By a solution to Eq. (12) we mean a function \(u \in W^2_p(\Omega)\), \(p > N\), which satisfies (12). (For the definitions and properties of the Sobolev spaces \(W^k_p(\Omega)\), (Sobolev) trace-spaces \(W^{k-p}_p(\partial \Omega)\) and Hölder spaces \(C^{0,\beta}(\partial \Omega)\), we refer for instance to [Bre11].)

The nonlinear problem (12) has received much attention in recent years. A few results on a disk \((N = 2)\) were obtained in the case of linear elliptic equations where the nonlinearity on the boundary was compared with the first Steklov eigenvalue. We refer to Klingelhöfer [Kli68]. The results in [Kli68] were significantly generalized to higher dimensions in [Ama76] in the framework of the sub- and super-solutions method.

Let us mention here that in [MP17], the authors proved multiplicity results for weak solutions (in \(H^1(\Omega)\)) for problems somewhat similar to (12) by using a priori estimates and bifurcation theory. Their results considered the case \(c \equiv 1\). The harmonic function situation, i.e., \(c \equiv 0\), was not included. In [Mav12, MN10] the authors proved the existence of weak solutions for elliptic equations with non-linear boundary conditions using variational arguments.

To obtain existence, multiplicity, and bifurcation from infinity results for equation (12), we impose the following general conditions on the (boundary) nonlinearity \(g\) and the nonhomogeneous term \(h\), and appropriately cast the problem in an abstract setting.

Conditions on the nonlinearity \(g\).

\((G1)\) \(g\) is asymptotically sublinear at infinity in \(u\), uniformly in \(x\); that is, \(\lim_{|u| \to \infty} \frac{g(x,u)}{u} = 0\) uniformly in \(x\) in the sense that for every \(\varepsilon > 0\) there is a constant \(r_\varepsilon > 0\) such that

\[|g(x,u)| \leq \varepsilon |u|\]

for all \(x \in \partial \Omega\) and all \(u \in \mathbb{R}\) with \(|u| \geq r_\varepsilon\).

\((G2)\) \(g\) satisfies a sign-like condition, i.e., there are functions \(A \in C(\partial \Omega)\) and \(B \in C(\partial \Omega)\) and constants \(r, R\) with \(r < 0 < R\) such that

\[g(x,u) \geq A(x) \quad \text{for all } x \in \partial \Omega\]

and all \(u \in \mathbb{R}\) with \(u \geq r\); and

\[g(x,u) \leq B(x) \quad \text{for all } x \in \partial \Omega\]

and all \(u \in \mathbb{R}\) with \(u \leq r\).

Conditions on the nonhomogeneous function \(h\). The nonhomogeneous function \(h\) satisfies the orthogonality-like conditions

\[(H)\]
\[\int_{\partial \Omega} B(x)\varphi_1 \leq -\int_{\partial \Omega} h(x)\varphi_1 \leq \int_{\partial \Omega} A(x)\varphi_1, \tag{13}\]

We would like to mention that the sublinearity condition \((G1)\) guarantees the existence of unbounded branches of solutions when the parameter \(\lambda\) approaches zero. These branches bifurcate from infinity in the sense of Rabinowitz; see [Rab73]. Conditions \((G2)\) and \((H)\) are used in connection with the so-called Landesman–Lazer resonance conditions.

Problem framework. We set up problem (12) in terms of the normal derivative trace equation on the boundary, and Nemyskii operators on trace-spaces. More specifically, we cast the problem as a nonlinear first-order differential equation “through” the boundary sub-manifold \(\partial \Omega\) (i.e., a normal derivative trace equation) along with homogeneous linear second-order partial differential equations (diffusion-type) “constraint” inside the domain \(\Omega\). Since the regularity conditions on the data may be significantly weakened as aforementioned, we indicate how we set up the problem in terms of Sobolev spaces.

We define the linear (Steklov) boundary operator

\[\mathcal{B} : \text{Dom}(\mathcal{B}) \subset W^{2-1/p}_p(\partial \Omega) \to W^{1/p}_p(\partial \Omega)\]

by

\[\mathcal{B}u := \frac{\partial u}{\partial \nu} - \mu_1 \sigma(x)u,\]

where \(x := \text{Dom}(\mathcal{B}) = \{u \in W^{2}_p(\Omega) : -\Delta u + c(x)u = 0 \text{ a.e. in } \Omega\}.\)

Since \(X \subset W^{2}_p(\Omega)\), we write symbolically \(W^{2}_p(\Omega) \hookrightarrow W^{1/p}_p(\partial \Omega)\) to simply mean that the trace-extension operator \(W^{2}_p(\Omega) \hookrightarrow W^{1/p}_p(\partial \Omega) \subset W^{1-1/p}_p(\partial \Omega)\) is a compact linear operator from \(W^{2-1/p}_p(\partial \Omega)\) into \(W^{1-1/p}_p(\partial \Omega)\) (see,
e.g., [Bre11] and references therein). Notice also that the second-order differential equation defines (or more precisely is included as a “constraint” in) the domain of the linear (boundary) operator \( B \), and that \( X \) is a closed subspace of \( W_p^2(\Omega) \).

Now, we define the nonlinear (Nemytskii) superposition-operator
\[
N : X \in W_{\rho}^{1−1/p}(\partial \Omega) \to W_{\rho}^{1−1/p}(\partial \Omega) \text{ by }
N u = g(\cdot, u(\cdot)).
\]

Eq. (12) is then equivalent to
\[
Bu = \lambda \sigma(\cdot)u + Nu + h, u \in X.
\]

This abstract setup on the trace-spaces together with a combination of degree theory (see, e.g., [Maw79]), continuation methods, and Rabinowitz bifurcation from infinity arguments [Rab73] are used to establish the existence and multiplicity of solutions and to provide the location and the behavior of the solution sets.

In order to apply degree theory, one should establish at least an a priori bound for all possible solutions to a homotopy associated with Eq. (12); see below.

**Proposition 1** (a priori estimate). Assume that the assumptions (G1)–(G2) and (H) hold true. Let \( \lambda_0 \in \mathbb{R} \) be a fixed constant such that \( 0 < \lambda_0 < \mu_2 - \mu_1 \). Then, there is a constant \( R_0 := R_0(\lambda_0) > 0 \) such that all possible solutions of Eq. (12), with \( 0 < \lambda \leq \lambda_0 \), satisfy
\[
|u|_{W_2^p(\Omega)} \leq R_0.
\]

That is, all possible solutions of Eq. (12) are (uniformly) bounded in \( W_2^p(\Omega) \) independently of \( \lambda \), provided \( 0 < \lambda \leq \lambda_0 \).

Let us mention that a similar result holds for all \( \lambda \) negative (and bounded away from zero). More precisely, we have the following a priori bound.

**Proposition 2** (a priori estimate). Let \( \lambda_0, \lambda_1 \in \mathbb{R} \) be (fixed negative) constants such that \( -\infty < \lambda_0 < \lambda_1 < 0 \). Suppose that the assumptions (G1) holds. Then there exists a constant \( R_0 := R_0(\lambda_0, \lambda_1) > 0 \) such that all possible solutions of Eq. (12), with \( \lambda_0 \leq \lambda \leq \lambda_1 \), satisfy
\[
|u|_{W_2^p(\Omega)} \leq R_0.
\]

That is, all possible solutions of Eq. (2) are (uniformly) bounded in \( W_2^p(\Omega) \) independently of \( \lambda \), provided that \( \lambda_0 \leq \lambda \leq \lambda_1 < 0 \).

Existence of solutions.

**Theorem 1** (Existence). Assume that the assumption (G1)–(G2) and (H) hold, then Eq. (12) has at least one solution for every \( \lambda < \mu_2 - \mu_1 \).

Moreover, for \( 0 < \lambda \leq \lambda_0 \), with \( \lambda_0 < \mu_2 - \mu_1 \), all solutions are uniformly bounded in \( W_2^p(\Omega) \), independently of \( \lambda \).

To prove Theorem 1, we first consider the case when \( \lambda \geq 0 \) is fixed. Picking \( \delta \in \mathbb{R} \) such that \( 0 < \delta < \mu_2 - \mu_1 \), and following the notation of the previous section, we consider the homotopy
\[
Bu - \delta \sigma(\cdot)u = \theta[(\lambda - \delta)\sigma(\cdot)u + Nu + h], u \in X,
\]
where \( \theta \in [0,1) \); which, when \( \theta = 0 \), reduces to the homogeneous linear problem \( Bu - \delta \sigma(\cdot)u = 0 \) that has only the trivial solution. (It would reduce to our original nonlinear problem (12) if \( \theta \) were equal to 1.) Since the linear operator \( B - \delta \sigma(\cdot)I \) defined by \( B - \delta \sigma(\cdot)I : X \to W_{\rho}^{1−1/p}(\partial \Omega) \) is bounded, one-to-one and onto (by the continuity of the trace operator and the Fredholm alternative), it follows that (15) is equivalent to the fixed point homotopy
\[
u = \theta(\mathcal{B} - \delta \sigma(\cdot)I)^{-1}((\lambda - \delta)\sigma(\cdot)u + Nu + h), \quad u \in X := \text{Dom}(\mathcal{B}).
\]

Therefore, by the compactness of the trace operator \( W_2^p(\Omega) \hookrightarrow W_{\rho}^{1−1/p}(\partial \Omega) \) and the topological degree theory (see, e.g., [Maw79]), it suffices to show that all possible solutions of the homotopy (16) are (uniformly) bounded in \( W_2^p(\Omega) \) independently of \( \theta \in [0,1) \). This proves the first part of Theorem 1. The second part of Theorem 1 follows readily from Proposition 1.

Now, to prove the existence of at least one solution for \( \lambda < 0 \) (fixed), we consider the homotopy (15) where \( \delta < 0 \) and now \( \theta \in [0,1] \). (Notice that \( \theta = 1 \) is included here.) Observing that \( \lambda_0 := \min[\lambda, \delta] \leq (1 - \delta)\lambda + \delta \lambda \leq \max[\lambda, \delta] := \lambda_1 < 0 \) for \( 0 \leq \theta \leq 1 \), it follows Proposition 2 that all possible solutions of Eq. (15) are (uniformly) bounded in \( W_2^p(\Omega) \) independently of \( \theta \in [0,1] \). The existence of at least one solution for each \( \theta \in [0,1] \) follows from topological degree arguments as above. (It should be noted that Assumptions (G2)–(H) do not matter when \( \lambda < 0 \), at least as far as existence of at least one solution is concerned.)

Recall that no multiplicity results occur when \( g \equiv 0 \) and either \( \lambda < 0 \) or \( 0 < \lambda < \mu_2 - \mu_1 \), since the Fredholm alternative guarantees uniqueness in this case! We claim that, by strengthening somewhat either (G2) or (H), we obtain multiplicity results and more importantly we describe the behavior of the solution set. The first result is motivated by the fact that one may allow the equality \( A(x) = B(x) \) for \( x \in \delta \Omega \) in (G2). We would like to point out that, in this instance, multiplicity may occur only for one value of \( \lambda \); more precisely at \( \lambda = 0 \) (even if \( g \not\equiv 0 \)), with the bifurcation branches in the \((\lambda, |u|)_{C(\partial \Omega)}\)-plane being only
(semi-infinite) straight line rays located on the vertical \(|u|_{C^{0,\alpha}(\partial \Omega)}\)-axis, as illustrated in the following remark.

**Remark.** Consider any (nonlinearity) \(g\) such that \(g(x,u) = 0\) for all \(x \in \partial \Omega\) and \(u \in \mathbb{R}\) with \(|u| \geq R\), where \(R > 0\) is a fixed number; i.e., the function \(g\) vanishes outside a "cylindrical shell" \(\partial \Omega \times [-R,R]\). For \(\lambda = 0\), it is easily seen that the function defined by \(u_i := \varphi_1\) is a solution to Eq.\((12)\) for every \(t \in \mathbb{R}\) that is such that \(|t| \min_{\partial \Omega} |\varphi(1)| \geq R\); provided the nonhomogeneous term \(h \equiv 0\) of course. An analysis of the proof of the above existence result (or the multiplicity results below) will indicate that, provided \(h\) is \(L^2(\partial \Omega)\)-orthogonal to \(\varphi_1\), \(\lambda = 0\) is the only parameter-value for which large solutions exist, and the bifurcation from infinity branches are (semi-infinite) straight line rays on the \(|u|_{C^{0,\alpha}(\partial \Omega)}\)-axis, in the plane, as described above. Therefore, the bifurcation from infinity parameter-interval collapses to just one-point interval \([\lambda] = \{0\}\).

Therefore, for the rest of the article, we will be interested in nonlinearities \(g\) that satisfy a sign-like condition and that are not identically null outside a compact \(u\)-interval in \(\mathbb{R}\).

**Bifurcation from infinity.** In addition to a (fairly) general existence result (see Theorem 1 above), our multiplicity results state that as long as the nonlinearity \(g\) satisfies a condition asymptotically, then when \(\lambda\) is in an appropriate interval on one side of zero, Eq.\((12)\) has at least two solutions, denoted \((\lambda^+_\varepsilon, u^+_\varepsilon)\) and \((\lambda^-_\varepsilon, u^-_\varepsilon)\), with \(-\varepsilon < \lambda^\pm_\varepsilon < 0\) such that for every \(\varepsilon \in (0,1)\). Eq.\((12)\) has at least two solutions, denoted \((\lambda^+_\varepsilon, u^+_\varepsilon)\) and \((\lambda^-_\varepsilon, u^-_\varepsilon)\), with \(-\varepsilon < \lambda^\pm_\varepsilon < 0\) such that for some \(0 < \alpha < 1\),

\[
\lim_{\varepsilon \to 0+} \min \left\{|u^+_\varepsilon|_{C^{0,\alpha}(\partial \Omega)}, |u^-_\varepsilon|_{C^{0,\alpha}(\partial \Omega)}\right\} = \infty;
\]

that is, they bifurcate from infinity since \(\lambda^\pm_\varepsilon \to 0\) as \(\varepsilon \to 0^+\),

Moreover, for \(0 \leq \lambda \leq \lambda_0\) with \(\lambda_0 < \mu_2 - \mu_1\), all solutions (which exist by Theorem 1) are uniformly bounded, independently of \(\lambda\). Therefore, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue \(\mu_1\). (In some sense, the "strong resonance" conditions "bend" the bifurcation branches; see Figure 1 below.)

A simple example to keep in mind here is the (continuous) function \(g\) given by \(g(x,u) := \eta_+(x)(1 + u^2)^{-1}\) for \(u \geq R > 0\) and \(g(x,u) := -\eta_-(x)(1 + u^2)^{-1}\) for \(u < -r < 0\), where \(\eta_\pm\) are smooth positive functions on \(\partial \Omega\), or nonbounded counterpart \(g(x,u) := 1/2\min(\eta_+ u^2(1 + u^2)^{-1}).\) Note that here, \(A(x) = B(x) = 0\); which by \((H)\) requires \(h\) to be \(L^2(\partial \Omega)\)-orthogonal to \(\varphi_1\). Observe that in either case \(\liminf_{u \to -\infty} g(x,u) = 0 = \limsup_{u \to -\infty} g(x,u)\) and \(\liminf_{u \to \infty} g(x,u) = 0 = \limsup_{u \to \infty} g(x,u)\); that is, no (linear) "decay rate" at infinity is required (see, e.g., [AA95] and references therein). Thus, the terminology (asymptotic) strong resonance used here! We also point out that the so-called Landesman–Lazer condition \((LL)\) (see below) is not satisfied for these nonlinearities since one has equalities in \((H)\), but we are still able to "locate" and "describe" the solution-branches.

Note that the "stronger" condition \((SS)\) may be used to establish that all possible solutions of Eq.\((14)\) are (uniformly) bounded in \(W^2_p(\Omega)\) when \(\lambda = 0\) as well; that is, the conclusion of Theorem 1 actually holds true for all \(\lambda \in [0,\lambda_0]\).

To prove Theorem 2, we consider the fixed point equation

\[
u = \theta(\mathbb{B} - \delta \sigma(1))^{-1}(\lambda - \delta)\sigma(1) I u + N u + h.
\]

Setting

\[
\mu := \lambda + \delta, Lu := [\mathbb{B} + \delta \sigma(1)]^{-1}\sigma(1) I u
\]

and

\[
K u := (\mathbb{B} + \delta \sigma(1))^{-1}(N u + h),
\]

it follows that the above fixed point equation is equivalent to the nonlinear "normal derivative trace" equation

\[
u = \mu L u + K(u), u \in C^{0,\alpha}(\partial \Omega), 0 < \alpha < 1.
\]

From this setup, it follows that \(\mu^{-1} = \delta^{-1}\), i.e., \(\lambda = 0\), is the principal eigenvalue of \(L\) and that, by the compactness of the trace operator, the solution-map (through the use of the "bootstrap" regularity argument as above)

\[
L : C^{0,\alpha}(\partial \Omega) \to C^{1,\alpha}(\mathbb{O}) \subseteq C^{0,\alpha}(\partial \Omega)
\]
is a compact linear operator when considered as an operator from $C^{0,\alpha}(\partial \Omega)$ into $C^{0,\alpha}(\partial \Omega)$. Using the regularity of $g$ and $h$ and a “bootstrap” argument again one shows that

$$K : C^{0,\alpha}(\partial \Omega) \rightarrow C^{0,\alpha}(\partial \Omega)$$

is a completely continuous mapping when viewed as a nonlinear operator from $C^{0,\alpha}(\partial \Omega)$ into $C^{0,\alpha}(\partial \Omega)$. Then using the sublinear growth condition $(G1)$, one can show that $K(u) = o(|u|_{C^{0,\alpha}(\partial \Omega)})$ as $|u|_{C^{0,\alpha}(\partial \Omega)} \rightarrow \infty$. Notice that $\text{Eq.}(18)$ has now an abstract form considered, e.g., in [Rab73] for bifurcation from infinity purposes. Therefore, $\lambda = 0$ is a bifurcation point from infinity since all assumptions of the bifurcation from infinity result are fulfilled (see, e.g., [Rab73, p. 465, Theorem 1.6 and Corollary 1.8]): that is, there exist two connected sets of solutions $\mathcal{C}^+, \mathcal{C}^- \subset \mathbb{R} \times C^{0,\alpha}(\partial \Omega)$ with $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$ which are such that for every (sufficiently) small $\varepsilon > 0$, $\mathcal{C}^+ \cap U_{\varepsilon} \neq \emptyset$, $\mathcal{C}^- \cap U_{\varepsilon} \neq \emptyset$ where $U_{\varepsilon} := \{ (\lambda, u) \in \mathbb{R} \times C^{0,\alpha}(\partial \Omega) : |\lambda| < \varepsilon, |u|_{C^{0,\alpha}(\partial \Omega)} > 1/\varepsilon \}$.

(Observe that, by the regularity of the solutions, $u \in C^{0,\alpha}(\partial \Omega) \cap X$ since it is a solution of the fixed point equation (18).) Since all solutions are uniformly bounded in $W^2_p(\Omega)$ for all $\lambda \in [0, \lambda_0]$ with $\lambda_0 < \mu_2 - \mu_1$ (see Proposition 1 and the bound in the case $\lambda = 0$) and for all $\lambda \in [\lambda_0, \lambda_1]$ with $\lambda_1 < 0$, there therefore exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_- < 0$, such that for every $\varepsilon > 0$ with $\varepsilon < |\lambda_-|$, there are two distinct solutions $(\lambda_+^\varepsilon, u_+^\varepsilon) \in \mathcal{C}^+$ and $(\lambda_-^\varepsilon, v_-^\varepsilon) \in \mathcal{C}^-$ with $-\varepsilon < \lambda_-^\varepsilon < 0$, $u_+^\varepsilon \neq v_-^\varepsilon$, and min $\{|u_+^\varepsilon|_{C^{0,\alpha}(\Omega)}, |v_-^\varepsilon|_{C^{0,\alpha}(\Omega)}\} > 1/\varepsilon$. It follows that $\Lambda_+^\varepsilon$ → 0 and min $\{|u_+^\varepsilon|_{C^{0,\alpha}(\Omega)}, |v_-^\varepsilon|_{C^{0,\alpha}(\Omega)}\}$ → $\infty$ as $\varepsilon \to 0^+$.

Theorem 3 (Bifurcation from infinity). Assume that $(G1)$–$(G2)$ hold and that

$$\int_{\partial \Omega} g_-(x) \varphi_1 < -\int_{\partial \Omega} h(x) \varphi_1 < \int_{\partial \Omega} g_+(x) \varphi_1,$$  \hspace{1cm} (19)

where $g_+(x) := \lim \inf_{u \to +\infty} g(x, u)$ and $g_-(x) := \lim \sup_{u \to -\infty} g(x, u)$.

Then $(0, \infty)$ is a bifurcation point from infinity; that is, the conclusion of Theorem 2 holds.

In the above result we strengthen “a little bit” the condition (H) by requiring strict inequalities while keeping $(G2)$ as it is given. This is the so-called Landesman–Lazer-type conditions; which was considered in the literature in some other setting. To show that all possible solutions of Eq.(14) are (uniformly) bounded in $W^2_p(\Omega)$ when $\lambda = 0$, we use the Landesman–Lazer condition (LL) and Fatou’s lemma. Then proceed as in the proof of Theorem 1.

A simple example to keep in mind here is the (smooth) function $g(x)$ independent of $x$ given by $g(u) = \eta_+ \tanh(u)$ for $|u| \geq R > 0$ with $\eta_+ > 0$ applying when $u > R$ and $\eta_- > 0$ applying when $u < -R$, or a nonbounded counterpart $g(u) = \sqrt{u} \sin^2(u) + \eta_+ \tanh(u)$ for $|u| \geq R > 0$.

**Figure 1.** Bifurcation diagram in the case of a “strong resonance” nonlinearity.

Notice that in either case $\lim \inf_{u \to -\infty} g(u) = \eta_-$ and $\lim \sup_{u \to +\infty} g(u) = -\eta_-$. Therefore the nonhomogeneous term $h$ has to satisfy the strict inequalities

$$-\eta_- \int_{\partial \Omega} \varphi_1 < -\int_{\partial \Omega} h(x) \varphi_1 < \eta_- \int_{\partial \Omega} \varphi_1.$$

Another aspect that we have not considered here, due in part to space limitation, but which is nonetheless important is the numerical analysis and simulation for these problems.

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**References**


Freeform Optics: Optimal Transport, Minkowski Method, and Monge–Ampère-Type Equations

Henok Mawi

1. Description and Background

A freeform optical surface, simply stated, refers to a surface whose shape lacks rotational symmetry. The use of such surfaces allows generation of complex, compact, and highly efficient imaging systems. Ever since lenses without symmetries were used in World War I in periscopes, the engineering and design of freeform optical surfaces have gone through remarkable evolution, with applications in a wide range of areas, including medical devices, clean energy technology, military surveillance equipments, mobile displays, remote sensing, and several other areas of imaging and nonimaging optics that can benefit from distributing light from a source to a target in a controlled fashion using spatially and energy-efficient systems. See [17] and the references therein.

Mathematically, the design of freeform optical surfaces is an inverse problem related to optimal transportation

Henok Mawi is an associate professor of mathematics at Howard University. His email address is henok.mawi@howard.edu. The author is partially supported by NSF grant HRD-1700236. Communicated by Notices Associate Editor Reza Malek-Madani. For permission to reprint this article, please contact: reprint-permission@ams.org. DOI: https://doi.org/10.1090/noti2613
theory and leads to a class of nonlinear partial differential equations (PDE) called generated Jacobian equations for which the Monge–Ampère equation is a prototype. The modeling of the problem is based on the systematic application of the laws of reflection and refraction in geometric optics along with energy conservation principles.

For completeness we recap the laws of geometric optics. Consider a reflecting or refracting surface $S$ and suppose a unit vector in the direction of $x$ is incident on the surface $S$ at a point where the normal to $S$ is given by the unit vector $\nu$. Reflection and refraction happen in tandem. (Fig. 1). However, for simplicity of the models we treat them separately. If $r$ is a unit vector in the direction of the reflected ray, the law of reflection (angle of incidence is equal to angle of reflection) in vector form says:

$$r = x - 2(x, \nu)\nu.$$  \hfill (1.1)

If medium I and medium II are two homogeneous isotropic media with respective refractive indices $n_1$ and $n_2$ and $m$ is the direction of the refracted ray into medium II, the law of refraction (Snell’s law), which states $\sin \theta_1 = \frac{n_2}{n_1} \sin \theta_t$, where $\theta_1$ is the angle between $x$ and $\nu$ (angle of incidence) and $\theta_t$ is the angle between $m$ and $\nu$ (angle of refraction) in vector form, states that:

$$m = \frac{1}{\kappa} (x - \lambda \nu)$$ \hfill (1.2)

where $\kappa = n_2/n_1$ is the relative refractive index, $\lambda = \lambda(\kappa, x, \nu)$ and $\nu$ points into medium II.

When medium I is optically denser than medium II or equivalently $\kappa < 1$, the refracted ray bends away from the normal. As a result, there is a critical value $\theta_c$ of the angle of incidence for which there is no value of $\theta_t$ unless $\sin \theta_t \leq \sin \theta_c = \kappa$. Thus for $\kappa < 1$ we impose the condition that

$$x \cdot m \geq \kappa.$$ \hfill (1.3)

Similar physical restriction applies for $\kappa > 1$.

By using equation (1.1) or (1.2) we can define a map

$$T_S : \Omega \rightarrow \Omega^*$$

that sends points in $\Omega$ (incident rays) to points in $\Omega^*$ (reflected rays if we use (1.1) or refracted rays if we use (1.2)) according to laws of geometric optics.

Generally, in an optical surface design problem the data consists of two bounded domains $\Omega$, $\Omega^*$ contained in $\mathbb{R}^n$ and two probability measures $\rho_1, \rho_2$ defined on $\Omega$ and $\Omega^*$ respectively, satisfying the energy conservation statement

$$\rho_1(\Omega) = \rho_2(\Omega^*).$$ \hfill (1.4)

$\Omega$ and $\Omega^*$ correspond to the directions of incident light and the directions of light after reflection or refraction respectively. Likewise $\rho_1$ and $\rho_2$ are the intensity of light from the source $\Omega$ and a desired illumination intensity on the target $\Omega^*$, respectively. We regard $(\Omega, \rho_1)$ as a source pair and $(\Omega^*, \rho_2)$ as a target pair.

The objective is then to find an optical surface (lens/mirror) $S$ such that

$$\rho_1(T_{S}^{-1}(F)) = \rho_2(F)$$ \hfill (1.5)

for all $F \subset \Omega^*$.

In practice, $S$ is a reflecting mirror or refracting lens and for $m \in \Omega^*$, $T_{S}^{-1}(m)$ represents the set of all directions from the source that will, after reflection or refraction off of the lens $S$, propagate in the direction of $m$. We use the notation $T_S$ for $T_{S}^{-1}$. The map $T_S$ is called the ray-tracing map of $S$.

The physical meaning of equation (1.5) is that the prescribed intensity of reflected or refracted light on the set $F \subset \Omega^*$, which is $\rho_2(F)$, is equal to the intensity of light that comes from the source, which is $\rho_1(T_S(F))$. Depending on whether $S$ is a reflector or a refractor, the design problem is called a reflector/refractor problem.

The models used to obtain numerical and analytical solutions for reflector/refractor problems are developed by using one of the following three approaches. One of the approaches is to use variational method where the problem is cast as an optimal mass transportation problem from $\rho_1$ to $\rho_2$ for an appropriate cost function and use Kantorovich duality, [20]. Another method is to derive the second order nonlinear PDE of Monge–Ampère-type satisfied by the surface defining function from the energy conservation relation (1.5) between the light energy emitted by the source and the light energy received by the target and analyze the PDE, [11]. A third approach is to exploit mainly the inherent geometric features of the problem and use a classical approach of Minkowski and Aleksandrov which was used in solving the Minkowski problem [16] of finding a closed convex surface whose Gaussian curvature is a given function of the exterior unit normal.

In the discussion below, we will focus on refractor problems and briefly describe the aforementioned approaches by using prototype refractor problems. We will state the problems explicitly and show how the available methods are used to solve the corresponding design problems. We will also mention problems that are worth studying in the future. We point out to the reader that unless mentioned otherwise, we consider $\kappa < 1$. 

![Figure 1. Laws of reflection and refraction.](image-url)
2. Optimal Transport: The Far Field Refractor Problem

Several freeform lens design problems have been successfully modeled by using optimal transport framework on the sphere. See [4, 10, 20]. The models have also been adopted by the optics community. Among these problems is the far field refractor problem.

2.1. Statement of the far field refractor problem with point source. In this problem, we are given two domains Ω, Ω∗ contained in the unit sphere S^{n-1} of R^n (in practice n = 3), |Ω| = 0; a punctual source of light located at the origin O, surrounded by media I, from which a monochromatic ray of light is issued in each direction x ∈ Ω with intensity density function given by g(x) for g ∈ L^1(Ω), g ≥ 0 a.e. on Ω and a prescribed intensity density distribution f ∈ L^1(Ω∗) which satisfies

\[ \int_{\Omega} g(x) dx = \int_{\Omega^*} f(x) dx. \]  
(2.1)

The objective is to find a refracting surface parametrized as S = {ρ(x)x : x ∈ Ω}, ρ > 0, between medium I and medium II such that all rays emitted from the point O with directions x ∈ Ω are transported via refraction by the surface into media II with directions in Ω∗ in such a way that the prescribed illumination pattern f ∈ L^1(Ω∗) is achieved on Ω∗.

In particular, if we let dρ1 = gdx and dρ2 = fdx where dx is the surface measure on S^{n-1}, this means ρ1 and ρ2 satisfy the mass balance condition ρ1(Ω) = ρ2(Ω∗) and a weak solution to the far field refractor problem is defined as an interface S so that the associated map T_S satisfies

\[ T_S(\Omega) \subseteq \Omega^* \quad \text{and} \quad \rho_2 = (T_S)_# \rho_1. \]  
(2.2)

We recall for F ⊂ Ω∗ that the measure \( (T_S)_# \rho_1(F) := \rho_1(T_S^{-1}(F)) \) is the push forward measure. Because of (1.3), we assume x ⋅ m ≥ κ for x ∈ Ω and m ∈ Ω∗.

By using Snell’s law (1.2) it can be shown that the semi-ellipsoid of revolution (ellipse in 2D case) with one focus at O, given in polar representation as (10)

\[ E(m, b) = \left\{ x \in S^{n-1}, \ m \cdot x \geq \kappa \right\} \]

for some b > 0 and m ∈ Ω∗, has a uniform refracting property. This means that if E(m, b) is an interface between media I and II, all rays issued from O in a direction x with x ⋅ m ≥ κ will be refracted in the direction m after hitting E(m, b). Motivated by this, a solution for the far field refractor problem is sought among surfaces \( S = \{ \rho(x)x : x \in \Omega \} \) which are supported from outside by semi-ellipsoids at each point. That is, for each point \( \rho(x_o)x_o \in \mathcal{S} \) there exists a semi-ellipsoid E(m, b) with m ∈ Ω∗ such that \( \rho(x) \leq \frac{b}{1 - \kappa m \cdot x} \) with equality at \( x_o \).

We now discuss how optimal transport technique is used to find such a solution. First, a brief review of optimal transport.

2.2. Review of optimal transport problems. Let \( X \) and \( Y \) be compact subsets of R^n and \( c : X \times Y \to [0, \infty] \). We refer to c as the cost function. Denote by \( \mathcal{P}(X) \) and \( \mathcal{P}(Y) \) the set of probability measures on X and Y respectively. Given \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \) a measurable map \( T : X \to Y \) satisfying \( T_#\mu = \nu \) is called a transport map from \( \mu \) to \( \nu \). Monge’s formulation of the optimal transport problem is to minimize

\[ T \rightarrow \int_X c(x, T(x)) d\mu(x) \]  
(2.3)

among all transport maps from \( \mu \) to \( \nu \). A minimizer of (2.3) is called an optimal transport map. A generalization due to Kantorovich [18] of Monge’s problem is stated as

\[ \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{X \times Y} c(x, y) d\gamma(x, y) \]  
(2.4)

among all transport plans \( \gamma \in \Gamma(\mu, \nu) \) where

\[ \Gamma(\mu, \nu) := \{ \gamma \in \mathcal{P}(X \times Y) : (\pi_1)_# \gamma = \mu, (\pi_2)_# \gamma = \nu \}. \]

Here \( \pi_1 : X \times Y \to X \) and \( \pi_2 : X \times Y \to Y \) are projection maps. If c is lower semicontinuous and bounded from below, problem (2.4) has a minimizer in \( \Gamma(\mu, \nu) \). A minimizer of (2.4) is called an optimal plan.

The c-transform of a given function \( \phi : X \to [-\infty, \infty) \) is defined as \( \phi^c : Y \to \mathbb{R} \) where

\[ \phi^c(y) = \inf_{x \in X} c(x, y) - \phi(x). \]

Also \( \zeta^c : X \to \mathbb{R} \), the c-transform of \( \zeta : Y \to [-\infty, \infty) \), is defined by

\[ \zeta^c(x) = \inf_{y \in Y} c(x, y) - \zeta(y). \]

A function \( \phi \in C(X) \) is called c-concave if \( \phi = \zeta^c \) for some \( \zeta \). We also define for any \( x \in X \) the c-superdifferential \( \partial_c \phi \) at x as

\[ \partial_c \phi(x) = \{ y \in Y : \phi(z) \leq c(z, y) - c(x, y) + \phi(x), \forall z \in X \}. \]

Problem (2.4) is a linear optimization problem with convex constraints and thus, it admits a dual problem of maximizing

\[ \left\{ \int_X \phi(x) d\mu + \int_Y \psi(y) d\nu : \phi(x) + \psi(y) \leq c(x, y) \right\} \]  
(2.5)

over \( \phi \in L^1(X) \) and \( \psi \in L^1(Y) [18] \). One of the basic results in optimal transport theory is that if c is continuous and X and Y are compact, then the dual problem (2.5) has a maximizer of the form \( \phi(x)^c \) where the map \( \phi \), which is referred to as Kantorovich potential, is a c-concave function. Furthermore, if \( \partial_c \phi \) is single valued for \( \mu - a.e. \) x, then \( T := \partial_c \phi \) induces the optimal plan \( \gamma \) for (2.4). That is, \( \gamma = (Id, T)_# \mu \) minimizes (2.4). Consequently, \( \partial_c \phi \) is an optimal transport map from \mu to \( \nu \).
2.3. The far field refractor problem as optimal transport. To transform the far field refractor problem (2.2) to an optimal transport problem, we introduce the logarithmic cost function $c : \Omega \times \Omega^* \to [0, \infty]$ given by

$$c(x, m) = -\log(1 - x \cdot m). \quad (2.6)$$

$c(x, m)$ is continuous and bounded from below since $x \cdot m \geq x$. Thus, if we consider the dual problem (2.5) corresponding to $c(x, m)$ given as

$$\sup_{u \in \mathcal{C}(\Omega), v \in \mathcal{C}(\Omega^*)} \left\{ \int_{\Omega} u(x)d\rho_1 + \int_{\Omega^*} v(m)d\rho_2 : u(x) + v(m) \leq c(x, m) \right\}, \quad (2.7)$$

there exists $\exists (\mathcal{C}(\Omega) \times C(\Omega^*), v \in \mathcal{C}(\Omega^*))$ a maximizer of (2.7). Let $\hat{\phi} \in \mathcal{C}(\Omega)$ be c-concave for the cost function $c(x, m) = -\log(1 - x \cdot m)$. If we set

$$\rho(x) = e^{\hat{\phi}(x)}, \quad (2.8)$$

then for each $x_0 \in \Omega$ there exists $b > 0$ and $m \in \Omega^*$ such that

$$e^{\hat{\phi}(x)} \leq \frac{b}{1 - x \cdot m} \quad \forall x \in \Omega \quad \text{and} \quad e^{\hat{\phi}(x_0)} = \frac{b}{1 - x_0 \cdot m_0}. \quad (2.9)$$

That is, at each point $\rho(x_0)x_0$ the surface $S = \{\rho(x)x : x \in \Omega\}$ is supported from outside by a semi-ellipsoid $E(m, b)$ for some $b > 0$ and $m \in \Omega^*$. If $S$ is smooth, then $S$ and $E(m, b)$ have the same normals at $\rho(x_0)x_0$. By the refraction property of $E(m, b)$ a ray emanating from $\partial \Omega$ in a direction $x_0$ will be refracted off $S$ in the direction $m$ by the surface $S = \{x \rho(x) : x \in \Omega\}$. Therefore, $T_S(x_0) = m$. Thus, $S$ satisfies, $T_S(\Omega) \subset \Omega^*$. Also, from (2.9) we get

$$\hat{\phi}(x) \leq -\log(1 - x \cdot m) + \hat{\phi}(x_0) \quad \forall \ x \in \Omega$$

and hence

$$T_S(x_0) = m \in \partial_c \hat{\phi}(x_0).$$

Furthermore, if $c$ is given by (2.6), for any c-concave $\hat{\phi}$ the c-superdifferential mapping $\partial_c \hat{\phi}$ is single valued for $\rho_1 - a.e. x$. Thus if $(\hat{\phi}, \hat{\phi}')$ is a maximizer of (2.7), $\partial_c \hat{\phi}$ is a measure preserving map from $\rho_1$ to $\rho_2$. Therefore $\partial_c \hat{\phi} = T_S \rho_1 - a.e.,$ and more importantly $\rho_2 = (T_S)\# \rho_1$ proving that $S = \{x \rho(x) : x \in \Omega\}$ will be a solution of the far field refractor problem. From this we deduce that solving the far field refractor problem corresponds to finding a maximizer of the dual problem to the optimal transport problem (2.7). It is further shown in [10] that if $S = \{x \rho(x) : x \in \Omega\}$ and $\tilde{S} = \{x \tilde{\rho}(x) : x \in \Omega\}$ are two solutions of the far field refractor problem, then $\rho = C\tilde{\rho}$ for some constant $C$.

The optimal transport approach can also be used in the case of anisotropic materials where optical properties vary according to the direction of propagation of light [6]. In this type of media modeling the design problem is more complicated due to issues such as birefringence, a situation where the incident rays may be refracted into two rays. It is also used to develop numerical methods to approximate solutions [4] to freeform design problems. Recently, a more efficient method called entropic regularization is applied to the the reflector problem [2] and it is likely that this approach can be applied to refractor problems.

Finally, it is worth noting that the far field problem has not been studied in its full generality using optimal transport method. In practice, if a ray of light hits an interface between two media which have different optical properties, the ray is partly reflected back and partly transmitted (Fig. 1). It is an open problem whether or not this energy imbalance between the source and target can be modeled as a variational problem by using the optimal transport approach.

3. Minkowski Approach: The Near Field Refractor Problem

The Minkowski problem involves finding a closed convex surface whose Gaussian curvature at a point with exterior normal $\xi$ is given by a continuous positive function $k(\xi)$ [16]. To solve this, the approach was first to define a set function on $S^{n-1}$ by $\sigma(M) = \int_M \frac{d\omega(\xi)}{k(\xi)}$ where $d\omega$ is a surface element on the unit sphere, and consider the general problem of finding a convex hypersurface for which $\sigma$ is a surface area function. That is, to find a convex hypersurface $F$ such that for a given subset $M' \subset F$, the area of $M'$ is equal to $\sigma(M)$ where $M' = \{x \in F : \text{the normal to } F \text{ at } x \text{ is in } M\}$. Then partition the sphere into smaller domains $g_k$ with area $\sigma_k$ and obtain an approximation $\sigma' = \sum_k \sigma_k \delta_{\tilde{\xi}_k}$ $\tilde{\xi}_k \in g_k$ to $\sigma$. Replacing $\sigma$ by $\sigma'$ gives a discrete problem of existence of convex polyhedron with faces of areas $\sigma_k$ and outward normal $\xi_k$. Let $P_k$ be a solution of the discrete problem. As $\ell \to \infty$ (or the diameter of $g_k$ to 0), it is proved that the sequence $P_k$ of surfaces converges to a surface which solves the Minkowski problem. This classical approach of Minkowski is used to obtain iterative methods to solve not only various geometric optics problems related to both reflection [3] and refraction, but also to develop numerical methods to solve PDEs [15]. The approach is also generalized to approximate solutions to semi-discrete optimal mass transport problems and generated Jacobian equations. See [1] and the reference therein. It is also one of the methods used by the optics community to design freeform optical surfaces [14]. Below we exhibit how this approach is used in the near field refractor problem.
3.1. Statement of the near field refractor problem with point source. For this problem we are given $\Omega \subset S^{n-1}, \Omega^* \subset \Sigma$, a hyperplane in $\mathbb{R}^n$ with $|\partial \Omega| = 0$ and the origin $O \not\in \Omega^*$. For each $x \in \Omega$, a light ray is issued in the direction of $x$ from a point source of light located at $O$. The illumination intensity of the source is given by a density function $g \in L^1(\Omega)$, $g > 0$ a.e. A prescribed irradiance distribution on the target set $\Omega^*$ is given by a nonnegative density function $f \in L^1(\Omega^*)$. $f$ and $g$ satisfy the mass balance (2.1). We assume that $\Omega$ is surrounded by medium I and $\Omega^*$ is surrounded by medium II. Additional constraints are imposed on $\Omega$ and $\Omega^*$ to make the problem physically feasible [9].

The near field refractor problem involves finding an interface $S$ between media I and II, parametrized radially as $S = \{\rho(x)x : x \in \Omega\}$, that redirects each ray with direction $x \in \Omega$ by refraction into $\Omega^*$ so that a prescribed irradiance distribution $f$ is obtained on $\Omega^*$. More precisely we will require

$$\int_{\Sigma(F)} g(x) d\sigma(x) = \int_F f(y) d\sigma(y) \quad (3.1)$$

holds for all Borel sets $F \subset \Omega^*$.

Similar to the semi-ellipsoids in the far field problem, there are surfaces that have uniform refracting property in the near field case. The Descartes ovoids will be used as the building blocks of the near field refractors and we look for a solution for the near field refractor problem among surfaces which are supported from above by an ovoid at each point.

It is known that the near field refractor problem can't be cast as an optimal mass transport problem. However, Minkowski's approach can be used to solve the problem.

### Figure 2. Refraction property of oval; $x = 0.7$, $P = (10,0), b = 8$.  

3.2. The near field refractor problem using Minkowski's approach. Once again let $d\rho_1 = gd\sigma(x)$ and $d\rho_2 = fd\sigma(y)$. The road map is as follows. First partition $\Omega^*$ and approximate the measure $\rho_2$ on the target by a sequence of discrete measures $\mu_\ell$ concentrated at finite points in $\Omega^*$ and that converge to $\rho_2$ as $\ell \to \infty$. Then solve the problem when the target distribution is $\mu_\ell$. Finally let $\ell \to \infty$ to obtain a solution to the problem.

Now, partitioning $\Omega^* = \bigcup_{j=1}^{\ell} \Omega_j$, one can define a discrete measure on $\Omega^*$ by $\mu_\ell = \sum_{j=1}^{\ell} f_j \delta_{P_j}$ where $P_j \in \Omega_j^*$, $\rho_2(\Omega_j^*) = f_j$. Clearly the mass balance $\mu_\ell(\Omega^*) = \sum_{j=1}^{\ell} f_j = \rho_2(\Omega^*)$ is satisfied. For the discrete measure $\mu_\ell$ on the target (see [8]), given $\epsilon > 0$, and $b_1 > 0$ satisfying $\kappa|P_1| < b_1 \leq |P_1| + r_0 \frac{(1-\kappa)^2}{1+\kappa}$, ($r_0$ that depends on the geometry of the problem), one can find through an iterative scheme a vector $b_\infty = (b_1, b_2, \cdots, b_\ell) \in \prod_{j=1}^{\ell} |(\kappa|P_j|, |P_j|)|$ such that the multifaceted refractor $S(b_\infty)$ defined by

$$S(b_\infty) = \{\rho(x)x : x \in \Omega \text{ and } \rho(x) = \min_{1\leq i\leq \ell} h(x, P_i, b_i)\}$$  

(3.3)

satisfies

$$\int_{\Sigma(F)} g(x) d\sigma(x) - f_j \leq \epsilon, \quad 1 \leq j \leq \ell. \quad (3.4)$$

This means that the surface $S(b_\infty)$ solves the discrete near field refractor (Fig. 3) problem with an error of $\epsilon$ for the intensity distribution. For a given error $\epsilon$, the convergence in finite number of steps of this scheme is proved by obtaining appropriate Lipschitz estimates for the refractor measure $\int_{\Sigma(b_\infty)\{P_i\}} g(x) d\sigma(x)$ as a function of $b$.

It can be shown that as $\epsilon \to 0$, $b_\infty \to b_\infty := (b_1, b_2, \cdots, b_\ell)$ and that the multifaceted refractor $S(b_\infty)$ defined by

$$S_\epsilon = \{\rho_\epsilon(x)x : x \in \Omega \text{ and } \rho_\epsilon(x) = \min_{1\leq i\leq \ell} h(x, P_i, b_i)\} \quad (3.5)$$

is an exact solution to the near field refractor problem if $\rho_2$ is given by the discrete measure $\mu_\ell$. It should be remarked...
here that the iterative process used to obtain \( b_\ell \) can also be implemented numerically. It is used to obtain an approximate solution for the near refractor problem when the irradiance distribution on the target is given by a discrete measure \([8]\). The convergence of this process in more generality is discussed in \([8]\). Notice that the freeform refracting surface which solves the near field refractor problem when \( \rho_2 \) is discrete measure is built from segments of Descartes ovoids.

Now that for each \( \ell \in \mathbb{N} \), a multifaceted near field refractor \( S_\ell \), parametrized by \( \rho_\ell \) is obtained corresponding to the discrete measure \( \mu_\ell \) on the target, the next step is to let \( \ell \to \infty \) and study the convergence. As discussed in \([9]\), \( \mu_\ell \) converges weakly to the measure \( d\rho_\ell = f d\sigma(y) \) and by using compactness argument, up to a subsequence \( \rho_\ell \) converges to \( \rho \). It is then shown that the surface \( S = \{ \rho(x)x : x \in \Omega \} \) is the desired solution to the near field refractor problem from \( d\rho_1 = g d\sigma(x) \) to \( d\rho_2 = f d\sigma(y) \).

The Minkowski approach, also known in literature as the supporting quadric method, has been used to treat several other beam shaping freeform design problems. The fundamental principle is similar for all the problems. One first determines the surfaces with uniform refraction, reflection property, like the semi-ellipsoids or the Descartes ovoids, along with the length parameters and then uses those surfaces as building blocks to obtain the required freeform lens. More general far field refractor problems such as the one accounting for loss of energy due to internal reflection \([7]\) and also in anisotropic media \([6]\) have been studied by using this technique. Whether this approach can be extended to propagation in anisotropic media for the near field refractor problem and for the parallel refractor problem discussed in the next section remains open.

4. PDE Methods: Regularity of Solutions

The existence of a solution to a refractor design problem can be shown in most cases by exploiting the physical and geometric features of the specific design problem and using the Minkowski approach. Equally important is the regularity properties of these solutions. The analysis of regularity of surfaces that are solutions to problems in geometric optics is not only of deep mathematical interest, but also has physical significance since non-smoothness of solutions has a physical interpretation of aberrations and diffraction of light. It should be noted that some regularity properties can be easily obtained from the geometry of the problems. For instance for \( \kappa < 1 \) the solutions for the far field refractor problem are Lipschitz continuous by the fact that they are supported by semi-ellipsoids at each point. Further analytical properties of solutions can be obtained by exploiting the relations between solutions of refractor problems and solutions of nonlinear PDEs.

For some design problems, optimal transport theory could be used to obtain the related PDE. Indeed, for appropriate assumptions on the cost function \( c(x,y) : X \times Y \to \mathbb{R} \), it is known that the optimal transport map \( T \) minimizing Monge’s problem solves a nonlinear PDE of Monge-Ampère-type. Since, as seen in Section 2, the far field refractor problem has an optimal transport formulation a similar nonlinear PDE can be deduced. In particular, it is known that the potential \( \phi(x) = \log \rho(x) \) in \((2.8)\) satisfies the corresponding equation, \([10]\)

\[
\det (\nabla^2_x \phi(x) + \nabla^2_x c(x, T(x))) = |\det(\nabla^2_x c(x, T(x)))| - \frac{g(x)}{f(T(x))}
\]

where for a function \( h \) on \( S^{n-1} \), \( \nabla_x h(x) \) is the tangential gradient of \( h \) at \( x \in S^{n-1} \). Ma-Trudinger-Wang in \([13]\) identified a key condition which depends on the fourth order derivatives of the cost \( c \), to obtain some regularity results. Using this condition in \([10]\) it was discussed that one can’t expect a \( C^1 \) regularity of the solution surface for the far field refractor problem when \( \kappa < 1 \). On the other hand for \( \kappa > 1 \) the solutions are smooth provided the density functions \( f \) and \( g \) are smooth functions which are bounded and away from zero.

Not all refractor problems can be formulated as optimal transport problems. In that case a corresponding PDE can still be obtained by using energy conservation and by computing the Jacobi determinant of the ray tracing map explicitly. For instance, consider the parallel near field refractor problem. In this problem we have a source \( \Omega \) which lies in medium I, and a target \( \Omega^* \) which lies in medium II and both are bounded domains in \( \mathbb{R}^n \). For brevity we assume that \( \Omega \subset \{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \} \) and \( \Omega^* \subset \{ x \in \mathbb{R}^{n+1} : x_{n+1} = k \} \) for some \( k > 0 \). From each \( x \in \Omega \), a ray of light is issued in a direction parallel to \( e_{n+1} \) the unit vector in the direction of \( x_{n+1} \) axis in \( \mathbb{R}^{n+1} \). The intensity of this light is given by a nonnegative density function \( g(x) \). The prescribed intensity at \( y \in \Omega^* \) is given by \( f(y) \). Also \( g \) and \( f \) satisfy \((2.1)\). A generalized solution to the problem is a refracting surface \( S = \{ u(x) : x \in \Omega \} \) with \( u : \Omega \to \Omega^* \) such that for the map \( T_S : \Omega \to \Omega^* \) we have \( T_S(\Omega) \subset \Omega^* \) and that the energy conservation

\[
\int_F g(x) \, dx = \int_{T_S(F)} f(x) \, dx
\]

is satisfied for any measurable subset \( F \) of \( \Omega \).

If \( dS_\Omega \) and \( dS_{\Omega^*} \) represent area elements on \( \Omega \) and \( \Omega^* \) respectively, and assuming the ray tracing map is smooth, the energy conservation condition along with change of variables requires that

\[
\det DT_S = \frac{dS_{\Omega^*}}{dS_\Omega} = \frac{g(x)}{f(T_S(x))}
\]

where \( DT_S \) is the Jacobian matrix of \( T_S \). \( DT_S \) can then be explicitly expressed as in \([11]\) in terms of \( u \) and its
gradient. In particular if \( u \in C^2(\Omega) \) is a solution of the parallel near field refractor problem, then \( u \) satisfies the Monge–Ampère-type PDE

\[
|\det(D^2u + A(x, x, u, Du))| = \left| \eta(x, x, u, Du) \right| \frac{g(x)}{f(T(x))}.
\]

The \( C^2 \) smoothness of weak solution when \( f \) and \( g \) are bounded and away from zero under some geometric conditions is proved in [11] by using standard technique from PDE theory.

A more comprehensive approach can be carried out by using what are called generated Jacobian equations.

Given \( \Omega, \Omega^* \) domains in \( \mathbb{R}^n \), a generating function \( G : \Omega \times \Omega^* \times \mathbb{R}^+ \to \mathbb{R}^+ \), with variables \( G(x, y, v) \), satisfies the following conditions:

(a) \( G_v < 0 \) for all \( (x, y) \in \Omega \times \Omega^* \);
(b) the map \( (y, v) \to (G(x, y, v), G_x(x, y, v)) \) is invertible for each \( x \in \Omega \);
(c) for each \( y \in \Omega^* \), \( G(x, y, v) \to +\infty \) as \( v \to 0^+ \), uniformly in \( x \in \Omega \);
(d) \( G \) is \( C^1 \) in \( v \), \( G \) in \( x, y \), and for each \( \alpha > 0 \),
\[
\sup_{\Omega \times \Omega^* \times [0,\alpha]} |G_x(x, y, v)| < \infty.
\]

By using the first condition, the dual generating function \( H \) is defined by \( G(x, y, H(x, y, v)) = v \) for all \( (x, y, v) \) when the expression is well defined.

