Peter Borwein: A Visionary Mathematician

David H. Bailey

Peter Borwein, former professor of mathematics at Simon Fraser University (British Columbia, Canada) and director of the university's Centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS), died on August 23, 2020, at the age of 67, of pneumonia, after courageously battling multiple sclerosis for over 20 years.

Peter was a prolific mathematician, with over 160 publications and ten books, by the present author's count. Many of these papers were co-authored with his brother Jonathan Borwein (deceased 2012, formerly at the University of Newcastle in Australia). Some were co-authored with his father David Borwein (retired from the Western Ontario University in Canada). What a prolific family! Peter was a co-recipient (1993) of the Chauvenet Prize and the Merten Hasse Prize, both awarded by the Mathematical Association of America, among other awards.

Figure 1. Peter Borwein.

1. Education and Career

Peter was introduced to mathematics at an early age by his mathematician father. He attended his first International Math Congress when he was only five years old (1958, in Edinburgh, Scotland). Peter retained his nametag from the conference; he wore it to his first day of school in St. Andrews [13]. Peter subsequently graduated from the University of Western Ontario in Canada, where his father was head of the Department of Mathematics, then completed an MSc and a PhD at the University of British Columbia in Vancouver.

Through his father, Peter was introduced to the legendary Paul Erdős—Peter was one of "Uncle Paul's epsilons." As Peter would write many years later, "My first published paper as a graduate student was on a problem of Erdős. As were at least a dozen subsequent papers. Erdős touched many mathematicians in this way. I often got 2:00 am phone calls that began in a distinctive Hungarian accent: 'This is Paul, I have a problem for you.'" [13].

In 1993, Peter and Jonathan joined Simon Fraser University in Burnaby, Canada (a suburb of Vancouver), where they established the Centre for Experimental and Constructive Mathematics (CECM). Later Peter helped found the Centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS), also at Simon Fraser, where he served as director for many years. Its mission was, according to Peter, "to host any scientist who uses computers as a tool in their research." He also served on the editorial boards of several journals, including the *The Ramanujan Quarterly* and *The Electronic Transactions on Numerical Analysis* [13].

David H. Bailey was a senior scientist at the Lawrence Berkeley National Lab (retired), Berkeley, California. His email address is `dhbailey@lbl.gov`.

DOI: https://doi.org/10.1090/noti2675

2. Research

Peter Borwein's published research spanned classical analysis, computational number theory, Diophantine number theory, symbolic computing, experimental mathematics and applied mathematics. This research includes several rather important results. As one example, Peter Borwein, together with Tamás Erdélyi, Ronald Ferguson, and Richard Lockhart, settled Littlewood's Problem 22 [12], which, in Littlewood's 1968 words, is the following:

Problem. If the n_j are integral and all different, what is the lower bound on the number of real zeroes of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less.

Borwein and his colleagues noted that the problem is equivalent to looking for reciprocal polynomials, with coefficients 0 or 1, with $2N$ terms, and with $N-1$ or fewer zeroes on the unit circle. To this end they programmed a computer search, which after some effort produced this example:

$$\sum_{0\le j\le 14,\, j\notin\{9,10,11,14\}} \left(z^j + z^{28-j}\right), \tag{1}$$

which has eight complex roots of modulus one; it corresponds to a cosine sum of 11 terms with only 8 zeroes in $[-\pi,\pi)$. A further search identified an example with 16 terms and 14 zeroes in $[-\pi,\pi)$, and, ultimately, an example with 140 terms and 52 zeroes in $[-\pi,\pi)$, thus demonstrating the existence of cosine sums with substantially fewer zeroes than the conjectured bound $N-1$.

