The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. This month we feature an article by the students of Karen Uhlenbeck and two excerpts from the book *Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics*. Next month's focus will be on careers at primarily undergraduate institutions.

In Celebration of Women's History Month

Advice from Our Advisor: Karen Uhlenbeck

Steven Bradlow and Magdalena Czubak

Introduction

When asked by the *Notices* to coordinate contributions from Karen Uhlenbeck's students for this "Advice from our Advisor" column we quickly agreed. The timing was perfect: perfect for the upcoming 2023 Women in History Month but also for a tribute to Karen on the occasion of her 80th birthday, which had just been celebrated in August 2022.

Karen has had nineteen PhD students. Below we present contributions from nine of them. Some clear common themes emerge which characterize Karen's role as an advisor: her infinite generosity with her time, her importance as a mathematical and personal anchor for her students, and of course the many ways she served as an inspiration for all.

Alexandre Casassola Gonçalves

"Go talk to other people. Meet other mathematicians, and their students; ask them about their problems, and show them your problem." This is one of Karen's most memorable pieces of advice. It was perfect for me at that time, for I used to be a person who resists asking for help, believing in the fallacy that "I should do it on my own." While most of the time, mathematicians try to work on their own, there are gaps that can only be filled by other people's contributions. Mathematics, after all, is a collaborative work, and lack of humbleness is not a virtue here.

Another piece of advice from Karen was to write down results, as soon as you get them. Don't just solve them in your mind, because you only know you understand something when you are able to tell other people about it, based on what you wrote. Karen used to ask for drafts of

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research articles we were writing, as well as for excerpts of our on-going dissertation. She would then give us valuable hints about mathematical academic writing, including "what not to write in detail, because people know it."

When she was not "on leave," Karen would meet her students once a week. She would spend at least one hour with each student, but this time could grow up to two hours when trying to figure out a particularly hard problem. These regular meetings with her PhD students seemed to stand on the same "priority level" as her other (important) professional commitments. I don't remember ever having our meeting canceled or even abbreviated because she had to receive some speaker for a conference, or some co-worker who just arrived.

One thing that stood out in our meetings was her patience at explaining a point I had not understood. She would talk, and then repeat, and then write on the board, and repeat again, over and over ... that could last for several meetings in a row, until I was able to achieve a breakthrough. Though, that does not mean she didn't leave a large bite for her students. Her famous advice after a long session—"Work out the details"—comprised the core of understanding we needed in order to arrive at where she was leading us.

This is not written down anywhere, but a large part of the work of an academic advisor surely is providing encouragement, and motivation to her students. How did Karen do that? By being available and open to talk about whatever your diculties were, academic or not. I remember that, as a student, most times when I left her office, I was feeling better than when I arrived simply because she had convinced me—again—that I would able to write my dissertation and earn my PhD degree. But it has to be pointed out that she did not consciously intend to convince me of this. It came out naturally from her attitude, during our conversations on the mathematical concepts and challenges I was facing. Metaphorically, she held my hand and carefully brought me out of the fog of my doubts.

Karen has been an outstanding prominent mathematician for many decades. I knew that when I first met her, at a conference. One year later I met her for the second time, in her office at UT Austin, after I had just arrived to start my PhD program. Obviously I was a bit nervous. For I didn't really known the person who would guide me and play a major role in my mathematical formation. All these (legitimate) concerns were quickly dismissed as soon as we started our scheduled weekly meetings, when I could observe and feel the way she talked to me, and to everybody else. I realized I had nothing to fear, for a very busy person with no time to spare, still can treat others with kindness.

In summary, we learn not only from what people tell us, but also from the example they set in their daily lives. I learned a lot from Karen (as well as from my former master's advisor, recently deceased), and I imagine them whenever I face the challenge of helping my current students: "work hard, be objective, write it down, talk to people, be patient, be kind, and don't ever think you are the best..."

Andrea Young

One of the great lessons that Karen taught me was not via advice but by modeling behavior. When I was a graduate student at the University of Texas at Austin, the geometry seminar was mostly attended by men. Indeed, more often than not, Karen and I would be the only women in the room. She would often sit quietly and listen to presentations by guest speakers, but almost without fail, she would ask a seemingly simple question that got to the very core of what was being discussed. And once she asked her question, she would continue to question the speaker until she was satised with the response. During these exchanges, I observed her male colleagues listening closely as they knew there was something interesting to learn if Karen was asking about it.