A generated Jacobian equation is a PDE of the form

\[
det(DY(x, u, Du)) = \psi(x, u, Du) / f(Y(x, u, Du)) \tag{4.1}
\]

where the vector field \( Y(x, u, p) \) is generated from \( G \) by
\[ D_xG(x, Y, Z) = p \]
and \( Z(x, u, p) = H(x, Y(x, u, p), v) \). For brevity consider the case where
\[
\psi(x, u, Du) = g(x) / f(Y(x, u, Du)) \tag{4.2}
\]

for \( g \in L^1(\Omega) \) and \( f \in L^1(\Omega^*) \).

For example, in an optimal transportation problem with cost \( c(x, y) \) which satisfies compatible conditions, setting \( G(x, y, v) = c(x) - v \) we get \( G_x(x, y, v) = c_x(x, y) \). The vector field \( Y(x, u, p) \) is generated by \( c_x(x, Y(x, p)) = p \) and the equation corresponding to (4.1) will be a generalized Monge–Ampère equation given as \( \det(D^2u - A(u, Du)) = B(u, Du) \) where \( A(x, z, p) = c_{xz}(x, Y(x, p)) \) and \( B(x, z, p) = det c_{xz}(x, Y(x, p)) \psi(x, z, p) \). See [19].

In particular if \( \mu = g(x) dx \) and \( v = f(y) dy \) and the cost function is the quadratic cost \( c(x, y) = \frac{1}{2} |x - y|^2 \), we obtain the vector field \( Y(x, p) = x - p \) from which (4.1) will be
\[
\det(D(x - Du(x))) = \psi(x, u(x), Du(x)). \tag{4.3}
\]

If \( T \) is any transport map from \( \mu \) to \( v \), we know from energy relations that
\[
\det DT = \frac{g(x)}{f(T(x))}. \tag{4.4}
\]

In particular if \( T \) is the optimal map and \( \phi \) is the potential, \( T_\phi(x) = x - D\phi(x) \). Using this relation we observe (4.4) is of type (4.1) with \( Y = x - p \). If we further express \( T_\phi = Du \) we get the standard Monge–Ampère equation
\[
\det(D^2u) = \frac{g(x)}{f(Du(x))}.
\]

A function \( \phi : \Omega \to \mathbb{R} \) is \( G \)-convex if for each \( x_0 \in \Omega \) there exists \( y_o \in \Omega^* \) and \( v_o \in \mathbb{R}^+ \) such that \( \phi(x) \geq G(x, y_o, v_o) \) for all \( x \in \Omega \) with equality at \( x = x_o \); i.e., \( G(\cdot, y_o, v_o) \) supports \( \phi \) at \( x_o \). For a \( G \)-convex function \( \phi \), the \( G \)-subdifferential of \( \phi \) at \( x_o \) is defined by
\[
\partial_G\phi(x_0) = \{ y_o \in \Omega^* : \text{there exists } v_o \in \mathbb{R}^+ \text{ such that } G(\cdot, y_o, v_o) \text{ supports } \phi \text{ at } x_0 \}.
\]

If \( \phi \) is differentiable, then \( \partial_G\phi(x_0) \) has exactly one element. Following [19] a \( G \)-convex function \( \phi \) is called a generalized solution of (4.1) with \( \psi \) given by (4.2) if
\[\int_E g(x) dx = \int_{\partial_G\phi(E)} f(x) dx \]
for all \( E \subset \Omega \).

For the near field refractor problem with point source, the corresponding generating functions will be
\[G(x, p, b) = \frac{1}{h(x, p, b)}\]
with \( h(x, p, b) \) defined as in (3.2). If \( S = \{ \rho(x) : x \in \Omega \} \) is a solution to the near field refractor problem and \( \hat{\phi}(x) = \frac{1}{\rho(x)} \) then the \( G \)-subdifferential of \( \phi \) and the inverse of the ray tracing mapping, \( T_\phi \) are related by
\[
\partial_G\phi(x_0) = T_\phi(x_0)
\]
and hence we have
\[\int_E g(x) dx = \int_{T_\phi(E)} f(x) dx = \int_{\partial_G\phi(E)} f(x) dx .\]

Thus, the near field refractor problem can be studied via the theory of generated Jacobian equations. For the particular case where \( x > 1 \) this is done in [12] where they proved the \( C^{2,\alpha} \) regularity of solutions. The solutions to the reflector problem with collimated beam, the near field reflector problem with point source and other related problems in geometric optics can also be expressed as generated Jacobian equations [19]. More discussion on the relationship between design of optical surfaces and generated Jacobian equations is in [5]. However, for design problems involving both reflection and refraction in anisotropic media, the regularity theory using PDE methods or otherwise is an untouched territory.
References


Credits

Opening image is courtesy of JARAMA via Getty. Figures 1–3 are courtesy of Henok Mawi. Photo of Henok Mawi is courtesy of Aaron Fagerstrom.
In this friendly survey, we look at some interactions between a certain class of infinite matrices called Riordan matrices and the class of combinatorial objects called lattice paths. There are a variety of interesting algebraic and combinatorial relationships between Riordan matrices and lattice path enumeration problems. This article attempts to introduce the reader to the concepts and techniques that may be used to explore these relationships.

The lattice paths considered in this article are presented as combinatorial objects which are subject to specific construction rules, and our interest is to enumerate and classify them with counting sequences. Riordan matrices are infinite matrices whose columns can be associated with a certain kind of sequence of generating functions. There are many interesting Riordan matrices that contain column entries or row sums associated with well-known counting numbers, such as the binomial coefficients, the ballot, Catalan, Motzkin, Schröder, Fine, Delannoy, and RNA numbers, and other counting numbers that can be found.
in Sloane’s On-line Encyclopedia of Integer Sequences [S1]. The aim of this article is to introduce the reader to Riordan matrices and to present examples of solutions to particular lattice path counting problems, where the lattice paths are enumerated by the column entries or row sums of Riordan matrices.

Algebraic combinatorics involves the use of techniques from algebra, topology, and geometry in the solution of combinatorial problems. Because of this interplay with many fields of mathematics, algebraic combinatorics is an area in which a wide variety of ideas and methods come together. Riordan arrays appear in algebraic combinatorics and are useful for proving combinatorial sums and identities. They also appear in various counting problems in enumerative combinatorics. A certain subset of Riordan arrays called proper Riordan arrays, otherwise known as Riordan matrices, form the Riordan group, an infinite noncommutative matrix group, which is the main combinatorial device reported in this article. The Riordan group can be characterized as being algebraic and combinatorial. Thus, as a subfield of enumerative and algebraic combinatorics, the Riordan group brings together a wide variety of combinatorial methods and mathematical ideas. There are also some interesting mathematical contributions outside of combinatorics that touch on geometry, group theory, Lie algebra and groups, functional analysis, representation theory, and queuing theory. For a more thorough survey of contributions of the Riordan group to other areas of mathematics and combinatorics, see the monograph by Shapiro et al. [SSB*].

The algebraic concepts subsequently reported in this article involve finding inverse relations and multiplying, inverting, and manipulating Riordan matrices. The concepts covered herein involve using recurrence relations, generating functions, combinatorial statistics and explicit bijections to solve certain lattice path counting problems. However, the authors do not claim to solve all lattice path counting problems associated with Riordan matrices.

The article proceeds with an overview of the type of lattice paths under consideration, followed by an elementary introduction to generating functions. Riordan arrays are then formally introduced, followed by a section devoted to the algebra of the Riordan group. The last two sections of the paper feature specific demonstrations of how the Riordan group can be applied to lattice path enumeration problems, especially as related to generalized Catalan paths.

Lattice paths. The subject of counting paths (walks) on the lattice in Euclidean space is one of the most important areas of combinatorics. The lattice paths described in this article are defined on the integral lattice \( \mathbb{Z}^d \), where \( d \geq 1 \), under the conditions that each path starts at some fixed origin, moves according to certain steps under specified rules and restrictions, and never crosses or touches certain coordinate axes or hyperplanes. For excellent introductions to lattice path combinatorics and enumeration, see the surveys by Humphreys [H] and Krattenthaler [K].

A lattice path is a sequence of contiguous (unit) steps which traverses the \( d \)-dimensional integral lattice \( \mathbb{Z}^d \). More precisely, a lattice path in \( \mathbb{Z}^d \) with \( k \geq 1 \) unit steps is a sequence \( s_1, s_2, \ldots, s_k \in \mathbb{Z}^d \) such that for each \( i, 1 \leq i \leq k, \ s_i - s_{i-1} \in \mathbb{S} \subseteq \mathbb{Z}^d \). In this case, we refer to \( \mathbb{S} \) as a step set. We will often refer to a lattice path simply as a path, where the path is encoded by a word based on an alphabet containing \( \mathbb{S} \). Geometrically, a lattice path is represented by the edges between the consecutive vertices of the path. A path with no steps is a point. The length of a path is the number of unitary steps.

In this article, the paths are considered to be in \( d \)-dimensional Euclidean space and never pass below a specified hyperplane. Typically, in \( \mathbb{Z}^d \), the paths never pass below the \( x \)-axis. The height of a path corresponds to the \( y \) value of the endpoint \((x, y)\) of the path. In three dimensions the paths are considered to be in three-dimensional Euclidean space and often never pass below the \( xy \)-plane. The height of each path corresponds to the \( z \) value of the endpoint \((x, y, z)\) of the path. Throughout this article, we will focus mainly on lattice paths in the plane, i.e., \( d = 2 \). However, the step sets of particular higher dimensional paths defined in [N2] will be mentioned in the last section. There are many types of path problems as well as methods and models for finding their solutions. Thus, there is a vast amount of literature on this topic from the self-avoiding walk problem to lattice path problems with infinite step sets.

Examples of lattice paths. This article will focus mainly on Dyck (Catalan) paths and related paths, such as Motzkin and Schröder paths, with a few other paths discussed briefly.

Example 1 (Dyck Paths). A Dyck (Catalan) path is a lattice path in the first quadrant of \( \mathbb{Z}^2 \) which begins at the origin \((0, 0)\), ends at \((2n, 0)\), has the step set \( \mathbb{S} = \{(1, 1), (1, -1)\} \), and never goes below the \( x \)-axis. These paths can be described as paths that consist of unit up steps with slope 1, denoted by \( U \), and unit down steps with slope \(-1\), denoted by \( D \). We refer to \( n \) as the semi-length of the path. A Dyck path of semi-length \( n \) is called a Dyck \( n \)-path. These paths are counted by the Catalan numbers, which are defined in the next section. The 5 possible paths of length 6 are depicted in Figure 1 and can also be encoded by the words \( UUUDDD, UUDUDD, UUDDUD, UDUDD, UUDUD \).

Example 2 (Motzkin Paths). A Motzkin path of length \( n \) is a lattice path in \( \mathbb{Z}^2 \) that begins at the origin \((0, 0)\), ends at \((n, 0)\), has step set \( \mathbb{S} = \{(1, 1), (1, 0), (1, -1)\} \) and never
passes below the x-axis. These paths consist of unit up steps with slope 1 denoted by $U$, unit down steps with slope $-1$ denoted by $D$, and unit horizontal (or level) steps with slope 0 denoted by $H$. Motzkin paths are counted by the Motzkin numbers that are defined in the next section. The 4 possible paths of length 3 are shown in Figure 2 and the 9 possible paths of length 4 can be encoded by the words

$UUDD, UHHD, UDUD, UDHH, UHDH, HUDD, HUHD, HHUD, HHHH$.

**Example 3 (Schröder Paths).** A Schröder path is a path in the first quadrant of $\mathbb{Z}^2$ that begins at the origin $(0,0)$, ends at $(2n,0)$, with step set $S = \{(1,1), (1,-1), (2,0)\}$ and never passes below the x-axis. These paths consist of $U$ steps, $D$ steps, and double horizontal or level steps denoted by $\tilde{H}$. These paths are counted by the large Schröder numbers which count various combinatorial objects including lattice paths. The 6 possible paths of length 4 are depicted in Figure 3.

**Generating functions.** In order to understand Riordan matrices, some working knowledge of generating functions is needed. In this article, Riordan arrays have been constructed to study these paths and many other types of lattice path counting problems.

Given a sequence $\langle b_n \rangle_{n \geq 0}$ of elements of a commutative ring, the ordinary generating function (GF) for $\langle b_n \rangle_{n \geq 0}$ is the formal power series

$$b(z) = \sum_{n \geq 0} b_n z^n = b_0 + b_1 z + b_2 z^2 + \cdots$$

where $z$ is an indeterminate (or auxiliary variable). In some instances “aerated” GFs are of interest. An aerated GF is a GF of the form

$$b(z^2) = \sum_{n \geq 0} b_n z^{2n} = b_0 + b_1 z^2 + b_2 z^4 + \cdots$$

where $\langle b_0,0,b_1,0,b_2,\ldots \rangle$ is the associated sequence of aerated coefficients. In this article, we will only refer to the concept of aerated GFs once in Example 13, but otherwise they may be encountered frequently in some lattice path enumeration problems.

Note that since GFs are defined algebraically as formal power series, and not as real-valued functions, series convergence is not necessary for the existence of GFs. Nonetheless, convergence of GFs is sometimes necessary for finding exact formulae and asymptotic estimates of the $n$th term of a sequence. See Wilf [W2], for more details on algebraic and analytic properties of GFs.

Sequences of coefficients in combinatorics are sometimes called counting sequences and are often computed by recurrence relations. A recurrence relation (or recursion) recursively defines a sequence where the $n$th term of the sequence is expressed in terms of some previous $k$ terms, $k < n$. In turn, GFs can be derived from their related recurrence relation. The famed Fibonacci numbers are computed by the following recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

with initial conditions $F_0 = F_1 = 1$. The GF for this sequence is perhaps one of the most recognized and studied GFs:

$$F(z) = \sum_{n \geq 0} F_n z^n = 1 + z + 2z^2 + 3z^3 + \cdots = \frac{1}{1-z-z^2}$$

where $\langle F_0, F_1, F_2, \ldots \rangle = \langle 1, 1, 2, \ldots \rangle$ are the popular Fibonacci numbers, which count various combinatorial objects including lattice paths. The $n$th Fibonacci number is given by

$$F_n = (\beta^{n+1} - \alpha^{n+1})/\sqrt{5}$$

where $\alpha = (1 - \sqrt{5})/2, \beta = (1 + \sqrt{5})/2$, and $\beta$ is known as the golden ratio.

Interestingly, with the same recurrence relation, if the initial conditions are $F_0 = 0, F_1 = 1$, then

$$F_n = (\beta^n - \alpha^n)/\sqrt{5}$$

Some suggested exercises for the reader are to use the recursion to derive $F(z)$, and then to use $F(z)$ to derive the exact formula of $F_n$ given above.

We now present three examples of GFs that are commonly encountered in lattice path enumeration.

**Example 4.** The Catalan GF is defined by

$$C(z) = \sum_{n \geq 0} c_n z^n = 1 + z + 2z^2 + 5z^3 + \cdots$$

$$= \frac{1 - \sqrt{1 - 4z}}{2z}$$

where $\langle c_n \rangle_{n \geq 0} = \langle 1, 1, 2, 5, \ldots \rangle$ are the Catalan numbers and $c_n = 1/(n+1)(2n\choose n}$ is the $n$th Catalan number. They are
computed recursively by \( c_0 = 1 \),
\[
c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}, \quad n \geq 1.
\]
The Catalan numbers appear in many combinatorial problems and have a variety of algebraic applications [S2].

Next, we present the Motzkin and Schröder GFs which have lattice path interpretations that are related to the Catalan numbers.

Example 5. The Motzkin \( GF \) is defined by
\[
m(z) = \sum_{n \geq 0} m_n z^n = 1 + z + 2z^2 + 4z^3 + \cdots
\]
\[
= 1 - z - \sqrt{1 - 2z - 3z^2}
\]
where \( \langle m_n \rangle_{n \geq 0} = \langle 1, 1, 2, 4, \ldots \rangle \) are the Motzkin numbers. They are computed recursively by \( m_0 = 1, \ m_1 = 1, \)
\[
m_n = m_{n-1} + \sum_{k=0}^{n-2} m_k m_{n-2-k}, \quad n \geq 2
\]
and are connected to the Catalan numbers by
\[
m_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k
\]
where \( \lfloor x \rfloor \) denotes the floor of \( x \) (i.e., the greatest integer \( \leq x \)).

Example 6. The Schröder \( GF \) is defined by
\[
s(z) = \sum_{n \geq 0} s_n z^n = 1 + 2z + 6z^2 + 22z^3 + \cdots
\]
\[
= 1 - z - \sqrt{1 - 6z + z^2}
\]
where \( \langle s_n \rangle_{n \geq 0} = \langle 1, 2, 6, 22, \ldots \rangle \) are the large Schröder numbers. They are computed recursively by \( s_0 = 1, s_2 = 2, \)
\[
s_n = 3s_{n-1} + \sum_{k=1}^{n-2} s_k s_{n-1-k}, \quad n \geq 2
\]
and are connected to the Catalan numbers by
\[
s_n = \sum_{k=0}^{n} \binom{2n-k}{k} c_{n-k}, \quad n \geq 0.
\]
The Catalan, Motzkin, and Schröder GFs appear in many combinatorial problems. For more properties and examples of GFs and recurrence relations, see [W2].

What are Riordan arrays? The Riordan array concept was introduced in 1991 by Shapiro et al. [SGWW]. A Riordan array is a special infinite lower-triangular matrix where the column entries consist of coefficients of certain formal power series. Representing infinite matrices by coefficients of (formal) power series is not new and goes back to papers by Schur [S] and Jabotinsky [J] on Faber polynomials. Thus, Riordan arrays depend upon certain formal power series and GFs.

Suppose \( g(z) = g_0 + g_1 z + g_2 z^2 + \cdots \) and \( f(z) = f_1 z + f_2 z^2 + \cdots \), where \( g_0 \neq 0 \) and \( g(z) \) and \( f(z) \) are formal power series \( \mathbb{C}[[z]] \). Then the infinite array
\[
L = (l_{n,k})_{n,k \geq 0},
\]
with entries in \( \mathbb{C} \) is called a Riordan array if the ordinary generating function for its \( k \)th column is the Cauchy (convolution) product of \( g(z) \) and \( f^k(z) \). That is, \( L \) is a Riordan array if the \( GF \) for the sequence \( \langle l_{n,k} \rangle_{n \geq 0} \) of numbers in the \( k \)th column of \( L \) is
\[
g(z) \cdot f^k(z)
\]
for all \( k \geq 0 \). The constant coefficient \( g_0 = 1 \) is commonly used for combinatorial convenience. Note that \( g(z) \) and \( f(z) \) are sometimes abbreviated, respectively, as \( g \) and \( f \).

Since the column-generating functions of a Riordan array \( L \) form a geometric sequence \( g, gf, \ldots, gf^k, \ldots \), we may denote the Riordan array \( L \) in pair form as
\[
L = (g(z), f(z)) = \left( \begin{array}{c}
g \\
gf \\
gf^2 \\
\vdots
\end{array} \right).
\]
The \((n,k)\)-th or generic element of \( L \) can be obtained by extracting the \( n \)th coefficient of \( gf^k \) and obtaining
\[
l_{n,k} = [z^n] g(z) \cdot f^k(z)
\]
where \([z^n] \) denotes the operator for extracting the \( n \)th coefficient of a generating function. For instance, \([z^4] F(z) = 5 = F_4 \). For rules that govern the actions of \([z^n] \), see [SSB⁺]. The rules are sometimes called the method of coefficients and sometimes involve using the binomial theorem and Lagrange inversion formula to extract the coefficients.

Following the definition of \( L \) and if, in addition, \( f_1 \neq 0 \) is satisfied, then \( L \) is invertible under matrix multiplication. The invertible Riordan arrays are called proper Riordan arrays or simply Riordan matrices. We note that Riordan matrices are also known as “recursive matrices” in the umbral calculus [MRSV]. Riordan matrices (arrays) are also defined by coefficients of exponential generating functions where one encounters the well-known derangement, partition (Bell), Bernoulli, Cauchy, Euler, and Stirling numbers; however, they are not described in this paper. For more information and examples of Riordan arrays and exponential formal power series see [B].
Examples of Riordan arrays. Here we provide examples of Riordan arrays related to the Pascal, Catalan, Motzkin, and RNA sequences.

Example 7 (Pascal’s Triangle). Pascal triangle, the most popular example, denoted by $P$ contains the well-known binomial coefficients. When written in infinite lower-triangular form,

$$
P = \left( \frac{1}{1-z}, \frac{z}{1-z} \right) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where

$$p_{nk} = [z^n] \frac{1}{1-z} \left( \frac{z}{1-z} \right)^k = \binom{n}{k}
$$
is the $(n,k)$-th element of $P$. More generally, for integers $t \geq 1$

$$P^t = \left( \frac{1}{1-tz}, \frac{z}{1-tz} \right)
$$
is a generalized Pascal-Riordan matrix with generic element $\binom{n}{k} t^{n-k}$. The entries of $P$ count certain lattice paths in $\mathbb{Z}^2$ with horizontal and diagonal steps that do not go above the line $y = x$ or below the $x$-axis [SSB+]. A suggested exercise for the reader is to show that the entries of $P^t$ count the lattice paths in $\mathbb{Z}^t$ with different kinds of horizontal steps that come in $t$ different colors.

Example 8 (Aigner-Catalan Array). The Aigner-Catalan array denoted by $C$ contains the well-known ballot numbers. $C$ is defined as

$$C = (c(z), zc(z)) = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where the leftmost column entries are the Catalan numbers, $c(z)$ is the Catalan GF,

$$c_{nk} = [z^n] c(z) (zc(z))^k = \binom{k+1}{n+1} \frac{2n-k}{n+1} \binom{n-k}{n-k}
$$
is the $(n,k)$-th element of $C$, and $c_{nk}$ are the well-known ballot numbers. The entries of $C$ count certain lattice paths in $\mathbb{Z}^2$ with horizontal and vertical steps that are restricted to lie on or below the line $y = x$ in the coordinate plane [SSB+].

Example 9 (Shapiro-Catalan Array). The Shapiro-Catalan array denoted by $B$ is defined as

$$B = (c^2(z), zc^2(z)) = \begin{pmatrix} 1 \\ 2 & 1 \\ 5 & 4 & 1 \\ 14 & 14 & 6 & 1 \\ 42 & 48 & 27 & 8 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where the leftmost column entries are the Catalan numbers (minus the leading 1) and

$$b_{nk} = [z^n] c^2(z) (zc^2(z))^k = \frac{k+1}{n+1} \frac{2n+2}{n-k}
$$
is the $(n,k)$-th element of $B$. The entries of $B$ count certain pairs of non-intersecting lattice paths in $\mathbb{Z}^2$ in the first quadrant.

Example 10 (Radoux-Catalan Array). The Radoux-Catalan array denoted by $\tilde{A}$ is defined as

$$\tilde{A} = (c(z), zc^2(z)) = \begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 3 & 1 \\ 5 & 9 & 5 & 1 \\ 14 & 28 & 20 & 7 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where the leftmost column entries are again the Catalan numbers and

$$a_{nk} = [z^n] c(z) (zc^2(z))^k = \frac{2k+1}{2n+1} \frac{2n+1}{n-k}
$$
is the $(n,k)$-th element of $\tilde{A}$. The entries of $\tilde{A}$ count certain Dyck paths [A1].

The Catalan arrays $\tilde{A}, B,$ and $C$ are of interest and appear in many combinatorial problems.

Example 11 (RNA Array [N1]). The RNA array, denoted by $R^*$, is defined as

$$R^* = (s(z), zs(z)) = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 3 & 1 \\ 4 & 6 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where the leftmost column entries are the RNA numbers (or generalized Catalan numbers) which count RNA secondary structures from molecular biology [S2], [W1]. The
For the RNA numbers is
\[ s(z) = \sum_{n\geq 0} s_n z^n = 1 + z + z^2 + \cdots \]
\[ = \frac{1 - z + z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}}{2z^2}, \]
and \((s_n)_{n\geq 0} = (1, 1, 1, 2, 4, 8, \ldots)\) [W1]. The RNA numbers are computed recursively
\[ s_{n+1} = s_n + \sum_{k=1}^{n} s_{k-1}s_{n-k}, \quad n \geq 2 \]
with \(s_0 = s_1 = s_2 = 1\). They are also computed by the following sum
\[ \sum_{k \geq 1} \frac{1}{n-k} \binom{n-k}{k} (n-k)(k < n). \]
The entries of \(R^\ast\) count certain Motzkin paths with certain restrictions [N1]. A problem of interest is to find a nice form for the generic element of \(R^\ast\)
\[ s_{n,k} = [z^n] s(z) (zs(z))^k. \]
Example 12 (An Improper Riordan Array). The array below denoted by \(Q^\ast\) is an example of an improper Riordan array [MRSV]
\[ Q^\ast = (q(z), z^2q(z)) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ 12 & 5 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
where \(q(z)\) is the generating function
\[ q(z) = \sum_{n \geq 0} q_n z^n = 1 + z + 2z^2 + 5z^3 + \cdots \]
\[ = \frac{1 - z - \sqrt{1 - 2z - 3z^2 - 4z^3}}{2z^2(1 + z)}. \]
Note that \([z^1] z^2q(z) = 0\). So, \(Q^\ast\) is not a proper Riordan array. However, the entries of \(Q^\ast\) count certain lattice paths in \(Z^2\) with steep and shallow diagonal steps [MRSV].

We conclude the examples by noting some well-known combinatorial triangles that are not (ordinary) Riordan arrays, as defined in this article. The Narayana array denoted by \(T\) contains the Narayana numbers \(\hat{T}_{n,k}, 0 \leq k < n\). \(T\) is typically given as an example of an invertible, infinite lower-triangular array that is not a Riordan array.
\[ T = \begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 6 & 6 & 1 \\ 1 & 10 & 20 & 10 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]
and the generic element is
\[ \hat{T}_{n,k} = \frac{1}{n} \binom{n}{k} (n + 1). \]
\(\hat{T}\) is not a Riordan array since the \(k\)th column is not defined by ordinary generating functions of the form \(g^k f^k\). Although \(T\) is not Riordan, there are interesting combinatorial interpretations of \(T\): the entries count the total number of Dyck \(n\)-paths with \(k\) peaks, the row sums are the Catalan numbers \(c_{n+1}\), and the diagonal sums are the RNA numbers \(s_n\) (minus the leading 1). Suggested exercises for the reader are to verify the row sums and diagonal sums. (The diagonal sums, sometimes called slices, move upward left-to-right within the array in a north-east direction.)

Finally, we make a brief note of the Stirling triangle, which contains the Stirling numbers of the second kind \(S(n,k)\), the number of partitions of an \(n\)-set into \(k\) blocks (nonempty subsets). The Stirling triangle is not a Riordan array, as we have defined in this article. However, since the exponential generating function for \(S(n,k)\) is of the form \(gf^{k}/k!\) where \(g\) and \(f\) are exponential generating functions and \(k \geq 0\), the Stirling triangle is a so-called exponential Riordan array. For more information on exponential generating functions and exponential Riordan arrays, see [B].

Algebra of the Riordan group. As mentioned earlier, the set of Riordan matrices form the Riordan group, which we will define and discuss in more detail in this section. We note that the Riordan group is also known as the Sheffer or \(1\)-umbral group [JLN]. See [SSB+] for a connection between the Sheffer and Riordan groups. For an excellent introduction to the Riordan group and Riordan arrays, see Barry [4].

We now introduce Riordan matrix multiplication, starting with multiplication by a column vector. If \(L = (g(z), f(z))\) is a Riordan matrix and \(h(z)\) is the \(GF\) associated with the infinite column vector \(h = (h_k)_{k \geq 0}^T\), where \(h^T\) denotes the transpose, then one can show that the matrix product \(Lh\) is an infinite column vector whose associated \(GF\) is \(g(z) \cdot h(f(z))\), where \(g \cdot h\) denotes a Cauchy product and \(h(f)\) denotes composition of \(GFs\) [SGWW]. This statement is known as the fundamental theorem of Riordan arrays (FTRA) and it allows us to define multiplication of Riordan arrays in the following way.

Suppose \((g(z), f(z))\) is a Riordan matrix and \(h(z)\) is a \(GF\). We define the product
\[ (g(z), f(z)) \otimes h(z) := g(z) \cdot h(f(z)). \]
Here, we have introduced the \(\otimes\) symbol for use when writing the matrix product of a Riordan matrix, in terms of its associated generating function pair, and a column vector, in terms of its associated generating function. Note that the product represents the usual matrix multiplication. Next, we give some examples of properties of
riordan matrix-vector multiplication before moving on to multiplication of two Riordan matrices.

For example, the product of Pascal’s Triangle $P$ and $h(z) = 1/(1-z)$ associated with column vector $(1,1,...)^\top$ is

$$P \otimes h(z) = (1/(1-z), z/(1-z)) \otimes 1/(1-z) = 1/(1-2z).$$

This example is the well-known result where the row sums of $P$ are

$$[z^n]1/(1-2z) = 2^n$$

for $n \geq 0$.

We now give some important GFs of various sums of Riordan matrices that are obtained by multiplying by different column vectors [SSB⁺].

**Row sums:**

$$(g(z), f(z)) \otimes 1/(1-z) = g(z)/(1-f(z))$$

**Diagonal sums:**

$$(g(z), zf(z)) \otimes 1/(1-z) = g(z)/(1-zf(z))$$

**Alternate sums:**

$$(g(z), f(z)) \otimes 1/(1+z) = g(z)/(1+f(z))$$

**Weighted sums:**

$$(g(z), f(z)) \otimes z/(1-z) = g(z)f(z)/(1-f(z))^2$$

Recall that diagonal sums move upward left-to-right within the array in a north-east direction. For instance, in $P$ the fifth diagonal sum is $8 = 1+4+3+0+0+0$. The diagonal sums of all entries of a Riordan array $L = (g(z), f(z))$ are computed by the row sums of the “vertically stretched” Riordan array $(g(z), zf(z))$. Vertically stretched Riordan arrays are not invertible because in this case the first three coefficients of $zf(z)$ are $f_0 = 0$, $f_1 = 0$ and $f_2 \neq 0$. See Example 12 for an example of a vertically stretched Riordan array, and for a broader introduction to stretched Riordan arrays, see [4]. Some suggested exercises are to show that the diagonal sums of $P$ are the Fibonacci numbers and the weighted row sums starting with leading coefficient 1 are given by the GF $g(z)/(1-f(z))^2$. Weighted row sums correspond to the first moments or the expected value of each row of the Riordan matrix. For example, the first moments of $R^*$ are the bisected Fibonacci numbers given by $GF 1/(1-3z+z^2)$ [N1]. A straight forward calculation shows

$$(s(z), zs(z)) \otimes 1/(1-z)^2 = 1/(1-3z+z^2)$$

where $s(z)$ is the RNA GF. However, the bisection formula can be applied to the Fibonacci generating function $F(z)$. For more information on the bisection formula, see [SSB⁺]. The result is

$$s(z)/(1-zs(z))^2 = F(\sqrt{z}) - F(-\sqrt{z})/2\sqrt{z}.$$ This gives a nice relationship between the Fibonacci and RNA GFs.

Since FTRA is based on the usual row-by-column matrix multiplication, it can be extended to the matrix product of two Riordan matrices. Let $L = (g(z), f(z))$ and $N = (h(z), l(z))$ be Riordan matrices. Then the matrix product $LN$ is also a Riordan matrix which can be described as follows:

$$LN = (g(z), f(z)) \otimes (h(z), l(z)) = (g(z) \cdot h(f(z)), l(f(z))).$$

Note that, in the context of this article, a product of Riordan matrices, expressed without the "⊗" symbol may be interpreted as the usual matrix multiplication.

**Example 13.** An exercise for the reader is to show that $PC_0 = M$ where $P$ is Pascal’s Triangle, $C_0 = (c(z^2), zc(z^2))$ is the “aerated” version of the Aigner-Catalan Riordan matrix from Example 8, and $M = (m(z), zm(z))$ is the Motzkin Riordan matrix from Example 5. The entries of $C_0$ and $M$ count certain partial Motzkin paths [4], [N2].

The set of all Riordan matrices forms a group under the operation of matrix multiplication [SGWW]. The identity element is $(1,z)$. This is the usual identity with ones along the main diagonal. The inverse of $(g(z), f(z))$ is $(1/g(\tilde{f}(z)), \tilde{f}(z))$ where $f(z)$ is the compositional inverse of $f(z)$ such that $f(\tilde{f}) = z = \tilde{f}(f)$. For example, the inverse of Pascal’s Triangle is

$$P^{-1} = (1/(1+z), z/(1+z))$$

and the $(n,k)$th element of $P^{-1}$ is

$$[z^n]\frac{1}{1+z}(\frac{z}{1+z})^k = (-1)^{n-k}\binom{n}{k}.$$ A slightly more complicated example, where the compositional inverse is not as obvious to compute, is

$$((1-z), z(1-z))^{-1} = C,$$

where $C = (c(z), zc(z))$ is the Aigner-Catalan array of Example 8. In many instances, finding $\tilde{f}(z)$ can be laborous and complicated.

**Example 14.** A nice application of $P$ and $P^{-1}$ involves the FTRA and the notion of binomial inversion (transform) of sequences. If $A(z)$ is the GF of the generic sequence $\langle a_n \rangle_{n \geq 0}$, then the GF of the binomial transform $\langle b_n \rangle_{n \geq 0}$ is given by

$$B(z) = P \otimes A(z) = (1/(1-z))A(z/(1-z)).$$

Similarly,

$$A(z) = P^{-1} \otimes B(z) = (1/(1+z))B(z/(1+z)).$$
As a result of this we have the classical binomial transform of sequences
\[ b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \iff a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \]

Example 15. Let \( S \) and \( W \) denote, respectively, column vectors associated with the sequences \( (s_n)_{n \geq 0} \) and \( (w_n)_{n \geq 0} \). Then \( PW = S \iff P^{-1} S = W \), where \( P \) is Pascal’s Triangle. Suppose we want to find \( W \) such that \( S \) is the RNA numbers described in Example 11. Then, by the FTRA
\[ P^{-1} \otimes s(z) = (1/(1+z)) s(z/(1+z)) \]
where \( s(z) \) is the RNA GF. Thus, \( P^{-1} \otimes s(z) \) equals
\[ 1 + z + z^2 - \sqrt{1 + 2z - z^2 - 6z^2 - 3z^4} = w(z) \]
where \( w(z) \) is the GF associated with
\[ (w_n)_{n \geq 0} = (1, 0, 0, 1, -1, 2, -1, 5, 3, -5, 13, -13, ...) \].
This curious sequence is not in the OEIS database [S1]. A suggested exercise is to show that \( P \otimes w(z) = s(z) \). This gives a nice relationship between Pascal’s triangle and the RNA numbers. An interesting open problem is to find a combinatorial interpretation of this multiplication that involves lattice paths or RNA secondary structures from molecular biology.

There are many important subgroups of the Riordan group. We list a few special subgroups below:
- Appell subgroup: \((g(z), z)\)
- Associated subgroup: \((1, f(z))\)
- Bell subgroup: \((g(z), zg(z))\)
- Checkerboard subgroup: \((g(z), f_0(z))\)
- Derivative subgroup: \((f'(z), f(z))\)
- Hitting time subgroup: \((zf'(z)/f(z), f(z))\)

Note that \( g_e \) and \( f_0 \) are even and odd functions, respectively. \( f'(z) \) is the first derivative of \( f(z) \). \( P, B, C, C_0, M, \) and \( R^* \) are elements of the Bell subgroup. In addition, \( C_0 \) is an element of the checkerboard subgroup and \( P \) the hitting time subgroup. Interestingly, one can use the fact that
\[ (g(z), f(z)) = (g(z), z) \otimes (1, f(z)) \]
to show that the Riordan group is a semi-direct product of the associated and Appell subgroups. For more algebraic properties of the Riordan group, see [JLN].

Finally, we mention that Riordan matrices have recursive formulas for their entries, known as "formation rules." A formation rule is a recurrence relation which describes how each entry in a Riordan matrix \( L \) can be expressed as a linear combination of entries in the preceding rows. One common formation rule is denoted by \([Z; A]\), which represents two recurrence relations that define the way entries of \( L \) are computed. "\( Z \)" is a sequence of coefficients for a recurrence relation that describes the entries of the leftmost or zeroth column of \( L \) and "\( A \)" is a sequence of coefficients for a recurrence relation that describes the entries of all other columns of \( L \).

More precisely, the \( Z \)-sequence \( Z = (z_0, z_1, ...) \) \((z_0 \neq 0)\) characterizes the zeroth column of \( L = (e_{n,k})_{n,k \geq 0} \). This means every element \( e_{n+1,0} \) can be expressed as a linear combination of all the elements in the preceding row by
\[ e_{n+1,0} = z_0 e_{n,0} + z_1 e_{n,1} + \cdots = \sum_{j \geq 0} z_j e_{n,j}. \]
The \( A \)-sequence \( A = (a_0, a_1, ...) \) \((a_0 \neq 0)\) characterizes the other columns of \( L \). In this case every element \( e_{n+1,k+1} \) can be expressed as a linear combination of the elements in the preceding row, starting from the preceding column by
\[ e_{n+1,k+1} = a_0 e_{n,k} + a_1 e_{n,k+1} + \cdots = \sum_{j \geq 0} a_j e_{n,k+j}. \]
The GFs of the \( A \)- and \( Z \)-sequences, respectively, are
\[ A(z) = z/f'(z) \text{ and } Z(z) = 1/f'(z) \left( 1 - 1/g(f'(z)) \right) \]
where \( f(z) \) is the compositional inverse of \( f(z) \) [4]. For example, the formation rule of the Shapiro-Catalan Riordan matrix \( B \) is \([2, 1, 1, 2, 1]\) where \( Z(z) = 2 + z \) and \( A(z) = 1 + 2z + z^2 \). In general, the entries are computed by \( e_{n+1,0} = 2e_{n,0} + e_{n,1} \) and \( e_{n+1,k+1} = e_{n,k} + 2e_{n,k+1} + e_{n,k+2} \). See [MRSV], [R] for more information on the \( A \)- and \( Z \)-sequences.

Riordan matrix method and lattice path counting. One can use Riordan matrices to solve many kinds of lattice path enumeration problems. A typical method for using Riordan matrices to count lattice paths, as well as other combinatorial objects, is outlined by the following steps:
1. Count a few cases for the objects and observe some counting numbers.
2. Set up the counting numbers as a lower-triangular matrix \( M \).
3. Find a pattern in the way the entries of \( M \) are computed.
4. Use the pattern (formation rules) to conjecture/define \( M \) as a Riordan matrix \( L \).
5. Find the generic element of \( L \).
6. Use the generic element or formation rules to prove combinatorially that \( L \) counts the objects.
7. Compute \( L \otimes (1/(1 - z)) \) to find the total number of combinatorial objects.

We now apply the Riordan matrix method to a particular lattice path counting problem. The problem was previously solved in [N2]. However, in this survey we provide more details and improve the presentation of the solution.
We observe that the leftmost (or zeroth) column entry with north bold. All other columns are formed for the step set
\[ S = \{N(0,1), S(0,-1), R(1,0), \overline{L}(-1,0)\} \]
with north (N), south (S), right (R), and left (\(\overline{L}\)) unit steps, and satisfy the following restrictions: (1) no path passes below the x-axis, (2) there are no paths with consecutive N and S steps, i.e., there are no NS steps, (3) no paths begin with an \(\overline{L}\) step, (4) all \(\overline{L}\) steps touch and remain on the x-axis. Then, we call these paths NSRL paths. Let \(R_1^\ast = (\rho_{n,k})_{n,k \geq 0}\) where \(\rho_{n,k}\) denotes the number of NSRL paths of length \(n\) that end at height \(k\). Recall that the height corresponds to the y value of the endpoint \((x,y)\) of the path. A typical path of length 10 and height 1 is denoted by the word

\[ R\overline{L}R\overline{R}R\overline{I}R\overline{N}R\overline{S}L \cdot N. \]

We want to find the total number of NSRL paths. Start by counting the number of NSRL paths of length \(n\) that end at height \(k\) for the first few cases where \(n,k \leq 5\) and obtain the following lower-triangular matrix

\[
R_1^\ast = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & \\
5 & 4 & 3 & 1 \\
12 & 10 & 7 & 4 & 1 \\
29 & 25 & 18 & 11 & 5 & 1
\end{pmatrix}.
\]

Then, we find the following patterns for the entries of \(R_1^\ast\). We observe that the leftmost (or zeroth) column entry 29 is computed by

\[
29 = 2 \cdot 12 + 1 \cdot 4 + 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0.
\]

All other leftmost column entries seem to follow this same pattern (formation rule), that is, the same linear combination of respective entries from the preceding rows. We now observe the other columns entries. Observe that 18 is computed by

\[
18 = 1 \cdot 10 + 1 \cdot 7 + 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0.
\]

All other bold entries in the other columns (not the leftmost) seem to follow this pattern. Thus, we conjecture this pattern (formation rule) and make the assumption that the \((n,k)\)-th entry of \(R_1^\ast\) is formed by the following recurrence relations. The leftmost or zeroth column is formed for \(\rho_{0,0} = 1, \rho_{1,0} = 1, \) and \(n \geq 1\)

\[
\rho_{n+1,0} = 2\rho_{n,0} + \sum_{k \geq 1} \rho_{n-k,k}.
\]

The other columns are formed for \(n, k \geq 1\)

\[
\rho_{n+1,k} = \rho_{n,k-1} + \sum_{j \geq 0} \rho_{n-j,k+j} = \rho_{n,k-1} + \rho_{n,k} + \cdots.
\]

As previously computed, one can easily confirm \(\rho_{5,0} = 29\) and \(\rho_{5,2} = 18\).

From the recurrence relations, we will derive explicit GFs whose coefficients are the column entries of \(R_1^\ast\). Following the matrix formation rules, the definition of a Riordan matrix, and making adjustments to properly align the coefficients, the \(k\)th column GF of \(R_1^\ast\) is defined for \(k \geq 1\) as

\[
g \cdot f_k = zg f_{k-1} + z (g f_k + z g f_{k+1} + z^2 f_{k+2} + \cdots).
\]

Solving for \(f\) gives

\[
f = z f^2 + (z - z^2) f + z.
\]

Now, we solve \(f\) in terms of \((z)\) and apply the quadratic formula to obtain \(f(z) = zs(z)\) where \(s(z)\) is the RNA GF. Similarly, the leftmost column GF of \(R_1^\ast\) is defined as

\[
g = 1 + z (2g + zg f^2 + \cdots).
\]

Using geometric series and simplifying,

\[
g = \frac{1 - 2z + z^2}{1 - 2z - z^2},\]

where \(\triangle = 1-2z-z^2-2z^3+z^4\). This confirms the pattern (formation rule) and we conjecture that \(R_1^\ast\) is the Riordan matrix defined by the GFs derived from the recurrence relations. Thus,

\[
R_1^\ast = \left(\frac{1 - z}{2z} \left(1 - 3z + \frac{z^2 - \sqrt{\triangle}}{-1 + 3z - z^2}\right), zs(z)\right).
\]

We now give combinatorial arguments that show that the entries of \(R_1^\ast\) model the lattice path counting problem earlier described in this section. We want to find the total number of NSRL paths of length \(n\) ending at height \(k\). We will connect the entries of \(R_1^\ast\) to the paths by showing the paths satisfy the recurrence relations. To do this we consider the following combinatorial arguments. Consider an NSRL path of length \(n\) ending at height \(k\). To form a new path of length \(n+1\) of height \(k\), we consider the following cases. Case 1: Given a path of length \(n\) ending at height \(k-1\), there is one choice to move the path up to height \(k\), a north step \(N\). Thus, if the last step is \(N\), there are \(1 \cdot \rho_{n,k-1}\) possibilities to construct a new path ending at height \(k\). In this case, all paths with last step \(N\) are counted by \(\rho_{n,k-1}\).

Case 2: Given a path of length \(n\) ending at height \(k\), there
is one choice for the path to remain at height $k$, a right step $R$. Thus, if the last step is $R$, there are $1 \cdot \rho_{n,k}$ possibilities to construct a new path ending at height $k$. In this case, all paths with last step $R$ are counted by $\rho_{n,k}$. Case 3: For $j \geq 0$, given a path of length $n - j$ ending at height $k + j$, there is one choice of $j$ south steps to move the path down to height $k$. Thus, there are $1 \cdot \rho_{n-j,k+j}$ possibilities to construct a new path ending at height $k$. In this case, all paths with $j$ south steps are counted by $\rho_{n-j,k+j}$. For $k > 0$ there are no paths with a last step $L$ since $L$ steps touch and remain on the x-axis. Now, summing all cases gives all possible ways to construct a new path of length $n + 1$ ending at height $k$. Therefore, the recurrence relation for the other columns of $R_1^n$ counts all NSRL paths of length $n$ ending at height $k \geq 1$. By similar arguments, we can show that the leftmost column of $R_1^n$ counts all paths of length $n$ ending at height $k = 0$. This proves the formation rule and gives $R_1^n$ a NSRL lattice path interpretation.

To find the total number of NSRL paths, we apply the FTRA and compute the row sums by

$$R_1^n \otimes (1/(1 - z)).$$

Simplifying, the total number of NSRL paths is given by

$$R_1^n \otimes (1/(1 - z)) = (1 - z)/(1 - 3z + z^2) = \sum_{n \geq 0} \rho_n z^n = BF(z)$$

where the $n$th coefficient is computed by the following sums [A2]

$$[z^n]BF(z) = 1 + \sum_{k=0}^{n} F_{2k} = \sum_{k=0}^{n} \binom{2n-k}{k}.$$ 

For example,

$$[z^3]BF(z) = 1 + \sum_{k=0}^{3} F_{2k} = 13 = \rho_3.$$ 

Thus, the total number of NSRL paths are counted by the bisected Fibonacci numbers with generating function $BF(z)$. The number of Dyck paths of length $2n$ and height at most 3 are also counted by these bisected Fibonacci numbers. A suggested exercise is to find an explicit bijection between these two sets of paths. See OEIS [S1] sequence number A001519 for other combinatorial objects enumerated by the counting numbers $(\rho_n)_{n \geq 0}$.

Generalized Catalan paths and Riordan matrices. The Riordan matrix method can be readily used to study statistics on lattice paths and other combinatorial objects. In this final section, we give some illustrations of how Riordan matrices can be leveraged to study statistical distributions of generalized lattice paths and to observe combinatorial relationships among higher-dimensional lattice paths.

A frequent example of lattice paths are the so-called Catalan paths, which are equivalent to ballot numbers and Dyck paths. A Catalan path is a path starting from the origin $(0, 0)$, using $n$ steps of the form $(1, 0)$ and $n$ steps of the form $(0, 1)$, that does not go above the line $y = x$. Catalan paths are equivalent to Dyck $n$-paths, as described in Example 1. The number of Dyck $n$-paths that end at height $k$ is given by the $(n, k)$th entry of the (aerated) Catalan matrix $C_0 = (c(z^2), zc(z^2))$.

Catalan paths can be generalized in a variety of ways to obtain other interesting lattice paths associated with well-known combinatorial sequences. For instance, $t$-Dyck paths are paths of length $tn$ from $(0, 0)$ to $(tn, 0)$, where $t > 0$, that use up steps $U$ of the form $(1, 1)$ and down steps $D$ of the form $(1, -t + 1)$ and do not go below the x-axis. When $t = 2$, we have Dyck paths while the paths we obtain when $t = 3$ are called ternary paths which are counted by the ternary numbers $\frac{1}{2n+1} \binom{3n}{n}$ with generating function $T(z)$ satisfying $T(z) = 1 + zT^3(z)$.

A $t$-Dyck path can also be thought of as a path starting from the origin $(0, 0)$ that uses $n$ steps of the form $(1, 0)$ and $(t-1)n$ steps of the form $(0, 1)$ that does not go above the line $y = (t-1)x$. In general, $t$-Dyck paths are counted by the Fuss–Catalan numbers $\frac{1}{(t-1)n+1} \binom{tn}{n}$.

One may be interested in enumerating certain properties of these generalized lattice paths, such as the number of peaks, valleys, returns, or hills. A return in a Dyck path is a non-origin point on the path that intersects the x-axis. The Riordan matrix $C = (c(z), zc(z))$, from Example 8, counts the number of Catalan paths of length $2(n + 1)$ having exactly $k + 1$ returns to the x-axis. A quick computation of $(c(z), zc(z)) \otimes \frac{1}{1-z}$ verifies that the row sums of $(c(z), zc(z))$ indeed count the total number of Catalan paths of length $2(n + 1)$. Moreover, the computation of

$$\frac{[z^n](c(z), zc(z)) \otimes \frac{1}{(1-z)^2}}{[z^n]c(z)}$$

produces the average number of returns among Dyck $n$-paths. An exercise for the reader is to perform this computation and show that the average number of returns among Dyck $n$-paths approaches 3 as $n \to \infty$.

Similarly, the total number of returns among ternary paths of length $3n$ is given by

$$\left(1, z(T(z))^2\right) \otimes \frac{z}{(1-z)^2} = z(T(z))^4.$$ 

Hence, the expected number of returns among ternary paths is

$$\frac{[z^n]z(T(z))^4}{[z^n]T(z)} = \frac{2n}{n+1}.$$
which approaches 2 as \( n \to \infty \). Furthermore, one can use these methods to show that the probability that a randomly chosen ternary path has exactly \( k \) returns approaches \( \frac{4k}{3k+1} \), and that the expected number of returns among nontrivial \( t \)-Dyck paths approaches \( \frac{t+1}{t-1} \) with variance \( \frac{2t}{(t-1)^2} \). Ultimately, these results lead to the fact that the number of returns among \( t \)-Dyck paths approaches \( \frac{t-1}{t-2} \).

A hill in a \( t \)-Dyck path is a subsequence of the form \( U^{t-1}D \) such that the preceding sequence of steps form a \( t \)-Dyck path. The number of Dyck \( n \)-paths having exactly \( k \geq 0 \) hills is given by the Riordan array \((\tilde{f}(z), z\tilde{f}(z))\) where
\[
\tilde{f}(z) = \frac{1 - \sqrt{1 - 4z}}{z (3 - \sqrt{1 - 4z})} = \frac{c(z)}{1 + zc(z)}
\]
is the GF for the Fine numbers \( \langle \tilde{f}_n \rangle_{n \geq 0} = \langle 1, 0, 1, 2, 6, 18, 57, 186, \ldots \rangle \). Dyck paths having no hills, known as Fine paths, are counted by the Fine numbers. The reader may verify as an exercise that the total number of hills among Dyck \( n \)-paths is given by
\[
[z^n] (\tilde{f}(z), z\tilde{f}(z)) \otimes \frac{z}{(1 - z)^2} = c(z) - 1,
\]
which implies that the expected number of hills among Dyck \( n \)-paths is exactly 1. This result suggests that there is a bijection between the set of Dyck \( n \)-paths and the set of hills among all Dyck \( n \)-paths. One nice bijection can be described as follows. Given the hill \( UD \) in a Dyck \( n \)-path of the form \( S_1UDS_2 \), where \( U \) is an up step, \( D \) is a down step, and \( S_1, S_2 \) are Dyck paths of semilength at most \( n \), map the hill \( UD \) to the Dyck \( n \)-path \( US_1DU S_2 \). It should be clear that the mapping is injective, and since every Dyck \( n \)-path can be described in the latter form, the mapping is surjective.

More generally speaking, similar methods on the generalized Fine array \((f_\ell(z), z\tilde{f}(z))\), where \( f_\ell(z) \) is the GF for the number of \( t \)-Dyck paths having no hills, can be used to show that the number of hills among \( t \)-Dyck paths of length \( tn \) has expected value
\[
\frac{2((t-1)n+1)^{t-2}}{t(tn-1)^{t-2}},
\]
where \( t\ell := tl(t-1) \cdots (t-j+1) \), and approaches a negative binomial distribution with parameters 2 and \( \frac{t^{t-1}}{t^{t-1} + (t-1)^{-1}} \) [CM].