Analysis of these computer-discovered examples suggested a route to a striking general result [12]:

Theorem. *There exists a cosine polynomial $\sum_{j=1}^{N} \cos(n_j\theta)$ with the n_j integral and all different so that the number of its real zeroes in the period $[-\pi,\pi)$ is $O\left(N^{5/6}\log N\right)$.*

In another notable example, Peter, in collaboration with Edward Dobrowolski and Michael J. Mossinghoff, solved a problem of Lehmer on polynomials with odd coefficients [10]. The Mahler measure of a polynomial, $M(f)$, is the product of the absolute values of roots outside the unit disk, multiplied by the absolute value of the leading coefficient. $M(f) = 1$ precisely when $f(x)$ is a product of cyclotomic polynomials and the monomial x. In 1933, Lehmer asked whether for every $\epsilon > 0$ there exists some polynomial satisfying $1 < M(f) < 1+\epsilon$. Lehmer himself had found that when $f(x) = x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$, then $M(f) = 1.176280\ldots$.

After substantial effort, Borwein and his colleagues obtained several significant results in this area, including the following for polynomials with odd coefficients [10]:

Theorem. *If $f(x) = \sum_{k=0}^{n-1} a_k x^k$ is a polynomial with no cyclotomic factors, all of whose coefficients are odd integers, then Mahler's measure of $f(x)$ satisfies*

$$\log M(f) \ge \frac{\log 5}{4}\left(1-\frac{1}{n}\right). \tag{2}$$

Further, these authors showed that if $f(x)$ has odd coefficients, degree $n-1$, and at least one noncyclotomic factor, then at least one root α of $f(x)$ satisfies $|\alpha| > 1 + (\log 3)/(2n)$, resolving an earlier conjecture of Schinzel and Zassenhaus.

3. Formulas and Algorithms for π

Peter Borwein is perhaps best known for discovering (with his brother Jonathan, in many but not all cases) new formulas and algorithms for π and other mathematical constants, many of which are extensions of results originally due to Ramanujan. One of the more striking of the Borwein formulas is [1, 7]:

$$\frac{1}{\pi} = \frac{12}{\sqrt{C}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!\,(A+Bk)}{(3k)!\,(k!)^3 C^k}, \tag{3}$$

where

$$\begin{aligned} A &= 212175710912\sqrt{61} + 1657145277365, \\ B &= 13773980892672\sqrt{61} + 107578229802750, \\ C &= \left(5280\left(236674 + 30303\sqrt{61}\right)\right)^3. \end{aligned}$$

Each additional term of the series (3) yields approximately 25 additional correct digits, provided that all computations are performed with at least the precision required for the final result.

Even more remarkable are the Borwein quadratically and higher order convergent algorithms for π, including p-th order convergent algorithms for any prime p, together with similar algorithms for certain other fundamental constants and functions [4–6, 8]. One of these algorithms is the following: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then iterate, for $k \ge 0$,

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}, \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned} \tag{4}$$

Then $1/a_k$ converges *quartically* to π: each iteration approximately *quadruples* the number of correct digits (provided again that each iteration is performed with at least the precision required for the final result). This algorithm, together with a quadratically convergent algorithm independently discovered by Brent and Salamin, has been employed in several large computations of π.

Another of the Borwein algorithms is the following [1]: Set $a_0 = 1/3$, $r_0 = (\sqrt{3}-1)/2$, $s_0 = (1-r_0^3)^{1/3}$. Then iterate, for $k \geq 0$,

$$\begin{aligned}
t_{k+1} &= 1 + 2r_k, \quad u_{k+1} = (9r_k(1 + r_k + r_k^2))^{1/3},\\
v_{k+1} &= t_{k+1}^2 + t_{k+1}u_{k+1} + u_{k+1}^2,\\
w_{k+1} &= 27(1 + s_k + s_k^2)/v_{k+1},\\
a_{k+1} &= w_{k+1}a_k + 3^{2k-1}(1 - w_{k+1}),\\
s_{k+1} &= \frac{(1-r_k)^3}{(t_{k+1} + 2u_{k+1})v_{k+1}}, \quad r_{k+1} = (1 - s_{k+1}^3)^{1/3}. \qquad (5)
\end{aligned}$$

Then $1/a_k$ converges *nonically* to π: each iteration approximately *nine times* the number of correct digits (provided again that each iteration is performed with at least the precision required for the final result).

Perhaps Peter's best-known result in this area is his 1997 paper "On the rapid computation of various polylogarithmic constants" [9]. This arose when Peter posed the question to some students and postdocs of whether there was any economical way to calculate digits in some base of a mathematical constant such as π, beginning at a given digit position, without needing to calculate the preceding digits.

