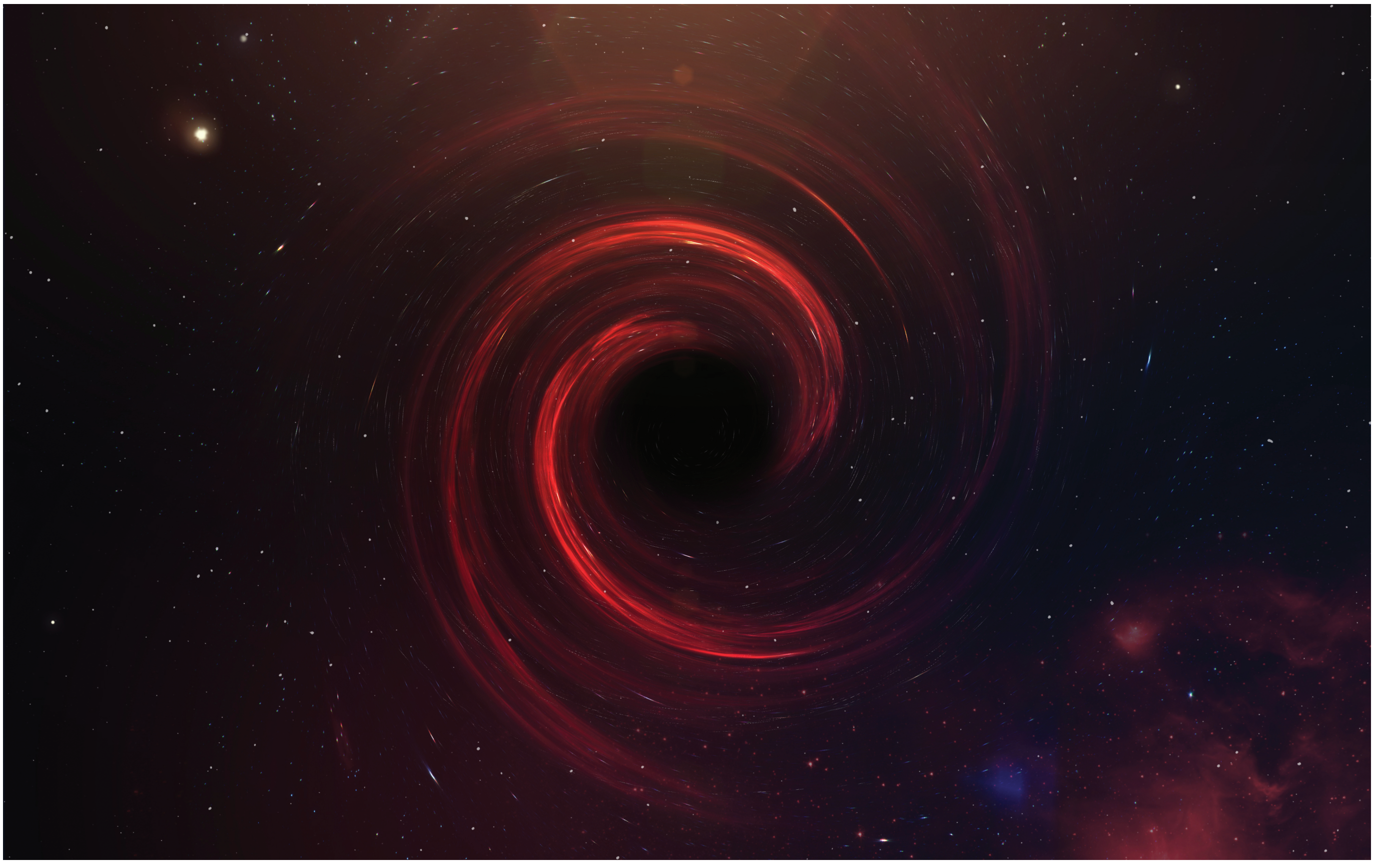
The Mathematics of Stable Black Holes



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1. Stability and Reality

The stability of a physical object is its ability to change in a controlled manner under small perturbations, but giving a definition of what "controlled" or "small perturbation" means is a subtle issue in general. The simplest scenario is the case of points of equilibrium, being physical phenomena or configurations that are independent of time (also said *stationary*), such as a ball on the bottom of a hill or a pendulum in its vertical position. For those phenomena it is natural to ask what happens if a small change is performed to them. Will the perturbed configuration remain at all times similar in structure to the initial one? Or stronger than that, will it return to be exactly identical to the original condition? These scenarios exhibit a stable behavior: in the former case, we call the point of equilibrium *orbitally stable*, in the latter *asymptotically stable*. This is the case of a ball on the bottom of a hill (see Figure 1) or a vertical pendulum pointing downward: when moved away from their positions they eventually return to their original state.

The opposite situation is when points of equilibrium that undergo small perturbations completely change their disposition, such as a ball on the top of a hill (see Figure 2) or a pendulum pointing upward: these are called *unstable*. Even though stable and unstable configurations are

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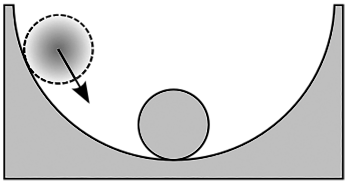

Figure 1. Stable equilibrium.

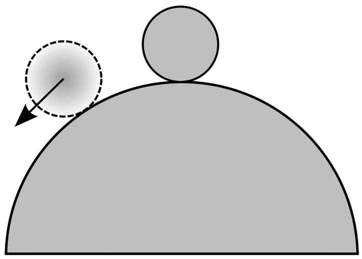

Figure 2. Unstable equilibrium.

symmetric from a mathematical point of view, they have a very different role in representing physical reality. In the physical world, the exact position of a point of equilibrium, down to all its decimal digits, can never be attained without a small error. Indeed, if it were possible to arrange a pendulum precisely on its upward position, it would remain still and not move for all times. On the other hand, since even in measuring the position of the equilibrium point one needs to use rational approximations, a physical object can only be put in a position which is very close to the equilibrium, but not on the exact one.

When a physical object is positioned in an unstable configuration one expects to see a quick change in its location, such as when a ball falls down a hill or a vertical pendulum pointing upward drops to its downward stable position. Even though it is rare to find natural objects in unstable configurations, unstable points of equilibium are of interest and used in real-life applications such as the stationing of satellite or telescopes that can correct the instability by continuously adjusting their position. Nevertheless, for idealized systems completely governed by gravity, the concept of stability is crucially related to the one of reality. Configurations which are detectable in the physical world are normally stable, and unstable points of equilibrium are rarely observable.

