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Notices

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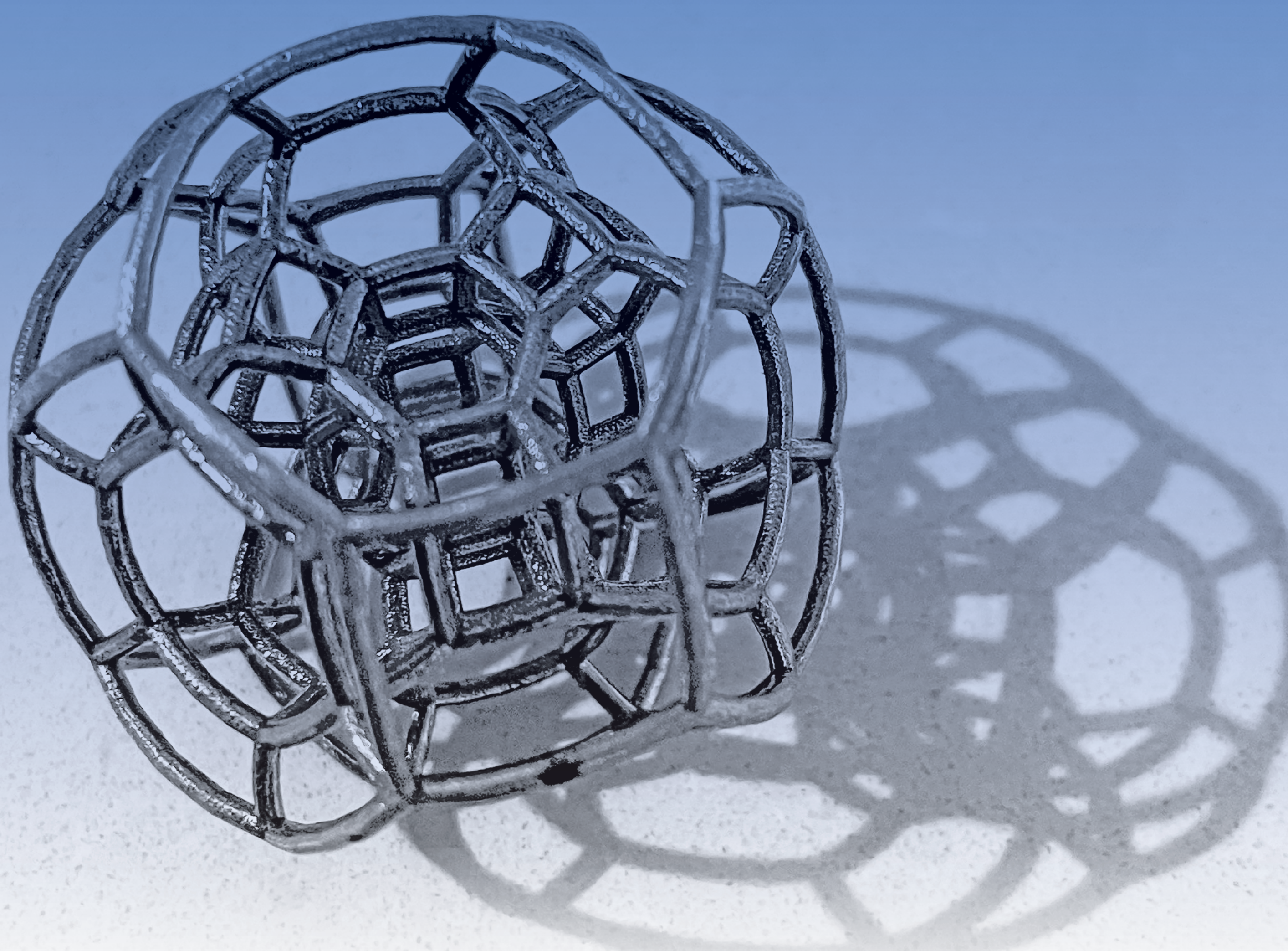

American Mathematical Society
Advancing research. Creating connections.

The cover design is based on imagery from "The Symmetric Group Through a Dual Perspective," page 897.

POSITION AVAILABLE

Executive Director

AMERICAN MATHEMATICAL SOCIETY

POSITION

The Trustees of the American Mathematical Society invite applications for the position of Executive Director of the Society. The Executive Director has the opportunity to strongly influence all activities of the Society, as well as the responsibility of overseeing a large and diverse spectrum of people, programs, and publications. **The desired starting date is February 1, 2024.**

DUTIES AND TERMS OF APPOINTMENT

The American Mathematical Society, founded in 1888 to further the interests of mathematical research and scholarship, serves the national and international community through its publications, meetings, advocacy, and other programs. The AMS promotes mathematical research and its communication and uses; encourages and promotes the transmission of mathematical understanding and skills; supports mathematical education at all levels; advances the status of the profession of mathematics, encouraging and facilitating the full participation of all individuals; and fosters an awareness and appreciation of mathematics and its connections to other disciplines and everyday life.

These aims are pursued mainly through an active portfolio of programs, publications, meetings, conferences, and advocacy. The Society is a major publisher of mathematical books and journals, including MathSciNet®, an organizer of numerous meetings and conferences each year, and a sponsor of grants and training programs. The Society's headquarters are located in Providence, Rhode Island, and the Executive Director is based there. The society also maintains a print shop in Pawtucket, Rhode Island; an office in Washington, DC, that houses the Office of Government Relations and the Office of Equity, Diversity, and Inclusion; and an office in Ann Arbor, Michigan, that publishes MathSciNet.

The Executive Director is the principal executive officer of the Society and is responsible for the execution and administration of the policies of the Society as approved by the Board of Trustees and by the Council. The Executive Director is a full-time employee of the Society and is responsible for the operation of the Society's offices in Providence, and Pawtucket, RI; Ann Arbor, MI; and Washington, DC. The Executive Director attends meetings of the Board of Trustees, the Council, and the Executive Committee, is an ex-officio (nonvoting) member of the policy committees of the Society, and is often called upon to represent the Society in its dealings with other scientific and scholarly bodies.

The Society employs a staff of over 200 in the four offices. The directors of the various divisions report directly to the Executive Director. Information about the operations and finances of the Society can be found in its Annual Reports, available at www.ams.org/annual-reports.

The Executive Director is appointed by and serves at the pleasure of the Trustees. The terms of appointment, salary, and benefits will be consistent with the nature and responsibilities of the position and will be determined by mutual agreement between the Trustees and the prospective appointee.

DESIRED QUALIFICATIONS

The successful candidate must be a leader, and we seek candidates who additionally have as many as possible of the following:

- A doctoral degree (or equivalent) in mathematics or a closely related field.
- Substantial experience and demonstrated visibility as a professional mathematician in academic, industrial, or governmental employment, with success in obtaining and administering grants.
- Extensive knowledge of the Society, the mathematics profession, and related disciplines and organizations, with a thorough understanding of the mission that guides the Society.
- Excellent communication skills, both written and oral, and an enthusiasm for public outreach.
- Demonstrated sustained commitment to diverse, inclusive, and equitable organizational environments and substantial experience in advancing equity, diversity, and inclusion priorities in the mathematical community.
- Demonstrated leadership ability supported by strong organizational and managerial skills.
- Familiarity with the mathematical community and its needs, and an ability to work effectively with mathematicians and nonmathematicians.
- Strong interest in engaging in fundraising and enjoyment of social interactions.

APPLICATIONS PROCESS

A search committee co-chaired by **Joseph Silverman** (joseph_silverman@brown.edu) and **Bryna Kra** (kra@math.northwestern.edu) has been formed to seek and review applications. All communication with the committee will be held in confidence. Suggestions of suitable candidates are most welcome.

Applicants should submit a CV and a letter of interest on MathJobs. The letter should be at most four pages, explaining your interest in being the Executive Director of the AMS and why you consider yourself to be a compelling candidate. The majority of the letter should discuss your major accomplishments and experiences that illustrate your leadership philosophy and address the desired qualifications for the position. **Applications received by September 15, 2023 will receive full consideration.**

AMERICAN MATHEMATICAL SOCIETY

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FEATURES

FROM THE AMS SECRETARY

INVITATIONS FROM THE AMS

ERRATA. On p. 340 of the February 2023 issue, the date of birth of Janos Aczel is given as December 26, 1964. The correct date is December 26, 1924.

On p. 1541 of "Remembrances of Ciprian Ilie Foias" in the October 2022 issue, "Luan Hoang" is misspelled as "Luong Hoang" on line 2 and in the last sentence of the fourth paragraph.

Notices of the American Mathematical Society

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Supported by the AMS membership, most of this publication, including the opportunity to post comments, is freely available electronically through the AMS website, the Society's resource for delivering electronic products and services. Use the URL **www.ams.org/notices** to access the *Notices* on the website. The online version of the *Notices* is the version of record, so it may occasionally differ slightly from the print version.

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The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to "advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals," and urge all members to conduct their professional activities with this goal in mind. (*as adopted by the April 2019 Council*)

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LETTERS TO THE EDITOR

To the Editor, *Notices of the AMS:*

The Association for Mathematical Research was organized in 2020 as a nonprofit organization with the mission of promoting mathematical research and scholarship. Over 250 founding members from across the globe joined in its initial formation, and total membership is now well over 500. The founding was motivated by the changes and tumult of the pandemic years, which revealed new opportunities for creating and communicating mathematical ideas.

Membership is free at `https://amathr.org/member`, and provides a periodic newsletter and the chance to serve on AMR committees and initiatives. The organization is run entirely by volunteers, with no staff anticipated in the near future.

We seek innovative approaches that apply advancing technologies to support mathematical research. Projects under way include a series of freely downloadable books (`https://amathr.org/books`), open access journals (`https://amathr.org/journals`), research reviews (`https://amathr.org/reviews`), and lecture series (`https://www.amathr.org/lectures`). Others are in the organizational stage. For an example, look at the reviews found at `https://amathr.org/reviews`. Important mathematical results are rapidly disseminated to the community, comprised of a research summary that can be accompanied by figures, illustrative videos, and links to lectures and references. These reviews will grow into a valuable resource for the community.

We hope to cooperate with and support established organizations such as the AMS. Please visit `https://www.amathr.org` to see our current activities. We welcome your participation in pursuing our mission of advancing mathematical research and scholarship. You can contact us by writing to `contact@amathr.org`.

Joel Hass
President, AMR

Royalty for Peer Reviewers: Some Thoughts

For quite some time many of us have been reviewing research papers for journals. Some of these journals charge money to their authors for publishing the accepted papers. These charges are often handsome. Further, on publication the articles earn money for the journal in various ways. The author also gets many benefits out of published papers which also carry monetary gains.

However, the reviewer, though a very crucial link in this whole process, stays dry.

I believe that reviewers should be included as rightful claimants of at least a tiny part of the monetary earning from the journal for their reviews.

With Kind Regards,
Dr. Ashish Kumar Upadhyay
Professor, Department of Mathematics
Institute of Science, Banaras Hindu University

**We invite readers to submit letters to the editor at* `notices-letters@ams.org`.

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A WORD FROM…

Keri Ann Sather-Wagstaff, Spectra Board Member & Political Committee Chair

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Transgender and Nonbinary Mathematics Pride

Pride month is a time for LGBTQIA+ (Lesbian, Gay, Bisexual, Transgender, Queer, Intersex, & Asexual inclusive) people to celebrate community, raise awareness of oppression, and work towards full human rights. It originates in the Stonewall riots of June 1969, led by Black and Latina transgender women.

This essay consists of eight consensual "elevator pitch" summaries of recent research of transgender and nonbinary mathematicians. Consent is important here as these are frightening times. Anti-trans bills and policies are working through legislatures and school/medical boards. Popular and political discourse includes transphobic hate speech, labeling us as mutilators and groomers of children. Sadly, conferences continue to be held in states where many of us feel unsafe or unwelcome [Ree23].

My goal is to inspire aspiring or practicing trans/nonbinary mathematicians, by featuring a proud and visible group of us. I also seek to make cis mathematicians aware of us, so they can work towards making conferences, classrooms, and other mathematics spaces more welcoming; see the article by Buckmire, et al. in this issue for concrete suggestions.

Jessamyn Dukes (any pronouns) is a genderqueer, nonbinary, transgender, nontraditional undergraduate math major at Rutgers (USA) with very broad mathematical interests. Their joint work [ABB+22], arising from the Polymath Jr 2022 summer program, investigates lattice models: discrete dynamical systems studied in statistical mechanics where one can evaluate local interactions, or states, at each vertex, in order to reconcile probabilities of the global multistate system. She uses the Yang–Baxter equation to determine if any given lattice model is solvable; such solutions are rare, with many applications. His work provides explicit algebraic conditions such that given two local states of an n-color lattice model, a third state or solution can be determined. Furthermore, they show that all solutions to the n-color lattice model can be reduced to this 3-color subcase, and give an explicit parametrization of all solutions accordingly. With $n = 2$ her result specializes to the "six-vertex" or "ice" model, which recovers Baxter's classical conditions for solvability of a lattice model.

Stacey G. Harris (she/her/hers) is a trans woman full professor at Saint Louis University (USA) working in the interface of geometry and mathematical physics. She investigates the geometry of spacetime: a manifold M with a notion of "causality" where points of M are events in the universe. Harris researches possible boundaries on M to understand its global behavior. One example, the Future Causal Boundary, consists of the sets of events that

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Keri Ann Sather-Wagstaff (she/her/hers) is a professor of mathematical and statistical sciences at Clemson University. Her email address is ssather@clemson.edu.

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causally precede other events. This construction yields a future-limit in the boundary for every physically endless causal curve of each particle and relies only on intrinsic properties of M. Until now, the Future Causal Boundary has been calculable only for M with a high degree of symmetry or special algebraic construction. Harris's paper [Har22] assumes only that M is foliated by a field of observers and offers up what physical properties those observers would need to measure so that the Future Causal Boundary is fairly simple topologically and so that the geometry of M has a continuous limit at the boundary for each observer.

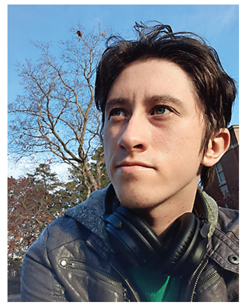
Faye Jackson (she/her/hers) is a trans woman undergraduate math major at the University of Michigan (USA) whose research is in combinatorics and number theory. She is a lesbian and the 2023 winner of the Alice T. Schafer Prize. Her recent preprint [JO23], with Otgonbayar from the UVA Number Theory REU, explores the arithmetic statistics of parts in partitions of n as n becomes large. It uses the circle method to derive an asymptotic formula for the number of parts congruent to r modulo t among all k-indivisible partitions, i.e., partitions where no part is divisible by k. The main term of this asymptotic does not depend on r, and so, in a weak asymptotic sense, the parts are equidistributed among congruence classes. However, inspection of the lower order terms indicates a bias towards different congruence classes modulo t. For large k, these biases are similar to work of Beckwith and Mertens for the class of all partitions, but they differ wildly for small k, and numerical evidence has led to several conjectures she is still working on.

Astra Kolomatskaia (she/her/hers) is a trans woman PhD student at Stony Brook University (USA) researching homotopy type theory and higher category theory.

She describes her work to her friends as follows: *Type theory is about putting math on a computer. Homotopy type theory is about letting the resulting system talk about shapes. In this setting, all I want to know is: "What is a triangle?"*

A higher category has objects, morphisms, and higher morphisms between morphisms with a sophisticated interdependence. In homotopy type theory, the barriers to talking about infinitary structures are known as higher-coherence issues, and the prototypical problem exemplifying this is the construction of semisimplicial types, higher triangles.

The fundamental insight of Kolomatskaia's work is that every definition of a mathematical structure should automatically induce a hierarchy of higher dependent structures living over it. In this framework, she is studying how such definitions can express cross-level interactions [Kol22].



Seppo Niemi-Colvin (he/him/his) is a transgender man postdoc at Indiana University (USA). He studies homology of 3-manifolds and knots, especially links of singularity and generalized algebraic knots. Links of singularity are 3-manifolds capturing the local topology around a normal complex surface singularity, and traditional algebraic knots use a singular complex curve in the complex plane. Generalized algebraic knots combine these two by allowing the curve and surface to be singular. His invariant of interest is Ozsváth–Stipsicz–Szabó's knot lattice homology which computes knot Floer homology combinatorially via resolutions of singularities. In his dissertation [NC22], he proved knot lattice homology's invariance under the choice of resolution for the singularity defining the knot, and he expressed this invariance on the level of a doubly filtered homotopy type. He hopes to compute this knot lattice homotopy type in more examples to better understand how these knots relate to each other and to knots in the 3-sphere.

Emily Quesada-Herrera (she/her/hers) is a Costa Rican trans woman postdoc at Graz University of Technology (Austria) working on the interface of number theory and harmonic analysis. Her recent paper [CQH22] with Chirre investigates the classical problem of representing integers and primes by quadratic forms using tools from analytic and algebraic number theory and Fourier analysis to obtain new estimates in this area. As an application, for primes of the form $x^2 + 27y^2$ studied by Euler and Gauss, assuming the generalized Riemann hypothesis, she shows that there is always such a prime in the short interval $[x, x+5.52\sqrt{x}\log x]$, for $x \gg 0$. She uses a broad toolbox that includes lattices in $\mathbb{R}^2$, ideals of imaginary quadratic fields, and summation formulas that connect these objects to Fourier analysis.

J. Daisie Rock (she/her/hers) is a trans woman BOF Postdoc at Ghent University (Belgium) who was a first-generation college student working in representation theory informed by analysis and category theory. Broadly speaking, her research is about generalizing historically discrete structures to continuous structures. For instance, consider mutation in cluster algebras, which has applications in high energy physics. One begins with a set $\mathcal{X}$ of *cluster variables* and special subsets of $\mathcal{X}$ called *clusters* that have the same cardinality and satisfy the *mutation* property, which allows one to replace cluster variables in a cluster while respecting the cluster property. Classically, this is a discrete process. Choose an $x \in T$, replace it, choose the next x', replace it, and so on. Her paper [Roc22] introduces a way to mutate a continuum of variables in order, a process that has exciting possibilities even for the classical situation.

Theresa Simon (she/her/hers) is a butch transgender woman and non-tenure-track faculty member at WWU Münster (Germany), working in applied mathematics. In her joint paper [MMSS22], she rigorously analyzes topologically nontrivial, particle-like patterns called magnetic skyrmions occurring in extremely thin magnets. They are modeled as minimizers of the harmonic map problem with a flat, 2D domain and target being the two-dimensional sphere, augmented by a lower order term and constant boundary conditions. She specifies the skyrmion's topology by requiring it to be homeomorphic to the identity map of the sphere after collapsing the boundary to a point. She proves that if the limit of the lower order term vanishes, then the skyrmion indeed behaves like a particle, and she fully determines its relevant properties.

References

[ABB⁺22] P. Addona, E. Bockenhauer, B. Brubaker, M. Cauthorn, C. Conefrey-Shinozaki, D. Donze, W. Dudarov, J. Dukes, A. Hardt, C. Li, J. Li, Y. Liu, N. Puthanveetil, Z. Qudsi, J. Simons, J. Sullivan, and A. Young, *Solving the n-color ice model*, `arXiv:2212.06404`, 2022.

[CQH22] A. Chirre and E. Quesada-Herrera, *Fourier optimization and quadratic forms*, Q. J. Math. **73** (2022), no. 2, 539–577, DOI 10.1093/qmath/haab041. MR4439798

[Har22] S. G. Harris, *Spacelike causal boundary at finite distance and continuous extension of the metric: A preliminary report*, Developments in Lorentzian Geometry: GeLoCor 2021, Cordoba, Spain, February 1–5, Springer, 2022, pp. 143–157.

[JO23] F. Jackson and M. Otgonbayar, *Unexpected biases between congruence classes for parts in k-indivisible partitions*, J. Number Theory **248** (2023), 310–342, DOI 10.1016/j.jnt.2023.01.006. MR4562070

[Kol22] A. Kolomatskaia, *Semi-simplicial types*, `https://github.com/FrozenWinters/SSTs`, 2022.

[MMSS22] A. Monteil, C. B. Muratov, T. M. Simon, and V. V. Slastikov, *Magnetic skyrmions under confinement*, `arXiv:2208.00058`, 2022.

[NC22] S. Niemi-Colvin, *Invariance of Knot Lattice Homology and Homotopy*, ProQuest LLC, Ann Arbor, MI, 2022. Thesis (Ph.D.)–Duke University. MR4463331

[Ree23] E. Reed, *Updated anti-trans legislative risk assessment map*, `https://tinyurl.com/2pvh3a4j`, 2023.

[Roc22] J. D. Rock, *Continuous quivers of type A* (IV)*: Continuous mutation and geometric models of E-clusters*, Algebras Rep. Thy. (2022), 34 pp.

Credits

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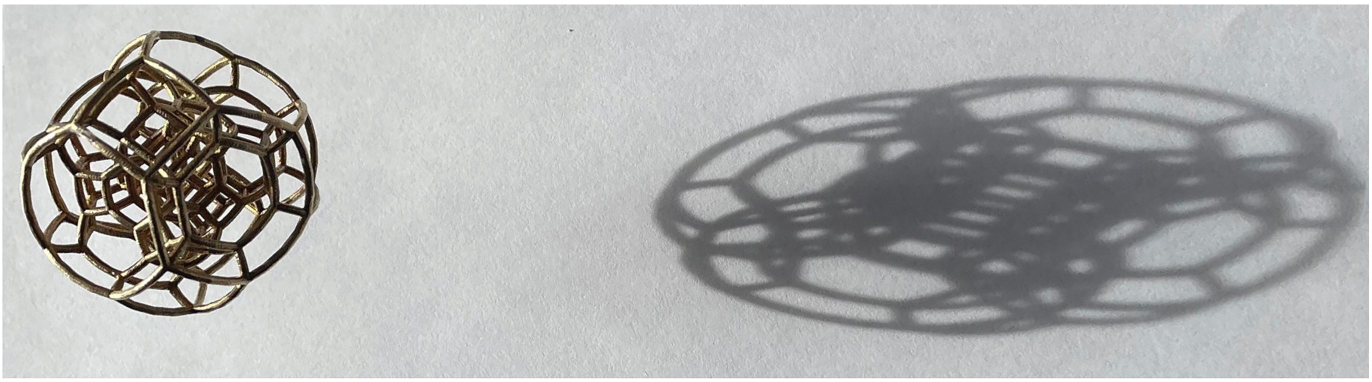
The Symmetric Group Through a Dual Perspective



Rosa Orellana and Mike Zabrocki

Mathematicians began the study of representation theory over a hundred years ago. Since then it has become a centerpiece technique in fields such as algebra, topology, number theory, geometry, mathematical physics, quantum information theory, and complexity theory. A premise of representation theory is that we can study groups and algebras from how they act on vector spaces.

In this article we take this a step further; to study actions of a group or algebra we study what commutes with the action. The collection of all linear transformations that commute with the action is called the commutant or the centralizer. The centralizer is itself an algebra which is called the *Schur–Weyl dual*.

A reason why this has become such an important technique is that it can lead to beautiful connections between seemingly different areas of mathematics. One example of this is the discovery of the Jones polynomial. This polynomial is a one variable invariant for oriented knots or links [6, 7]. The polynomial was discovered while studying linear functionals of the Temperley–Lieb algebra, an example of a centralizer algebra. Following this work, Jones received his Fields Medal for discovering deep connections between representation theory, topology, and theoretical physics [7].

In this article we will present how centralizer algebras can be used to study the representation theory of the symmetric group. Although we know a lot about its representation theory, there are still open problems that are out of reach such as the *Kronecker problem* and the *restriction problem* that we discuss below. Our approach to these problems has been to use centralizer algebras to develop combinatorial tools to study them.

All of the vector spaces in this article are over the complex numbers $\mathbb{C}$, and GL_n will denote the general linear group of invertible $n \times n$ matrices with complex entries.

What is a representation? Representation theory is the art of studying abstract algebraic objects, such as groups and algebras, by understanding how they act on vector spaces.

When acting on a vector space, each element in the algebra or group is represented by a linear transformation (or more concretely, a matrix). The set of resulting linear transformations is a representation of the algebra or group. Moreover, multiplying two elements in the algebra or group corresponds to composition of the corresponding linear transformations. It is common to refer to the vector space together with the action as the representation, or if the action is understood to just the vector space as the representation.

In this article we are interested in the representation theory of the symmetric group, S_k, the group of bijections from the set $\{1, 2, \ldots, k\}$ to itself. There are many ways to represent the elements of S_k, in this article we will think

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of them in cycle notation or as diagrams. For example, $(1,3,4)(5,6) \in S_6$ corresponds to the diagram in Figure 1.

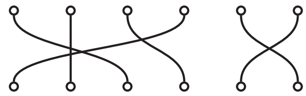


Figure 1. A diagram depicting the permutation $(1,3,4)(5,6) \in S_6$.

We will use the following example of a representation to illustrate the ideas introduced in this article. Consider

$$S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\},$$

where e is the identity and the other elements are written using cycle notation (with one-cycles omitted). The symmetric group S_3 acts on the vector space $\mathbb{C}^3$. If we choose the basis of standard column vectors for $\mathbb{C}^3$, $\{e_1, e_2, e_3\}$, then an element $\sigma \in S_3$ acts by $\sigma \cdot e_i = e_{\sigma(i)}$. As an example, $(1,2) \cdot e_1 = e_2$, $(1,2) \cdot e_2 = e_1$, and $(1,2) \cdot e_3 = e_3$. Then each $\sigma \in S_3$ is represented by a permutation matrix:

$$e \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1,2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(2,3) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, (1,2,3) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$(1,3) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, (1,3,2) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We will refer to this representation as the *permutation representation* of S_3.