As mentioned previously, lattice paths from \((0, 0)\) to \((n, 0)\) that use up steps of the form \((1, 1)\), down steps of the form \((1, -1)\), and level steps of the form \((1, 0)\) are known as Motzkin paths, which are counted by the Motzkin numbers. When we allow two possible colors for the level step, the resulting paths, which we will call 2-colored Motzkin paths, are counted by the Catalan numbers with generating function \( c^2(z) \). In fact, the \((n, k)\)th entry of the Bell subgroup Riordan matrix \( B = (c^2(z), zc^2(z)) \), from Example 9 is the number of these 2-colored Motzkin paths of length \( n \) having terminal height \( k \). Recall that a formation rule for \((c^2(z), zc^2(z))\) is \([2, 1, 1, 2, 1]\). Another lattice path interpretation of the array \((c^2(z), zc^2(z))\) is the number of Dyck paths of length \( 2n + 1 \) that go from \((0, 0)\) to \((2n+1, 2k+1)\).

On the other hand, ternary paths of length \( 3n + 1 \) that go from \((0, 0)\) to \((3n+1, 3k+1)\) are counted by the Bell subgroup matrix \((T^3(z), zT^3(z))\) where \( T(z) \) is the GF for the ternary numbers. The formation rule for \((T^3(z), zT^3(z))\) is \([3, 3, 1; 1, 3, 3, 1]\). Given this formation rule, it is not hard to see that the \((n, k)\)th entry of \((T^3(z), zT^3(z))\) also counts paths from \((0, 0)\) to \((n, k)\) that never go below the \(x\)-axis and use the following 8 types of steps: \( U(1, 1), L_1(1, 0), L_2(1, 0), L_3(1, 0), D_1(1, -1), D_2(1, -1), D_3(1, -1) \). There is a natural bijection between these paths and the aforementioned ternary paths.

More generally, the Bell subgroup matrix whose formation rule is determined by the \( t \)-th row of Pascal’s triangle will count lattice paths of length \( tn + 1 \) that go from \((0, 0)\) to \((tn + 1, tk + 1)\) and use up steps of the form \((1, 1)\) and down steps of the form \((1, -t + 1)\) that never go below the \(x\)-axis. And furthermore, this same matrix will count paths from \((0, 0)\) to \((n, k)\) that never go below the \(x\)-axis and use \( 2^t \) possible steps, each of the form \( L_t(1, -k+1) \) where \( k = 0, 1, \ldots, t \) for some \( i \) between 1 and \( \binom{t}{i} \). Given that there are natural bijections between these two different path interpretations, it would be interesting to explore through the Riordan matrix method how certain statistics may translate between these lattice path interpretations and how the resulting statistical distributions compare.

As a final illustration of the vast connections between lattice path enumeration and Riordan matrices, we present two open problems related to higher-dimensional lattice paths and generalized Riordan arrays.

Consider the generalized Riordan matrix
\[
R_t^* = \left( ((1 - z) / (1 - z s(z)))^t s(z), z s(z) \right)
\]
where \( s(z) \) is the RNA GF. Note that \( R^* \) and \( R_t^* \) are special cases where \( t = 0 \) and \( t = 1 \), and for \( t = 2 \), we have
\[
R_2^* = \left( (1 - 2z + z^2) / (1 - 3z + z^2), z s(z) \right).
\]

Interesting open problems are to find lattice path interpretations of the generalized Riordan matrix \( R_t^* \) for \( t \geq 2 \). In Figure 4, there is a nice example of an infinite two-dimensional array made up of products of Riordan matrices that involve Pascal’s triangle \( P \) and the “aerated” Aigner-Catalan matrix \( C_0 = (c(z^2), zc(z^2)) \). The infinite array denoted by \( E_{t,\nu} = P^t C_0 E_\nu \) constructed in [N2] is a triple product of Riordan matrices where \( E_\nu = \left( 1 / (1 - z)^\nu, z \right) \). Some
Figure 4. Each entry is a product of Riordan matrices $\ell_{t,v} = P^t C_0 E^v$, where $P$ is Pascal’s triangle, $C_0$ is a special Shapiro-Catalan array and $E$ is the lower-triangular array whose entries equal 1.

interesting matrices that appear in $\ell_{t,v}$ are the Pascal, Catalan, Motzkin, hexagonal (Hex), and directed animal arrays. Solutions of certain higher-dimensional lattice path counting problems are known for the first two columns of this infinite two-dimensional array, but less is known about the remaining columns. Thus, $\ell_{t,v}$ is of combinatorial interest. See [N2] for more information on Riordan matrices and generalized lattice paths. $\ell_{t,v}$ is the Riordan matrix

$$\ell_{t,v} = \left( k_1(z)/(1-zk_1(z))^v, zk_1(z) \right)$$

where

$$k_1(z) = \frac{(1-tz - \sqrt{1-2tz + (t^2 - 4)z^2})/2z^2}{1/(1-tz)c((z/(1-tz))^2)}$$

and $c(z)$ is the Catalan GF. Thus, $\ell_{t,v}$ is a generalized Catalan Riordan matrix. See Figure 4 for the first few entries of $\ell_{t,v}$. Surprisingly, by moving down the leftmost column of $\ell_{t,v}$, a solution of Sands’s problem is found in [N2]. In addition, Sands’s problem was extended to higher dimensions in $\mathbb{Z}^d$ for $d > 3$ where there are countably many step directions and the paths never pass below the $(d-1)$th hyperplane $x_1 + \cdots + x_{d-1} = 0$. The height of each higher-dimensional path corresponds to the value $x_d$ of the endpoint $(x_1, \ldots, x_d)$ of the path. The step sets of the higher-dimensional paths are defined in [N2]. Another subclass of paths called generalized (or partial-t) Motzkin walks are also counted by the entries of the leftmost column. In the same paper [N2], higher-dimensional path results were also obtained by moving down the first column of $\ell_{t,v}$. By moving down the first column, surprisingly, a more restrictive subset of Sands type paths called power walks are obtained. These paths are generalized to higher dimensions and a Motzkin analog is given [N2]. An interesting open problem is to find a higher-dimensional lattice path interpretation of the matrix entries $P^t C_0 E^2$ of the second column of $\ell_{t,v}$. There is not much known about $\ell_{t,v}$ for column entries for $v \geq 2$.

ACKNOWLEDGMENT. The authors would like to thank the associate editor and anonymous referees for useful comments that improved this article.

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The Early Career Section provides information and suggestions for early career mathematicians and those who mentor them. Angela Gibney serves as the editor of this section with assistance from Early Career Intern Katie Storey. In this issue we are printing excerpts from the new soon-to-be-published book Justice Through the Lens of Calculus. Next month we will feature articles in celebration of Women’s History Month.

From Justice Through the Lens of Calculus

Building an Evolving Framework: A Clarion Call / Manifesto

Gregory V. Larnell

A volume such as this one represents both an abiding concern and an urgent call to the field: No longer is it possible (if ever, at least not in good conscience) to look beyond longstanding, abysmal, systemic patterns of exclusion attributable to the curricular trajectory to and through calculus—patterns that stubbornly, chronically, yet persistently

Gregory V. Larnell is an associate professor in the College of Education of the University of Illinois, Chicago. His email address is glarnell@uic.edu.

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The idea for creating this volume emerged from an NSF-funded research project entitled “Progress through Calculus” (DUE I-USE #1430540). The focus of the Progress through Calculus (PtC) project was to examine the Precalculus through Calculus II course sequence and associated supports in United States (U.S.) colleges and universities. The project consisted of two phases of data collection and analysis. In the first phase, a national census survey was sent to mathematics departments offering a graduate degree (master’s or Ph.D.) in mathematics. The survey gathered information to better understand the characteristics of successful calculus programs. In the second phase of the project, in-depth longitudinal case studies were conducted at 12 colleges and universities. The case studies investigated models of the Precalculus through Calculus II sequence, their implementations, and their impact on student outcomes. More details about the entire project are located at: https://maa.org/ptc.

As a result of this research, Editors Hagman, Voigt, and Gehrtz formed a thematic research team examining issues of diversity, equity, and inclusion (DEI) across the national census survey data and the 12 case study universities. It became clear that while many of the members of mathematics departments valued issues of DEI, most did not yet have actionable ideas or strategies for addressing these ideas locally within their departments. Additionally, many departments pointed towards broader university-wide programs for addressing issues of DEI, with only a select few having local initiatives within the purview of the mathematics department.

Matthew Voigt is an assistant professor in the Department of Engineering & Science Education at Clemson University. His email address is mkvoigt@clemson.edu.

Rachel Levy is a professor of mathematics at the North Carolina State University. Her email address is rlevy@ncsu.edu.

Jess Ellis Hagman is an associate professor of mathematics at the Colorado State University, Fort Collins. Her email address is jess.ellis@colostate.edu.

Jessica Gehrtz is an assistant professor of mathematics at the University of Texas at San Antonio. Her email address is Jessica.gehrtz@utsa.edu.

Brea Ratliff is the immediate past-president of the Benjamin Banneker Association, Inc. (BBA). Her email address is bcr0028@auburn.edu.

Nathan Alexander is an assistant professor in the Department of Mathematics at Morehouse College. His email address is professornaite@gmail.com.

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As such, the Editors saw a clear need to gather a multitude of ideas, works in progress, and creative solutions to systematically and centrally address topics of diversity, equity, and inclusion in mathematics programs. Through generous support from the National Science Foundation, we were able to address this need through the development and creation of a resource attending to DEI in Calculus programs that could be made available to the broader mathematics community. This volume began with a call for individual case studies (see Figure 1) from math departments attending to DEI issues, which became the 30 case studies of this volume. The case study submissions were generally written by mathematics faculty engaged in teaching and administering the Calculus programs.

Upon analysis of the case studies, we identified salient topics within the submissions and invited Mathematics Educators to author cross-cutting thematic chapters. We asked this group of thematic authors to read the case studies, situate them in extant literature, and create a future vision for a more diverse, equitable, inclusive, and justice-oriented field of mathematics. The thematic authors also pose questions to the readers of this volume to allow for further exploration and insights into local contexts. In addition to the cross-cutting thematic authors, we recruited an author team to bring in student voices and another author team to share relevant data to problematize how we measure DEI efforts and to situate this volume in its historical period. The authors and Editors met several times to choose and discuss the cross-cutting theme chapters of interest that were relevant to the ideas discussed in the case studies.

We want to acknowledge our own struggles as we curated a collection of voices, the concurrent privilege and burden of doing this work, and all the power dynamics that can come into play. Our goal was mindful, respectful, and collaborative work. At the same time, during the development of this volume, we were challenged to think about our positionality and how to ensure the presence of diverse voices within this volume. Even with the best intentions of promoting diversity, equity, and inclusion, our push to develop the volume needed to be balanced and paused at times to ensure that multiple perspectives were being given space. As such, we began a re-envisioning of our project. We altered authorship and editorial teams, included student voices in authorship, and explicitly reframed and changed how we presented the case studies and chapters in this book to promote an anti-deficit framing.

Our work on this volume occurred during some extraordinary moments of national focus on race, violence, disinformation, and the disruption caused by a global pandemic. Many of us experienced personal trauma and loss while we were in the process of creating this work. The Editors feel grateful that this community has been a source of hope, support, and friendship. We hope that you will find ideas, solace, and discomfort in this book as you engage on a journey of supporting issues of diversity, equity, and inclusion in your own context.

Acknowledgments and Dedication
The Editors would like to acknowledge support from the National Science Foundation (DUE-1432381), the Mathematical Association of America, and the American Mathematical Society. In addition, we would like to thank the MAA Notes Volume Editor and Reviewers, the Progress through Calculus research team, and the case study and chapter author teams. We would also like to thank each other as well as our families, friends, and colleagues who have walked this road with us. A special thank you to Kiara Edwards for her expertise in preparing the grant supplement, Victoria Barron for her support in updating references and citations, and Destinee Cooper for her assistance in formatting and reviewing the volume.

We dedicate this volume to anyone who has received the message that mathematics was not for them. We dedicate this volume to anyone who felt they did not belong in

Scope: The MAA Notes Volume on Diverse Equitable and Inclusive (DEI) Issues in Calculus Programs encourages a broad array of submissions that highlight issues of DEI in introductory mathematics programs with special attention to precalculus, differential calculus, and integral calculus and surrounding departmental programs to support students in these courses. For this volume we are soliciting “illustrative case studies” that showcase ways in which departments and instructors are attending to promoting diverse introductory mathematics programs, achieving, or monitoring equitable student outcomes and experiences, and promoting inclusive teaching practices. Achieving and promoting DEI issues in introductory mathematics programs is not an easy undertaking so we encourage submissions of models in progress, discussions of potential obstacles, challenges, and what departments/instructors have done to overcome barriers to address these issues. We also encourage collaborations between mathematics department members and people outside the department involved in programs, such as individuals in administrative positions or working with student support centers. Submissions might address (but are not limited to) the following topics: a) How coordination can support fairness and also justness b) How placement procedures can value multiple ways to demonstrate readiness c) How professional development (for faculty, instructors, and GTAs) can address DEI d) How changes to the curriculum respond to an increasingly diverse student population e) How departments/universities collect and use data to inform changes related to DEI f) How centers or programs operate to support inclusion and student success in STEM.

Figure 1. Call for Case Studies included in this volume.
mathematics. And finally, we dedicate this volume to you, the reader, for taking the time and energy to engage with these issues.

CALCULUS: Crossing the Bridge to Success in STEM

Elaine A. Terry

Introduction

A private Jesuit Catholic university, Saint Joseph’s is located in Lower Merion and Philadelphia counties. Founded in 1851 as Saint Joseph’s College for men today the university is a coeducational institution with a student population of approximately 8,300 including undergraduate (day and evening), graduate and doctoral degree students. The undergraduate day program is approximately 77% white and 14% from an underrepresented group which includes African-American and Hispanic (non-Black). In any one academic year more than one hundred first-year students declare a STEM (natural sciences, mathematics, computer science) major. Many first-year STEM students experience difficulty handling the challenges of taking a college mathematics course (pre-calculus or calculus) and one or two lab science courses simultaneously. By the end of their freshman year some of them have made the decision to switch to a non-STEM major. Those that remain in STEM have the false belief that low and even failing grades will not prevent them from entering medical, professional or graduate school. While these are issues that may affect all STEM students, it is especially a difficulty for underrepresented students, who represent a small but significant number of STEM majors. Studies suggest that with early intervention underrepresented students can successfully complete an undergraduate STEM program. CALCULUS: Crossing the Bridge to Success in STEM (CB-STEM) was developed in order to equip first-year underrepresented students with the tools necessary for success in a STEM major. CB-STEM is a four-week summer pre-college non-residential program. It is the first intervention program at the University to address the gap in STEM education specifically for underrepresented (African-American, Hispanic (non-Black) and First-Generation) students at the pre-freshman level. As a multi-faceted program, the primary goal is to provide incoming underrepresented students with the tools and resources that are beneficial for their first year in college as a STEM major.

To be considered for the program, students must meet the following criteria:

- Be admitted as a full-time student to the University.
- Be a first-time incoming college freshman with a declared STEM major.
- Be classified as a member of an underrepresented group which includes African-American, Hispanic, and First-Generation college students.
- Have completed pre-calculus or calculus in high school.

Students that are accepted into the program are expected to participate in academic and informational workshop classes. There is no cost to students to participate in the program.

The primary objectives of CB-STEM are:

- To increase participants’ chances of passing their first college calculus course.
- To introduce students to: lecture format, classroom technology, laboratory class, faculty expectations, and college-level exams.
- To help students gain an understanding of good study habits, techniques, and skills.
- To introduce students to STEM faculty.
- To expose students to available academic resources at the University.
- To help students to connect with and build community with other first-year STEM students.

There are three academic workshops that are designed to make students aware of the rigorous requirements of the Saint Joseph’s STEM major. The foundation of the program is the mathematics workshop, which uses Previews to Calculus worksheets to help teach and reinforce early calculus concepts including the limit and the derivative. More information about the mathematics workshop as well as a brief overview of the biology and chemistry workshops follow. A fourth seminar is designed to give students information that will be beneficial to them socially, academically, and professionally.

Natural Science and Informational Workshops

The three one-week academic workshops introduce students to the rigorous course expectations of the Saint Joseph’s STEM curriculum. The workshops were organized in conjunction with two University professors from the departments of biology and chemistry. The three of us met to discuss the program and agree upon its structure. Each workshop met for sixty-minutes for four consecutive days. We discussed the difficulties that many students have with these courses as first-year STEM students. It was agreed that students would benefit by learning about course expectations, how to take notes, homework, and write lab reports. Students were also given information about studying for tests and test taking strategies. We all cited the lack of these skills as the primary reasons for lack of success as a first-year STEM student.

A fourth workshop is conducted by a University administrator who has experience working with underrepresented
student populations. These are fifty-minute workshops that meet at least four times throughout the duration of the program. The aim of the workshop is to empower students with knowledge, skills and self-awareness that is necessary for success as a first year undergraduate STEM student. Topics of discussion are geared towards achieving success in the classroom and the importance of finding opportunities at the university that are relevant to STEM education. In addition, students are given information that will aid in the transition to college by helping them to find opportunities to become immersed in the university community as well as preparation for professional and graduate school and career opportunities.

**The Mathematics Workshop**

By the first day of the CB-STEM program all of the students have taken the University mathematics placement test. Because they have declared a STEM major, they have either placed into pre-calculus or calculus. In most cases, the students who placed into pre-calculus have not had calculus in high school. Those that place into calculus have a choice of two different University calculus courses based upon their declared major. The CB-STEM mathematics seminar exposes the pre-calculus students to the early concepts and language of calculus including limit, tangent line, derivative, and area below a curve. For the calculus students, the workshop serves as a review of calculus that gives them a deeper understanding of calculus concepts besides the usual drill and practice problems that many of them are accustomed to from high school calculus. The math workshop meets every day for four weeks for one hour fifteen minutes. Lectures are limited to thirty and no more than forty minutes. Students are required to take notes and encouraged to ask questions.

In order to reinforce the mathematical concepts from the brief lectures students are given *Previews to Calculus* worksheets. The worksheets are intended to challenge students to rethink how mathematics problems are solved. The exercises are written so as to reinforce conceptual understanding from the lectures and to stimulate interest in learning calculus in a less routine manner. There are worksheets that cover topics including the infinitesimal, slopes of graphs, rate of change, and area below a graph. The applications in the worksheets illustrate how calculus can be applied to other disciplines including the natural sciences, business, and economics. Teaching assistants, current undergraduate STEM students, serve as tutors to help students with any problems they may experience while working on the worksheet.

The first worksheet, Preview #1, is presented below. It opens with a brief description of calculus; discussing the meaning of calculus as the study of change and how the tools in calculus are used to describe numerically how something is changing at a given instance. To see this the worksheet begins with a problem that students should be able to complete. They are encouraged to work the problem and write a brief explanation of what the value they obtain means. Average rate of change (ARC) is defined and discussed. Two problems are given that encourage students to use an average rate of change formula that is appropriate for the given problem and to discuss the values obtained. Units must be included with the answer in order to get students to understand the idea of time dependent rate of change.

**Preview #1: What is Calculus**

In short, calculus is the study of change. You might ask: What is meant by change in mathematics? Change of what? The most important concept in calculus involving change is known as the derivative. Derivatives help to compute the rate of change of something; they help to answer the questions: How fast or slow is something changing at a given instant? What is the rate at which something is increasing or decreasing at a given instant?

In this first preview an example is given that you should be able to work through. Following this example, two questions are asked that will illustrate the need for a mathematical tool that models movement, change.

**Example 1.1**

The number \( N \) of rabbits in a colony can be modeled by the polynomial function below:

\[
N = f(t) = 120t - 0.4t^4 + 1000
\]

where \( t \) is the time in months since observing began.

(a) What is the number \( N \) of rabbits in the colony when observing began? Briefly explain what your answer means.

(b) What is the number \( N \) of rabbits in the colony 4 months from when observing began? Does this represent an increase or decrease in the number of rabbits in the colony? Briefly explain what your answer means.

(c) What is the number \( N \) of rabbits in the colony 5 months from when observing began? Does this represent an increase or decrease in the number of rabbits in the colony? Briefly explain what your answer means.

**Definition:** Recall that for a given function \( y = f(t) \), that the average rate of change, \( ARC \), of \( f \) over an interval \([t_1, t_2]\) can be found using a difference quotient as follows:

\[
ARC = \frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]

Use the ARC difference quotient for the following problems.

(d) What is the average rate of change of rabbits from month 3 to month 4? Include units with your answer and briefly explain what your answer means.

(e) What is the average rate of change of rabbits from month 5 to month 6? Include units with your answer and briefly explain what your answer means.
In calculus we would ask the following questions as it relates to the increase/decrease of rabbits in the colony:

Q1: What is the RATE OF CHANGE of the number of rabbits increasing/decreasing in the colony AT 3 months?

Q2: What is the RATE OF CHANGE of the number of rabbits increasing/decreasing in the colony AT 5 months?

The two questions at the end of the worksheet are discussed. What do they mean? How do we find the answer? Students are then asked to complete another worksheet that contains tables with the values that will aid in getting a better idea of rate of change at a given instance. They are then encouraged to review the results that they obtained and to use them to try answering Q1 and Q2. Discussion is held about the difference between average rate of change (ARC) and (instantaneous) rate of change. Again, units for instantaneous rate of change are discussed to reinforce that we are not just looking for the number of rabbits but the rate at which the number of rabbits is changing with respect to a certain time.

Use the rabbit colony function \( N = f(t) = 120t - 0.4t^4 + 1000 \) to complete the following tables with the appropriate values. Use the tables to help answer Q1 and Q2. Approximate to four decimal places.

<table>
<thead>
<tr>
<th>Table 1A</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_n ), months</td>
<td>( N_n ), # of Rabbits</td>
<td>Time Interval</td>
<td>Average Rate of Change</td>
</tr>
<tr>
<td>3.9</td>
<td>3.9999</td>
<td>3.9999, 4</td>
<td>3.9999</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 1B</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_n ), months</td>
<td>( N_n ), # of Rabbits</td>
<td>Time Interval</td>
<td>Average Rate of Change</td>
</tr>
<tr>
<td>4.9</td>
<td>4.9999</td>
<td>4.9999, 5</td>
<td>4.9999</td>
</tr>
</tbody>
</table>

**2019 CALCULUS: Crossing the Bridge to Calculus COHORT**

While we were unable to recruit the 12–14 students that we proposed, we had four very astute students in the first cohort, one male and three females. Of the four, two had taken calculus in high school with the other two having taken pre-calculus. Three of them enrolled in a calculus course as they were declared STEM majors. The psychology major had the option of following the calculus path or taking a different general education math course. She chose to enroll in *The Whole Truth About Whole Numbers*, which is described as a number theory course for non-math majors.

As the table below indicates, from an academic standpoint, the students had a successful first year. However, there is some concern about the physics major due to the drop in GPA.

<table>
<thead>
<tr>
<th>Student</th>
<th>Major</th>
<th>Fall GPA</th>
<th>Fall Math</th>
<th>Sp GPA</th>
<th>Sp Math</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>CS</td>
<td>3.81</td>
<td>Pre-Cal B</td>
<td>3.71</td>
<td>Calc. I A-</td>
</tr>
<tr>
<td>#2</td>
<td>Psych</td>
<td>3.80</td>
<td>Whole Truth A-</td>
<td>3.90</td>
<td>NA</td>
</tr>
<tr>
<td>#3</td>
<td>Physics</td>
<td>3.14</td>
<td>Calc. I B</td>
<td>2.93</td>
<td>Calc. II B-</td>
</tr>
<tr>
<td>#4</td>
<td>Biology</td>
<td>3.93</td>
<td>Calc. I A</td>
<td>3.88</td>
<td>NA</td>
</tr>
</tbody>
</table>

A survey was administered via Survey Monkey. Following are results from some of the questions that were asked.

1. **How helpful was the CB-STEM program to you adjusting to the academic requirements of SJU?** Results: 1 responded a great deal; 2 responded a lot; 1 responded a little

2. **How likely are you to recommend the CB-STEM program to other Freshmen STEM students?** Results: 2 responded very likely and 2 responded likely

3. **How satisfied are you with the STEM major you have chosen and the department in which your major is housed at SJU?** Results: 3 responded satisfied and 1 responded neither satisfied nor dissatisfied

4. **How likely are you to recommend SJU to someone seeking to study a STEM subject?** Results: 3 responded likely and 1 responded neither likely nor unlikely

There are various reasons for the small number of participants in the program. Including the fact that we are unable to provide housing, students tend to work during the summer and/or may have other academic opportunities that are preferred to our program. Also, the program ran during the week of orientation for first-year students. This is a time when students are on campus for registration and other activities. We plan to be more cognizant of these issues in the future when planning. Two students that had been accepted into the program but chose not to participate confided in me that they regret that they did not participate in the program given the difficulties they were experiencing as biology majors. Both students have changed their major.
Conclusion
The primary aim of CB-STEM is to ensure that underrepresented students have the opportunity to successfully complete their degree in a STEM field at Saint Joseph’s University. Beyond helping students feel academically prepared for calculus and a STEM major, CB-STEM also helps students acquire social-emotional skills, identify personal behaviors that may affect their success at the University, and identify and reflect on their goals for the future. Overall CB-STEM is important to creating a sense of belonging, a shared identity and stressing the importance of academic success in STEM.

References

Course Redesign: Pathways Towards Transformation
Geillan Aly

Build a community to help inspire and thrive
The drive towards enhancing diversity is not just about righting sociohistorical wrongs, it is also about calling upon the strength of multiple perspectives to make larger advances towards a goal. We are smarter together and greater strides are made when multiple ideas, experiences, and knowledge bases come together to tackle a problem. A diverse group of individuals unencumbered by bias or asymmetrical power relationships are more creative, innovative, and productive (Smith-Doerr, Alegria, & Sacco, 2017; Wooley, Chabris, Pentland, Hashmi, & Malone, 2010). Thus, faculty looking to revamp their courses should look to others for guidance, advice, and input. Furthermore, if the overall goal is to support nonhegemonic students, faculty would benefit from learning to listen to, work with, and implement the suggestions of others who can provide valuable insights they may not have previously considered.

Build Communities Among Faculty and other Stakeholders
The larger university community can be a strong resource to tap as you endeavor to change your mathematics classes. As many mathematics classes, and in particular Calculus classes, are intended to serve other departments, these departments should be consulted to provide input into specific course objectives, programming, and to help provide real-world applications and problems for students (Kilty et al. (CS 15); Terry (CS 26); Zobitz et al. (CS 30)). Look to your institution’s central resource for teaching to help develop revitalized pedagogical practices or to more objectively evaluate your program (Bennett et al. (CS 3); Canner et al. (CS 6); Chang & Chen (CS 7); Fuller et al. (CS 1)). Aligning with and educating academic advisors on the affordances of specific programs or special sections can increase student enrollment and participation (Mawhinney et al. (CS 18)). Your Office of Institutional Effectiveness and administrative support staff can help assess the effectiveness of policies on a longitudinal basis and potentially raise the profile of related endeavors. This in turn may lead to generating more institutional support and resources (Benken et al. (CS 2); Canner et al. (CS 6); Chang & Chen (CS 7); Oliver et al. (CS 21)). Bringing together a team of stakeholders, instructors, and experts from the mathematics department and the school of education can maximize individual investment and minimize objections to proposed changes (Johnson et al. (CS 14)).

Developing a community of practice strengthens and supports the pedagogical skills of participating mathematics faculty (Fuller et al. (CS 11); Jensen-Vallin et al. (CS 13)). When instructors work collectively, such as through weekly meetings, everyone has the opportunity to contribute to the process and feel invested in the changes. This also gives adjunct professors and instructors the opportunity to share concerns and suggestions, giving voice to those who often teach students but may have limited agency in the structure of their courses. At Lamar University, community interaction has been strengthened through social teas and luncheons with topical themes that are aimed at improving teaching, learning technology, and teaching towards equity (Jensen-Vallin et al. (CS 13)). Participants in California State University Channel Islands’ professional development program gave adjunct professors who felt invisible and voiceless a supportive community in which they could interact and learn (Soto et al. (CS 22)). Having new or inexperienced instructors co-teach allowed instructors to master new teaching techniques and develop camaraderie (Byrne et al. (CS 5)).

When multiple sections of a course are offered, it is more difficult to standardize students’ learning experience. Differences in instructor styles and expectations are certainly expected. However, it is only when these differences result in significant inconsistencies such as rigor or covered content, that students can be subject to an inequitable educational experience (Akin & Viel (CS 1); Chang & Chen (CS 7); Jensen-Vallin et al. (CS 13); Mingus et al. (CS 20)). Inequitable class structures may also result in an inequitable workload among the instructors. For example, one instructor may be favored because students have a better classroom experience with them or because they have lower grading.
expectations. Streamlining and synchronizing course content is an opportunity to make the course more uniform among sections. Coordinating a course can also provide an opportunity to develop a community among faculty if they are a part of the coordination effort by co-creating assessments, rubrics, and coordinating pacing (Mingus et al. (CS 20)). Variation across instructors provides an opportunity to collaborate, share best practices, and present a cohesive course to students. A dynamic calendar tracking course content, which can be edited easily by instructors and contain links to resources associated with each lesson, encourages instructors to coordinate schedules and lessons across different sections in a class (Oliver et al. (CS 21)). Materials specific to fostering equity in the classroom and ways to introduce “just in time” review topics (where prerequisite content is reviewed as needed) can also be highlighted and shared. Creating a repository of materials for instructors allows for more consistency across instructors and is an easy way to reference and share resources (Chang & Chen (CS 7); Johnson et al. (CS 14); Oliver et al. (CS 21)). This system works particularly well for inexperienced instructors or adjuncts who may not have the time to develop their own resources, be familiar with the nuances of the institution’s specific needs and content or meet with faculty or course coordinators in their department.

Regular meetings among instructors can be used to discuss issues of pedagogy, assessment, pacing, design and use of activities, how best to provide effective support for instructors, and share resources (Bennett et al. (CS 3); Byrne et al. (CS 5); Chang & Chen (CS 7); Golden et al. (CS 12); Oliver et al. (CS 21)). These meetings are an opportunity to explore issues of equity including opportunity gaps in classes with multiple sections or differences in instructor grading criteria, and to discuss common readings on inclusive teaching and general strategies around creating a more equitable learning environment. These meetings can also allow instructors to celebrate their successes and support one another through challenges.

**Build community between faculty and students and among students**

There are various types of effective pedagogical practices in mathematics such as complex instruction, project-based learning, inquiry-based learning, and other forms of student-centered instruction. One unifying characteristic in how these practices compare with an instructor-centered lecture is the bilateral nature of the student-instructor relationship. Traditionally, an instructor lectures at students and students’ responses are in passive forms of communication such as written homework, quizzes, and exams. In the aforementioned alternatives, the instructor works with the student to nurture their curiosity and foster the development of their knowledge.

One of the most common changes that was made in the Calculus classes in the above case studies is the inclusion of a program or aspect which recognized the importance of including the students as an active member of the classroom community. Giving students a voice, space, and agency to control their experience is a major component of many redesign endeavors. This may be as “simple” as redeveloping content so that classroom content is presented in a more student-centered manner, or as “radical” as giving students a larger role in building the overall mathematics community.

Student satisfaction and input can provide an indication that change is necessary. At times, students can provide insight into problematic policies or situations. At Duke University, students indicated that the grading scheme was both non-transparent and inequitable because final grades did not accurately reflect each student’s individual knowledge or abilities. This left students feeling demoralized and discouraged from pursuing their intended STEM majors (Akin & Viel (CS 1)). Mentoring students and listening to them can illuminate some of the challenges and misunderstandings that students have which can result in students believing that they are not capable of completing a STEM degree (Terry (CS 26)). Personal interactions provide frank discussions on some of the limitations experienced by students. The resulting summer bridge program was designed to address problems that contributed to BIPOC students abandoning their STEM major. These conversations can initiate significant change. In another aforementioned example, a student was the catalyst for change when faculty realized that some students did not feel welcome in their mathematics class (Stacy (CS 23)). Student feedback can also help divert resources to address students’ actual learning needs rather than simply focusing on mathematical remediation (Golden et al. (CS 12)). The department gave students agency through a bottom-up approach to change.

Messaging is an important component to community building. Emphasizing the creative aspects of mathematical practices can increase the self-efficacy of someone who does not believe they are a “math person” but can nevertheless succeed in mathematics based on their creative abilities (Stacy (CS 23)). Demonstrating students’ worth demonstrates that the students are valued and are a part of the community—a critical factor in STEM attrition (Miller, Williams, & Silberstein, 2019). Small, conscious acts such as learning students’ names and encouraging them to use chat features during online meetings, make students comfortable and can significantly increase their sense of belonging (Oliver et al. (CS 21)). Social events such as holding a tea can foster community (Soto et al. (CS 22)). In one case, students and teaching assistants are invited to a professor’s home (Starbird et al. (CS 24)). Faculty can also help students navigate the hidden curriculum.2

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2The Hidden Curriculum is the concept that students learn more than the formal content when in a classroom. There are unwritten, unofficial, and at times unintended ideas, values, and perspectives that students learn. To further understand the hidden curriculum, see Portelli (1993). To see how the hidden curriculum is manifested in the classroom see Anyon (1980).
Early Career

At the University of Texas at Austin, the students’ ability to succeed is a fundamental belief for instructors. Students in their program are explicitly and repeatedly encouraged to major in mathematics and struggling students are actively supported (Starbird et al. (CS 24)). Professional development around student power dynamics also can help instructors reflect on how subtle messaging can have significant ramifications (Johnson et al. (CS 14)). One such exercise, which helps instructors consider which students they connected with, was designed to demonstrate how intentional and unintentional actions could affect students’ sense of belonging, achievement, and persistence in mathematics. In other cases of professional development, instructors can be made aware of how their actions engage or alienate students (Oliver et al. (CS 21)). If an instructor calls on the same students, they deny others an opportunity to participate, illustrating how actions such as cold-calling should be discouraged.

Students can also feel valued if they are looked to as a source of knowledge and support. Programs that incorporate peer learning in and out of the classroom help build community and show the peer tutors that they are a valued resource (Benken et al. (CS 2); Chang & Chen (CS 7); Fuller et al. (CS 11); Mingus et al. (CS 20)). Peers can also support students’ emotional well-being by acting as a role model and by helping others navigate their institution’s hidden curriculum (Mingus et al. (CS 20); Zobitz et al. (CS 30)). There is also the added benefit that the tutors also gain academically from their assignments (Leung, 2019). The stretch Calculus class at Washington College assigns each student a learning objective from the course. Each student is then responsible for reviewing this objective with the class in the manner of their choice (Stacy (CS 23)). Culturally responsive teaching likewise underscores the personal worth of a student’s knowledge base and experience and makes mathematical content more relevant, accessible, and meaningful (Fuller et al. (CS 11)).

A welcoming environment can take various forms. Recognizing that there is a lack of community in an HSI composed mostly of commuter students can provide insight into the types of small changes which can make significant differences to students’ engagement and achievement (Benken et al. (CS 2); Chang & Chen (CS 7)). Other forms of messaging can be more subtle such as incorporating kindness cues (Soto et al. (CS 22)), cultivating a sense of pride in students’ linguistic and cultural heritage and acknowledging that being bilingual is an asset and a source of strength can be very beneficial (Villalobos et al. (CS 28)). A welcoming environment can also recruit students to support programs. Specific language that lets students know they are included in a support program can be helpful. For example, inclusion based on a “nomination” process indicates that their presence is important and honored. Negatively-toned and deficit-based language about placement or progress should be eliminated in lieu of messaging that highlights benefits and the positive aspects of programs (Deshler et al. (CS 9); Mawhinney et al. (CS 18)).

Having students consistently work in groups can help advance mathematical knowledge while developing a sense of community (Deshler et al. (CS 9); Fuller et al. (CS 11); Mawhinney et al. (CS 18); Mingus et al. (CS 20); Zobitz et al. (CS 30)). Community can be further strengthened when students are assessed as a group and must rely on one another to be successful (Starbird et al. (CS 24)). Fostering community can also be a long-term endeavor such as at the University of West Virginia where one instructor dedicated themselves to teach all four Calculus courses in a sequence to a cohort (Deshler et al. (CS 9)). These faculty also provided small acts of meaningful support such as helping students with registration changes in their first semester, removing challenges faced by new students who are unfamiliar with how to navigate registration systems. Policies which hinder the development of community should also be reconsidered, as was done at Duke University when evaluations determined that the curved grading scheme increased competition among students and decreased student camaraderie and sense of belonging (Akin & Viel (CS 1)).

The mathematics classroom can also be a way to help students feel welcome in the larger campus community. Students can be required to complete assignments where they interact with various services on campus, such as the university writing center and library (Starbird et al. (CS 24)). Such purposeful acts teach students that success comes from taking advantage of all opportunities and available resources.

References


Increasing Inclusion in Large Enrollment, Uniform Math Courses: Instructor Training and Course Assessment

Hanna Bennett, Susan J. Cheng, Paul Kessenich, Elaine Lande, and P. Gavin LaRose

Dimension of Change 1: Training for New Instructors and Ongoing Teaching Support

To support instructors who are new to teaching in our introductory math courses, we run a training program during the week before classes start in the fall term. The overarching goal of this program is to motivate instructors to teach in an interactive, inquiry-focused manner, and to provide them with the experience and tools they need to do this effectively. The program format and topics have evolved over the years, and currently focus on experiential practice. The participants practice and receive feedback on lecturing, experience a model class as students to see how it can promote learning, and practice running this type of class themselves. There are also sessions on the logistical and administrative aspects of teaching. The program has also a strong emphasis on getting buy-in from the participants and on how they can get student buy-in when they go into their classes.

We have added an inclusive teaching workshop to the first day of this program and made complementary changes to the content of the programming in the remainder of the week. The two-hour workshop, run by our campus Center for Research on Learning and Teaching, follows a theatrical skit and discussion that bring out issues of inclusive teaching. The workshop starts by introducing instructors to the idea of teaching inclusively and why it matters, emphasizing that as inclusive teachers they must be deliberate and aware of the impact of students’ identities and systemic inequities. It then provides concrete strategies that they can use to increase transparency about expectations and evaluation criteria, cultivate a sense of academic belonging in their classrooms, and increase the structure of classroom interactions to promote an inclusive classroom community. Changes in the programming in the remainder of the week include use of these strategies in sessions on group work, and a focus on norms and expectations for classrooms and student teams.

Recognizing that developing and refining one’s teaching philosophy takes time, we also provide ongoing support to our instructors. In their first semester teaching, all new instructors receive one or two class visits to provide constructive feedback on their teaching and on teaching inclusively. Additional support is provided by weekly course meetings to discuss course material and the logistics of teaching it, and in our course preceding calculus we have added to these two follow-up workshop sessions on inclusive teaching for instructors. We also solicit student feedback mid-semester for the instructors to use to improve their teaching.

There are several key lessons from this dimension of change. First, graduate students found activity-based sessions in the new instructor training program (especially those pertaining to group work and inquiry-based learning), and informal discussions with other instructors and with course coordinators, to be particularly valuable. Second, these informal supports continue to be rated as helpful for instructors with more experience. And third, additional mentorship and feedback on teaching, as well as more opportunities to build a peer support community, were the most frequent suggestions for further support. Finally, we note also an ongoing challenge for our program: while many of our instructors are fully invested in the work of teaching (and teaching inclusively), this competes with the academic norm of the preeminence of research. As such, a challenge we face is to shift the mindsets of (some of) our new instructors so that being an effective and inclusive instructor is considered a fundamental part of being a successful mathematician.

This work to help new instructors teach inclusively exists in the context of a wider effort in our department to create a community of instructors who teach inclusively, and to build a consensus in the Department that this is necessary. We have an ongoing Learning Community on Inclusive Teaching (LCIT) that meets on a monthly basis to provide an informal space in which instructors are able to discuss selected readings on inclusive teaching, raise questions they

3The instructional team in charge of this program includes coordinators and graduate student co-coordinators of the introductory courses, undergraduate directors, and some other associated faculty.

4Because this is the course on which we focused initially, these workshop sessions were done for those instructors. We expect that this will expand as we focus also on Calculus I and II.
have about teaching, and recommend things they have
done in their classrooms that have worked well. In many
respects, this provides an existent space in which some
of the informal support that instructors have as they start
to teach can continue, with a specific focus on inclusive
teaching. The LCIT meets over lunch, and for its first two
years of existence has been supported by small university
grants that have allowed us to provide lunch for the partic-
ipants. While participation in the learning community is
voluntary, it accomplishes—for its participants—the goals
of providing a venue for continued exploration and support
of inclusive teaching in the department. It has engaged a
significant number of instructors who teach our introd-
tory courses. Reaching all instructors in the Department and
building a consensus that this work is essential presents
a much bigger problem that requires systemic change at
a larger scale than the courses in which we are currently
working. This is work we must do but is beyond the scope
of the project we describe here.

Dimension of Change 2: Course Assessment
Core to the philosophy and pedagogy of our introductory
courses is an emphasis on conceptual understanding, learn-
ing as a growth process, and a small class, active-learning
instructional model that has been shown to be supportive
of all students in mathematics. To better align the primary
assessment in our course preceding calculus with this set
of core principles, we have developed and pilot-tested a
new assessment structure for the course that includes a
significant mastery learning component. This replaces
a model in which 95% of students’ course averages was
generated by three timed (and necessarily high-stakes)
exams and a scaled grade distribution, which has been
shown to put women and underrepresented minorities in
STEM at a disadvantage (Madeaus & Clarke, 2001; Piontek,
2008). By contrast, a repeatable, low-stakes assessment that
allows students to build understanding and ownership of
the course material addresses many of the factors that have
been shown to drive these students away from STEM fields
(Seymour & Hewitt, 1997).

In the new assessment model, 50% of the points deter-
mining students’ grades are earned on repeatable mastery
assessments, and a further 10% from work done in class
and class assignments. The remaining 40% is derived from
two timed but much lower stakes exams that are shorter and
significantly less formidable than the previous three. This
model has two key features. First, it constitutes a compre-
"hensive revision to the assessment in the course, in which
the components are realigned for consistency with the
course, other assessments, and course goals. And second,
it aligns the grading structure of the course with the core
goal that students learn in a growth mindset (Dweck, 2006)
manner, which aligns with our core course goals.

The development of the mastery assessments followed
a model characteristic of such work: we determined first a
comprehensive list of learning objectives for the course, and
then created a set of repeatable assessment problems with
randomized parameters that allow us to determine whether
students have mastered each. These are grouped into eight
assessments, each having five questions, which we admin-
ister using our on-line homework and testing system (WeB-
WoK, n.d.). Because our goal is for students to master the
learning objectives, we require that students answer four or
five of the five questions correctly to receive partial or full
credit for the assessment. In total, students can earn 35% of
the points in the course by completing these assessments.
At the end of the semester, we have a final mastery assessment
which includes topics from all preceding assessments and
is ten questions long. These scores contribute an additional
15% of the points towards students’ course grades. For all
mastery assessments, students can practice as often as they
like online, and receive credit by obtaining a high enough
score when taking it in a proctored lab. The lab is open 40
hours a week and students have approximately two weeks
to complete each mastery assessment. We have also allowed
students to choose two assessments to reopen at the end
of the semester.

To complement the use of the mastery assessments, we
made two additional changes to the course. First is an in-
crease in the assessment weight of the work that students
do in and for their class section. In the past, students’ work
on the web homework for the course was the last 5% of
the credit in their course grade. Now, 10% is determined by
their work on the web homework, quizzes in their section,
team homework done with other students. The team
work homework requires the solution of significant, conceptual
problems, which students submit in carefully written solu-
tion papers describing their work and the mathematics they
used. Both the problem work and the solution paper are
produced collaboratively by the team. The final change to
the course assessment is, of course, in the timed exams. We
changed from two midterms and a final to two exams, one a
midterm and the other a non-comprehensive exam toward the end of the semester. While the exams were in the past 8–12 problems long, with a 90-minute (for midterms) or 120-minute (for the final) time limit, the exams are now about 6 problems long, with a 90 minute time limit. The exams also focus specifically on the learning objectives that we are unable to evaluate easily on the mastery assessments: higher-order problem solving, graph sketching, and the description of mathematics in written form.

The grading scheme for the course then sets thresholds for each letter grade (80 points for an A, 65 for a B, etc.) and a minimum number of points obtained on the mastery assessments (30 for an A or B, 24 for a C, etc.) and the final mastery assessment (9/10 for an A or B, 8/10 for a C, etc.).

Overall, this change in the course appears to have worked very well, even with the disruption caused by the onset of the COVID-19 pandemic midway through the first semester in which it was implemented. The pandemic necessitated an abrupt shift to online learning (with a corresponding and dramatic increase in the support needed by both instructors and students), which undermined our assessment plans for the project. The change to remote learning also makes it difficult to draw quantitative comparisons between semesters with and without mastery assessment: not only did the instructional mode for the course change, but the change and the pandemic itself have had a disproportionately negative impact on many of the populations that are underrepresented in STEM. That said, we have qualitative data that suggest that the changes have had a significant positive impact.

Key among these qualitative measures is that all students in the pilot who put in sufficient work to persist to the end of the semester passed the course. With the previous, exam-centered, assessment we have not been able to make this claim about any semester (even in the absence of a pandemic and its impact on the course). A related feature of the new assessment model is that now students have much better information, from a range of assessments making up almost half the credit in the course, to use when they are making the decision to continue in or to drop the course. In the previous format many students would either drop the course or give up after receiving a low grade on the first midterm, while others would stick with the course because the first midterm was only 25% of their grade—even though from the instructor’s perspective it was unlikely that the student could succeed.

The new assessment structure also is better able to deal with exceptions necessitated by student circumstances or other factors external to the course: if a student is unable to complete a mastery assessment because of illness or other factors, it is straightforward to reopen it for them. This is not the case when the preponderance of the assessment is by synchronous, timed exams. The change has an indirect impact on how equitable the course assessment is, in that it makes it easier to create a course policy that is responsive to all students’ needs and external constraints.

Anecdotally, instructors reported far fewer complaints from students about the exams, and students—while feeling at times overwhelmed by the work demanded by the mastery assessments—were strongly in favor of the change. Instructors also reported that students gained confidence as they passed mastery assessments. The feeling of accomplishment, that they could do the course work, and that their performance was in their control were all positive changes in student attitude. There is evidence supporting these anecdotal reports: students’ persistence, as measured by the number of attempts that they made on each of the assessments, continued at a high level throughout the semester (the average number of attempts per student on the last four assessments was slightly higher than that for the first four). This change is important: when faced with three exams some students felt that their grade was not in their control but would instead depend on what the exam looked like or how the grading scale turned out. This premise carries with it significant negative implications for student learning: for example, rather than truly figuring out a part of a difficult topic, students would either hope it did not come up on the exam or hope that others would also struggle with it so that they could get by on the grading scale. Or they would guess or omit the solution to one part of the topic and bank on losing at most a few points. By contrast, with mastery assessments, students aiming for an A or B in the course can’t just skip over a topic, because of the minimum mastery point requirements for the different letter grades. Further, students can explicitly identify the ideas they were struggling with and continue to work on them until they attain mastery. This overall evidence of a transition to a growth mindset is another, though indirect, measure of our potential at supporting underrepresented students, as adoption of growth mindsets in classrooms has been shown to have a positive impact on student success, particularly for groups that are traditionally underrepresented (Hill et al., 2010; Sisk et al., 2018).

References
Supporting Underrepresented Minority Students in STEM Through In-Class Peer Tutoring

Tara C. Davis

Description of the Program

Development of the Program

We will begin with the history of the program. Our institution has four concentrations within the math major, one of which is a Math Education concentration. This is more like a pure math major, and does not include an education dual degree or teaching certification, but it does include one required math tutoring class. This class, Math Education Practicum, is open to students from all concentrations, and it focuses on providing students with practical classroom tutoring experience, alongside the investigation of issues of teaching and learning mathematics. Originally, the student tutors worked in developmental math labs. Organizational restructuring and changes in leadership over the years lead to the eventual removal of these developmental math labs. Placements were required for the math tutors enrolled in the Practicum class. This led to the practice of utilizing the tutors during class time, rather than in an outside of class tutoring lab.

There is a second dimension to the development of this program. In addition to requiring placements for the Math Education majors enrolled in the Practicum class, we required employment for our students funded by the Louis Stokes Alliance for Minority Participation (LSAMP) [IOA-LSAMP NSF grant #HRD 1826864]. At the same time as the developmental math labs were being phased out, we were redefining the scope of student work under the LSAMP grant. Students meeting the criteria were eligible to be paid for tutoring work, and these LSAMP students constituted our second pool of potential in-class peer tutors. To qualify for LSAMP, students must be American citizens or permanent residents, maintain a 3.0 GPA, have declared a major in a STEM field, and be an underrepresented minority (African American, Alaskan Native, American Indian, Hispanic or Pacific Islander). The main objective of HPU's LSAMP is to support Native Hawaiian and other Pacific Islander students.

Current Status of the Program

In the current iteration of this program, students are assigned to work with a math faculty mentor as peer tutors in active learning, or inquiry-based math classrooms. It should be noted that on rare occasions there are some in-class peer tutors working at our institution who come from the Math Practicum pool, who are from groups over-represented in STEM (e.g., Japanese males. However, all LSAMP students, and the vast majority of the non-LSAMP students, are women, Hispanic, African American, Native American, or Native Hawaiian). For example, in 2018, the Math Practicum class had five students enrolled, three of whom were Pacific Islander LSAMP students, one was Hispanic, and the other was international. Another semester the class was over 65% Native Hawaiian. The program as it relates to this article will focus on the use of all student tutors, not exclusively on those from any racial or ethnic category, though as noted above, the majority of our peer tutors belong to groups that are historically underrepresented in STEM.

The author, who oversees the peer tutoring program and serves as the LSAMP Campus Coordinator, would send an annual solicitation to the math faculty, looking for those who wish to support students, and who agree to utilize active learning pedagogy (so that the tutors have work to do during class). Those faculty wishing to supervise and mentor a tutor were provided with one. Student tutors were selected through a formal application process for the LSAMP program, and to be eligible to participate, the student must be from a racial or ethnic group the NSF identifies as being underrepresented in STEM. In addition to the requirements for the grant, the student should have earned a high grade in the class they would be tutoring. Students were trained either through the Practicum class, or by their faculty mentor; oftentimes the tutor already had a relationship with their mentor, and had taken the class they would be tutoring so they were familiar with the material and teaching methodology. The tutors were paid an hourly stipend. The author coordinated the employment with the HR and grants office, and placement with a faculty mentor, who managed the tutor’s daily activities and assignments. There are approximately 5–10 student participants in the program per year, where each tutor is able to reach dozens more students through their daily tutoring work.

The tutors worked for several different math instructors, in classes from developmental math and pre-calculus, through mid-level calculus and linear algebra, to upper-level classes like proof writing and abstract algebra. The student tutors met with their faculty mentor prior to the start of the semester to have all expectations and duties...
would also be available to help the small groups. Therefore, students were given individual attention each class period and would have the opportunity to ask questions in their small group to either the instructor or the peer tutor. The students often did ask questions of the tutors, who could then relay them to the instructor, who could reconvene a whole class discussion.

The program serves different math populations by focusing on the needs of the enrolled students and the faculty mentor. A developmental math class, for example, will focus more on students practicing routine problems, and the tutors may be required to help students check their answers, or answer questions in small groups. In a calculus or linear algebra class, the enrolled students may be engaged in a more complex task, which could take a large portion of a class period to complete, and in this scenario the tutor would be expected to facilitate small group discussion by listening to and giving advice on strategy, in addition to helping with basic computations and formulas.