Black holes are point of equilibrium of the *Einstein equation*, the master equation of Albert Einstein's General Relativity. Even though the first images of black holes have been obtained (by the Event Horizon Telescope) and gravitational waves from their mergers have been detected (by LIGO-Virgo), black holes are mathematical solutions whose stability has not been fully proved yet. In those physical discoveries, mathematics is used to represent black holes through a family of mathematical solutions, called the *Kerr solution*, parametrized by their mass

Figure 3. The first image of a black hole, using Event Horizon Telescope observations of the center of the galaxy M87.

and angular momentum, whose stability is one of the most important open problems in the field of Mathematical General Relativity. Establishing such a result is fundamental to confirm the relevance of the Kerr black hole as a realistic physical object, as only stable configurations are expected to naturally exist in the physical world.

What happens when a black hole solution is perturbed away from its stationary state? Leaving the realm of stationarity, the configuration will become dynamical, with its properties changing over time. The way it changes depends on the physics: in the case of the ball or the pendulum it oscillates in virtue of the gravitational force. The dynamics of perturbations of black holes is also due to gravity, as encoded by the Einstein equation.

The field of Mathematical General Relativity was given a tremendous boost by Choquet-Bruhat in 1952 [FB52] when she solved the Cauchy problem for the Einstein equation using that its dynamics has a wave character: perturbations of a black hole do oscillate like a wave. The wave-like behavior predicted by Choquet-Bruhat's theorem does not imply that the perturbation will necessarily radiate away: waves can for example interact with each other and create resonances, or concentrate their energy in a finite region of space for a long time.

Choquet-Bruhat's theorem is a local result, describing the behavior of solutions to the Einstein equation for a short period of time. The proof of the *Kerr stability conjecture* would instead need a global result to show that the Kerr family of black holes is an asymptotically stable point of equilibrium for the Einstein equation. This means that a perturbation of a member of the Kerr black hole family will eventually settle down to another member of the same family, with possibly a different mass and angular momentum, after a very long time.

2. Black Hole Solutions

Einstein formulated his theory of General Relativity in 1915. According to General Relativity, a spacetime is a 4-dimensional manifold equipped with a Lorentzian metric

g, i.e. with signature $(+++-)$, that satisfies the *Einstein equation*[1]:

$$\mathrm{Ric}(g) - \frac{1}{2}\mathrm{R}(g)g = \mathrm{T}, \tag{1}$$

where

- $\mathrm{Ric}(g)$ is the Ricci curvature of g,
- $\mathrm{R}(g)$ is the scalar curvature of g,
- T is called the stress-energy tensor, and contains information about the matter fields present in the spacetime.

The Einstein equation (1) is the mathematical translation of the words by the American physicist John Wheeler "spacetime tells matter how to move; matter tells spacetime how to curve": the left hand side of the equation is a particular combination of curvatures of the spacetime, which encodes how curved the spacetime is, while the right hand side describes the behavior of the matter.

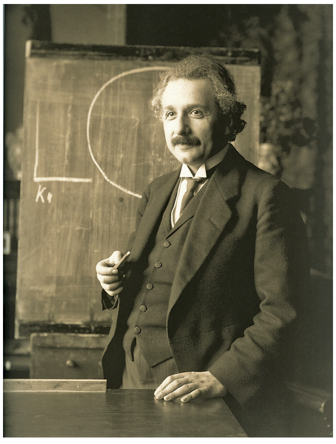

Figure 4. Einstein 1921 by F. Schmutzer.

The Einstein equation (1) is an equation for the unknown metric g and the distribution of the matter. The combination of curvatures on the left hand side of (1) takes the form of a system of second order partial differential equations in the metric g. Those second order PDEs are sourced by the right hand side of the equation, which is a given expression containing the information about the matter present in the universe, whose dynamics is also governed by the Einstein equation, in the form of $\mathrm{div}(T) = 0$.

Because of the involved character of the Einstein equation, contrary to Newtonian gravity, General Relativity is not trivial even in the absence of matter, i.e. if $\mathrm{T} \equiv 0$. In this case the Einstein equation (1) reduces to the *Einstein vacuum equation*[2]:

$$\mathrm{Ric}(g) = 0. \tag{2}$$

The Einstein vacuum equation is the benchmark of the study of General Relativity, as it encodes the behavior of the gravitational field as well as many of the difficulties of the general case. One example of a non-vacuum Einstein equation is when interaction of the gravitational field with electromagnetic radiation is allowed. In that case, the Ricci curvature is sourced by a quadratic expression in terms of the electromagnetic field, which satisfies itself the equations of electrodynamics. More precisely, one obtains the *Einstein-Maxwell equation*:

$$\mathrm{Ric}(g) = 2\,\mathrm{F}\cdot\mathrm{F} - \frac{1}{2}|\,\mathrm{F}\,|^2 g, \tag{3}$$

where F is a 2-form on the manifold, called the electromagnetic tensor, satisfying the Maxwell equations:

$$d\,\mathrm{F} = 0, \qquad \mathrm{div}_g\,\mathrm{F} = 0. \tag{4}$$

Black holes are special solutions to the Einstein equation. The most general explicit black hole solution to the vacuum equation (2) is the *Kerr solution*, obtained by Roy Kerr in 1963. In coordinates (t, r, θ, ϕ) the Kerr metric is explicitly given by

$$\begin{aligned} g_{M,a} = &-\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2 \end{aligned} \tag{5}$$

where

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

The Kerr metric $g_{M,a}$ is a 2-parameter family of solutions to $\mathrm{Ric}(g) = 0$. The two real parameters M and a represent the mass and the angular momentum of the black hole respectively, with[3] $|a| \leq M$. For each pair of parameters satisfying $|a| \leq M$, called the sub-extremal ($|a| < M$) and extremal ($|a| = M$) range, the Kerr metric $g_{M,a}$ describes a stationary black hole which rotates around its axis.

In the case of vanishing angular momentum $a = 0$, the Kerr solution reduces to the spherically symmetric *Schwarzschild metric*, a 1-parameter family of solutions given by

$$\begin{aligned} g_M = &-\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ &+ r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \tag{6}$$

Having a high degree of symmetry, the Schwarzschild metric was found much earlier than the Kerr metric, right after Einstein wrote his equation.