Seeing her do this every Thursday afternoon taught me two non-mathematical lessons. First, it taught me the value of asking questions and of not being satisfied with a simple answer to my questions. Secondly, and most importantly, she showed me how to use my voice unapologetically and how to take up space in a male-dominated environment. Karen could be assertive—aggressive, even—in her questions, but it always seemed to me to be from a place of genuine interest and love of mathematics and knowledge. I am as grateful for these lessons as I am for all the math she taught me. They have been transferable skills that have served me even as I have moved out of mathematics and into higher education administration.

Antonella Marini

I recall many moments with Karen, the first dating back to when she was at the University of Chicago, and I was a first-year graduate student, sweet and tough ones. I cherish both.

Tough love. Of my undergraduate paper on gauge theories she said "nice, but not an earth-shaking result." A severe comment, a compliment, and motivational, all bundled in one. Of a proof she said "I won't go over details, but I'll tell you it is wrong." And so I learned to not trust signs and factors of two and to try and develop a healthy mathematical intuition instead.

A novel feminine way of being a mathematician. Karen described mathematical realities making them appear concrete. She once said "the hardest mathematical concepts are not something one ever understands. One simply accepts them." Because of her teachings, I try to pass on to

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my students the notion of a mathematician as a humble artisan who learns intuition through sweat, and with that, the skill of uncovering the statue from inside the block of stone. The aspiration is to acquire familiarity with the mathematical landscape and move freely. Karen is a true feminist, a woman of incredible mathematical intuition, who invented her unique way of being a mathematician, as a person and as a female.

A sense of humor. Of her article, "Solutions to Yang–Mills equations that are not self-dual," co-authored with the Sibners, often referred to as a "landmark paper," she once said "I still don't understand the topology."

It took time to understand her. Thomas Otway, who was at UT as part of her team in the late eighties, once said to me "When you talk with her about our work, write down what she says, verbatim. It'll make sense LATER." In one of her visits, Leslie Sibner exclaimed, "You assigned the Dirichlet problem for Yang–Mills as a 'WARM-UP' problem!?" That ended up being my dissertation (who knows what the follow-up problem was supposed to be).

Personal advice. Once she said: "you've grown up to sound like a mathematician." And, "marry a mathematician!" (She thought it hard for a female mathematician to be understood by a non-mathematician.) When I claimed I was not ambitious, she called "you are, you climbed Mont Blanc."

A devoted advisor. When Karen moved to UT Austin, her Chicago students followed her. The world around was competing for Karen's attention at lunchtime, but she made sure that we had a "reserved spot" at her table. She was adamant that we should receive her undivided mathematical attention, while at our dedicated meetings, and be left independent afterward. She scheduled weekly meetings with each of us.

A "*PDE person*" and much more. She is an analyst at heart (one of her favorite results is "Regularity for a class of non-linear elliptic systems," *Acta Math.*, 1977), but truly also one of the founders of modern geometric analysis. I am filled with appreciation for having come in contact with multifaceted mathematical and human greatness. Watching her doing mathematics, like an apprentice learns at shop from the master, has been the greatest gift.

Brendan Guilfoyle

I studied for my doctorate under Karen's supervision at UT from 1991 to 1997. I can honestly say that, without her support at key moments, I would not have been able to graduate.

When I arrived, she had something of a fearsome reputation amongst the graduate students—at one time a computer game was going around featuring various characters with superpowers representing different members of the faculty and Karen's avatar had the ability to turn you to stone with its intense gaze. Anyone who has talked mathematics with Karen will know that gaze!

However, in person she was always friendly and helpful and I was delighted when she accepted me as a doctoral candidate. As a foreign student, I was excluded from the loan programs that my fellow US graduate students were tapping into to make ends meet. While I did have a teaching assistantship, this was a period of massive fee increases at UT—it went from hundreds of dollars a semester to thousands of dollars over the space of a few years.

As a consequence, I became involved in graduate student activism, helping to found the Graduate Student Assembly (which still exists at UT) and a graduate student newspaper. At the time we were campaigning for fee waivers for TAs, as it was manifestly unfair that you had to give back a month's wages every semester, mostly for nine "dissertation hours" a week which were in reality a one- to two-hour meeting with your supervisor.

In any event, my high-profile involvement was frowned upon by the powers that be in the mathematics department and I recall being threatened Mafioso-style in an RLM elevator one day by a disgruntled senior Faculty member. Luckily for me, Karen took a different view and liked the fact that I was involved in student politics. Karen's student days had been in Berkeley in the 60s, so she had witnessed tear gas going off in the classrooms etc. Her main concern was that I would give up mathematics and become a full-time activist.