To define a representation, we need a vector space and an action of the group or algebra on the vector space. Another way to think of a representation is as a group homomorphism to the general linear group, GL_n. For example, the permutation representation of S_3 is the homomorphism $\rho : S_3 \to GL_3$ shown above.

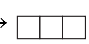
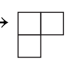
Each finite group or finite-dimensional algebra has an infinite number of matrix representations, but we only need a finite number of them to express all of the representations. A common theme in mathematics is to identify the basic building blocks in the theory. For example, in number theory the building blocks are the primes. Applying this idea to representation theory leads to the concept of an *irreducible representation*, which is a representation that does not contain a subspace that is closed under the action. For instance the permutation representation mentioned above has a subspace $W = \mathrm{span}\{e_1 + e_2 + e_3\}$ which is closed under the action and hence the permutation representation is not irreducible. The subspace W is an irreducible representation of S_3.

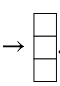
In this article, we will restrict our attention to semisimple representations. Just as composite numbers can be written using primes, semisimple representations can be decomposed into direct sums of irreducible ones. Many open problems in combinatorial representation theory ask for algorithms for decomposing representations into irreducible ones.

Representations of the symmetric group. A nontrivial and beautiful fact is that the irreducible representations of a finite group are in bijection with the conjugacy classes of that group. In the case of the symmetric group, S_n, the conjugacy classes are determined by cycle type (the lengths of the cycles in cycle notation). Since the cycle type is a partition of n, then the irreducible representations are also indexed by these.

Recall that a *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of nonnegative integers that add up to n. We use $|\lambda|$ for the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. We will think of a partition as a Young diagram, an array of boxes with λ_i boxes in the i-th row that are left-justified. We will use the English convention in which we write the boxes corresponding to λ_1 in the top row, λ_2 in the second row, and so on.

For instance, S_3 has three irreducible representations, which are indexed by partitions of 3,

$(3) \to$ , $\quad (2,1) \to$ ,

and $\quad (1,1,1) \to$ .

We use $\mathbb{S}^\lambda$ to denote the irreducible representation indexed by λ. One way to describe $\mathbb{S}^\lambda$ is by giving a basis and the action of S_n on this basis. Every representation of S_n can be written as a direct sum of irreducible ones. This fact is known as Maschke's theorem and is a property that is true for representations of finite groups. For example, the permutation representation of S_3 is isomorphic ($\cong$) to the direct sum of two irreducible representations, namely

$$\mathbb{C}^3 \cong \mathbb{S}^{(2,1)} \oplus \mathbb{S}^{(3)},$$

where $\mathbb{S}^{(2,1)} \cong \mathrm{span}\{e_3 - e_1, e_2 - e_1\}$ and $\mathbb{S}^{(3)} \cong \mathrm{span}\{e_1 + e_2 + e_3\}$.

Character tables. One of the downsides of thinking about representations in terms of matrices is that the matrices depend on the basis chosen for the vector space. Changing basis produces an isomorphic representation. Fortunately, for complex representations the representations are determined up to isomorphism by their character. The *character* of a representation $\rho : G \to GL_d$ is the function $\chi^\rho : G \to \mathbb{C}$ defined by

$$\chi^\rho(g) = trace(\rho(g)), \quad \text{for every } g \in G.$$

Recall that in linear algebra the trace of a matrix is the sum of its diagonal entries. Using properties of the trace function, we can show that two conjugate elements have the same trace and that isomorphic representations have the same character. Therefore, the essence of the representation of a group can be stored in a vector with entries equal to the trace at a representative of a conjugacy class.

For example, the permutation representation of S_3 has $\chi^\rho(e) = 3$,

$$\chi^\rho((2,1)) = \chi^\rho((1,3)) = \chi^\rho((2,3)) = 1,$$

$$\chi^\rho((1,2,3)) = \chi^\rho((1,3,2)) = 0.$$

Therefore, we can think of its character as the vector $\chi^\rho = \langle 3,1,0\rangle$ with one value for each conjugacy class.

The irreducible complex characters of a finite group can be stored compactly in a square matrix where each row corresponds to an irreducible representation and each column corresponds to a conjugacy class, often indexed by a conjugacy class representative. For instance, the character table of S_3 is given in Figure 2.

	e	$(1,2)$	$(1,2,3)$
$\chi^{(1,1,1)}$	1	-1	1
$\chi^{(2,1)}$	2	0	-1
$\chi^{(3)}$	1	1	1

Figure 2. The irreducible character table for the symmetric group S_3.

Writing a representation as a direct sum of irreducibles is the same as writing a character as a vector sum of irreducible characters. In our running example, the character of the permutation representation is the sum of two irreducible characters,

$$\chi^\rho = \chi^{(2,1)} + \chi^{(3)} = \langle 2,0,-1\rangle + \langle 1,1,1\rangle = \langle 3,1,0\rangle.$$

Characters contain all essential information about representations; in fact, Frobenius developed the representation theory of finite groups completely in terms of their characters.

The Kronecker product. The tensor product of two representations for any group is also a representation. In the special case of the symmetric group, given two irreducible representations, $\mathbb{S}^\lambda$ and $\mathbb{S}^\mu$ with λ and μ both partitions of n, $\mathbb{S}^\lambda \otimes \mathbb{S}^\mu$ has underlying vector space the tensor product of the vectors spaces for $\mathbb{S}^\lambda$ and $\mathbb{S}^\mu$. If $v \otimes w \in \mathbb{S}^\lambda \otimes \mathbb{S}^\mu$, then $\sigma \in S_n$ acts *diagonally*, i.e., $\sigma \cdot v \otimes w = \sigma \cdot v \otimes \sigma \cdot w$.

The character of $\mathbb{S}^\lambda \otimes \mathbb{S}^\mu$, written $\chi^{\lambda\otimes\mu}$, is the point-wise product of the characters of $\mathbb{S}^\lambda$ and $\mathbb{S}^\mu$, i.e., for $g \in S_n$,

$$\chi^{\lambda\otimes\mu}(g) = \chi^\lambda(g)\chi^\mu(g).$$

For example, using character vectors with $\lambda = \mu = (2,1)$ (see the character table for S_3)

$$\chi^{(2,1)\otimes(2,1)} = \langle 2,0,-1\rangle\langle 2,0,-1\rangle = \langle 4,0,1\rangle.$$

An interesting challenge is to write a tensor product such as $\chi^{(2,1)\otimes(2,1)}$ as a sum of irreducible characters. In this case, by playing around with the character table in Figure 2, we can see that

$$\chi^{(2,1)\otimes(2,1)} = \chi^{(3)} + \chi^{(2,1)} + \chi^{(1,1,1)}.$$

In general, the coefficients of the irreducible characters, $g(\lambda,\mu,\nu)$, are the nonnegative integers which describe the number of times that the irreducible character χ^ν occurs in the decomposition of $\chi^{\lambda\otimes\mu}$ when written as a sum of irreducibles,

$$\chi^{\lambda\otimes\mu} = \sum_\nu g(\lambda,\mu,\nu)\chi^\nu.$$

In combinatorial representation theory we are interested in finding combinatorial algorithms to compute the coefficients and tie them to enumerable set of objects. Then, we use this set to deduce properties of the coefficients. The following is a well-known open problem in this area:

The Kronecker problem. Find a set of objects depending only on λ, μ, and ν with cardinality $g(\lambda,\mu,\nu)$. We call this a *combinatorial interpretation.*

This problem has motivated decades of research since the early 1900s. Most recently this is due to deep connections with quantum information theory [3] and the central role it plays within Geometric Complexity Theory [11]. This is an approach that seeks to settle the celebrated P versus NP problem, one of the several Millennium Prize Problems set by the Clay Mathematics Institute.

Why a combinatorial interpretation? Combinatorial interpretations of the multiplicities often lead to the discovery of new properties, a better understanding, and in some cases to proofs of longstanding open problems. For example, Knutson and Tao found a combinatorial model for the multiplicities occurring in the tensor product of representations of the general linear group. They used their model to prove the saturation property of these coefficients and this led to a proof of Horn's conjecture from 1962 characterizing the spectrum of the sum of two Hermitian matrices [9].

Stability of Kronecker coefficients. The Kronecker product of symmetric group representations satisfies a stability property first discovered by Murnaghan [2, 12]. Murnaghan observed that for sufficiently large n, the decomposition of $\chi^{(n-|\alpha|,\alpha)\otimes(n-|\beta|,\beta)}$ only depends on the parts of α and β and not on n. For example, for $n \geq 7$, we always get the following decomposition when $\alpha = (1)$ and $\beta = (2,1)$:

$$\begin{aligned}\chi^{(n-1,1)\otimes(n-3,2,1)} = \chi^{(n-2,1,1)} &+ \chi^{(n-3,1,1,1)} \\ &+ \chi^{(n-2,2)} + 2\chi^{(n-3,2,1)} \\ &+ \chi^{(n-3,3)} + \chi^{(n-4,2,1,1)} \\ &+ \chi^{(n-4,2,2)} + \chi^{(n-4,3,1)}.\end{aligned}$$

In general, we get nonnegative integer coefficients, $\overline{g}^{\gamma}_{\alpha,\beta}$ that depend on three partitions α, β, and γ. These coefficients are called *reduced (or stable) Kronecker coefficients*.

A duality between S_k and GL_n. In general, a representation of GL_n is a homomorphism, $\rho : GL_n \to GL_d$. However, the representation theory of GL_n can get pretty wild. In algebraic combinatorics, we often restrict our attention to polynomial representations. This means that the matrices $\rho(A)$ have polynomial entries in the entries of the matrix $A \in GL_n$. The irreducible polynomial representations are indexed by partitions with at most n parts. The polynomial representations of GL_n were first studied by Issai Schur in his 1901 thesis under the supervision of Frobenius.

By fixing a basis of a three-dimensional vector space, an example of a polynomial representation $\rho : GL_2 \to GL_3$ is given by the following matrix:

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

In 1927, Schur reformulated his thesis results in what today is known as the *Schur–Weyl duality*. This duality defines a correspondence between irreducible representations of the symmetric group S_k and irreducible, homogeneous, polynomial representations of GL_n of degree k. Letting $V = \mathbb{C}^n$, GL_n acts diagonally on $V^{\otimes k}$ (k-fold tensor product of V), that is, for $A \in GL_n$ and $v_1 \otimes \cdots \otimes v_k$ in $V^{\otimes k}$,

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = Av_1 \otimes Av_2 \otimes \cdots \otimes Av_k.$$

Concretely, Av_i is the product of the matrix A with the column vector v_i. The symmetric group S_k also has a right action of $V^{\otimes k}$ by permuting the tensor factors. For example, for $\sigma = (1,3,4) \in S_4$, $v_a \otimes v_b \otimes v_c \otimes v_d \cdot (1,3,4) = v_d \otimes v_b \otimes v_a \otimes v_c$, where $1 \le a,b,c,d \le n$, which can be visualized in Figure 3.

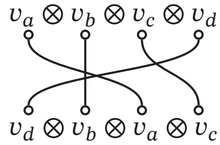


Figure 3. Visualization of the right action of $(1,3,4)$ in S_4 on a tensor in $V^{\otimes 4}$.

The basic observation that Schur made is that the diagonal action of GL_n and the permutation action of S_k commute. This implies there is a well-defined action of the group $GL_n \times S_k$ (direct product) on $V^{\otimes k}$. When we decompose $V^{\otimes k}$ in terms of irreducible representations of $GL_n \times S_k$ we get

$$V^{\otimes k} \cong \bigoplus_{\lambda} \mathbb{V}^{\lambda} \otimes \mathbb{S}^{\lambda}, \tag{1}$$

where $\mathbb{V}^{\lambda}$ is an irreducible, homogeneous, polynomial representation of GL_n and λ runs over all partitions of k with at most n parts. This gives a correspondence between representations of S_k and polynomial representations of GL_n.

Consider the case when $k = 3$ and $n = 9$. Combinatorics can help to visualize $V^{\otimes 3}$ and how it decomposes following Equation (1). Figure 4 represents basis elements of $V^{\otimes 3}$ as (i_1, i_2, i_3) in three-dimensional space with $1 \le i_1, i_2, i_3 \le 9$ and organizes them so that the irreducible GL_9 components are compact. The blue points represent $\mathbb{V}^{(3)} \otimes \mathbb{S}^{(3)}$, the red and the green points together represent $\mathbb{V}^{(2,1)} \otimes \mathbb{S}^{(2,1)}$ and the yellow points represent $\mathbb{V}^{(1,1,1)} \otimes \mathbb{S}^{(1,1,1)}$. The Robinson–Schensted algorithm [16] gives a way of making this assignment in general.

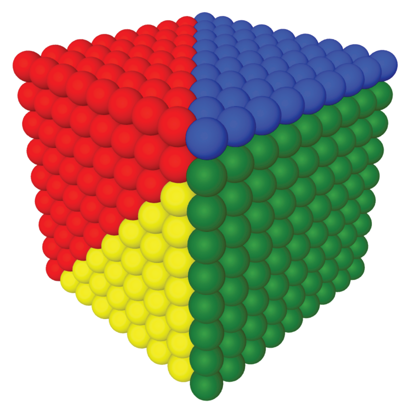


Figure 4. A combinatorial view of the decomposition of $V^{\otimes 3}$ into GL_9 representations.

Characters of GL_n. A polynomial in commuting variables $x_1, \dots, x_n$ is *symmetric* if any permutation of the variables leaves the polynomial invariant. For every polynomial representation of GL_n, there exists a symmetric polynomial $f(x_1, \dots, x_n)$ such that if $A \in GL_n$ has eigenvalues $\theta_1, \dots, \theta_n$, then the character value at A is $f(\theta_1, \dots, \theta_n)$. In the previous section we saw that for every partition λ with at most n parts, there exists an irreducible polynomial representation of GL_n which we refer to as $\mathbb{V}^{\lambda}$. Schur showed the character corresponding to $\mathbb{V}^{\lambda}$ are obtained by evaluations of a polynomial $s_\lambda(x_1, \dots, x_n)$ which is constructed combinatorially as follows:

1. In the boxes of the Young diagram of λ insert numbers $1, 2, \dots, n$ so that the numbers increase weakly along

each row from left to right and strictly from top to bottom in each column. This is called a *semistandard Young tableau* (SSYT for short).

For example, if $\lambda = (2)$ then its Young diagram is $\boxed{}\boxed{}$. If $n = 3$, these are the possible tableaux:

$$\boxed{1}\boxed{1},\ \boxed{1}\boxed{2},\ \boxed{1}\boxed{3},\ \boxed{2}\boxed{2},\ \boxed{2}\boxed{3},\ \boxed{3}\boxed{3}.$$

2. For each tableau, T, in part (1), define a monomial $x^T = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$, where i_j is the number of times that j occurs in T. For example, the corresponding monomials for the SSYT above are x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, and x_3^2, respectively.
3. The *Schur polynomial* $s_\lambda(x_1, \dots, x_n)$ is defined by summing all monomials possible:

$$s_\lambda(x_1, \dots, x_n) = \sum_T x^T,$$

where the sum is over all SSYT constructed using λ and $1, 2, \dots n$.

In the case of our running example, we get

$$s_{(2)}(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2.$$

The interested reader may check that the polynomial is symmetric since permuting the indices 1, 2, and 3 in any way gives the same polynomial. In addition one may verify by listing the tableaux of shape $(1, 1)$ that

$$s_{(1,1)}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3.$$

If $A \in GL_n$ has eigenvalues $\theta_1, \dots, \theta_n$, then the character value for the representation $\mathbb{V}^\lambda$ when acted on by A is obtained by substituting $x_i = \theta_i$, for all i, in $s_\lambda(x_1, \dots, x_n)$. The number $s_\lambda(\theta_1, \dots, \theta_n)$ is the trace of the matrix representing A when A acts on a basis of $\mathbb{V}^\lambda$. For example, if A has eigenvalues $1, -1$, and 2, then setting $x_1 = 1$, $x_2 = -1$, and $x_3 = 2$ in $s_{(2)}(x_1, x_2, x_3)$ gives the character of the representation $\mathbb{V}^{(2)}$ of GL_3 at the matrix A. In this case, $s_{(2)}(1, -1, 2) = 5$ is the character value.

Restricting representations. Any polynomial representation of GL_n is a representation for any subgroup G of GL_n. In particular, the symmetric group S_n, thought of as the group of $n \times n$ permutation matrices, is a subgroup of GL_n. Therefore, for any λ, $\mathbb{V}^\lambda$ is a representation of S_n. We write $\mathrm{Res}_{S_n} \mathbb{V}^\lambda$ for this restricted representation. The following is a well-known open problem, for more details and references see [13].

The Restriction Problem: Given an irreducible polynomial representation of GL_n, $\mathbb{V}^\lambda$, give a combinatorial algorithm to compute the coefficients, $r_{\lambda,\mu}$, that occur when restricted to the symmetric group in the equation

$$Res_{S_n} \mathbb{V}^\lambda \cong \bigoplus_\mu r_{\lambda,\mu} \mathbb{S}^\mu.$$

We can obtain the character of the restricted representation $Res_{S_n} \mathbb{V}^\lambda$ by evaluating $s_\lambda(x_1, x_2, \dots, x_n)$ only at eigenvalues of permutation matrices.

For example, if $n = 3$ we can restrict $\mathbb{V}^{(2)}$ with character

$$s_{(2)}(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

to S_3. To get the character, for each conjugacy class of S_3 choose a representative and compute its eigenvalues. The identity, e, has eigenvalues $1, 1, 1$, a two-cycle has eigenvalues $1, 1, -1$, and $1, \xi, \xi^2$ are the eigenvalues for a three-cycle, where ξ is a primitive third root of unity. Then evaluate, $s_{(2)}(1, 1, 1) = 6$, $s_{(2)}(1, 1, -1) = 2$, and $s_{(2)}(1, \xi, \xi^2) = 0$. Thus, again using the character table in Figure 2, the character of $\mathrm{Res}_{S_3} \mathbb{V}^{(2)}$ is the vector $\langle 6, 2, 0 \rangle = 2\,\chi^{(3)} + 2\,\chi^{(2,1)}$. We can see that $r_{(2),(3)} = 2$ and $r_{(2),(2,1)} = 2$.

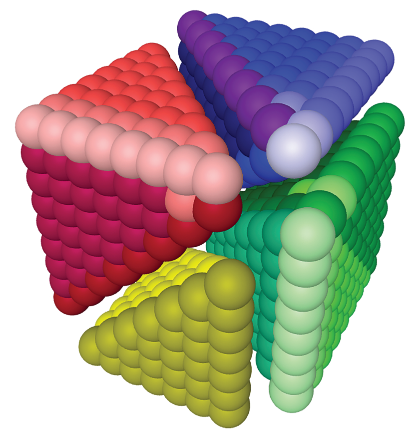


Figure 5. A combinatorial view of the decomposition of $V^{\otimes 3}$ into GL_9 representations by primary colors and then the restriction of those into S_9 representations by the different shades of those regions.

A dual approach to restriction. Schur used the diagonal action of GL_n on $V^{\otimes k} = (\mathbb{C}^n)^{\otimes k}$ and computed its centralizer in order to study the polynomial representations of GL_n. As we said above, the commutant or centralizer in this case is the symmetric group S_k, which can be visualized in terms of diagrams as in Figure 1. Schur's work inspired others to use this techniques to study representations of subgroups G of GL_n using centralizers. For example, if G is the orthogonal group, the centralizer algebra is the Brauer algebra. The Temperley–Lieb algebra is a subalgebra of the Brauer algebra and itself a centralizer of the quantum group of type A, $U_q(\mathfrak{sl}_2)$. The study of diagram algebras, centralizer algebras, and connections with topology and physics has become a subfield in combinatorial representation theory.

A key centralizer algebra in our story arises when $G = S_n$ is realized as the subgroup of permutation matrices in GL_n. Jones [8] and Martin [10] (independently) computed the centralizer algebra of the diagonal action of S_n. For $\sigma \in S_n$, the diagonal action is

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes \cdots \otimes \sigma v_k,$$

where σv_i is the product of the permutation matrix σ with the column vector v_i. For example, $(1,2) \cdot v_1 \otimes v_1 = v_2 \otimes v_2$. Notice that this is different from the (right) action of S_k on $V^{\otimes k}$ that permutes tensor factors. Under the action illustrated in Figure 3, $(1,2)$ would leave $v_1 \otimes v_1$ invariant. The algebra consisting of the linear transformations $D : V^{\otimes k} \to V^{\otimes k}$ which commutes with this action is known as the *partition algebra*, $P_k(n)$.

The partition algebra $P_k(n)$ has a linear basis that is in bijection with set partitions of the set $\{1, \dots, k\} \cup \{\overline{1}, \dots, \overline{k}\}$. These set partitions can be visualized as graphs and hence are often referred to as *partition diagrams*. We draw these graphs by arranging the vertices in two rows: $1, \dots, k$ appear from left to right in the top row; and $\overline{1}, \dots, \overline{k}$ from left to right in the bottom row. The connected components in the graph correspond to the blocks of the set partition, therefore many graphs can be used to represents the same linear transformation in the partition algebra. An example of a diagram in the partition algebra $P_5(n)$ is given in Figure 6.

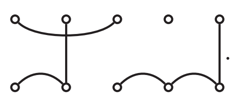
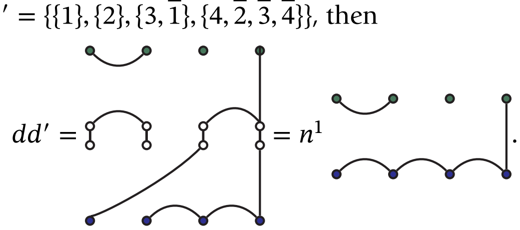


Figure 6. The partition algebra diagram corresponding to the set partition $\{\{1,3\},\{2,\overline{1},\overline{2}\},\{4\},\{\overline{3},\overline{4},\overline{5},5\}\}$.

The product in $P_k(n)$ is completely described using diagrams. Given two set partitions, d and d', to compute their product dd', we put the diagram of d on top of the diagram of d'. We count the number of connected components that use only middle vertices; call this number m. Then dd' consists of the diagram consisting of the components containing only top vertices of d and bottom vertices of d' in the concatenated graph, ignoring middle vertices, and there is a coefficient of n^m multiplied by the resulting diagram. As an example consider $d = \{\{1,2\},\{\overline{1},\overline{2}\},\{3\},\{4,\overline{3},\overline{4}\}\}$ and $d' = \{\{1\},\{2\},\{3,\overline{1}\},\{4,\overline{2},\overline{3},\overline{4}\}\}$, then



The partition algebra $P_k(n)$ is the $\mathbb{C}$-span of the partition diagrams with this concatenation product. The algebra is associative, has an identity $\{\{1,\overline{1}\},\{2,\overline{2}\},\dots,\{k,\overline{k}\}\}$ and its dimension is the Bell number $B(2k)$.

When $n \geq 2k$, the irreducible representations of $P_k(n)$ are indexed by partitions of n, $(n - |\lambda|, \lambda_1, \dots, \lambda_\ell)$, such that $\lambda_1 + \cdots + \lambda_\ell \leq k$. Jones [8] described the duality between representations of the partition algebra, $P_k(n)$, and those of the symmetric group S_n. He showed that the direct product $S_n \times P_k(n)$ acts on $V^{\otimes k}$ and this representation decomposes as follows

$$V^{\otimes k} \cong \bigoplus \mathbb{S}^\lambda \otimes \mathbb{L}^\lambda, \tag{2}$$

where $\mathbb{L}^\lambda$ is an irreducible representation of $P_k(n)$ and the sum is over all partitions $\lambda = (n - |\lambda|, \lambda_1, \dots, \lambda_\ell)$ such that $\lambda_1 + \cdots + \lambda_\ell \leq k$. In [1], the authors studied this duality and connections to the Kronecker coefficients. In particular they showed that restricting representations of the partition algebra gives an alternate way to study the Kronecker coefficients.

In Figure 5 we have taken the decomposition of $V^{\otimes 9}$ into GL_9 irreducible representations shown in Figure 4 and used finer shadings of colors to indicate how Equation (2) is related to the restriction problem by breaking each of the components further into S_9 irreducibles.

The character of $V^{\otimes k}$. The action on $V^{\otimes k}$ is a polynomial representation of $GL_n \times S_k$. Its character at an element $(A, \sigma) \in GL_n \times S_k$ is the *power symmetric polynomial*, $p_\mu(x_1, \dots, x_n)$ where μ is a partition representing the sizes of the cycles of σ and $x_1, \dots, x_n$ are the eigenvalues of A.

For a positive integer r, $p_r = x_1^r + \dots + x_n^r$ and for a partition $\mu = (\mu_1, \dots, \mu_l)$, $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_l}$.

From the isomorphism in (1) we obtain the following equation of symmetric polynomials, known as the Frobenius formula,

$$p_\mu(x_1, \dots, x_n) = \sum_\lambda s_\lambda(x_1, \dots, x_n) \chi^\lambda(\sigma_\mu) \tag{3}$$

where $\chi^\lambda(\sigma_\mu)$ is the irreducible character of S_k evaluated at an element σ_μ with cycle structure μ. The sum runs over all partitions λ of k with at most n parts.