One can notice that the metric (5) has a singular behavior at the root of $\Delta = 0$ (at $r = 2M$ in the case of Schwarzschild). Nevertheless, along the hypersurface $\Delta = 0$ there is no geometrical invariant which becomes singular, but the hypersurface is rather a coordinate singularity: there is a change of coordinates which modifies the metric into one which is perfectly well-behaved at $\Delta = 0$. Still, such hypersurface has an interesting geometrical property: curves which are timelike or null (i.e. whose tangent vector satisfy $g(X, X) \leq 0$) are not able to leave the region $\Delta \leq 0$ once they enter it. Since according to General Relativity, any massive object is only allowed to travel slower than

[1] *Here we set the speed of light and the gravitational constant to unity, i.e.* $c = G = 1$. *In addition, we only consider the case of zero cosmological constant,* $\Lambda = 0$, *or asymptotically flat solutions.*

[2] *In the case of* $\mathrm{T} = 0$ *by taking the trace of* (1) *with respect to* g *one can deduce that* $\mathrm{R}(g) = 0$.

[3] *If* $|a| > M$ *the spacetime is said to contain a naked singularity.*

the speed of light along timelike trajectory while light rays follow null ones, it follows that not even light is able to escape such a region of spacetime. It is a *black hole* region, and the hypersurface $\Delta = 0$ is called its *event horizon*.

A region of spacetime where not even light can escape was a very difficult concept to grasp for the physics community at the time, and for a long time the Schwarzschild solution was only considered to be valid outside the black region, i.e. for $r > 2M$. It was believed that the presence of the black hole region was an artifact of the symmetry of the solution: as in the real world there is no perfectly spherically symmetric massive object, so there would not be any black hole region. The derivation of the Kerr metric in 1963, as well as Penrose's "singularity theorem", suggested that the black hole region could be much more general than an artifact of high degree of symmetry and did not have to be just a mathematical property of an unrealistic physical object: it could in fact be real. The Kerr solution is now considered to be the most fundamental black hole solution, and it is believed to represent the astrophysical black holes present in the universe.

In the case of the Einstein-Maxwell equation (3)(4), the most general known explicit black hole solution is called the *Kerr-Newman* metric $g_{M,a,Q}$, a 3-parameter family given by the same expression of the Kerr metric as in (5), with

$$\Delta = r^2 - 2Mr + a^2 + Q^2,$$

where Q represents the charge of the black hole for $a^2 + Q^2 \leq M^2$. The spherically symmetric case of charged black hole is called the *Reissner-Nordström* spacetime $g_{M,Q}$ for $a = 0$.

The family of Kerr and Kerr-Newman black holes are stationary solutions to the Einstein equation, as it can be seen from the fact that the expression in (5) does not depend on time. They are points of equilibrium for the Einstein equation.

3. The Kerr Stability Conjecture

We can now formulate the conjecture of stability of the Kerr solution, affirming that the Kerr[4] family of black hole solutions is an asymptotically stable point of equilibrium for the Einstein equation.

Conjecture 1 (Stability of the Kerr family). *Initial data for the Einstein vacuum equation which are sufficiently close to a Kerr black hole evolve asymptotically in time to another member of the Kerr family.*

According to the Kerr stability conjecture, if a Kerr black hole with a certain mass and angular momentum gets physically perturbed, the perturbed spacetime will still be represented by a member of the Kerr family, with in general a different mass and rotation, after possibly a long time. In the language of PDEs, the stability of the Kerr family can be formulated as follows. Let g_λ represent the Kerr family of stationary solutions, for some parameter $\lambda = (M, a)$, with $\mathrm{Ric}(g_\lambda) = 0$. Proving that the family of solutions g_λ is stable under small perturbations as a solution to the Einstein vacuum equation consists in proving that a solution with initial data close to g_λ converges asymptotically in time to $g_{\lambda'}$ with λ' close to λ.

There are various levels of increasing difficulty for the stability problem:

1. Prove that general solutions to the linearized equation
$$(d\,\mathrm{Ric})|_{g_\lambda}(\dot{g}) = 0 \tag{7}$$
are bounded and decay in time: this is the *linear stability*.
2. Prove that solutions to the fully non-linear Einstein vacuum equation with initial data close to the Kerr family are bounded and decay in time: this is the *non-linear stability*.

Observe that the conjecture does not claim that one member of the Kerr family is stable, but that rather the whole family is stable. This is due to one source of instability of the Einstein equation, which is consequence of the fact that the linearized Kerr metrics given by

$$\dot{g}_\lambda(\dot{\lambda}) = \frac{d}{ds}\Big|_{s=0} g_{\lambda + s\dot{\lambda}}$$

are also solutions to the linearized Einstein equation (7). As mentioned above, this corresponds to a physical change of the mass or rotation of the black hole upon the perturbation. In particular, any result of linear stability such as boundedness or decay would have to hold up to linearized Kerr solutions of the above form.

Another source of instability of the Einstein equation is due to its tensorial character: for any metric g solution of $\mathrm{Ric}(g) = 0$, its pullback through any diffeomorphism is still a solution of the equation. The choice of the diffeomorphism or the particular form of the metric is called *gauge*. In particular, any result of linear stability would have to hold up to a choice of gauge.

The mechanism through which the dispersion of the perturbation in the stability problem takes place needs to be formulated in terms of the Cauchy problem for the Einstein equation, which describes the dynamics of black hole solutions.

4. The Cauchy Problem

What kind of PDEs are hidden behind the compact expression (1) of the Einstein equation? Already early works of Einstein suggested that his master equation had the character of a wave, but it was only in 1927 that Darmois formulated the Einstein equation as a quasilinear hyperbolic

[4] *A similar conjecture can be formulated for the Einstein-Maxwell equation and the stability of the Kerr-Newman family.*

Figure 5. Yvonne Choquet-Bruhat.

system of PDEs, which in wave coordinates $\Box_g x^\mu = 0$ takes the form

$$\Box_g g = N(g, \partial g), \tag{8}$$

called the *reduced Einstein equation*, where $\Box_g$ is the D'Alembertian operator associated to the metric g and $N(g, \partial g)$ denote non-linear terms in g and its first derivative. The initial data for the Einstein equation consists of a Riemannian 3-metric and its second fundamental form given on a spacelike hypersurface Σ_0, which satisfy some compatibility conditions, called the constraint equations. The D'Alembertian operator $\Box_g$ is a generalization of the standard wave operator on $\mathbb{R}^4$ given by

$$\Box_{g_0} = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2,$$

associated to the constant coefficient metric

$$g_0 = -dt^2 + dx^2 + dy^2 + dz^2,$$

called *Minkowski spacetime*.

In her seminal work in 1952, Choquet-Bruhat proved the existence of solutions to the Cauchy problem for the Einstein equation with smooth initial data using the quasilinear formulation for the reduced vacuum Einstein equation (8).