As it was, I finished in a reasonable amount of time, made a career in mathematics, and have kept in touch with Karen down through the years. Without her support throughout this period, I very much doubt that it would have been possible for me to finish the programme.

Caio José Colletti Negreiros

Karen Uhlenbeck was my PhD's advisor at the University of Chicago. After attending a Seminar run by Karen I was moved by her contagious enthusiasm for mathematics. We met once a week in her office (as far as I recall it was Thursday about three o'clock in the afternoon). It was an unforgettable experience to have learned with her how to produce new results in mathematics.

The fundamental very first advice she gave me was her suggestion for my thesis problem. As she predicted, this problem opened a way to many related mathematical questions. In this period at Chicago she gave me a lot of important mathematical ideas and advice related to my thesis.

She taught me how to face a specic mathematical question from several viewpoints in order to possibly overcome some mathematical difficulty.

After five years in Brazil I spent one year with her at the University of Texas at Austin as a post-doc starting

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September 1992. In this post-doc I learned many possibly new directions to move forward in mathematics. But months after my return to Brazil I became very ill and it took me almost three years to fully recover.

She has taught me that we must see our students in a broader sense. Karen was concerned with me as well with my family and acted effectively in a very human way to support us in some difficult situations that we faced in our lives.

Other fundamental advice from her was to make my professional decisions considering mathematics the first thing to be taken into account. This advice was very decisive in my professional life.

All my mathematical career is based on questions that have some relation to my thesis. After years a few of these related questions were solved mainly togethe with colleagues or PhD students at University of Campinas. By the way, my original thesis problem, as far as I know, remains unsolved.

In brief, it was a very memorable experience to have learned and worked in mathematics during those years in Chicago and Austin with a mathematician like Karen. From my own experience I can assure you that she was very demanding with respect to mathematics and very human concerning other issues.

Georgios Daskalopoulos

I am very happy to have the opportunity to express my deep gratitude and appreciation to Karen Uhlenbeck for her continuous mentorship and support throughout the years.

I first met Karen in 1985 when I joined the University of Chicago as a PhD student. At the time, Karen was giving lectures on the new exciting developments of gauge theory. After taking her class on variational problems (I still have the notes from this class), it became clear to me that I wanted to become her student. In 1986, after she returned from a leave in San Diego, I asked her for a thesis problem. She suggested that I work on making rigorous the approach of Atiyah–Bott on the Morse theory of the Yang–Mills functional on Riemann surfaces. I was very excited to work on this problem under her supervision and when she left Chicago I followed her to UT Texas where I completed my thesis. During this time, I also wrote a small paper with her on infinite-dimensional transversality and a few years later (jointly with R. Wentworth) another paper on stable extensions of holomorphic bundles. Recently, I have been extremely lucky to collaborate with Karen on a long-term project and experience again the unique way she does mathematics.

Throughout the years, Karen has been an incredible source of inspiration for me. From very early on I noticed that her mathematical thinking is based on simple geometric arguments, a great wealth of intuitive small steps fitting together perfectly to create a complex puzzle. One important piece of advice she gave me from the start was always to think about a special case of a problem before moving to the general case. "Never try to solve something nonlinear," she reiterated, "unless you fully understand the linear case." Evidently, most would agree that this is a common principle about any type of research. However, it was only after observing Karen work through different questions that I realized her ingenious capability for taking a complicated, maybe even intractable problem, and by slightly changing a few parameters, making it into one that is both tractable and interesting.

Karen has always been interested in physics and encouraged her students to study physics. The notion of symmetry plays a big role in her mathematical thinking. I recall that during my graduate studies in Chicago, she asked me to take a class by the theoretical physicist Prof. Nambu (who got the Nobel Prize a few years later for his discovery of the mechanism of spontaneous broken symmetry). The class was all about infinite-dimensional Lie algebras. I didn't get too much out of the class, and it was only years later that I understood Karen's prospective about the role of symmetry in mathematics and physics. More specically, it was when she explained to me the work of Hilbert and Noether on conservation laws in geometry and physics and applied these ideas in our joint work to study certain types of variational problems.

Another striking aspect about Karen's style of doing research is that, unlike most mathematicians of my generation who have learned geometric analysis from its tail end, Karen knows the origins and motivation of each problem very well. She rarely needs to go to references as she prefers to derive everything from first principles. This may not always mean giving a complete argument, but certainly enough to convince herself of the validity or invalidity of the claims at stake. I recall she always encouraged me to read less and think more.