The vector space $V^{\otimes k}$ is also a representation of $S_n \times P_k(n)$ and its character at an element $(\sigma, d_\mu) \in S_n \times P_k(n)$ can be obtained from $p_\mu(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the eigenvalues of the permutation matrix σ. We will not explicitly define d_μ here, but it is a generalized conjugacy class representative in $P_k(n)$, for details see [4]. Hence from the isomorphism in (2) we also have

$$p_\mu(x_1, \dots, x_n) = \sum_\lambda \chi^\lambda(\sigma) \chi^\lambda_{P_k(n)}(d_\mu), \tag{4}$$

where $\chi^\lambda_{P_k(n)}(d_\mu)$ are irreducible characters of the partition algebra. Since the left-hand side of (4) is a symmetric function, we conjectured that there should be symmetric functions that evaluate the irreducible characters of the symmetric group. More precisely, there should exist polynomials $\tilde{s}_\lambda$ such that

$$p_\mu(x_1, \dots, x_n) = \sum_\lambda \tilde{s}_\lambda(x_1, \dots, x_n) \chi^\lambda_{P_k(n)}(d_\mu),$$

where $x_1, \dots, x_n$ are eigenvalues of the permutation matrix σ.

Characters of symmetric groups as symmetric polynomials. The Schur polynomials, $\{s_\lambda \mid \lambda \text{ a partition}\}$, form a basis for symmetric polynomials with the following properties:

(A) When we evaluate s_λ at the eigenvalues of $A \in GL_n$, we get characters of irreducible polynomial representations of GL_n.
(B) When we multiply two Schur polynomials the coefficients are the same as those which occur when we decompose tensor products of irreducible polynomial representations of GL_n.

In [13], we defined a new basis of symmetric polynomials $\{\tilde{s}_\lambda \mid \lambda \text{ a partition}\}$ that connects the ideas mentioned in this article through the following properties:

(1) For any partition λ and $n \geq |\lambda| + \lambda_1$, $\tilde{s}$ evaluates to the irreducible characters of the symmetric group,

$$\tilde{s}_\lambda(x_1, \dots, x_n) = \chi^{(n-|\lambda|,\lambda)}(\sigma),$$

where $x_1, \dots, x_n$ are the eigenvalues of the permutation matrix σ.
(2) Recall $\overline{g}^{\nu}_{\lambda,\mu}$ are the reduced Kronecker coefficients which occur as stable limits of Kronecker coefficients. Then,

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\nu \overline{g}^{\nu}_{\lambda,\mu} \tilde{s}_\nu.$$

(3) If $r_{\lambda,\mu}$ are the restriction coefficients when a polynomial representation $\mathbb{V}^\lambda$ of GL_n is restricted to S_n. Then,

$$s_\lambda = \sum_\mu r_{\lambda,\mu} \tilde{s}_\mu.$$

Observe that properties (1) and (2) of the polynomials $\tilde{s}_\lambda$ are analogous to properties (A) and (B) of the Schur polynomials. In addition, property (3) connects these two bases.

For example, when $n = 3$,

$$\tilde{s}_{()} = 1, \qquad \tilde{s}_{(1)} = x_1 + x_2 + x_3 - 1,$$

and

$$\tilde{s}_{(1,1)} = x_1x_2 + x_1x_3 + x_2x_3 - x_1 - x_2 - x_3 + 1.$$

As mentioned above, the eigenvalues of the identity matrix are $1, 1, 1$, the permutation matrix of a two-cycle has eigenvalues $1, 1, -1$ and the permutation matrix of a three-cycle has eigenvalues $1, \xi, \xi^2$, where ξ is a primitive third root of unity. The interested reader can evaluate these three polynomials at the three sets of eigenvalues to recover the character table from Figure 2.

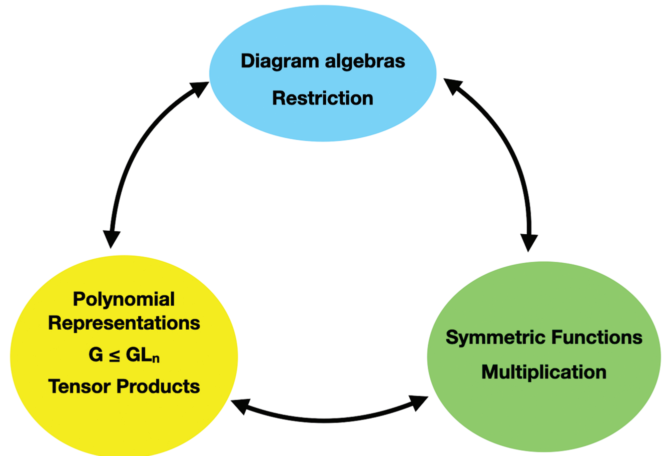


Figure 7. A diagram representing how the mathematical ideas mentioned in this paper are related.

Conclusion and further reading. To make progress on open problems related to the combinatorial representation theory of the symmetric group, we can study representations of GL_n, diagram algebras, or symmetric functions.

A good resource to learn about the representation theory of the symmetric group is [16]. Chapter 7 in [17] gives a combinatorial introduction to symmetric functions. For a nice survey on the representation theory of the partition algebra see [5]. For details on the properties of the basis $\{\tilde{s}_\lambda\}$ see [13, 14] and references therein. For progress on the Kronecker coefficients related to the basis $\{\tilde{s}_\lambda\}$ see [15].

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References

[1] C. Bowman, M. De Visscher, and R. Orellana, *The partition algebra and the Kronecker coefficients*, Trans. Amer. Math. Soc. **367** (2015), no. 5, 3647–3667, DOI 10.1090/S0002-9947-2014-06245-4. MR3314819
[2] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas, *The stability of the Kronecker product of Schur functions*, J. Algebra **331** (2011), 11–27, DOI 10.1016/j.jalgebra.2010.12.026. MR2774644
[3] Matthias Christandl, Aram W. Harrow, and Graeme Mitchison, *Nonzero Kronecker coefficients and what they tell us about spectra*, Comm. Math. Phys. **270** (2007), no. 3, 575–585, DOI 10.1007/s00220-006-0157-3. MR2276458
[4] Tom Halverson, *Characters of the partition algebras*, J. Algebra **238** (2001), no. 2, 502–533, DOI 10.1006/jabr.2000.8613. MR1823772
[5] Tom Halverson and Arun Ram, *Partition algebras*, European J. Combin. **26** (2005), no. 6, 869–921, DOI 10.1016/j.ejc.2004.06.005. MR2143201

[6] Vaughan F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111, DOI 10.1090/S0273-0979-1985-15304-2. MR766964

[7] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) **126** (1987), no. 2, 335–388, DOI 10.2307/1971403. MR908150

[8] V. F. R. Jones, *The Potts model and the symmetric group*, Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, NJ, 1994, pp. 259–267. MR1317365

[9] Allen Knutson and Terence Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090, DOI 10.1090/S0894-0347-99-00299-4. MR1671451

[10] Paul Martin, *Temperley-Lieb algebras for nonplanar statistical mechanics—the partition algebra construction*, J. Knot Theory Ramifications **3** (1994), no. 1, 51–82, DOI 10.1142/S0218216594000071. MR1265453

[11] Ketan D. Mulmuley and Milind Sohoni, *Geometric complexity theory. I. An approach to the P vs. NP and related problems*, SIAM J. Comput. **31** (2001), no. 2, 496–526, DOI 10.1137/S009753970038715X. MR1861288

[12] F. D. Murnaghan, *The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group*, Amer. J. Math. **60** (1938), no. 3, 761–784, DOI 10.2307/2371610. MR1507347

[13] Rosa Orellana and Mike Zabrocki, *Symmetric group characters as symmetric functions*, Adv. Math. **390** (2021), Paper No. 107943, 34, DOI 10.1016/j.aim.2021.107943. MR4295089

[14] Rosa Orellana and Mike Zabrocki, *The Hopf structure of symmetric group characters as symmetric functions*, Algebr. Comb. **4** (2021), no. 3, 551–574, DOI 10.5802/alco.170. MR4275829

[15] Rosa Orellana and Mike Zabrocki, *Products of symmetric group characters*, J. Combin. Theory Ser. A **165** (2019), 299–324, DOI 10.1016/j.jcta.2019.02.019. MR3944530

[16] Bruce E. Sagan, *The symmetric group*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Representations, combinatorial algorithms, and symmetric functions. MR1093239

[17] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, DOI 10.1017/CBO9780511609589. MR1676282

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From Optimization to Sampling Through Gradient Flows

N. García Trillos, B. Hosseini, and D. Sanz-Alonso

Optimization and sampling algorithms play a central role in science and engineering as they enable finding optimal predictions, policies, and recommendations, as well as expected and equilibrium states of complex systems. The notion of "optimality" is formalized by the choice of an objective function, while the notion of an "expected" state is specified by a probabilistic model for the distribution of states. Optimizing rugged objective functions and sampling multimodal distributions is computationally challenging, especially in high-dimensional problems. For this reason, many optimization and sampling methods have been developed by researchers working in disparate fields such as Bayesian statistics, molecular dynamics, genetics, quantum chemistry, machine learning, weather forecasting, econometrics, and medical imaging.

State-of-the-art algorithms for optimization and sampling often rely on *ad-hoc* heuristics and empirical tuning, but some unifying principles have emerged that greatly facilitate the understanding of these methods and the communication of algorithmic innovations across scientific communities. This article is concerned with one such principle: the use of gradient flows, and discretizations thereof, to design and analyze optimization and sampling algorithms. The interplay between optimization, sampling, and gradient flows is an active research area and a thorough review of the extant literature is beyond the scope of this article.[1] Our goal is to provide an accessible and lively introduction to some core ideas, emphasizing that gradient flows uncover the conceptual unity behind several existing algorithms and give a rich mathematical framework for their rigorous analysis.

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[1] AMS Notices *limits to* 20 *the references per article; we refer to* [GTSA20, CLGL$^+$20, GIHLS20] *for further pointers to the literature.*

We present motivating applications in section 1. Section 2 is focused on the gradient descent approach to optimization and introduces fundamental ideas such as preconditioning, convergence analysis, and time discretization of gradient flows. Sampling is discussed in section 3 in the context of Langevin dynamics viewed as a gradient flow of the Kullback–Leibler (KL) divergence with respect to (w.r.t) the Wasserstein geometry. Some modern applications of gradient flows for sampling are discussed in section 4, followed by concluding remarks in section 5.

1. Motivating Applications

We outline two applications in Bayesian statistics and molecular dynamics that illustrate some important challenges in optimization and sampling.

1.1. Bayesian statistics. In Bayesian statistics [GCSR95], an initial belief about an unknown parameter is updated as data becomes available. Let θ denote the unknown parameter of interest belonging to the parameter space Θ and let $\pi_0(\theta)$ denote the *prior* distribution reflecting our initial belief. Furthermore, let y be the observed data also belonging to an appropriate space $\mathcal{Y}$. Then Bayes's rule identifies the distribution of θ conditioned on the data y:

$$\pi(\theta|y) \propto \pi(y|\theta)\pi_0(\theta), \tag{1.1}$$

where $\propto$ indicates that the right-hand side should be normalized to define a probability distribution. Here $\pi(y|\theta)$ is called the *likelihood* function and $\pi(\theta|y)$ is called the *posterior* distribution. Bayesian inference on θ is based on the posterior, which blends the information in the prior and the data.

The choice of prior and likelihood is a modeling task which, perhaps surprisingly, is often not the most challenging aspect of Bayesian inference. The main challenge is to extract information from the posterior since (i) it typically does not belong to a standard family of distributions, unless in the restrictive case of conjugate models [GCSR95]; (ii) the parameter θ can be high dimensional; and (iii) the normalizing constant $\int_\Theta \pi(y|\theta)\pi_0(\theta)\,d\theta$ in (1.1) (known as the *marginal likelihood*) is rarely available and it can be expensive to compute. These practical hurdles inform the design of optimization and sampling algorithms to find posterior statistics.

Of particular importance is the posterior mode or *maximum a posteriori (MAP)* estimator

$$\theta_{\text{MAP}} := \arg\max_\theta \pi(\theta|y).$$

Many optimization algorithms for MAP estimation start from an initial guess θ_0 and produce iterates $\{\theta_n\}_{n=1}^N$ by discretizing a gradient flow with the property that $\theta_n \approx \theta_{\text{MAP}}$ for large n. Such gradient flows in parameter space will be discussed in section 2.

Computing MAP estimators is closely related to classic regularization techniques such as penalized least squares and Tikhonov regularization [SAST18]. In order to fully leverage the Bayesian framework, it is often desirable to consider other posterior statistics such as mean, variance, credible intervals, and task-specific functional moments, which can be written in the form

$$\mathbb{E}^{\pi(\cdot|y)}[\phi(\theta)] := \int \phi(\theta)\pi(\theta|y)\,d\theta,$$

where ϕ is a suitable test function. Since θ is often high dimensional, the standard approach to compute these expectations is to use Monte Carlo [Liu01]: obtain N samples $\{\theta_n\}_{n=1}^N$ from the posterior $\pi(\theta|y)$, and then approximate

$$\mathbb{E}^{\pi(\cdot|y)}[\phi(\theta)] \approx \frac{1}{N}\sum_n \phi(\theta_n).$$

While Monte Carlo integration is in principle scalable to high dimensions, the task of generating posterior samples is still highly nontrivial. To that end, one may consider sampling $\theta_n \sim \rho_n$, where the sequence $\{\rho_n\}_{n=1}^N$ arises from discretizing a gradient flow with the property that $\rho_n \approx \pi(\cdot|y)$ for large n. Such gradient flows in the space of probability distributions will be discussed in sections 3 and 4.

To relate the discussion above to subsequent developments, we note that dropping the data y from the notation, the posterior density can be written as

$$\pi(\theta) = \frac{1}{Z}\exp\bigl(-V(\theta)\bigr), \tag{1.2}$$

where $Z = \int_\Theta \pi(y|\theta)\pi_0(\theta)\,d\theta$ is the marginal likelihood and $V : \Theta \times \mathcal{Y} \to \mathbb{R}$ is the negative logarithm of the posterior density.

1.2. Molecular dynamics. Another important source of challenging optimization and sampling problems is statistical mechanics, and in particular the simulation of molecular dynamics (see chapter 9 in [Liu01]). According to Boltzmann and Gibbs, the positions q and momenta p of the atoms in a molecular system of constant size, occupying a constant volume, and in contact with a heat bath (at constant temperature), are distributed according to

$$\pi(q,p) = \frac{1}{Z}\exp\Bigl(-\beta\bigl(U(q)+K(p)\bigr)\Bigr), \tag{1.3}$$

where Z is a normalizing constant known as the partition function, β represents the inverse temperature, U is a potential energy describing the interaction of the particles in the system, and K represents the kinetic energy of the system. Letting $\theta := (q,p)$, we can write the Boltzmann–Gibbs distribution (1.3) in the form (1.2), with

$$V(\theta) = -\beta\bigl(U(q)+K(p)\bigr). \tag{1.4}$$

As in Bayesian statistics, it is important to determine the most likely configuration of particles (i.e., the mode of π), along with expectations of certain test functions w.r.t.

the Boltzmann distribution. These two tasks motivate the need for optimization and sampling algorithms that acknowledge that the potential U is often a rough function with many local minima, that the dimension of q and p is large, and that finding the normalizing constant Z is challenging.

2. Optimization

In this section we discuss gradient flows for solution of the unconstrained minimization problem

$$\text{minimize } V(\theta) \quad \text{s.t. } \theta \in \Theta, \tag{2.1}$$

where $V(\theta)$ is a given objective function. Henceforth we take $\Theta := \mathbb{R}^d$ unless otherwise noted. As guiding examples, consider computing the mode of a posterior or Boltzmann distribution by minimizing V given by (1.2) or (1.4). The methods described in this section are applicable beyond the specific problem of finding the modes, however, this interpretation will be of particular interest in relating the material in this section to our discussion of sampling in section 3.

2.1. Gradient systems. One of the most standard approaches to solve (2.1) is *gradient descent*, an optimization scheme that is based on the discretization of the gradient system

$$\dot{\theta}_t = -\nabla V(\theta_t), \quad t > 0, \tag{2.2}$$

with user-defined initial value θ_0; throughout this article $\nabla V(\theta_t)$ will denote the gradient of the function V at the point θ_t, which will be tacitly assumed to exist wherever needed.

While equation (2.2) is perhaps the most popular formulation of the continuous-time gradient descent dynamics, the equivalent integral form below reveals more transparently some of its properties:

$$V(\theta_t) = V(\theta_s) - \frac{1}{2}\int_s^t |\nabla V(\theta_r)|^2 dr - \frac{1}{2}\int_s^t |\dot{\theta}_r|^2 dr, \tag{2.3}$$

for all $t \geq s > 0$. Indeed, notice that from (2.3) it is apparent that $V(\theta_t) \leq V(\theta_s)$ for $s \leq t$, i.e., the value of the function V decreases in time, and in all but a few trivial situations the decrease is strict. Another advantage of the reformulation (2.3) (or its inequality form (2.5) below) is that it can be adapted to more general settings with less mathematical structure than the one needed to make sense of (2.2). In particular, (2.3) can be used to motivate a notion of gradient flow in arbitrary metric spaces; see [AGS08] for an in-depth discussion of this topic.

Proposition 2.1. *Suppose V is a C^1 function. Then* (2.2) *and* (2.3) *are equivalent and they both imply*

$$|\nabla V(\theta_t)|^2 = |\dot{\theta}_t|^2. \tag{2.4}$$

Proof. By Cauchy–Schwarz and Young's inequalities, for any $t > 0$ it holds that

$$\begin{aligned} -\langle \nabla V(\theta_t), \dot{\theta}_t \rangle &\leq |\nabla V(\theta_t)||\dot{\theta}_t| \\ &\leq \frac{1}{2}|\nabla V(\theta_t)|^2 + \frac{1}{2}|\dot{\theta}_t|^2, \end{aligned}$$

and both inequalities are equalities iff $-\nabla V(\theta_t) = \dot{\theta}_t$. Therefore,

$$\begin{aligned} V(\theta_t) &= V(\theta_s) + \int_s^t \langle \nabla V(\theta_r), \dot{\theta}_r \rangle dr \\ &\geq V(\theta_s) - \frac{1}{2}\int_s^t |\nabla V(\theta_r)|^2 dr - \frac{1}{2}\int_s^t |\dot{\theta}_r|^2 dr, \end{aligned}$$

and equality holds iff $-\nabla V(\theta_t) = \dot{\theta}_t$ for all $t > 0$. The identity $|\nabla V(\theta_t)|^2 = |\dot{\theta}_t|^2$ follows directly from (2.2). □

Notice that the relationship $\dot{\theta}_t = -\nabla V(\theta_t)$ is only required in proving the *energy dissipation inequality*

$$V(\theta_t) \leq V(\theta_s) - \frac{1}{2}\int_s^t |\nabla V(\theta_r)|^2 dr - \frac{1}{2}\int_s^t |\dot{\theta}_r|^2 dr, \tag{2.5}$$

since the reverse inequality is a consequence of Cauchy–Schwarz. Notice further that (2.2) implies (2.4), but in general the converse statement is not true. For example, the flow $\dot{\theta}_t = \nabla V(\theta_t)$ satisfies (2.4), but in general does not satisfy (2.2). Likewise, (2.2) implies $\frac{d}{dt}V(\theta_t) = -|\nabla V(\theta_t)|^2$ (which follows directly from the chain rule), but not conversely. Indeed, in $\mathbb{R}^2$ we may take A to be any orthogonal matrix and consider $\dot{\theta}_t = -A^2\nabla V(\theta_t)$ so that $\frac{d}{dt}V(\theta_t) = -|\nabla V(\theta_t)|^2$ but (2.2) is not, in general, satisfied. This digression illustrates that equation (2.3) captures in one single identity of *scalar* quantities the *vectorial* identity (2.2), even if it is not as intuitive as other scalar relations.

2.2. A note on convergence. Despite the fact that gradient descent satisfies the energy dissipation property, it is in general not true that as time goes to infinity the dynamics (2.2) converge to a global minimizer of (2.1). This could happen for different reasons. First, the problem (2.1) may not have a minimizer (e.g., take $V(\theta) = e^{-\theta}$ for $\theta \in \mathbb{R}$). Second, ∇V may have critical points associated with saddle points or local optima of V as we illustrate in the next example.

Example 2.2. Consider the double well potential

$$V(\theta) = \frac{3}{8}\theta^4 - \frac{3}{4}\theta^2, \quad \theta \in \mathbb{R}, \tag{2.6}$$

so that $\nabla V(\theta) = \frac{3}{2}\theta(\theta^2 - 1)$. Notice that $\theta = 0$ is an unstable equilibrium of (2.2), which corresponds to a local maximum of V. For each local minima $\theta = \pm 1$ of V there is an associated subregion in the space of parameters (known as

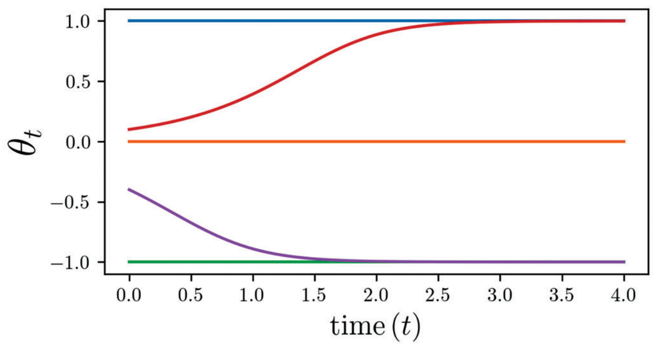

Figure 1. Five trajectories of the one-dimensional gradient system (2.2) with double well potential (2.6). The objective V has critical points at $\theta = 0$ and $\theta = \pm 1$. The intervals $(0, \infty)$ and $(-\infty, 0)$ are basins of attraction for $\theta = 1$ and $\theta = -1$, respectively.

a basin of attraction) such that any initial condition θ_0 chosen in this subregion leads the gradient dynamics toward its corresponding local minimizer, see Figure 1.

Suitable assumptions on V prevent the existence of local minimizers that are not global and also imply rates of convergence. One such assumption is the Polyak–Lojasiewic (PL) condition [KNS16]:

$$\alpha(V(\theta) - V^*) \leq \frac{1}{2}|\nabla V(\theta)|^2, \quad \forall \theta \in \Theta, \tag{2.7}$$

where $V^* = \inf_{\theta\in\Theta} V(\theta)$ and $\alpha > 0$ is a constant. Note that the PL condition readily implies that any stationary point θ^* of V is a global minimizer, since

$$\alpha(V(\theta^*) - V^*) \leq \frac{1}{2}|\nabla V(\theta^*)|^2 = 0.$$

Under the PL condition we can easily obtain a convergence rate for continuous-time gradient descent.

Proposition 2.3. *Suppose V satisfies* (2.7). *Then,*

$$V(\theta_t) - V^* \leq (V(\theta_0) - V^*)\exp(-2\alpha t), \quad \forall t \geq 0. \tag{2.8}$$

Proof. Using condition (2.7) in equation (2.5) and recalling (2.4) we conclude that

$$V(\theta_t) - V^* \leq (V(\theta_0) - V^*) - 2\alpha \int_0^t (V(\theta_r) - V^*)dr.$$

The result follows by Gronwall's inequality. □

One can verify the PL condition under various assumptions on the function V [KNS16]. Here we present, as an important example, the case of α-strong convexity. For $\alpha > 0$, one says that V is *α-strongly convex* if for any $\theta, \theta' \in \Theta$ it holds that

$$V\big(t\theta + (1-t)\theta'\big) \leq tV(\theta) + (1-t)V(\theta') - \frac{\alpha}{2}t(1-t)|\theta - \theta'|^2,$$

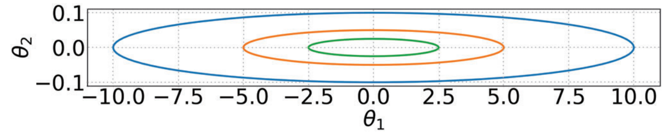

Figure 2. Level curves of a two-dimensional potential of the form $V(\theta) = \alpha_1\theta_1^2 + \alpha_2\theta_2^2$ with $0 < \alpha_1 \ll \alpha_2$. The anisotropy of this potential causes the gradient system (2.2) to converge slowly.

for all $t \in [0,1]$. This condition can be shown to be equivalent to

$$V(\theta') \geq V(\theta) + \langle \nabla V(\theta), \theta' - \theta\rangle + \frac{\alpha}{2}|\theta' - \theta|^2, \ \ \forall \theta, \theta' \in \Theta,$$

from which we can see, after minimizing both sides w.r.t. θ', that

$$V^* \geq V(\theta) - \frac{1}{2\alpha}|\nabla V(\theta)|^2, \tag{2.9}$$

which is equivalent to (2.7). From this we conclude that α-strong convexity implies the PL condition with the same constant α.

Note that strong convexity is a stronger condition than the PL condition. For example, the function $V(\theta) = \frac{1}{2}\theta_1^2$ (where $\theta = (\theta_1, \theta_2)$) satisfies the PL condition with $\alpha = 1$, but it is not strongly convex.