Theorem 1 (Choquet-Bruhat [FB52], 1952). *Any initial dat set satisfying the constraint equations gives rise to a local in time smooth solution to the full Einstein equation and such solution is locally*[5] *geometrically unique.*

The most important properties satisfied by solutions to the standard wave equation, such as finite speed of propagation and continuous dependence on the initial data, also hold for solutions to the general covariant wave operator $\Box_g$ for any Lorentzian metric g. Moreover, Theorem 1 implies local well-posedness and continuous dependence on the initial data for the Einstein equation: given a set of compatible initial data for the Einstein equation on some initial time slice Σ_0, one can uniquely solve the Einstein equation locally in time in the future of Σ_0.

On the other hand, in order to explore the issue of stability for stationary solutions to the Einstein equation such as black holes one needs to perform a global analysis, i.e. showing that the solution exists for long time after the initial data has been given and describing what its behavior is then. This is a question that Theorem 1 cannot answer and is in general very difficult. For example, consider a case which is in principle much simpler than the Einstein equation: a non-linear scalar wave equation such as

$$\Box_{g_0} \psi = (\partial_t \psi)^2. \tag{9}$$

If we consider equation (9) with initial data

$$\psi|_{t=0} = \partial_t \psi|_{t=0} = 0,$$

then by uniqueness the solution is given by $\psi(t) = 0$ for all $t \geq 0$. It is therefore global, smooth and bounded in time. On the other hand, if we consider equation (9) with initial data

$$\psi|_{t=0} = \partial_t \psi|_{t=0} = \epsilon,$$

for some $\epsilon > 0$, there exists a finite time T such that $\psi \to \infty$ for $t \to T$. The solution is not defined globally in time and in fact blows up in finite time. We say that the trivial solution to equation (9) (which is also stationary as it is independent of time) is not stable under small perturbations: if the initial data changes of size ϵ, for any small ϵ, then the behavior of the solution changes dramatically, passing from one which is bounded for all times to one which blows up in finite time.

Since the (reduced) Einstein equation (8) has in principle the same schematic structure as equation (9) (albeit more complicated as it involves tensorial quantities and quasilinear terms), one may worry that perturbations of even the trivial solution, the Minkowski spacetime, could blow up in finite time.

Surprisingly, this does not happen, thanks to the special structure of the non-linearities of the Einstein equation, as proved by the seminal work of Christodoulou-Klainerman in 1993.

Theorem 2 (Christodoulou-Klainerman [CK93], 1993). *The Minkowski spacetime g_m is globally non-linearly stable as solution to the Einstein vacuum equation.*

The Einstein equation satisfies a version of the *null condition*, introduced by Klainerman to prove global existence for solutions to the non-linear wave equation in Minkowski space. The null condition prescribes that the quadratic linearities are expressed as a null form, which is a quadratic form $Q(\partial\psi, \partial\psi)$ that satisfies $Q(\xi, \xi) = 0$ for all null vectors ξ, i.e. for which $g(\xi, \xi) = 0$. In the case of the

[5] *In 1969, Choquet-Bruhat and Geroch proved global uniqueness for the maximal developement of initial data.*

Einstein equation, the null condition identifies the mechanism for the nonlinear stability, despite the slow decay of the perturbations.

The monumental proof of Christodoulou-Klainerman's theorem is a geometrical construction of the spacetimes obtained as perturbations of Minkowski spacetime. In addition to the null condition, the proof is based on some important advances in the study of partial differential equations, such as a robust vectorfield approach, used to derive quantitative decay based on generalized energy estimates and commutation with (approximate) Killing and conformal Killing vectorfields. The proof is also based upon an elaborate bootstrap argument according to which one makes educated assumptions about the behavior of the solutions and then shows, through a sequence of a-priori estimates, that they are in fact satisfied. In particular, in the case of Minkowski space, the control of the non-linearities of the equations are the predominant difficulty.

Christodoulou-Klainerman's theorem answers affirmatively to the question of stability of the trivial solution to the Einstein equation. In expanding our attention to the question of stability of non-trivial solutions to the Einstein equation, such as the Kerr family of black holes, the ingredients of Christodoulou-Klainerman's proof mentioned above are still needed to deal with the non-linearities of the problem. Nevertheless, in the case of perturbations of black hole solutions, the linear aspect of the equations become very challenging by itself. In what follows, we will mostly concentrate on the analysis of the linear wave equation and the linearized gravity in black hole solutions.

5. The Wave Equation on Black Holes

From the reduced Einstein equation in wave coordinates one can see that the linearized Einstein equation in perturbations of Kerr black holes is related to the study of the linear wave equation

$$\Box_{g_{M,a}} \psi = 0, \tag{10}$$

where $\Box_{g_{M,a}}$ is the D'Alembertian operator associated to the Kerr metric $g_{M,a}$ given by (5).

We now give an overview on how to study the wave equation (10), and how that is related to the stability of black hole solutions.

5.1. The vectorfield method. As is common in the analysis of PDEs, the study of general solutions to (10) aims to obtain a-priori energy estimates and spacetime integral estimates of the form

$$E[\psi](t) + \mathrm{Mor}[\psi](0,t) \leq CE[\psi](0), \tag{11}$$

where $E[\psi](t)$ denotes some positive definite energy of ψ on a spacelike slice at time t, such as $E[\psi](t) = \int_{\Sigma_t} |\partial\psi|^2$, $\mathrm{Mor}[\psi](0,t)$ denotes some positive definite integral on the spacetime region $\mathcal{M}(0,t)$ bounded by spacelike slices Σ_0

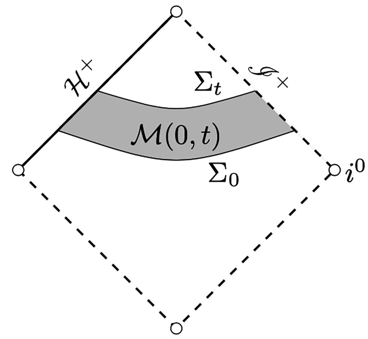

Figure 6. Penrose diagram of a black hole solution.

at time 0 and Σ_t at time t and C is a uniform constant. Statements such as (11) which combine energy estimates and integrated local energy decay estimates are normally referred to as *energy-Morawetz estimates*, in honor of the pioneering work of Cathleen Morawetz.

Recall that conservation of the energy for the standard wave equation in Minkowski space can be established by multiplying the wave equation by $\partial_t \psi$ and integrating by parts. Schematically one obtains:

$$\begin{aligned} 0 &= \Box_{g_0}\psi \cdot \partial_t\psi \\ &= \big(-\partial_t^2\psi + \triangle\psi\big)\cdot\partial_t\psi \\ &= -\partial_t^2\psi\cdot\partial_t\psi - \partial_t\nabla\psi\cdot\nabla\psi + \text{boundary terms} \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\nabla\psi|^2) + \text{boundary terms,} \end{aligned} \tag{12}$$

which, upon integration on a spacetime domain, implies conservation of the energy density $|\partial_t\psi|^2 + |\nabla\psi|^2$.