Several times in my mathematical career, when stuck on a problem, I have gone back to her for advice. She has always been very eager to think about my problem and has often given me a response which at first has left me mystified. It took me a lot of effort, sometimes several months before I discovered what she meant to say, but at the end I always realized the incredible wealth of her answer. Like the oracle of Delphi with its famous $\chi\rho\eta\sigma\mu o\iota$ open to different interpretations, yet full of enigmatic wisdom, Karen's prophecies have enlightened my mathematical life on numerous occasions.

Magdalena Czubak

"People publish too much," is one of the most memorable comments from Karen during graduate school. "One good idea per year is very good," is another one.

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Karen was generous with her time, and we always had a weekly meeting.

I wanted to do everything by myself, and Karen let me, even at a point when I do not think I would have been upset if she helped. Toward the end of writing up my thesis, I realized there was an issue with one lemma, and I was stuck on it for what felt like forever at the time: a whole month. I would go into our meeting and tell Karen what I tried but this is why it did not work, but now I had a new idea to try. Karen would say nothing, and when I finally figured it out (in a shower) and was so happy to tell her, and said that I was getting worried, she said, one month to be stuck is not that long. This is obvious now, and I am grateful that Karen let me find a solution on my own as this built the foundation for future math endeavors.

In retrospect, I wish I had asked more questions back then, so I could have benefited more from the vast knowledge and deep understanding Karen has, even if I probably would not have initially understood what Karen would have said. Karen does have a "nonlinear" way of thinking, and certainly in grad school I was very linear, and have been trying to embrace nonlinearity only in recent years. Karen was also famous for half-finished sentences, so this did not help me, but I was elated that by the time I was graduating, I knew what the end of the sentence was!

I also learned from Karen, not necessarily from any single comment, but from our interactions, to always put math first. What is best for math? That was how I chose where to go for my postdoc, and how I make other professional decisions to this day.

It took me several years after graduation to start addressing Karen as Karen instead of Prof. Uhlenbeck. Before it happened, I was thinking that I could always address Karen as Prof. Uhlenbeck; there are few people that I respect as much, both as a mathematician and a human being.

Michael Gagliardo

Karen Uhlenbeck had an incredibly important impact on the trajectory of my life even before she was my PhD advisor. After leaving UT Austin in 2000 with my master's degree mostly because I believed I was not graduate school material, I found myself working as an actuary at an insurance company. I hated it and while looking for teaching opportunities, I asked Karen to be a letter writer for some community college positions. Instead of just saying yes (or even no) and moving on, Karen told me that I should think of returning to graduate school and that she felt I was capable of producing good work. Due to the timing, it took me another year to return to UT Austin, but Karen looked out for me (and many other graduate students) while I was there, eventually becoming my thesis advisor. Every time I have a chance to think about how much I enjoy my position at California Lutheran University and the opportunities it provides me both personally and professionally, I am so thankful Karen thought to say, "you should come back." I get to be where I am today because of her mentorship.

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Steven Bradlow

I became Karen's student in 1985 at the University of Chicago. At that time Karen already had three other PhD students and during my time she took on two more. One graduated soon after I arrived, one was working on harmonic maps, and the rest of us were all working on gauge theory projects. We were a reasonably close-knit group. I think we were all very aware of how extraordinary our advisor was and we realized how lucky we were to be working with her. Undoubtedly we also bonded over our common struggles with some of the special challenges posed by working with Karen, including:

1. Her inexhaustible energy. I (and I think all of my mathematical siblings) had weekly meetings with Karen. At these meetings I would present my progress of the week at the blackboard and we would discuss—sometimes in minute nitty-gritty detail—the latest obstacles and what should come next. I forget what the scheduled length was for these meetings, but invariably they would run well past two hours and I would always run out of energy long before Karen did. Karen understood well that she had more energy than most other human beings. I recall her saying that one thing she had noticed about most MacArthur fellows was that, like her, they were all extremely high-energy people. Also, she once described how as a young postdoc she was unable to settle down to work unless she first did some vigorous physical activity to burn off her excess energy.

2. Her cryptic communication style. I'm sure it comes as no surprise to anyone who knows Karen if I report that unfinished sentences and rapid jumps made it hard to follow her train of thought as she hop-scotched from one peak to the next in the mathematical terrain. We knew that whatever she said was profound and certain to be helpful, but often I left my meetings with Karen wondering what exactly it was that I had just been told. I recall many sessions with my fellow student Georgios Daskalopoulos puzzling together over messages we got from Karen, trying to crack the code. We didn't always succeed but when we did, we were always richly rewarded—and rather proud of our selves for having done it.