2.3. Choice of the metric. Let us consider a function of the form $V(\theta) = \alpha_1\theta_1^2 + \alpha_2\theta_2^2$, where $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. Suppose that $0 < \alpha_1 \ll \alpha_2$ and that α_1 is very close to zero, as in Figure 2. We can now apply Proposition 2.3 with $\alpha = \alpha_1$, but since we assumed α_1 is small we see that the right-hand side of (2.8) decreases very slowly. This suggests that gradient descent may take a long time to reach V's global minimum when initialized arbitrarily.

The poor behavior of gradient descent described above arises whenever there are regions of points away from the minimizer at which the gradient of V is very small. One approach to remedy this issue is to introduce a more general version of gradient descent that accelerates the dynamics in those regions where the gradient of V is small. This is the goal of *preconditioning*. Let $H : \Theta \to \mathcal{S}_{++}^d$ be a continuous field of positive definite matrices, i.e., a function that assigns to every point $\theta \in \Theta$ a $d \times d$ positive definite matrix $H(\theta)$. The *preconditioned gradient descent* dynamics induced by H is defined as:

$$\dot{\theta}_t = -H(\theta_t)^{-1}\nabla V(\theta_t), \quad t > 0. \tag{2.10}$$

Observe that (2.10) coincides with the original gradient descent dynamics (2.2) when H is constant and equal to the $d \times d$ identity matrix. On the other hand, when V is convex and twice differentiable, choosing $H(\theta) = \nabla^2 V(\theta)$, the Hessian of V, results in a continuous-time analog of *Newton's algorithm*. In the example in Figure 2, we can directly compute $\nabla^2 V = \begin{pmatrix} 2\alpha_1 & 0 \\ 0 & 2\alpha_2 \end{pmatrix}$, i.e., the Hessian is a fixed matrix since the potential V is quadratic. Substituting this choice of H in (2.10) for that example gives the dynamics $\dot{\theta}_t = -\theta_t$, a scheme that achieves a much faster convergence rate.

The reader may wonder if we could have in fact chosen $H = \frac{1}{r}D^2V$ for large constant r in order to induce a system that converges to equilibrium at a faster rate. However, as implied by our discussion in section 2.4 and, specifically, Remark 2.6 there is no benefit in doing so, as the cost of discretizing becomes correspondingly higher; rescaling the Hessian may be simply interpreted as a change of units. In general, there is a natural tension between accelerating continuous-time dynamics by changing the metric of the space, and producing accurate time discretizations for the resulting flows; see [GTSA20] for a related discussion in the context of sampling algorithms. In a similar vein, we notice that the superior convergence rate and affine invariance of Newton's algorithm comes at the price of utilizing the Hessian matrix $\nabla^2 V$, which in many applications can be prohibitively costly to compute or store. To this end, constructing matrix fields $H(\theta)$ that are good proxies for the Hessian and that can be computed efficiently is the goal of preconditioning. Perhaps the most well-known family of such algorithms is the family of *Quasi-Newton* algorithms [NW99] and in particular the *Broyden–Fletcher–Goldfarb–Shanno* (BFGS) algorithm, which approximates the Hessian using gradients calculated at previous iterates.

2.3.1. *Geometric interpretation.* As a step toward introducing the material in section 3, here we give a geometric interpretation of equation (2.10). Specifically, we will show that (2.10) can still be understood as a gradient descent equation but w.r.t. a different metric on the parameter space Θ. For this purpose it is convenient to recall that a *Riemannian manifold* $(\mathcal{M}, g)$ is a manifold $\mathcal{M}$ endowed with a family of inner products $g = \{g_\theta\}_{\theta\in\mathcal{M}}$ (often referred to as the *metric*), one for each point on the manifold, and which can be used to measure angles between vectors at every tangent plane of $\mathcal{M}$. We will use $\mathcal{T}_\theta\mathcal{M}$ to denote the tangent plane at a given $\theta \in \mathcal{M}$. The standard example of a Riemannian manifold is $\mathcal{M} = \mathbb{R}^d$ with g_θ the Euclidean inner product at every point. More general examples of Riemannian manifolds with $\mathcal{M} = \mathbb{R}^d$ can be generated from a field of positive definite matrices H. Consider the family of inner products:

$$g_\theta(u, v) \coloneqq \langle H(\theta)u, v\rangle, \quad u, v \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product. Notice that g_θ is indeed an inner product since $H(\theta)$ is positive definite. In what follows we often suppress the dependence of g on θ for brevity.

We now proceed to define the notion of the gradient of a function F defined over an arbitrary Riemannian manifold. Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $F : \mathcal{M} \to \mathbb{R}$ be a smooth enough function. The gradient of F at the point $\theta \in \mathcal{M}$ relative to the metric g, denoted $\nabla_g F(\theta)$, is defined as the vector in $\mathcal{T}_\theta\mathcal{M}$ for which the following identity holds:

$$\frac{d}{dt}F(\gamma(t))\big|_{t=0} = g(\nabla_g F(\theta), \dot{\gamma}(0)) \tag{2.11}$$

for any differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $\gamma(0) = \theta$; by $\dot{\gamma}(0)$ we mean the velocity of the curve γ at time 0, which is an element in $\mathcal{T}_\theta\mathcal{M}$. In the example of $\mathcal{M} = \mathbb{R}^d$ with inner products induced by a field H (which we denote with g_H), we have that

$$\begin{aligned}\frac{d}{dt}V(\gamma(t))\big|_{t=0} &= \langle\nabla V(\theta), \dot{\gamma}(0)\rangle\\ &= \langle H(\theta)H(\theta)^{-1}\nabla V(\theta), \dot{\gamma}(0)\rangle\\ &= g(H(\theta)^{-1}\nabla V(\theta), \dot{\gamma}(0)),\end{aligned}$$

for any curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ with $\gamma(0) = \theta$, from where it follows that $\nabla_{g_H} V(\theta) = H(\theta)^{-1}\nabla V(\theta)$, where we recall ∇ denotes the usual gradient in $\mathbb{R}^d$.

In this light, (2.10) can be interpreted as a gradient descent algorithm, only that the gradient is taken w.r.t. a metric that is different from the standard Euclidean one. In section 3, where we discuss sampling, we will return to some of the insights that we have developed in this section. In particular, in order to define gradient descent dynamics of a functional over a manifold we need to specify two ingredients: 1) an energy V to optimize, and 2) a metric g_θ under which we define the gradient. For the last item, it will be convenient to have a clear understanding of how to represent smooth curves in the manifold of interest and characterize their velocities appropriately. Indeed, equation (2.11) explicitly relates the metric g_θ of the manifold, the target energy V, the rate of change of the energy along arbitrary smooth curves γ, and the gradient $\nabla_g V$ of the energy relative to the chosen metric.

2.3.2. *Geodesic convexity.* There are analogous conditions to the PL and strong convexity assumptions discussed in section 2.1 that guarantee the convergence of the flow (2.10) toward global minima of V. First, write (2.10) as an energy dissipation equality of the form

$$V(\theta_t) = V(\theta_s) - \frac{1}{2}\int_s^t |\nabla_g V(\theta_r)|^2_{\theta_r} dr - \frac{1}{2}\int_s^t |\dot{\theta}_r|^2_{\theta_r} dr, \tag{2.12}$$

where we have used $|\cdot|^2_\theta$ to denote $g_\theta(\cdot,\cdot)$. The equivalence between (2.10) and (2.12) follows from an identical argument as in Proposition 2.1 applied to an arbitrary inner product. The analogous PL condition in the preconditioned setting takes the form:

$$\alpha(V(\theta) - V^*) \leq \frac{1}{2}|\nabla_g V(\theta)|^2_\theta, \tag{2.13}$$

which generalizes the PL condition in the Euclidean setting to general inner products and gradients.

To introduce an appropriate notion of convexity that allows us to generalize the results of section 2.1 we need to introduce a few more ideas from Riemannian geometry. Given a Riemannian manifold $(\mathcal{M}, g)$, we define the geodesic distance d_g induced by the metric g as:

$$d_g^2(\theta, \theta') = \inf_{t \in [0,1] \mapsto (\gamma_t, \dot{\gamma}_t)} \int_0^1 |\dot{\gamma}_t|_{\gamma_t}^2 \, dt. \tag{2.14}$$

We will say that $\gamma : [0,1] \to \mathcal{M}$ is a constant speed geodesic between θ and θ' if γ is a minimizer of the right-hand side of the above expression. Equivalently, a constant speed geodesic between θ and θ' is any curve with $\gamma(0) = \theta$ and $\gamma(1) = \theta'$ such that $d_g\big(\gamma(s), \gamma(t)\big) = |t - s| d_g(\theta, \theta')$ for all $s, t \in [0,1]$. The advantage of the latter definition is that it is completely described in terms of the distance function d_g and in particular does not require explicit mention of the Riemannian structure of the space.

We can now define the notion of *α-geodesic convexity*. We say $V : \mathcal{M} \to \mathbb{R}$ is α-geodesically convex if for all $\theta, \theta' \in \mathcal{M}$ there exists a constant speed geodesic $\gamma : [0,1] \to \mathcal{M}$ between them, such that

$$V\big(\gamma(t)\big) \leq tV(\theta) + (1-t)V(\theta') - \frac{\alpha}{2} t(1-t) d_g(\theta, \theta')^2, \tag{2.15}$$

for all $t \in [0,1]$.

Notice that α-geodesic convexity reduces to α-strong convexity when $(\mathcal{M}, g)$ is an Euclidean space. Also, it can be shown that α-geodesic convexity for $\alpha > 0$ implies the generalized PL condition (2.13) (see Lemma 11.28 in [Bou23]), which in turn implies, following the proof of Proposition 2.3, exponential decay rates for the energy V along its gradient flow, in direct analogy with Proposition 2.3.

Remark 2.4. Equation (2.14) relates the distance function d_g with the family of inner products g. This formula is very useful as it allows us to recover the metric g from its distance function d_g. This observation will be relevant when discussing the formal Riemannian structures on spaces of probability measures in the context of sampling in section 3.3.1.

2.4. Time discretizations.

2.4.1. *Standard gradient descent.*

In this section we discuss how to obtain practical optimization algorithms by discretizing the gradient system (2.2) in time. First, the *explicit Euler* scheme gives the standard gradient descent iteration

$$\theta_{n+1} = \theta_n - \tau \nabla V(\theta_n). \tag{2.16}$$

In numerical analysis of differential equations, $\tau > 0$ is interpreted as a small time-step; then, if (2.16) and (2.2) are initialized at the same point θ_0, it holds that $\theta_n \approx \theta_t$ for $t = n\tau$. In the optimization context of interest, τ is referred to as a learning rate and it is insightful to note that (2.16) can be defined variationally as

$$\theta_{n+1} = \operatorname{argmin}_\theta \Big(\langle \nabla V(\theta_n), \theta - \theta_n \rangle + \frac{1}{2\tau} |\theta - \theta_n|^2 \Big).$$

Thus, θ_{n+1} is found by minimizing $V(\theta_n) + \langle \nabla V(\theta_n), \theta - \theta_n \rangle + \frac{1}{2\tau}|\theta - \theta_n|^2$, noticing that the first two terms form the first order approximation of the objective V around the most recent iterate θ_n. In practice, the learning rate may be chosen adaptively using a line search [NW99].

Compared to the continuous-time setting, energy dissipation and convergence of the explicit Euler scheme require further assumptions on the function V. The following proposition is analogous to Proposition 2.8 but relies on a *smoothness* assumption on the gradient of V additional to the PL condition.

Proposition 2.5. *Suppose that V has L-Lipschitz gradient, has minimum V^*, and satisfies the PL condition. Then the gradient descent algorithm defined by* (2.16) *with step-size $\tau := \frac{1}{L}$ has a linear convergence rate. More precisely, it holds that*

$$V(\theta_{n+1}) - V^* \leq \Big(1 - \frac{\alpha}{L}\Big)^n \big(V(\theta_0) - V^*\big). \tag{2.17}$$

Proof. A classical result in convex analysis ensures that the assumption that ∇V is L-Lipschitz implies

$$V(\theta_{n+1}) \leq V(\theta_n) + \langle \nabla V(\theta_n), \theta_{n+1} - \theta_n \rangle + \frac{L}{2} |\theta_{n+1} - \theta_n|^2.$$

Using (2.16), we then deduce that

$$V(\theta_{n+1}) \leq V(\theta_n) - \frac{1}{2L} |\nabla V(\theta_n)|^2,$$

which combined with the PL condition gives

$$\begin{aligned} V(\theta_{n+1}) - V^* &\leq V(\theta_n) - V^* - \frac{\alpha}{L}\big(V(\theta_n) - V^*\big) \\ &= \Big(1 - \frac{\alpha}{L}\Big)\big(V(\theta_n) - V^*\big). \end{aligned}$$

The result follows by induction. □

From the proof of Proposition 2.5 we see that, under the L-smoothness condition and assuming that the step size τ is sufficiently small, the explicit Euler scheme dissipates the energy V. Moreover, this condition helps us quantify the amount of dissipation in one iteration of the scheme in terms of the norm of the gradient of V at the current iterate.

As an alternative discretization, one can consider the *implicit Euler* scheme:

$$\theta_{n+1} = \theta_n - \tau \nabla V(\theta_{n+1}), \tag{2.18}$$

which coincides with the first order optimality conditions for

$$\theta_{n+1} = \operatorname{argmin}_\theta \Big(V(\theta) + \frac{1}{2\tau} |\theta - \theta_n|^2 \Big). \tag{2.19}$$

It follows directly from the definition of the implicit Euler scheme that it satisfies a dissipation inequality analogous to (2.5) without imposing any additional smoothness conditions on V. However, an important caveat is that determining θ_{n+1} from θ_n requires solving a new optimization problem (2.19) or finding a root for the (in general) nonlinear equation (2.18). On the other hand, from a theoretical perspective the implicit Euler scheme, or *minimizing movement scheme* as it is called in [AGS08], is an important tool for proving existence of gradient flow equations in general metric spaces; see Chapters 1–2 in [AGS08].

2.4.2. *Discretizations and preconditioning.* One possible time discretization for (2.10) is given by

$$\theta_{n+1} = \theta_n - \tau H(\theta_n)^{-1}\nabla V(\theta_n),$$

which is a direct adaptation of (2.16) to the preconditioned setting. Proposition 2.5 can be readily adapted using the PL condition and L-smoothness of V relative to the geometry induced by the field H.

Remark 2.6. In line with the discussion at the end of section 2.3, we notice that the effect of scaling the field H by a constant $1/r$ is to scale the constants in both the PL condition and the L-smoothness condition by a factor of r. The net gain in (2.17) from rescaling the metric is thus null.

Another possible time discretization for (2.10) when the function H is the Hessian of a strictly convex function $h : \Theta \to \mathbb{R}$ (not necessarily equal to the objective function V) is the *mirror descent* scheme:

$$\begin{cases} z_{n+1} = z_n - \tau\nabla V(\theta_n), \\ \theta_{n+1} = (\nabla h)^{-1}(z_{n+1}). \end{cases} \tag{2.20}$$

The idea in mirror descent is to update an associated mirror variable (a transformation of θ by a mirror map, in this case ∇h) using a gradient step, as opposed to directly updating the variable θ as in the standard explicit Euler scheme. Using a Taylor approximation of $(\nabla h)^{-1}$ around z_t we see that

$$\begin{aligned} \theta_{n+1} &= (\nabla h)^{-1}\big(z_n - \tau\nabla V(\theta_n)\big) \\ &\approx \theta_n - \tau H(\theta_n)^{-1}\nabla V(\theta_n), \end{aligned}$$

revealing why mirror descent can be regarded as an approximation of (2.10) when H is the Hessian of h.

It is worth remarking that the update rule (2.20) has the following variational characterization:

$$\theta_{n+1} := \operatorname{argmin}_{\theta\in\Theta}\Big(\langle\nabla V(\theta_n),\theta\rangle + \frac{1}{\tau}D_h(\theta\|\theta_n)\Big), \tag{2.21}$$

where the function $D_h(\theta\|\theta')$ has the form

$$D_h(\theta\|\theta') := h(\theta) - h(\theta') - \langle\nabla h(\theta'),\theta-\theta'\rangle,$$

and is often referred to as *Bregman divergence*. This variational characterization was discovered and used in [BT03] to deduce convergence properties of mirror descent. Notice that the strict convexity of h guarantees that the function $D_h(\cdot\|\cdot)$ is non-negative and zero only when both of its arguments coincide. Bregman divergences thus play a similar role to the one played by the quadratic function $\frac{1}{2}|\cdot - \cdot|^2$ in the variational form of the standard explicit Euler scheme.

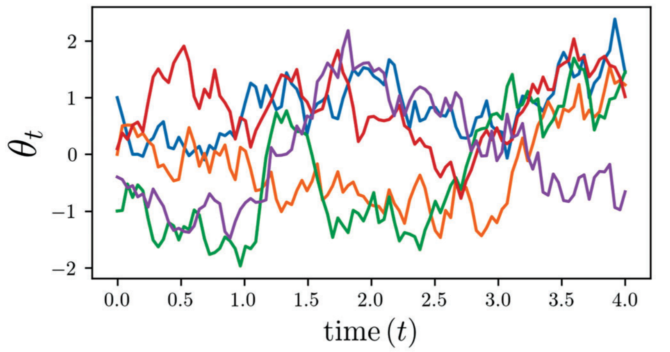

Figure 3. Five trajectories of Langevin dynamics with double well potential V given by (2.6).

3. Sampling

While direct sampling from certain distributions, e.g., Gaussians, may be rather straightforward, sampling from a general target distribution can be challenging, especially in high-dimensional settings. In this section we consider the problem of sampling a target density $\pi(\theta) \propto \exp\big(-V(\theta)\big)$. As guiding examples, one may consider sampling a posterior or Boltzmann distribution, see (1.2)–(1.3). We start by introducing Langevin dynamics in section 3.1, a stochastic differential equation that resembles the gradient system (2.2), but which incorporates a Brownian motion that makes the solution trajectories $\{\theta_t\}_{t\geq 0}$ random. In section 3.2 we present some results that state that, under suitable assumptions, the density ρ_t of θ_t converges to the desired target density π as $t \to \infty$. In section 3.3 we discuss how the Langevin dynamics define a gradient flow in the space of probability distributions. Finally, section 3.4 discusses how to use discretizations of Langevin dynamics to obtain practical sampling algorithms. Our presentation here parallels that of section 2.

3.1. **Langevin dynamics.** Consider the *overdamped Langevin* diffusion [Pav14]

$$d\theta_t = -\nabla V(\theta_t)\,dt + \sqrt{2}\,dB_t, \tag{3.1}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion on $\Theta = \mathbb{R}^d$. Langevin dynamics can be interpreted as a stochastic version of the gradient descent dynamics (2.2). This is illustrated in the following example, which also provides intuition on the connection between Langevin dynamics and sampling.

Example 3.1. Consider Langevin dynamics with the double well potential V introduced in (2.6). Figure 3 shows five trajectories, initialized as in Figure 1. For each $t > 0$, θ_t is now a random variable, whose Lebesgue density will be denoted by ρ_t in what follows. Figure 4 shows an

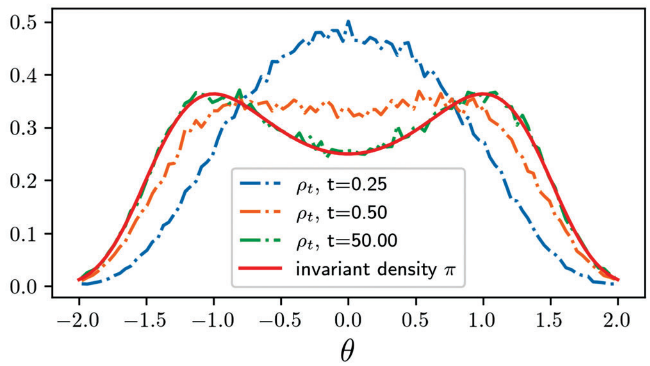

Figure 4. Histograms of ρ_t at $t = 0.25, 0.5, 50$. For large t, ρ_t is close to the target density $\pi \propto e^{-V}$.

approximation of ρ_t for $t \in \{0.25, 0.5, 50\}$ obtained by simulating $N = 10^5$ solution trajectories. Notice that at $t = 50$, ρ_t is exceedingly close to the target density $\pi(\theta) \propto \exp(-V(\theta))$, so that θ_t can be viewed as a sample from π. Thus, while θ_t is random due to the Brownian motion, the density $\rho_t(\theta)$ is larger at points θ where $V(\theta)$ is small. □

3.2. **A note on convergence.** It is natural to ask whether the law $\{\rho_t\}_{t\geq 0}$ of a given stochastic process converges to an *invariant* distribution π. For the Langevin diffusion, the positive answer illustrated in Example 3.1 holds under suitable assumptions on V that are analogous to the PL and strong convexity conditions in section 2. A natural way to study the long-time behavior of ρ_t is to derive a differential equation for its evolution. To that end, one may characterize the time derivative of the action of ρ_t on suitable test functions $\phi : \Theta \to \mathbb{R}$. More precisely, we compute $\frac{d}{dt}\int \phi(\theta)d\rho_t(\theta)$, which is a standard derivative of a function from the real line to itself. For the Langevin diffusion (3.1) it can be proved that:

$$\frac{d}{dt}\int_\Theta \phi(\theta)d\rho_t(\theta) = -\int_\Theta \nabla\phi\cdot\nabla\big(V+\log(\rho_t)\big)d\rho_t(\theta), \quad (3.2)$$

$\forall t > 0, \forall \phi \in C_c^\infty(\Theta)$. The above condition is the *weak formulation* of the Fokker–Planck equation:

$$\partial_t\rho_t = \operatorname{div}\big(\rho_t\nabla\big(V+\log(\rho_t)\big)\big) =: \mathcal{L}\rho_t. \quad (3.3)$$

From now on we interpret (3.3) in its weak form (3.2).

We observe that $\pi \propto e^{-V}$ is a stationary point of the dynamics (3.3). That is, if we initialize the dynamics at $\rho_0 = \pi$, then $\rho_t := \rho_0$ for all $t > 0$ is a solution to the equation. The next result describes the long-time behavior of a solution to the Fokker–Planck equation when initialized at more general ρ_0.

Theorem 3.2. *Let ρ_t be the solution to the Fokker–Planck equation with $\rho_0 \in L^2(\pi^{-1})$, where $L^2(\pi^{-1})$ is the L^2 space with the weight function π^{-1}. Suppose that there is $\alpha > 0$ such that the following Poincaré inequality holds: for every $f \in C^1 \cap L^2(\pi)$ that has zero mean under π, it holds that $\alpha\|f\|^2_{L^2(\pi)} \leq \|\nabla f\|^2_{L^2(\pi)}$. Then it holds that*

$$\|\rho_t - \pi\|_{L^2(\pi^{-1})} \leq e^{-\alpha t}\|\rho_0 - \pi\|_{L^2(\pi^{-1})}.$$

Proof. Define u_t by $\rho_t = u_t\pi$. We can verify that

$$\partial_t u_t = -\nabla V\cdot\nabla u_t + \operatorname{div}(\nabla u_t), \quad u_0 = \rho_0\pi^{-1}.$$

Therefore, the zero-mean function $u_t - 1$ satisfies

$$\frac{\partial(u_t-1)}{\partial t} = \mathcal{L}(u_t - 1). \quad (3.4)$$

Multiplying by $(u_t - 1)\pi$, integrating, and using that by assumption $\alpha\|u_t-1\|^2_{L^2(\pi)} \leq \|\nabla u_t\|^2_{L^2(\pi)}$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_t-1\|^2_{L^2(\pi)} \leq -\alpha\|u_t-1\|^2_{L^2(\pi)}. \quad (3.5)$$

Gronwall's inequality gives the desired result. □

The above result implies that, as time t goes to infinity, the distribution ρ_t converges toward the target density $\pi \propto e^{-V}$ exponentially fast. The notion of convergence implied by Theorem 3.2, however, is not as strong as other notions such as the convergence in KL divergence that will be discussed in the next section. In particular, Theorem 3.2 should be contrasted with the discussion in section 3.3.2.

3.3. **Choice of objective and metric.** Here we discuss a concrete variational interpretation for the Langevin system (3.3). In essence, this entails viewing sampling as an optimization algorithm (in particular, as a gradient flow) that aims at recovering the target density π. As discussed toward the end of section 2.3, to realize this interpretation it is important to identify precisely the geometric objects involved in the definition of a gradient flow (energy, metric, etc.). In all subsequent sections, $\mathcal{M}$ will be the space of probability measures over Θ, which we will denote with $\mathcal{P}(\Theta)$, and F will be the *KL divergence* relative to π, which we recall is defined as:

$$D_{\mathrm{KL}}(\nu\|\pi) := \int_\Theta \log\Big(\frac{\nu(\theta)}{\pi(\theta)}\Big)\nu(\theta)\,d\theta, \quad (3.6)$$

whenever ν is absolutely continuous w.r.t. π and $D_{\mathrm{KL}}(\nu\|\pi) = \infty$ otherwise. Different choices of metric over $\mathcal{P}(\Theta)$ will induce different evolution equations (recall our discussion of preconditioning in the context of optimization over the parameter space Θ), and thus different optimization schemes. In section 3.3.1 we discuss a specific geometric structure for $\mathcal{P}(\Theta)$ that realizes the Fokker–Planck equation of the Langevin diffusion as a gradient flow of F, and in section 4 we discuss other gradient flow structures that motivate other sampling algorithms.