While conservation of energy through a simple integration by parts is robust to perturbations of the Minkowski metric, decay was initially derived either using the Kirchhoff formula or by Fourier methods, which are manifestly not robust. Later, Morawetz used the *abc method* of Friedrichs, consisting of multiplying a PDE for ψ by a multiplier of the form $a\partial_t\psi + b\nabla\psi + c\psi$ for well chosen functions a, b, c and regrouping terms through integration by parts upon integration on a spacetime domain. Morawetz made use of a radial vectorfield in the *abc* method to obtain integrated local energy decay estimates for solutions to the wave equation outside of an obstacle in Minkowski space. Morawetz also introduced weighted multipliers with weight growing in t to derive pointwise decay estimates in Minkowski.

The use of weighted multipliers was expanded by Klainerman in the *classical vectorfield method*, first developed for the wave equation in Minkowski space, in which weighted multipliers were combined with weighted commuting vectorfields, called commutators. The *modern vectorfield method*, introduced later by Dafermos-Rodnianski, makes use of weighted multipliers with weights which do not grow in t and combines them to weighted commutators to obtain integrated local energy decay estimates in general geometries, such as black hole solutions. To compensate for the the lack of enough Killing and conformal Killing vectorfields on a Schwarzschild or Kerr background, the vectorfield method has been extended in the last fifteen years to include also vectorfields whose deformation tensors have coercive properties in different regions of spacetime, not necessarily causal.

The vectorfield method is based on applying the divergence theorem in a causal domain of the manifold to certain energy currents, which are constructed from the *energy-momentum tensor*. The energy-momentum tensor associated to a general wave equation $\Box_g \psi = 0$ is given by

$$\mathcal{Q}[\psi]_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \psi \partial^\lambda \psi. \qquad (13)$$

The wave equation is satisfied if and only if the divergence of the energy-momentum tensor $\mathcal{Q}[\psi]$ vanishes. We say that a vectorfield X is used as a multiplier if one constructs the current

$$\mathcal{P}^{(X)}_\mu = \mathcal{Q}[\psi]_{\mu\nu} X^\nu,$$

and applies the divergence theorem to a spacetime region $\mathcal{M}(0,t)$ region bounded by two spacelike hypersurfaces Σ_0 and Σ_t to obtain

$$\int_{\Sigma_t} \mathcal{P}^{(X)}_\mu n^\mu_{\Sigma_t} + \int_{\mathcal{M}(0,t)} D^\mu \mathcal{P}^{(X)}_\mu = \int_{\Sigma_0} \mathcal{P}^{(X)}_\mu n^\mu_{\Sigma_0}, \qquad (14)$$

where n_Σ denotes the future directed timelike unit normal to Σ. A standard computation implies that

$$D^\mu \mathcal{P}^{(X)}_\mu = \frac{1}{2} \mathcal{Q}[\psi] \cdot {}^{(X)}\pi$$

where ${}^{(X)}\pi_{\mu\nu} = D_{(\mu} X_{\nu)}$ is the deformation tensor of the vectorfield X. Recall that if X is a Killing vectorfield, then ${}^{(X)}\pi = 0$.

By combining vectorfields X for which the boundary energies $\int_{\Sigma_t} \mathcal{P}^{(X)}_\mu n^\mu_{\Sigma_t}$ are positive definite, such as for $X = \partial_t$ (which is Killing) in Minkowski space, and those for which the spacetime energies $\int_{\mathcal{M}(0,t)} D^\mu \mathcal{P}^{(X)}_\mu$ are positive definite, such as modifications of $X = \partial_r$ in Minkowski space, one can obtain Morawetz estimates for the solutions.

5.2. Trapping and superradiance.

In order to prove boundedness and decay for general solutions ψ of (10) in Kerr spacetime one encounters the following serious difficulties.

- *Trapped null geodesics.* This concerns the existence of null geodesics in the spacetime which are neither crossing the event horizon nor escaping to null infinity, along which solutions to the wave equation can concentrate for arbitrary long times. In the Schwarzschild case, these geodesics orbit the *photon sphere* at $r = 3M$. Remarkably, those trapped null geodesics are unstable: geodesics which are perturbed away from the photon sphere tend to leave the photon sphere (and possibly escape to infinity)[6]. As a result of the trapping properties, the Morawetz estimates have to degenerate at the photon sphere, as shown by Sbierski.
- *Trapping properties of the event horizon.* The horizon itself is ruled by null geodesics, which do not communicate with null infinity and can thus concentrate energy. This problem was solved by understanding the so called *red-shift* effect associated to the event horizon by Dafermos-Rodnianski, which counteracts this type of trapping.
- *Superradiance.* This is the failure of the stationary Killing field ∂_t to be everywhere timelike in the exterior of the black hole, which is timelike only outside of the so-called *ergoregion*. As a consequence, the associated conserved energy fails to be positive definite.

The starting and most demanding part of the analysis of the wave equation on black holes is the derivation of a global Morawetz estimate which degenerates in the trapping region. Once such estimate is established, one can commute with the Killing vectorfields of the background, and the so called red-shift vectorfield, to derive uniform bounds for solutions. The most efficient way to also get decay is based on the presence of family of r^p*-weighted*, quasi-conformal vectorfields defined in the far r region of spacetime, as introduced by Dafermos-Rodnianski [DR10].

A complete proof of boundedness and decay for solutions to (10) was only obtained in the past decade, culminating in the work by Dafermos-Rodnianski-Shlapentokh-Rothman [DRSR16] which covers the case of the full sub-extremal range $|a| < M$.

Theorem 3 (Dafermos-Rodnianski-Shlapentokh-Rothman [DRSR16], 2014). *General solution ψ of* (10) *on the exterior of a Kerr black hole background in the full sub-extremal range $|a| < M$, arising from bounded initial energy on a suitable Σ_0, have bounded energy flux through a global foliation of the exterior. In particular, ψ satisfies uniform pointwise bounds.*

Prior to Dafermos-Rodnianski-Shlapentokh-Rothman's work in sub-extremal Kerr, boundedness and decay properties for the scalar wave equations in Schwarzschild black holes ($a = 0$) were obtained by Blue-Soffer [BS03] and Blue-Sterbenz, and in very slowly rotating Kerr black holes ($|a| \ll M$) by Tataru-Tohaneanu [TT11], Dafermos-Rodnianski and Andersson-Blue [AB15]. We shall now give an overview of these works, which use various techniques in the analysis of the wave equation: mode

[6] *The particles of light which orbit around the black hole close to the photon sphere and then get scattered away are the ones detected by the Event Horizon Telescope, and they form the luminous ring outside the black hole depicted in their images.*

decomposition, pseudo-differential operators and physical-space estimates.