Peter himself subsequently found the following surprisingly simple scheme for binary digits of $\log 2$, based on the formula $\log 2 = \sum_{k\geq 1} 1/(k2^k)$, due to Euler. First note that binary digits of $\log 2$ starting at position $d+1$ can be written $\operatorname{frac}(2^d \log 2)$, where $\operatorname{frac}(x) = x - \{x\}$ is the fractional part. Then

$$\begin{aligned}
&\operatorname{frac}(2^d \log 2)\\
&= \operatorname{frac}\left(\sum_{k=1}^{\infty} \frac{2^d}{k2^k}\right) = \operatorname{frac}\left(\sum_{k=1}^{d} \frac{2^{d-k}}{k} + \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right)\\
&= \operatorname{frac}\left(\sum_{k=1}^{d} \frac{2^{d-k} \bmod k}{k}\right) + \operatorname{frac}\left(\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right), \qquad (6)
\end{aligned}$$

where "$\bmod k$" has been added to the numerator of the first term of (6) since we are only interested in the fractional part after division by k. The key point here is that the numerator expression, namely $2^{d-k} \bmod k$, can be computed very rapidly by the binary algorithm for exponentiation mod k, without any need for extra-high numeric precision, even when the position d is very large, say one billion or one trillion. The second sum can be evaluated as written, again using standard double-precision or quad-precision floating-point arithmetic. The final result, expressed as a binary floating-point value, gives a string of binary digits of $\log 2$ beginning at position $d+1$.

In the wake of this observation, Peter and others searched the literature for a formula for π, analogous to Euler's formula for $\log 2$, but none was known at the time. Finally, a computer search conducted by Simon Plouffe numerically discovered this formula, now known as the BBP formula for π [9]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right). \qquad (7)$$

Indeed, this formula permits one to efficiently calculate a string of base-16 digits (and hence base-2 digits) of π, beginning at an arbitrary starting point, by means of a relatively simple algorithm similar to that described above for $\log 2$. Nicholas Sze used a variation of this scheme to calculate binary digits of π starting at position two quadrillion [2].

4. Mentoring

In July 1984, a relatively obscure mathematician-computer scientist at the NASA Ames Research Center in California opened the latest issue of *SIAM Review* to find an article coauthored by Peter Borwein and his brother, wherein they sketched several of their recently discovered fast algorithms for computing constants such as π and a variety of basic transcendental functions [4]. Intrigued, this researcher implemented several of these algorithms on a supercomputer, using a software package he had written to perform very high-precision computations. After some success, he contacted Peter Borwein, who, together with Jonathan, graciously invited him to participate with the Borweins in some larger computations and other explorations in the area of experimental mathematics. This led to a very productive collaboration extending over 30 years, with both Peter and Jonathan.

As the reader may have guessed, this relatively obscure NASA researcher was the present author, who is very deeply indebted to both Peter and Jonathan Borwein for their mentorship and collaboration over so many years.

Peter in particular was an inspiration for a number of researchers in the field. Veselin Jungic, a professor of mathematics at Simon Fraser University, remarked that Peter was "my friend, mentor, and a role model" [13]. Kevin Hare, of the University of Waterloo, recalls once, when attending a symbolic computing conference in Vancouver, Peter came to him saying, "This afternoon looks really boring; do you want to go hiking instead?" Along the way, Peter started asking math questions. Hare recalls [3],

> I found his approach to supervising was basically, toss lots of questions at me and hope that one of them sticks. If anybody has ever sat beside him during a conference talk, they know exactly what I mean by that. Or, I guess anybody that has been sitting with him in a pub/coffee shop while conference talks are going on also knows what I am talking about.

Karl Dilcher of Dalhousie University, who was one of Peter's first postdoctoral fellows, recalls [3],

> What I admired (and still admire) most about Peter is the fact that he always has a problem, or problems, on his mind; he will ask you, prod you, share insights with you and be very persistent. When he has found a truly exciting and worthwhile problem, he will not let go until it is solved. He will hack away at it from all directions, will try to get others interested and involved (and often succeed in this), and more often than not he will eventually make substantial progress, either alone, or in collaboration with others.