Remark 3.3. It holds that $D_{\mathrm{KL}}(\nu\|\pi) \geq 0$ for all ν, and equality holds if and only if $\nu = \pi$. In particular, the unique minimizer of $\nu \in \mathcal{P}(\Theta) \mapsto D_{\mathrm{KL}}(\nu\|\pi)$ is the target π. Although trivial, this observation motivates *variational inference*—see [LCB+22] and references therein—a method for producing tractable proxies for π that relies on the minimization of $D_{\mathrm{KL}}(\cdot\|\nu)$ over a user-chosen family of tractable distributions. □

Remark 3.4. If π has a density w.r.t. the Lebesgue measure that is proportional to e^{-V}, then $D_{\mathrm{KL}}(\nu\|\pi) < \infty$ implies that ν is also absolutely continuous w.r.t. the Lebesgue measure. In that case we will abuse notation slightly and use ν to also denote ν's corresponding density. In particular, when we write $\log(\nu)$ it is understood that ν is interpreted as the density function w.r.t. Lebesgue measure of the measure ν.

3.3.1. *KL and Wasserstein.* Following a series of seminal works that started with a paper by Jordan, Kinderlehrer, and Otto in the late 90s (see sections 8.1. and 8.2. in [Vil03]) we will interpret equation (3.3) as the gradient flow of the energy F w.r.t the *Wasserstein* metric. Given $\rho, \rho' \in \mathcal{P}(\Theta)$ with finite second moments, their Wasserstein distance $W_2(\rho, \rho')$ is given by

$$W_2^2(\rho, \rho') := \min_{\Upsilon \in \Gamma(\rho, \rho')} \int_{\Theta\times\Theta} |\theta - \theta'|^2 d\Upsilon(\theta, \theta'), \tag{3.7}$$

where $\Gamma(\rho, \rho')$ is the set of couplings between ρ and ρ', i.e. the set of Borel probability measures on the product space $\Theta \times \Theta$ with first and second marginals equal to ρ and ρ', respectively.

Formula (3.7), although simple, does not reveal the infinitesimal geometric structure of the Wasserstein space to define gradients of functionals over $\mathcal{P}(\Theta)$. What is missing is a representation of the distance W in a form similar to (2.14). The next result by Benamou and Brenier (see section 8.1. in [Vil03]) provides the missing elements.

Proposition 3.5. *Let $\rho, \rho' \in \mathcal{P}(\Theta)$. Then*

$$\begin{aligned} W_2^2(\rho, \rho') = \inf_{t\in[0,1]\mapsto(\gamma_t, \nabla\varphi_t)} & \int_0^1 \int_\Theta |\nabla\varphi_t(\theta)|^2 d\gamma_t(\theta) dt \\ & s.t. \partial_t\gamma_t + \operatorname{div}(\gamma_t \nabla\varphi_t) = 0, \\ & \gamma(0) = \rho, \gamma(1) = \rho'. \end{aligned} \tag{3.8}$$

The infimum in (3.8) is taken over all maps $t \in [0,1] \to (\gamma_t, \nabla\varphi_t)$, where each pair $(\gamma_t, \nabla\varphi_t)$ consists of a measure $\gamma_t \in \mathcal{P}(\Theta)$ and a vector field of the form $\nabla\varphi_t$ for a smooth $\varphi_t : \Theta \to \mathbb{R}$, that satisfy the *continuity equation*:

$$\partial_t\gamma_t + \operatorname{div}(\gamma_t \nabla\varphi_t) = 0, \tag{3.9}$$

interpreted in weak form. Notice that the Fokker–Planck equation (3.3) is a particular case of the continuity equation with $\varphi_t = V + \log(\rho_t)$. Inspired by equation (2.14) (see also Remark 2.4) we can provide a geometric interpretation of identity (3.8): the continuity equation (3.9) provides a representation of curves in the formal manifold $\mathcal{M} = \mathcal{P}(\Theta)$. In this representation, the velocity of a curve (tangent vector) at each point in the curve can be identified with a vector field (over Θ) of the form $\nabla\varphi$. Furthermore, (3.8) motivates introducing an inner product at each $\nu \in \mathcal{P}(\Theta)$ of the form:

$$g_\nu(\nabla\varphi, \nabla\varphi') := \int_\Theta \nabla\varphi(\theta) \cdot \nabla\varphi'(\theta)\, d\nu(\theta).$$

With the above geometric interpretation in place, we may follow equation (2.11) and identify the gradient of $F(\cdot) = D_{\mathrm{KL}}(\cdot\|\pi)$ at an arbitrary point ν. For this purpose take an arbitrary solution $(\gamma_t, \nabla\varphi_t)$ to the continuity equation (3.9) (i.e., take an arbitrary curve in $\mathcal{M}$) for which $F(\gamma_t) < \infty$ and compute:

$$\begin{aligned} \frac{d}{dt}F(\gamma_t) &= \frac{d}{dt}\int_\Theta \log\left(\frac{\gamma_t}{e^{-V}}\right) d\gamma_t(\theta) \\ &= \int_\Theta \nabla\varphi_t \cdot \nabla\big(V + \log(\gamma_t)\big)\, d\gamma_t(\theta) \\ &= g_{\gamma_t}\big(\nabla\varphi_t, \nabla(V + \log(\gamma_t))\big). \end{aligned}$$

In the above we have gone from the first line to the second one using the weak form of the continuity equation; to go from the second to third line we have used the definition of g_{γ_t}.

From the above computation we conclude that the gradient of F (w.r.t. to the Wasserstein metric) at a point ν for which $F(\nu) < \infty$ takes the form $\nabla\big(V + \log(\nu)\big)$. In particular, the curve in $\mathcal{M} = \mathcal{P}(\Theta)$ whose velocity vector agrees with the negative gradient of the functional F takes the form of the Fokker–Planck equation (3.3). In other words, (3.3) can be interpreted as the gradient flow of $F(\cdot) = D_{\mathrm{KL}}(\cdot\|\pi)$ w.r.t. the Wasserstein metric.

Remark 3.6. To some extent, the computations in this section have been formal and some of the above derivations have been left unjustified. These computations rely on a formal adaptation of formula (2.2) to the setting of the Riemannain manifold $\mathcal{P}(\Theta)$ endowed with the Wasserstein distance. For a rigorous treatment of the topics discussed in this section the reader is referred to the second part of the book [AGS08]. There, the notion of gradient flow in $\mathcal{P}(\Theta)$ is motivated by the dissipation identity (2.3) and adapted to the metric space $(\mathcal{P}(\Theta), W_2)$.

Remark 3.7. At a high level, the ideas discussed in this section can be used to propose flows aimed at solving variational inference problems like the ones briefly mentioned in Remark 3.3. Indeed, following the geometric intuition from projected gradient descent methods, where one uses the projection of the negative gradient of the objective onto the tangent plane of the constrained set to define the projected gradient descent flow, one may consider the projection of the negative gradient of the energy F (w.r.t Wasserstein) onto the tangent planes of the submanifold $\mathcal{G}$;

naturally, in this setting the notion of (orthogonal) projection is taken w.r.t. the Riemannain metric underlying the Wasserstein space. This idea has been recently explored in [LCB+22] for certain families $\mathcal{G}$ of tractable distributions.

3.3.2. *Geodesic convexity of the relative entropy in the Wasserstein space.* The definition of geodesic convexity introduced in (2.15) can be readily adapted to the setting of an energy defined over an arbitrary metric space, and in particular to the setting of $\mathcal{P}(\theta)$ endowed with the Wasserstein distance. Indeed, notice that equation (2.15) is completely determined by the energy of interest, the distance function, and the notion of constant speed geodesic, which in turn can be defined in terms of the distance function.

We have the following theorem by McCann relating the convexity of the function V with the α-geodesic convexity of $D_{\mathrm{KL}}(\cdot\|\pi)$ when $\pi \propto e^{-V}$; see Theorem 5.15. in [Vil03].

Theorem 3.8. *Suppose that V is α-strongly convex and let $\pi \propto e^{-V}$. Then $D_{\mathrm{KL}}(\cdot\|\pi)$ is α-geodesically convex w.r.t. the Wasserstein distance.*

With Theorem 3.8 in hand, the α-strong convexity of V implies that

$$D_{\mathrm{KL}}(\rho_t\|\pi) \leq e^{-2\alpha t} D_{\mathrm{KL}}(\rho_0\|\pi), \quad \forall t \geq 0,$$

along a solution $\{\rho_t\}_{t\geq 0}$ of the Fokker–Planck equation (3.3). This notion of exponential contraction to equilibrium is not implied by the Poincaré condition from Theorem 3.2.

3.4. **Time discretizations.** This section describes how to obtain sampling algorithms from time discretization of the Langevin dynamics (3.1). In analogy to the explicit Euler scheme (2.16) for (2.2), the Euler–Maruyama discretization for (3.1) is given by $\theta_0 \sim \rho_0$, and

$$\theta_{n+1} = \theta_n - \tau \nabla V(\theta_n) + \sqrt{2\tau}\xi_n, \quad \xi_n \overset{\text{i.i.d.}}{\sim} N(0,1). \quad (3.10)$$

For $t = n\tau$, the law ρ_n of θ_n approximates the law ρ_t of θ_t given by (3.1). However, the error introduced by time discretization causes ρ_n to not converge, in general, to π as $n \to \infty$. In other words, the probability kernel $q(\theta_n, \cdot) = \mathrm{law}(\theta_{n+1}|\theta_n)$, defined by the Markov chain (3.10), does not leave π invariant.

To remedy this issue, one may consider using (3.10) as a *proposal kernel* within a Metropolis–Hastings algorithm [Liu01], leading to the *Metropolis Adjusted Langeving Algorithm*, often referred to as MALA. The basic idea is to use an accept/reject mechanism to turn the proposal kernel q into a new Markov kernel that leaves π invariant. Given the current state θ_n, one proposes a move $\theta_n \mapsto \theta^*_{n+1}$ by sampling $q(\theta_n, \cdot)$; the move is accepted with a probability

$$a = \min\left(1, \frac{\pi(\theta^*_{n+1})}{\pi(\theta_n)} \frac{q(\theta^*_{n+1}, \theta_n)}{q(\theta_n, \theta^*_{n+1})}\right). \quad (3.11)$$

If the move is accepted, we set $\theta_{n+1} \coloneqq \theta^*_{n+1}$. Otherwise, we set $\theta_{n+1} \coloneqq \theta_n$. The Metropolis–Hastings acceptance probability (3.11) is chosen in such a way that π is the invariant distribution of the new chain $\{\theta_n\}_{n=1}^{\infty}$. Notice that the two steps of the algorithm, namely, sampling from the proposal kernel defined by (3.10) and evaluating (3.11), can be implemented without knowledge of the normalizing constant of π. We refer to [RT96] for further details on the convergence of Langevin diffusions and their discretizations and we refer to [CLGL+20] for a sampling analog of the mirror descent optimization algorithm in (2.20).

4. Modern Twists on Langevin

In this section we outline recent extensions of the gradient flows of section 3.3.1 aimed at sampling. The idea is to employ gradient flows of the energy $F = D_{\mathrm{KL}}(\cdot\|\pi)$ w.r.t. metrics beyond the Wasserstein distance. One hopes that the new dynamics lead to faster convergence to minimizers and, in turn, to more efficient sampling algorithms.

4.1. **Ensemble preconditioning.** In analogy with section 2.3, we consider preconditioned variants of the Langevin diffusion (3.1),

$$d\theta_t = -H(\theta_t)^{-1}\nabla V(\theta_t)\,dt + \sqrt{2H(\theta_t)^{-1}}\,dB_t, \quad (4.1)$$

where $H(\theta)$ is once again the preconditioning matrix field. The intuition from subsection 2.3 carries over in this setting: by choosing an appropriate preconditioner we can speed up the convergence of the Langevin diffusion to the target density. However, in contrast to (3.1), choosing H as a function of θ_t can in general lead to a nonlinear evolution for the law of the process. Moreover, without additional structure, the target $\pi \propto e^{-V}$ may not be an invariant distribution for (4.1). This motivates discussing suitable choices for H. One approach to constructing the matrix H is to consider an ensemble of particles evolving according to (4.1) and use the location of the particles to construct an appropriate preconditioner. Following [GIHLS20], consider an ensemble of $J \geq 1$ interacting Langevin diffusions $\theta_t \coloneqq \{\theta_t^{(j)}\}_{j=1}^J$ that are evolved according to the coupled system

$$d\theta_t^{(j)} = -H(\theta_t)^{-1}\nabla V(\theta_t^{(j)}) + \sqrt{2H(\theta_t)^{-1}}\,dB_t^{(j)},$$

where $\{B_t^{(j)}\}_{j=1}^J$ are i.i.d. Brownian motions, and

$$H(\theta_t) = \frac{1}{J}\sum_{j=1}^{J}\left(\theta_t^{(j)} - \bar{\theta}_t\right)\left(\theta_t^{(j)} - \bar{\theta}_t\right)^{\top},$$

for $\bar{\theta}_t$ the ensemble mean of the $\theta_t^{(j)}$. In other words, the preconditioner is chosen to be the empirical covariance matrix of the ensemble at each point in time, a quantity that is convenient to compute in practice.

As in previous sections, one can show that the ensemble preconditioned Langevin dynamics has a gradient flow structure. Taking the mean-field limit, i.e., letting $J \to \infty$, we formally obtain the following diffusion for the evolution of the ensemble

$$d\theta_t = -H(\rho_t)^{-1}\nabla V(\theta_t)\,dt + \sqrt{2H(\rho_t)^{-1}}\,dB_t,$$

where ρ_t denotes the distribution of θ_t as before and $H(\rho_t)$ is the covariance matrix of ρ_t. The Fokker–Planck equation for ρ_t is given by

$$\partial_t\rho_t = \operatorname{div}\Big(\rho_t H(\rho_t)^{-1}\big(\nabla V(\theta_t) + \nabla \log \rho_t\big)\Big), \tag{4.2}$$

which has the target $\pi \propto e^{-V}$ as a stationary point. The divergence form in (4.2) suggests a gradient flow structure as shown in [GIHLS20]. Indeed, the above equation is a gradient flow of F w.r.t. the *Kalman–Wasserstein* distance W_K defined as

$$\begin{aligned} W_K^2(\rho,\rho') &:= \inf_{\gamma_t,\varphi_t}\int_0^1\int\langle\nabla\varphi_t, H(\rho_t)\nabla\varphi_t\rangle\rho_t\,d\theta\,dt,\\ &\text{s.t. } \partial_t\gamma_t + \operatorname{div}(\gamma_t H(\gamma_t)\nabla\varphi_t) = 0,\\ &\gamma(0)=\rho,\quad \gamma(1)=\rho'. \end{aligned}$$

4.2. **Langevin with birth-death.** One of the shortcomings of the Langevin dynamics is that its convergence suffers when sampling from multimodal target densities: the process may get stuck around one of the modes and it may take a long time to cross the energy barrier between various modes or to overcome entropic bottlenecks. To ameliorate this metastable behavior [LLN19] proposes to consider the *birth-death accelerated Langevin (BDL)* dynamics

$$\begin{aligned} \partial_t\rho_t &= \mathcal{L}\rho_t + \rho_t(\log\pi - \log\rho_t)\\ &\quad - \rho_t\int_\Theta(\log\pi - \log\rho_t)d\rho_t(\theta), \end{aligned}\tag{4.3}$$

where $\mathcal{L}$ is as in (3.1). Notice that compared with the Fokker–Planck equation (3.3), equation (4.3) contains two additional terms. The second term on the right-hand side favors increasing (resp. decreasing) $\rho_t(\theta)$ whenever $\rho_t(\theta) < \pi(\theta)$ (resp. $\rho_t(\theta) > \pi(\theta)$), hence the name birth-death. The third term is only included to ensure that ρ_t remains a probability distribution through the birth-death process. The birth-death terms make the equation nonlocal (due to averaging) and allow the dynamics to explore the support of a multimodal π more efficiently: if the dynamics get stuck in one mode, the birth-death process can still transfer some mass to another mode that has not yet been thoroughly explored.

It turns out that, akin to the Langevin dynamics, BDL is also a gradient flow of the KL divergence but w.r.t. a modification of the Wasserstein distance referred to as the *Wasserstein–Fisher–Rao (or the Spherical Hellinger–Kantorovich)* distance:

$$\begin{aligned} &W_{\mathrm{FR}}^2(\rho,\rho')\\ &\quad = \inf_{\gamma_t,\varphi_t}\int_0^1\left(\int|\nabla\varphi_t|^2 + |\varphi_t|^2 d\gamma_t - \Big(\int\varphi_t d\gamma_t\Big)^2\right)dt \end{aligned}$$

subject to the constraints

$$\partial_t\gamma_t + \operatorname{div}(\gamma_t\nabla\varphi_t) = -\gamma_t\Big(\varphi_t - \int\varphi_t d\gamma_t\Big), \tag{4.4a}$$

$$\gamma(0)=\rho,\quad \gamma(1)=\rho'. \tag{4.4b}$$

As before, the BDL continuity equation (4.4a) is understood in the weak sense and plays the same role as the continuity equation in the Benamou–Brenier formulation (3.8). Equation (4.4a) thus provides an alternative representation for admissible curves in the space $\mathcal{P}(\Theta)$. Motivated by this, we will thus think of a pair $(\varphi,\nabla\varphi)$ as tangent to a given point $\nu\in\mathcal{P}(\Theta)$ and introduce the inner product at the point ν

$$\begin{aligned} g_\nu\big((\varphi,\nabla\varphi),(\varphi',\nabla\varphi')\big) &:= \int\nabla\varphi\cdot\nabla\varphi' d\nu\\ &\quad + \int\varphi\varphi' d\nu - \int\varphi d\nu\int\varphi' d\nu. \end{aligned}$$

With this geometric structure for the space $\mathcal{P}(\Theta)$ endowed with the WFR metric, we can proceed to carry out an analogous computation to the one at the end of section 3.3.1 and show that the BDL dynamics is the gradient flow of the energy functional $D_{\mathrm{KL}}(\cdot\|\pi)$ w.r.t. the W_{FR} geometry; see [LLN19, Thms. 3.2 and 3.3]. Furthermore, the rate of convergence of BDL is at least as good as that of Langevin dynamics and is asymptotically independent of the negative-log-density V, making it suitable for exploration of multimodal landscapes. However, an important caveat is that turning BDL into a sampling algorithm requires utilizing an ensemble of interacting Langevin trajectories to empirically approximate the mean-field dynamics (4.3) using a kernel density estimation, which may be forbiddingly expensive in high-dimensional settings.

5. Conclusion and Discussion

This article provided a gentle introduction to gradient flows as a unifying framework for the design and analysis of optimization and sampling algorithms. Three key elements are involved in specifying a gradient flow: the space of interest (parameter or distribution); an energy function defined on that space to be minimized; and a geometric notion of a gradient. After discussing different versions of gradient flows for optimization, we mostly focused on Langevin dynamics as a sampling analog to gradient descent and discussed its generalizations through ensemble preconditioning and addition of nonlocal terms.

The flexibility of the gradient flow framework gives significant freedom in the choice of the energy function and geometry to design new algorithms beyond the examples discussed in this article. For instance, in [AKSG19] the gradient flow of the maximum mean discrepancy (as opposed to KL) w.r.t. the Wasserstein distance is used to analyze an ensemble sampling algorithm. As another example, the *Stein Variational Gradient Descent* or SVGD algorithm [Liu17] can be viewed as a gradient flow of KL w.r.t. to a modified Wasserstein distance defined from an appropriate reproducing kernel Hilbert space.

To close, we point out that many alternative approaches that do not rely on direct reference to gradient flows can be used to enhance the convergence of Langevin dynamics and ameliorate their metastable behavior. For example, *Hamiltonian Monte Carlo* or HMC (see [Liu01] Chapter 10) utilizes a proposal kernel obtained by discretization of (deterministic) Hamilton equations as opposed to (stochastic) Langevin dynamics; theoretical and empirical evidence suggests that HMC scales favorably to high-dimensional settings. As another example, momentum methods may be used to accelerate the convergence of optimization and sampling algorithms; these methods may be interpreted as arising from time discretization of higher-order systems that approximate a gradient flow structure in certain limiting regimes [KS21]. Nonreversible variants of the Langevin diffusion can potentially achieve faster convergence (see section 4.8 in [Pav14]), while umbrella, tempering, and annealing sampling strategies (see chapter 10 in [Liu01]) enable the efficient traversing of multimodal targets. Finally, recent machine learning techniques such as normalizing flows [KPB20] offer an alternative approach to sampling and density estimation by direct parameterization of transport maps using neural networks. Employing the so called neural ODE models leads to formulations that closely resemble gradient flows and the continuity equation (3.9) with the vector field $\nabla\varphi_t$ replaced by a neural network.

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References

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. MR2401600

[AKSG19] M. Arbel, A. Korba, A. Salim, and A. Gretton, *Maximum mean discrepancy gradient flow*, Advances in Neural Information Processing Systems **32** (2019).

[Bou23] N. Boumal, *An introduction to optimization on smooth manifolds*, Cambridge University Press, Cambridge, 2023. MR4533407

[BT03] A. Beck and M. Teboulle, *Mirror descent and nonlinear projected subgradient methods for convex optimization*, Operations Research Letters **31** (2003), no. 3, 167–175.

[CLGL+20] S. Chewi, T. Le Gouic, C. Lu, T. Maunu, P. Rigollet, and A. Stromme, *Exponential ergodicity of mirror-Langevin diffusions*, Advances in Neural Information Processing Systems **33** (2020), 19573–19585.

[GCSR95] A. Gelman, J. B. Carlin, H. S. Stern, and D. B. Rubin, *Bayesian data analysis*, Texts in Statistical Science Series, Chapman & Hall, London, 1995. MR1385925

[GIHLS20] A. Garbuno-Inigo, F. Hoffmann, W. Li, and A. M. Stuart, *Interacting Langevin diffusions: gradient structure and ensemble Kalman sampler*, SIAM J. Appl. Dyn. Syst. **19** (2020), no. 1, 412–441, DOI 10.1137/19M1251655. MR4059375

[GTSA20] N. Garcia Trillos and D. Sanz-Alonso, *The Bayesian update: variational formulations and gradient flows*, Bayesian Anal. **15** (2020), no. 1, 29–56, DOI 10.1214/18-BA1137. MR4050876

[KNS16] H. Karimi, J. Nutini, and M. Schmidt, *Linear convergence of gradient and proximal-gradient methods under the Polyak-łojasiewicz condition*, Joint European Conference on Machine Learning and Knowledge Discovery in Databases, 2016, pp. 795–811.

[KPB20] I. Kobyzev, S. J. D. Prince, and M. A. Brubaker, *Normalizing flows: An introduction and review of current methods*, IEEE Transactions on Pattern Analysis and Machine Intelligence **43** (2020), no. 11, 3964–3979.

[KS21] N. B. Kovachki and A. M. Stuart, *Continuous time analysis of momentum methods*, J. Mach. Learn. Res. **22** (2021), Paper No. 17, 40, DOI 10.4995/agt.2021.13084. MR4253710

[LCB+22] M. Lambert, S. Chewi, F. Bach, S. Bonnabel, and P. Rigollet, *Variational inference via Wasserstein gradient flows*, arXiv preprint arXiv:2205.15902 (2022).

[Liu17] Q. Liu, *Stein variational gradient descent as gradient flow*, Advances in Neural Information Processing Systems **30** (2017).

[Liu01] J. S. Liu, *Monte Carlo strategies in scientific computing*, Springer Series in Statistics, Springer-Verlag, New York, 2001. MR1842342

[LLN19] Y. Lu, J. Lu, and J. Nolen, *Accelerating Langevin sampling with birth-death*, arXiv preprint arXiv:1905.09863 (2019).

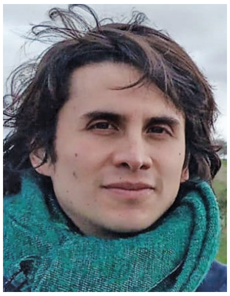
[NW99] J. Nocedal and S. Wright, *Numerical Optimization*, Vol. 35, Springer Science, 1999.

[Pav14] G. A. Pavliotis, *Stochastic processes and applications: Diffusion processes, the Fokker-Planck and Langevin equations*, Texts in Applied Mathematics, vol. 60, Springer, New York, 2014, DOI 10.1007/978-1-4939-1323-7. MR3288096

[RT96] G. O. Roberts and R. L. Tweedie, *Exponential convergence of Langevin distributions and their discrete approximations*, Bernoulli **2** (1996), no. 4, 341–363, DOI 10.2307/3318418. MR1440273

[SAST18] D. Sanz-Alonso, A. M. Stuart, and A. Taeb, *Inverse Problems and Data Assimilation*, arXiv preprint arXiv:1810.06191 (2018).

[Vil03] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003, DOI 10.1090/gsm/058. MR1964483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Banach Limits and Their Applications

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In 1904, Henri Lebesgue constructed what we now call the Lebesgue integral. It satisfies the following properties:

– linearity: for every real scalars α and β it holds $\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$;

– positivity: $\int_a^b f(x)\,dx \geq 0$ if $f(x) \geq 0$ for all $a \leq x \leq b$;

– normalization: $\int_a^b 1\,dx = b - a$;

– shift-invariance: $\int_a^b f(x)\,dx = \int_{a+h}^{b+h} f(x-h)\,dx$;

– additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$;

– normality: $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$, $n \to \infty$ provided that $f_n \uparrow f$ monotonically as $n \to \infty$.