The mode decomposition refers to the analysis of mode solutions to the wave equation, i.e. solutions of the separated form

$$\psi(r, t, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{15}$$

where $\omega \in \mathbb{C}$ is the time frequency, and $m \in \mathbb{Z}$ is the azimuthal mode. The separated form for $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$ is related to the Fourier transform of the solution with respect to the symmetries of the spacetime (the stationary vectorfield ∂_t and the axially symmetric ∂_ϕ), and therefore corresponds to its frequency decomposition.

In addition to the two Killing vectorfields, the Kerr metric admits a Killing tensor, called the *Carter tensor*, which represents a hidden symmetry of the spacetime and which provides a third constant of motion, allowing for the full integrability of the geodesic flow. Because of such integrability, functions of the form (15) are solutions to the wave equation as long as $R(r)$ and $S(\theta)$ respectively satisfy a radial and an angular ODE. The Carter separation introduces, in addition to the frequency ω and m, a real frequency parameter $\lambda_{m\ell}(a\omega)$ parametrized by $\ell \in \mathbb{N}_0$, which are the eigenvalues of an associated elliptic operator, whose eigenfunctions are called oblate spheroidal harmonics $S_{\omega m \ell}$.

By physical-space estimates we refer to an analysis of the wave equation which does not require a mode or frequency decomposition as in (15). A physical-space approach has the usual advantages of the classical vectorfield method, such as being robust with respect to perturbations of the metric.

5.2.1. *The case of Schwarzschild, $a = 0$.* In the spherically symmetric Schwarzschild solution, the Killing vectorfield ∂_t is timelike everywhere in the exterior region, and the trapped null geodesics are contained in a physical-space timelike cylinder, called the photon sphere. The first property implies that superradiance is not present, as the vectorfield ∂_t defines a positive definite conserved energy. The second property implies that the trapping region of the spacetime does not depend on the frequency of the solution.

As the orbital null geodesics are restricted to a physical-space hypersurface, given by $r = 3M$, spacetime energy estimates can be obtained through a vectorfield of the form $\mathcal{F}(r)\partial_r$, with $\mathcal{F}$ vanishing at $r = 3M$, and the analysis of the wave equation can be performed in physical-space. The first Morawetz estimates due to Blue-Soffer [BS03] and Blue-Sterbenz are based on a modified version of the classical Morawetz integral energy decay estimate. Further developments appear in the works of Dafermos-Rodnianski [DR10], and Marzuola-Metcalfe-Tataru-Tohaneanu.

5.2.2. *The case of slowly rotating Kerr, $|a| \ll M$.* In Kerr spacetimes, the analysis of the wave equation is complicated by the presence of an ergoregion and the dependence of the trapping region on the frequency of the solution. In addition to the fact that the conserved energy associated to ∂_t is not positive definite everywhere, the trapped null geodesics of Kerr are not confined to a hypersurface in physical-space, but rather cover an open region of the spacetime which depends on the energy and angular momentum of the geodesics. For this reason, the derivation of a Morawetz estimate in Kerr requires a more refined analysis involving both the vectorfield method and mode decompositions or pseudo-differential operators, as obtained by Tataru-Tohaneanu [TT11] and Dafermos-Rodnianski.

In the case of small angular momentum, $|a| \ll M$, both problems due to superradiance and trapping simplify. In fact, the superradiant part of the separated solution (15) is not trapped, and it satisfies a local energy decay identity obtained by perturbing the one in Schwarzschild. Moreover, the problem of capturing the trapping region can be overcome using frequency-localized generalizations of the Morawetz multipliers obtained in Schwarzschild. Even though the null geodesics are not localized on a physical-space hypersurface, they are localized in frequency-space, as the potential of the radial ODE has a unique simple maximum in the trapped frequency range, whose value $r_{trap}(a\omega, m, \lambda_{m\ell})$ depends on the frequency parameters. This allows for the construction of a frequency-space analogue of the current $\mathcal{F}(r)\partial_r$ which vanishes at r_{trap} for each triple of trapped frequencies.

In slowly rotating Kerr spacetime, Andersson-Blue [AB15] obtained the first analysis of solutions exclusively in physical-space by extending the classical vectorfield method to include second order operators. This approach also makes use of the Carter hidden symmetry in Kerr, but not through the separation of the solution as in (15), but rather as a physical-space commutator to the wave equation. Andersson-Blue's method [AB15] supplements the two Killing vectorfields of the spacetime with a second order operator

$$\mathcal{K} = D_\alpha (K^{\alpha\beta} D_\beta)$$

where $K^{\alpha\beta}$ is the Carter tensor of the Kerr metric. As Killing tensors commute with the D'Alembertian operator in Ricci-flat spacetimes, the second order differential operator $\mathcal{K}$ can be used as a symmetry operator in addition to ∂_t and ∂_ϕ. This allows to obtain a local energy decay identity for $|a| \ll M$ at the level of three derivatives of the solution which degenerate near $r = 3M$, as trapped null geodesics lie within $O(|a|)$ of the photon sphere $r = 3M$.

5.2.3. *The case of sub-extremal Kerr, $|a| < M$.* In passing from the slowly rotating case $|a| \ll M$ to the full sub-extremal range, there is an intermediate step which present many simplifications, which is the case of axially symmetric solutions to the wave equation, i.e. solutions of (10) satisfying $\partial_\phi \psi = 0$. For those solutions, superradiance is effectively absent and the trapping region collapses to a physical-space hypersurface. Although ∂_t still fails to be everywhere timelike, its associated energy is non-negative. This is due to the fact that the null generator of the horizon, $\widehat{T} := \partial_t + \frac{a}{r^2+a^2}\partial_\phi$, is timelike everywhere in the exterior region. In particular, for axially symmetric solutions, the Hawking vectorfield $\widehat{T}$ behaves like the Killing vectorfield ∂_t. As the dependence on the azimuthal frequency m becomes trivial, the axially symmetric trapped null geodesics all asymptote towards a hypersurface $\{r = r_{trap}\}$ in physical-space, where r_{trap} is defined as the largest root of the polynomial

$$\mathcal{T} := r^3 - 3Mr^2 + a^2 r + Ma^2, \qquad (16)$$

and the construction of the current $\mathcal{F}(r)\partial_r$ simplifies. The decay for axially symmetric solutions in Kerr spacetime was first derived by Dafermos-Rodnianski in frequency-space and it was then obtained by Ionescu-Klainerman and Stogin in physical space. The derivation of Morawetz estimates in physical space is known to fail for general solutions in Kerr, as shown by Alinhac.