Similarly, Michael Mossinghoff of Davidson College recalls [3],

> Peter seems to have an incredible knack for suggesting just the right sort of problem to his collaborators—problems that are not only irresistible, but very well-suited to the listener's interests. He's a lot like Erdős in this respect. Each time I arrived in Vancouver, Peter always had a fascinating new project that was just irresistible to join.

5. Multiple Sclerosis

No account of Peter Borwein's career would be complete without mentioning the remarkable grace with which he faced his condition of multiple sclerosis. Initially diagnosed prior to the year 2000, the disease eventually left him confined to a wheelchair, increasingly dependent on family and caregivers, and, sadly, increasingly unable to pursue research or to effectively collaborate with colleagues. The present author recalls visiting Peter in January 2019 at his home in Burnaby, British Columbia. In spite of his paralysis and infirmity, Peter's pleasant demeanor and humor were on full display. Would that we could all bear our misfortunes with such equanimity!

References

[1] D. H. Bailey, J. M. Borwein, P. B. Borwein, and S. Plouffe, *The quest for pi*, Math. Intelligencer **19** (1997), no. 1, 50–57, DOI 10.1007/BF03024340. MR1439159

[2] J. Palmer, "Pi record smashed as team finds two-quadrillionth digit," *BBC News*, 16 Sep 2020, `https://www.bbc.com/news/technology-11313194`.

[3] J. M. Borwein, "Peter Borwein revisited," Computer Assisted Research Mathematics and Its Applications, University of Newcastle, Australia, 2008, `https://carmamaths.org/resources/jon/pbb.pdf`.

[4] J. M. Borwein and P. B. Borwein, *The arithmetic-geometric mean and fast computation of elementary functions*, SIAM Rev. **26** (1984), no. 3, 351–366, DOI 10.1137/1026073. MR750454

[5] Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1987. A study in analytic number theory and computational complexity; A Wiley-Interscience Publication. MR877728

[6] J. M. Borwein and P. B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 691–701, DOI 10.2307/2001551. MR1010408

[7] J. M. Borwein and P. B. Borwein, *Class number three Ramanujan type series for $1/\pi$*, J. Comput. Appl. Math. **46** (1993), no. 1-2, 281–290, DOI 10.1016/0377-0427(93)90302-R. Computational complex analysis. MR1222488

[8] J. M. Borwein, P. B. Borwein, and D. H. Bailey, *Ramanujan, modular equations, and approximations to pi, or How to compute one billion digits of pi*, Amer. Math. Monthly **96** (1989), no. 3, 201–219, DOI 10.2307/2325206. MR991866

[9] David Bailey, Peter Borwein, and Simon Plouffe, *On the rapid computation of various polylogarithmic constants*, Math. Comp. **66** (1997), no. 218, 903–913, DOI 10.1090/S0025-5718-97-00856-9. MR1415794

[10] Peter Borwein, Edward Dobrowolski, and Michael J. Mossinghoff, *Lehmer's problem for polynomials with odd coefficients*, Ann. of Math. (2) **166** (2007), no. 2, 347–366, DOI 10.4007/annals.2007.166.347. MR2373144

[11] P. Borwein, *An efficient algorithm for the Riemann zeta function*, Constructive, experimental, and nonlinear analysis (Limoges, 1999), CRC Math. Model. Ser., vol. 27, CRC, Boca Raton, FL, 2000, pp. 29–34. MR1777614

[12] P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials: solution to a problem of Littlewood*, Ann. of Math. (2) **167** (2008), no. 3, 1109–1117, DOI 10.4007/annals.2008.167.1109. MR2415396

[13] V. Jungic, [untitled], *Notes of the Canadian Mathematical Society*, vol. 52 (2020), no. 2, `https://notes.math.ca/en/article/peter-borwein-1953-2020/`.

David H. Bailey

Credits

Figure 1 is courtesy of Lara Dauphinee.
Photo of David H. Bailey is courtesy of Linda J. Bailey.