Lebesgue himself raised the question of whether the normality can be deduced from the other properties listed above. This question was answered in the negative by Stefan Banach in 1923. He discovered the way to associate with any bounded real-valued function f on (a, b) a real number such that for any Riemann integrable function it coincides with its Riemann integral. However, this number does not necessarily coincide with the Lebesgue integral of the Lebesgue integrable function. Banach's example was based on transfinite induction. Later on, he developed a more systematic approach to this construction based on the Hahn–Banach extension theorem [Ban93]. Employing this approach, he not only constructed a nonnormal integral, but "nonnormal" limits on bounded functions and those on sequences as well. The latter ones (now called Banach limits) are the subject of this note.

Banach Limits

Let us denote by ℓ_∞ the space of all real-valued bounded sequences $x = (x_0, x_1, ...)$ with the uniform norm

$$\|x\|_{\ell_\infty} := \sup_{n\in\mathbb{N}} |x_n|,$$

where $\mathbb{N} = \{0, 1, 2, ...\}$. We denote by c_0 the space of all vanishing sequences, i.e.,

$$c_0 = \{x \in \ell_\infty : x_n \to 0\}.$$

We also denote by c the space of all convergent sequences in ℓ_∞.

Definition. A linear functional B on ℓ_∞ is said to be a Banach limit if it is

(i) positive: $Bx \geq 0$ for $x \geq 0$,
(ii) normalized: $B\mathbb{1} = 1$, where $\mathbb{1} = (1, 1, ...)$,
(iii) shift-invariant: $BTx = Bx$ for all $x \in \ell_\infty$, where T is a left-shift operator defined as follows

$$T(x_0, x_1, ...) = (x_1, x_2, ...).$$

We denote the set of all Banach limits by $\mathfrak{B}$. For every $x \in \ell_\infty$ it follows from the definition that

$$\inf_{n\geq m} x_n \leq Bx \leq \sup_{n\geq m} x_n \text{ for all } m \in \mathbb{N}.$$

Thus,

$$\lim_{m\to\infty} \inf_{n\geq m} x_n = \liminf_{n\to\infty} x_n \leq Bx \leq \limsup_{n\to\infty} x_n = \lim_{m\to\infty} \sup_{n\geq m} x_n$$

for every $x \in \ell_\infty$ and every Banach limit B. In particular, $Bx = \lim_{n\to\infty} x_n$ for any $x \in c$ and the set $\mathfrak{B}$ is the closed convex subset of the unit sphere of the dual space ℓ_∞^* — the space of all bounded linear functionals on ℓ_∞. Also, one has $y_n := (\underbrace{1, ..., 1}_{n}, 0, 0, ...) \uparrow \mathbb{1}$ monotonically as $n \to \infty$ and (by definition) for every $n \in \mathbb{N}$ and every $B \in \mathfrak{B}$ one has $By_n = 0$ while $B\mathbb{1} = 1$. Thus, Banach limits are nonnormal functionals.

A positive functional γ on ℓ_∞ that coincides with the usual limit on the subspace of convergent sequences (i.e., $\gamma(x) = \lim_{n\to\infty} x_n$ for any $x \in c$) is said to be an *extended limit*. Thus, Banach limits are extended limits. Extended limits indeed extend a limit functional defined on the space of all convergent sequences to the space of all bounded sequences. Like the limit superior and limit inferior, extended limits can be applied in situations where one wants to algebraically manipulate with limit equations or inequalities, even when it is not assured beforehand that the limit (in the classical sense) exists.

Ultrafilters. There is another approach to extended limits, which is based upon ultrafilters. One sees it more frequently in set-theoretical expositions. It is worth to mention that ultrafilters in their turn are nonconstructive objects. Their existence is based on Zorn's lemma.

Definition. A collection $\mathcal{F}$ of subsets of the set X is said to be a filter if it is:

(i) upward closed: every subset of X, containing a set from $\mathcal{F}$, belongs to $\mathcal{F}$;
(ii) closed under intersections: if two subsets of X belong to $\mathcal{F}$, then so does their intersection;
(iii) nondegenerate: the empty set of X does not belong to $\mathcal{F}$.

Example. (i) The collection of all subsets of $X \neq \emptyset$, containing a fixed nonempty subset $A \subset X$, is a filter. Such filters are called principal;
(ii) The collection of all neighborhoods of a nonempty subset of a topological space is a filter;
(iii) Let X be an infinite set. The complements to finite subsets of X form a filter. Such a filter is called a Fréchet filter.

Definition. A filter $\mathcal{F}$ on the set X is said to be an ultrafilter if there is no filter $\mathcal{F}'$ on X, such that $\mathcal{F} \subsetneq \mathcal{F}'$.

A filter is said to be free if the intersection of all its elements is empty. For example, the Fréchet filter is free. Note that an ultrafilter on an infinite set X is free if and only if it contains the Fréchet filter on a set X.

Definition. Let f be a mapping from the set X to a topological space Y and let $\mathcal{F}$ be a filter on the set X. An element $y \in Y$ is said to be a limit of f along the filter $\mathcal{F}$ (we write $y = \lim_{\mathcal{F}} f$) if $f^{-1}(V) \in \mathcal{F}$ for every neighborhood V of y.

Note, that each $x \in \ell_\infty$ can be considered as a map $x : \mathbb{N} \to \mathbb{R}$.

Proposition. *The limit* $\lim_{\mathcal{F}}$ *along any free ultrafilter* $\mathcal{F}$ *on* $\mathbb{N}$ *is a multiplicative extended limit on* ℓ_∞*, i.e.,*

(i) *for each* $x \in \ell_\infty$ *the expression* $\lim_{\mathcal{F}} x$ *exists;*

(ii) $\lim_{\mathcal{F}} x = \lim_{n\to\infty} x_n$ *for any* $x \in c$;

(iii) $\lim_{\mathcal{F}} x \cdot \lim_{\mathcal{F}} y = \lim_{\mathcal{F}}(x \cdot y)$ *for every* $x, y \in \ell_\infty$, *where* $x \cdot y = (x_n \cdot y_n)_{n\geq 0}$.

Thus, both Banach limits and limits along free ultrafilters are extended limits on ℓ_∞. However, these two classes of extended limits are distinct, since shift-invariance and multiplicativity of a functional are mutually exclusive properties. Indeed, for $x = (0,1,0,1,\dots)$ and any Banach limit B we have $Bx \cdot BTx = (Bx)^2 = (\frac{Bx+BTx}{2})^2 = \frac{(B\mathbb{1})^2}{4} = 1/4$ and $B(x \cdot Tx) = 0$. Thus, the identity $B(x \cdot y) \neq Bx \cdot By$ does not hold in general and, so, Banach limits are not multiplicative.

The set EL of all extended limits is convex, i.e., for any $\gamma, \omega \in EL$ the element $t\gamma + (1-t)\omega$ belongs to EL for every $t \in [0,1]$. An element $\gamma \in EL$ is said to be an *extreme point* if γ does not lie in any open line segment joining two points of EL. The set of Banach limits does not contain any extreme points of the set of all extended limits. Moreover, one can prove the following result.

Proposition. *For every extreme point* $\gamma \in EL$ *and every* $B \in \mathfrak{B}$ *one has* $\|\gamma - B\|_{\ell_\infty^*} = 2$.

One can construct Banach limits using ultrafilters. Consider the Cesàro operator $C : \ell_\infty \to \ell_\infty$ given by the formula

$$(Cx)_n = \frac{1}{n+1}\sum_{k=0}^{n} x_k,\ n \in \mathbb{N}.$$

Proposition. *The functional*

$$l(x) := \lim_{\mathcal{F}} Cx,\ x \in \ell_\infty$$

is a Banach limit for every free ultrafilter $\mathcal{F}$ *on* $\mathbb{N}$.

Note that the construction of the above proposition yields only Banach limits of a special type. They are termed *factorizable*. The set of all factorizable Banach limits is a proper subset of the set of all Banach limits. It can be easily seen from Example (iv) below.

There is a construction of Banach limits via the tools of nonstandard analysis. This discussion is beyond the scope of this note. For more details we refer the reader to [Lux92].

Almost convergence. It follows from the Hahn–Banach theorem that there are infinitely many Banach limits. G. Lorentz [Lor48] introduced a special class of sequences where all Banach limits agree.

Definition. A sequence $x \in \ell_\infty$ is said to be almost convergent (to $a \in \mathbb{R}$) if $Bx = a$ for every Banach limit B.

The linear space of all almost convergent sequences is denoted by ac. The following characterisation of the space ac was proved by G. Lorentz [Lor48].

Theorem. *A sequence* $x \in \ell_\infty$ *is almost convergent to* $a \in \mathbb{R}$ *if and only if*

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=m}^{m+n} x_k = a$$

uniformly in $m \in \mathbb{N}$, *i.e.,*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N$$
$$\left[\left|\frac{1}{n+1}\sum_{k=m}^{m+n} x_k - a\right| < \varepsilon\right]\ \forall m \in \mathbb{N}.$$

The Lorentz theorem implies that every almost convergent sequence (to $a \in \mathbb{R}$) is Cesàro convergent, in other words $(Cx)_n \to a$. The converse statement does not hold in general (see, e.g., Example (iv) below).

The previous theorem was strengthened by L. Sucheston [Suc67] as follows.

Theorem. *For every* $x \in \ell_\infty$ *we have*

$$\{Bx : B \in \mathfrak{B}\} = [q(x), p(x)],$$

where

$$q(x) = \lim_{n\to\infty} \inf_{m\in\mathbb{N}} \frac{1}{n+1}\sum_{k=m}^{m+n} x_k,$$

$$p(x) = \lim_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{1}{n+1}\sum_{k=m}^{m+n} x_k.$$

Thus, $x \in ac$ *if and only if* $p(x) = q(x)$.

We present several examples to clarify the notion of almost convergence.

Example. (i) A sequence $x_n = (-1)^n$, $n \geq 0$ is almost convergent to zero. Moreover, if x is a periodic sequence and its period is $r > 0$, then x is almost convergent to a number $\frac{1}{r}\sum_{n=0}^{r-1} x_n$.

(ii) The sequence $\{\sin(nt)\}_{n\geq 0}$ is almost convergent to zero for every $t \in \mathbb{R}$.

(iii) Consider the Rademacher system given by

$$\{r_n(t) = \operatorname{sign}\sin(2^n \pi t)\}_{n\geq 0},\ t \in [0,1].$$

Using probabilistic methods it can be shown that $q(\{r_n(t)\}_{n\geq 0}) = -1$ and $p(\{r_n(t)\}_{n\geq 0}) = 1$ for almost every $t \in [0,1]$ (with respect to Lebesgue measure on $[0,1]$). Hence, the sequence $\{r_n(t)\}_{n\geq 0}$ is not almost convergent for almost every t. On the other hand, for every dyadic rational t (that is, $t = i2^{-j}$, $0 \leq i \leq 2^j$, $j \in \mathbb{N}$) the sequence $\{r_n(t)\}_{n\geq 0}$ is eventually zero and, so, trivially is almost convergent to zero.

(iv) Consider the sequence $x = \{x_n\}_{n\geq 0}$ defined as follows:

$$x_n = \begin{cases} 1, & 2^k \leq n \leq 2^k + k, k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

A direct verification shows that the sequence Cx is convergent to zero. Since limits along free ultrafilters are

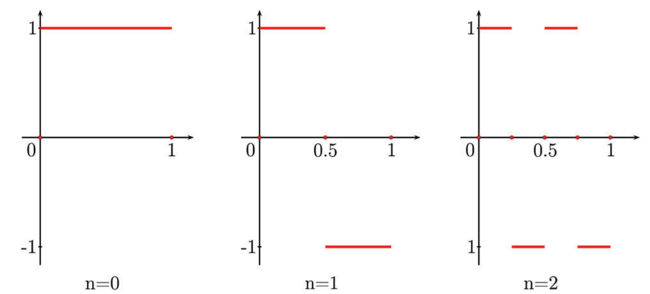

Figure 1. The first three Rademacher functions.

extended limits, it follows that every factorizable Banach limit vanishes on x. However, $p(x) = 1$ and $q(x) = 0$ and, so, $\{Bx : B \in \mathfrak{B}\} = [0, 1]$.

The space ac possesses very interesting properties. In particular, it is neither separable nor dense in ℓ_∞. In 2006, E. Alekhno proved that the space ac is not complemented in ℓ_∞.

Invariant Banach limits. Every Banach limit is shift-invariant by definition. In 1950, W. Eberlein noticed that there are some Banach limits that are additionally invariant with respect to regular Hausdorff transformations. Later, in 1953, his result was extended by R. Cooke to a wider class of matrix transformations. More recently, motivated by the theory of singular traces these results were even further extended [SS10].

Denote by Γ the set of all positive (i.e., $Hx \geq 0$ provided that $x \geq 0$) and regular (i.e., H preserves c_0 and $H\mathbb{1} = \mathbb{1}$) linear operators H in ℓ_∞ satisfying the following condition: $\limsup_{j\to\infty}(A(I - T)x)_j \geq 0$ for all $x \in \ell_\infty$, $A \in R$, $R = R(H) = \mathrm{conv}\{H^n, n \in \mathbb{N}\}$, where conv stands for the convex hull of a set.

The set Γ is nonempty. Indeed, the left-shift and Cesàro operators defined above and dilation operators

$$\sigma_n(x_0, x_1, \dots) = (\underbrace{x_0, x_0, \dots, x_0}_{n}, \underbrace{x_1, x_1, \dots, x_1}_{n}, \dots),$$

$n \in \mathbb{N}, n \geq 2$ satisfy all the conditions mentioned above; that is $T, C, \sigma_n \in \Gamma$ for every $n \in \mathbb{N}, n \geq 2$. We denote the set of H-invariant Banach limits by $\mathfrak{B}(H) = \{B \in \mathfrak{B} : B \circ H = B\}$. Following this notational agreement, we should be denoting the set $\mathfrak{B}$ as $\mathfrak{B}(T)$, however, we will simply write $\mathfrak{B}$ as before. The interest to invariant Banach limits is in particular motivated by the theory of singular traces, where they are employed to study subclasses of Dixmier traces.

Diameter and radius of the set $\mathfrak{B}$ in ℓ_∞^* equal 2, that is

$$d(\mathfrak{B}, \ell_\infty^*) = \sup_{B_1, B_2 \in \mathfrak{B}} \|B_1 - B_2\|_{\ell_\infty^*} = 2,$$
$$r(\mathfrak{B}, \ell_\infty^*) = \inf_{B_1 \in \mathfrak{B}} \sup_{B_2 \in \mathfrak{B}} \|B_1 - B_2\|_{\ell_\infty^*} = 2.$$

Moreover, for every $B_1 \in \mathfrak{B}$ there exists $B_2 \in \mathfrak{B}$ such that $\|B_1 - B_2\|_{\ell_\infty^*} = 2$. In turn, the sets $\mathfrak{B}(C)$ and $\mathfrak{B}(\sigma_n)$ are maximally removed from the set $\mathrm{ext}\,\mathfrak{B}$ of extreme points of $\mathfrak{B}$. More precisely we have the following result.

Theorem. *For every $n \in \mathbb{N}$, $n \geq 2$ and every $B \in \mathrm{ext}\,\mathfrak{B}$, $B_1 \in \mathfrak{B}(C)$ and $B_2 \in \mathfrak{B}(\sigma_n)$ one has*

$$\|B - B_1\|_{\ell_\infty^*} = \|B - B_2\|_{\ell_\infty^*} = 2.$$

It was shown by C. Chou that the cardinality of the set $\mathrm{ext}\,\mathfrak{B}$ is maximal and equals $2^{\mathfrak{c}}$. Despite the fact that sets $\mathfrak{B}(C)$ and $\mathfrak{B}(\sigma_n)$ are far from the set $\mathrm{ext}\,\mathfrak{B}$, the same result holds for their extreme points.

Proposition. *The cardinality of the sets $\mathrm{ext}\,\mathfrak{B}(C)$ and $\mathrm{ext}\,\mathfrak{B}(\sigma_n)$, $n \geq 2$ equals $2^{\mathfrak{c}}$.*

The interrelation between the sets $\mathfrak{B}(C)$ and $\mathfrak{B}(\sigma_n)$ are of particular interest. Although there is no a priori reason for this, the authors, jointly with D. Zanin, proved that the inclusion $\mathfrak{B}(C) \subset \mathfrak{B}(\sigma_n)$ holds for every $n \in \mathbb{N}$, $n \geq 2$. The key point of the proof is so-called marriage theorem from graph theory. On the other hand, there exist Banach limits that are invariant with respect to all dilation operators, but not Cesàro invariant. All the sets $\mathfrak{B}(\sigma_n)$, $n \in \mathbb{N}$ are distinct. One has $\mathfrak{B}(\sigma_n) \subset \mathfrak{B}(\sigma_{n^k})$ for every $n, k \in \mathbb{N}$, $n, k \geq 2$. This inclusion is proper. There are no other inclusions in the class $\mathfrak{B}(\sigma_n)$. Indeed, for every $n \in \mathbb{N}$, $n \geq 2$ there exists $B \in \mathfrak{B}(\sigma_n)$ such that $B \notin \mathfrak{B}(\sigma_m)$ for every $m \in \mathbb{N} \setminus \{n, n^2, n^3, \dots\}$.

It is conjectured that for every $H \in \Gamma$ the distance from any $B \in \mathfrak{B}(H)$ to the set $\mathrm{ext}\,\mathfrak{B}$ equals 2 and the cardinality of the set $\mathrm{ext}\,\mathfrak{B}(H)$ equals $2^{\mathfrak{c}}$.

Geometry and topology of $\mathfrak{B}$. Consider the space ℓ_∞^* equipped with weak* topology. Since the convergence in weak* topology is pointwise, it follows from the definition that the set $\mathfrak{B}$ is convex and weak* closed. Moreover, it follows from the Krein–Millman theorem that

$$\mathfrak{B} = \overline{\mathrm{conv}}^{w^*}\,\mathrm{ext}\,\mathfrak{B},$$

that is the set $\mathfrak{B}$ coincide with the closure of the convex hull of extreme points of Banach limits in the weak* topology. On the other hand, as shown by R. Nillsen the set $\mathrm{ext}\,\mathfrak{B}$ is not closed in the weak* topology [Nil76]. M. Talagrand established a stronger version of this result: the closure of $\mathrm{ext}\,\mathfrak{B}$ in the weak* topology contains no G_δ subsets of $\mathfrak{B}$ [Tal76]. It is conjectured that the analogues of Nillsen's and Talagrand's results hold for $\mathrm{ext}\,\mathfrak{B}(H)$ for every $H \in \Gamma$.

Every countable family $\{B_i\}_{n\in\mathbb{N}}$ of distinct elements of $\mathrm{ext}\,\mathfrak{B}$ spans a subspace isometric to ℓ_1, that is

$$\left\| \sum_{n=0}^{\infty} x_i B_i \right\|_{\ell_\infty^*} = \sum_{n=0}^{\infty} |x_i|,$$

provided that $x = (x_0, x_1, \dots) \in \ell_1$ [SS13, Proposition 3]. Informally speaking, this means that the set of Banach limits is a simplex of dimension $2^{\mathfrak{c}}$ in the unit sphere of ℓ_∞^*.

Given $B \in \mathfrak{B}$ and $x \in \ell_\infty$ define the function

$$F_{B,x}(t) = B\chi_{\{n \in \mathbb{N}:\, x_n \le t\}},\ t \in \mathbb{R},$$

where χ_A denote a characteristic function of a set $A \subset \mathbb{N}$. This function is termed the distribution function (of B and x) [Ale15]. The function $t \mapsto F_{B,x}(t)$ is nondecreasing in t and, so, is continuous except possibly countably many points. The following result relates the continuity of the distribution function to the extremality of a Banach limit [Ale15, §5.2].

Theorem. *For every $B \in \mathfrak{B}$ the following assertions are equivalent:*

(i) *there exists $x \in \ell_\infty$ such that the function $F_{B,x}$ is continuous;*

(ii) *for every $n \in \mathbb{N}$ there exists a partition $\mathbb{N} = N_1 \cup \cdots \cup N_n$ such that $B(x\chi_{N_k}) = \frac{1}{n}$ for every $k = 1, \dots, n$;*

(iii) *for every $A \subset \mathbb{N}$ and $\lambda \in [0, B(\chi_A)]$ there exists $A_1 \subset A$ such that $B(x\chi_{A_1}) = \lambda$.*

Any of these conditions implies that B is not an extreme point.

Applications of Banach Limits

In this section we discuss several applications of Banach limits to analysis. There are plenty of other areas that are out of scope of this note. We mention some of them: orthogonal series, number theory, ergodic theory, probability and study of the Navier–Stokes equation.

Singular traces. One of the most important characteristics of a (square) matrix is its trace, given by the sum of diagonal elements. A natural extension of matrices is the operators on separable Hilbert spaces. The matrix trace also extends to this case in the following way:

$$\mathrm{Tr}(A) = \sum_{n=0}^{\infty} \langle Ae_n, e_n \rangle,$$

where A is a compact linear operator acting on a Hilbert space and $\{e_n\}_{n\ge0}$ is any orthonormal basis of the Hilbert space (note that $\mathrm{Tr}(A)$ is finite only for some compact operators A). This standard trace is a unitarily invariant functional, that is $\mathrm{Tr}(A) = \mathrm{Tr}(UAU^*)$ for every unitary operator U.

Equivalently, this trace can be evaluated as a sum of eigenvalues:

$$\mathrm{Tr}(A) = \sum_{n=0}^{\infty} \lambda_n(A),$$

where A is a compact linear operator acting on a Hilbert space and the eigenvalues are counted according to their multiplicities. In the case of matrix trace this fact is a straightforward consequence of the unitary invariance and the existence of the Jordan form of A. The general formula is due to Victor Lidskii.

The trace class $\mathcal{L}_1$ is the collection of all operators on a Hilbert space for which the standard trace is finite. The standard trace is *normal* in the following sense: $\mathrm{Tr}(A) = \sup_{i\in I} \mathrm{Tr}(A_i)$ for every net of positive operators $\{A_i \in \mathcal{L}_1\}_{i \in I}$ such that $A_i \uparrow A$, $i \in I$.

There are other nontrivial traces (that is, linear unitarily invariant functionals) on $\mathcal{L}_1$. However, all of them are proportional to the standard trace and, thus, are normal. The longstanding problem was whether nonnormal traces exist. It was answered by Jacques Dixmier in 1966. We explain his construction in the slightly simpler form than the original one.

Consider the algebra $B(H)$ of all bounded linear operators on a Hilbert space H and the ideal $\mathcal{L}_{1,\infty}$ of all compact operators in $B(H)$,

$$\mathcal{L}_{1,\infty} := \{A \in B(H) :\ s_n(A) = O(\frac{1}{n+1}),\ n \to \infty\}.$$

Here, $\{s_n(A)\}_{n\ge0}$ stands for the sequence of singular values of a compact operator $A \in B(H)$, that is the eigenvalues of $|A| := (A^*A)^{1/2}$ taken in the nonincreasing order counted according to their multiplicities. For $A \in \mathcal{L}_{1,\infty}$, the sequence

$$\left\{\frac{1}{\log(2+n)} \sum_{k=0}^{n} s_k(A)\right\}_{n\ge0}$$

belongs to ℓ_∞.

Further, for an arbitrary extended limit ω on ℓ_∞ J. Dixmier constructed the functional

$$\mathrm{Tr}_\omega(A) := \omega\left(\left\{\frac{1}{\log(2+n)} \sum_{k=0}^{n} s_k(A)\right\}_{n\ge0}\right),$$

$0 \le A \in \mathcal{L}_{1,\infty}$. He showed that this functional is positively homogeneous and additive on the positive cone of $\mathcal{L}_{1,\infty}$. Thus, Tr_ω can be extended by linearity to a linear functional on $\mathcal{L}_{1,\infty}$. Let us briefly list several properties of this extended functional.

(i) Tr_ω is a trace, since singular values are unitarily invariant; that is $s_k(A) = s_k(UAU^*)$ for every $k = 0, 1, \dots$ and for every unitary operator U;

(ii) Tr_ω is nontrivial: $\mathrm{Tr}_\omega(\mathrm{diag}\left\{\frac{1}{n+1}\right\}_{n\ge0}) = 1$;

(iii) $\mathrm{Tr}_\omega(A) = 0$ for every finite rank operator A.

Therefore, Tr_ω is a nontrivial trace on $B(H)$ which is not a scalar multiple of the standard trace. Moreover, it fails to be normal (this can be deduced from (ii) and (iii) above). That was the first example of singular (that is, nonnormal) trace. For the next couple of decades the singular (Dixmier) traces were nothing but "pathological monsters."[1] The theory experienced its rebirth some thirty years ago, when Alain Connes made singular traces a cornerstone of his quantized calculus. In particular, he used singular traces to extend the classical Yang–Mills action,

[1] *J. Dixmier's letter to the conference "Singular traces and their applications" at Luminy, 2012 (see* [LSZ13]*).*

M. Wodzicki's noncommutative residue and an integral of differential forms.

The Dixmier consruction outlined above, has some essential drawbacks. First, it produces traces of a very specific type, rather than all singular traces on $\mathcal{L}_{1,\infty}$. Second, distinct extended limits can generate the same trace. Hence, this construction is neither surjective nor injective.