Finally, in the work of Dafermos-Rodnianski-Shlapentokh-Rothman [DRSR16] local energy decay estimates are obtained in the full sub-extremal range $|a| < M$ for the wave equation in Kerr using frequency-space analysis. They perform a careful construction of frequency-dependent multiplier currents for the separated solutions, and crucially make use of the structure of trapping, i.e. the existence of a simple maximum $r_{trap}(a\omega, m, \lambda_{m\ell})$ for the radial potential, and the fact that superradiant frequencies are not trapped, which they show it holds in the full sub-extremal range $|a| < M$. They then make use of a continuity argument in a to justify the future integrability necessary to perform the Fourier transform in time.

5.2.4. *The case of extremal Kerr, $|a| = M$.* The geometry of the extremal Kerr spacetime satisfying $|a| = M$ exhibits several qualitative differences from the sub-extremal case, most notably the degeneration of the red-shift effect at the horizon due to the vanishing surface gravity. In extremal Kerr for generic solutions to the wave equation certain higher order derivatives asymptotically blow up along the event horizon as a consequence of conservation laws discovered by Aretakis, in what is now known as the *Aretakis instability*. This generic blow up is unrelated to superradiance and holds even for axially symmetric solutions. For axisymmetric solutions, decay results have been obtained by Aretakis in frequency space. The understanding of the behavior of general non-axially symmetric solutions to the scalar wave equation on the extremal black holes is an open problem in General Relativity.

6. Linearized Gravity

While the scalar wave equation (10) represents the first simplified toy problem in studying the Einstein equation, the stability problem for black hole solutions concerns the long-time behavior of solutions to the full Einstein vacuum equation $\mathrm{Ric}(g) = 0$. As the first step in the study of the Einstein equation, one can consider the case of the linearized equation (7), or *linearized gravity*.

6.1. **Mode analysis.** Historically, two versions of linearization for the Einstein equation have been considered: perturbations at the level of the metric or at the level of curvature. The first important results concerning both linearizations were obtained by the physics community based on the classical method of separation of variables, with works by Regge-Wheeler, Zerilli, Moncrief in Schwarzschild. In the case of Kerr, Teukolsky [Teu73] extended the curvature perturbation approach to Kerr, and Chandrasekhar introduced a transformation theory relating the two approaches, which was further elucidated by Wald. The *mode stability*[7] result in Kerr is due to Whiting.

6.2. **The Teukolsky equation.** The linearized Einstein equation, as formally written in (7), is a coupled system of equations in the components of the metric. As such, it is quite difficult to solve or obtain a-priori estimates for. Nevertheless, in perturbations of Kerr there is a way to extract one decoupled equation from the system and this equation represents the starting point for the analysis of the full linearized gravity. The equation, found by Teukolsky in 1972 [Teu73] and called since then *Teukolsky equation*, is satisfied by two curvature components of the metric perturbations, more precisely, the curvature components

$$\alpha_{ab} := W(e_a, e_4, e_b, e_4) = W_{a4b4},$$
$$\underline{\alpha}_{ab} := W(e_a, e_3, e_b, e_3) = W_{a3b3},$$

where W is the Weyl curvature, e_3 and e_4 are the null vectorfields

$$e_3 = \frac{r^2+a^2}{\rho^2}\partial_t - \frac{\Delta}{\rho^2}\partial_r + \frac{a}{\rho^2}\partial_\phi, \qquad e_4 = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi, \qquad (17)$$

and e_a, e_b for $a, b = 1, 2$ denote orthonormal vectorfields on the tangent space orthogonal to e_4 and e_3. The null frames defined in (17) are called *principal null frames* of the Kerr metric as they diagonalize the Weyl curvature, i.e. with respect to those the Weyl curvature becomes

$$W_{a4b4} = W_{a3b3} = W_{a343} = W_{a434} = 0,$$

with only $W_{abcd}, W_{3434} \neq 0$ non vanishing. The Weyl curvature components α_{ab} and $\underline{\alpha}_{ab}$ vanish on the Kerr metric, and therefore the equations satisfied by them govern

[7] *Mode stability consists in proving that separated solutions of the form* (15) *with finite initial energy do not exponentially grow in time.*

the linear perturbation of the Kerr black hole. Observe that α_{ab} and $\underline{\alpha}_{ab}$ are traceless symmetric 2-tensors on the horizontal structure associated to the null frame (e_3, e_4), i.e. $\alpha(X,Y) = \alpha({}^{(h)}X, {}^{(h)}Y)$ where ${}^{(h)}X$ is the projection of X into the orthogonal space to (e_3, e_4). Another important property of the tensors α and $\underline{\alpha}$ is that they are *gauge-invariant*, i.e. they change quadratically to a linear change of gauge in the perturbation[8].

The Teukolsky equation [Teu73] satisfied by the 2-tensor α is a wave-like second order PDE, schematically given by

$$\begin{aligned}&\square_{g_{M,a}}\alpha - V(r,\theta)\alpha\\&+ c_1(r,\theta)\nabla_{\partial_r}\alpha + c_2(r,\theta)\nabla_{\partial_t}\alpha + c_3(r,\theta)\nabla_{\partial_\phi}\alpha\\&= 0,\end{aligned} \tag{18}$$

and similarly for $\underline{\alpha}$. The highest order term of the Teukolsky equation is a D'Alembertian operator, followed by first order terms in ∂_r, ∂_t, ∂_ϕ, and a potential. One may therefore expect that the techniques developed for the standard wave equation (10) can be applied to the Teukolsky equation to obtain estimates for its general solutions.

Unfortunately, this is not the case. The presence of the first order derivative terms prevents to treat the Teukolsky equation by energy-type estimates. Recall from (12) that the boundedness of the energy for $\square_{g_0}\psi = 0$ is derived by multiplying the wave equation by $\partial_t\psi$ and integrating by parts. Similarly this holds for a wave equation with a (time-independent) positive potential V, i.e. $\square_{g_0}\psi - V\psi = 0$. For a general Teukolsky equation instead, because of the presence of the first order terms $c_1\partial_r + c_2\partial_t + c_3\partial_\phi$, one cannot directly obtain boundedness of the energy as for the standard wave equation. For this reason the absence of exponentially growing modes was the only version of stability that was known to hold for the Teukolsky equation for a long time, even in Schwarzschild spacetime.