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Benjamin Moynihan, Robin T. Wilson, Joan Wynne, Lee J. McEwan, Mary M. West, Frank E. Davis, Herbert Clemens, Greg Budzban, Edith Aurora Graf, and Aidan Soguero

Benjamin Moynihan is the interim executive director of The Algebra Project. His email address is ben@algebra.org.
Robin T. Wilson is a professor of mathematics at Loyola Marymount University. His email address is robin.wilson@lmu.edu.
Joan Wynne is cofounder of the Bob Moses Research Center for Math Literacy Through Public Education at Florida International University. Her email address is wynnej@fiu.edu.
Lee J. McEwan is a professor emeritus of mathematics at The Ohio State University at Mansfield. His email address is mcewan.1@osu.edu.
Mary M. West is an education researcher and national research coordinator for the Algebra Project. Her email address is marymawest@gmail.com.
Frank E. Davis is an educational researcher and former president of TERC, Inc. His email address is frankedavis@icloud.com.
Herbert Clemens is a professor emeritus of mathematics at The Ohio State University. His email address is clemens.43@osu.edu.
Greg Budzban is a dean and a professor of mathematics emeritus in the College of Arts and Science at Southern Illinois University at Edwardsville. His email address is gbudzba@siue.edu.
Edith Aurora Graf is a senior research scientist at Educational Testing Service. Her email address is aagraf@ets.org.
Aidan Soguero is a communications consultant with The Algebra Project. His email address is Aidansoguero@gmail.com.

Communicated by Notices Associate Editor Asamoah Nkwanta.
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Introduction
In an address titled, “Algebra, The New Civil Right,” presented to the Strengthening Underrepresented Minority Mathematics Achievement Intervention Programs Conference II on November 6, 1993, Bob Moses commented, “I find it ironic that mathematicians should be so centrally placed in a national issue. It seems that history has done you a disservice. It has put to you a task for which you are not prepared. Mathematicians would be the last people that I would turn to to organize the country. And yet, that’s what it seems you have to do.” The essays that follow are by people who took up his challenge—mathematicians, as well as math educators, teachers, and community activists who founded and sustained initiatives that have transformed K–12 mathematics teaching and learning since 1982.

Robert P. “Bob” Moses pushed for mathematics literacy as a civil right for all public school children. He infused this struggle for math literacy with lessons he learned during his pivotal role organizing with Black Mississippians seeking to fulfill their Federal right to vote (1961–1965),

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which interrupted his doctoral studies in Philosophy of Mathematics at Harvard University (1956–1957; 1976–1982). He taught high school mathematics first at the Horace Mann School (1957–1961), then at Samé School for the Ministry of Education in Tanzania (1969–1976), and founded the Algebra Project in 1982. For the rest of his life he engaged with students, teachers, school system administrators, parents, community leaders, university faculty in several disciplines, and national leaders in math education, bringing diverse perspectives together to “raise the floor of math literacy” for America’s most vulnerable students. He taught middle and high school classes as a visiting teacher in Jackson, Mississippi (1993–2006), and supported hundreds of math teachers in districts across the nation, always connecting with members of the community, schools, and universities to codevelop demonstrations of effective mathematics learning.

Bob initiated the Algebra Project in 1982 with a five-year MacArthur Foundation Fellowship recognizing his work in civil rights, education, and philosophy. He later received honorary degrees from 17 colleges and universities, was Principal Investigator on eight NSF awards totaling over $11 million dollars, and received support and honors from other foundations and professional organizations, including election to the American Academy of Arts and Sciences in April, 2021.

His book with Charles Cobb, Radical equations: Civil rights from Mississippi to the Algebra Project [1], chronicles his civil rights work in the 1960s and how this connected with his development of the Algebra Project. Other writings include a book coedited with Theresa Perry, et al., Quality Education as a Constitutional Right: Creating a Grassroots Movement to Transform Public Schools [4]; and a March 2011 article in AMS Notices coauthored with the late Ed Dubinsky, Philosophy, Math Research, Math Ed Research, K–16 Education, and the Civil Rights Movement: A Synthesis [3].

Laura Visser-Maessen’s 2016 biography Robert Parris Moses: A Life in Civil Rights and Leadership at the Grassroots provides a detailed and multifaceted portrait [5].

In a posthumously published opinion article, ‘Returning to ‘Normal’ in Education is Not Good Enough,” Moses wrote, “In the 1960s, voting was our organizing tool to demolish Jim Crow and achieve political impact. Since then, for me, it has been algebra. What’s math got to do with it?—you ask. Everything, I say. Amidst the planet-wide transformation we are undergoing, from industrial to information-age economies and culture, math performance has emerged as a critical measure of equal opportunity.” [2]

From Bob’s address to mathematicians in 1993 to this final opinion article about the importance of mathematics education, he saw an important need for mathematicians to collaborate with others to provide equal opportunity for all.

We believe readers may best explore Moses’s legacy through the voices of his collaborators as they share their experiences and perspectives in the pages that follow.

Janet Jemmott Moses

At the end of 1968, Bob and I were employed by the Tanzanian Ministry of Education to teach in Samé Secondary School in rural Tanzania. It was here that we worked and began our family. Bob taught math, and I English. Although the medium of instruction was Kiswahili in elementary schools, English continued to be the medium of instruction in secondary schools, thereby enabling us to teach there.

Samé Secondary, snuggled in the plains of the Upare mountains, 65 miles south of the majestic and daunting Mount Kilimanjaro and 300 miles north of Dar es Salaam, the unofficial capital of Tanzania and the home of Kiswahili culture that connects the East African coast and the southern swath of India, was a boys boarding school. In response to British colonial education which left the new country with 16 college graduates, and an elementary school system which only could accommodate 13% of its elementary students, the Ministry of Education faced the daunting task of building a system of free, universal education, and of creating a pipeline of Tanzanian teachers.

Armed with Willard Van Orman Quine’s assertion that the language of mathematics gets off the ground on the regimentation of ordinary discourse, Bob set himself on a journey to become fluent in the ordinary discourse of Kiswahili. He remarked that Kiswahili brought a clarity to the operation of $3 \times 2$ that was absent in English. The verbatim translation of the expression $3 \times 2$ in Kiswahili (tatu mara mbili) reads 3 two times. This expression in English reads 2 three times. I think that this insight provided a conceptual basis for what later became the Algebra Project’s first curricular module, with the experiential event that requires students to answer “how much” and “which way” queries embedded in the number line.

In Tanzania we experienced an educational system that out of necessity was created to harness the intellectual and creative talents of its children. This necessity infuses the values of the Algebra Project, which measures its success against a goal post of creating a pipeline of math literate students prepared to engage the economic requirements of first-class citizenship in the 21st century.

Bob loved math. And his standards for his students were as rigorous as the standards of mastery to which he

Janet Jemmott Moses is a former pediatrician in the MIT Medical Department, and the wife of Bob Moses.
held himself accountable. He believed that these standards were the floor of an earned insurgency by students who had been abandoned at the bottom of the well of academic achievement, and that relief of their condition was to be found in the classroom and in the organizing efforts for a quality education in their communities.

David J. Dennis Sr.

I do believe that my relationship with Bob, one of the greatest human beings I have ever met, was destined. I first met Bob in Baton Rouge in 1962. He was heading a project for the Student Nonviolent Coordinating Committee (SNCC) in McComb, MS, and I was heading a project for the Congress of Racial Equality (CORE) in Baton Rouge, LA. Bob and a group of SNCC workers came to Baton Rouge to recruit some of the students from Southern University who had been suspended for participating in demonstrations in downtown Baton Rouge to work with them in McComb. That was my first meeting and it was at a time when I was at a crossroads as to what we should be doing. Bob shared with me his ideas and future plans for Mississippi, which later resulted in me asking CORE to transfer me to Mississippi, which they did in April of 1962. I remained in Mississippi with Bob until we both left in 1965.

After 1965, I did not see Bob again until 1989, when Mississippi veterans of the Civil Rights Movement met in Jackson to protest the movie “Mississippi Burning” which we believed was an inaccurate story of the realities of Mississippi at that time. At the same time, I was undergoing serious psychological issues that were related to my past work in the Civil Rights movement. I was at a loss as to what I should be doing with my life.

During our four days in Jackson, Bob introduced me to the Algebra Project. Bob and the Algebra Project probably saved my life. Bob had to sense my need and he did not let go until I joined him in 1992, promising him three years of my life with the Algebra Project. I am still here. (David Dennis Sr. has been the leader of organizing the Southern Initiative Algebra Project since 1992, assisting the project’s introduction into rural and urban settings of several Southern states including Mississippi and also developing partnerships with several historically Black colleges and universities (HBCUs) including Xavier, Dillard, Virginia State University, and the University of the District of Columbia.)

During my 30 years with the Algebra Project, Bob and I grew closer and our families bonded. We could sense when one needed the other’s touch, smile, and soothing words. He will forever live as long as I live.

I love and miss my Brother.

David J. Dennis Sr. is the founder and director of the Southern Initiative Algebra Project.

Lynne Godfrey

I first met Bob in the early 1980s when he and his wife, Dr. Janet Moses, enrolled their children at the Dr. Martin Luther King, Jr. Open School (King Open) in Cambridge, Massachusetts, a school organized around principles of equity and social justice. When Bob’s oldest daughter, Maisha, was a student in my 5–6th grade classroom, Bob began working with the school to take a closer look at our math data, which shed a harsh light on the school’s practices of tracking and discovered that this was creating segregated math paths. This practice led to disparities in access to conceptually challenging and rigorous math for Black, Latinx, and economically under-resourced students. This was the beginning of a lifelong commitment to learning and activism for myself and many of my colleagues at the King Open School.

For two years, I apprenticed with and observed Bob’s unwavering attention to children’s thinking and the nurturing of their mathematical ideas and identities. Materials for the Algebra Project’s first instructional materials grew out of the work Bob did in the classroom. Bob was as patient and committed to the adults’ learning as he was to the children’s, often meeting with teachers on weekends and hosting evening sessions for parents and caregivers. He dedicated his time to instilling a deeper understanding of the mathematics and the sociopolitical implications of math that serve as its gatekeeper. Bob’s message of math as a civil right has been the organizing force for many educators, young people, and their families. His commitment to a quality math education exemplifies the “NCTM Principles to Actions” [13], ensuring mathematical success for all.

Charles Payne

It is hard for me to think about Bob Moses without smiling. He was one complicated man. I thought of him as something of an older brother, beloved and respected but annoying because he was right too often.

I first met the Moses family in the late 1970s, when I had recently started teaching a course on the Civil Rights Movement. Bob agreed to speak with my class, but only if he could bring his whole family. I spent as much time watching the family interact with one another as I did talking with Bob and Janet about the movement. It was very clear that doing family was, for them, doing the movement.

You could see movement values in the way they talked to each other and to the children. I think that was a major

Lynne Godfrey is an educational math consultant in Boston, Massachusetts. Charles Payne is a professor of African American Studies at Rutgers University.
part of what a great many people got from being around Bob and Janet, a sense that the movement wasn’t something over yonder and back then. It was something that was wherever you were, if you insisted on standing on movement values.

Years later, I found myself in Chicago, helping to establish the Algebra Project there. Two things stand out about the early days of the work. I got to watch Bob and Bill Crombie teach a lesson to parents in a community room in the justly infamous Robert Taylor Homes. Parents needed to appreciate what their children would be experiencing. The lesson started with a chart representing the Red Line, the train line serving Robert Taylor Homes, something the residents knew more about than Bob and Bill did. The graph of the train line became, as discussion went on, a number line, and talking about changing direction on the number line became a discussion of positive and negative numbers, so people found themselves talking about a key algebraic concept in ways that were rooted in experiences they owned.

People can learn things that are thought to be beyond them if you start by respecting what they know and do. The first schools were scattered across the city, but kids at different schools started carrying their books so that the Algebra Project materials were on the outside where they could be seen. They knew perfectly well that they weren’t the kinds of kids who were expected to learn tough subjects like algebra, and they were announcing to the world that it had underestimated them.

Being in the Algebra Project amounted to what Theresa Perry (professor emerita of Africana Studies and Education at Simmons University) would call a counter-narrative. In my own teaching now, I am acutely aware that getting the technical aspects of teaching right is often going to be less important than giving young people tools to fight the labels imposed on them. The Algebra Project got both right.

**Alan Shaw**

In the Spring of 1984, while I was the president of the Black Student Association at Harvard University, Bob Moses was our authentic example of a compassionate, courageous, and conscientious Black intellectual. Bob’s compassion, especially for those steeped in deep poverty and equally deep oppressive political and institutional systems, led him into the Deep South in the early 1960s. Bob spoke to us about his work in the Student Nonviolent Coordinating Committee (SNCC) and Council of Federated Organizations (COFO), and about Freedom Summer and the Mississippi Freedom Democratic Party (MFDP). We learned of his courage as he saw friends die, and as he himself was beaten and shot and faced constant death threats for efforts to register Black people to vote in a country that is supposedly one of the world’s great authentic democracies.

And we were profoundly challenged by his conscientious admonition that those of us with economic, cultural, or political capital should work to build up and empower those who are less fortunate. He explained that he would not be empowering Black sharecroppers in Mississippi if they were dependent on some Harvard educated northerner to lead their political movement. If, on the other hand, in the ethos of a community organizer, he helped to support local leadership, like that of Fannie Lou Hamer in the Mississippi Freedom Democratic Party, then he was truly empowering Black sharecroppers. When Bob finished his talk in the spring of 1984, we were in awe.

Yet, it was not just the work that he had done in the past that was so inspiring. Bob’s work at the time was a new twist on a new type of community organizing, but this time it was inside of the schools. He challenged us, as college students, to join this work of empowerment that focused specifically on our children, but also indirectly on the adults willing to come together and form an empowering communal context to support our children. We spent many hours in classrooms at the King Open School as the first of what Bob would later call College Math Literacy Workers (CMLWs). We also spent countless hours in his living room and the living rooms of other fellow organizers as he did the intellectual work of developing the Algebra Project pedagogy on numerous flipcharts, and as he constructed the “five-step” curricular process itself, that is still central to the Algebra Project curricular process to this day.

Bob was not just bringing a new way of thinking into an elementary school, Bob was bringing a new way of thinking onto a college campus, and it transformed us. One of the fondest memories I have of him is when he heard that I was an amateur gymnast, and he urged me to do a backflip in the cafeteria for the kids at the King Open School, so the kids could use that as a shared event to study how math is involved in everything we do. I did as I was told, and afterward that afternoon we talked with the kids about the conservation of angular momentum and the significance of what happens when the one doing the backflip pulls into a tight tuck. But more than that, because of Bob, Michelle and I got connected to a community of young people that afternoon, who saw how engaging and fun experiential mathematics could be. Our community is in desperate need of more leaders like Bob Moses. He may no longer be with us, but his spirit and his shining light leading the way forward, lives on.

*Alan Shaw is an associate professor of computer science and information systems at Kennesaw State University.*
**Nell Cobb**

I came to the Algebra Project in Chicago in the early 1990s as a veteran secondary mathematics teacher in the Chicago Public Schools. I thought that I was a good mathematics teacher, one who reached students by helping them understand what I was trying to teach from the mathematics textbook.

I really never thought about why the material was organized in a particular way or really how I could teach other than to show students how to solve the problem, do several examples with them, and then have students practice the process, known as the “gradual release strategy.”

Here is how the students who have experienced the Algebra Project Trip Line unit would approach a task that asks them to find a solution to the equation $2x + 5 = 11$. We would interpret the equation in terms of the context of movement on the number line. We start at location $2x$, move 5 units to the right and end at location 11. On the trip line, it looks like Figure 1.

![Figure 1. Trip Line.](image)

**Figure 1.** Trip Line.

A person can provide a context for this problem in ordinary language as “Mary went on a run starting at $2x$ Street which is twice the distance from the center of the city (the origin) as $x$ Street.”

“She runs 5 units to the right and finishes at 11th Street. What is the street name for $2x$ Street? What is the street name for $x$ Street?” A picture could look like the one shown in Figure 1.

From 11th Street if we move 5 units left to determine $2x$ street, we would be at location 6th Street. Since 6th Street is twice the distance from the city center as $x$ street, we find that $x$ has to be 3rd street. Which can be interpreted as in Figure 2.

![Figure 2. Trip Line.](image)

This is an interpretation of the abstract representation, $2x + 5 = 11$, that helps students understand mathematics within a certain context as a result of a shared experience.

During my career I had the privilege to work with college students in mathematics who were placed in remedial classes as well as with preservice secondary and elementary teachers. My teaching changed when I realized that in the Algebra Project one of the first goals was to teach by helping people make sense of the mathematics they study. I had no idea what that meant until I was able to observe Bob work with our Algebra Project Teachers in Chicago. Now this is ingrained in my brain, and I find ways to help people make sense of mathematics and to be open minded about learning in general.

**Florence Fasanelli**

In 1988, Dr. Arnold Strassenburg, Division Director of Teacher Enhancement Programs at the National Science Foundation (and my boss at the time), introduced me to Bob with the goal of exploring whether NSF could fund the nascent Algebra Project. This was not possible at that time, as education was only beginning to gain strength at NSF. However, I then introduced Bob to Dr. Dorothy Strong, Director of Mathematics for Chicago Public Schools. By 1990, with Dorothy’s support, the Algebra Project was underway in Chicago schools and Bob’s work became a larger part of the broader mathematics education community.

Bob’s methods were new to me, but after 1992, when I took a job at the Mathematical Association of America (MAA) designing and directing the Strengthening Underrepresented Minorities in Mathematics (SUMMA) Intervention Program, I experienced the depth of the connection to the Civil Rights Movement. In March 1993, while making a site visit to mathematician Charles Alexander’s project at the University of Mississippi, we visited the Delta Algebra Project in Greenwood, MS. Middle school students hung posters they had made to demonstrate the commutative property by walking back and forth in the neighborhood. We then rode a bus through Greenwood with the students and learned about the Freedom Riders.

I accepted Bob’s invitation to drive me to Jackson to spend the night with his friend, Dave Dennis, the lawyer who directed the Algebra Project initiative in the Mississippi Delta. For the entire drive Bob spoke about his training at Harvard in philosophy and how his professor Willard Van Orman Quine had impressed him with the concept of “regimentation of ordinary discourse.” Quine’s position that the language we use in mathematics is not a natural language came to underscore Bob’s approach to understanding mathematics.

Florence Fasanelli is a retired AAAS mathematician in residence.
learning and teaching mathematics. At meetings in later years, Quine’s book was often held in Bob’s hand. Listening and learning from discussions with Bob changed my approach to teaching teachers and helping others develop projects for funding with the fundamental idea that quality education is a civil right—as is voting.

Luckily, Bob accepted invitations from the SUMMA Intervention Program to bring his methods to a larger group of mathematicians and mathematics educators. In October 1993, Bob outlined his thinking with his usual elegance for the SUMMAC (Consortium) of 90 mathematicians who conducted out-of-school programs for minority students on their campuses. He compared the struggle for literacy and voting as the way to political power with the struggle for mathematical education as the way to economic power. He made it clear that if these listening educators were to be successful, they must organize their communities to assist in bringing the empowerment of mathematics to all. In the lively follow-up, Bob discussed deep issues of philosophy and the epistemology of mathematics contrasting the Frege–Quine position of mathematical existence with Hilbert–Church’s mere consistency of mathematical objects. As one mathematician later reflected: for intervention projects and university enrichment programs to succeed, the leaders must speak the students' language, and then broaden their experience to the power of the generality of mathematics. After this powerful talk, a member of the Mathematical Sciences Education Board commented: “Now I know what it is we have been trying to do.”

Uri Treisman wanted to meet Bob to be able to link his work with the Dana Foundation to Bob’s work. We met in Cambridge on May 18, 1994 where Uri expressed the Foundation’s concern about what might be useful to American education with innovations that touch on social justice. These two MacArthur Fellows sought to share and support each other’s way of thinking about leadership and the foundations of their approaches to learning mathematics. By that time, the Algebra Project was in 110 schools in seven additional states forcing the questions: “What are the values that would underlie a national project, since the Algebra Project is a grassroots structure?”; “How do you have a curriculum that is alive and meets the needs of the customized local version?”; “How do you take on the history and politics of the school system as we use local materials and hook up with local groups?” Several hours later there was enough material to propose a planning conference for expansion across multiple sites.

My own teaching changed. I learned new techniques and shared them with my graduate students at George Washington University, who then used the techniques and shared them with others. While not officially part of the Algebra Project, all of the people using his techniques have assisted in raising the floor for the bottom quartile through Bob.

**John Belcher**

“Students spend much of their time in mathematics class trying to figure out what the teacher is thinking and very little time on their own thinking.” Bob spoke words to this effect early on in my tenure with the Algebra Project. My mathematics education journey up to that point had been circuitous. I graduated from Brown University with a ScB degree in Applied Mathematics. I had come to jokingly say that I applied the mathematics of this degree by being a drummer and studying West African drumming traditions.

It took some years for me to recognize the profoundness of this observation. I have witnessed how students become laser focused on cues from the teacher. Comments such as, “great question” or “wonderful answer” provide insights into what the teacher is looking for—what the teacher is thinking. It contributes to the phenomenon of how, typically, one category of questions dominates students’ participation in conventional mathematics classes—variations of the question, “Am I right?” (e.g., “Am I doing this right? Is this the right answer?”) These manners of privileging the teacher as authority have the unfortunate consequence of disconnecting students from their own thinking to an extent that is not the case in other subject matter classes.

Bob’s focus was upon ensuring that all students, particularly students of color given their histories of denied access, achieve a “floor” of mathematics literacy—an acceptable standard of competency defined by local communities. In [1], Bob cites Ella Baker in providing a frame for understanding “radical” in the context of the Algebra Project’s work: “It means facing a system that does not lend itself to your needs and devising means by which you change that system.” The operative words are “your” and “you,” in that the “target population” must operate with agency—in the case of the Algebra Project’s work, young people of color (and their families and communities) need to make the demand for mathematical literacy and act on those demands in order for the work to be considered successful or radical.

Bob possessed the power of imagination, heightened by his ability to compel others to imagine along with him. Just imagine transformative possibilities that might emerge from investigating mathematics in environments that feel like “home,” in the sense that discourse and other cultural patterns feel familiar and that one’s “whole self” feels welcome. Just imagine producing new, field-expansive mathematics knowledge based upon the ways

*John Belcher is a mathematician and a curriculum developer for the Algebra Project.*
that time, space, pattern, and arrangement are explored and manipulated within African, African-American, and other African Diasporic cultural traditions. Just imagine “a federal civil rights bill for education, akin to the Civil Rights Act of 1957” [2] asserting the constitutional right to a quality education for all Americans. Just imagine…

**Frank E. Davis and Mary M. West**

For the past thirty years, we have had the privilege of conducting research and evaluation on the Algebra Project, usually through grants from the NSF. Although we have heard Moses say “I really don’t know what researchers do” with a mischievous look in his eye, his belief in the importance of research and evaluation as a tool for social change was unquestionable. Moses’s contribution to this work was always visible and substantial, and his innovations were in the forefront of major national shifts in research and evaluation of mathematics education.

For example, his doctoral studies focused on philosophers who sought to develop theories of knowledge as aspects of both individual and collective social experiences. He was drawn to the work of Willard van Orman Quine, who saw the work of science as the development from ordinary language to a more restricted language that provides a clarity and simplicity of theory and supporting evidence. Moses linked this study to his experiences as a mathematics teacher and students’ difficulties learning mathematics, resulting in a pedagogical process called the “five-step” curricular process.

This process was grounded in experiential learning and described several steps for facilitating students’ reconstruction (and construction) of a new language out of their ordinary language. “People talk” was regimented to create “feature talk”—a language of key concepts and symbolic representations developed in the science and mathematics communities of practice. The important first step was engaging in a shared experience that was designed to gain students’ attention and motivate their examination of how we represent objects in pictorial, linguistic, and symbolic forms. This process provided a coherent approach to considering questions about what to teach, how to teach, and what and how to assess in their learning.

Bob’s vision of the work needed to demonstrate this approach led to several major research and evaluation projects, some still continuing. In our studies of interventions in schools we found the Algebra Project was successful when implemented with support of the school and district leadership, teachers, and community members. With classroom support from the Algebra Project’s professional development specialists for a few years, parents’ buy-in, time for teachers to plan together, and ideally with support from local university faculty, students and teachers thrived. Each site was different, so these interventions required work sensitive to local communities and state requirements. But students of middle and high school teachers using project classroom materials and pedagogy who had this support, advanced though Algebra and higher math courses and to high school graduation at higher rates than groups of similar non-Algebra Project students in the same schools or districts.

Research is also continuing in areas of the development of learning progressions and trajectories and more precise understandings of how students develop mathematics concepts such as functions, the development of students’ identities as mathematics learners, and the use of technology in mathematics learning, for example in computational thinking, modeling, and assessments. Another significant contribution of Moses’s was how to propel work on a social problem and agenda that is embedded in the complex social structures of our society. Moses saw a “problem space” requiring a “solution space” that linked together those who lived the problem, who have the necessary insights into its solution and who must participate in its solution (for example, students and teachers in local school communities) with allies who have the necessary expertise and potential to reshape the system of education, to produce a shared vision and collaborations to achieve solutions that propel change.

Bob saw his work as guided by the purpose of creating learning opportunities for students who have been denied the opportunity to achieve the mathematics literacy necessary for 21st century economic and civic participation. Research that is both fundamental and use-inspired is said to be in “Pasteur’s Quadrant” [15]. Pasteur provided us with fundamental knowledge about the origins of human diseases as well as possible treatments, in the service of public health. We think that maybe Moses’s poignant question about “what researchers do” may be linked to his foregrounding the goal of solving a fundamental and use-inspired problem of achieving a quality mathematics education for all.

Over the years, Moses brought researchers together with many others who were working to overcome educational inequity linked to race, ethnicity, and class. In these settings, all of us were required to broaden our professional and personal frameworks on the problem to include many more actors and collaborators. His question “what do researchers do,” while not simply answered, has led to critical and fundamental research that informs a civic goal of equitable educational opportunities.
Jay Gillen

Bob said “It wasn’t radical to do voter registration per se in the 1960s. What was radical was to insist that sharecroppers in Mississippi had the right to vote. Today, it isn’t radical to teach algebra in schools. What is radical is to teach algebra to the students that the country has decided to throw away.”

I want to explain one of the contexts for this statement that I think Bob had in mind and that might not be obvious.

I remember Bob saying that in a certain way the Klan had won, because in the past they were an external force that could be identified and fought, but now they are inside of people’s heads—though we deny it—and so they are harder to identify and harder to confront.

No matter our race or ethnicity, we need to understand the structural obstacles within our own heads. Bob wants us to think about how radical it is to see all Black children as mathematicians. How radical it is to see all Native children as mathematicians. All immigrant children and children of immigrants as mathematicians. It would not have been radical to demand the vote only for a few sharecroppers. It is not radical or even decent to “rescue” a talented tenth from poverty or to extract some bright children from the humiliating circumstances of the schools that we condemn millions to attend.

The unique circumstances of the information age expand the requirements for full citizenship. Each person must be imagined today as capable of doing sophisticated, college-level math, because political and economic power in the 21st century hinges on the ability to make decisions with data, about technology, about systems represented quantitatively and symbolically.

Failing to democratize the vast power of science and math will continue to result in the hardening of caste and will leave a whole segment of the population in near servitude, just as millions are locked today in a caste status defined literally by the iron bars of prisons and by the intricate mazes of punitive state monitoring and supervision. Bob used to say that we have only a short time before the transition to the information age is complete and the requirements for full citizenship. Each person to vote in the 1960s pried open the country’s closed, hooded eyes. As Bob repeated again and again, “If we can do it, then we should.”

Greg Budzban

“What are you working on?”

The question came from Bob Moses. As the department colloquium coordinator for Southern Illinois University, Carbondale, I had invited Moses to campus in April 2001. He had recently published [1], describing his founding of the Algebra Project and its connection to his historic voter registration work in Mississippi. The Algebra Project is dedicated to a national mathematics literacy effort so that underserved minority students can gain the skills necessary to succeed in today’s increasingly technological society. He posed the question over coffee after stopping for lunch on the way to the airport after his visit.

I grabbed a pen and some napkins and began to describe the Road Coloring Problem (RCP).

The RCP first appeared in published form in a 1977 paper of Adler, Goodwyn, and Weiss [9]. Suppose one is given a directed graph and, for simplicity, that each vertex has outdegree two. A road-coloring is a one-to-one assignment of two labels (red and blue, or r and b) to the edges leaving a vertex. Once a graph is labeled, any finite
sequence of labels is an \textit{instruction} and a transformation on the set of vertices of the graph. The RCP focused on identifying the properties of the directed graph that ensured one of the colorings would have a \textit{synchronizing instruction} mapping all vertices to a single vertex. At that time, the problem was still unsolved. It has since been solved by Avraham Trahtman [8].

Moses listened intently and asked if he could take the napkins with him. I dropped him off at the airport and told him what an honor it was to have met him. Two weeks later, I received a message saying he wanted to talk about the RCP and an NSF proposal. When I called, he explained that he had shown his students at Lanier High School in Jackson, Mississippi the RCP and challenged them to find synchronizing instructions. His students began to construct “cities” and search out the different ways to color the roads. They loved “solving the puzzle” and were fascinated by the unsolved nature of the problem. He wanted to know if I had ever considered what high school-level mathematics could be extracted from what he called the “experience.”

I was intrigued and drove to Jackson to see for myself. The students at Lanier are almost exclusively African-American. Each morning they walked through metal detectors in front of uniformed police to enter school. It was the middle of May when I visited, and the summer heat was oppressive. Air-conditioning vents leaked water into rusted trash cans in the hallways. Bob met me and said that he was going to bring his 9th grade students to the library. More than a hundred students came and sat in groups at tables. I watched as they laid large sheets of paper on the tables and placed poker chips on each of the vertices. Spontaneously, a student at a table would say something like, “Try this…” and then say an instruction, “red-red-blue.” As I listened, I heard ideas emerge. They spoke of the “buildings” they were “leaving from” and the buildings they were “going to.” They almost always moved the pieces at the same time, coordinating their actions with their fellow tablemates. I realized what Bob meant by an “experience.” The students were having a physical experience of functions and their compositions.

I noticed that some of the graphs had the property that each vertex also had indegree two. These structures permitted a road-coloring in which both \(r\) and \(b\) were permutations. Such a coloring would never produce a synchronizing instruction. I listened as they struggled with this problem and finally had an idea of what to discuss with them. I went to the front of the library, asked them to stop what they were doing, and on the left side of a sheet of chart paper drew a graph having a road-coloring of two permutations. Next to this I took the same graph and road-colored it in a way that would produce a synchronizing instruction (Figure 3). I asked the students if they could see and explain the difference in the two road-colorings.

A young man near the front raised his hand. “You could never solve the puzzle with the one on the left,” he said confidently. “Why not?” I asked. “Each person goes to a separate building. You never have two people going to the same building, so you’ll never be able to get everyone together.”

Bob stepped forward and asked the student to turn around to the class and repeat what he had said. I later learned how important it was to Bob to nurture students’ ability to articulate their ideas to a group. The student repeated what he had said, and I saw many students nodding their heads.

My life was transformed at that exact moment, and over the next 20 years I worked with Bob Moses and colleagues at the Algebra Project to build curricula based on mathematically rich experiences making important ideas accessible for wide ranges of educational backgrounds. I learned so much from him as a mentor. We have used road-coloring with students as young as 3rd grade, but there are still research-level problems within the experience. For Bob, it was critical to find ways to engage students with real mathematics. As he said, “What’s math got to do with it?—you ask. Everything, I say.”

\textbf{Joan Wynne}

Some might say Bob Moses was a “Renaissance Man.” He was a mathematician, but also a philosopher who studied Camus, Sartre, and Fanon. He loved music, the arts, and literature. As a radical thinker, Bob intuited that Black liberation, ultimately, would be the way to extricate the grass-roots from elitist clutches on those domains and from the perils of second-class citizenship.

Because I embraced literature and social justice, I came to Bob’s work not as a lover of math, but as a devotee of his courageous history in the Civil Rights Movement. Little did I know, then, that I would learn and profess later, that math literacy, in the 21st century, is essential to our children’s economic, legal, and civic survival.
My conversion began while watching Bob, who often stayed in our home, prepare math lessons for teaching in Miami classrooms. He used student reflections on that day’s chart paper to probe how to design new lessons for the next day’s instruction. I marveled at his precise attention to every detail of students’ conceptual conjuring. And watched as he used the same care to devise metaphors for teachers that might inspire them to dig deeper into the granular and the gestalt of mathematics.

Sometimes, as Bob listened to jazz late in the night, he wrote articles using the cadences, and often the lyrics, to explicate a new policy direction for the nation’s Congress; or the consequences of ignoring America’s caste system that denies disenfranchised children equitable structured opportunities for education; or professors’ need to invest time in K–12 classrooms.

During Bob’s seventeen-year partnership with Florida International University, he often pondered renowned authors’ works, especially James Baldwin’s, later writing essays where he used authors’ words to explain the gravity of the nation’s obligation to right the wrongs of its history, especially in education. These deliberations grounded me in a slice of a new Civil Rights Movement, the demand to make math literacy a 21st-century constitutional right.

I share some of Baldwin’s words that capture Bob’s work as a mathematician and as a freedom fighter. In 1964, serendipitously, while Bob was organizing Freedom Summer in Mississippi, to shine the light on the horrors of the killing of “Black Mother’s Sons,” Baldwin was writing these words:

One discovers the light in the darkness, that is what darkness is for; but everything in our lives depends on how we bear the light. It is necessary, while in darkness, to know that there is a light somewhere, to know that in oneself, waiting to be found, there is a light. [11]

Bob Moses refused to let the light go out for the nation’s children. With a dolphin’s grace, he swam into the depths of the darkness, danger, and meanness that surround our children. With generosity and humility, he brought the light—the moral and intellectual light, the light of justice, courage, and liberation to everyone he touched; every community he entered; the youth he deeply listened to and nurtured; the teachers he guided to pursue excellence; the scholars he inspired to take their talents to schools and communities; the institutions he challenged to dismantle the oppressive structures that strain our youth; the American Congress he chided for deliberately denying quality education to African American children; to every crawl space that he dared to refuse him entry, he brought the light.

And now Bob Moses expects us all to carry that light forward, whether mathematicians, artists, nation builders, youth, or elders. Like Baldwin, Bob expects us to never break faith with one another, so the light does not go out, because we are responsible to the generations yet to be born. Bob’s life echoed Baldwin’s words: “The light, the light, one will perish without the light.”

Bob showed us all, even Baldwin, “how one bears the light.”

Marta Civil

I think I met Bob for the first time in November 2004, at a conference on Culturally Responsive Mathematics Education at what was then the NSF headquarters in Arlington, VA. I knew of Bob’s work in very general terms. For example, quite a few years earlier, I remember being in Boston and talking to a friend and colleague about Bob’s idea of using the T in Boston (the subway) to engage students with positive and negative numbers based on their lived experience with the transportation system. While I was not very sure about the details, the message to me was clear: students have lots of everyday experiences that can serve as grounds for mathematical explorations. This approach resonated with my own work in the Funds of Knowledge for Teaching project in Tucson: students have rich mathematical experiences that often go unnoticed because they do not match those provided in school curricula. Bob’s work in the Algebra Project aimed to change this situation.

At the conference, Bob engaged us in a mathematical exploration. I think it had to do with how to make sense of \(a – b = a + (–b)\) [6]. I remember getting caught into a mathematical/philosophical conversation with one of my group members as we worked on this activity, but to be
honest, I also remember wondering "what does this have to do with culturally responsive mathematics education?" Yet I also remember Bob's genuine interest in all the participants' contributions. Here was this well-known civil rights activist and mathematician, someone that many of us had heard about but had not met till then, spending his time with us, listening to our ideas in his characteristic humble and kind way. To me the lasting impact of Bob's legacy is his kindness and his interest and respect for others' ideas while always moving forward towards dismantling social injustices. The way he interacted with others is an example of culturally responsive mathematics education—I see it now!

**Bob Megginson**

I first heard Bob Moses speak in the 1990's at a mathematics conference and later at a session for another mathematics conference. In the second conference, Moses spoke about mathematics education as a civil right, and how the importance of that came through loud and clear in something he had read in that morning's newspaper (which he brought with him, and waved over his head as he spoke). The talk was terrific, and it was immediately after that session that I first actually met him.

Some years later, while I was serving a term as deputy director of the Mathematical Sciences Research Institute in 2002–2004, the Institute established an Educational Advisory Committee. To our delight, Bob agreed to serve on the committee, and was a frequent and active participant in the resulting MSRI Critical Issues in Mathematics Education workshops.

Bob's influence did not end with mathematicians but also expanded to politicians. Congressman Jamie Raskin (D, Maryland) described how Bob's organizing methods inspired his own in his first run for public office, in a primary campaign for the Maryland State Senate, saying

"But how could we compete against a long-term incumbent backed by machine politics, big money, and major organizational endowments?…"

"I had this deep instinct because of a book written by my hero Bob Moses, the philosopher-activist who helped turn the Student Nonviolent Coordinating Committee in the 1960s into an historic force for sweeping change in Mississippi and throughout the South…"

"In his book *Radical equations* (coauthored with Charles Cobb [1]), Moses asks the question 'How do you organize?' His answer is: you bounce a ball.

"You bounce a ball, and some little kids come by to play, and then some bigger kids arrive, and then some high school and college kids, and you begin to talk issues with their parents, and then you organize.

"Bob's bouncing ball and remarkable human-scale organizing led to Freedom Summer; the Mississippi Freedom Democratic Party; the great challenge to Dixiecrat politics at the 1964 Democratic National Convention in Atlantic City, New Jersey; the Civil Rights Act of 1964; the Voting Rights Act of 1965; and the transformation of the Democratic Party.…"

"In our campaign, I told Tommy (Raskin's son), we already had lots of adult volunteers, but we needed to recruit young people, too.

"We needed to bounce a ball." [10]"

Raskin then named his campaign Democracy Summer, and the ball that Bob metaphorically handed him took quite a bounce, since Raskin, against expectations, won that primary and then prevailed in the subsequent election.

Just yesterday, I spoke at the University of Michigan's Honors Convocation, which this year had the theme "ethical leadership." It seemed fitting that I should end my talk with some remarks about my own personal hero:

"We lost an icon of ethical leadership when the great Civil Rights leader Bob Moses passed away last July. I learned a lot about ethical leadership from him, particularly when it seemed like nothing I was doing was making a difference. After all the beating, intimidation, and violence he faced during the Civil Rights Movement of the 1960s, it might seem that he would have every right to throw up his hands and say he'd done enough. But when people would express discouragement despite working hard at what they knew was right, his quiet response would be something like, 'Don't ever think that what we do isn't making a difference. It does.' And then people would go out and keep on making that difference."  

I miss him very much, and will always cherish the difference he made for me.

**Deborah Loewenberg Ball**

In Jewish tradition, when one speaks of a person who has passed, it is customary to say after their name, *zichrono livrachah*. Translated, it means, "May the person's memory be for a blessing." I am always struck by the unusual grammar of the phrase "for a blessing." The grammar implies an important question. What does it mean for a person's memory to be for a blessing?

From the first time I met Bob Moses, I felt his essence as a teacher. He was striking in his capacity to use at one moment, language and speech, and at another, silence, to

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Bob Megginson is an Arthur F. Thurnau Professor of Mathematics at the University of Michigan.

Deborah Loewenberg Ball is a William H. Payne Collegiate Professor of Education at the University of Michigan.
focus others on an essential point. He was deliberate and intentional. He was a teacher who focused not on being the star, or the entertainer, or the director, but on building in others the necessary commitment, focus, desire, and the capacity to push, to insist, and to not settle.

One such moment of seeing him in action as the teacher he was took place about 20 years ago in a large rustic open building, deep in the trees, at the Asilomar Conference Center south of California’s Bay Area, not far from the Pacific Ocean. After supper, as the light faded, Bob had been invited to gather participants who had come together in their commitment to work for equity in mathematics education. He had been asked by the organizers to frame the issues and the work ahead. I remember sitting on wooden chairs, while Bob was sitting, facing us, deeply quiet. When he spoke, he talked about the urgency of raising the floor to put a stop to the way the system fails young people—Black and Brown, poor—and denies them mathematics education. The quiet of his voice only put more power behind his message. The silences required us to think, to take into ourselves and contemplate what he was saying.

Another such moment was when Bob was the commencement speaker and honorary degree recipient at the University of Michigan. One evening while he was in Ann Arbor for this event, I hosted a dinner for the wide range of people who sought to hear and learn from Bob. The large open space that is our main room and the adjoining kitchen and open dining room were packed with people, sitting on the floor with plates of food, standing against walls, perching in pairs or threes on chairs and stools. Bob himself sat on a stool and narrated the trajectory of his work as an organizer and fighter for civil rights, from voting rights to mathematics. He held the room. I have never felt the thickness of human intent listening in quite the same way. The next day, his commencement speech before several thousand people, was just one sentence. It stunned the university president. Commencement speeches are often too long, and sometimes trite. His was neither. He said that his purpose was that we remember what he said, this single imperative: “I want everyone here to accept as a common mission to guarantee quality public education to everyone in America as a matter of right guaranteed by the federal constitution.” In its unadorned power, this single point was what he sought to impress upon every person in that hall that day.

Once again, I saw the power in Bob’s deliberateness. He carried that urgent message in different forms to different groups. I saw him actively recruit others who could contribute their power, their work, their muscle, and their commitment, to this fundamental purpose. For him, the work was always collective, about organizing, about people of all ages and identities, and it was for the young people. He always talked about “young people,” not children, not kids, not kiddos. He respected young people, believed them capable of demanding more, and worked tirelessly to build what he called “the demand side” of the need for quality public education.

For three summers, Bob taught a group of young people here on our campus. During the year, they were in high school in a nearby community, but, as Black learners, their sense of themselves as math learners and more had been distorted by the deficit narratives that swirled around them, narratives that told them that they could not be doers of mathematics, and that they did not need it. Watching Bob, once again, working from his deep convictions about the necessity of organizing for change, insist that the young people think, try, learn, and expect more, demand more. The young people were essential to the push for a quality public education, even as it was the right he believed they should claim. His teaching mirrored what I had seen him do in other settings—a deliberate design for the problems they would explore, silences that pulled them in and got them to talk, to ask, to push back. As a teacher myself, I was moved by the power of his immutable faith in the young people’s possibilities and the insight that they were essential to the work of raising the floor and guaranteeing the right to a quality public education.

Organizing, whether for the sharecroppers’ right to vote or for young people’s right to a mathematics education—organizing depended on teaching. And teaching depended on organizing people to demand, to want, to insist. For Bob, organizing and teaching went hand in hand.

These memories of Bob Moses and so many more have given each of us much to be grateful for. But Bob did not do his work for us to be grateful for him. For Bob, it was not—and must not—be about him. For Bob the teacher, the people were the center, the end goal, the point. And what he has given us from the way he lived and taught are the stuff of learning to do the work of pursuing and ensuring justice.

Memory, and the act of remembering, are special human capacities. Remembering is active. It is also a gift, a resource. We can keep getting to know Bob even now. We thereby make his memory for a blessing. We can commit to actively continue to learn more about what Bob was teaching us and to act in this world on those teachings. We can retrace and remember examples of things Bob said, did, and quietly showed—maybe quietly but incredibly powerfully. Being inspired by his life, his ways of relating to others, his fierce commitments and his steadfast commitment to acting upon them—these provide curriculum for us as we continue the struggle for justice. As we continue to fight for the fundamental human and civil right
for a quality education, Bob has taught us each so much to support our work. We can still learn from him. And in doing so, we can, collectively and individually, make Bob Moses’s memory for a blessing as we move forward.

**Herbert Clemens**

Very occasionally in one’s life, that first encounter with an individual strikes some inner chord that elicits a quiet but powerful “oh yes.” Even more rarely, that initial affirmation does not disappoint, weeks, years, even decades later. So it was in my own life when a selfless example led me to first awakenings about social justice, so it has been with a couple of mathematicians over the years who understood things I wanted to understand, in the way I wanted to understand them, but much better than I ever could, and so it was with Bob Moses.

With an almost Lincolnesque scope and vision, he was able in his soft-spoken way to put the great issues of his day and his people and by inference the rest of us in an ineluctable moral and historical context. He made one feel reverence for the nobility of the concept of a nation “so conceived and so dedicated” and also feel the reality of how cravenly, even criminally, we have often behaved in the avoidance of those noble ideals.

I first heard Bob Moses speak in the early days of the MSRI Critical Issues in Mathematics Education program, and like many others, I was blown away. He made the seemingly unlikely parallel between the plight of those who were denied the vote in the 20th century and those whose circumstances of poverty and marginalization effectively deny them math education in the 21st century. He spoke of “earned insurgency” as a necessity in both settings. Just like in political struggle, when young people set out to learn math, personal struggle, commitment, and trust in the worthiness of the goal are all required.

There is almost certainly no one way to teach mathematics, it is probably at its best when it is attuned to each teacher’s and each student’s individual way of coming to terms with concepts of “quantity” and “space” in their own heads. But there are a couple of things that are universally necessary, namely student trust in the teacher and willingness of both to do the work. We as a profession and our country as a society would probably be a lot further along in the trust and mutual respect department if we tried harder to live and work as Bob Moses did.

As I wrote this, I tried to reflect a bit on what Bob Moses’s example has meant for me in my own political and professional life. I was good at math but not so good that dedicating my professional life exclusively to math would pass my personal “worthiness of the goal” test. Bob Moses’s leadership, over the last couple of decades that I have known him, continually reminded me of the need to do more, to contribute more to others and to society. He said it eloquently with words but he said it even more persuasively with the way he lived, all the way up to his very last days. His unwavering vision and his awe-inspiring personal courage, especially during the years in Mississippi in the 1960s, are worthy of a singular place in the arc of our nation’s history, bending it a little bit more towards justice.

**Michael Nettles**

Bob Moses and Educational Testing Service (ETS) began getting acquainted in the fall of 2011. At the time, Bob was a visiting professor in the Department of African American Studies at Princeton University. Bob’s host at the university, Professor Eddie Glaude, thought that ETS would be an ideal place for Bob to address challenges concerning equity and quality mathematics education, so he introduced Bob to me. Bob and the Algebra Project turned out to be highly productive and beneficial collaborators for ETS throughout the remaining years of Bob’s life.

After his initial visit to ETS, I began calling colleagues who were mathematics, language assessment, and research experts to join our conversation and to listen to Bob as he spoke about the language and social class barriers that needed to be overcome in teaching, learning, and assessment in mathematics in the United States, citing seemingly unlimited examples and bringing to bear his unique, vast experiences.

Bob, the Algebra Project, and ETS found common ground with these challenges, and ETS began realizing the potential of the Algebra Project as a vehicle for connecting with the bottom quartile of learners. Bob came with

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Michael Nettles is senior vice president and the Edmund W. Gordon Chair for Policy Evaluation & Research at ETS.
theories, a rich and deep history of experience, and expertise in mathematics, politics, and human interaction. At ETS, we had much to learn and were eager to collaborate. Our initial project together was a joint NSF grant in 2013–2014 to develop a logic model and theory of action for the Algebra Project. From there, in the Atlanta Public Schools in the school year 2015–2016, we joined the Algebra Project’s teaching of mathematics to cohorts of high school students by producing the first large-scale standardized assessments for selected Algebra Project modules.

**Edith Aurora Graf**

I was introduced to Bob by Michael and his colleague Jon Rochkind at a meeting at ETS. He asked me if I had considered the role of language in mathematics instruction and assessment. I had read some influential literature on the role of vocabulary and language on mathematics assessment performance, and we discussed this a bit. Over the years, I have reflected on Bob’s question and now interpret it much more broadly: I think he was asking me to reflect on the role of language in the development of mathematical thinking—his question was not limited to the role of language in performance on a single assessment event. It has often been the case that Bob’s words had one meaning for me when I first read or heard them, but that they took on a deeper meaning as I learned more about the history of the Algebra Project or the Algebra Project’s “five-step” curricular process.

This short story parallels the experience in working on two NSF-supported projects. In the first project, colleagues at the Algebra Project, ETS, Southern Illinois University Edwardsville, and the Young People’s Project (YPP) collaborated on designing tasks that would assess student understanding with respect to a learning progression for the concept of function. As part of that work, YPP staff conducted focus groups and cognitive interviews to learn more about how students were interpreting the tasks. Per my initial interpretation of Bob’s question, we learned a lot about the role of language and vocabulary in the interpretation of mathematics tasks.

One of my ETS colleagues, Jessica Andrews-Todd, is an expert in collaborative problem-solving. We reached out to her to join us in proposing a second project that would support small-team collaboration on the mathematics tasks. Such work would, we felt, be aligned with the Algebra Project’s approach of encouraging group work. In addition, we proposed that some proportion of the team conversations be facilitated by college math literacy workers (CMLWs) from YPP. And so, our second grant was awarded and has been ongoing for about a year now. We hope that this project can support Bob’s legacy and provide some answers to his broader question about how language and discourse can play a role in mathematics instruction and assessment.

**Julia Aguirre**

It is not every day that you get to talk with someone whose work affirmed and transformed your work as a justice-focused math educator. It was 2005, and I was attending my first MSRI Critical Issues in Mathematics Education conference. Bob Moses was a conference organizer and presenter. He was also scheduled to do a series of talks in the Monterey Bay area. I was a biracial Chicana and assistant professor at UC Santa Cruz, mother of a toddler and an infant, and focused on my work as an equity-scholar in mathematics education. I was fortunate enough to get a poster accepted at the conference and was asked if I would bring Bob to Monterey for his next speaking engagement. This was a 2-hour car ride. What would we talk about?

A couple of years before I had read the book, *Radical Equations* [1] while on a camping trip with my future husband. I knew about the Algebra Project, but the book helped me understand its origins. The community organizing principles applied in the 1960s to register Black sharecroppers who could not read to vote were being applied to teaching disenfranchised young people Algebra, a highly coveted domain of mathematics that was a gatekeeper in American education. I was taken by the metaphor of the “bouncing ball” to attract middle school students to come and learn mathematics, while at the same time arguing to adults in power that mathematics literacy, and algebra specifically, was a civil rights issue of the 21st century. I remember feeling empowered with this because it was an argument that I had previously heard. It provided me with a powerful tool to create counter narratives to the dominant view held by many educators, scholars, politicians, and parents questioning whether all students should or could learn algebra.

There was a profound moment for me at the 2005 conference where I witnessed the power of generational love, that Bob had for young people. As part of the conference, a group of Black and Latinx high school students from a public school whose math department was de-tracking were invited to do a live math class in front of 200 mathematicians, math educators, and teachers. The young people walked in the auditorium and proceeded to the stage area where tables and chairs were set up. As the lesson was about to start, a young Black male student with headphones on, walked slowly down the stairs. Bob Moses

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*Julia Aguirre is a professor and faculty director for the teacher certification program in the school of education at the University of Washington, Tacoma.*
stopped him and asked him to think for a moment, to recognize the implications of his actions, to consider what it meant to his classmates, teacher, and those waiting. Everyone was stunned. Discomfort filled the air as the young man looked apologetically to Bob. Then, his classmates began to applaud, all of them. They recognized what happened, for this was not the first time this student made a late entrance. But this time, someone was holding him accountable for his actions—a valuable lesson. Bob had calmly yet firmly called this child in with generational love of an elder, a grandfather, mentor, and as a Black man. He was the only person in the room who could have done that. It was beautiful.

As the conference ended, I escorted Bob to my car. I don’t remember what we talked about but I remember playing jazz music. I remember watching him stare out the window as we passed the strawberry and artichoke fields of Watsonville, California with farmworkers stooped over in the afternoon sun. He was quiet, listening as I explained that the Central Coast was where organizing for farmworker rights happened under the leadership of Cesar Chavez and Delores Huerta. I wondered what he was thinking at that moment. Were there echoes of Mississippi in the sunlight?

The next day he arrived to a room full of professors, graduate students, undergraduates, and community members eager to hear him talk about mathematics as a civil right. I was asked to introduce him. I remember sitting next to him and starting. As I got to the part of how his work influenced mine, I was overwhelmed with tears. Still to this day, I tear up when I think about it. The tears reflect a profound respect, faith, patience, and perseverance needed for this justice work. He lived through and fought for the right to vote and now he was helping me and future generations learn again that the fight for justice is based in community and love.

I met Bob again at the TODOS Math For All Conference. The board of directors offered him a slot to share his ideas about a national campaign to make quality education a constitutional right with participants and start to get folks organizing. Bob also asked if he could participate in the leadership session on equity and justice in mathematics education that I and my colleagues were facilitating. While I am no stranger to facilitating professional development to mathematics teacher educators and leaders, Bob’s presence was like having the great Ella Fitzgerald coming to watch you sing jazz. It was an honor to have him there and I think we hit the right notes.