The work of A. Pietsch in 80th helped to remove this disadvantage. He established a bijective correspondence between traces on double-sided operator ideals and shift-invariant functionals on sequence spaces. Recently, this approach was further developed by the authors and their colleagues. It enabled certain advances in the theory of singular traces and its applications in noncommutative geometry and analysis.

We describe the Pietsch construction in the case of the ideal $\mathcal{L}_{1,\infty}$. Consider the mapping

$$P : \ell_\infty \to \mathcal{L}_{1,\infty}$$

defined as follows:

$$Px = \operatorname{diag}\Big(\frac{x_0}{2^0}, \underbrace{\frac{x_1}{2^1}, \frac{x_1}{2^1}}_{2^1}, \underbrace{\frac{x_2}{2^2}, \dots, \frac{x_2}{2^2}}_{2^2}, \dots, \underbrace{\frac{x_n}{2^n}, \dots, \frac{x_n}{2^n}}_{2^n}, \dots\Big).$$

Theorem. *In the notations above, we have*

(i) *for every trace $\tau : \mathcal{L}_{1,\infty} \to \mathbb{C}$, the linear functional $\theta_\tau = \tau \circ P : \ell_\infty \to \mathbb{C}$ is left-shift-invariant;*

(ii) *for every left-shift-invariant functional $\theta : \ell_\infty \to \mathbb{C}$, the mapping*

$$A \to \theta\Big(\Big\{\sum_{k=2^n-1}^{2^{n+1}-2} s_k(A)\Big\}_{n\geq 0}\Big), \quad 0 \leq A \in \mathcal{L}_{1,\infty},$$

is positively homogeneous and additive on the positive cone of $\mathcal{L}_{1,\infty}$ and, so, extends to a trace on $\mathcal{L}_{1,\infty}$. We denote it by τ_θ;

(iii) *the correspondences $\tau \to \theta_\tau$ and $\theta \to \tau_\theta$ are bijective and are inverses of each other.*

The following formula defines a quasi-norm on the ideal $\mathcal{L}_{1,\infty}$:

$$\|A\|_{1,\infty} = \sup_{n\geq 0}(n+1)\cdot s_n(A).$$

When we specialize the above theorem to the case of bounded traces on the quasi-normed ideal $(\mathcal{L}_{1,\infty}, \|\cdot\|_{1,\infty})$, we obtain that a trace τ on $\mathcal{L}_{1,\infty}$ is bounded if and only if the corresponding functional θ_τ is bounded on ℓ_∞.

A trace τ on $\mathcal{L}_{1,\infty}$ is said to be normalized if $\tau(A) = 1$ for every $0 \leq A \in \mathcal{L}_{1,\infty}$ such that $s_n(A) = \frac{1}{n+1}$, $n \geq 0$.

Theorem. *A trace τ on $\mathcal{L}_{1,\infty}$ is positive and normalized if and only if the corresponding functional θ_τ is a Banach limit on ℓ_∞.*

This surprising interrelation led to the resolution of a number of problems in the theory of singular traces on $\mathcal{L}_{1,\infty}$. In particular, by observing that the cardinality of the set of Banach limits is $2^{\mathfrak{c}}$, Pietsch has inferred that the cardinality of the set of all positive normalized traces on $\mathcal{L}_{1,\infty}$ is $2^{\mathfrak{c}}$ (see [Pie14, Theorem 9.6]). Further, an elegant description of the set of all Dixmier traces and its various subclasses in terms of invariant Banach limits were obtained in [SSUZ15, Section 5]. Interesting applications of singular traces, defined by extended limits with nontrivial invariances, to fractals can be found in [CMSZ19] and [KSZ17].

Operator theory. Using Banach limits one constructs operators in nonseparable Banach spaces with unusual properties.

Consider the space $L_{p,\infty}$, $1 < p < \infty$ of all integrable functions on $[0,1]$, such that the norm

$$\|x\|_{L_{p,\infty}} = \sup_{e\subset[0,1],\text{mes}e>0} (\text{mes } e)^{\frac{1}{p}-1} \int_e |x(s)|\,ds$$

is finite.

For $n \in \mathbb{N}$ and $u \in (0,1)$, let $I(u,n) := (u, u+2^{-n})$ if $u < 1-2^{-n}$ and $I(u,n) := \emptyset$ otherwise. For every Banach limit B and every $x \in L_{p,\infty}$, consider the following expression:

$$(A_B x)(u) = B\left\{2^{\frac{p-1}{p}n}\int_{I(u,n)} x(t)\,dt\right\}_{n\geq 0}, \qquad (1)$$

$0 < u < 1$.

Consider the subspace $Q = \overline{\operatorname{span}}\{z_k : k \in \mathbb{N}\} \subset L_{p,\infty}$, where

$$z_k(t) = \begin{cases} \frac{p-1}{p}(t-2^{-k})^{-\frac{1}{p}}, \ 2^{-k} < t < 2^{-k+1}, \\ 0, \ \text{otherwise.}\end{cases}$$

Consider the space $\ell_p(0,1)$ of all functions on $(0,1)$, which take at most countable number of nonzero values, equipped with the norm

$$\|y\|_{\ell_p(0,1)} = \left(\sum_{t\in(0,1)} |y(t)|^p\right)^{1/p}.$$

Theorem ([SS85]). *Let $p \in (1,\infty)$. Formula* (1) *defines a bounded norm-one operator from $L_{p,\infty}$ to $\ell_p(0,1)$. Its restriction to the subspace Q is an isometry.*

In [FHST12] this operator was used to construct a bounded in $L_{p,\infty}$, $p \neq 2$ strictly singular operator R such that R^3 is noncompact.

Orthogonal series. The notion of almost convergence is a useful tool to describe the asymptotic behavior of bounded, but divergent sequences. Here, we discuss asymptotics of Fourier–Haar coefficients of functions from $L_{p,\infty}$ spaces introduced in the above subsection.

Set $\Delta_n^k = (\frac{k-1}{2^n}, \frac{k}{2^n})$ and $\bar{\Delta}_n^k = [\frac{k-1}{2^n}, \frac{k}{2^n}]$ for $n = 0, 1, \dots$, $1 \leq k \leq 2^n$ and define the Haar system to be the collection of functions $\chi_{0,0} \equiv 1$,

$$\chi_{n,k}(t) = \begin{cases} 2^{n/2}, & t \in \Delta_{n+1}^{2k-1}, \\ -2^{n/2}, & t \in \Delta_{n+1}^{2k}, \\ 0, & t \notin \bar{\Delta}_{n+1}^{2k-1} \cup \bar{\Delta}_{n+1}^{2k}. \end{cases}$$

The values at the endpoints of $[0,1]$ are chosen by continuity and those at the points of discontinuity are set to be zero. Sometimes it is more convenient to consider the single-index Haar system $\{\chi_m\}_{m=1}^{\infty}$, where $m = 2^n + k$.

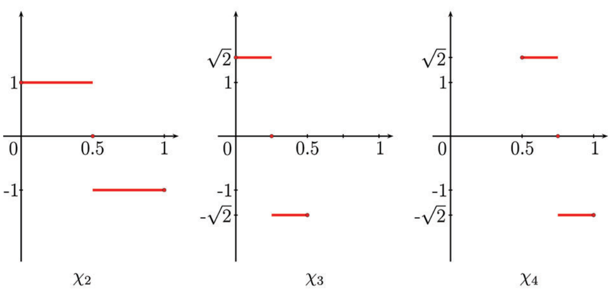

Figure 2. The 2nd, 3rd, and 4th Haar functions.

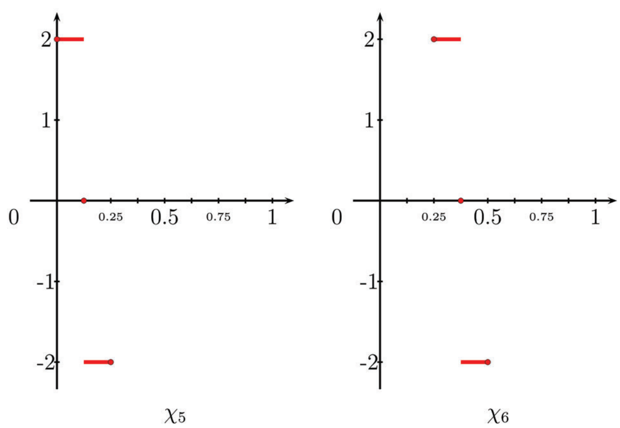

Figure 3. The 5th and 6th Haar functions.

For every function $x \in L_1$ we consider the sequence

$$c_m(x) = \int_0^1 x(s)\chi_m(s)\,ds, \quad m \geq 1$$

of its Fourier coefficients with respect to the Haar system. For $1 < p < \infty$ and $x \in L_{p,\infty}$ the sequence $\{c_m(x)m^{\frac{1}{2}-\frac{1}{p}}\}_{m\geq 1}$ is bounded, but not necessarily convergent. This naturally raises the question of whether this sequence is almost convergent. In 2009, the first and the third author showed that the sequence $\{c_m(x)m^{\frac{1}{2}-\frac{1}{p}}\}_{m\geq 1}$ is almost convergent to zero. In particular, if $x \in L_{2,\infty}$, then $\{c_m(x)\}_{m\geq 1}$ is almost convergent to zero. This result can be considered as an analogue of the classical Mercer theorem (see, e.g., [Bar64, Chapter I, §19]) for the Haar system and the space $L_{2,\infty}$.

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References

[Ale15] Egor A. Alekhno, *On Banach-Mazur limits*, Indag. Math. (N.S.) **26** (2015), no. 4, 581–614, DOI 10.1016/j.indag.2015.04.001. MR3371963

[Ban93] Stefan Banach, *Théorie des opérations linéaires* (French), Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original. MR1357166

[Bar64] N. K. Bary, *A treatise on trigonometric series. Vols. I, II*, A Pergamon Press Book, The Macmillan Company, New York, 1964. Authorized translation by Margaret F. Mullins. MR0171116

[CMSZ19] A. Connes, E. McDonald, F. Sukochev, and D. Zanin, *Conformal trace theorem for Julia sets of quadratic polynomials*, Ergodic Theory Dynam. Systems **39** (2019), no. 9, 2481–2506. MR3989126

[FHST12] Julio Flores, Francisco L. Hernández, Evgueni M. Semenov, and Pedro Tradacete, *Strictly singular and power-compact operators on Banach lattices*, Israel J. Math. **188** (2012), 323–352, DOI 10.1007/s11856-011-0152-z. MR2897735

[KSZ17] A. Konn, F. A. Sukochev, and D. V. Zanin, *Trace theorem for quasi-Fuchsian groups*, Mat. Sb. **208** (2017), no. 10, 59–90. MR3706885

[Lor48] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190. MR27868

[LSZ13] Steven Lord, Fedor Sukochev, and Dmitriy Zanin, *Singular traces*, De Gruyter Studies in Mathematics, vol. 46, De Gruyter, Berlin, 2013. Theory and applications. MR3099777

[Lux92] W. A. J. Luxemburg, *Nonstandard hulls, generalized limits and almost convergence*, Analysis and geometry, Bibliographisches Inst., Mannheim, 1992, pp. 19–45. MR1274473

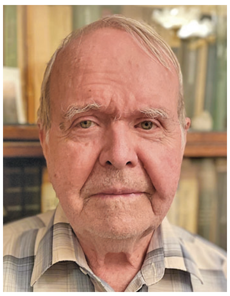
[Nil76] Rodney Nillsen, *Nets of extreme Banach limits*, Proc. Amer. Math. Soc. **55** (1976), no. 2, 347–352. MR407530

[Pie14] Albrecht Pietsch, *Traces on operator ideals and related linear forms on sequence ideals (part II)*, Integral Equations Operator Theory **79** (2014), no. 2, 255–299, DOI 10.1007/s00020-013-2114-9. MR3206148

[SS10] Evgenii M. Semenov and Fedor A. Sukochev, *Invariant Banach limits and applications*, J. Funct. Anal. **259** (2010), no. 6, 1517–1541, DOI 10.1016/j.jfa.2010.05.011. MR2659770

[SS13] E. Semenov and F. Sukochev, *Extreme points of the set of Banach limits*, Positivity **17** (2013), no. 1, 163–170. MR3027652

[SS85] E. M. Semënov and A. M. Shteĭnberg, *Estimates of norms of operator blocks in Banach lattices* (Russian), Mat. Sb. (N.S.) **126(168)** (1985), no. 3, 327–343, 431. MR783950
[SSUZ15] Evgeniy Semenov, Fedor Sukochev, Alexandr Usachev, and Dmitriy Zanin, *Banach limits and traces on* $\mathcal{L}_{1,\infty}$, Adv. Math. **285** (2015), 568–628, DOI 10.1016/j.aim.2015.08.010. MR3406510
[Suc67] Louis Sucheston, *Banach limits*, Amer. Math. Monthly **74** (1967), 308–311. MR225145
[Tal76] Michel Talagrand, *Moyennes de Banach extrémales* (French, with English summary), C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 23, Aii, A1359–A1362. MR420131



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Credits

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EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Applying for Grants and Jobs.

Communicating in Math

Letter from Editors

Ben Jaye and Krystal Taylor

Let us introduce ourselves: we are Ben Jaye and Krystal Taylor, the new editors of the Early Career section of the *Notices of the AMS*. We both work in analysis:

- Ben is an assistant professor at Georgia Tech University. He graduated from the University of Missouri, and previously worked as a postdoc at Kent State University and a tenure-track assistant professor at Clemson University;
- Krystal is an associate professor at the Ohio State University. She graduated from the University of Rochester and held postdocs at the IMA & Technion in Israel. Her area of research centers on geometric measure theory on fractals and harmonic analysis. In addition to her pure math pursuits, she is invested in applications of math to real-world problems and started the Math to Industry program at OSU.

Angela Gibney began the Early Career section and developed it into an amazing resource for people navigating a career in math. Personally, we have sought advice in planning talks, writing grant applications, and knowing what to expect in job interviews, from articles in the Early Career. To quote Tim Gowers, *"The most important thing that a young mathematician needs to learn is of course mathematics. However, it can also be very valuable to learn from the experiences of other mathematicians."*

We are delighted to be taking over from Angela as editors, and will do our best to continue to build the Early Career's library[1] of experiences and perspectives while also keeping early career researchers aware of professional opportunities in academia and beyond.

Our first issue is on the theme of communication, and contains the following articles:

- Bianca Viray and John Voight write on the role of storytelling when constructing mathematics talks;
- Elizabeth Matsui and Roger Peng write advice on building effective collaborative research relationships;
- Roman Holowinsky writes about some proactive steps that anyone can take to put themselves in a position to get a job that is the best fit for them;
- Andrew Hwang writes advocating for semantic coding in TeX, which can greatly reduce typing, and typing errors, while providing notational flexibility and making source files more easily maintainable.

Over the coming year we intend to include issues with themes such as:

- both sides of the grant application process;
- handling disappointment and failure;
- alternative journeys to academic positions;

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[1] *The AMS maintains the complete list of Early Career articles:* https://www.ams.org/cgi-bin/notices/amsnotices.pl?article_id=career&article_type=gallery&gallery_type=career.

in addition to articles celebrating Hispanic Heritage Month, Black History Month, and Women's History Month.

Finally, the Early Career can only thrive if people want to contribute! If you are reading this and have even the vaguest idea for an article that you would like to write, or are interested in contributing an article to one of the themes listed above, please get in touch with us. We would be delighted to hear your ideas! With this in mind, we would also like to thank everyone in advance who will be writing an article for the Early Career over the next year.

Yours,

Ben and Krystal

Ben Jaye and Krystal Taylor

Credits

Photo is courtesy of Krystal Taylor.

The Value of Mathematical Storytelling: Our Perspective on Giving Talks

Bianca Viray and John Voight

What is the purpose of giving a mathematical talk? As speakers, we hope to impart informative, intriguing, and sometimes entertaining mathematical ideas, perspective, and intuition. For example, a talk about a paper should be *complementary* to the main body of the paper (and go beyond what may be set up in its introduction). After all, a reader has the freedom to jump around a paper, but a listener wants to be meaningfully led. And some things are just too complicated for a talk, even if they are necessary in a paper. (If you make a definition that no one understands, does it make a sound?) Talks should be audience-focused, seeking to impart digestible material in real time.

We have found that one of the best ways to achieve this goal is to frame talks as a form of *mathematical storytelling*. Indeed, the art of storytelling—an innate human talent—provides a natural, clarifying lens when preparing a talk. People excel at following and remembering stories—we do it all the time! Our understanding is more robust when notions are stitched into context. If we were to give one broad piece of advice about giving a good talk, it would be this: focus on the story.

In what follows, we will expand on this advice based on our (mostly US-based) experience. The usual caveats apply: the advice is not meant to delineate a "one size fits all" recipe, the details are always under revision, and the advice reflects our trials and attempts. Sometimes—maybe often—we fail! But we do hope that something we suggest here will be useful to you as you reflect on how to give a stimulating math talk.

This note is organized as follows. We begin in section 1 by surveying some of the basic aspects of mathematical storytelling. In section 2, we elaborate further and propose storyboarding as an implementation. We conclude in sections 3 and 4 with further things to keep in mind and a few final remarks.

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1. Elements of Mathematical Storytelling

So what makes a good mathematical story anyway? We don't mean *literally* a story, like a crime drama involving elliptic curves (*Law and Order: ECU*?). We just mean that the essential elements that underlie a good narrative are also what makes for a good math talk.

Story arc and key moments. Every good story has a thoughtful *arc*, the path that a story follows. For a mathematical story, this arc might have *constant slope*: start at the beginning with the definitions, establish preliminaries like laying bricks one after the other, and state your theorem at the end. This can work as a talk—indeed, it is quite common—but we contend that it is not always the best way to tell a memorable mathematical story.

In fact, there are many possibilities for a good arc. The best choice will depend on the specific topic, your intentions, and your audience. One *mountain trek* arc might be as follows:

(1) Kickoff with something to grab the audience's attention: a tantalizing question, a quip, a puzzle or riddle, or an old but enduringly potent observation.

(2) Head on a gondola up the mountain (i.e., with good pacing and not expecting your audience to do all the work) to some kind of vista: state the motivating conjecture, give the bigger picture or larger context, or paint a broad theme or principle you are trying to address.
(3) Follow along the ridge to the statement of your main results or key points.
(4) Enjoy and comment on the view. Maybe look back on the path up the mountain by giving some ideas of the proof. Point out the dangerous or intense parts, or highlight an ingenious shortcut. Or instead discuss a side trip where there is another interesting vista nearby (e.g., examples or applications).

(Further elaboration and some other arc possibilities are provided in §2.)

The denouement or key dramatic moment in the talk is probably in the presentation and digestion of your theorem. But each step along the way can provide an essential or memorable part of the story. Being explicit about these key steps helps to ensure your audience knows where they are on the trip. (Otherwise, they might start murmuring from the back of the car: "Are we there yet?")

The arc clarifies what is essential to your story. A good storyteller is mindful about what to include and what to exclude, and a story arc lays out a minimal path that captures the essential elements. More is not always more. A short story is not a compressed version of a novel to be read at top speed, and the same is true for mathematical talks. The best storytellers have their own rhythm and pace, balancing between keeping the audience interested, but not so much that it overwhelms—and adapting as needed, based on how the audience responds.

Present your perspective. Just as a story is told in the voice of the author, so too for a mathematical story, you should showcase your mathematical voice. After all, it was you who proved your theorem! In addition to appreciating the statement itself, the audience surely wants to hear how you thought about the problem and came to its resolution. So do that! In particular, you should feel empowered to present material in your own way. If you prefer an equivalent definition to a more "standard" choice because it is more useful for your context, then use your preferred one. If you have a cartoon image or some visual way to understand how the proof works, sketch it. If there's a relevant anecdote which can help provide a bridge across some more technical material, share it.

(It is also a good idea to literally use your voice: modulate your pitch and speed to indicate what is exciting and what you consider background, using pauses for emphasis. No one knows what is important if you speak in a monotone way.)

Introduce your characters. The mathematical objects, definitions, and theorems are the characters of your story. Give them identities, make them memorable. Can anyone remember a slew of characters introduced in the first five minutes of a movie, with nothing differentiating them? So let's not introduce a whole lot of new notation all at once without giving the audience a way to understand the objects.

What makes mathematical objects, concepts, or results memorable? Perhaps the relevant objects are mysterious, and your theorem acts as a microscope to see them in detail? Or maybe your theorem says that certain objects are impenetrable or behave randomly and you can't obtain more structural understanding than already exists? Does your theorem serve as a telescope revealing a hint of structure in an unexplored part of the universe? Is there an aspect which is unexpectedly subtle or particularly conniving? Does an object deserve a nickname, perhaps because it is known for behaving a certain way?

2. Strategies for Building a Story

Now that we've brainstormed some possible aspects of mathematical storytelling, let's discuss some strategies to translate this into action.

Decide on your story arc. Above, we presented the mountain trek arc, but you should think of what is best suited for your story. Here are some story arcs that can also be quite effective:

- Your talk could have a "three act" structure, with Act I an opening, Act II a description of conflict or progress toward a goal, and Act III the resolution.
- Some tales are best understood as conquering a foe: the theorem statement is like naming or knowing the enemy (and, in the end, making it a friend). How did you hem in your adversary? Perhaps you want to briefly survey near-miss attempts? When cornered, did it vanish from sight, or did you just need to give it a hug?
- You may have a single thread or idea that you want to explain using three different examples or cases, so you could set out the thematic statement at the start and repeatedly return to it.
- Instead of presenting your theorem toward the end, consider presenting it right up front. You may not be able to define every term, so you might have to substitute rough ideas with precise definitions. In this way, your theorem is the hook. Then, you can flash back ("three weeks earlier") to explain terms, elaborate, and justify.
- A survey talk might situate your work into the larger story of a developing area.
- Perhaps you have a "tale of two theorems," which explains a result from two different angles, or addresses

a tension between two sides (like computational versus theoretical).

Consider adding an interlude for relief by providing an example, a fun tangent, or a compelling bit of history.

Just be sure that your arc is simple enough to fit nicely into the time slot of your talk. We find that key points in the arc might be spaced out approximately every 10 minutes. The choice of a story arc should also be informed by the audience: to know potential interests, points of connection, and their stamina for arduous parts of the journey. Your story arc should compel the audience to want to follow your path. You may not want to start the talk with a detailed outline: not every story starts out by saying exactly what is going to happen and when!

Thematic possibilities. Some themes in mathematics reappear across time and area, and these can be useful as a template to get your story started. For example: the juxtaposition of structure and randomness, or how topic X informs topic Y (the different ways in which analysis and algebra inform arithmetic, for example). Or maybe these are "axes" on which a talk could land, like problem solving versus theory building, the concrete versus the abstract, general versus specific. Referring to these archetypes can provide a quick connection to your audience, a touchstone which can be particularly useful when giving a talk to a group outside your research specialty.

Storyboard. Now that you have your arc, it's time to flesh out what your audience will take home from the arc. For key steps, your audience might appreciate a "postcard memory." For example, in a 50 minute talk, you might have time for four key moments, corresponding to the points in the mountain trek arc. You might try to fill in your talk first in postcard form, then filling four sheets of paper with more details (answering questions like: when is the right time to introduce this character?).

We also find it helpful sometimes to think of *slogans*, i.e., summaries given in a few words that avoid jargon, like "geometry determines arithmetic." What is your guiding philosophy, motivating question, or summary? It can be helpful to lay these out on the storyboard. These slogans should be as free as possible from specialized mathematical jargon—these are the parts of your talk your audience should be able to recount to someone else over lunch. (Well, to whatever extent this is possible, anyway!) If you are a more visual thinker, you could sketch a cartoon panel for each key point; often this exercise can give you great ideas for illustrations to use during your talk.

A key part of making this storyboard is experimenting with your definitions and theorems. Mathematicians often state theorems and definitions to prioritize ease of use over comprehension. For example, while the definition of a one-to-one function ("if $f(a) = f(b)$ then $a = b$") makes it very easy to prove that the composition of two one-to-one functions is one-to-one, we have found that some students better grasp the concept when it is introduced with the slogan "*not* many-to-one" (from the contrapositive). We encourage you to try out several different ways of stating your theorems and definitions, keeping your audience in mind. For example, consider writing out or explaining your results in special cases. Would a pictorial or diagrammatic representation help? What about a slogan that gets at the key reason why your theorem is surprising or powerful or useful, without being bogged down by mathematical symbols and unnecessary terminology? Remember that the exact, strongest, precise version of your theorem can be found in your paper (which, when finalized, can be made available on `arXiv.org`) and does not need to be reproduced verbatim in your talk. Although you can, and should, mention that there is a stronger or more general version.

3. Things to Keep in Mind

When trying out mathematical storytelling, we have found a few broad suggestions to be helpful.

Focus on the why. It is very easy for a talk to primarily answer "what" questions. What is the background needed? What has been proved before? What is the definition of this object? What is the main theorem? But for someone who is not already an expert in the area—and sometimes even for experts!—the answers to these questions may feel like a jumble of facts *without context*, which can be very challenging to remember.

It can be helpful to focus on answering "why" questions. Why is this question interesting? Why is this problem difficult? Why is this answer expected or unexpected? Why is this hypothesis necessary? Why is the theorem true? Why is it believable in the first place? A proof shows *that* a theorem is true, but it does not always make clear *why* the theorem is true. Conversely, the reason why we expect a theorem to be true does not always yield a proof. In that case, it is also valuable to explain why the heuristic falls short.