6.3. The Chandrasekhar transformation. The first quantitative proof of the linear stability of Schwarzschild was obtained by Dafermos-Holzegel-Rodnianski [DHR19b], where they extended the *Chandrasekhar transformation*, a transformation relating curvature and metric perturbations previously known only for mode solutions, to general solutions in physical-space. By applying such transformation to the Teukolsky equation for the curvature components α and $\underline{\alpha}$ in Schwarzschild spacetime, Dafermos-Holzegel-Rodnianski [DHR19b] obtained a wave equation of the form

$$\square_g\psi - V(r)\psi = 0,$$

with positive potential V, called a *Regge-Wheeler equation*, for a quantity at the level of two null derivatives of α. To this equation one can apply techniques developed for the standard wave equation and deduce boundedness and decay properties for solutions to the original Teukolsky equation. Once control of the gauge-invariant curvature components α and $\underline{\alpha}$ is obtained, the remaining work in [DHR19b] is to derive decay for the other curvature components and linearized Ricci coefficients associated to the their choice of gauge, given by the double null foliation, by making carefully chosen gauge conditions.

A similar method to the above can be found in the case of the Kerr spacetime. Ma [Ma20] and Dafermos-Holzegel-Rodnianski [DHR19a] obtained generalizations of the Chandrasekhar transformation to Kerr which takes the Teukolsky equations to a generalized version of the Regge-Wheeler (gRW) equation. Such gRW equation has the form [Ma20]

$$\square_g\psi - V(r,\theta)\psi - i\frac{4a\cos\theta}{\rho^2}\partial_t\psi = L_\psi[\alpha], \tag{19}$$

for a complex-valued scalar function ψ, where V is a positive real potential and $L_\psi[\alpha]$ denotes linear terms in up to two derivatives of α which vanish for $a = 0$. Even though the gRW equation (19) has a first order term in ∂_t, it satisfies good divergence properties and for $|a| \ll M$ both generalizations in [Ma20] and [DHR19a] can be studied using standard techniques proper of the wave equation, such as a combination of mode decomposition and vectorfield techniques. Due to the presence of the linear terms in α on the right hand side of (19), one has to view the wave equation in (19) as coupled with the defining equations for ψ through the Chandrasekhar transformation, that is coupled with second order transport type equations in α. The frequency analysis of the gRW and the Teukolsky equation has been recently extended to the sub-extremal range $|a| < M$ by Shlapentokh-Rothman-Teixeira da Costa. The stability results for the full linearized Einstein vacuum equations for $|a| \ll M$ have been obtained by Andersson-Blue-Bäckdahl-Ma and Hintz-Häfner-Vasy [HHV21], using adaptations of the Newman-Penrose formalism and wave coordinates respectively.

6.4. Beyond vacuum. The most important properties of the Kerr black hole, such as the presence of a trapping region and superradiance, and the existence of the Carter tensor, also hold for the Kerr-Newman spacetime, the charged rotating black hole, solution to the Einstein-Maxwell equations. Even though astrophysical black holes are normally expected to be represented by the Kerr spacetime, there has been numerical evidence that the collapse to a rotating black hole in the presence of a magnetic field (such as the one expected to surround astrophysical black holes) would cause the remnant to be electrically charge. Moreover, recent studies suggest that the gravitational waves

[8] *In addition, they fully encode the gravitational radiation: if a linear perturbation satisfies $\alpha = \underline{\alpha} = 0$ then it is only given by a change of mass/rotation parameters and a gauge solution.*

detected by LIGO-Virgo are compatible with a non-negligible (of about $|Q|/M \leq 0.3$) charge-to-mass ratio.

In the case of perturbations of charged black holes, the linearized dynamics is complicated by the interaction between gravitational and electromagnetic fields, resulting in a *coupled system* of Teukolsky and generalized Regge-Wheeler equations, whose derivation and analysis has only recently been explored in Reissner-Nordström [Gio20] and in slowly rotating Kerr-Newman [Gio22]. Most notably, a symmetric structure appears in the coupling operators of the system allowing for a definition of a combined energy-momentum tensor for the coupled system, which can be analyzed in physical-space by relying on an extension of Andersson-Blue's [AB15] method. This is particularly important as the mode decomposition had been shown by Chandrasekhar to be incompatible with the analysis of perturbations of Kerr-Newman: in fact its mode stability had not been obtained[9], in contrast to the other black hole solutions.

7. The Non-linear Stability of Kerr

The full non-linear stability of black hole solutions involves the construction of the global evolution of initial data which are perturbations of the stationary solution, and therefore requires a continuity argument and involved dynamical mechanisms to modulate the gauge conditions and the final parameters. Even though some aspects of the linearized gravity are used in the proofs of non-linear stability, such as the Teukolsky equation and Chandrasekhar transformation, other techniques, especially those relying on the symmetries of the perturbed solution such as the mode decomposition, are less applicable to the general case.

The non-linear stability of the Schwarzschild family was obtained (i) for axially polarized perturbations by Klainerman-Szeftel [KS20] in 2018, and (ii) for a codimension-3 submanifold of the moduli space of initial data by Dafermos-Holzegel-Rodnianski-Taylor [DHRT21] in 2021. In the case of positive cosmological constant, the slowly rotating Kerr-de Sitter family has been proved to be stable by Hintz-Vasy [HV18] in 2016.

The proof of the non-linear stability of Kerr black holes for $|a| \ll M$ has recently been obtained as a combination of results by Klainerman-Szeftel in 2019 (on the choice of gauge conditions called Generally Covariant Modulated, or GCM, spheres), Klainerman-Szeftel [KS21] in 2021 (on the set-up of the proof and control of the gauge-dependent quantities), G.-Klainerman-Szeftel [GKS22] (on the analysis of the wave equations) and Shen (on the construction of GCM hypersurfaces) in 2022. The results in [KS21], [GKS22] and Shen are currently under peer-review.

[9] *Mode stability for Kerr-Newman is nevertheless supported by numerical evidence.*

An extension of the formalism by Christodoulou-Klainerman [CK93] to include non-integrable horizontal structures and a far-reaching application of Andersson-Blue's [AB15] method to derive estimates in physical-space are among the ingredients of the proof. As the full discussion of the non-linear stability problem goes beyond the intention of this article, we refer to the introduction of the above works for more details.

The resolution of the Stability of the Kerr family for small angular momentum, a major open problem ever since the discovery of the Kerr metric in 1963 and to which so many physicists and mathematicians have greatly contributed, constitutes a milestone development in general relativity. In a fascinating interchange between mathematics and physics, the recent mathematical proofs of the physically-motivated stability problem of black holes are crucial and definitive evidence of their reality as physical objects.

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