Thank you for the lessons and the opportunities. I am forever grateful.

Lee J. McEwan

My first encounter with Bob Moses happened in the evening on a cold day in December 2008. I picked him up at the Cleveland airport and drove him an hour south to a hotel in Mansfield, Ohio. We were both introverts, and Bob initiated most of the conversation. Although I was in my fifties, talking to Bob brought flashbacks of graduate school anxiety. He was exquisitely precise, soft voiced, profoundly serious, and seemed to carefully evaluate everything I said. Over the next decade I got used to the fact that when Bob was with you, he was entirely present and never drifted off in his own thoughts.

When we got to the hotel, I made sure Bob was checked in and prepared to bid goodnight. But Bob started telling me about how his work proceeded. Standing in the cold evening air, I received a half-hour master class in organizing, which began a decades-long chapter in my life. “When a person gets a problem, usually he starts to think about how to solve the problem. When an organizer gets a problem, he thinks about ‘Who can I get to work on this problem?’”

Regarding education in the current century, Bob told me “The mathematicians have been dealt a surprising hand.” He viewed math literacy as a sine qua non of citizenship and access to the economy, and believed that the lack of such literacy disproportionately hurts marginalized communities. In subsequent years, as I supported the implementation of the Algebra Project in a local high school, my ideas about teaching changed. I no longer think in terms of lecturing as a way to teach mathematics to non-majors. “There is no cookie cutter” for curriculum, and Bob preferred that they be shared and adapted to different local settings. My years with Bob Moses convinced me that professional mathematicians need to take on the problem of how to teach mathematics to future teachers, that they might teach students to think mathematically, and not regard mathematics as a collection of algorithms to be mastered.

Erica Walker

I first met Bob Moses when I was leaving Cambridge to join the Program in Mathematics at Teachers College,

Erica Walker is the dean of the Ontario Institute for Studies in Education at the University of Toronto.
Columbia University. Ben Moynihan had a lot to do with it—Ben had been my classmate at the Harvard Graduate School of Education. When Ben found out I was in mathematics education and I figured out that he worked with the great Bob Moses, Ben very kindly introduced me to him. It was a moment for which I am so grateful, and that has inspired much of my work in mathematics education.

The book *Radical Equations* [1], had a profound effect on me, as a daughter of the South. Bob and his coauthor Charlie Cobb captured in words, recommendations, and actions, the feel and the strength of the Black community’s love for education that I had always known in my church, my neighborhood, and my schools in Atlanta, Georgia. Always a teacher, his connecting the work of the Civil Rights Movement and his strategies for educating fellow Americans about their rights to a movement for math literacy and power was groundbreaking, and proof that education in its many forms transcends place and our often limited notions of how and where we learn and what it means to learn. Bob clearly saw people of all ages for their talent and understood that the work of a teacher is to help cultivate, nurture, and channel their dreams and interests for learning. His conceptions of (mathematics) education give freedom to us as educators and researchers to broaden our own views. As someone who approaches research in mathematics education from multiple interdisciplinary perspectives, his thinking and theorization of education has been an inspiration.

Over the years, my path crossed several times with Bob’s and his beloved Algebra Project. He was tremendously kind and always gave you his undivided attention with an engaged intensity for listening to what you had to say. Children especially loved him. Whenever I saw him engage with young people I was reminded of those wonderful times growing up when a neighbor or teacher demonstrated such care and interest in our learning and development. He gave several talks at Teachers College, and he and his team invited me several times to be part of Algebra Project conveings and conversations where I had the chance to meet other incredible colleagues and collaborators. In the summer of 2011, he very kindly wrote a powerful foreword to my first book, *Building Mathematics Learning Communities in Urban High Schools*, and used a lovingly familiar pattern of refrain (in this case, the phrase “Worthy of our attention”) in the traditions of Black storytelling. I love reading it. “Worthy.” It reminds us of the moral imperative of education, something that is too often lost in all of our un-nuanced discourse about schools and teaching and learning, and that advancement often requires challenge and confrontation to make things right. “Worthy.” He reminds us of young people’s power, and the power of their energy and passion and how they must be channeled for educational advancement.

Later that year, I visited Bob when he was Distinguished Visiting Fellow in the Center for African American Studies at Princeton. On a crisp October day, in a warm sunlit office, Bob shared with me his recent work and plans to craft a new movement about quality education as a constitutitional right. We talked and thought and delved into pamphlets and books and papers about what rights mean in this country and Bob’s vision for strategies for this (new) movement. Hours later, I left to take the train back to New York, my mind spinning with new and old ideas, connected in exciting and transformative ways. That always happened when and wherever I saw Bob.

Bob’s work in mathematics education has been tremendously influential, generative, and inspirational to generations of students, teachers, administrators, researchers, and policymakers and will continue to be for generations to come. He was committed to mathematics and making it a better discipline, with access and opportunity for all. He will always be an influential part of the discourse about mathematics education within and beyond schools, pre-K through postsecondary.

These are among the many gifts that Bob has so generously shared with us, rooted in a deep and abiding love of Black people. We can’t forget that, and we must continue to press on. He would expect that, and lovingly demand it.

**Robert Berry**

Bob Moses has influenced my work by pushing me to use mathematics as a tool for social justice to support the development of mathematical literacy among young people. His work took on a social justice stance rooted in the Civil Rights Movement’s legacy. Consequently, as a mathematics educator engaged in social justice and mathematical literacy work, I owe him a debt of gratitude.

Robert Berry is a dean in the college of education at the University of Arizona.
Bob’s work helped me understand that the power of education is in supporting people to think critically about the issues impacting our communities. Critical thinking is at the heart of mathematical literacy. Bob stated, “that the absence of math literacy in urban and rural communities throughout this country is an issue—as urgent as the lack of registered Black voters in Mississippi was in 1961” [1].

I have learned that mathematical literacy provides access to opportunities that would enable people to improve the social and economic conditions of their communities. For example, mathematical literacy helps us understand the economic impact of livable wages, the political influence of gerrymandering, and the social impact of racial profiling. Mathematics literacy underlies much of the fabric of society, such as economic justice, environmental rights, health, immigration, gender equity, racial equity, and civil rights. Critical thinking and mathematics literacy allow people to explain inequities and offer possible solutions, and participate in a democratic society.

In [1], Moses connects civil rights with mathematics by stating, “Algebra is a civil right.” As a civil right, mathematics provides college access, thus a possible way to help people out of poverty. Moses is unique in his perspective because he used his experiences and tactics as a civil rights leader to create the Algebra Project, a culturally responsive curriculum that organizes communities around mathematical literacy.

In Radical Equations, Bob Moses discusses the importance of amplifying the voices of the youth voice and activism through mathematics. He said, “We don’t listen to kids enough. Really listen. It is a difficult thing for grown-ups to do—listen and actually pay attention to what young people are saying. In the Algebra Project, we are still learning how to do this also. It is the voices of young people I hear every day, more than anything, that gives me hope.”

Much of my work has been to create spaces where young people can speak, be heard, and be seen. Therefore, I strongly appreciate Bob’s perspective on listening to and learning from young people. Young people provide insightful perspectives for reading, writing, mathematizing, and critiquing the issues impacting them and the broader communities.

Bob Moses’s legacy is represented in the many mathematics teachers, mathematics educators, mathematicians, and youth leaders whose work sits at the intersection of mathematics, social justice, and civil rights. Many of us believe that one of the civil rights issues of our day is mathematical literacy because it contributes to people’s ability to work and participate in a democratic society. We all are thankful for Bob Moses, who provided us with a framework to engage in our work and a model of excellence in very modest ways. Bob Moses’s legacy lives on through our work.

Robin T. Wilson

I first met Bob Moses in my second year in the doctoral program in Mathematics at UC Davis, when I was having a hard time finding the motivation to finish. In many ways, he came into my life at just the right time. It wasn’t that I doubted I could do it, but I was having some real doubts about why I was pursuing an advanced degree in mathematics and whether this was just to pad my own ego, or if it could be something more—something that could be of service to my community in a meaningful way.

One day I got a call from my mother inviting me to hear Bob speak at a Sacramento City Unified School District teacher Professional Development workshop. After the event, to my delight I ended up giving him a ride to the airport. In the car he invited me to visit Lanier High School in Jackson, Mississippi where he was teaching. I took him up on the offer.

Lanier was the lowest performing high school, in the lowest performing county, in the lowest performing state in terms of the Nation’s math literacy metrics. And at Lanier, Bob worked with the lowest performing students. He felt like those were the students that mattered most, the ones that were the most underserved by the system. I got to watch him in action as a teacher with these students and I saw how much he believed in them, and how unafraid he was to present them with a challenge. And then he passed the reigns to me, and I had to teach these students with him there. I had to find the same deep belief in their ability, and I had to find that same deep belief in myself that teaching these students was something I could do. For Bob, doing mathematics with these students, holding these students to high expectations while holding onto an unshakeable faith in their brilliance, was a radical act. And it was an act of organizing, a political act to be doing mathematics with these students at this high school in this state. I learned so much from him that afternoon.

After that trip, I kept getting invited back to Algebra Project events. In the 2000s, I attended several meetings in Miami for the Algebra Project’s NSF grant to develop high school curricula. There I met other mathematicians like Greg Budzban, Ed Dubinsky, David Henderson, and Staffas Broussard, who were senior to me but had come to the Algebra Project for the same reasons. I met lots of math teachers and students too. In the Algebra Project, we all found something more that our mathematics training could be used for. By teaching mathematics to the students most underserved we could contribute to the movement and the struggle for the future of this nation. Bob helped
us see the role we mathematicians had to play in this struggle and convinced us of its importance. We heard his call and were willing to put our training as mathematicians in service of a different kind of calling.

He showed us that as mathematicians we had an important role to play in the outcome of America’s great democratic experiment. He showed me personally that mathematics was not just about theorems, beauty, and creativity. Bob showed me that, in addition to all of these things, mathematics could be a tool that we can use to organize for social justice. It was a tool that, if given to the children, they could use to continue this fight after we are gone.

Bob knew he would likely not live to see the fruits of his labor, and he was ok with that. It’s really something to live knowing that you will never see the freedom that you are fighting for, but at the same time keep fighting like victory is just around the corner. The Algebra Project is celebrating 40 years of existence, founded in 1982 when Bob won a MacArthur “genius award.” And he felt like we might have 50 years to go from here before we find victory in this movement. There is only one way to prepare for that kind of fight. It’s to turn to the children, to the youth, to prepare them to take up the baton and lead the way. That’s why Bob turned his attention to the youth. And that’s what Bob saw in me, when I was still a young person in my 20s.

Perhaps the most valuable lesson I learned from Bob was that “there could be life worth living in the struggle.” I have never seen the kind of commitment to a cause that Bob had for this movement. It was his life which became a central part of his family’s life as well, and he brought me into that and made me feel like family. He made me feel that I was a part of something bigger than myself. Bigger than the institutions I was involved with. Bigger even, than mathematics itself.

Maisha Moses

When I think about Dad’s work and his legacy, I go back to my high school days, when I became more aware of his history. I remember asking him what the Movement had meant to him. He thought for a second, and said, “Freedom.” And he said it so strongly and so powerfully, it just resonated with me.

In retrospect, I think Dad found himself in the Movement. His decision to go South in 1960 and give his life to the Movement really changed and shaped the arc of his whole life. This was the big theme for him—freedom. As an African child growing up in America, my dad seemed always to be figuring out how to find his place in America. Then later, as a Black man in the 1950s and 1960s, he began discovering how to carve his space in this American society. The Movement was an answer to a very deep searching that he underwent across his childhood and young adulthood.

Much later, he started the Algebra Project (AP) partly because his passion and studies were mathematics, the philosophy of mathematics—he possessed that very natural love—yet his life’s mission was about freedom. So, the Algebra Project became a marriage of those two big themes in his life. He always told my siblings and me that you can live a meaningful life in struggle—a life which included facing the weight of our country’s history. Dad developed this recognition about Black life primarily through the Movement and its elders, and then he kept it alive in his AP work.

Subsequently, Dad taught me that what it takes to make change in this country is an ongoing, enduring, consistent commitment to struggle; discovering how to struggle; knowing who your friends are in the struggle and how to work together; knowing how to name a problem and how to craft a solution to that problem. As I think about the arc of his life and his legacy, those are the broad strokes that sit with me now—AP and the Freedom Struggle.

His AP work began while he, as a parent, was trying to ensure that we got a good education. Because I was tired of doing math with him at home, Dad came into my classroom to teach me math. Then and there, he again saw the ways education functions to sort people—and how it continues to be a deep manifestation of America’s caste system. So, he became my algebra teacher in eighth grade. And because he was teaching me, I became the only Black student in the highest-level math class. We had been sorted into ability groups, and the ability groups just fell along race and class lines. Though this was decades ago, Cambridge public schools and the nation are still struggling with this issue—and it’s very explicit in mathematics education.

Dad’s Movement eyes saw the ways in which Black, Latino, and poor children were not getting access to the highest levels of math education that their school districts were offering others. But he also understood, way back in the 1980s when he was starting the Algebra Project, that math literacy was becoming, on the eve of the 21st Century, as important a literacy as reading and writing. I believe he was ahead of his time because he was one of the earliest people to consider math literacy as a civil right in a 21st Century, knowledge-driven, technology-driven economy.

Moreover, he explained that young people today, who don’t get mathematics education, will be like serfs on a plantation during the first half of the 20th century. When we talk now about the obstacles that this generation is
facing around freedom, access, and equity, I continue reflecting on how Dad recognized very early that math literacy is key to students’ success in gaining first-class citizen rights—most people just weren’t thinking about it in that way.

But most of what I learned from Dad, I learned from spending a long time around him as he worked. And always, it was his patience that struck me. Dad just had a way of being with people, working with and connecting to people where they are, wherever that is. His patience enabled most of us to connect to this big vision, this epic struggle that his life was about—he gave people time to discover how to connect their lives to this broader struggle. The way he worked with a child one-on-one doing math, or with student groups, teachers; or the way he gave talks—I would say that’s a lot of what I carry with me and shapes how I do my work.

His communicating to us his understanding of this movement that we’re in today helped shape the Young People’s Project’s (YPP) philosophy. He insisted that the movement around math literacy requires the people who are most directly affected by the problem, make demands on their own behalf while they struggle and organize to create solutions to the problem. And in education, that’s the children, the young people. And so, he drummed that message into my brother, Omo, who started YPP; Albert Sykes; and his classmates who then were in middle school—insisting that their role, their work, their energy in this math literacy work was critical, even essential.

And, that’s the essence of what the Young People’s Project does. It is figuring out how to create spaces with that intent in mind, that young people must be at the center of making change in education. Obviously, not the only people. It takes everybody, but without the students, it’s not sufficient nor successful.

YPP grew from the Algebra Project. And AP has made a huge contribution to the world of math education, through its curricular process and pedagogy, which takes ordinary events or experiences that people have and goes through a process, the AP “five-step” process, to abstract the mathematics out of those events. When I was training and teaching with AP, so many teachers said, “If I had learned math this way, then I would understand it.” That process was a significant intellectual contribution to the field of math education. And, of course, connecting it to citizenship and broader issues of equity was genius.

So, the Young People’s Project inherited Dad’s thinking about algebra as a right; its approach to grounding math in experiences, games and activities; and using student language to talk about math. Dad insisted that “Mathematics is not a language that anybody speaks.” So, the question becomes how to help children learn to relate their natural language, whatever it is, to the language of mathematics.

YPP applied these concepts to developing young people as teachers of mathematics. We expanded AP’s idea of knowledge work, that there’s a powerful role young people can uniquely play in learning bits and pieces of math well enough to teach it to their peers. But that idea challenges the more traditional approaches to math teaching and learning, approaches that aren’t working for far too many. AP/YPP process is a tool for democratizing who has a voice in the learning space; it allows youth to become teacher and learner and agents of change, making a demand on a system that historically silences them.

And for teachers, it creates a process to step out of that position as “the source” of all knowledge. School districts, teachers, communities don’t necessarily look at the kids, especially the kids who are struggling, and see them as assets and think, “How do I tap into all of the strengths that they’re bringing to the table, to help us work on these problems that we’re having.” Rather, they continue to see the kids themselves as the problem.

Finally, my dad was the first person that made me aware of the ways education functions to perpetuate a caste system. For a long time, he talked about sharecropper education. And then he came across a book, *Slums and Suburbs*, by James Bryant Conant, a former Harvard president and founder of ETS [14]. Conant’s research led him to the understanding that the North and South after the Civil War, agreed to keep “the Negro” in a tight caste system, and Conant said, “the clearest manifestation of America’s caste system is in its education system.”

And so, central to our work is exploring how we play a role through mathematics education in breaking down that system of caste. In that work, we ask how do teachers and students reckon with the perpetuation of caste in education? How do they understand their everyday relationship to it? How in such a system could every teacher, student, professional developer become a freedom fighter?
And could it be that the two themes of freedom and mathematics in my dad’s life calls to us all?

Figure 9. Bob and his daughter, Maisha Moses.

References


Credits

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Remembering Professor Aderemi O. Kuku (1941–2022), an Internationally Recognized Mathematician and Scholar

Johnny L. Houston

It was announced in February that Aderemi O. Kuku, a distinguished mathematician and internationally recognized scholar, was no longer with us. A dual citizen of Nigeria and the United States, Professor Kuku had a long, brilliant, and productive career in Nigeria and around the world as a mathematics professor, scholar, prolific author, and a highly requested presenter.

On February 15, 2022, in Nigeria’s National Newspaper, The Daily Post, Nigeria’s President Muhammadu Buhari stated that he had condoled with the family of the renowned mathematician, Professor Aderemi O. Kuku, over the passing of this distinguished scholar and administrator on February 13, 2022.

In a condolence message by his spokesman, Femi Adesina, on February 15, President Buhari acknowledged that the late Professor Kuku’s work would continue to stand as a testimony of his diligence, scholarship, and brilliance. He, therefore, joined the academic and mathematical sciences communities in Nigeria and abroad, in mourning the researcher and author who shared his knowledge and talents with so many, in so many different arenas and in so many different countries.

Aderemi Oluyomi Kuku, the son of Busari Adeoye Kuku and Abusatu Oriaran Baruwa, was born in Ijebu-Ode, Ogun State in the African country of Nigeria on March 20, 1941 as a member of the famous Kuku family in Ijebu-Ode. Aderemi Oluyomi Kuku is a descendant of Oba Ofiran. The mother of Aderemi’s father (Busari Adeoye Kuku) was Barikisu Kuku, a great granddaughter of Oba Ofiran and Otunba Kuku (a Professor) who was a member of the Adetuyo Ruling House of Owa of Okun-Owa.
With this background, it is easy to understand why Aderemi’s parents had a strong commitment to educate their children. Aderemi’s two older brothers became experts in telecommunications and electrical technology, respectively, and his younger brother became a professor of Electronics. Aderemi had an affinity for mathematics as a career and he pursued it. Aderemi Kuku’s first school was Bishop Oluwole Memorial School, in Agege, Lagos State, Nigeria. He then attended St James, an Anglican primary school, Oke-Odan, in Ogun State, becoming the first in the school to leave with a certificate. He then attended the Eko Boys High School in Lagos, Nigeria from 1955 to 1959. In his final year of study, 1959, he was Head Boy of the High School. This school had been founded in 1913 and when it celebrated its centennial birthday on 13 January 2013, Kuku was honored with a special award for distinguished accomplishments at the centenary dinner held at the Banquet Hall of the Nigerian Law School.

After graduating from Eko Boys High School, he went to Abeokuta Grammar School where he obtained the Highest School Certificate in Mathematics. At this stage of his education, Kuku had options. He was awarded a scholarship from the African Scholarship Program of American Universities. These scholarships were for highly qualified African secondary school students to go to the United States to study for a first degree. This program was administered by the United States Agency for International Development (USAID), which also had a program to take Nigerian students to Makerere University College in Kampala, Uganda, to study for a first degree. Makerere was part of the University of East Africa which was an independent external college of the University of London in England. Kuku was advised that a British degree was more highly valued in Nigeria than one from the United States, so he chose to study at Makerere University College (which became the independent Makerere University in 1970). A University of London degree was highly valued, but it was difficult to achieve. This was because a student had to take the same examination at the end of the three-year courses as did students studying in London and yet the teaching at Makerere was not of the quality of that in London.

Kuku entered Makerere University College in 1962 and, in 1965, was awarded a BSc degree with Special Honors in Mathematics. He achieved the Special Honors degree by doing lots of studying on his own, in addition to the material taught in the courses. After graduating, Kuku was appointed Assistant Lecturer at the University of Ife in Ile-Ife, Osun State, Nigeria. This university was founded in 1961 and teaching began in October of the following year. We note that it was renamed Obafemi Awolowo University in 1987, the name it currently has in 2022. Despite having a university position, Kuku had no plans to end his education at the BSc degree level. He enrolled at the University of Ibadan to study for a master's degree. This university had been established in 1948 as a college of the University of London, being the first university to be established in Nigeria. It became an independent university in 1962, after Nigerian independence.

Kuku’s MSc degree was supervised at the University of Ibadan by Joshua Leslie who had studied at the University of Chicago. Leslie then went to the University of Paris and did research to earn a doctorate. When Kuku enrolled in the University of Ibadan, Leslie had just returned from a research visit to the Institute for Advanced Study at Princeton. Although Leslie was not an expert in algebra, the area that interested Kuku, Leslie had met Hyman Bass at Princeton and realized that there were important new developments happening in algebra. Kuku went to the library and began to read recent algebra papers, which was somewhat of a challenge since Leslie was not able to help him. Moreover, Kuku had to teach courses at the University of Ife. He submitted an MSc thesis “A Survey of Algebraic K-Theory.” At Professor Leslie’s request, Kuku’s MSc thesis was supervised by Professor Hyman Bass as an external examiner. In 1968, Kuku was awarded an MSc degree from University of Ibadan.

After receiving an invitation from Hyman Bass to come to Columbia University to do research for a doctorate degree, Kuku arranged a leave from the University of Ibadan and spent the year 1970–71 at Columbia University with
Hyman Bass as his thesis advisor. Kuku returned to the University of Ibadan in 1971, submitted his thesis “On the Whitehead group of p-adic integral group-rings of finite p-groups,” and was awarded a PhD by the University of Ibadan.

His distinguished career started at the University of Ibadan in Nigeria in 1968 when he accepted the position of Lecturer II in Mathematics at the University of Ibadan, replacing Joshua Leslie who had joined the faculty at Northwestern University. He was promoted to Senior Lecturer in Mathematics in 1976, to Reader in Mathematics in 1980, to full Professor of Mathematics in 1982 at the age of 41, to Head of the Mathematics Dept. from 1983–86, to Dean of the Postgraduate School from 1986–90, and Committee Chair of all Postgraduate Schools in Nigerian Universities from 1986–90. From 1995–2003, he was Professor of Mathematics at the International Centre for Theoretical Physics (a UN sponsored research centre for mathematics and physics), retiring because of mandatory age requirements. From 2009–2014, he was the William W. S. Claytor Endowed Professor of Mathematics at Grambling State University in Louisiana.

Dr. Kuku's first publications were “Some algebraic K-theoretic applications of the LF and NF functors” (1973), “Whitehead group of orders in p-adic semi-simple algebras” (1973) and “S finiteness theorems in the K-theory of orders in p-adic algebras” (1976). For the next forty years, he worked in various aspects of algebraic K-theory, with connections to commutative and non-commutative algebra, number theory, geometry, and representation theory. Algebraic K-Theory is a contemporary and multidisciplinary subject that has applications in mathematical physics, dynamical systems, econometrics, and control theory. Altogether, he produced over 75 publications including research articles, 10 books/monographs, and articles on issues in STEM. Professor Kuku’s first book was Abstract algebra (1980). He also published the books (with E Thoma and J H Rawnsley) Group Representations and its Applications (1985), Basic Commutative Algebra (1997), Topics in Algebraic K-theory (1997), and Representation Theory and Higher Algebraic K-theory (2007), as well as other books.

The author first met Dr. Bass and Dr. Kuku in Africa where all three presented at COPAM, the Fourth Pan African Congress of Mathematicians, Sept. 18–26, 1995 in Ifrane, Morocco. After which, Professor Kuku and the author became professional and personal friends.

Professor Kuku held many visiting positions at highly reputable universities and research centers in the USA, Canada, Europe, Hong Kong, and China, including the following: Member, Institute for Advanced Study, Sept. 2003–Aug. 2004; Visiting Research Professor, MSRI, Aug.–Dec. 2004; Visiting Professor, The Ohio State University, 2005; Distinguished Visiting Professor, Miami University, Oxford, OH, 2005–2006; Visiting Professor, Universitat Bielefeld, Germany, 2006; Visiting Professor, IHES, Paris, France, 2006; Visiting Professor, Max Planck Inst. Fur Mathematik, Bonn, Germany, 2007; Visiting Professor, University of Iowa, 2007–2008; Distinguished Visiting Professor of Mathematics, Institut de Mathematiques et de Sciences Physiques, Porto Novo, Benin Republic, Nov.–Dec. 2015; Distinguished Professor of Mathematics, National Mathematical Center, Abuja, Nigeria, 2015.

University, 2005; University of Lausanne, 1996; University of Ljubljana, 1999; Queen’s University, Kingston, Ontario, 1982, 1993; University of Western Ontario, London, Ontario, 2001; University of Hong Kong, 1993; University di Genova, 1996; University of Singapore, 1985; Institute of Math/Systems Science, Chinese Academy of Science, Beijing, 1993; East China Normal University Shanghai, 1993; Northwestern Polytechnical University of Xian, 2002; Nanjing University, 2002; Tongji University, Shanghai, 2002; Indian Statistical Institute, Delhi, 2002; Instituto de Matematicas, Unidad Morella, Mexico, 2005; University of Buenos Aires, 2013; Sheriff University of Technology, Tehran, 2000; University of West Indies at Kingston, Jamaica, 1993; University of Technology, LAE, Papua New Guinea, 2013; Universities of Abidjan, Cote d’Ivoire, 1986, 1987, 1990, 1995; Dakar, Senegal, 1987, 1989; Ouagadougou, Burkina Faso, 1997; Yaoundé, Cameroon, 1990, 1992; Brazzaville, Congo, 1987, 1989; Nairobi, Kenya, 1986, 1991; and several universities in Nigeria; Universities of Cape Town, Port Elizabeth, Stellenbosch, Pretoria; University of Natal, Pietermaritzburg; University of Witwatersrand, Johannesburg; University of Western Cape, Bellville; University of the North, Pietersburg; University of the Free State, Bloemfontein; Rand Afrikaans University, Johannesburg; Rhodes University, Grahamstown, all in 1997. He also delivered special invited addresses at meetings of mathematical societies including: American Mathematical Society; Canadian Mathematical Society; Mathematical Association of America; National Association of Mathematicians; Hong Kong Mathematics Society Annual Lecture, 1993.


An international conference on "Algebraic K-theory and its applications" March 17–21, 2011, was organized by Nanjing University, China in honor of his 70th birthday, and the Proceedings of the Conference were published in a special issue of the Journal of K-theory, Volume 12, no. 1, published by Cambridge University Press.

The photo (Houston, Texas, 2019) was the last in-person visit the author had with Professor Kuku. We both made presentations at the NAM 2019 Faculty Conference on Research and Teaching Excellence. Professor Kuku was an active and cherished Life Member of NAM for two decades.

Professor Kuku contributed immensely to the academic and administrative development of the University of Ibadan, earning the University’s International Distinction in Academia. Professor Kuku’s large mathematical footprint remains indelible and perpetually inspiring. He was a globally respected mathematician in algebraic K-theory. His presence and stimulating contributions will be very
much missed. However, his publications and other contributions will forever remind us of his diligence, his scholarship, and his brilliance. Professor Aderemi Oluymoi Kuku will be forever remembered by those who knew him.

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National Association of Mathematicians, Inc. (NAM) 
Passing the Torch 
A Reflection of NAM’s Development and Growth 
by NAM’s Leaders/Contributors—
the First Five Decades 
Johnny L. Houston

The National Association of Mathematicians, Inc. (NAM) is a nonprofit professional organization in the mathematical sciences with membership open to all persons interested in the mission and purpose of NAM, which are:

- Promoting excellence in the mathematical sciences for all Americans, and
- Promoting the mathematical development of all underrepresented American minorities, especially African Americans.

NAM was founded under the principles of inclusivity, diversity, and equity at a time when major American mathematical sciences organizations were excluding underrepresented American mathematicians of color from their membership, editorial boards, research symposia, and other professional activities. NAM continues to promote and encourage inclusivity, diversity, and equity by all mathematical sciences organizations, all institutions of higher learning, and all other arenas and levels where persons are teaching or engaging in the mathematical sciences. NAM is committed to the promotion of equal opportunity and treatment for all NAM members and participants in NAM-sponsored events, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, veteran status, or any other reason not related to scientific merit or qualifications. NAM was established in January 1969 by 17 American underrepresented minority mathematicians. These persons would have preferred to be included in the major American mathematical sciences organizations and their activities. Unfortunately, these organizations neither embraced diversity nor exhibited any genuine efforts towards inclusivity and equity. This being the case, NAM Founders and other American
underrepresented minorities decided to create other paths to success by establishing parallel organizations, inviting all who agreed with their mission and purpose to become members or at least support their goals. Five decades later, much has been accomplished. However, much more needs to be done. This article presents some of the major accomplishments of NAM during its first five decades by its leaders and major contributors and provides a glimpse of some of their future challenges.

1. The Origin of NAM

NAM began at the Joint Mathematics Meetings (JMM) in 1969 in New Orleans, Louisiana. This city, known as “the Big Easy,” is also the Mardi Gras capital of the USA. However, due to systemic racism and discrimination, African Americans were not permitted to participate in many of the Mardi Gras activities. As in so many similar situations in the USA, African Americans formed their own organization, the Zulu Krewe, and decided how they would participate in Mardi Gras. The 17 NAM Founders assembled in New Orleans on January 26, 1969. They reflected on the philosophy and vision that was articulated by the Reverend Martin Luther King, Jr., and that the MLK Eternal Flame symbolizes how his tragic assassination in 1968 did not destroy his vision and dreams. Instead, Dr. King’s vision and dreams inspired the Founders to focus on a new future in the mathematical sciences in the USA. They focused on a future of inclusivity, diversity, and equity for all Americans seeking to learn math.

Founders of NAM.
- James A. Donaldson
- Samuel Douglas
- Henry Eldridge
- Thyrsa Frazier-Svager
- Richard Griego
- Johnny Houston
- Curtis Jefferson
- Vivienne Mayes
- Theodore Portis
- Angelia Rodriquez
- Charles Smith
- Robert Smith
- Beauregard Stubblefield
- Henry Taggert
- Walter Talbot
- Harriett Walton
- Scott Williams

The Founders vicariously (mentally and collectively) lit a torch in 1969, but first they did the following:
- They shared their experiences about past JMM behaviors toward them, underrepresented American minority mathematicians (UAMM).
- They noted the same or similar experiences at JMM 1969.
- They discussed the past/present and noted the blatant exclusion of UAMM from anything noteworthy, including speakers, committees, and boards, and including the absence of any historically Black colleges/universities (HBCUs)/minority-serving institutions (MSIs) at any level of involvement.

These discussions led to a “Kujichagulia (self determination) moment,” where a “collective torch” was ignited and the 17 gathered gave birth to NAM! The Founders outlined and initiated some “invigorating actions” that have developed over five decades and will continue for future decades.

2. NAM is a Nationally/Internationally Recognized Professional Organization in the Mathematical Sciences

NAM is a voting member of the Conference Board of the Mathematical Sciences (CBMS), as are AMS, MAA, SIAM, and other nationally recognized mathematical sciences groups. NAM’s operations, programs, and activities are supervised by its Board of Directors, consisting of 12 standing committees ([https://www.nam-math.org/governance.html](https://www.nam-math.org/governance.html)), which function according to the guidelines of NAM’s constitution and bylaws. Governance was established when NAM became incorporated as a nonprofit organization in 1972, early in its first decade, and was modified in later decades.
In 2022, NAM’s programs and activities consisted primarily of the following.

- Monthly meetings of NAM’s standing committees.
- Monthly meetings of NAM’s Board of Directors.
- NAM’s quarterly Newsletter is distributed once each season.
- NAM’s winter events: NAM’s National Meeting at the JMM.
- NAM’s spring events: NAM’s Faculty Conference on Research and Teaching Excellence (FCRTE) at an HBCU/MSI, which includes NAM’s Albert T. Bharucha-Reid Lecture.
- NAM’s summer events: NAM’s David Blackwell Lecture and other items at the Mathematical Association of America summer MathFest.
- NAM’s fall events: NAM’s Undergraduate (UG) MATHFest on an HBCU/MSI campus, which includes the J. Ernest Wilkins Lecture.

NAM’s quarterly Newsletter is NAM’s primary method of communicating with its members, friends, and supporters. NAM’s past and current Newsletters are available on NAM’s website nam-math.org.

NAM’s winter events are held at NAM’s National Meeting, in conjunction with the JMM, where NAM is now a partner. Because of NAM’s strong commitments to the principles of diversity, inclusivity, and equity, NAM leaders decided early in its first decade to hold its annual meetings at the same site and at the same time that the JMM is held each January. NAM wanted its members to be able to participate in most of NAM’s National Meeting events as well as attend and participate, when feasible, in the events of other majority mathematical sciences organizations. Moreover, NAM leaders chose the dates and times so that NAM events did not conflict with major events of the majority mathematical sciences organizations. It was NAM’s hope that an impactful number of members of the majority mathematical sciences organizations would attend some of NAM’s events, helping to demonstrate and encourage dual interest in inclusivity, diversity, and equity.

3. NAM’s Winter Events at JMM

- William W. S. Claytor–Dudley W. Woodard Lecture (research presentation), named in honor of the research legacies of the third and second African Americans, respectively, known to have earned PhDs in mathematics.

- Elbert F. Cox–Walter R. Talbot Address (issues/topics in math/math education), named in honor of the first and fourth African Americans, respectively, known to have earned PhD degrees in mathematics, for their legacies of challenging the system.
- Haynes, Granville, and Browne Presentations by recent PhDs, named in honor of the first three African Americans known to have earned PhDs in mathematics.
American women known to have earned their PhDs in mathematics for their legacies teaching and mentoring young African Americans to do graduate study and earn PhDs in mathematics.

- NAM’s one-hour Panel Discussion on a variety of topics.
- NAM’s Annual Business Meeting, receiving reports from the Board and taking votes on issues.
- NAM’s Annual Banquet (the featured gathering of NAM’s members and friends), which includes NAM’s awards and recognitions listed below.
  - Lifetime Achievement Award(s), (possibly more than one), which is an award given annually to an underrepresented American mathematician who has exhibited an exemplary career for a period of at least 25 years that is worthy of emulation by younger mathematicians (passing the torch). The award may be given posthumously.
  - Clarence F. Stephens–Abdulalim A. Shabazz Teaching Excellence Award, which is given annually to minority mathematics professors who seek to emulate their legacies. It includes a monetary prize.

4. NAM’s Spring Events

- Faculty Conference on Research and Teaching Excellence (FCRTE), held at an HBCU/MSI. This is an annual weekend event for underrepresented minority faculty, which includes presentations by faculty on research topics and/or topics on teaching techniques/issues, as well as a Faculty Panel Discussion on topics such as:
  - Tutoring and mentoring techniques, the importance of perseverance
  - Summer opportunities for faculty and students—fellowships/internships
- Albert Turner Bharucha-Reid Lecture (at FCRTE). This lecture by an invited speaker is in honor of the legacy of Albert Turner Bharucha-Reid, who was a great teacher and a world-class researcher.

5. NAM’s Summer Event at MAA MathFest

NAM’s David Blackwell Lecture was established in 1994, in honor of the legacy of this great professor and distinguished world-class researcher and scholar. In 2019, Johnny L. Houston gave the 25th David Blackwell Lecture.
Houston’s presentation was followed by NAM’s 50th anniversary party (sponsored by MAA).

6. NAM’s Fall Events

- Undergraduate MATHFest (UG MATHFest) at an HBCU/MSI is a weekend event (Friday–Sunday) for underrepresented American minority students in the mathematical sciences at the sophomore–senior college level. They are exposed to similar college students, especially from other HBCUs/MSIs. They meet underrepresented American graduate students in the mathematical sciences, graduate faculty from Research I institutions seeking to recruit underrepresented American minority students to pursue graduate degrees, and seasoned underrepresented minority mathematicians who serve as role models for the underrepresented American students attending. This unique gathering can significantly inspire the undergraduate students’ future goals and careers. It is a networking haven of opportunities to learn, participate, and plan one’s future. It exhibits a real sense of diversity, equity, and inclusivity. The students are challenged to be involved.
- At UG MATHFest an invited speaker gives the J. Ernest Wilkins Lecture in honor of the legacy of this world-class mathematician who enjoyed inspiring youth to pursue careers in mathematics and other STEM fields. Dr. Wilkins gave the inaugural lecture in 1994.

7. An Overview of NAM’s First Four Decades


- 1969: The founding of NAM.
- Founding leaders: Walter R. Talbot, Johnny Houston, Scott Williams.


- 1970 Executive Committee: Irvine Vance, Chair; Etta Falconer, Secretary; Benjamin Martin, Articles of Incorporation/Constitution-ByLaws.
- Presidents elected during NAM’s first decade:
  - Frank James, 1971–1973
  - Theodore Sykes, 1973–1975
  - Japheth Hall, 1975–1976
- 1972: NAM incorporated in the state of Georgia, after adoption of constitution/bylaws.
- 1973: NAM’s first Annual Business Meeting and Annual Panel established.
- 1974: NAM’s national office established in Atlanta.
- 1975: NAM began having its National Meetings at JMM.
- 1975: NAM selected Johnny L. Houston as its first Executive Secretary.
- 1979: NAM’s tenth anniversary celebrated in Boulder, Colorado, supported with NAM’s first grant, from NOAA, secured by Beauregard Stubblefield (who worked at NOAA), one of NAM’s Founders.


- NAM’s National Meetings grew in complexity and attendance.
- 1980: NAM’s Claytor–Woodard Lecture was established by President Samuel H. Douglas and Executive Secretary Johnny L. Houston.
- 1980: NAM’s Cox–Talbot Address was established by President Samuel H. Douglas and Executive Secretary Johnny L. Houston. Professor J. Arthur Jones, Florida A & M University, gave the inaugural lecture.
- 1981: NAM produced and published the first NAM proceedings.
- 1984: Rogers J. Newman was elected NAM’s fifth President.
• Distinguished Speakers who gave the Claytor–Woodard Lecture in the second decade:
  ○ 1984: Professor David H. Blackwell
  ○ 1985: Professor Albert Turner Bharucha-Reid
  ○ 1986: Professor J. Ernest Wilkins
• The following Board members were elected: Sylvia T. Bozeman, Vice President; Harriett Walton (serving more than a decade), Secretary-Treasurer; James Donaldson, Newsletter Editor.

• NAM Undergraduate MATHFest (UG MATHFest) was established in 1992 as a NAM activity under the leadership of NAM’s fifth President, Rogers J. Newman (1984–1994) and Executive Secretary Johnny L. Houston (1975–2000). Professor Aderemi O. Kuku was UG MATHFest 1992 Invited Distinguished Speaker.
• The following NAM annual major events and signature programs were established under the leadership of NAM’s sixth President, John W. (Jack) Alexander (1994–2004) and Executive Secretary Johnny L. Houston.
  ○ Annual Faculty Conference on Research and Teaching Excellence (FCRTE)
  ○ The Albert Turner Bharucha-Reid Lecture (at FCRTE), 1994
  ○ The David Blackwell Lecture at MAA summer MathFest, 1994
  ○ The J. Ernest Wilkins Lecture at NAM’s UG MATHFest, 1994
  ○ NAM’s Lifetime Achievement Award, 1994
• NAM’s Executive Secretary, Johnny L. Houston, began a quarterly column in NAM’s Newsletter titled “Spotlight on a Mathematician,” containing a one-page biography of an African American mathematician. Altogether, it provided over 25 biographies.

• During this decade, NAM’s expanded programs, lecture series, and annual activities led to more joint-venture activities within the mathematical sciences community.
• NAM’s seventh President, Nathaniel (Nate) Dean (2004–2014), Executive Secretary, Leon Woodson, and Vice President, Dawn Lott obtained grant funds to enhance programs and activities, and initiated joint-venture activities that further expanded NAM’s influence. Dean’s background in the mathematical sciences inspired more NAM involvement with research, producing PhD students, and participation with corporate America.
• It was at the beginning of this decade that Johnny L. Houston retired as NAM’s Executive Secretary after 25 years of service (1975–2000). Houston was selected to be Executive Secretary Emeritus by NAM’s general membership and to be an Ex-Officio Member on the NAM Board. Houston wrote a book, The History of NAM, 1969–1999, in 2000. Johnny L. Houston is the only NAM member who has served as an active member starting as one of its Founders in 1969, through 2022 and counting, as well as being on the NAM Board of Directors for 48 years.

8. Three NAM Presidents Who Served Ten Years
All of NAM’s leaders and major contributors who helped “pass the torch” contributed immensely to NAM’s development and growth. Particularly impactful were three who served as President of NAM for ten years each. We now present a detailed biography of each.

“My first JMM meeting was the one held in Washington, DC in January 1961. I had just picked up my degree from the University of Michigan. I wanted to know more about this JMM meeting of which so many Michigan professors had discussed” —R. J. Newman, 1994.

Rogers J. Newman was born in Ramar, near Montgomery, Alabama, on December 22, 1926, as the only child of Jonathan Newman, a farmer and insurance agent, and Vera Primos Newman, a school principal. He married Dorothy Alice Willis. They had three sons: Rogers Joseph, Jr., Roy Oliver, and Robert Marion. He passed at 89 on January 9, 2016.

Newman received his high school diploma from Alabama State College Laboratory School (Montgomery). He enrolled in Morehouse College in Atlanta, Georgia, in 1944, earning a BA degree in mathematics in 1948; Martin Luther King, Jr., was in his graduation class. Immediately following graduation, he enrolled at Atlanta University (now Clark Atlanta University), earning an MA degree in mathematics in 1949. Newman began his illustrious teaching career at Bishop College in Marshall, Texas (1949–1950). In January 1961, he received his PhD in mathematics from the University of Michigan in complex variables. He did further formal studies at Imperial College, London (1970–1971) and at Louisiana State University (summer 1970, summer 1971).

During his career of more than 40 years, Professor Newman taught and influenced scores of established mathematicians. His list of students who earned doctorates
includes Delores Spikes, Stella Ashford, Juanita Bates, Roosevelt Calbert, and Preston Dinkins. He was selected as a Danforth Teacher (1995) and Teacher of the Year (1981). He produced several publications and made a number of scholarly presentations. He was active in several organizations, but foremost he was a distinguished leader of NAM—President (1984–1994)—and he marketed NAM to all areas of the mathematical sciences community. He gave NAM’s first scholarly presentation at a NAM National Meeting (1971). He helped to enhance NAM’s National Meetings with quality presentations. He engaged more HBCUs to participate in NAM annually. He and Executive Secretary Houston established NAM’s UG MATHFest in 1992. He was active with the MAA Board of Governors (1986–1989), as well as with the AMS, NSF, and Math Reviews. He received NAM’s Distinguished Service Award (1994) and other awards/honors. He will be forever remembered by those who know NAM’s history.

8.2. Legacy of John W. Alexander (1938–2022). John W. (Jack) Alexander, Jr., was born May 17, 1938, in Salem, Ohio. He passed January 13, 2022, in Miami, Florida, at the age of 83. Dr. Alexander was a very scholarly individual, having earned five degrees in higher education. He earned a BS in mathematics (1961, Boston University); an MA in mathematics (1965, Bowling Green State University); an EdD in mathematics education (1985, Boston University); an MBA (1987, California Coast University [CCU]); and a PhD in management science/operation research (1989, CCU). He retired as a Professor of Mathematics at Miami Dade College in 2020.

Dr. Alexander’s interest in the broad applications of numerical knowledge is reflected by the diversity of his professional positions and activities. His professional positions included: Mathematics Consulting Director to the West African Regional Mathematics Program of the State Department (1970–1977); Actuary for Connecticut Mutual Life Insurance Co. (1978–1981); Chief Statistician, Futures Group Think Tank, Glastonbury, Connecticut (1981–1982); Associate Professor, Wentworth Institute of Technology (1982) and later Dean of the College of Arts & Sciences (1984–1990); and faculty member at the University of the District of Columbia (1990–1997) where he served a term as Mathematics Department Chair. Later he served as Staff Officer and Research Mathematician for the Board on Mathematical Sciences (1995–1996) and Director of the Board (1996–1997) at the National Academy of Sciences. He returned to academia as Professor of Mathematics at Atlanta Metropolitan College and Spelman College (1998–2002) and ended his academic career as Professor of Mathematics at Miami Dade College (2002–2020).

Dr. Alexander had a long and distinguished relationship with NAM. In 1992, Dr. Alexander was elected to NAM’s Board of Directors as Vice President. Later he served a productive ten-year term as President of NAM (1994–2004). At the beginning of his presidency, NAM celebrated its 25th anniversary year, during which the Board examined its past activities and made future plans. The Board then charged President Alexander and Executive Secretary Johnny Houston with critically reviewing the ideas that had emerged and proposing plans to guide the organization’s activities for the next 5–25 years. The major elements of the 1994 Alexander–Houston Proposal, which were adopted by the Board, included: a five-year Strategic Plan; establishment of an annual Regional Faculty Conference with a Bharucha-Reid Lecture; establishment of named lectures for notable mathematicians David Blackwell and J. Ernest Wilkins, Jr.; establishment of the annual Haynes–Granville–Browne Recent PhDs Presentations; creation of the NAM Lifetime Achievement Award; and upgrading NAM’s Newsletter. In 2004, Dr. Alexander received the NAM Lifetime Achievement Award in recognition of his extraordinary leadership and service to NAM. Dr. Alexander was a master teacher of mathematics, a visionary of ideas for the use of mathematics, and an effective and impactful leader of NAM. He will be forever remembered by those who know NAM’s history.

8.3. Legacy of Nathaniel Dean (1956–2021). Nathaniel (Nate) Dean, an American mathematician and educator who made significant contributions to abstract and algorithmic graph theory, as well as data visualization and parallel computing, was born in Mississippi on January 9, 1956, and passed in Texas on February 18, 2021. He received a BS in mathematics and physics from Mississippi State University (1978), an MS in applied mathematics from Northeastern University (1983), and a PhD in mathematics from Vanderbilt University (1987).

He had a stellar career in both industry and higher education. After receiving his PhD in graph theory in 1987, Dean worked for 11 years in the Software Production Research Department of Bell Laboratories, producing over 30 scientific publications. In 1997, he received the President’s Silver Award from Bell Labs. In 1998, Dr. Dean became an Associate Professor of Computational and Applied Mathematics at Rice University. While at Rice, he supervised four PhD students with thesis topics ranging from algorithmic graph theory to biological computing. In 2003, he moved from Rice to Texas Southern University (TSU), becoming a Full Professor and Chair of Mathematics. He also served as Director of an NIH Computational Research Laboratory.
at TSU. Dean’s departure to Texas State University (TX-St) occurred in 2006. At TX-St he supervised his fifth PhD student and served as Chair of the mathematics department for several years. He was highly respected by faculty and by students. He retired in 2016, following the last active years of his career.

Dean’s research focused on creating mathematical models of complex systems and developing computer tools to visualize, design, and analyze such systems. His research areas included discrete mathematics, optimization, data mining, and network visualization. He produced over 60 publications in these fields, and some of his work in data mining was highlighted in the PBS television series *Life by the Numbers* (1998), including data mining software that he had developed to teach discrete mathematics at the K–12 level. Dean posed a conjecture to the second neighborhood problem in 1995 which led to progress. The problem is still open (2021).

Dean also worked on mathematics education and outreach throughout his career. He was always recruiting students (especially underrepresented American minorities) to study mathematics. One of his aims was to help address the issue of the very low numbers of underrepresented American minorities in the workforce in the mathematical sciences.

Dean became Vice President of NAM (2001) and served as President (2004–2014), leading NAM with excellence. He served as an Associate Editor of the Notices (AMS) and served on the Board of Governors (MAA). He was Managing Editor of the Journal of Graph Theory, co-organizer of several mathematics conferences, and served as Editor of four volumes for the AMS: *Computational Support for Discrete Mathematics*, *African Americans in Mathematics I and II*, and *Robust Communication Networks*. He was active with SIAM and with CAARMS. He received several recognitions for his achievements and outreach, including NAM’s Lifetime Achievement Award, and was recognized as a great graph theorist. He held a 2nd-degree black belt in martial arts. He enhanced many aspects of NAM’s programs and activities. He will be forever missed by those who know the history of NAM.


Under NAM’s eighth President, Edray Goins (2015–2020), a younger generation received/passed the torch. Here is what they did.

- Improved NAM’s website and database.
- Marketed NAM’s image with merchandise.
- Increased collaboration with other mathematical sciences organizations.
- Established the Clarence F. Stephens–Abdulalim A. Shabazz Teaching Award.
- Renamed the Executive Secretary position to that of Executive Director.
- Attracted a diverse group of younger mathematicians to the membership/Board.
- Established NAM’s Historical and Archival Committee (HAC), an ad hoc subcommittee under NAM’s Publication Committee, with Johnny L. Houston as Chair and Sylvia T. Bozeman as Vice Chair. Robert W. Woodruff Library and Research Center (Atlanta U. Center) was selected as NAM’s physical repository for HAC.
- Developed Attendees/Network Directories that were distributed at UG MATHFest.
- Launched NAM’s second endowment campaign.
- Leon Woodson, Executive Secretary, passed the torch to Leona Harris, NAM’s first Executive Director.
- Talithia Washington, Vice President, passed the torch to Naomi Cameron, Vice President.
- Roselyn Williams, Treasurer, passed the torch to Cory Colbert, Treasurer.

Omayra Ortega became NAM’s ninth President and first female President in 2021. She had previously been NAM’s Newsletter Editor for several years.

NAM current Board of Directors, 2021–2022.

- Omayra Ortega, President
- Rhonda Fitzgerald, Vice President
- Aris Winger, Executive Director
- Cory Colbert, Treasurer
- Shea Burns, Secretary
- Haydee Lindo, Newsletter Editor
- Johnny Houston, Executive Secretary Emeritus, Ex Officio Member
- Chinenye Ofodile, Region A Member
- Terrence Blackman, Region B Member
- Brittany Mosley, Region C Member
- Brett Jefferson, Outside of Academia Member
• Robin Wilson, Majority Institution Member
• Karen Taylor, Community College Member

This Board conducted all NAM programs and activities virtually during Covid (2020–2021). They have conducted in-person programs and activities since 2022. In 2022, NAM received grant funds to support UG MATHFest in-person for 2022 and 2023.

10. Challenges, Solutions, and Honoring Our History

10.1. The NAM challenge. According to the Pew Research Center, in America, the White, non-Hispanic population, which was 199 million in 2005, will grow to 207 million in 2050, a 4% increase. In 2050, 47% of the US population will be non-Hispanic White, compared with 67% in 2005. The Hispanic population, which was 42 million in 2005, will rise to 128 million in 2050, tripling in size. Latinos will be 29% of the population, compared with 14% in 2005. The Black population, which was 38 million in 2005, will grow to 59 million in 2050, a rise of 56%. In 2050, the nation’s population will be 13.4% Black, compared with 12.8% in 2005. The Asian population, which was 14 million in 2005, will grow to 41 million in 2050, nearly tripling in size. In 2050, the nation’s population will be 9% Asian, compared with 5% in 2005.

Thus, the Black population is projected to remain 13% of the US population from 2005 until 2050! From 2005 to 2020, only around 2% of the Americans who earned a PhD in mathematics, annually, were Black! What is required to get the percentage of Black PhDs earned annually in mathematics to be near 10% by 2050?

10.2. Suggested solutions by J. L. Houston.

• A nationwide, strong advocacy and commitment to implement genuine diversity, equity, and inclusivity practices at every level of teaching and learning in America, especially in STEM areas.
• A nationwide, strong advocacy for more diverse STEM expertise produced at the PhD level. There are more than 50 R1 universities with PhD programs in mathematics (an average of more than 1 per state) that have never produced a single Black PhD student in math; all R1 universities should consider recruiting and producing at least one Black PhD graduate in math every 10 years.
• Every Black mathematics professor at an R1 university who supervises PhD students is urged to recruit and produce more than one Black PhD student.