Among the many aspects I found remarkable about Karen was her clear-eyed view of her place in the mathematical firmament, and her acceptance of the obligations that followed from that. In the late 1980s (when I was her student) Karen was unquestionably a mathematical superstar. She clearly understood, without any hint of hubris, that as a result of her status wherever she went people would be clamoring for her attention. She seemed to accept the ensuing claims on her time in the best possible spirit of

noblesse oblige (where in this context the term should not be burdened with any negative connotations!).

I would like to mention just one more of the many pieces of wisdom that I gleaned from Karen. I forget the precise circumstances—maybe it was in the aftermath of a seminar in which some piece of mathematics was not clearly explained—but Karen made an offhand remark about how, after years of constructing them herself, "I know what a proof is." I remember my sense of puzzlement—surely anyone with a basic grasp of propositional logic would know a proof when they saw it—but the remark struck a chord. Only later did I figure it out: logical correctness is a necessary but not a sufficient component of an argument that needs to be engaging, understandable, and persuasive, i.e. of a proof.

The glass ceiling for women persists in mathematics, perhaps less overtly than in times past. But academic mathematics retains cultural practices, norms, and conventions that act to hinder women's (and other groups') advancement. As long as these barriers remain, those women who wish to persist will need role models, mentors, and useful advice. *Aspiring and Inspiring: Tenure and Leadership in Academic Mathematics*, a collection of essays from successful female mathematicians, provides some of those role models and some of that advice. The chapter authors were asked to reflect on their own experiences and help point younger colleagues toward success. Several themes emerge strongly from the collection: Networks of support are important; feeling a sense of belonging is critical; individual mentorship matters; one should formulate one's own definition of success. Of course, the strongest theme is that academic mathematics has not achieved equality of opportunity. Until we do achieve it, we'll need books like this.

The essays were gathered and edited by Rebecca Garcia, Pamela Harris, Dandrielle Lewis, and Shanise Walker. Below we reprint excerpts from the collection. The full volume will be available from AMS Books in Spring 2023.

Excerpts from "Intersectionality as Impetus and Impediment"

Erica Graham

Community

To successfully navigate the mathematical world, I needed to surround myself by people who collectively provided me with connections, counsel, and confidence. At every stage of my career, beginning with choosing a graduate program, I have had to adjust who and what filled these roles for me.

The advice (or warning) I often given students looking to pursue graduate degrees is, "Grad school is going to suck, so you might as well surround yourself by people who don't suck." The same can be said of the road to tenure. Sure, some joy and passion makes it easier, but in order to deal with the externalities associated with pursuing tenure and establishing oneself in mathematics, it is helpful to have a community of people who make the process significantly less daunting. Here's how I built my community of connectors, counselors, and confidence-builders.

. . .As graduate student. The summer prior to beginning my graduate program, I participated in the Enhancing Diversity in Graduate Education (EDGE) summer program. Remember that calculus 2 professor? It turns out, Rhonda Hughes was also the co-founder, along with Sylvia Bozeman from Spelman College, of this program for women in mathematics,[1] which was designed to provide an extensive and sturdy support network to facilitate the success of women in graduate programs in the mathematical sciences. At the time, EDGE was a way for me to get back into the swing of doing mathematics—I'd been working in the real world for two years—before undertaking the momentous task of getting a PhD. That summer, I experienced mathematics in a completely different way. There was nothing about the program's coursework that was easy, but the network that EDGE gave me is something that has kept me going throughout all stages of my career. I'll come back to this.

In the process of deciding where to go to grad school, I was self-aware enough to know that the environment in which I planned to immerse myself for the next several years was considerably more important to me than anything else. Recruitment visits provided an invaluable window into what life might be like in my prospective program. As a Black woman originally from Queens, no one could have predicted that I'd end up attending graduate school in Utah. But, the feeling I got from visiting the math biology group, a subset of the graduate mathematics program, at the University of Utah told me what I needed to know. At the time, the community dynamics I witnessed among the faculty and students within the program instantly felt familiar. And, thankfully, my gut was right.

As expected, grad school was difficult, to say the least. But, the community I created for myself made it doable. My personality and working style were more or less compatible with my advisor, who provided many examples of 'how to be a research mentor' that I am constantly working to emulate. I haven't gotten there yet. I had a great collaborative group of fellow classmates, with whom I spent hours toiling on homework, studying for prelims, and later, practicing thesis defenses and job talks. We had older graduate

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[1] https://www.edgeforwomen.org/