10.3. Can NAM/other smaller nonprofit math groups exist on volunteerism and dues alone?

• NAM has depended 100% on volunteer contributors, including 100% nonpaid Board of Director members, standing committee members, and other participants, with the recent exception of a quarterly stipend for the NAM Newsletter Editor.
• NAM has been completely nonprofit with a nonpaid staff and has been quite successful.
• Hundreds of contributors have voluntarily shared directly or indirectly:
  o Their time, talents, and influence to support and plan NAM’s programs/activities
  o Their time, talents, and resources to successfully implement NAM’s programs/activities

Planning and implementing quality programs and activities are the lifeline of nonprofit organizations. Having dependable staff and finances to achieve these requirements are the critical foundations upon which their existence depends. The professorate of the future is likely to demand more teaching and scholarly activities and give less credit for service. This means that volunteerism and dues alone will not be sufficient for survival.

NAM and other small nonprofit organizations in mathematics must find ways to constantly secure outside funding and pay some hired staff. It is not impossible but a challenge! Some groups are considering asking industry or “think tank groups” to donate one of their professional personnel to spend one month during a given year to do fundraising and planning to help the organization to acquire sufficient grant funds/endowment funds and philanthropic donations to support paid staff and to support its major programs/activities, annually, for a minimum period of three to five years each.

10.4. Honoring the 50+ persons who have given birth, development, and growth to NAM during its first five decades by passing the torch now that they have passed away. The listing of people below is a memoriam from NAM to express its highest esteem for their being the first to pass the torch. There are several more who passed the torch for NAM whose names are not included because we have not been able to confirm whether or not they are still alive. In a future article being planned, we hope to have a more complete “memorial list.”

We sincerely apologize to any relatives and/or friends if names that certainly should have been listed were omitted unintentionally. May NAM members forever appreciate the contributions of those listed!

10.4.1. Distinguished firsts and NAM Presidents who passed the torch.

• Patrick Francis Healy (1834–1910), first known Black man to earn a PhD, Louvain, Belgium, 1865 (in any discipline)
• Edward Alexander Bouchet (1852–1918), first Black man to earn a PhD from an American university, physics, Yale, 1876
10.4.2. Early pioneers/researchers who passed the torch for NAM vicariously, indirectly, or by direct participation.

- Dudley Weldon Woodard (1881–1965)
- Euphemia Lofton Haynes (1890–1980)
- William W. Schiellifin Claytor (1908–1967)
- Joseph A. Pierce (1902–1969)
- Joseph J. Dennis (1905–1977)
- Marjorie Lee Browne (1914–1979)
- Elgy Johnson (1915–1987)
- Alfred D. Stewart (1916–1987)
- Edward M. Carroll (1916–1997)
- Lillian K. Bradley (1918–1995)
- Clarence F. Stephens (1918–2018)
- Katherine G. Johnson (1918–2020)
- David Blackwell (1919–2011)
- Beauregard Stubblefield (1923–2013)
- J. Ernest Wilkins (1923–2013)
- Lloyd K. Williams (1925–2001)
- Albert Turner Bharucha-Reid (1927–1985)
- Abdulalim A. Shabazz (1927–2014)
- John Ewell (1928–2007)
- Charles B. Bell (1928–2010)
- Thyrsa Frazier Sager (1930–1999)
- Vivienne Malones-Mays (1932–1995)
- William T. Fletcher (1934–2017)
- James A. Donaldson (1941–2018)
- Aderemi O. Kuku (1941–2022)
- Manuel Kepler (1942–1999)
- Arthur Grainger (1942–2017)
- Charles Dwight Lahr (1944–2016)
- Amassa C. Fauntleroy (1945–2017)
- Abdul-Aziz Yakubu (1958–2022)
- Rudy L. Horne (1968–2017)

10.4.3. Some of the many other distinguished contributors who passed the torch for NAM, vicariously, directly, or by direct participation.

- Gerald Chachere (1944–2001)
- Ronald Biggers (1945–2005)
- M. Solveig Espelie (1940–1984)
- Don Hill (1944–2009)
- Lee Lorch (1915–2014)
- Vernice Steadman (1946–2015)
- Wilbur Smith (1941–2020)
- Stella Ashford (1942–2018)
- Irvin Vance (1928–2018)
- Boyd Coan (1949/1950–2019)
- Janis Oldham (1956–2021)
- Shirley McBay (1935–2021)
- Della Bell (1942–2021)
- Genevieve Knight (1939–2021)
- Frank Hawkins (1935–2020)
- Llayron Clarkson (1924–2022)
- James E. Joseph (1937–2022)

References


Credits

The NAM logo is courtesy of NAM. The torch image is courtesy of imagedepotpro via Getty. The image of King Zulu is courtesy of Miguel Discart. All other images are courtesy of Johnny L. Houston.
Joseph Carter Corbin: Arkansas’s “Profound Mathematician”

Jesse Leo Kass

While Joseph Carter Corbin (1833–1911) is a celebrated figure remembered for his impact in higher education and politics within the state of Arkansas,¹ he is absent from many modern historical accounts of American mathematicians.² However, during his lifetime Corbin was very much part of the mathematical community. He was described as a “Profound Mathematician” in Men of Mark, an influential anthology of accomplished Black men [Sim87, pp. 829–832]. He regularly contributed to mathematical periodicals like the American Mathematical Monthly. Upon his death, the Monthly commemorated him by publishing his obituary [11].

Corbin’s omission from historical accounts of American mathematicians is neither surprising nor unusual. Most mathematicians of his generation are omitted because accounts tend to focus on research activities. Only towards the end of Corbin’s life did significant numbers of American math professors begin to do research.

Certainly Corbin is worthy of inclusion in the history of American mathematics. His life was remarkable. The son of freed slaves, Corbin was the founding head of the University of Arkansas at Pine Bluff, Arkansas’s public Historically Black University. While working there, for over three decades, he regularly published in mathematical periodicals.

With the goal of introducing Corbin to modern mathematicians, we survey his life in this article. We focus on his experiences in higher education and mathematics and, in particular, give an overview of his mathematical publications.

Early life. Corbin was born in Ohio in 1833. His parents had been enslaved in Virginia but had moved to Ohio about a decade before Corbin’s birth. Little is known about their move, but it was a common one: most Black Ohioans had come from the bordering slave states of Virginia and Kentucky. Corbin’s mother, Susan, had moved after being emancipated by her enslaver. It is unknown whether the father, William, was also emancipated or if he had fled enslavement.

Even though Corbin never personally experienced enslavement, his life was significantly constrained by state

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¹Corbin’s life is the subject of a book [Fin17] and an article [Rot71]. The present account of Corbin’s life draws heavily from those publications, and we provide citations only for those facts not documented in those sources.

²Two notable exceptions which discuss Corbin’s publications in the American Mathematical Monthly are [Mea59, NB92].

³We will use the adjective “Black” to refer to enslaved Americans and their descendants. This is somewhat ahistorical. Corbin was often regarded as being of mixed race even though both parents had been enslaved. For example, federal census takers recorded Corbin’s race as “mulatto” in every census between 1850 and 1910 except for 1900 (when it was recorded as “Black”) and possibly 1890 (this record is unavailable). See also the discussion of Corbin’s time at Ohio University in the section titled “Education.”
lives and social practice. For example, Black Ohioans were not allowed to attend public schools, and many private schools were racially segregated. Despite the obstacles facing the family, Corbin’s parents were able to achieve considerable personal success; both were literate, and William worked as a “sewer of clothes” (a skilled profession).4

Education. When Corbin was growing up, his family lived in Chillicothe, a midsize town of a few thousand people in southern Ohio which was a regional center for Black life (Blacks residents made up 10% of Chillicothe’s population but only 1% of the state’s). Corbin attended school in Chillicothe until he was fifteen years old (around 1848). He then moved to Louisville, Kentucky, to take advantage of the greater educational opportunities afforded by a larger city. He returned to Ohio in 1850 to attend Ohio University in Athens.

Black students had studied at OU since the 1820s, and Corbin was the third student to enroll. While the presence of Black students on college campuses was sometimes a source of tension, Corbin did not seem to have attracted any special attention. It is unclear whether Corbin was even regarded as Black by faculty and students. For example, an 1853 newspaper article on the annual commencement exercises at Ohio University mentions Corbin by name but makes no reference to his race, a standard journalistic practice at this time [53]. Similarly, an 1885 list of university alumni [OU85, p. 86] describes Corbin simply as a teacher in Louisville4 but describes another alumnus (John Newton Templeton) as “the only Alumnus...of African descent” [OU85, p. 78].

When Corbin arrived at OU, the university was functioning as a standard 19th century US university. It offered a B.A. degree for students who completed a 4-year sequence of college courses. The university also maintained a Preparatory Department which offered a 2-year sequence designed to prepare students for the college courses.5 At the start of Corbin’s first semester, enrollment stood at 64 students, divided more or less evenly between college and preparatory students. They were taught by five professors, each being responsible for all coursework in a given subject. The graduation requirements differed from those of a modern American university in that students did not select majors. Instead, all college students took the same fixed sequence of courses.

The required coursework heavily emphasized the study of Greek and Roman literature, but it also included a mathematics curriculum that covered algebra and elementary geometry in the first year; trigonometry, analytic geometry, and differential calculus in the second year; and integral calculus in the third year.7

At OU, Corbin was taught by two different math professors, William J. Hoge and Addison Ballard. As was common for American math professors at the time, neither Hoge nor Ballard had any specialized training in math.6 Corbin’s first math classes were taught by Hoge, an alumnus of the university who had graduated with an undergraduate degree in 1843. Hoge left the university while Corbin was still a student (in 1851), and he was replaced by Ballard. Ballard had received a bachelor’s degree from Williams College and had been serving as OU’s Latin Professor since 1848.

Career. When Corbin graduated from Ohio University in 1853, his formal education came to an end,7 and he spent the next two decades working as a teacher, bank clerk, and newspaper editor in Louisville and Cincinnati.

In 1872, Corbin moved to the former slave state of Arkansas. This was not an uncommon move for talented

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4See William’s entry in the 1850 US census. He was described as a “porter” in the 1860 census.
5The description of Corbin as a teacher in Louisville suggests that his entry in the list was based on old information collected shortly after Corbin graduated (during the 1850s or 1860s, when he was in Louisville). At the time the list was published, Corbin had been living in Arkansas for over a decade.
6While uncommon today, many 19th century universities maintained similar departments. Indeed, doing so was often necessary as schooling along the lines of modern K–12 education was often only available on a limited basis.
7Corbin’s analytic geometry and calculus classes were taught using Albert E. Church’s books Elements of the Differential and Integral Calculus and Elements of Analytical Geometry, while the algebra and geometry classes were taught using textbooks by Charles Davies. Algebra was taught from Elements of Algebra: Translated from the French of M. Bourdon. It is not entirely clear which geometry textbook was used as Davies wrote several. The assigned text is listed as “Davies’ Legendre” in the 1851–50 university catalogue and as “Davies” in 1852–53 university catalogue. The first reference is to Elements of Geometry and Trigonometry from the works of A. M. Legendre, but the second reference could either be to that book or to another one such as Elements of Descriptive Geometry.
8See [Bur11, pp. 659–669] for a more detailed discussion of the education of a typical math professor in nineteenth century America. Formal advanced education in mathematics was out of reach for most as it typically necessitated studying abroad. American universities only began awarding earned doctorate degrees in 1861.
9Corbin would later receive two A.M. degrees from OU and a PhD from an unknown Baptist college, but these were honorary degrees.
Black men like Corbin, as the political changes brought about by the Civil War created unprecedented opportunities for change. After working briefly as a newspaper reporter and a post office clerk, Corbin was elected to oversee Arkansas’s public education system as superintendent of public instruction. His election made Corbin the head of the board of trustees for the state’s newly founded public university—Arkansas Industrial University (now the University of Arkansas).

The university was technically “open to all, without regard to race, sex, or sect,” but Black students were not allowed to attend classes with white students. Instead, they received private tutoring. To address the issue of college education for Black Arkansans, the board of trustees, including Corbin, petitioned the state legislature to create a school for educating Black teachers. The legislature responded favorably to the petition and created the Branch Normal College (now the University of Arkansas at Pine Bluff).

Corbin’s term in office was cut short when Conservative Democrats, who were largely hostile towards Black Arkansans, gained control over the state government and vacated many political offices, including Corbin’s. Corbin left the state and moved to Jefferson City, Missouri, where he taught at the newly founded Lincoln Institute (now University). However, after a brief time there, he returned to Arkansas to serve as the first principal of the Branch Normal College.

Since the 1873 legislative act, no real progress had been made in establishing the college. Under state law, the Branch Normal College was supposed to provide the same education as the Normal Department at Arkansas Industrial University. A normal department or school was a common 19th-century institution that offered postsecondary education for school teachers which typically involved a truncated college curriculum along with specialized coursework in pedagogy. However, achieving parity with Arkansas Industrial University was an unrealistic goal for the Branch Normal College as many of the college’s students were facing rural poverty and political disenfranchisement. To further complicate the situation, Corbin was given few resources—the college had no permanent facilities and no faculty aside from Corbin, who was even responsible for menial tasks like cleaning classrooms. Nevertheless, Corbin was able to make major improvements. By the 1880s, a college building and dormitories had been constructed, and the faculty had increased to three. By 1883, student enrollment had reached over 200 students.

The Branch Normal College offered an academic education through three different departments: (a) the Preparatory Department which offered remedial classes to prepare students for college work, (b) the Normal Department which awarded a Licentiate of Instruction (a teaching certificate), and (c) the Collegiate Department which awarded a Bachelors of Arts. The majority of students were enrolled in the Preparatory Department. The preparatory curriculum consisted of three years of courses on subjects such as grammar, geography, drawing, and math. The math education consisted of six terms of elementary arithmetic and one of algebra.

According to the 1882 college catalogue, arithmetic was taught using books from Robinson’s Series of Mathematics, a popular series of textbooks. It is unclear precisely which books were used as they are only described as “Arithmetic,
Students in the Normal Department completed two years of remedial work followed by two years of college courses. In addition to classes on Latin, English grammar, and pedagogy, students studied algebra and geometry in their third year and then plane trigonometry in their fourth. Students in the Collegiate Department completed the courses taken by the normal students and then two additional years of college classes in subjects like foreign languages, English literature, and mathematics. The math classes offered in the last two years were a class on analytic geometry and an optional course on calculus.

The curriculum that Corbin promoted at the Branch Normal College came into conflict with an increasingly influential educational philosophy that focused on industrial education. This philosophy endorsed only a very limited academic education for Black students and instead emphasized the value of manual labor as a means to impart values like self-discipline and self-reliance. Many of those who promoted industrial education scorned Black students who saw academics as a way of obtaining personal advancement or fulfillment. Booker T. Washington, the most prominent promoter of industrial education, expressed the views of many in his autobiography. In an account of teaching in Alabama, he wrote that, "The students who came [to his school] first seemed to be fond of memorizing long and complicated ‘rules’ in grammar and mathematics, but had little thought or knowledge of applying these rules to the everyday affairs of their life" [Was07, p. 122]. To illustrate his point, he offered the example of a student whose fondness for math led him to study methods for computing cubic roots. Washington set students like him to activities that he deemed more appropriate: farm work and learning how to properly set a dinner table.

Seeking to implement industrial education in Arkansas, state legislators in the 1890s created a Department of Mechanical Arts at the Branch Normal and hired as department head William S. Harris, a white man from Virginia. He was effectively made head of the college by the board of trustees when they transferred key responsibilities from Corbin to Harris.

Corbin nominally remained the principal of the Branch Normal for approximately another decade. In 1902, the board of trustees replaced him by hiring Isaac Fisher, a recent graduate of Booker T. Washington’s Tuskegee Institute. Fisher’s hire was part of a plan to expand industrial education at the Branch Normal, but Fisher’s efforts to transform the college faltered in the face of mixed support from trustees and strong opposition by the Black community. While Corbin’s removal was a major setback to efforts to run the Branch Normal as an academic institution, the college continued to offer advanced courses taught by Corbin’s former student James C. Smith.

After his removal from the college, Corbin remained in Pine Bluff and served as principal of Merrill Public School (the local Black high school) until his death in 1911, at the age of seventy-seven.

Corbin’s mathematical work. Corbin first published in a mathematical periodical in 1882, when he was almost fifty years old. At this point in his life, Corbin had achieved remarkable success in politics and education, but this publication is the first record of his intellectual engagement with mathematics. This was a point of pride for Corbin, but it is unclear if these publications were the expression of a life-long interest or one developed late in life. His publication record shows both a level of sophistication and a depth of knowledge that extended beyond both what he’d have been able to identify this book.

[11] states that, from 1902 to January 1904, he served as president of Ouachita Baptist College (now University) in Camden. While newspaper records from the time record Corbin’s presence in Camden, the rest of this statement seems to be erroneous. During this period, the college’s president was one John W. Conger, not Corbin. The college was also not located in Camden. It has always been located in Arkadelphia, a town about 40 miles away. In fact, the college has no record of an association with Corbin. His employment, in any capacity, would have been highly unusual as Ouachita was a whites-only college during the 19th century. The author thanks Ouachita Professor and University Archivist Lisa Speer for correspondence on this issue.
been taught at university and what he was teaching at the Branch Normal College. For example, the calculus textbook he used as a student contained essentially no proofs and even omitted basic definitions like the definition of the definite integral as a limit of Riemann sums. Only the geometry courses he took went beyond mechanical work and included detailed proofs as part of a treatment of plane geometry.

The periodicals that Corbin contributed to were similar to, and in fact included, the *American Mathematical Monthly* which is currently published by the Mathematical Association of America. A typical periodical included a problems and solutions section together with book reviews and expository articles. Compared to today’s research journals, these periodicals served a very different audience and performed a different function. They were not intended as long-term records of original mathematical research. Instead, they were published to stimulate activity among people who otherwise had few outlets for their mathematical interests. Most readers were individuals studying math in relative isolation, with only limited access to mathematical literature and other resources. They included both amateur mathematicians and math professors, who were often the sole mathematicians at their universities. In an era when mathematicians were only beginning to organize themselves into professional societies, these periodicals performed a similar function in fostering a mathematical community.

Corbin’s first published problem is the following one: Separate the fraction $a/b$ into two parts, $c/d$ and $e/f$ such that $c + e = d + f$.

This problem appeared in a 1882 issue of *The Mathematical Magazine* [Cor82a]. The subsequent issue contained two methods for finding the solution $c/d = (a - b - 1)/(b - 1)$ and $e/f = (b^2 + b - a)/(b^2 - b)$.

The differential equation is $y^{(3)} - 7y' + 6y = 0$, and it can be solved using the techniques from Johnson’s book—standard techniques still taught to undergraduate students. Despite its routine nature, this problem appears to have interested readers (a number of solutions were submitted [CLD98]). Among those whose solutions were published was Princeton University professor Edgar Odell Lovett, who gave a complete solution even though he remarked that it was “a familiar one to students of differential equations.”

Lovett’s solution is particularly notable as it demonstrates that, despite their elementary nature, Corbin’s publications were even reaching American research mathematicians with an international reputation. Lovett had completed a doctorate in Germany, published in research journals, and was a member of several European professional societies.

Corbin’s differential equation problem must have been the product of significant self-study as the subject was not taught at Ohio University when Corbin was a student. Other problems drew on the plane geometry that Corbin

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21The MAA did not exist during Corbin’s lifetime, and the Monthly was privately published by the math professor Benjamin Finkel.

22For example, in addition to university professors, the contributors to the first volume of the Monthly included a counsellor at law (John Domon) and an examiner for the US Civil Service Commission (Theodore L. DeLand).

23While the MAA did not exist during Corbin’s lifetime, the American Mathematical Society was formed six years after Corbin’s first publication, in 1888. It was originally a New York-based society named the New York Mathematical Society; but it was given its current name and reorganized as a national society in 1894. Corbin was never an AMS member, and in general, the AMS was run as an elite and exclusive organization during his lifetime. Admission to the society required receiving a majority of ballots by members in an election after being proposed by two members and recommended by the AMS Council. In contrast, the periodicals Corbin published in were run in a democratic spirit. For example, in the introduction to the inaugural issue of the Mathematical Visitor, the editor wrote that he aimed to produce an “unpretending periodical” that inspired a love of mathematics, and he invited contributions from “professors, teachers, students, and all lovers of the ‘bewitching science.’”

24Although this is the only solution that was published, there are others. For example, $c/d = (an - bn - 1)/(nb - 1)$ and $e/d = (nb^2 + b - a)/(nb^2 - b)$ is a solution for any integer $n \in \mathbb{Z}$.
The difference (sum) of the products containing
While this method was certainly not original to Corbin, it
was not usually taught at American universities in the 19th
century. This topic is the subject of Corbin's only expository article, "Note on Elimination." This 1896 Monthly
publication [Cor96] is a short one-page note in which
Corbin explains a method for solving a system of two lin-
ear equations in two variables, \( x \) and \( y \). The method is
effectively Gaussian elimination, i.e., solve for one vari-
able by eliminating the other. He summarizes the rule by:
"The difference (sum) of the products containing \( x \) \( (y) \)
is equal to the difference (sum) of the numerical products."
While this method was certainly not original to Corbin, it
appears that it was not well known to American mathe-
maticians as he presents it as an alternative to the "deter-
minant method" (perhaps a version of Cramer's rule).

Corbin also published two problems on determinants.
The first appeared in a 1895 issue [Cor95] of the Monthly.\(^{27}\)

Find the quotient of
\[
\begin{pmatrix}
(s - a_1)^2 & a_1^2 & \ldots & a_1^2 \\
(s - a_2)^2 & a_2^2 & \ldots & a_2^2 \\
\vdots & \vdots & \ddots & \vdots \\
(s - a_n)^2 & a_n^2 & \ldots & a_n^2 \\
\end{pmatrix}
\div
\begin{pmatrix}
s - a_1 & a_1 & \ldots & a_1 \\
s - a_2 & a_2 & \ldots & a_2 \\
\vdots & \vdots & \ddots & \vdots \\
s - a_n & a_n & \ldots & s - a_n \\
\end{pmatrix}
\]

The desired expression for the quotient is
\[
\frac{1}{s^{n-1}} \left( s + \sum \frac{a_i^2}{s - 2a_i} \right) \cdot \frac{1}{1 + \sum \frac{a_i}{s - 2a_i}}.
\]
The solution is obtained by applying matrix manipula-
tions that preserve the determinant. First, observe that the
given ratio of determinants equals:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( 1 )</th>
<th>0</th>
<th>0</th>
<th>\ldots</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( 1 )</td>
<td>( s - a_1 )</td>
<td>( a_1 )</td>
<td>\ldots</td>
<td>( a_1 )</td>
<td></td>
</tr>
<tr>
<td>( 1 )</td>
<td>( s - a_2 )</td>
<td>( a_2 )</td>
<td>\ldots</td>
<td>( a_2 )</td>
<td></td>
</tr>
<tr>
<td>( 1 )</td>
<td>( s - a_3 )</td>
<td>( a_3 )</td>
<td>\ldots</td>
<td>( a_3 )</td>
<td></td>
</tr>
<tr>
<td>( 1 )</td>
<td>( s - a_n )</td>
<td>( a_n )</td>
<td>\ldots</td>
<td>( a_n )</td>
<td></td>
</tr>
</tbody>
</table>

Each column now has the property that almost all entries
are the same, and thus it can be simplified by subtracting
a suitable multiple of the first column. This yields:

<table>
<thead>
<tr>
<th>(-a_1^2)</th>
<th>(-a_2^2)</th>
<th>\ldots</th>
<th>(-a_n^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(s - 2a_1) )</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( s(s - 2a_2) )</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( 1 )</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td>( 1 )</td>
<td>0</td>
<td>\ldots</td>
<td>( s(s - 2a_n) )</td>
</tr>
</tbody>
</table>

\(^{27}\)The expression we give corrects a typo in [Cor95]. In the original printed
version, the bottom-right-most entry in the first matrix is \( s - a_n^2 \), and we have
changed this to \( s - a_n^2 \). To see that the entry \( s - a_n^2 \) is a typo, see the pub-
lished solution in the April 1896 issue of the Monthly.
Computing each determinant as the Laplace expansion along the first row, we get that the first determinant is
\[ s^n \prod (s - 2a_i) + a_1^2 s^{n-1} \prod (s - 2a_i) + \ldots \]
\[ + a_1^2 s^{n-1} \prod (s - 2a_i) + \cdots + a_n^2 s^{n-1} \prod (s - 2a_i), \]
and the second is
\[ \prod (s - 2a_i) + a_1 \prod (s - 2a_i) + a_2 \prod (s - 2a_i) + \ldots + a_j \prod (s - 2a_i) + \cdots + a_n \prod (s - 2a_i). \]

Factoring out \( \prod (s - 2a_i) \), we get the desired solution.

Corbin’s second published problem on determinants was his last publication. In Corbin’s book A Treatise on the Theory of Determinants, Corbin asked:

Muir gives the following problem: Prove:

\[
\begin{bmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c' \\
1 & d & dd' \\
\end{bmatrix} = (a - b) \begin{bmatrix}
1 & ab & a + b \\
1 & cd' & c + d' \\
1 & c'd & c' + d \\
\end{bmatrix}
\]

which, of course, can be solved by finding the terms of both determinants. Is there any method of changing from one form to the other which is direct?

The reference to “Muir” is a reference to Thomas Muir’s book A Treatise on the Theory of Determinants. The problem appears as one of the exercises in the chapter “Determinants in general” on the general properties of the determinant. The solution is similar to the one for the previous problem. The first matrix is manipulated using operations that preserve the determinant until its determinant is visibly equal to the determinant of the second matrix.

**Corbin’s legacy.** American mathematical culture underwent major changes around the time Corbin’s career came to an end. Four years after his death, the Mathematical Association of America was formed, in large part to provide a permanent institution to support the American Mathematical Monthly. The Monthly became an official journal of the MAA, and the loosely knit readership of math enthusiasts, which Corbin had been a part of, began to transform into an organized group of professionals. Black mathematicians continued to be members of this community. The inaugural charter members of the MAA included James T. Cater, a graduate of the HBCU Atlanta University and a math professor at Straight University (now part of Dillard University). At the Branch Normal College, events like Corbin’s removal were major setbacks to efforts to run the college as an institute of higher education. However, they did not end those efforts. Faculty continued to participate in national mathematical culture. In 1938, faculty at the college (then renamed the Kansas Agricultural, Mechanical & Normal College) joined the ranks of the American Mathematical Society when college professor William Louis Fields was elected to membership. Fields joined the MAA three years later.

In 1903, a year after he was removed from the Branch Normal College, Corbin delivered an address at the annual meeting of the Arkansas Negro State Teachers’ Association. The meeting was attended by over 100 school teachers and was reported in the press, so it provided Corbin an important forum for voicing his views on education. His speech reads as a rebuke of the educational philosophy represented by the decision to make Isaac Fischer principal of the Branch Normal College. Corbin said

[A] need of the negro is a supply of men with the necessary equipment and capacity to carry on his vast national enterprises. It may be a surprising thing to many that I claim that the negro has any enterprises of this nature; but the claim is easily verified.

As evidence for his claim, Corbin proceeded to give examples of achievements in professions like law, education, and farming. Were Corbin to give the speech today, one imagines that he would also include mathematics.

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28Specifically, it appears on page 66 of Muir’s book as Exercise 27 in Exercise Set IX in Chapter II.

29Other Branch Normal faculty who were later elected to AMS membership include Morris Edward Mosley and Willie E. Clark.

30By 1951, four additional Branch Normal faculty members (Mrs. Willie E. Clark, Garland D. Kyle, Nathan T. Seeby, Jr., and H. B. Young) had joined the MAA.
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References

[OU85] Ohio University, General Catalogue of the Ohio University from the Date of its Charter in 1804 to 1885, Athens, Ohio, 1885.
[53] Trip to Athens, Pomeroy Weekly Telegraph (Pomeroy, OH) (June 3, 1853).

Jesse Leo Kass

Credits

Figure 1 is from page 26 of the Ohio University Bulletin for the Year 1909: Alumni Number, courtesy of Ohio University Libraries, and is in the public domain.
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Spelman College: A Model of Success for Producing Black Women in Mathematics

Sylvia T. Bozeman, Emille Davie Lawrence, Yewande Olubummo, and Monica Stephens

At Spelman College, a private, historically Black, liberal arts college for women in Atlanta, Georgia, educating women to reach their fullest potential is a way of life. Since it was founded in 1881, Spelman has been a leading producer of women of African descent who earn bachelor’s degrees and go on to graduate and professional schools and successful careers in all fields of endeavor.

Mathematics is no exception. Over the past 20 years, Spelman, with annual enrollments averaging 2,100 students, has graduated about 18 students per year with bachelor’s degrees in mathematics. Almost 60 percent of those women continued their education beyond the undergraduate level. Even more impressive is that 10 percent of those who pursued advanced degrees earned a PhD in various fields. This includes 27 alumnae who earned doctorates in mathematics, statistics, biostatistics, or math education. The remaining graduates obtained employment in various areas such as education, engineering, technology, business, finance, and consulting, or entered masters or professional degree programs.

Indeed, Spelman is recognized as a powerhouse among undergraduate institutions that educate female mathematicians. In 2019, Diverse: Issues in Higher Education listed Spelman as the number 1 producer, among all US institutions, of African Americans with bachelor’s degrees in mathematics and statistics.¹ A 2013 study by the National Science Foundation ranked Spelman number 2 among institutions with the highest number of Black students with bachelor’s degrees who go on to receive doctorates in science and engineering, including mathematics.²

Spelman’s success in educating and supporting female mathematicians is a testament to its commitment to a unique educational mission. But given that, nationwide, women—and especially women of color—are underrepresented in a discipline historically dominated by white men, other colleges and universities can also play a role in expanding the presence of women in mathematics.

How, then, can institutions of higher education rethink and re-invigorate efforts to uncover and cultivate the mathematical talents of women? And how can they ensure that more women are prepared to use that expertise for society’s benefit? Spelman’s tradition of providing a mathematics


education tailored to its students may offer ideas worth emulating on other campuses.

**Founding Faculty Leaders Lay the Groundwork for STEM Learning**

Spelman College established its mathematics department in the mid-1920s. In 1929, Alma Ferguson became the first student to receive a bachelor’s degree in mathematics from the college, and she went on to earn a master’s degree from the University of Wisconsin-Madison. Ferguson taught briefly at Spelman and later taught and was chair of the mathematics department at the Atlanta University Laboratory High School from 1932 to 1939.

Over the past century, almost 1,000 Spelman students have earned bachelor’s degrees in mathematics. The department owes its success in producing female mathematicians to the leadership of highly accomplished and committed educators, both male and female, who foster an environment for STEM learning that sets high standards and provides the necessary support to encourage students to meet those standards. Among these faculty is one of the department’s first faculty members and its first chair—Georgia Caldwell Smith, the fourth African American woman to receive a PhD in mathematics from a US institution.

In the 1950s and 1960s, the department strengthened its faculty with the addition of several educators who held or earned doctorates in mathematics: Shirley Mathis McBey, Gladys Thomas Glass, and Etta Zuber Falconer. They took up the cause to increase the number of African American women in the mathematical sciences, added new courses in abstract algebra and real analysis, and encouraged more students to study under their tutelage.

In the 1970s, the Department of Mathematics changed leadership and added faculty who brought new expertise. In 1972, McBey moved from the department to become the inaugural Chair of the Division of Natural Sciences, which housed the departments of biology, chemistry, and mathematics. In that same year, Falconer, an outstanding mathematician and administrator, started her tenure as Chair of the Department of Mathematics, and Nagambal Shah, a statistician, joined the faculty. Two years later, Sylvia Trimble Bozeman joined the department. In 1975, Falconer transitioned to Chair of the Division of Natural Sciences as McBey left to join the National Science Foundation. In this role, they spearheaded faculty collaborations to attract funding and created new programs to serve all science and mathematics students. Under the direction of McBey and Falconer, the sciences at Spelman began to flourish.

**Meeting Students’ Individual Needs**

It soon became apparent that the College needed to meet a wide range of student needs in the sciences. A combination of curricular and co-curricular efforts were created to assist in the retention, development, and graduation of science and mathematics students. In addition to recruiting outstanding educators to provide quality instruction and developing a curriculum that prepares students for graduate school and careers, Spelman’s approach to mathematics education has focused on understanding and addressing its students’ individual needs.

As a highly selective institution, Spelman enrolls students who rank at the top of their high school classes. Still, new students arrive on campus with vastly different levels of academic ability and preparation. Some students, because they find schoolwork easy, have poor study skills. Others, because of a lack of encouragement and advising by teachers and counselors, are dissuaded from considering a STEM major in college (notwithstanding their obvious love and aptitude for science and mathematics). Still others, because of the level of prior instruction, lack skills foundational to their studies.

Many Spelman students face financial challenges that can have a negative impact on their learning, especially if they must work part-time jobs to help pay for educational costs. Students who are first in their families to attend college may have no one at home who understands and can help them navigate their new environment. And, as studies have shown, low visibility of female mathematicians, scientists, and engineers as role models is among the reasons many women do not pursue STEM education and careers or drop out of STEM disciplines along the way.

The Pre-Freshman Summer Science program was created in 1972 to address issues around academic preparation and study skills, the absence of role models, and the need to create community among science majors. It was designed for incoming first-year students majoring in biology with an interest in health science careers. In the 1980s, Spelman launched the Summer Science and Engineering Program (SSEP) to extend the same support to students interested in other STEM areas.

During these six-week programs, entering students could take science, mathematics, or engineering courses with the option of earning a few hours of college credit. Along with a central academic component, SSEP also sought to expose students to female role models with scientific careers, offer opportunities to develop relationships with other students and faculty, and provide an introduction to Spelman’s culture.

For more than 20 years, these programs produced students who were confident and ready to excel as science, mathematics, and engineering majors. The positive
outcomes of these first summer programs also led to the creation of the Women in Science and Engineering Scholars Program (WISE) at Spelman in 1987. Funded by NASA, the WISE Program provided incoming students majoring in STEM a six-week summer science session, academic support and counseling, and summer research experiences at NASA sites. Many WISE students earned graduate degrees in various STEM areas, including mathematics.

For many Spelman students, one of the most pressing needs is that of financial support. In recognition of this fact, scholarships and other forms of financial support are often included in initiatives providing academic support. At the department and the division levels, considerable attention is given to the creation of named scholars programs supported through external funding from companies, foundations, and federal agencies. Some scholars may have on-campus jobs as tutors or supplemental instructors for “bottleneck” STEM courses. Others may have a requirement to engage in independent study or research.

In 1995, Spelman embarked on comprehensive initiatives to address a broad range of student needs and to promote student academic success. For example, an Office of Science, Engineering, and Technical Careers was created and offered a suite of support activities, including assistance with the identification of summer research programs nationally and support for travel to conferences. The new Freshman Success Office focused on retention strategies such as understanding learning styles, note taking, test taking skills, time management, and other skills needed to ensure early success. It worked with faculty to provide tutorial support and supplemental instruction in gateway courses. These initiatives and others all lasted for a period and were replaced by variations as needed.

As a result of its institutional efforts, the college became one of six minority-serving institutions to receive a Model Institutions for Excellence grant from the National Science Foundation in 1995. This funding facilitated long-term infrastructure enhancements, including faculty development and undergraduate research, and provided scholarships for STEM students, particularly those entering science and engineering graduate programs and careers. In 2017, 26% of Spelman students received degrees in STEM compared to 16% of students at other historically Black colleges and universities and 17% at other liberal arts colleges.

Mentoring helps students see possibilities. Spelman’s mathematics faculty believe that in order for students to see possibilities for themselves, they need to see successful people in the field who look like them or with whom they can identify. By the late 1980s, Spelman had five African American female faculty members who held PhDs in mathematics—a distinction that was extraordinary at the time. Today, the department’s faculty—77 percent of whom have doctorates—reflects international, as well as racial and gender diversity.

These educators are at the heart of Spelman’s mentoring strategy. They invest time in getting to know the mathematics majors, hold them to extremely high academic standards, and encourage and nurture them to believe in their ability to do math—regardless of the disparate educational experiences and preparation they may have received prior to coming to Spelman. Furthermore, they understand that an effective mentor must convey confidence in the student’s ability to succeed. Students consider faculty members as “the experts,” not only on mathematical content, but often with respect to the student’s learning capacity. In order to help students advance, faculty must understand how much their opinions matter, whether spoken or not. Oftentimes, they must encourage a student to take on challenges beyond what the student feels they are capable of achieving. One result of mentoring at Spelman is the growth in the number of students who go on to a graduate program after earning a bachelor’s degree in mathematics.

Community building creates a supportive network. Many mathematics majors at Spelman say they were attracted to the department because of the sense of belonging and closeness they felt. To build community, the department engages students early and continuously, giving them ample opportunities to get to know their professors and peers. Students are assigned to a faculty adviser in the discipline (rather than an adviser in an administrative unit) who is regularly available to them for formal and informal meetings.

An Introductory Seminar in Mathematics course and mandatory math majors meetings foster relationships among students and faculty in the department. The seminar, a course for first- and second-year students, introduces majors to the diversity of people in the field, including alumnae and other women and people of color; helps them identify career opportunities available to mathematics majors; and enhances their appreciation of mathematics through exploring solutions to historical problems. The mandatory meetings, which the department hosts several times during the year, are forums where faculty share information about opportunities, alumnae return and speak, and upper-level students discuss their summer research experiences.

Mentoring, Community Building, and Research Are Pillars of Success

Spelman has a comprehensive approach to mathematics education that has a significant impact on the success of mathematics majors at the college. The most transformative elements of this approach—mentoring, community building, and research opportunities—are reflected in how the department serves its students.
The Mathematics Laboratory at Spelman is a central location for the creation of community among students. It is a multipurpose space where advanced students tutor peers in lower-level courses; students gather in study groups for upper-level courses; and faculty provide mathematics reference books and magazines. Students consider the lab, staffed by students with faculty or staff oversight, “our space.”

Research opportunities help students feel like mathematicians. Spelman’s mathematics faculty are strong advocates of student research. They understand that because many students have not been exposed to Black scientists or mathematicians, it can be difficult for them to relate to what it means to work in the field. To help students get a sense of how it feels to be a mathematician, the department has created research-based initiatives designed to give students “real world” experience, with support from outside funding sources.

In 1991, the department established Scholars in Mathematics at Spelman (SIMS), a program funded by the National Science Foundation for five years. The primary goal of SIMS was to prepare students for a graduate degree in pure or applied mathematics. Program activities included supervised research, a research issues course, participation in seminars, and attendance at conferences.

In 1993, Spelman launched the Center for Scientific Applications of Mathematics (CSAM). It operated for 10 years, with initial support from the Kellogg Foundation and later support from the Eastman Kodak Company. CSAM supported student research on and off campus, interdisciplinary faculty research, and the development of new interdisciplinary curricula. In addition to increasing the number of students involved in research, CSAM facilitated the production of several issues of the *Spelman Science and Mathematics Journal*, an undergraduate publication that featured research-based technical articles authored or co-authored by Spelman students, with some assistance from their mentors. The Center’s programming also included outreach in the form of summer sessions for local high school science and mathematics teachers that emphasized hands-on instruction and the use of technology.

The Mathematics Research and Mentoring Program (RaMP) is the most recent of several scholars programs the department has sponsored to prepare students with high potential for graduate education in the mathematical sciences. It includes faculty-mentored research for juniors and seniors, and opportunities for students to attend and present research at regional and national conferences. Originally funded by the National Science Foundation in 2011, Math RaMP is sustained through private donors, industry partners, and professional organizations such as the Mathematical Association of America. Over the past 10 years, the program has served more than 50 mathematics majors, including 21 who have gone on to graduate programs and three who have completed PhDs in mathematics-related areas.

**The Spelman Math Curriculum Continues to Evolve**

Given that there are a growing number of math-related graduate programs and careers open to students today, including mathematical biology, mathematical finance, data science, and other applied areas, Spelman’s mathematics department continues to progress, to ensure that graduates have the greatest flexibility in making choices about their future.

Recent hires include educators with experience in graph theory, combinatorics, mathematical biology, and data science. As a result, the department has added a new...
elective course in graph theory and, in summer 2021, led a research experience for undergraduates (REU) program in mathematical biology with five students. Because of the expertise new faculty bring, the department is considering additional elective offerings in applied statistics, applied linear algebra, and machine learning, as well as updates to existing electives.

Increasing opportunities through changes in the curriculum. The mathematics major at Spelman has been largely traditional. The core requirements include the calculus sequence for three semesters and one semester each of linear algebra, foundations of math, abstract algebra, real analysis, and computer programming. Students also select three elective courses and must take a second semester of either abstract algebra or real analysis.

Changes have been made to the curriculum recently that require all math majors to take the Introductory Seminar in Math course. Other changes in course offerings include new elective courses such as complex variables, linear algebra II, graph theory, and a special topics course on codes and cryptography. In addition, the required proof-based courses have added a group problem day; and the requirements for the BS have expanded to include more math electives. These changes to the curriculum have helped improve persistence in the major by giving students a good mathematical foundation and increasing their chances to succeed in required courses, especially at the upper level.

In consultation with industry and graduate school partners and with Spelman alumnae, the department is currently redesigning the major. The goal is to target skills and streamline courses to better prepare students for graduate school and career placement by giving them a competitive advantage in an area of their choice. Possible new concentrations within the major, to be implemented in the next two years, include pure math, applied math, statistics/data science, and education.

Increasing opportunities through new programs and initiatives. The college and the department are providing math majors with additional options and experiences through new programming. In 2019, Spelman, in collaboration with Clark Atlanta University, Morehouse College, and the Morehouse School of Medicine, launched the Atlanta University Center Data Science Initiative (DSI). The DSI provides training in data science to faculty and students and is encouraging the creation of data science and data analytics minors and majors at the partnering institutions.

In 2019, Spelman was designated as a US Department of Defense Center of Excellence. The college’s Center of Excellence for Minority Women in STEM honors Spelman’s legacy of preparing women for graduate school and careers in the STEM disciplines. Its goals are to strengthen faculty and student research in emerging fields, such as artificial intelligence and machine learning, and provide scholarships to mathematics majors who show potential to obtain graduate degrees in STEM. The center hosts conversations showcasing cutting-edge, interdisciplinary STEM research being conducted by prominent women of color such as Kizzmekia S. Corbett, PhD, who is a scientist in the Vaccine Research Center of the National Institute of Allergy and Infectious Diseases (NIAID). Dr. Corbett collaborated on the development of the Moderna COVID vaccine, and has inspired students with her accomplishments. Through this Center, mathematics majors are able to understand how mathematics connects to current interdisciplinary areas of research.

Spelman’s mathematics department is also benefiting from recent partnerships with PhD-granting universities and governmental agencies. Students have engaged in research in machine learning, discrete math, and mathematical finance through collaborations with the Carnegie Mellon University Department of Mathematics; Michigan State University Department of Computational Mathematics, Science and Engineering; and the Army Research Labs. Spelman and Michigan State have partnered on a 3+2 bachelor’s to master’s program for Spelman STEM majors interested in pursuing advanced study in data science. This partnership will also help Spelman move forward with plans to create a data science minor.

What Can Be Learned from the Spelman Model? As a leading college for women, Spelman’s mission includes being a champion for women to develop to their highest potential and to serve as leaders and change agents in every area of life. And, of course, the institution has a historic charge to meet the educational needs of Black women. Even so, are there elements of the Spelman story that can be adapted to other settings?

Ensure institutional commitment at all levels. Faculty are usually the visionaries who generate the ideas, create the programs, and lead the day-to-day engagement with students. However, to make significant and sustainable progress, faculty commitment must be backed by institutional commitment. Senior administrators must clearly communicate that faculty efforts are a priority, ensure adequate funding for programs, and enforce policies and practices that make it possible for initiatives to be born and grow to fruition.

Among the most important administrative policies and practices are intentional efforts to diversify the faculty. Institutions can send a strong message about their support for women and people of color interested in mathematics...
and the sciences by recruiting, retaining, and supporting the development of STEM faculty from these populations.

Clearly, faculty are in the best position to recognize a student’s interests and abilities and encourage them to take courses and participate in other academic pursuits. However, co-curricular programming is needed to take students to the next step by helping them identify and participate in special activities beyond the classroom, such as summer research or internship opportunities. Students also need assistance with developing key professional skills like resume writing and interview training, as well as needing help identifying career options in STEM. Investment in career centers and staff who can assist students in this way is critical.

**Set high expectations for all students.** Faculty can attest to the fact that many students will try to reach or exceed the level of achievement expected of them. When it comes to encouraging women to pursue and persist in mathematical studies and careers, setting high expectations is particularly important because, as mentioned earlier, many have not had role models to inspire them or champions to validate them.

First, faculty can foster a teaching and learning environment where students are accepted for the people they are. This means not just acknowledgement, but also affirmation of differences in race, gender, ability, and other markers of identity. Faculty can affirm differences, for example, by addressing students by their preferred pronouns or by inviting a diversity of role models to the department to engage with students. It also means demonstrating, in word and deed, the belief that all students, regardless of their backgrounds, can develop skills in mathematics and have fulfilling careers in the field.

Second, faculty can make an investment in students by serving not just as teachers, but also as coaches and mentors. By taking a personal interest in students, understanding their strengths and struggles, and giving them affirming feedback and guidance, faculty can play a significant role in helping students thrive.

**Pursue partnerships.** In working to increase the number of women in mathematics, colleges and universities should keep in mind that they do not have to “go it alone.” Those with limited resources to start or sustain programs can seek support and collaboration with other academic institutions, businesses and industries, government agencies, and foundations. Such initiatives may be essential to a mathematics department’s ability to sponsor activities, hire faculty and staff, purchase equipment, provide internships, and fund student scholarships.

A unique collaboration between Spelman College and Bryn Mawr College served both institutions and the mathematics community well. In 1998, faculty at the two women’s colleges found that they were facing a common challenge. Although the colleges sent well-prepared students to graduate programs in mathematics, many of those students dropped out for seemingly nonacademic reasons. To help address this issue, one mathematics faculty member from each of these two colleges joined efforts to create Enhancing Diversity in Graduate Education (EDGE). Beginning with funding from the National Science Foundation and early funding from the Mellon Foundation, the EDGE Program has become a comprehensive mentoring and training program dedicated to strengthening the ability of women to successfully complete graduate programs in the mathematical sciences. While introducing students to the rigor and pace of graduate courses, EDGE recognizes that academic preparation alone may not be enough. The program also encourages students to overcome isolation by integrating themselves into their new institution and departmental community – learning its culture, developing relationships with its people, and understanding its unwritten rules – all while maintaining their individual identities and values.

Known for its commitment to diversity of students, faculty, and staff and for the inclusion of students traditionally underrepresented in the mathematical sciences, the EDGE summer session has served more than 300 students, including 21 Spelman graduates. More than 100 EDGE participants have earned PhDs in a mathematical science, and many more are currently pursuing graduate degrees.

Another example of a productive collaboration is evident with MATHFest, a math conference for undergraduates sponsored by the National Association of

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**Figure 3.** Spelman Math Jeopardy team at the 2013 MAA Southeastern Section Meeting. Left to right: Kayla Echols, Lynesia Taylor, Raven Smith.
Mathematicians (NAM). Well attended by Black and Hispanic mathematics majors, MATHFest provides students with an opportunity to present their research, engage in problem-solving with peers, and meet mathematicians of color, and it encourages students to pursue further study and careers in the mathematical sciences. Spelman College faculty have been closely connected with NAM since Etta Falconer became the first NAM Secretary in 1970. The College hosted the second annual MATHFest in 1992 and most recently hosted it in 2018.

Securing a bright future for women in mathematics. For the past century, Spelman’s mathematics department has worked deliberately to develop and support women in the mathematical sciences. As a result, it has earned an international reputation for the high caliber of its students and their outstanding achievements. The department’s efforts have drawn energy and impetus from the college’s mission to provide an exceptional liberal arts education for young Black women and its promise that “every Spelmanite graduates with a competitive edge.”

There is every reason to believe that many other institutions are equally capable of developing and supporting women in the mathematical sciences. The choices and strategies that influence Spelman’s success—including visionary faculty leadership, a supportive administration, a focus on role models and mentoring, the acquisition of resources through strategic partnerships, and a commitment to continuous improvement—may also provide ideas for other colleges and universities that take on this challenge.

Spelman’s STEM programs and initiatives have evolved and will continue to evolve with the times, but several factors remain constant: dedicated faculty who have high expectations of their students and mentor them in college and beyond; a rigorous curriculum that challenges students in traditional and emerging areas; strategic partnerships that provide valuable student development opportunities; and a supportive, nurturing environment that helps students build confidence and grow to achieve their potential.

We hope other institutions that want to contribute to increasing the number of women—and especially women from underrepresented groups—who study mathematics and pursue mathematics-related careers, may find inspiration and ideas from Spelman College’s success.

Spelman Alumnae Success Stories

Over 500 Spelman mathematics graduates have earned advanced degrees and are making contributions in teaching and leadership positions at schools, colleges, and universities or in various mathematics-related roles in the corporate, government, and nonprofit sectors. The following are some achievements of a few of these graduates.

Cynthia Wallace (C’1993)
- MS in Statistics, University of North Florida
- Current: North Carolina State Banking Commission Member
- Career Highlights: A 25-year career in the financial services industry; Synchrony (formerly GE CAPITAL), VP, Credit Acquisition; 2020 Democratic Congressional Candidate (North Carolina)

Talithia Williams (C’2000)
- PhD in Statistics, Rice University
- Current: Associate Professor of Mathematics at Harvey Mudd College
- Career Highlights: honorary Doctorate of Humane Letters from Fielding Graduate University; Mathematical Association of America Pólya Lecturer; Mathematical Association of America’s Adler Award for Distinguished Teaching
EDUCATION

Shelby Wilson (C’2006)
- PhD in Applied Math, University of Maryland
- Current: Senior Data Scientist, Johns Hopkins University Applied Physics Lab.
- Career Highlights: Associate Professor of Mathematics at Morehouse College; Co-founder, Mathematically Gifted & Black

Anisah Nu’Man (C’2009)
- PhD in Mathematics, University of Nebraska
- Current: Assistant Professor of Mathematics, Spelman College
- Career Highlight: MSRI ADJOINT Program, Co-Director

Asia Wyatt (C’2013)
- PhD in Applied Mathematics, Statistics, and Scientific Computation, University of Maryland
- Current: Senior Professional Staff at The Johns Hopkins Applied Physics Laboratory (APL)
- Awarded an APL Innovation Initiative grant to explore autonomous flocking algorithms

Janelle Jones (C’2006)
- MA in Applied Economics, Illinois State University
- Current: Chief Economist, US Department of Labor
- Career Highlight: Economic analyst, Economic Policy Institute

Tanya Moore (C’1995)
- PhD in Biostatistics, UC Berkeley
- Current: Founder and Managing Partner of Intersecting Lines, LLC
- Career Highlights: VP of Mission Advancement for Goodwill of San Francisco, San Mateo, and Marin; Co-founder of The Infinite Possibilities Conference

Victoria Seals (C’1991)
- Ed.D in Educational Leadership, University of Georgia
- Current: President, Atlanta Technical College
- Career Highlight: Dean of Academic Affairs, Gwinnett Technical College

Ché Smith (C’2005)
- PhD in Biostatistics, University of North Carolina, Chapel Hill
- Current: Senior analytics engineer, Netflix
- Career Highlight: Mathematical statistician, US Food and Drug Administration

Jennika Gold Thomas (C’2001)
- MS in Computational Finance, Carnegie Mellon University
- Current: Global Head of Data, Morningstar
- Career Highlight: VP and Associate Director of Risk and Fixed Income Analytics at FactSet

Michelle Craddock Guinn (C’2004)
- PhD in Mathematics, University of Mississippi
- Current: Associate Professor, Belmont University
- Career Highlight: Postdoctoral Fellow at the US Military Academy (Davies Fellowship)

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Interfacing Music and Mathematics: A Case for More Engagement

Lawrence C. Udeigwe

The opinions expressed here are not necessarily those of the Notices or the AMS.

Introduction

Music is as old as humankind and, since the time of ancient civilization, there have been knowledge seekers interested in exploring various elements of music. Mathematics, in both elementary and advanced forms, has been used in the analysis of lower-level elements of music such as tempo, pitch, timing, chord formation, and meter. Music-related physical phenomena, such as acoustics, and their implication on the development of musical instruments have readily employed mathematics for analysis. Essential information-age appliances, such as computers and cell phones, that have played a substantial role in the dissemination and preservation of music rely heavily on mathematics for their implementation and continual improvement. These are but a few areas of interface between mathematics and music, yet the subject of “mathematics of music,” which I will henceforth refer to as musical mathematics, has been relegated to the fringe and seems to only be touchable to bold algebraists and number theorists who do not mind being labeled hobbyists. The assertion of this article is that from the classroom through private and government organizations interested in STEM, there is a need to create more room for the exploration of the interface between music and mathematics.

There is Historical Precedent for Engagement

The Pythagoreans in ancient Greece were the first researchers to link musical scales and the principles of numbers [Pla74], although there are records showing that the ancient Chinese, Indians, and Egyptians studied the mathematical principles of sound [Bri87]. Pythagoras’s apocryphal experiment with vibrating strings showed him that if he plucked two strings of equal tension, in which the length of one is of a certain proportion to the length of the other, he would get a certain type of harmonious sound, referred to today as consonance. In particular, if the ratio of the lengths of the two strings is 2:1, one gets an octave, that is, two of the same note with the pitch frequency of the shorter string being double that of the longer string; if their ratio is 3:2, the pitch frequency of shorter one is $1\frac{1}{2}$ times that of the longer one, and their combined sound—simultaneously or temporally apart—is referred to as a perfect fifth; and if their ratio is 4:3, their combined sound is referred to as a perfect fourth. Pythagoras’s vibrating string experiment ushered the incorporation of musical sound into the philosophical framework of the Pythagoreans, ultimately helping to shape their central doctrine that “all nature consists of harmony arising out of numbers” [Jea68].
Although some may argue that this music-driven doctrine of the Pythagoreans had no quantifiable significant positive influence on the development of science and mathematics, it inspired later scientists like Johannes Kepler and Galileo Galilei; Kepler attempted to find consonant music intervals in the orbits of the planets [Kep97].

There is Foundational Work in Modern Mathematics to Facilitate Engagement

Even though earlier researchers, such as the Pythagoreans, devoted efforts to studying sound and harmony in relation to numbers, their efforts did not materialize into documented axiomatic underpinnings for music in modern mathematics. Nevertheless, the mathematical exploration of musical sound and structure has found a home in many areas of modern mathematics.

While discussing the geometric series in a calculus class a few summers ago, a student of mine pointed out a subtle use of this series in determining the duration of a musical note or rest with dots. The majority of the class, including two students who are musicians, expressed surprise that a concept used in calculus could be easily applied to reading musical notes. In conventional Western music, it is common for a whole note to get four beats; a half note gets two beats; a quarter note gets one beat; an eighth note gets a half beat, etc. (see Fig 1). If a note has a dot behind it, its duration is extended by one-half its original length, that is, the dot multiplies the original duration by $\frac{3}{2}$. For instance, the duration of a dotted eighth note is given by

$$\frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right) = \frac{3}{4}.$$  

Sometimes, however, composers use multiple dots to effectively convey their duration on a note. Digital automated notations also readily use multiple dots to efficiently notate the precise duration that sometimes only a computer can capture. A second dot beside a note implies an additional duration of one-fourth the original duration, and so on. Fig 2 shows how different numbers of multiple dots modify the half-note. This means an eighth-note with two dots would have a duration of

$$\frac{1}{2} + \frac{1}{2} \left(\frac{1}{2}\right) + \frac{1}{4} \left(\frac{1}{2}\right) = \frac{7}{8}.$$  

Thus $D_N$, the length of a note of duration $D$ followed by $N$ dots is given by

$$D_N = D \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^N\right) = D \sum_{k=0}^{N} \left(\frac{1}{2}\right)^k = D \frac{1 - \left(1/2\right)^{N+1}}{1 - \left(1/2\right)} = D \left(2 - \left(1/2\right)^N\right).$$  

One way in which most music strives to interest and please the listener is through the presentation of temporal patterns with acoustic cohesion. Mathematicians have used set theory and abstract algebra to explore this compositional attribute of music. Starting at a C, the set of pitches in the chromatic scale is

$$\{C, Db, D, Eb, E, F, Gb, G, Ab, A, Bb, B\}.$$  

The chromatic scale has a free and transitive action of the cyclic group $\mathbb{Z}/12\mathbb{Z}$, with the action being defined via transposition of notes. So the chromatic scale can be thought

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Basic musical notes and their temporal durations.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Modification of a half-note by dots.}
\end{figure}
of as a torsor for the group. If we label these notes with integers, i.e., \{0,1,2,3,4,5,6,7,8,9,10,11\}, we can then define an X major chord, in its simplest form, as the collection \{X, X+4, X+7\} modulo 12\mathbb{Z}. For instance, the C major chord \{C,E,G\} can be expressed as \{0,4,7\}. A transposition in music is analogous to a translation in mathematics, and in fact, the latter can be used to model the former. Transpositionally related chords are the same up to translation. Hence, the C major chord is transpositionally related to the G major chord, \{G,B,D\} or \{7,11,2\}, because \{7,11,2\} \equiv \{0 + 7, 4 + 7, 7 + 7\} modulo 12\mathbb{Z}. An inversion in music is analogous to a reflection in mathematics and corresponds to subtraction from a constant value. Chords related by an inversion are the same up to reflection; hence, the C major chord is related to the C minor chord, \{C,F,A\} or \{0,3,7\}, by inversion because \{0,3,7\} \equiv \{-7,-7,-4,7-0\} modulo 12\mathbb{Z}. Just like translation and reflection preserve the characters of a map in mathematics, musical transposition and inversion are important because they preserve the character of a chord.

Starting with elementary foundational formulations involving mathematical transformations (such as the ones described in the preceding paragraph) for organizing motives and chords, one can analyze a piece of music to discover deep structures in it. One example is [Tym06], in which Dmitri Tymoczko's goal is to model composers' assessment of the relative sizes of voice leadings. Even though one cannot assume that these assessments will be consistent with any mathematical norm or metric, the fact that transposition and inversion can be treated as preserving musical distance allowed him to require that voice-leading comparisons be invariant under transposition and inversion of any individual musical voice.

Another example is the representation of regular temperaments. In music, a regular temperament is any tempered system of tuning such that each frequency ratio is obtainable as a product of powers of a finite number of generators, or generating frequency ratios. In twelve-tone equal temperament (12-TET), the tuning system used in most Western music, the octave is divided into 12 parts, all of which are equally tempered (equally spaced) on a logarithmic scale. The theory of regular temperaments has been extensively developed with a wide range of sophisticated mathematics, for example by associating each regular temperament with a rational point on a Grassmannian [Maz18] (The Grassmannian \(G_r(k, V)\) is a space that parameterizes all \(k\)-dimensional linear subspaces of the \(n\)-dimensional vector space \(V\)). In essence, this formulation allows one to replace sound events in a musical piece with operators that act on all possible musical parameterizations.

These examples, though far from exhaustive, illustrate that there is enough foundational work and a fertile ground in modern mathematics to facilitate research work towards a rich and sustainable interface between mathematics and music. A comprehensive compilation on the use of mathematics in music can be found in Guerino Mazzola et al’s four volume book, “The Topos of Music” ([Maz17a], [Maz17b], [MGMD0509], [Maz02]), and some recent advances in the area can be found in the Journal of Mathematics and Music published by Taylor and Francis Online.

The Field of Music is Vast and Every Part of it Needs Mathematical Engagement

There is a wide array of acoustic endeavors that can be classified as music. From the perspective of the listener, the classification of these endeavors may be synonymous with the mainstream music genres (i.e., classical, jazz, Afrobeats, techno, etc.). A utilitarian, however, may choose to classify musical endeavors as dance, elevator music, children’s music, film music etc. The practitioners in each of these fields of music have needs that open avenues for more interactions with the mathematical world, just as mathematical interactions have emerged for each field of say, biology. The use of number theory and abstract algebra in music is currently restricted to classical music—jazz, with its complex use of dissonance, for instance, has not fully benefited from these activities—and even at that, most of the work done in this regard is hardly given the exposure (say, at conferences and in textbooks) that it deserves. One common reason that is usually given for this is that the language of mathematics is already difficult and that adding the extra task of learning the language of music theory makes research work in musical mathematics inaccessible. While there is some truth to this, other fields that were once deemed very far from mathematics in terms of function and language, have successfully created niches inside mathematics. A good example is biology: while biology as a scientific field has existed for almost as long as mathematics, the subject of mathematical biology experienced a surge in interest only in the 1900s, and mathematicians have managed to identify the different ways to use mathematics in ways that serve the particular need of each subfield. Today we have niches such as theoretical neuroscience, computational immunology, mathematical epidemiology, etc. In sum, this same phenomenon can also happen in musical mathematics, though it would require a comprehensive identification, assessment, and inclusion of all the possible niches that make up musical mathematics.

Mathematics Has a Role in Interfacing Science and Music

Music clearly provides great examples of many interesting phenomena in hearing, and as such is a constant source of inspiration for basic hearing cognition research (see [GMTB18] and [AHV+21] for examples). There are also
active clinical endeavors that employ musical sounds to directly improve lives. Clinical music therapy, for instance, has been linked to stress reduction, mood improvement, and improvement in self-expression; and is regarded by many in the health field as an evidence-based therapy (see [MST+20] and [GLW+06] for examples). Hence, there are long-standing interests in the scientific use of music within several scientific fields (such as neuroscience, psychology, and cognitive science). Broad research questions in this line of work have included: what makes music pleasurable? Why do some music pieces sound good and others do not, and why is this judgment subjective? And why do we have music to begin with? There also have been very targeted questions like how neurons in a certain area of the auditory cortex (the part of the brain responsible for processing auditory signals) become selective to music over time, even without musical training [BNHMK21]. These are big questions that cannot be answered scientifically alone. It requires the efforts of the interested scientist to guide the right experiments as well as those of a music mathematician to perform the right theoretical analysis. Furthermore, being able to tap into these questions, I believe, will help unravel some emergent musical phenomena. For instance, the intricate interaction between speed, harmony, and melody that jazz improvisation presents yearns for explorations from every tangential field – psychology, neuroscience, and anatomy. More engagement from mathematicians will help to develop a better and more complete framework of exploration.

The Music Community is Ready and Looks Up to Mathematicians

My conversations with musicians inform me that they are very aware of the connection between mathematics and music. Most of them express the desire to explore this connection and truly believe that they stand to gain much if they can only take a step in the right direction. Even though many of them do not have the mathematical expertise to meaningfully contribute to the mathematics of music, a number of them have contributed to the discussion of the mathematics-music interface in artistically substantial ways. For instance, in music theory, the circle of fifths is a way of organizing the 12 chromatic pitches as a sequence of perfect fifths (see Fig 3). A well-known story in the jazz community is that the famous saxophonist John Coltrane had a long-lasting fascination with the connection between mathematics and music. It is said that he constructed “Coltrane’s Circle of Tones” (see https://www.openculture.com/2017/04/the-tone-circle-john-coltrane-drew-to-illustrate-the-theory-behind-his-most-famous-compositions-1967.html) in an attempt to formulate an axiomatic connection between geometry and music. Many musicians believe that there is still a lot to uncover mathematically with the circle of fifths and other examples of musicians making efforts to interface mathematics and music abound. Martin Scherzinger, an ethnomusicologist, has devoted a good amount of effort towards mathematically analyzing southern African dance music through the lens of Zimbabwean Matepe and Mbira music [Sch17]. In 2016, the Herbie Hancock Institute of Jazz began the first phase of Math, Science, and Music, an initiative that uses music as a tool to teach math and science to young people in public schools across the United States. Initiatives of this type, though non-technical, could make good use of mathematicians’ insight for success.

The NSF Needs to Get Involved

The National Science Foundation (NSF) could do a lot to amplify the mathematics of music. This can start with a long-term workshop that brings together a diverse group of experts in the areas of STEM that intersect with music. The overarching goal should be to address mathematical (and scientific) issues that are relevant to this research community and to expose possible areas of opportunity for multidisciplinary music-related research deeply rooted in mathematics and science. A workshop of this nature would be instrumental in identifying the most substantive research questions that can be addressed by researchers in the field of musical mathematics. It will also help recognize the community needs, knowledge gaps, and barriers to research progress in this area. Furthermore, it will help in identifying future directions that cut across disciplinary boundaries that are likely to lead to transformative multidisciplinary research in musical mathematics.

Conclusion

We do not need a survey or study to prove that exploring music is a meaningful and gratifying experience for anyone engaging in it. Even so, studies continue to show that music education enhances learning skills, creativity, teamwork, discipline, cultural awareness, and self-esteem...
Opinion

Mathematics has a big void to fill in the quest to continue exploiting all the benefits of music to humankind.

References


Credits

Figures 1–3 are courtesy of Lawrence C. Udeigwe. Photo of Lawrence C. Udeigwe is courtesy of Ty Smith.
Almost Periodic and Almost Automorphic Functions in Abstract Spaces
Reviewed by Sorin G. Gal

Almost Periodic and Almost Automorphic Functions in Abstract Spaces
By Gaston N’Guérêkata

It is known that a periodic function is a function that repeats its values at regular intervals. But the sum of two periodic functions is not always a periodic function. For example, \( f(x) = \sin(x) + \sin(\sqrt{2}x) \) is not periodic.

In mathematics, an almost periodic function is, loosely speaking, a function whose value is approximately repeated when its argument is increased by properly selected constants (called almost periods) and the above mentioned example is an almost periodic function.

In 1925–1926, Harald Bohr published a series of foundational articles on almost periodic functions. Formally speaking, an almost periodic function is a continuous function \( f : \mathbb{R} \to \mathbb{C} \) such that for every \( \epsilon > 0 \), there are infinitely many \( \tau = \tau(f, \epsilon) \) satisfying \( \sup_{t \in \mathbb{R}} |f(t + \tau) - f(t)| < \epsilon \) such that the size of the gaps between successive \( \tau \)'s is bounded. Each such \( \tau(f, \epsilon) \) is said to be an \( \epsilon \)-period.

In 1961, Solomon Bochner [2] (see also Bochner [3] in 1964) introduced “a weakened concept of almost periodicity” (defined below) for which he said “we will designate this weakened concept as almost automorphy because we have been encountering it first, and on several occasions then, in the (differential geometric) study of automorphic functions on real and complex manifolds.”

In the following decades, the theory was extended to functions with values in a Banach space, with all the results collected together in the books: L. Amerio and G. Prouse, Almost Periodic Functions and Functional Equations, Van Nostrand Reinhold, New York, 1971 and C. Corduneau, Almost Periodic Functions, Chelsea Publishing Co., New York, 1989, and to functions with values in locally convex spaces by G. M. N’Guérêkata in his 1980 PhD thesis [5].

The singular event which ignited a surge of interest in the theory of almost automorphic functions and applications to evolution equations was the 2001 publication of Gaston N’Guérêkata’s Almost Automorphic and Almost Periodic Functions in Abstract Spaces (Kluwer Academic / Plenum, 2001).

This second edition of the book represents a remarkable effort in laying down the foundation of the theory of almost automorphic functions in Banach spaces and the theory of almost periodic functions in locally convex and nonlocally convex spaces and their applications to evolution equations.

The publication of the first edition of this book aroused widespread interest in the theory of almost automorphic functions and its applications to evolution equations. As a consequence, it was one of the most cited analysis books.

Sorin G. Gal is a professor of mathematics at the University of Oradea, Romania. His email address is galso@uoradea.ro.
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of the year 2001. In the years to follow, several generalizations of almost automorphic functions were introduced in the literature, including the study of almost automorphic sequences. In addition, the interplay between almost periodicity and almost automorphy has been explored for the first time in the context of operator theory, complex variable functions, and harmonic analysis methods. Readers will welcome the second edition of this book not only because it clarifies and improves on what was in the first edition, but because it presents new results, methods, and references. Also, this edition answers the fundamental question: “What is the number of almost automorphic functions that are not almost periodic in the sense of Bohr?”

Open problems for almost periodic and almost automorphic functions with values in nonlocally convex spaces (p-Fréchet spaces) are also presented. For example, one major problem discussed is the lack of an established theory of integration of continuous functions with values in nonlocally convex spaces. This chapter includes background material on topological vector spaces and operators which is used later in the text.

Chapter 2 gives a lovely introduction to the theory of Bochner–almost automorphic functions with values in a Banach space X. More precisely, it deals with continuous functions \( f: \mathbb{R} \to X \) such that for every sequence of real numbers \((s_n)\), there exists a subsequence \((s'_n)\) such that
\[
\lim_{m \to \infty} \lim_{n \to \infty} f(t + s_m - s_n) = f(t)
\]
for each \( t \in \mathbb{R} \). This is equivalent to saying that a continuous function \( f: \mathbb{R} \to X \) is said to be almost automorphic if for every sequence of real numbers \((s_n)\), there exists a subsequence \((s'_n)\) such that \( g(t) = \lim_{n \to \infty} f(t + s_n) \) exists for each \( t \in \mathbb{R} \) and
\[
\lim_{n \to \infty} g(t - s_n) = f(t)
\]
for each \( t \in \mathbb{R} \). The function \( g \) above is measurable but not necessarily continuous. When the convergence above is uniform in \( t \in \mathbb{R} \), one can show that the function is almost periodic in the sense of Bohr. The function
\[
f(t) = \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{2} t}\right)
\]
is a typical example of an almost automorphic function which is not almost periodic.

Chapter 2 and the two that follow present basic properties of almost automorphic functions in a unified, homogeneous, and masterly way which lays the groundwork for applications to differential equations and dynamic time scales. These chapters should help researchers fill gaps in the theory and develop further extensions and applications in various fields.

Since it is hard to write down examples of almost automorphic functions, for a long time people wondered “how many” almost automorphic functions are not almost periodic. The answer was given by Z-M Zheng, H-S Ding and G.M. N’Guérékata in their remarkable 2013 paper which proves that the collection of all almost periodic functions is a set of first category in the space of all almost automorphic functions. This means that almost automorphic functions exist in plentitude. This result constitutes a major reason to study almost automorphic functions and consequently increases the interest in this book.

Another fundamental question is the following: if a function \( f: \mathbb{R} \to X \) is almost automorphic, when is the integral \( F(t) = \int_0^t f(\xi) d\xi \) also almost automorphic? In the case of almost periodicity, the answer was given by Kadets [4]. It states that a necessary and sufficient condition is that \( X \) does not contain any subspace isomorphic to \( c_0 \), the Banach space of all numerical sequences \((u_n)\) such that \( \lim_{n \to \infty} u_n = 0 \), equipped with the supnorm. B. Basit [1] was able to extend Kadets’s theorem to the almost automorphic case. It’s too bad that the author does not present the proof of the important result of Basit in [1]. He gives a nice proof of this result in the case of a general Banach space \( X \). This result states that if a function \( f: \mathbb{R} \to X \) is almost automorphic, then the integral \( F(t) \) is almost automorphic if and only if the range of \( f(t) \) is relatively compact in \( X \).

Chapter 7 of the book deals with semigroups of bounded linear operators that behave like almost automorphic functions at infinity. Some of their topological and asymptotic properties based on the Nemyskii and Stepanov theory of dynamical systems are included. It is unfortunate that the author does not give a more in-depth presentation of the material in Chapter 7, specifically the study of asymptotically almost automorphic motions of dynamical systems and possible stability in the sense of Poisson motion. However, the exposition of this topic may lead to further research in this direction.

Chapters 8 and 9 deal with almost periodic functions with values in locally convex spaces and with values in nonlocally convex spaces. For example, in their 2007 paper in the Global Journal of Pure and Applied Mathematics, the reviewer and G.M. N’Guérékata investigated the theory of Bohr–almost periodic functions in \( p \)-Fréchet spaces, \( 0 < p < 1 \), which are nonlocally convex spaces. More precisely, for \( 0 < p < 1 \), a vector space \( X \) over \( \mathbb{R} \) or \( \mathbb{C} \) is called a...
$p$-Fréchet space, if it is endowed with a so-called $p$-norm $\| \cdot \|$ and $X$ is a complete metric space with respect to the metric $d(x, y) = \| x - y \|$. Recall that a $p$-norm satisfies the following conditions: $\| x + y \| \leq \| x \| + \| y \|$, $\| x \| = 0$ if and only if $x = 0$, $\| \lambda x \| = |\lambda| \| x \|$.

The challenging question was to start with an appropriate definition. The authors were able to show that several results in the locally convex spaces setting hold in $p$-Fréchet spaces, $0 < p < 1$. However, due to the geometry of $p$-Fréchet spaces, $0 < p < 1$, especially because the Hahn–Banach theorem does not hold in such spaces, the fundamental theorem of calculus fails to be true. This implies the nonexistence of mean-values of such functions, a very important property of almost periodic functions with values in Banach spaces, which constitutes an obstacle in studying differential equations in nonlocally convex spaces. These open problems are presented in Chapter 9 of the book.

Chapters 10 and 11 discuss the applications of the previous results to differential equations in finite and infinite dimensional spaces.

An advantage of this second edition is that it corrects and gives a nice proof of the important result by D. Bugajewski and G.M. N’Guérékata stating that if $E$ is a Fréchet space, then the space of almost periodic functions $f : \mathbb{R} \to E$ is also a Fréchet space.

The Appendix is also a significant strength of the book. In particular, it presents a useful and simplified graph, showing the various relations between the classes of functions studied.

This book can serve as a reference for seminars and research on almost automorphy and almost periodicity in abstract spaces. It is accessible to anyone who is familiar with graduate-level functional analysis and operator theory and is easy to read. The exposition is clear and the proofs are given in detail. At the end of each chapter, there is a bibliography for the chapter content that an interested reader can use to further explore topics of interest. Also, the open problems presented in Chapter 9 on almost periodic and almost automorphic functions with values in nonlocally convex spaces ($p$-Fréchet spaces) are a unique asset of this second edition.

References


New and Noteworthy Titles on our Bookshelf
February 2023

**Political Geometry**
Rethinking Redistricting in the US with Math, Law, and Everything In Between
Edited by Moon Duchin and Olivia Walch

In recent years, issues regarding gerrymandering and how mathematics can be applied to help address this have been brought to the forefront. Many may remember Moon Duchin’s inspiring and thought-provoking 2018 JMM plenary talk on this topic and be immediately curious about her new book *Political Geometry* as a result.

Gerrymandering occurs when the lines of a voter district are drawn unnaturally to benefit one particular group. *Political Geometry* contains articles on the topic of gerrymandering from multiple perspectives. It was written by authors with a wide range of expertise including mathematics, political science, law, and philosophy. The collection discusses how to spot gerrymandering, how various voting laws have impacted redistricting, and how race and political party are involved. An entire chapter is devoted to explaining how mathematics and computer science can be used to fight gerrymandering, using tools such as entropy, spatial measures, and algorithmic redistricting. The authors even propose policy changes that may help to combat gerrymandering.

Aimed at a general audience, this book includes something for everyone. It is filled with real-world examples and wonderful graphics. The wide array of authors offers a wholistic and complete treatment of gerrymandering and redistricting issues, including how mathematics can be used to help with this ongoing problem while factoring in legal limitations and philosophical questions of fairness.

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**The Development of Mathematics Between the World Wars**
Case Studies, Examples and Analyses
Edited by Martina Bečvářová

World War I caused shockwaves that rippled through Europe (and around the world), having a profound impact on all aspects of life. The mathematics of the time is no exception: mathematical minds were lost to war, collaborations between citizens of different nations or members of different faiths became tense and often crumbled, and people struggled to recover from the destruction around them. As a result, the mathematics that was developing prior to WWI was stunted and new mathematics began to be developed and flourish in new regions of Europe.

*The Development of Mathematics Between the World Wars* explores the impact that WWI had on mathematics, specifically in Central and Eastern Europe. It contains chapters dedicated to countries such as Germany, Poland, and Czechoslovakia. Some countries, like Germany, lost some of their mathematical prowess. Conversely, other countries, such as Poland, emerged as new mathematical hubs where fields like measure theory and probability theory were developed by mathematicians from a diverse collection of backgrounds. This is a very interesting and thorough exploration of how the war altered mathematics that would be highly recommended to mathematicians and math historians alike. While the focus is on Europe, the concluding chapters explore the global effects of the war on fields such as combinatorics and potential theory.
Mathematics for Social Justice: Focusing on Quantitative Reasoning and Statistics
Edited by Gizem Karaali and Lily Khadjavi

The state of Connecticut keeps very good data on traffic stops in the state. Is the fact that non-White drivers accounted for 38% of the stops but 63% of the vehicle searches indicative of bias? How about the fact that 61% of the White drivers were released with just a warning, but that only 53% of the non-White drivers were treated that way?

It is easy to imagine creating an exciting course module investigating this data source for your Introductory Statistics or Quantitative Reasoning course. Actually you don’t have to imagine it; it is Chapter 8 in Mathematics for Social Justice: Focusing on Quantitative Reasoning and Statistics. It is one of seventeen projects designed to be dropped into your statistics or QR course that investigate themes of social justice including income inequality, gerrymandering, and policing.

The modules share a common structure. There is extensive background information to introduce the social, political, or economic issue addressed. The authors describe the students and course they teach, indicate other courses for which the module might be appropriate, and describe the mathematical outcomes and prerequisites of the activity.

This volume (and its companion published in 2019) powerfully makes the case that we have a responsibility as citizen-teachers to engage more seriously with the problems of social science and politics.

Welcome to Real Analysis: Continuity and Calculus, Distance and Dynamics
By Benjamin Kennedy

Welcome to Real Analysis is designed for use in an introductory undergraduate course in real analysis. Much of the development is in the setting of a general metric space; the book makes substantial use not only of the real line and n-dimensional Euclidean space, but also sequence and function spaces. The more abstract ideas come to life in meaningful and accessible applications: for example, the contraction mapping principle is used to prove an existence and uniqueness theorem for solutions of ordinary differential equations, and the existence of certain fractals; the continuity of the integration operator on the space of continuous functions on a compact interval paves the way for some results about power series.

The exposition is exceedingly clear and well-motivated. There are a wide variety of exercises and many pedagogical innovations: for example, each chapter includes “Reading Questions” which are meant to probe students’ understanding of the text and are designed to spark classroom discussion of subtle or important points. The first eight chapters comprise the standard topics in a first real analysis course in the setting of general metric spaces. The last two chapters really show off the power of working at this level of abstraction. Chapter 9 investigates dynamical systems and completely describes the conjugacy between particular intervals maps and subshifts of finite type. In Chapter 10, Kennedy introduces the Hausdorff metric on compact subsets of the plane in order to study fractals. If you want your real analysis course to incorporate metric spaces from the beginning and you want your students to be able to read their textbook—this book is an excellent choice.

The AMS Bookshelf is prepared bimonthly by AMS Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is skennedy@amsbooks.org.

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The Role of Mathematics in Today’s Movement for Racial Justice

Evelyn Lamb, Omayra Ortega, and Robin Wilson

Introduction

“Racism may wear a new dress, buy a new pair of boots, but neither it nor its succubus twin fascism is new or can make anything new. It can only reproduce the environment that supports its own health: fear, denial, and an atmosphere in which its victims have lost the will to fight.”

—Toni Morrison

“But all our phrasing—race relations, racial justice, racial profiling, white privilege, even white supremacy—serves to obscure the fact that racism is a visceral experience, that it dislodges brains, blocks airways, rips muscle, extracts organs, cracks bones, breaks teeth. You must never look away from this. You must always remember that the sociology, the history, the economics, the graphs, the charts, the regressions, all land with great violence on the body.”

—Ta-Nehisi Coates

“Never forget that justice is what love looks like in public.”

—Cornel West

Using mathematics as a tool to critically analyze systemic racism has a long history in the United States. In 1900 W. E. B. Du Bois predicted, quite prophetically, that “the problem of the 20th century is the problem of the color line.” Du Bois was also among the first to invoke mathematics and statistics to analyze issues of racial injustice through his Data Portraits Visualizing Black America. The first Black person known to have earned a graduate degree in mathematics in the U.S. was Kelly Miller, who went on to use what he had learned as a graduate student at Johns Hopkins to challenge the flawed statistics of eugenics in Fredrick Hoffman’s 1896 book Race Traits and Tendencies of the American Negro and, as a faculty member at Howard University, taught mathematics as a tool for understanding social issues. The first known Black woman to enter graduate school in mathematics was Anna Julia Cooper, who later dedicated her life to the struggle for racial justice. Almost a century later, former mathematics teacher and civil rights leader Bob Moses declared that “mathematics literacy is the literacy of the 21st century,” and that the failure to provide equitable mathematics education for all has helped maintain the color line that still threatens our democracy.

The 2021 MSRI Workshop on Mathematics and Racial Justice, which took place online in June 2021, was born out of an effort from members of the mathematics community to engage with the national conversation about racial justice that came to the forefront in the summer of 2020 after the video recording of the brutal murder of George Floyd by a Minnesota police officer captured the world’s attention. The protests and demonstrations in the months that followed brought the issues of police violence, racial injustice, and anti-Blackness into classrooms, boardrooms,
and dinner tables around the country and around the world.

In response, many institutions began to ask whether there was more that they could do. MSRI has a history of support for diversity and inclusion in the mathematical sciences dating back to the work of former MSRI Director William Thurston and Deputy Director Lenore Blum that led to MSRI hosting the first Conference for African-American Researchers in the Mathematical Sciences (CAARMS) in 1995. So, it was not a surprise when MSRI Deputy Director Hélène Barcelo reached out to members of the organizing committee with an offer to provide a platform for a conversation about race and the math community. They were not initially certain how they should act; they were certain, however, in their conviction that mathematicians can bring unique skills and abilities to the struggle for racial justice that are urgently needed in the world. They decided, with the support and assistance of the MSRI leadership, on organizing a mathematics research workshop that would provide an opportunity for the broader mathematical sciences community to learn about how scholars use mathematics as a tool for understanding and exploring issues of racial injustice.

Our country’s history of racism endures despite gains made during the Civil Rights Era and this legacy continues into our present. Educational, financial, medical, judicial, and political institutions are not without culpability in committing injustices towards Black people. Through the workshop, the organizers intended to engage mathematicians and other scholars within the mathematical sciences who might work to dismantle this legacy.

The murders of unarmed Black people at the hands of the state are disturbing echoes of the history of the extreme violence of state-sanctioned Jim Crow laws and the lynching of Black people in the U.S. In fact, while we were drafting this introduction, 10 people were murdered in an anti-Black terrorist attack in a grocery store in a predominantly Black section of Buffalo, NY, and the algorithms that govern social media are suspected to have played a role in inciting this violent act.

As mathematicians who see every day the potential for analyzing and guiding solutions to some of the world’s most pressing problems, we recognized that it was time for the mathematics community to rally around scholars using mathematics to analyze, study, and guide solutions for the problems of racial justice and to educate and inspire mathematics educators and researchers to take an active role in creating and implementing those solutions.

The MSRI Workshop on Mathematics and Racial Justice focused on the following guiding question: How can mathematics be used to identify and dismantle the ways in which biases manifest in social constructs, specifically those constructs concerning racial justice? The organizers of the workshop selected four key areas on which to focus: Bias in Algorithms and Technology; Fair Division, Allocation, and Representation; Public Health Disparities; and Racial Inequities in Mathematics Education. In this article, we share examples of the thought-provoking talks from the workshop. For a more extensive summary of the workshop events, read the workshop compendium available for download at http://library.msri.org/msri/Math-and-Racial-Justice.pdf.

**Bias in Algorithms and Technology**

In her plenary talk about the sources and consequences of algorithmic bias, University of Texas at Austin researcher Maria De-Arteaga described a case study she and colleagues had done about bias in crime-reporting statistics and its effect on predictive policing using data from Bogotá, Colombia. (Note: We wish to bring attention to the ongoing police brutality in both Bogotá and Colombia as a whole, where police are violently repressing civilians’ protests and disproportionately targeting Black and indigenous communities.)

Predictive policing is increasingly deployed in cities and countries around the world and has come under scrutiny due to a lack of transparency and concern about biased outcomes. One focus of these critiques has been the potential for dangerous feedback loops when using arrest data as a basis of crime prediction. The issue is that the proxy, arrests, may not represent where crimes are being committed due to biases in arrests. In a paper about this phenomenon, Lum and Isaac write, “predictive policing is aptly named: it is predicting future policing, not future crime.” They demonstrate that using data on drug arrests in Oakland, California as inputs to the model used by PredPol, one of the large companies providing predictive policing systems, would result in high concentrations of policing in racial and ethnic minority neighborhoods. In another paper, Ensign, et al. used a generalized Pólya urn model to theoretically analyze how feedback loops in arrest-based predictive policing systems arise.

In response, proponents of predictive policing have agreed that there is a concern there but say that they do not use this type of data in their models. PredPol, for example, says that their algorithms are unbiased by nature due to the fact that the data collected and analyzed is primarily victim data. When they do use arrest data, they exclude certain types of drug-related offenses and traffic citation data, which are usually police-initiated, because it is known to reflect officer bias.

In a study conducted in Bogotá, Colombia, Akpinar, De-Arteaga, and Chouldechova demonstrate how differential victim crime reporting can lead to geographical
outcome disparities in hotspot predictions, even when the predictive policing algorithms do not use arrest data. These disparities result in both over- and under-policing. They found that there were large disparities in crime reporting between districts. In one district, Chapinero, 9% of people were victims of a crime, and 28% of them reported it. In Usaquén, 18% of people were victims of a crime, and 13% of them reported it. When trained only on reported crime data, some districts required more than double the crime rate of other districts to be selected as hotspots.

The problem is not confined to the researchers’ choice of model, a self-exciting process model, which is being considered for deployment in Bogotá and already used by PredPol. They looked at other types of models that are used in predictive policing and found very similar results. Therefore, the problem will be present in any of the predictive policing models currently in use. In summary, differences in victim crime-reporting rates can lead to geographical bias in common hotspot prediction algorithms, even when no data from arrests or police-initiated contact is used. These algorithms can therefore lead to misallocation of police patrols in the form of both over-policing of some neighborhoods and under-policing of others. Victim crime-reporting rates are known to be driven by socio-economic factors, types of crime, and other demographics. More work is needed for an in-depth discussion of the interplay between predictive disparities and these factors in the Bogotá context.

**Fair Division, Allocation, and Representation**

The fairness of voting systems is a racial justice issue. In the early history of the U.S., Black people were not allowed to vote at all. After the Civil War, unjust voting laws in many states kept them from exercising that right. Today, the gutting of the Voting Rights Act, the dispropor- tionate number of Black people who cannot vote due to felony convictions—especially for nonviolent drug-related crimes—and congressional districts that distort the will of the people continue to dilute the political power of Black people and other marginalized groups. Stephanie Somersille, founder of Somersille Math Education Services, spoke about the role mathematicians can play in combatting gerrymandering (the process of drawing congressional districts that favor one party) and drawing fairer congressional districts, highlighting a new metric for quantifying gerrymandering that she has been involved in developing.

One of the obstacles to addressing gerrymandering in court is that there is no widely agreed-upon metric defining gerrymandering. The “you know it when you see it” definition of gerrymandering is unsatisfying. Being able to measure the problem is crucial in addressing it.

Currently popular metrics fall into two camps: map data metrics and election data metrics. The map metrics, such as the Polsby-Popper ratio, the Roeck ratio, and the convexity coefficient, and the convex hull, are based on the geometric irregularity of the shapes of districts. The Polsby-Popper ratio, for example, is proportional to the ratio of the area of the district to the square of its perimeter. The convexity coefficient is based on the probability that the straight line between any two points in the district is itself entirely contained within the district.

Map data metrics have a few weaknesses. For one, the physical geography of a district influences the metric. A river, mountain range, or coastline can make a district seem gerrymandered when it is not. Furthermore, with modern technology, it is easy to generate thousands of maps with the same score on any metric of interest and choose the one that best suits one’s agenda.

Election data metrics, on the other hand, are based on voting patterns in recent elections and tend to assume the existence of two dominant political parties. These metrics include the mean-median difference, efficiency gap, partisan bias metric, and declination function. They include data such as the number of wasted votes — votes for the losing candidate and votes for the winning candidate beyond the majority necessary to win — and comparisons between statewide and district-wide election margins. These metrics can have shortcomings as well, such as flagging proportional outcomes as gerrymandered in regions that are dominated by one party.

Somersille and colleagues have developed a new metric, the Geography and Election Outcome metric (GEO metric), that uses both geography and election data to measure gerrymandering. The basic idea is to look at wasted votes and determine whether swapping some voters with a neighboring district would cause the relevant party to gain an additional seat. (For more details about the metric, see the compendium.)

The GEO metric is promising for several reasons. It is understandable, which is helpful for use in court. It recognizes the importance of both geography and election data in determining whether a district is gerrymandered. Finally, it flags districts for two of the primary forms of gerrymandering: “packing” and “cracking.” A politician who is drawing a map in their own interest often “pack” some voters for the opposing party into a few districts that the opposition will win by a large majority and “crack” others into several districts that the map-drawer’s own party will win by slim margins.

One new metric for assessing gerrymandering will not solve the problem on its own. Mathematicians who wish
to use their skills and training to work for fairer elections and congressional maps need to develop a strong background in election history and laws in addition to mathematics.

**Public Health Disparities**

Emma Benn, a statistician at the Icahn School of Medicine at Mount Sinai, spoke about the need for a paradigm shift in the way public health researchers attribute and discuss race and racism as contributors to health outcomes. Every year, researchers publish hundreds of studies that demonstrate race- and ethnicity-related health disparities.

Researchers have the tendency to attribute these health disparities to biological differences between races, but race is a social construct only tenuously rooted in genetics. The fact that study after study finds race/ethnicity-related health differences but that race has little biological meaning creates difficulty in attributing a causal effect to race. Paul Holland, a statistician working for the Educational Testing Service, raised this point in a 2003 paper. He argued that the measured effects of race do not have a causal interpretation. He believed that causes of outcomes should be experiences that individuals undergo, not attributes that they possess. Causal variables themselves must reflect the possibility of manipulation. Skin bleaching and plastic surgery aside, race is not mutable. Race therefore does not fit into an inferential framework, although it may play an important role in causal studies for descriptive reasons. Holland writes, “In my opinion, RACE can play an important descriptive role in identifying important societal differences such as those in wealth, education, and health care. The attribution of cause to RACE as the producer of these differences is, to me, the most casual of causal talk and does not lead to useful action.”

Race may play a role as an effect modifier; that is, an intervention or exposure may have a different effect on an outcome across racial/ethnic groups. Findings of that nature can help researchers delve deeper into the effects of discrimination and bias. But many studies stop at race and never delve into naming racism, rather than race, as a cause of health disparities. Unless the amount of melanin in skin can cause an outcome, caution should be used when ascribing a causal role to race rather than to racism or its downstream effects. Furthermore, the overarching goal of medical research is not to describe differences; it is to reduce disease and improve health. So a focus on race as a causal factor is less helpful than a focus on causal factors that can be changed.

If researchers are to move from describing racial differences to identifying mutable targets for intervention, then race cannot be the endpoint. In 2020, the convergence of the COVID-19 pandemic and widespread acknowledgement of the crisis of systemic racism in the U.S. motivated researchers to move beyond individual-level associations between race and health to studies that look at broader systems and structures. When researchers suspect that biological and genetic differences between races may indeed contribute to different health outcomes, genetic ancestry data would be a better variable to include than the more blunt tool of race, as a 2021 article by Akinyemi Oni-Orisan and coauthors argues. They write,

We do not believe that ignoring race will reduce health disparities; such an approach is a form of naive “color blindness” that is more likely to perpetuate and potentially exacerbate disparities. Although ignoring race could improve equality (by leading to identical medical treatment for everyone), we believe that equity is necessary to address disparities. We urge our colleagues in medicine and science to refrain from haphazardly removing race from clinical algorithms and treatment guidelines in response to recent initiatives attempting to combat anti-Black racism. The ultimate goal, we believe, would be to replace race with genetic ancestry in an evidence-based manner. But we have not yet reached a point where genetic-ancestry data are readily available in routine care or where clinicians know what to do with these data. Until we do, ignoring race and thereby reverting to the United States’ outdated system of health care, in which clinical research findings are generated in populations of European descent and extrapolated to the treatment of non-European populations, is neither equitable nor safe for those other populations.

**Racial Inequities in Mathematics Education**

Brittany Mosby, the Director of Historically Black Colleges and Universities (HBCU) Success programs and initiatives at the Tennessee Higher Education Commission, spoke about creating mathematics classrooms that are liberatory for students.

Liberatory education is education that becomes “the practice of freedom,” to use a term coined by Paulo Freire, educator, philosopher, and author of the book *Pedagogy of the Oppressed*. Liberatory education can be transgressive and disruptive of the status quo in the classroom because its goal is to create beings who are able to change their society, not to create employable workers. Liberatory education is community- and dialogue-centered rather than centered on individualism. It is an anti-oppressive, humanizing pedagogy whose goal is self-actualization, not only of the students but also of their instructors. It engages with the world outside the classroom. To practice freedom
within education requires tearing down the walls between the classroom and the outside world.

Some people use nationalistic aims, whether military or economic, as motivation for increasing the participation of Black and Brown students in STEM fields. Education as the practice of freedom, however, sits in direct opposition to those aims. Instead of focusing on creating people who can produce more and more economic value for the country, liberatory education focuses on creating holistic, lifelong learners who are able to question and challenge society and who have the tools necessary to do so.

There are several key elements of a liberated mathematics classroom. First, it affirms students’ existing cultural knowledge and mathematical intuition. Teachers are not overly concerned with pointing out where students are wrong but with working in tandem with them to deepen their knowledge of a subject and work towards more complete understanding.

A liberated mathematics classroom also highlights the utility of mathematics as a language to understand problems across multiple fields. When students ask, “when will I use this” or “why do I need to know this,” a teacher should be able to point to the utility of mathematics as one reason. Teachers should be incorporating problems from multiple other fields into their classrooms to demonstrate the versatility and power of the techniques students are learning.

A liberated mathematics classroom encourages metacognition and agency in the learning process. When a teacher is in community with students, the students are able to take responsibility for their learning, which requires them to think about how they are learning and whether they are learning effectively. They should also be equipped to fix problems they have with learning.

A liberated mathematics classroom does not rely solely on lecturing to impart knowledge. This assertion may be the point of biggest pushback on the framework of liberating education because mathematics educators feel so much pressure to get through a large number of learning objectives. Lectures may be necessary sometimes, but even in lectures, students should be engaged, active participants, not empty vessels.

A liberated mathematics classroom balances rote, single-skill practice with complex, contextualized, multi-step problems. Fluidity with some rote skills is a helpful, and often necessary, step for students who want to apply mathematics to more complex problems, but even when focusing on rote skills, teachers can incorporate student discovery into the process. For example, instead of merely memorizing the multiplication table, students can find patterns and symmetry in the multiplication and figure out why. Then teachers can incorporate contextualized problems that use multiplication.

Finally, a liberated mathematics classroom is decolonized. It decenters whiteness, maleness, and European-ness. It introduces students to non-Western foundations of mathematics, such as those in the Arab world and Africa, and includes the history of women’s contributions to mathematics.

A classroom where mathematics is the practice of freedom is a classroom that centers mathematics as a process of discovery. The mathematics classroom is a place of collaboration, among students, between students and instructors, and between the students and the content as co-creators of that content. Rather than being overly concerned with smacking down wrong ideas, it empowers students along their journeys. This process is radical and often uncomfortable, sometimes even to students, who may be used to being passive recipients of information rather than active co-creators.

**Conclusion**

Historically, mathematics has been used as both an instrument of oppression and an instrument of liberation; mathematics education has reinforced racial hierarchies, but it has also been a gateway for freedom and opportunity. Algorithms and statistics can perpetuate or identify and mitigate racism; mathematical tools can be used to create fairer elections or entrench unjust power dynamics. Mathematics has a role to play in today’s movement for racial justice, and mathematicians can choose how to use their skills to advance justice. The speakers at the MSRI Workshop on Mathematics and Racial Justice showed how they use their expertise in a broad range of mathematical fields, including geometry, statistics, and data science, to document injustice and develop solutions for it. Using their presentations as jumping-off points, participants joined breakout rooms to generate ideas and concrete plans to take action in their own classrooms and communities. For example, participants who focused on bias in algorithms and technology proposed the development of an “FDA for algorithms,” outlining the way such a regulatory body could prevent harmful algorithms from being employed in high-stakes decisions affecting consumers and citizens, while participants in the room dedicated to public health discussed working with hospitals, NGOs, and universities conducting medical research to provide support for more robust and complex studies of racial disparities in health. For a more detailed exposition of the topics addressed in the workshop, see the workshop compendium: [http://library.msri.org/msri/Math-and-Racial-Justice.pdf](http://library.msri.org/msri/Math-and-Racial-Justice.pdf)
References


Credits

Photo of Evelyn Lamb is courtesy of Evelyn Lamb. Photo of Omayra Ortega is courtesy of Timothy Archibald. Photo of Robin Wilson is courtesy of Tom Zasadzinski, University Photographer at Cal Poly Pomona.
In the elections of 2022, the Society elected a vice president, a trustee, five members at large of the Council, three members of the Nominating Committee, and two members of the Editorial Boards Committee. The membership also adopted four bylaws changes.

**Vice President**

Bianca Viray  
*University of Washington*  
Term is three years  
*(February 1, 2023–January 31, 2026)*

**Board of Trustees**

Judy L. Walker  
*University of Nebraska-Lincoln*  
Term is five years  
*(February 1, 2023–January 31, 2028)*

**Members at Large of the Council**

Term is three years *(February 1, 2023–January 31, 2026)*

Christine Berkesch  
*University of Minnesota Twin Cities*

William A. Massey  
*Princeton University*

Sam Payne  
*University of Texas at Austin*

Emily Riehl  
*Johns Hopkins University*

Cynthia Vinzant  
*University of Washington*
Election Results
FROM THE AMS SECRETARY

Nominating Committee

Term is three years (January 1, 2023–December 31, 2025)

Jayadev Athreya
University of Washington

Kathryn Leonard
Occidental College

Alan W. Reid
Rice University

Editorial Boards Committee

Term is three years (February 1, 2023–January 31, 2026)

Robert Guralnick
University of Southern California

Kate Juschenko
University of Texas at Austin

All four bylaws amendments passed. These are described in the election section of the September 2022 issue of the Notices (p. 1401, available at https://www.ams.org/journals/notices/202208/rnoti-p1401.pdf). The resulting current version of the bylaws is available at https://www.ams.org/bylaws.
Vice President or Member at Large

One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2023. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures below.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures below.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and procedures, which are described below. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.

Editorial Boards Committee

Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The president will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Nominating Committee

Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The president will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Rules and Procedures

Use separate copies of the form for each candidate for vice president, member at large, or member of the Nominating or Editorial Boards Committees.

1. To be considered, petitions must be addressed to Secretary, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213, USA, and must arrive by 24 February 2023.

2. The name of the candidate must be given as it appears in the American Mathematical Society’s membership records and must be accompanied by the member code. If the member code is not known by the candidate, it may be obtained by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).

3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.

4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.

5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.

6. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot.
Nominations by Petition

The undersigned members of the American Mathematical Society propose the name of
_________________________________________________ as a candidate for the position of (check one):

☐ Vice President (term beginning 02/01/2024)
☐ Member at Large of the Council (term beginning 02/01/2024)
☐ Member of the Nominating Committee (term beginning 01/01/2024)
☐ Member of the Editorial Boards Committee (term beginning 02/01/2024)

of the American Mathematical Society.

Return petitions by February 24, 2023 to:
Secretary, AMS, 201 Charles Street, Providence, RI 02904-2213, USA

Name, address, and AMS member code, if available (printed or typed)

_________________________________________________
Signature

_________________________________________________
Signature

_________________________________________________
Signature

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Signature

_________________________________________________
Signature

_________________________________________________
Signature
AMS Prizes & Awards

Leroy P. Steele Prize for Lifetime Achievement

About this Prize
The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students. The amount of this prize is US$10,000.

Next Prize: January 2024

Nomination Period: 1 February – 31 March 2023

Nomination Procedure: https://www.ams.org/steele-prize

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Leroy P. Steele Prize for Seminal Contribution to Research

About this Prize
The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:
1. Open (2025)
2. Analysis/Probability (2026)
3. Algebra/Number Theory (2027)
4. Applied Mathematics (2028)
5. Geometry/Topology (2029)
6. Discrete Mathematics/Logic (2024)

Next Prize: January 2024

Leroy P. Steele Prize for Mathematical Exposition

About this Prize
The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper. The amount of this prize is US$5,000.

Next Prize: January 2024

Nomination Period: 1 February – 31 March 2023

Nomination Procedure: https://www.ams.org/steele-prize
Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prizes for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

**Chevalley Prize in Lie Theory**

The Chevalley Prize is awarded for notable work in Lie theory published during the preceding six years; a recipient should be at most twenty-five years past the PhD.

**About this Prize**
The Chevalley Prize was established in 2014 by George Lusztig to honor Claude Chevalley (1909–1984). Chevalley was a founding member of the Bourbaki group. He made fundamental contributions to class field theory, algebraic geometry, and group theory. His three-volume treatise on Lie groups served as standard reference for many decades. His classification of semisimple groups over an arbitrary algebraically closed field provides a link between Lie’s theory of continuous groups and the theory of finite groups, to the enormous enrichment of both subjects.

The current prize amount is US$8,000, awarded in even-numbered years, without restriction on society membership, citizenship, or venue of publication.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Submit a letter of nomination, complete bibliographic citations for the work being nominated, and a brief citation that might be used in the event that the nomination is successful.

To make a nomination go to [https://www.ams.org/chevalley-prize](https://www.ams.org/chevalley-prize).

**Frank Nelson Cole Prize in Algebra**

This prize recognizes a notable research work in algebra that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

**About this Prize**
This prize (and the Frank Nelson Cole Prize in Number Theory) was founded in honor of Professor Frank Nelson Cole upon his retirement after twenty-five years as secretary of the American Mathematical Society. Cole also served as editor-in-chief of the *Bulletin* for twenty-one years. The original fund was donated by Professor Cole from moneys presented to him on his retirement, and was augmented by contributions from members of the Society. The fund was later doubled by his son, Charles A. Cole, and supported by family members. It has been further supplemented by George Lusztig and by an anonymous donor.

The current prize amount is US$5,000, and the prize is awarded every three years.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

To make a nomination go to [https://www.ams.org/cole-prize-algebra](https://www.ams.org/cole-prize-algebra).

**Levi L. Conant Prize**

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

**About this Prize**
Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon the death of his wife.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006.

The Conant Prize is awarded annually in the amount of US$1,000.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Nominations with supporting information should be submitted online. Nominations
Calls for Nominations & Applications

FROM THE AMS SECRETARY

should include a letter of nomination, a short description of the work that is the basis of the nomination, and a complete bibliographic citation for the article being nominated.

To make a nomination go to https://www.ams.org/conant-prize

Ulf Grenander Prize in Stochastic Theory and Modeling

The Grenander Prize recognizes exceptional theoretical and applied contributions in stochastic theory and modeling. It is awarded for seminal work, theoretical or applied, in the areas of probabilistic modeling, statistical inference, or related computational algorithms, especially for the analysis of complex or high-dimensional systems.

About this Prize

This prize was established in 2016 by colleagues of Ulf Grenander (1923–2016). Professor Grenander was an influential scholar in stochastic processes, abstract inference, and pattern theory. He published landmark works throughout his career, notably his 1950 dissertation, Stochastic Processes and Statistical Interference at Stockholm University, Abstract Inference, his seminal Pattern Theory: From representation to inference, and General Pattern Theory. A long-time faculty member of Brown University’s Division of Applied Mathematics, Grenander received many honors. He was a Fellow of the American Academy of Arts and Sciences and the National Academy of Sciences and was a member of the Royal Swedish Academy of Sciences.

The current prize amount is US$5,000, and the prize is awarded every three years.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included.

To make a nomination go to https://www.ams.org/grenander-prize

Albert Leon Whiteman Memorial Prize

The Whiteman Prize recognizes notable exposition and exceptional scholarship in the history of mathematics.

About this Prize

This prize was established in 1998 using funds donated by Mrs. Sally Whiteman in memory of her husband, Albert Leon Whiteman.

The US$5,000 prize is awarded every three years.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included.

To make a nomination go to https://www.ams.org/whiteman-prize

Bertrand Russell Prize

About this Prize

The Bertrand Russell Prize of the AMS was established in 2016 by Thomas Hales. The prize looks beyond the confines of the profession to research or service contributions of mathematicians or related professionals to promoting good in the world. It recognizes the various ways that mathematics furthers fundamental human values. Mathematical contributions that further world health, our understanding of climate change, digital privacy, or education in developing countries are some examples of the type of work that might be considered for the prize.

The current prize amount is US$5,000, awarded every three years.

Next Prize: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Include a short description of the work that is the basis of the nomination, including complete bibliographic citations. A curriculum vitae should be included.

To make a nomination go to https://www.ams.org/russell-prize

Award for Distinguished Public Service

The Award for Distinguished Public Service recognizes a research mathematician who has made recent or sustained distinguished contributions to the mathematics profession through public service.
Calls for Nominations & Applications

FROM THE AMS SECRETARY

About this Award
The AMS Council established this award in response to a recommendation from its Committee on Science Policy.

The US$4,000 award is presented every two years.

Next Award: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Submit a letter of nomination describing the candidate’s accomplishments, a CV for the nominee, and a brief citation that explains why the work is important.

To make a nomination go to https://www.ams.org/public-service-award.

Award for an Exemplary Program or Achievement in a Mathematics Department

This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university’s undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

About this Award
This award was established in 2004. For the first three awards (2006–2008), the prize amount was US$1,200. The prize was endowed by an anonymous donor in 2008, and starting with the 2009 prize, the amount is US$5,000.

This US$5,000 prize is awarded annually. Departments of mathematical sciences in North America that offer at least a bachelor’s degree in mathematical sciences are eligible.

Next Award: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: A letter of nomination may be submitted by one or more individuals. Nomination of the writer’s own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements which make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). Where possible, the letter and documentation should address how these successes 1) came about by systematic, reproducible changes in programs that might be implemented by others, and/or 2) have value outside the mathematical community. The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

To make a nomination go to https://www.ams.org/department-award.

Award for Mathematics Programs that Make a Difference

The Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS’s Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are potentially replicable models.

About this Award
This award brings recognition to outstanding programs that have successfully addressed the issues of underrepresented groups in mathematics. Examples of such groups include racial and ethnic minorities, women, low-income students, and first-generation college students.

One program is selected each year by a Selection Committee appointed by the AMS President and is awarded US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level. Nomination of one’s own institution or program is permitted and encouraged.
Calls for Nominations & Applications

FROM THE AMS SECRETARY

Next Award: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: The letter of nomination should describe the specific program being nominated and the achievements that make the program outstanding success. It should include clear and current evidence of that success. A strong nomination typically includes a description of the program’s activities and goals, a brief history of the program, evidence of its effectiveness, and statements from participants about its impact. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages. Nomination of the writer’s own institution or program is permitted. Non-winning nominations will automatically be reconsidered for the award for the next two years.

To make a nomination go to https://www.ams.org/make-a-diff-award.

Award for Impact on the Teaching and Learning of Mathematics

This award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.

Priorities of the award include recognition of:
(a) accomplished mathematicians who have worked directly with pre-college teachers to enhance teachers’ impact on mathematics achievement for all students, or
(b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

About this Award
The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen.

The US$1,000 award is given annually.

Next Award: January 2024

Nomination Period: 1 February – 31 May 2023

Nomination Procedure: Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages, and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included. The non-winning nominations will automatically be reconsidered, without further updating, for the awards to be presented over the next two years.

To make a nomination go to https://www.ams.org/impact.

Fellowships
Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of twelve members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at https://www.ams.org/ams-fellows.

Joint Prizes

George David Birkhoff Prize in Applied Mathematics (AMS-SIAM)

The Birkhoff Prize is awarded for an outstanding contribution to applied mathematics in the highest and broadest sense.
Calls for Nominations & Applications
FROM THE AMS SECRETARY

About this Prize
The prize was established in 1967 in honor of Professor George David Birkhoff, with an initial endowment contributed by the Birkhoff family and subsequent additions by others. The American Mathematical Society (AMS) and the Society for Industrial and Applied Mathematics (SIAM) award the Birkhoff Prize jointly.

The current prize amount is US$5,000, awarded every three years to a member of AMS or SIAM.

Next Prize: January 2024
Nomination Period: 1 February – 31 May 2023
Nomination Procedure: To make a nomination go to https://www.ams.org/birkhoff-prize.

Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student (AMS-MAA-SIAM)

Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student (AMS-MAA-SIAM) is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient’s research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

About this Prize
The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is US$1,200, awarded annually.

Next Prize: January 2024
Nomination Period: 1 February – 31 May 2023
Nomination Procedure: To nominate a student, submit a letter of nomination, a brief description of the work that is the basis of the nomination, and complete bibliographic citations (or copies of unpublished work). All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student’s research.

To make a nomination go to https://www.ams.org/morgan-prize.

JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

About this Award
This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.

Next Prize: January 2024
Nomination Period: open
Nomination Procedure: Nominations should be submitted on mathprograms.org. Note: Nominations collected before September 15th in year N will be considered for an award in year N+2.
An Interview with Bryna Kra

Scott Hershberger

Every other year, when a new AMS president takes office, the Notices publishes interviews with the outgoing and incoming presidents. Bryna Kra's two-year term as president will begin on February 1, 2023. Kra is the Sarah Rebecca Roland Professor of Mathematics at Northwestern University. Notices contributing writer Scott Hershberger spoke with her in June 2022. An edited version of that interview follows.

**Notices:** Are there any takeaways from your time on the AMS Board of Trustees that you'll bring to your role as president?

**Kra:** I have learned that change is very slow in the math community, but there is momentum for change. In some sense, I think, this is a very reasonable position to have, although it may frustrate people that sometimes the AMS is not quicker to react when there are problems. The AMS is a huge and intricate organization that serves many different cohorts. This means that it doesn't serve any of them perfectly, but it actually does serve many of them quite well. I think we are moving in a good direction. We are serving broader, more diverse cohorts in a better way. But this will not happen overnight.

**Notices:** What will be your priorities as AMS president?

**Kra:** My priority is to support a broader swath of the community, especially to support them in their research. For example, people at primarily undergraduate institutions don’t have great access to research funds, and even small funds could help quite a bit. Many institutions—especially historically Black colleges and universities, Hispanic-serving institutions, minority-serving institutions—do not have sufficient access to our journals and MathSciNet, which is an integral tool for doing research these days.

There’s often a drop-off in support for mid-career mathematicians. You’ve gotten tenure, and now you’re supposed to be suddenly completely independent and able to get everything done. It doesn’t always work that way. Finding ways to support this group of mathematicians is important, and the AMS needs to work to identify what types of programs would help this group keep research momentum going.

**Notices:** The COVID-19 pandemic has influenced everyone's lives in so many ways. Thinking about its impact on the math community, what do we need to change moving forward?

**Kra:** The pandemic has highlighted the importance of remote work and virtual meetings. The AMS needs to ensure that all members have access to these tools, regardless of their location. Additionally, the pandemic has underscored the need for flexible working arrangements and the importance of mental health support for mathematicians.

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Scott Hershberger is a master's student in science communication at the University of Wisconsin–Madison. When he conducted this interview, he was the communications and outreach content specialist at the AMS. His email address is scotthersh42@gmail.com.

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Figure 1. Bryna Kra is the incoming AMS president.
Kra: It’s had a huge impact, especially on students and postdocs, many of whom were denied opportunities to meet more senior people in the field and to establish new collaborations. There’s not going to be a one-size-fits-all solution because different people were affected in different ways. But the AMS can help. I think part of it is in-person meetings and small grants so that people can set up collaborations that were put on hold or never happened.

We’ve all learned that we can live with some amount of online interaction. It might be that we need to have some online meetings regularly and take advantage of better technologies for them. Hybrid meetings might be a possibility when we figure out how to make them work. And in-person meetings, I think, are still at the moment the easiest to really make work. But we need to find ways to make them work better and be more inclusive.

Notices: What specific steps do you hope the AMS takes during your presidency to live up to the recommendations of the March 2021 report from the Task Force on Understanding and Documenting the Historical Role of the AMS in Racial Discrimination?

Kra: We need to be involved more heavily with training the next generation. No one program will be able to address all of the issues that the Task Force raised, and we need to think about many ways to effect changes. As an example, Northwestern has a post-baccalaureate program aimed at historically underrepresented groups in mathematics, giving them an extra year of courses and professional support to help them go off to graduate school. I think there need to be more programs like this across the community; the AMS can act as an incubator and a facilitator, making connections and supporting these programs. I’m sure there are many ways we have yet to try to advance the recommendations of the Task Force.

Notices: In her interview as outgoing president, Ruth Charney mentioned declining membership as a challenge that the AMS faces. To you, what are the key reasons why being an AMS member is valuable today?

Kra: The resources that the AMS offers only exist because of the hard work of many people. By being a member of the AMS, one is indirectly supporting MathSciNet. The AMS (along with Duke) supports MathJobs, one central listing place both for faculty and for applicants on the job market. All of the books and publications, the many conferences that we organize, the large number of fellowships that we give—all of that is only possible because we have such a large membership, and a large number of people who volunteer and work for the AMS.

Figure 2. Bryna Kra in her office at Northwestern University.

In addition, the AMS has a voice in advocacy for mathematics. Our large number of members gives us some clout with policymakers, making our lobbying for the support of mathematics and mathematicians more effective.

Notices: What do you view the AMS’s role to be in advancing sustainability within the math community and in the world in general?

Kra: Some pure mathematics research will be used in sustainability. It might not be used today, or even next year, or even 10 years from now, but some of it should have applications in the future. The AMS’s role is to support research that will eventually produce valuable outputs that affect the environment and sustainability.

And I do think that the AMS needs to be thinking about our footprint, our travel being the major one that we incur. It’s a careful balance between the benefits of meeting in person versus the environmental cost.

Sectional meetings used to be more local than they are now. They have become national meetings. And I think there’s a lot of reason to move them back toward a more local approach. Not everybody needs to attend every meeting in person, and holding hybrid versions of our sectional meetings might make it possible for more people to participate with a lower environmental impact.

Notices: The AMS has been working to build connections with mathematical scientists in business, entrepreneurship, government, industry, and non-profit (BEGIN) sectors. Can you talk about the vision for those efforts?

Kra: Mathematics is a unique way of training people. We are not just training people to reproduce academia—we’re training people to go out and interact with the world and
work in business and industry. Lots of places are looking for the analytic skills that come with being trained as a mathematician, even if it’s not the exact research that one worked on for a doctorate. The AMS has a crucial role to play in making connections between the research community and industry.

**Notices:** How can we make that vision a reality?

**Kra:** The reimagined JMM is an opportunity to provide more professional development and more opportunities for employers to be looking for employees. I think the AMS can play a role as a matchmaker and as a builder of connections by running further programs to match the needs in industry with mathematicians who have the skills and the analytical expertise.

**Notices:** More broadly, what do you hope for the future of the reimagined JMM?

**Kra:** It’s a really exciting time because the JMM has a huge impact on many people. I had job interviews at the JMM a long time ago. I’ve met people and worked with them because of special sessions there, so it’s impacted my career throughout. I hope the reimagined JMM is going to do this for more people. It’s a great place for people to get support that they might not be able to get locally.

The reimagined JMM will have more than 15 partner organizations, meaning that there will be many options for participants. Having all of this in one meeting creates more opportunities for networking and for collaboration than any single society meeting could offer.

**Notices:** The 2022 ICM was originally planned to be held in Russia—a very controversial choice from the start due to Russia’s human rights abuses. What lessons should the AMS and the entire math community draw from this situation?

**Kra:** Just because we’ve been doing something some way for a long time does not mean that it’s the only way to do things. And perhaps this was an example of that.

As an international mathematics community, we need to think broadly about how we design meetings from start to finish. I hope that future ICMs will be in places where it is easier for everybody to feel comfortable participating. Location matters. And perhaps even more important than just thinking about the location, it is important that every organizing committee bring diverse perspectives, ensuring that every meeting is a welcoming and productive experience for all participants.

**Notices:** Do you think the public perception of math has shifted in recent years, especially since the pandemic started?

**Kra:** Unfortunately not. I think math’s public image continues to take a beating. Some of it is not our fault, and it’s beyond our control. But some of it is within our control. We can explain what we do in our research without using technical jargon. We really need to make a better effort to communicate the importance of what we do.

**Notices:** To you, what should be the role of the AMS as an organization in trying to change that public perception?

**Kra:** Some of the AMS’s publications are for a broad audience. Some of the posters that we make explain where math is used, and you see them hung up sometimes in a high school. We support a lecture for high school students, congressional briefings, things like this. But I think we can do more to bring a broader understanding of why mathematics needs to be supported.

An interview with Ruth Charney, the outgoing AMS president, appeared in the January 2023 issue of Notices of the AMS.

Scott Hershberger

**Credits**

Figure 1 is courtesy of Bryna Kra.
Figure 2 is courtesy of Antonio Auffinger.
Photo of Scott Hershberger is courtesy of Scott Hershberger.
NEWS

AMS Updates

2023 AMS Fellows

Thirty-nine mathematical scientists from around the world have been named Fellows of the American Mathematical Society (AMS) for 2023, the program’s eleventh year.

To be recognized by their peers as Fellows of the AMS, these scientists have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

The AMS is pleased to honor excellence by presenting the class of 2023 Fellows. View the names, institutions, and citations of the full 2023 class of Fellows at http://www.ams.org/ams-fellows.

—AMS Programs Department

New AMS-Simons Research Grants Open for Applications

Grant applications are open for the AMS-Simons Research Enhancement Grants for PUI Faculty, designed to meet the needs of mathematics faculty members at primarily undergraduate institutions (PUIs) who have an active research portfolio.

These new grants will foster and support research collaboration by mathematicians employed full-time at institutions which do not award doctoral degrees in mathematics. Applications on https://www.mathprograms.org will be open until March 20, 2023.

Each year for three years, awardees will receive $3,000 to support research-related activities. In addition, annually for three years, the awardee’s institution will receive $300 for administrative costs and the awardee’s department will receive $300 in discretionary funds.

“Faculty at all types of institutions actively engage in mathematics research, and modest amounts of grant funding can have an outsized effect on the success of a project,” said Bryna Kra, AMS president. “I’m excited to see the AMS partner with the Simons Foundation to support the research of mathematicians at primarily undergraduate institutions with the creation of flexible grants funding visits for collaborations, conference travel, and other research-related expenses.”

Starting in 2023, the AMS expects to award at least 40 grants annually, funded by the Simons Foundation. Funds for the first round of grants will be disbursed in July 2023. Tenured and tenure-track mathematicians with an active research program, who have earned a PhD degree at least five years before the start of the grant will be eligible to apply.

"Impactful mathematics research is being conducted by faculty at primarily undergraduate institutions across the country, and the AMS is so pleased that the Simons Foundation is supporting this work," said Catherine Roberts, AMS executive director.

For more information, please email ams-simons@ams.org.

—AMS Programs Department

AMS.org Updated with New Functionality

New navigation and mathematical content on https://www.ams.org will connect you more easily to AMS resources. See https://www.ams.org/news?news_id=7000.

Mathematical content, news, upcoming meetings, and ways to get involved are featured on the home page.

Drop-down navigation menus now appear atop every page. With drop-down navigation, we have streamlined the footer as well.

MathSciNet and Bookstore links have been repositioned to the upper right corner.

—AMS Communications

DOI: https://doi.org/10.1090/noti2633
Inaugural Fall Graduate School Fair Draws More Than 300 Students

At the AMS’s first Fall Graduate School Fair, held October 12, 2022, representatives of more than forty graduate programs in the mathematical sciences hosted virtual tables to answer student questions.

The online platform Gather provided a dynamic way to interact, replicating an in-person experience for more than 300 undergraduates and master’s students. The students were provided with avatars that allowed them to navigate and interact within the virtual fair, much like an online video game.

In addition to the virtual tables, the Graduate School Fair kicked off with a panel of students and graduate chairs, who shared ways that students can explore graduate school options and can pose insightful questions of program staff.

“The panel provided exactly what I had hoped: an opportunity to empower students,” says Sarah Bryant, AMS director of programs. “It also, I hope, prompted department representatives to think deeply about how they are recruiting to their programs.” She added that the panel discussion was recorded and is available online on the AMS YouTube channel at https://www.youtube.com/c/AmsOrg.

The AMS Fall Graduate School Fair will recur annually in addition to the Graduate School Fair at the Joint Mathematics Meetings. “I wish this resource had been available when I was applying to graduate school,” says panelist Christopher L. Cox, mathematics professor and department head, University of Tennessee at Chattanooga.

—AMS Communications

AMS Email Support for Frequently Asked Questions

A number of email addresses have been established for contacting the AMS staff regarding frequently asked questions.

The following is a list of those addresses together with a description of the types of inquiries that should be made through each address:

- abs-coord@ams.org to contact the AMS Headquarters in Providence, Rhode Island.
- amsdc@ams.org to reach the AMS Washington Office about government relations and advocacy programs.
- amsfellows@ams.org to inquire about the Fellows of the AMS.
- amsmem@ams.org to request information about membership in the AMS and about dues payments or to ask any general membership questions; may also be used to submit address changes.
- ams-mrc@ams.org for questions about the AMS Mathematics Research Communities.
- ams-simons@ams.org for information about the AMS Simons Travel Grants Program and AMS Simons Research Enhancement Grants for PUI Faculty.
- ams-survey@ams.org for information or questions about the Annual Survey of the Mathematical Sciences or to request reprints of survey reports.
- bookstore@ams.org for inquiries related to the online AMS Bookstore.
- classads@ams.org to submit classified advertising for the Notices.
- com-staff@ams.org to reach the AMS Communications Department.
- consortia@ams.org for information on consortia subscriptions and MathSciNet for Developing Countries Program.
- cust-serv@ams.org for general information about AMS products (including electronic products), to send address changes, or to conduct any general correspondence with the Society's Customer Service Department.
- development@ams.org for information about charitable giving to the AMS.
- edi-query@ams.org for questions about AMS equity, diversity, and inclusion efforts.
- education@ams.org for questions and suggestions for AMS education programs and advocacy.
- emp-info@ams.org for information regarding AMS employment and career services.
- eprod-support@ams.org for technical questions regarding AMS electronic products and services.
- exdir@ams.org to contact the AMS executive director.
- gradprg-ad@ams.org to inquire about a listing or ad in the Find Graduate Programs online service.
- mathcal@ams.org for questions regarding posting on the Mathematics Calendar.
- mathjobs@ams.org for questions about the online job application service Mathjobs.org.
- mathprograms@ams.org for questions about the online program application service Mathprograms.org.
Deaths of AMS Members

Janos (John) Aczel, of Canada, died on January 1, 2020. Born on December 26, 1964, he was a member of the Society for 56 years.

Samuel F. Barger, of Charlotte, North Carolina, died on October 11, 2022. He was a member of the Society for 53 years.

Robert G. Cawley, of Columbia, Maryland, died on August 27, 2022. Born on January 29, 1936, he was a member of the Society for 38 years.

Jane P. Coffee, of Staten Island, New York, died on September 23, 2022. Born on April 10, 1944, she was a member of the Society for 51 years.

William S. Cohn, of Detroit, Michigan, died on July 8, 2022. Born on June 19, 1952, he was a member of the Society for 39 years.

Chandler Davis, of Canada, died on September 24, 2022. Born on August 12, 1926, he was a member of the Society for 74 years.

Francois Fricker, of Switzerland, died on August 31, 2022. Born on September 26, 1939, he was a member of the Society for 45 years.

George Giordano, of Canada, died on September 24, 2021. Born on July 16, 1957, he was a member of the Society for 33 years.

F. T. Howard, of Winston-Salem, North Carolina, died on September 26, 2022. Born on May 17, 1939, he was a member of the Society for 59 years.

Winfried Just, of Athens, Ohio, died on November 2, 2022. Born on February 20, 1958, he was a member of the Society for 34 years.
Herbert C. Kranzer, of Jericho, New York, died on August 5, 2022. Born on April 10, 1932, he was a member of the Society for 68 years.

Gary J. Kurowski, of Davis, California, died on July 26, 2022. Born on March 22, 1931, he was a member of the Society for 63 years.

Marian Kwapisz, of Poland, died on August 6, 2022. Born on January 6, 1935, he was a member of the Society for 33 years.

William Livingston, of Joplin, Missouri, died on July 20, 2022. Born on October 26, 1940, he was a member of the Society for 59 years.

Narahari Umanath Prabhu, of Ithaca, New York, died on December 14, 2022. Born on April 25, 1924, he was a member of the Society for 57 years.

Richard J. Turyn, of Brookline, Massachusetts, died on October 24, 2022. Born on May 26, 1930, he was a member of the Society for 71 years.

Bodo Volkmann, of Germany, died on August 18, 2022. Born on April 16, 1929, he was a member of the Society for 70 years.

Wilbur Whitten, of Forest, Virginia, died on December 6, 2022. Born on March 16, 1931, he was a member of the Society for 61 years.

A. C. Woods, of Worthington, Ohio, died on September 5, 2022. Born on September 25, 1929, he was a member of the Society for 57 years.

Steven M. Zelditch, of Evanston, Illinois, died on September 11, 2022. Born on September 13, 1953, he was a member of the Society for 43 years.
Mathematics People

Sznitman, Scott Awarded Blaise Pascal Medals

The European Academy of Science awarded 2022 Blaise Pascal Medals to Alain-Sol Sznitman, professor of mathematics at ETH Zurich, and to Susan Scott, professor of theoretical physics in the Centre for Gravitational Astrophysics at Australian National University. Scott is the first Australian to receive the Blaise Pascal medal.

Sznitman received the Mathematics medal in recognition of his contributions to probability theory. The Academy wrote of Sznitman, “He is one of the main players who have transformed probability theory into one of the most active and important branches of mainstream mathematics—both directly via their own work, but also by creating a sense of community.

“In this last decade, Sznitman has again crafted a deep subject. With the ‘interlacement’ questions, one looks at questions of the connectivity properties of ‘the complement of a random structure’ rather that of the random structure itself.

“This turned out to be a deep topic, with relations to many other currently active areas of probability theory (maxima of random fields), where again, the ideas that he developed turn out to be central,” according to the Academy.

Sznitman has been a professor of mathematics at ETH Zurich since 1991; he became emeritus in 2021. He received his PhD in 1983 from the Université Pierre-et-Marie-Curie—Paris VI under the supervision of Jacques Neveu.

Scott received the Physics medal in recognition of her contributions to the advances of physics. “Distinguished Professor Susan Scott is an internationally recognized mathematical physicist who has made groundbreaking discoveries in general relativity, cosmology, and gravitational wave science spanning more than three decades,” according to the Academy.

“She played a leading role in Australia’s participation in the first detection of gravitational waves in 2015, and the development of the field of gravitational wave science in Australia following on from that discovery.”

Scott received her PhD in 1991 from the University of Adelaide [Australia] under the supervision of Peter Szekeres. In 2022, she became the first Australian to be elected Fellow of the International Society on General Relativity and Gravitation (ISGRG), joining such scientists as Stephen Hawking and Nobel Laureates Roger Penrose and Kip Thorne.

—AMS Communications

Nicole Joseph Receives 2023 Louise Hay Award

The Association for Women in Mathematics is pleased to announce the 2023 Louise Hay Award to Nicole Joseph, associate professor of mathematics education at Vanderbilt University. Joseph is honored for contributions to mathematics education that reflect the values of taking risks and nurturing students’ academic talent that are central to Louise Hay’s legacy.

“Professor Nicole Joseph’s research is centered on the experiences and narratives of Black girls and women in STEM,” the citation reads. “… As one of her recommenders stated, Joseph’s research exhibits ‘scholarship in action.’ In other words, Joseph both investigates hard and retracted questions while doing the work necessary to undo these patterns.”

Joseph responds, “I am deeply honored to join the list of distinguished awardees, including Dr. Virginia Warfield from the University of Washington, who was on my dissertation committee. Throughout my career I have aimed to carry out similar commitments as Louise Hay, specifically related to mentorship, advocacy, and leadership.

“I started this journey as a young Black girl who found herself in advanced mathematics courses in middle and high school alone. No one else looked like me, and that was a problem. I was young and did not have the words, but I knew as a young person that it was not right to not have other students in mathematics that looked like me.”

DOI: https://doi.org/10.1090/noti2632
Established in 1991, the Hay Award recognizes outstanding achievements in any area of mathematics education. Louise Hay was recognized widely for her contributions to mathematical logic; for her strong leadership as head of the Department of Mathematics, Statistics, and Computer Science at the University of Illinois at Chicago; for her devotion to students; and for her lifelong commitment to nurturing the talent of young women and men.

The annual presentation of this award is intended to highlight the importance of mathematics education and to evoke the memory of all that Hay exemplified as a teacher, scholar, administrator, and human being. The award will be presented at the Joint Mathematics Meetings, January 4–7, 2023 in Boston, MA.

—Association for Women in Mathematics

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†Apply up to 20 AMS points to these rates. One point = $1 discount.

If you are searching for a job but are not yet employed,* you can still be an AMS member. Choose the rate option that is comfortable for your budget. Then use your benefits to assist your search.

New to the AMS:
www.ams.org/join

Current eligible members who have not yet paid 2023 dues:
www.ams.org/account

*Annual statement of unemployed status is required.
Classified Advertising

Employment Opportunities

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn. For more information, please visit cam.tju.edu.cn or contact Mr. Albert Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.

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New Books Offered by the AMS

Logic and Foundations

A Bridge to Advanced Mathematics

From Natural to Complex Numbers

Sebastian M. Cioabă, University of Delaware, Newark, DE and Werner Linde, Friedrich-Schiller University Jena, Germany

Most introduction to proofs textbooks focus on the structure of rigorous mathematical language and only use mathematical topics incidentally as illustrations and exercises. In contrast, this book gives students practice in proof writing while simultaneously providing a rigorous introduction to number systems and their properties. Understanding the properties of these systems is necessary throughout higher mathematics. The book is an ideal introduction to mathematical reasoning and proof techniques, building on familiar content to ensure comprehension of more advanced topics in abstract algebra and real analysis with over 700 exercises as well as many examples throughout. Readers will learn and practice writing proofs related to new abstract concepts while learning new mathematical content. The first task is analogous to practicing soccer while the second is akin to playing soccer in a real match. The authors believe that all students should practice and play mathematics.

The book is written for students who already have some familiarity with formal proof writing but would like to have some extra preparation before taking higher mathematics courses like abstract algebra and real analysis.

This item will also be of interest to those working in algebra and algebraic geometry and analysis.

Pure and Applied Undergraduate Texts, Volume 58

January 2023, 525 pages, Softcover, ISBN: 978-1-4704-7148-4, LC 2022034218, 2010 Mathematics Subject Classification: 00-01, 00A05, 00A06, 05–01, List US$89, AMS members US$71.20, MAA members US$80.10, Order code AMSTEXT/58

bookstore.ams.org/amstext-58

New AMS-Distributed Publications

Differential Equations

The Yang-Mills Heat Flow and the Caloric Gauge

Sung-Jin Oh, University of California, Berkeley, CA, and Korea Institute for Advanced Study, Seoul, Korea and Daniel Tataru, University of California, Berkeley, CA

This is the first part of the four-paper sequence which establishes the Threshold Conjecture and the Soliton Bubbling vs. Scattering Dichotomy for the energy critical hyperbolic Yang-Mills equation in the $(4+1)$-dimensional Minkowski space-time.

The primary subject of this paper, however, is another PDE, namely the energy critical Yang-Mills heat flow on the 4-dimensional Euclidean space. The authors’ first goal is to establish sharp criteria for global existence and asymptotic convergence to a flat connection for this system in $H^1$, including the Dichotomy Theorem (i.e., either the above properties hold or a harmonic Yang-Mills connection bubbles off) and the Threshold Theorem (i.e., if the initial energy is less than twice that of the ground state, then the above properties hold). The authors’ second goal is to use the Yang-Mills heat flow in order to define the caloric gauge which will play a major role in the analysis of the hyperbolic Yang-Mills equation in the subsequent papers.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 436


bookstore.ams.org/ast-436
Meetings & Conferences of the AMS
February Table of Contents

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit [www.ams.org/cgi-bin/abstracts/abstract.pl]. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS
Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: stovall@math.wisc.edu; telephone: (608) 262-2933.
Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: (610) 758-3717.
Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: (706) 542-2547.
Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: mamanes@hawaii.edu; telephone: (808) 956-4679.

Meetings in this Issue

2023

March 18–19 Atlanta, Georgia p. 347
April 1–2 Spring Eastern Virtual p. 349
April 15–16 Cincinnati, Ohio p. 350
May 6–7 Fresno, California p. 352
September 9–10 Buffalo, New York p. 353
October 7–8 Omaha, Nebraska p. 353
October 13–15 Mobile, Alabama p. 354
October 21–22 Albuquerque, NM p. 354
December 4–8 Auckland, New Zealand p. 354

2024

January 3–6 San Francisco, California (JMM 2024) p. 355
March 23–24 Tallahassee, Florida p. 355
April 6–7 Washington, DC p. 355
May 4–5 San Francisco, California p. 355
July 23–26 Palermo, Italy p. 356
October 26–27 Riverside, California p. 356

2025

January 8–11 Seattle, Washington (JMM 2025) p. 356

2026

January 4–7 Washington, DC (JMM 2026) p. 356

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.
Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Atlanta, Georgia
Georgia Institute of Technology

March 18–19, 2023
Saturday – Sunday

Meeting #1184
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 44, Issue 2

Deadlines
For organizers: Expired
For abstracts: January 17, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Betsy Stovall, University of Wisconsin-Madison, Title to be announced.
Blair Dowling Sullivan, University of Utah, Title to be announced.
Yusu Wang, University of California San Diego, Title to be announced.
Amie Wilkinson, University of Chicago, Title to be announced (Erdős Memorial Lecture).

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Topics in Graph Theory and Combinatorics (Code: SS 1A), Songling Shan, Illinois State University, and Guangming Jing, Augusta University.

Advances in Applied Dynamical Systems and Mathematical Biology (Code: SS 25A), Chunhua Shan, The University of Toledo, and Guihong Fan, Columbus State University.
Advances in Mathematical Finance and Optimization (Code: SS 24A), Ibrahim Ekren, Arash Fahim, and Lingjiong Zhu, Florida State University.

Algebraic Methods in Algorithms (Code: SS 16A), Kevin Shu and Mehrdad Ghadiri, Georgia Institute of Technology.

Combinatorial Matrix Theory (Code: SS 22A), Zhongshan Li, Marina Arav, and Hein Van der Holst, Georgia State University.

Combinatorics, Probability and Computation in Molecular Biology (Code: SS 38A), Christine Heitsch and Brandon Legried, Georgia Institute of Technology.

Commutative Algebra and its Interactions with Algebraic Geometry (Code: SS 10A), Michael Brown and Henry K. Schenck, Auburn University.

Contact and Symplectic Topology in Dimensions 3 and 4 (Code: SS 34A), Akram Alishahi, Peter Lambert-Cole, and Gordana Matic, University of Georgia.

Discrete Analysis (Code: SS 31A), Giorgis Petridis, Neil Lyall, and Akos Magyar, University of Georgia.


Diversity in Mathematical Biology (Code: SS 36A), Daniel Alejandro Cruz, University of Florida, and Margherita Maria Ferrari, University of Manitoba.

Dynamics of Partial Differential Equations (Code: SS 18A), Gong Chen, Georgia Institute of Technology, Hao Jia, University of Minnesota, and Dallas Albritton, Princeton University.

Fractal Geometry and Dynamical Systems (Code: SS 11A), Mrinal Kanti Roychowdhury, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and Scott Kaschner, Butler University.

Geometric and Combinatorial Aspects of Lie Theory (Code: SS 40A), William Graham, University of Georgia, Amber Russell, Butler University, and Scott Larson, University of Georgia.

Geometric Group Theory (Code: SS 4A), Ryan Dickmann, Georgia Institute of Technology, Sahana H. Balasubramanya, University of Münster, and Abdoul Karim Sane, Georgia Institute of Technology.

Harmonic Analysis (Code: SS 14A), Betsy Stovall, University of Wisconsin-Madison, Benjamin Jaye, Georgia Tech, and Manasa Vempati.

High-dimensional Convexity and Probability (Code: SS 13A), Galyna Livshyts and Orli Herscovici, Georgia Institute of Technology, and Dan Mikulincer, MIT.


Logic, Combinatorics, and their Interactions (Code: SS 27A), Anton Bernshtein, Georgia Institute of Technology, and Robin Tucker-Drob, University of Florida.

Macdonald Theory at the Intersection of Combinatorics, Algebra, and Geometry (Code: SS 37A), Olya Mandelshtam, University of Waterloo, Sean Griffin, UC Davis, and Andy Wilson, Kennesaw State University.

Mathematical Modeling and Simulation Techniques in Fluid Structure Interaction Problems (Code: SS 17A), Pejman Sanaei, Georgia State University.

Mathematical Modeling of Populations and Diseases Transmissions (Code: SS 33A), Yang Li, Georgia State University, Jia Li, University of Alabama in Huntsville, and Xiang-Sheng Wang, University of Louisiana at Lafayette.

Multiscale Approaches to Modeling Ecological and Evolutionary Dynamics (Code: SS 28A), Daniel Brendan Cooney, University of Pennsylvania, Denis Daniel Patterson, Princeton University, Olivia Chu, Dartmouth College, and Chadi M Saad-Roy, University of California, Berkeley.

Qualitative Aspects of Nonlinear PDEs: Well-posedness and Asymptotics (Code: SS 23A), Atanas G. Stefanov, University of Alabama Birmingham, Fazel Hadaifard, University of California - Riverside, and Jiuhong Wu, Oklahoma State University.

Quasi-periodic Schrödinger Operators and Quantum Graphs (Code: SS 35A), Fan Yang, Louisiana State University, Matthew Powell, UCI, and Burak Hatinoglu, UC Santa Cruz.

Recent Advances and Applications in Imaging Sciences (Code: SS 39A), Carmeliza Luna Navasca, University of Alabama at Birmingham, Fatou Sanogo, Bates College, and Elizabeth Newman, Emory University.

Recent Development in Advanced Numerical Methods for Partial Differential Equations (Code: SS 21A), Seulip Lee and Lin Mu, University of Georgia.

Recent Developments in Commutative Algebra (Code: SS 5A), Thomas Polstra, University of Alabama, and Florian Enescu, Georgia State University.

Recent Developments in Graph Theory (Code: SS 32A), Guantao Chen, Georgia State University, Zhiyu Wang, Georgia Institute of Technology, and Xingxing Yu, Georgia Tech.

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Recent Developments in Mathematical Aspects of Inverse Problems and Imaging (Code: SS 20A), Yimin Zhong and Junshan Lin, Auburn University.


Recent Trends in Structural and Extremal Graph Theory (Code: SS 29A), Joseph Guy Briggs and Jessica McDonald, Auburn University.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 6A), Daniel K. Nakano, University of Georgia, Chun-Ju Lai, Institute of Mathematics, Academia Sinica, Taipei 10617 Taiwan, and Weiqiang Wang, University of Virginia.


Spectral Theory (Code: SS 19A), Rudi Weikard, University of Alabama at Birmingham, and Stephen P. Shipman, Louisiana State University.

Stochastic Analysis and its Applications (Code: SS 7A), Parisa Fatheddin, Ohio State University, Marion, and Kazuo Yamazaki, Texas Tech University.

Stochastic Processes and Related Topics (Code: SS 15A), Ngartelbaye Guerngar, University of North Alabama, and Le Chen, Erkan Nane, and Jerzy Szulga, Auburn University.

Topological Persistence: Theory, Algorithms, and Applications (Code: SS 12A), Luis Scoccola, Northeastern University, Hitesh Gakhar, University of Oklahoma, and Ling Zhou, The Ohio State University.

Topology and Geometry of 3- and 4-Manifolds (Code: SS 3A), Miriam Kuzbary, Georgia Institute of Technology, David Gay, University of Georgia, Jon Simone, Georgia Institute of Technology, and Nur Saglam, Georgia Tech.

Undergraduate Mathematics and Statistics Research (Code: SS 30A), Leslie Julianna Meadows, Georgia State University, Tsz Ho Chan and Asma Azizi, Kennesaw State University, and Mark Grinshpon, Georgia State University.

Spring Eastern Virtual Sectional Meeting

Meeting virtually, hosted by the American Mathematical Society

April 1–2, 2023
Saturday – Sunday

Meeting #1185
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 44, Issue 2

Deadlines
For organizers: Expired
For abstracts: January 31, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Kirsten Eisenträger, Pennsylvania State University, Title to be announced.
Jason Manning, Cornell University, Title to be announced.
Jennifer L. Mueller, Colorado State University, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.


Analysis of Markov, Gaussian and Stationary Stochastic Processes (Code: SS 4A), Alan C. Krinik, California State Polytechnic University, Pomona, and Randall J. Swift, Cal Poly Pomona.

Cybersecurity and Cryptography (Code: SS 10A), Lubjana Beshaj, Army Cyber Institute, Shekeba Monshref, IBM, and Angela Robinson, NIST.
**Fractal Geometry and Dynamical Systems** (Code: SS 6A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, **Sangita Jha**, National Institute of Technology Rourkela, India, and **Saurabh Verma**, Indian Institute of Information Technology Allahabad.


**Hypergeometric Functions, q-series and Generalizations** (Code: SS 5A), **Howard Saul Cohl**, National Institute of Standards and Technology, **Robert Maier**, University of Arizona, and **Roberto Costas-Santos**, Universidad Loyola de Andalucía.

**Modeling, Analysis, and Control of Populations Impacted by Disease and Invasion** (Code: SS 1A), **Rachel Natalie Leander** and **Wandi Ding**, Middle Tennessee State University.

**Quasiconformal Analysis and Geometry on Metric Spaces** (Code: SS 8A), **Dimitrios Ntalampekos**, Stony Brook University, and **Hrant Hakobyan**, Kansas State University.

**Recent Advances in Differential Geometry** (Code: SS 2A), **Bogdan D. Suceava**, California State University Fullerton, **Adara M. Blaga**, West University of Timișoara, Romania, **Cezar Oniciuc**, “Al.I.Cuza” University of Iași, Romania, **Marian Ioan Munteanu**, “Al.I.Cuza” University of Iași, Romania, **Shoo Seto**, California State University, Fullerton, and **Lihan Wang**, California State University, Long Beach.


**Recent Advances in Ion Channel Models and Poisson-Nernst-Planck Systems** (Code: SS 11A), **Zilong Song**, Utah State University, and **Xiang-Sheng Wang**, University of Louisiana at Lafayette.

**Recent Progress in Chromatic Graph Theory** (Code: SS 12A), **Hemanshu Kaul** and **Samantha Dahlberg**, Illinois Institute of Technology.

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**Cincinnati, Ohio**

University of Cincinnati

**April 15–16, 2023**  
Saturday – Sunday

**Meeting #1186**  
Central Section  
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced  
Issue of Abstracts: Volume 44, Issue 2

**Deadlines**  
For organizers: Expired  
For abstracts: February 14, 2023

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**

Johnny Guzman, Brown University, *Title to be announced.*

Lisa Piccirillo, MIT, *Title to be announced.*

Krystal Taylor, The Ohio State University, Department of Mathematics, *Title to be announced.*

Nathaniel Whitaker, University of Massachusetts, *From Segregation to Research Mathematician* (Einstein Public Lecture in Mathematics).

**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

**Advances in Dispersive Partial Differential Equations I** (Code: SS 13A), **William R. Green**, Rose-Hulman Institute of Technology, **Mehmet Burak Erdogan**, University of Illinois at Urbana Champaign, and **Michael J. Goldberg**, University of Cincinnati.


Arithmetic Statistics I (Code: SS 19A), Brandon Alberts, Eastern Michigan University, and Soumya Sankar, Ohio State University.

Brauer Groups in Algebraic Geometry and Arithmetic I (Code: SS 25A), Jack Petok and Sarah Frei, Dartmouth College.

Cluster Algebras, Positivity and Related Topics I (Code: SS 28A), Eric Bucher, Xavier University, John Machacek, The University of Oregon, and Nicholas Ovenhouse, Yale University.

Combinatorial and Geometric Knot Theory I (Code: SS 11A), Micah Chrisman, The Ohio State University, Sujoy Mukherjee, University of Denver, and Robert G. Todd, Mount Mercy University.


Ends and Boundaries of Groups: On the Occasion of Mike Mihalik’s 70th Birthday I (Code: SS 2A), Craig R. Guilbault, University of Wisconsin-Milwaukee, and Kim E. Ruane, Tufts University.

Extremal Graph Theory I (Code: SS 18A), Neal Bushaw, Virginia Commonwealth University, Puck Rombach and Calum Buchanan, University of Vermont, and Vic Bednar, Virginia Commonwealth University.

Geometric and Analytic Methods in PDE I (Code: SS 20A), Dennis Kriventsov, Rutgers University, Mariana Smit Vega Garcia, Western Washington University, and Mark Allen, Brigham Young University.

Growth Models, Random Media, and Limit Theorems I (Code: SS 6A), Magda Peligrad, Wlodek Bryc, and Xiaqin Guo, University of Cincinnati.

Harmonic Analysis and its Applications to Signals and Information I (Code: SS 12A), Dustin G. Mixon, The Ohio State University, and Matthew Fickus, Air Force Institute of Technology.

Homological Methods in Commutative Algebra I (Code: SS 27A), Michael DeBellevue, Syracuse University, and Josh Pollitz, University of Utah.

Inequalities in Harmonic Analysis I (Code: SS 26A), Ryan Gibara, University of Cincinnati, Kabe Moen, University of Alabama, and Leonid Slavin, University of Cincinnati.

Interactions between Analysis, PDE, and Probability in Non-smooth Spaces I (Code: SS 1A), Nageswari Shanmugalingam, University of Cincinnati, Luca Capogna, Smith College, and Jeremy T. Tyson, National Science Foundation.

Interactions between Noncommutative Ring Theory and Algebraic Geometry I (Code: SS 3A), Jason Gaddis, Miami University, and Robert Won, George Washington University.

Mathematical Modeling in Biosciences I (Code: SS 29A), Sookkyung Lim, University of Cincinnati, Jeungeun Park, SUNY at New Paltz, Yanyu Xiao, University of Cincinnati, Hem R. Joshi, Xavier University, Cincinnati, and David Gerberry, Xavier University.


Nonlinear Partial Differential Equations from Variational Problems and Fluid Dynamics I (Code: SS 16A), Tao Huang, Wayne State University, Hengrong Du, Vanderbilt University, and Changyou Wang, Purdue University.

Probabilistic and Extremal Combinatorics I (Code: SS 14A), Jozsef Balogh, University of Illinois at Urbana-Champaign, and Tao Jiang, Miami University.

Quantitative Aspects of Symplectic Topology I (Code: SS 15A), Jun Li, University of Dayton, Olguta Buse, IUPUI, and Richard Keith Hind, University of Notre Dame.

Recent Advances in Finite Element Methods: Theory and Applications I (Code: SS 21A), Tamas L. Horvath, Oakland University, and Giselle Sosa Jones, University of Houston.

Recent Developments in the Study of Fluid Flows, Turbulence, and its Applications I (Code: SS 30A), Vincent Martinez, CUNY Hunter College & Graduate Center, and Samuel Punshon-Smith, Tulane University.

Recent Trends in Graph Theory I (Code: SS 23A), Adam Blumenthal, Westminster College, and Katherine Perry, Soka University of America.

Recent Trends in Integrable Systems and Applications I (Code: SS 5A), Deniz Bilman and Robert J. Buckingham, University of Cincinnati.

Representation Theory, Geometry and Mathematical Physics I (Code: SS 22A), Daniele Rosso, Indiana University Northwest, and Jonas T. Hartwig, Iowa State University.

Stochastic Analysis and its Applications I (Code: SS 9A), Po-Han Hsu, University of Cincinnati, Tai-Ho Wang, Baruch College, CUNY, and Ju-Yi Yen, University of Cincinnati.

The Interface of Geometric Measure Theory and Harmonic Analysis I (Code: SS 10A), Eyvindur Ari Palsson, Virginia Tech, and Krystal Taylor, The Ohio State University, Department of Mathematics.

MEETINGS & CONFERENCES

Fresno, California

California State University, Fresno

May 6–7, 2023
Saturday – Sunday

Meeting #1187
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of Abstracts: Volume 44, Issue 3

Deadlines
For organizers: Expired
For abstracts: March 7, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Sami H. Assaf, University of Southern California, To be announced.
Natalia Komarova, UCI, To be announced.
Joseph Teran, University of California, Los Angeles, To be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances by CSU Student Scholars (Code: SS 22A), Jessica De Silva, California State University, Stanislaus, Andrea Arauza Rivera, California State University, East Bay, and Mario Banuelos, California State University, Fresno.

Advances in Functional Analysis and Operator Theory (Code: SS 6A), Igor Nikolaev, St. John’s University, Marat V. Markin, California State University, Fresno, and Michel L. Lapidus, University of California Riverside.

Algebraic Structures in Knot Theory (Code: SS 4A), Carmen L. Caprau, California State University, Fresno, Sam Nelson, Claremont McKenna College, and Neslihan Gügümcü, Izmir Institute of Technology in Turkey.

Algorithms in the Study of Hyperbolic 3-manifolds (Code: SS 26A), Maria Trnkova, University of California In Davis, and Robert C. Haraway, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application (Code: SS 20A), Bhuvaneswari Sambandham, Dixie State University, and Aghalaya Vatsala, University of Louisiana At Lafayette.

Artin-Schelter Regular Algebras and Related Topics (Code: SS 27A), Ellen E Kirkman, Wake Forest University, and James J. Zhang, University of Washington.

Combinatorics and Representation Theory (associated with the Invited Address by Sami Assaf) (Code: SS 16A), Nicolle Gonzalez, University of California, Berkeley, and Sami H. Assaf, University of Southern California.

Complexity in Low-Dimensional Topology (Code: SS 14A), Jennifer Schultens, University of California Davis, and Eric Sedgwick, DePaul University.

Data Analysis and Predictive Modeling (Code: SS 8A), Earvin Balderama, California State University, Fresno, and Adriano Zambom, California State University, Northridge.

Inverse Problems (Code: SS 5A), Robert M. Owczarek, University of New Mexico, and Hanna E. Makaruk, Los Alamos National Laboratory, Los Alamos, NM.

Math Circle Games and Puzzles that Teach Deep Mathematics (Code: SS 13A), Maria Nogin, Agnes Tuska, Yaomingxin Lu, and Gábor Molnár-Sáska, California State University, Fresno.

Mathematical Biology: Confronting Models with Data (Code: SS 21A), Erica Marie Rutter, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova) (Code: SS 1A), Jesse Kreger, University of California, Irvine, and Natalia Komarova, UCI.

Mathematics in Data Science (Code: SS 29A), Elena S. Dimitrova, California Polytechnic State University, San Luis Obispo, and Ruriko Yoshida, Naval Postgraduate School.

Methods in Non-Semisimple Representation Categories (Code: SS 11A), Paul Sobaje, Georgia Southern University, Eric Friedlander, University of Southern California, Los Angeles, and Julia Pevtsova, University of Washington.

Modeling and Analysis of Cellular Processes in Biomedical Problems (Code: SS 31A), Joyce Lin and Warren Roche, Cal Poly State University.
MEETINGS & CONFERENCES

Nonlinear PDEs in Fluid Dynamics (Code: SS 30A), Juhi Jang, Igor Kukavica, and Linfeng Li, University of Southern California.

Recent Developments in Mathematical Biology (Code: SS 28A), Lihong Zhao, University of California, Merced, and Christina Edholm, Scripps College.

Research in Mathematics by Early Career Graduate Students (Code: SS 7A), Marat V. Markin, Doreen De Leon, and Khang Tran, California State University, Fresno.

Scientific Computing (Code: SS 19A), Changho Kim, University of California, Merced, and Roummel F. Marcia, University of California, Merced.

The Use of Computational Tools and New Augmented Methods in Networked Collective Problem Solving (Code: SS 18A), Agnes Tuska and Mario Banuelos, California State University, Fresno, and Andras Benedek, Research Centre for the Humanities, Institute of Philosophy, Hungary.

Women in Mathematics (Code: SS 12A), Doreen De Leon, Katherine Kelm, and Oscar Vega, California State University, Fresno.

Zero Distribution of Entire Functions (Code: SS 9A), Khang Tran and Tamás Forgács, California State University, Fresno.

Buffalo, New York
University at Buffalo (SUNY)

September 9–10, 2023
Saturday – Sunday

Meeting #1188
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Invited Addresses
Jennifer Balakrishnan, Boston University, Title to be announced.
Sigal Gottlieb, University of Massachusetts, Dartmouth, Title to be announced.
Samuel Payne, University of Texas, Title to be announced.

Deadlines
For organizers: February 9, 2023
For abstracts: July 18, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Omaha, Nebraska
Creighton University

October 7–8, 2023
Saturday – Sunday

Meeting #1189
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Invited Addresses
Jennifer Balakrishnan, Boston University, Title to be announced.
Sigal Gottlieb, University of Massachusetts, Dartmouth, Title to be announced.
Samuel Payne, University of Texas, Title to be announced.

Deadlines
For organizers: March 7, 2023
For abstracts: August 8, 2023

Program first available on AMS website: July 27, 2023
Issue of Abstracts: To be announced
Mobile, Alabama

University of South Alabama

October 13–15, 2023
Friday – Sunday

Meeting #1190
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: August 24, 2023
Issue of Abstracts: To be announced

Deadlines
For organizers: March 13, 2023
For abstracts: August 15, 2023

The invited addresses listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Theresa Anderson, Carnegie Mellon University, Title to be announced.
Laura Miller, University of Arizona, Title to be announced.
Cornelius Pillen, University of South Alabama, Title to be announced.

Special Sessions
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Mathematical Modeling of Problems in Biological Fluid Dynamics (Code: SS 1A), Laura Miller, University of Arizona, and Nick Battista, The College of New Jersey.

Albuquerque, New Mexico

University of New Mexico

October 21–22, 2023
Saturday – Sunday

Meeting #1191
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: August 31, 2023
Issue of Abstracts: To be announced

Deadlines
For organizers: March 21, 2023
For abstracts: August 22, 2023

Auckland, New Zealand

December 4–8, 2023
Monday – Friday

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
San Francisco, California
Moscone West Convention Center

**January 3–6, 2024**
Wednesday – Saturday
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Tallahassee, Florida
Florida State University in Tallahassee

**March 23–24, 2024**
Saturday – Sunday
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia
Howard University

**April 6–7, 2024**
Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

San Francisco, California
San Francisco State University

**May 4–5, 2024**
Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of Abstracts: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Special Sessions**
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**Recent Advances in Differential Geometry,** Zhiqin Lu, University of California, Shoo Seto and Bogdan Suceavă, California State University, Fullerton, and Lihan Wang, California State University, Long Beach.
Palermo, Italy

July 23–26, 2024
Tuesday – Friday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Riverside, California

University of California, Riverside

October 26–27, 2024
Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025
Wednesday – Saturday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026
Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
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A Conversation on Professional Norms in Mathematics

Mathilde Gerbelli-Gauthier, Institute for Advanced Study, Princeton, NJ, Pamela E. Harris, Williams College, Williamstown, MA, Michael A. Hill, University of California, Los Angeles, CA, Dagan Karp, Harvey Mudd College, Claremont, CA, and Emily Riehl, Johns Hopkins University, Baltimore, MD, Editors

The articles in this volume grew out of a 2019 workshop, held at Johns Hopkins University, that was inspired by a belief that when mathematicians take time to reflect on the social forces involved in the production of mathematics, actionable insights result. Topics range from mechanisms that lead to an inclusion-exclusion dichotomy within mathematics to common pitfalls and better alternatives to how mathematicians approach teaching, mentoring, and communicating mathematical ideas.

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