What should the audience remember? It is easy to think of a talk as accomplishing the important but limited objective: what do I need to tell the audience so that they will understand and appreciate my theorem? In our experience, that can often lead to a talk that is jam-packed with technical definitions and background.

A more helpful prompt might be: What do I want my audience to remember after the talk? These can inform the key moments in your storyboard. Imagine an audience member leaving your talk and running into a colleague who could not attend. How would the audience member, who is probably not an expert in the area, recount your talk?

Show, don't tell. Instead of saying that a hypothesis is necessary, provide an example where the argument or theorem fails. Instead of saying that a question is central or difficult, give a simple example that illustrates its subtlety or an intriguing special case that remains open. Capture your audience's sense of wonder and surprise instead of asserting that your result is surprising. Use a heuristic explanation which gives a good reason to believe your theorem, even if it is not actually used in the proof.

4. Concluding Remarks

Almost everything we have discussed here is most useful *before* we even start writing notes or preparing slides. It may seem like a lot, but we have found that taking these steps actually makes the writing or slide preparation easier and more efficient, because we have greater clarity about what is essential and what can be omitted.

Of course, there are many good places to read about "tips and tricks" for preparing good slides and for giving good presentations. We suggest considering these *after* thinking about the story, but they are still very important! There are several Early Career articles [Ker19, Dev19, Kra22, Leh20] and online resources [Ell05, Tao, Raw] that give advice about giving talks, and we have also compiled a (long) list of our favorite tips [VV].

Two final caveats. First, trying to improve all aspects of your talk at once can be counterproductive (and paralyzing). After deciding on your mathematical story, we suggest choosing at most three presentation-related aspects to focus on improving (e.g., remembering to pause before advancing to the next slide). Once you have made improvement on one, you can always pick another. Second, our advice reflects our personalities and cultural upbringing. You may have a different style, so interpret our advice in a way that works for you!

Storytelling can be a powerful technique for developing mathematical talks. Before preparing a talk, decide on story arc; then flesh it out using a storyboard, keeping in mind the need for economy and focusing on what the audience should remember from the talk. We hope this will be helpful in sharing your mathematical insights and results in a meaningful and memorable way.

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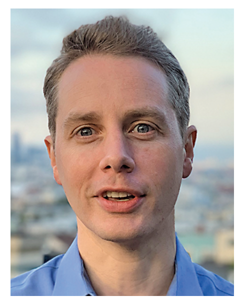
References

[Dev19] Satyan L. Devadoss, *Giving Good Talks*, 2019.
[Ell05] Jordan Ellenberg, *Tips on giving talks* (2005), `https://quomodocumque.files.wordpress.com/2010/09/talktipsheet.pdf`.
[Ker19] Alec Kercheval, *10 Ancient Rules for Giving a Conference/Seminar/Research Talk in Mathematics*, 2019.
[Kra22] Bryna Kra, *Giving a Talk*, 2022.
[Leh20] Brian Lehmann, *Giving a Job Talk* (2020).
[Tao] Terence Tao, *Talks are not the same as papers*, `https://terrytao.wordpress.com/career-advice/talks-are-not-the-same-as-papers/`.
[Raw] Steve Rawling, *Storytelling Techniques for Product Teams*, `https://pipdecks.com/pages/storyteller-tactics`. Online; accessed 25 May 2022.
[VV] Bianca Viray and John Voight, *Practical tips for giving and preparing a talk*, `https://jvoight.github.io/cme/TalkImplementationTips.pdf`. Online; accessed 16 November 2022.

Bianca Viray



John Voight

Credits

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Photo of John Voight is courtesy of Brian Kennedy.

Collaborative Scholarship: Why and How

Elizabeth C. Matsui and Roger D. Peng

Collaboration is not typically included as a topic in the formal curriculum of graduate school or as a part of career training activities for post-doctoral fellows and early career faculty, but for many faculty, collaboration is the richest aspect of their professional lives. This may come as a surprise to some, who may wonder if collaboration is truly worth the effort.

Why collaborate [1]. Collaboration is a mechanism for learning. Life-long learning is generally highly valued

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among academics, and collaboration offers a different avenue for learning that complements activities like reading, attending seminars, etc. Expanding your "routine" ways of learning to include collaboration intellectually enriches your professional life. Collaboration can also greatly enrich the creativity and impact of the scholarly work. The addition of diverse perspectives and expertise to a project increases the likelihood of new discoveries. From a practical perspective, collaboration can increase productivity. In a successful collaboration, projects will naturally spin off from the initial project, and having multiple collaborators allows each to take the lead on a different project. Beyond these professional benefits, collaboration is a natural way to build a support network. An academic career is full of twists and turns that are often specific to the academy or even the field and having partner(s) to share the experience with over the short and long term can be valuable for providing career support and guidance. Last, but not least, collaboration can lead to life-long friendships. So much so, that we coined a new term "collabofriend" in episode 89 of The Effort Report [2], our podcast about life in academia.

There are certainly disincentives for collaborating, which are important to consider. For example, relationships are complicated and many collaborations fizzle out or, even worse, are contentious. The incentives in academia tend to reward individuals over teams, as tenure is focused on the individual's scholarly contributions to their field and its impact. Although institutions have recognized the need to revisit incentive structures to reward collaborative scholarship—often referred to as "team science" in science—it's not clear that there has been a major shift in the promotions and tenure evaluation processes to reflect this goal. Collaboration may also take more time—at least in the early stages when relationship building, establishing processes and structures (such as the cadence and format of meetings and roles and responsibilities), and learning the language of other fields for interdisciplinary collaboration, must be done before being able to meaningfully embark on the project at hand.

How to collaborate. *What research says*. While for some people, collaboration seems to come naturally, for many, it does not. This observation is reflected in the scholarly work on collaboration, which has characterized key elements of successful collaboration but also highlighted that faculty are often unprepared for successful collaboration [3]. These key elements include:

- A common goal or mutual benefit
- Voluntary engagement
- Agreed upon process and structure for the work, such as the cadence and format of meetings and roles and responsibilities of each team member
- Shared decision-making
- Shared understanding of a problem domain

Without these key elements, it's hard to imagine a path that exists for successful collaboration.

What our experience says. About a decade ago, we shared our perspectives on a specific type of collaboration—between clinician scientists and biostatisticians—in a pair of blog posts [4, 5], and much of the advice is generalizable to scholarly collaborations. One thing that has become clear over years of participating in and observing collaborations is that having the key elements of collaboration outlined above in place is far from a guarantee of successful collaboration. Perhaps the biggest stumbling block to the success of a project is when there is the perception that you share a common goal, when you do not. A shared goal is critical to a collaborative project's success. While a shared goal is important for the success of a collaborative project, shared values are critical to long standing productive collaborations. For example, in our field, the ultimate goal is to improve public health and we do that through advancing the science. A pitfall is assuming that the value of advancing science is shared among colleagues. For example, a colleague may be focused on short-term career advancement goals while you are focused on the long-term goal of advancing your field because of you value advancing science more than short-term career success. Sometimes these two types of goals are aligned, but often they are not because someone with their eye on short-term career advancement may be focused on getting a paper published as quickly as possible, even if it means the work is less rigorous or impactful, to build their CV. This disconnect between short-term goals related to career advancement and long-term goals that reflect values can stymie progress, interfere with the creative process, and cause unhealthy conflict. If you think of your job as advancing the field, then you end up with long lasting collaborations. Why? Because you look for colleagues who share this value—that the long-term goal of our jobs is to advance our fields. Having shared values helps to ensure that the collaboration is not bounded by a contractual arrangement where one team member provides a service in exchange for a defined and tangible compensation. Such contractual, or transactional, arrangements can stifle the freedom needed for creativity and establishing a long-lasting partnership.

Finding colleagues who share this value goes a long way toward ensuring a successful collaboration. However, a second ingredient is also required: mutual respect. This sounds obvious, but how does mutual respect play out in practice?

First, all good collaborations involve some teaching and some learning. Although your major contribution is your expertise, your collaborator(s) still need to have some understanding of your area of expertise so that you can communicate effectively. Similarly, you will need to have some understanding of your collaborator(s) area(s) of expertise. When teaching, be patient and don't patronize. To learn, don't hesitate to ask questions when you don't understand something.

Second, good collaborations are self-sustaining as one project leads to another (or multiple projects). However, they are only self-sustaining if collaborators generously share data and ideas with each other.

Third, be respectful of time. It's discouraging to get last-minute requests from a collaborator, which sends a message that your time is less valuable than theirs. If you are leading the project, thoughtfully plan out the timeline together. Be realistic about the timeline and build in "cushion" for the inevitable unexpected hiccups.

Fourth, understand that your collaborator is not a technical tool or instrument. In our field, some can fall into the trap of viewing the biostatistician as someone who "crunches numbers" or a biologist as someone who runs assays on biologic specimens and sends the results or a clinician as someone who collects biospecimens. This reductive view leads to a major missed opportunity—which is that collaborators who bring technical skills to the team also bring intellectual skills to the team, which are likely more valuable than the technical skills. This reductive view also sends the message that you view the collaboration as a transactional partnership rather than an intellectual partnership.

Fifth, "go where they are." You might read this and think that you should be sure to go to their office for meetings, and while this is a nice gesture, really what we mean is to go to the place where they do their work. For collaborations between people in the same field, this of course is not necessary, but when collaborating across fields, a first-hand look at how your collaborator does their work is illuminating and you will learn things you could not learn simply through talking with your collaborator.

Sixth, their business is your business, so pitch in. If you narrowly define the scope of your role, you risk sending the message that you view the collaboration as a transactional relationship; that you approach the relationship as if there is a contract with a scope of work and you won't deviate from that. In a mutually respectful collaboration, colleagues appreciate that the broader body of work that their collaborator is pursuing is directly connected to their own work, so that helping a collaborator on a project that might not have been explicitly spelled out as part of the agreed upon collaborative work helps the collaboration.

In the end, it's the relationship that drives the scholarly work and not vice versa. If there's mutual respect and genuine interest in each other's work, it's more likely than not that collaborative scholarly work will naturally emerge from the relationship. On the other hand, if colleagues are brought together because of a shared interest in a particular area of scholarship—which happens when people are brought together to participate in workshops or other activities to catalyze collaboration—it's unlikely a fruitful collaboration will emerge without these relationship fundamentals. Perhaps the best advice is to find the people you want to work with and the scholarship will follow.

References

[1] Diane Sonnenwald, *Scientific Collaboration: A Synthesis of Challenges and Strategies*, ARIST **41** (2007), DOI 10.1002/aris.2007.1440410121.

[2] `https://effortreport.libsyn.com/89-the-one-about-collabofriends`; accessed November 29, 2022.

[3] Catherine Newell and Alan Bain, *Academics' perceptions of collaboration in higher education course design*, Higher Education Research & Development **39** (2020), no. 4, 748–763, DOI 10.1080/07294360.2019.1690431.

[4] `https://simplystatistics.org/posts/2013-10-09-the-care-and-feeding-of-your-scientist-collaborator/`; accessed November 29, 2022.

[5] `https://simplystatistics.org/posts/2013-10-08-the-care-and-feeding-of-the-biostatistician/`; accessed November 29, 2022.


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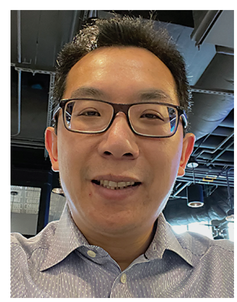
Roger D. Peng

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Technical & Social Components of Getting a Job

Roman Holowinsky

We've helped hundreds of graduate students, postdocs, and graduate alumni launch and advance their careers outside of academia over the past five years at the Erdős

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Institute. Here are my top takeaways for both the technical and social aspects of getting a job.

Setting the Stage

You are extremely employable. Less than 2% of the world's population holds a doctoral degree. You have a unique skill set that has been honed by going to graduate school and successfully completing a PhD. Some of your skills are:

- Identifying problems and proposing solutions
- Clearly communicating difficult/technical concepts
- Being responsible, organized, and able to manage big projects

However, having **a PhD is not a golden ticket**. Trying to get hired can be a full-time job on its own and requires plenty of preparation.

Have a broad network and communicate well. Getting a job you love and successfully advancing your career requires community and the support of your peers, mentors, and professional network at every stage. The more diverse your network is, the greater your set of opportunities will be. Furthermore, the business world values collaboration and teamwork. Your colleagues and superiors will judge you both on what you know and how well you work with others. Being a strong communicator and team player will give you a competitive edge.

Explore your options and don't be afraid to create your own opportunities. What do you enjoy doing? What do you want to get out of a job? What do you want to get out of life? You can take your career in several different directions. The nonacademic career path is more fluid than the tenure track. Getting your first job is the hardest part. After that, if you keep your options open, new and exciting opportunities will regularly present themselves. Have you come across a small or medium size business that you think could be a perfect fit, but don't see a job posting? Write to the CEO and founder(s). Also, keep in mind that entrepreneurship and creating your own job is always an option.

Faculty are usually not the subject matter experts on getting a job outside of academia. However, your peers and departmental PhD alumni working outside of academia are. They are the ones you should be turning to for help and guidance on nonacademic careers. Academic departments benefit tremendously when they understand the value of their alumni network and properly utilize their collective experience, knowledge, and resources.

There's no one-size-fits-all perfect strategy. As you speak with others about their experiences and processes in getting a job, you will hear varying opinions on application strategies—like how many places you should apply to and whether you should always include a cover letter. Some applicants cold-applied with a blanket resume to hundreds of positions while others applied to only a dozen positions but spent days refining their resume and cover letter for each individual position. Eventually, after enough effort and help from their network, they all got a job. Choose an initial strategy that you feel most comfortable with. Then you can continue to refine your approach based on what is working for you.

Getting Started

Aligning with organizations, like the Erdős Institute, that support PhDs and connect them with a professional network, will give you access to career development resources and services that go beyond what is available to you in your department and university.

You need to know what jobs are out there. Find and attend career exploration seminars, panels, and events where your PhD alumni peers share their experiences and career paths. They may offer advice on what you could be doing now to be better prepared. Research their organization and follow up with the speakers if you are interested in learning more. Most are willing to have a conversation, if their schedule permits, and you clearly demonstrate that you have taken time in advance to learn more about them and their employer.

Put together a resume. Resumes are your marketing document. They succinctly say who you are, what kind of work you are looking for, and what qualifying skills you have for the job you are applying to. The more you tailor your resume to a specific job posting's stated requirements and desired skills, the better. You need to pass the "eight-second glance test," since some recruiters are reviewing hundreds of resumes. Have a clear summary section at the top and follow it up with your most relevant skills. Make it easy for the recruiter to quickly find what they're looking for. This will encourage them to review your resume in more detail after the initial glance.

I see graduate student resumes as typically falling into four different categories depending on how many of the following conditions have been met:

1. **The PhD is nearing completion**—this demonstrates resilience and the ability to take on and complete huge tasks.
2. **Portfolio projects outside of the dissertation research (and comprehensible to a broader audience) have been completed**—this demonstrates interest in other topics and the ability to work on more "practical" problems.
3. **Having prior nonacademic work experience (like an internship)**—this demonstrates that you are "employable" outside of academia. This could seem inconsequential, but it may just give you that competitive edge. Also, some recruiters who aren't qualified to

evaluate your technical skills feel more comfortable knowing that someone else has already evaluated you.

4. **Unique bonus experience**—this demonstrates personality, leadership, and depth. Include positions and activities like being president of a student organization or professional society chapter. Established your university's/state's first-ever conference on "topic x" which brought together academics and nonacademics? Even better. Varsity captain of your university's soccer team in undergrad and led the team to a national championship two years in a row? Fantastic. These types of experiences show potential employers that you are both smart and a natural leader.

Now of course, there are certain industry sectors and research positions outside of academia where PhDs have been hired for decades and something closer to a traditional CV is more appropriate. Having both a resume and CV is valuable regardless of whether you are applying to academic positions.

Get ready to interview. Preparing for interviews takes time—like preparing for a qualifying or candidacy exam. Many alumni will tell you to set aside a few months to practice. You will need to be ready both for technical interviews and behavioral interviews. Practice paired coding and whiteboarding exercises, review case studies, and practice answering behavioral interview questions using the Situation, Task, Action, Result (STAR) method. At the Erdős Institute we have collected resources, developed interview content with our alumni, and have coaches on hand to help you with every type of interview.

Understand that interview procedures vary widely. Different companies have different hiring processes. Larger companies need to implement standardized processes to handle the volume of applications, while smaller companies have greater flexibility.

Accepting an Offer and Advancing Your Career

Eventually, the offers will start coming in and you will need to weigh your options and begin the negotiation process. Congratulations! Navigating all of this can be tricky and you should aim for only accepting competitive offers. Consider working with a career coach. Coaches can help build up your confidence, help you self-reflect, and help map out the steps you need to take for career success and advancement.

The nonacademic career path is more fluid than the tenure track. Getting your first job is the hardest part. After that, if you keep your options open and continue fostering your personal and professional networks, new and exciting opportunities will regularly present themselves to you!

Want more personalized help? Please feel free to reach out: `roman@erdosinstitute.org`.

The Erdős Institute is a multi-university collaboration focused on helping PhDs get jobs they love at every stage of their career. Founded in 2017, the Institute helps train and place a diverse pool of graduate students, postdocs, and graduate alumni.

Graduate students, postdocs, and graduate alumni from member institutions have free access to all of the programming and resources that the Institute provides. Examples include the "Invitation to Industry" seminar series, Data Science Bootcamps, Alumni-Led Mini-Courses, Interview Prep Workshops, and Career Coaching.

Roman Holowinsky

Credits

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Semantic Coding in LaTeX

Andrew D. Hwang

LaTeX-aware platforms such as Stack Exchange and Overleaf have made mathematical typesetting convenient and widely available to internet citizens. While numerous tutorials explain how to achieve typographical effects, few emphasize basic principles of semantic coding, macro design, and code organization. This note introduces considerations, coding habits, and other practices that enhance document flexibility and maintainability, sometimes substantially.

1. Semantic Coding

In this electronic era, authors do not write documents per se, but *source files*, which are *instructions for creating documents*. Source files are converted into documents by software such as LaTeX or a web browser using MathJaX. This abstraction layer provides opportunities that might not be immediately apparent.

Authors of source files are well-advised to recognize and utilize the distinction between *visual appearance* on the display medium, and *semantics*, the meaning of notation.

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Consider the linear algebra snippet

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \mathbf{v},$$

which expresses the Euclidean dot product as a matrix product if vectors are written as columns. One way to code this is *visually*:

```
\langle\mathbf{u}, \mathbf{v}\rangle
  = \mathbf{u}^t \mathbf{v},
```

treating LaTeX as a command-based word processor. Widely-read tutorials teach visual coding. The expediency is understandable, but the outcome is code that does not utilize the full power of LaTeX.

Our snippet may instead be coded *semantically* as

```
\Inner{\Vu}{\Vv} = \Trans{\Vu}\Vv,
```

together with a few lines of preamble code:

```
\newcommand{\Vector}[1]{\mathbf{#1}}
\newcommand{\Vu}{\Vector{u}}
\newcommand{\Vv}{\Vector{v}}

\newcommand{\Inner}[2]{
  \left\langle#1, #2\right\rangle
}
\newcommand{\Trans}[1]{#1^{t}}
```

Semantic coding represents organizational and mathematical meaning in LaTeX macros, and beneficially factors out visual decisions about notation.

Sectioning commands, and LaTeX environments such as `theorem` and `equation`, are semantic. Most LaTeX novices learn to use native sectioning commands, the features of the `amsmath`, `amssymb`, and `amsthm` packages, the `geometry` package to control the text block, and so forth. This note advocates judiciously extending these good authorial habits to document content itself.

Some LaTeX symbols, such as common binary relations, or summation and integration signs, have such a fixed meaning that the native LaTeX command *is* effectively semantic. On the other hand, many common notations are not universal: $\mathbf{R}$ versus $\mathbb{R}$ for the set of real numbers, or dx versus the new (and in the author's opinion, regrettable) ISO standard $\mathrm{d}x$. For common but nonuniversal notation, semantic coding conveys numerous benefits, some of them substantial.

2. Flexibility

A source file for a journal article must be compiled by the author(s) and by the journal. A source file for a book or set of course handouts is likely to be revised and recompiled over time. Human-readable code that minimizes manual editing is highly desirable.

Journals and other publishers often have a house style. Semantic coding allows a LaTeX source file to be easily edited to match house style. Suppose our linear algebra snippet (part of a 200-page book) is handled by a publisher who insists on the notation

$$\vec{u} \cdot \vec{v} = {}^t u\, v.$$

If the source files are coded visually, the author faces a tedious search-and-replace in each source file. If the source files are coded semantically, however, the publisher accomplishes the same effect by changing three lines of preamble code.

Suppose an author has visually coded, and also used a single symbol such as $\mathbf{R}$ to denote more than one thing, perhaps the set of real numbers and a commutative ring, a radial vector field, or a curvature tensor, because "Context always makes clear which is meant." The publisher insists on blackboard bold for sets of numbers, or on fraktur for rings. In that circumstance, every instance of the symbol in the source file requires manual attention by a skilled reader.

The `amsmath` manual [AMS02] (under "Recommended use of math font commands") advocates using LaTeX's macro capabilities to build lightweight abstraction layers. For example:

```
\newcommand{\Number}[1]{\mathbf{#1}}
\newcommand{\Reals}{\Number{R}}
\newcommand{\Natls}{\Number{N}}
```

A real sequence is therefore a mapping from `\Natls` to `\Reals`. Note the two levels of indirection, one to select a single font for sets of numbers, and one to define the macros actually used in the document body. If boldface is also used for vectors, that fact should not be coded with `\Number`, but with a separate `\Vector` macro. The names `\Number` and `\Vector` are chosen merely to be self-descriptive, and are capitalized to avoid collision with native LaTeX macros.

In a semantic vein, one might define differentials with

```
\DeclareMathOperator{\Diff}{d}
\newcommand{\dx}[1][x]{\Diff\! #1}
```

The document body contains only `\dx` for $\mathrm{d}x$, or (e.g.) `\dx[t]` for $\mathrm{d}t$. There is a small overhead to define this differential macro, but a substantial benefit to avoiding the inflexible manual keying of dozens or hundreds of `\mathrm{d}x` commands.

This use of author-designed macros permits painless changes of notation. Mathematical writing, from organization to expository style to selecting notation, is an art. Notation is best chosen to accommodate the reader's expectations, and to capture the relevant dependencies of a concept. While writing a book or series of course handouts, an author's notational wishes may change. An author writing about metric spaces might initially denote the open ball of radius r and center x_0 by $B_r(x_0)$, or by $N(x_0; r)$, or by something other. Visually coding any particular choice is inflexible. Semantic coding beneficially factors out the notational choice:

```
% \Nbhd{center}{radius}
\newcommand{\Nbhd}[2]{B_{#2}(#1)}
```

To summarize, judicious semantic coding facilitates alignment of a source file and its mathematical content. Semantic LaTeX code is generally easier to read than visual code, an asset discussed in more detail below. Visual consistency of the output is more easily attained, usually with fewer characters of input. The choice of notation remains flexible until the final document is compiled.

More abstractly but also beneficially, semantic coding dovetails with mindful use of notation. In order to design suitable macros, an author must consider a document's aims and readership, its mathematical content and typographical requirements. These are habits of clear, effective exposition.

2.1. Spacing. Typographical alignment can help highlight parallel concepts. Books typeset in metal often utilized alignments difficult to achieve in LaTeX, such as centered displayed equations set on the same line as explanatory text, or aligned ditto marks. While semi-semantic spacing commands such as `\quad` and `\qquad` have their places in a document body, hard-coded native space commands to achieve alignment are *brittle*: subject to breakage under changes of wording, font, and text block size.

One robust way to align in LaTeX is to measure and store the desired width, then to set material at specified size. The code below typesets a snippet of text with width equal to a particular piece of text:

```
\newlength{\TempLen}
\newcommand{\PadText}[3][c]{%
  \settowidth{\TempLen}{#2}%
  \makebox[\TempLen][#1]{\text{#3}}%
}
```

For example, `\PadText{associativity}{"}` sets a centered ditto mark in a box as wide as the word "associativity." An optional first argument `l` or `r` controls the alignment.

3. Readability

LaTeX source files are converted to human-friendly documents by a computer. What difference does it make if the source file is easy to read? As with semantic coding, there are multiple answers. First, human authors and editors must read, modify, and maintain source files. Especially with displayed mathematics, but even with plain prose, a file is easier to read if the source is broken into lines of about 60–70 characters, if spaces are used judiciously between code tokens, if code is consistently indented and line-broken to highlight its syntactic and semantic structure, and if macro arguments are enclosed in braces. Compare

```
\begin{align*}
  \lim_{n \to \infty}
    \left(1 + \frac{x}{n} \right)^{n}
    &= \lim_{h \to 0^{+}}
    \left(1 + xh\right)^{1/h} \\
    &= \lim_{h \to 0^{+}} \exp\left[
      \frac{1}{h} \log(1 + xh)
    \right] \\
```

with the LaTeX-equivalent

```
\ba\lim_{n\to\infty}\left(1+
\frac xn\right)^n&=\lim_{h\to0^+}
\left(1+xh\right)^{1/h} \\
&=\lim_{h\to0^+}\exp\left[\frac1h
\log(1+xh)\right] \\
```

The first is easier for the eye to parse: The author has used spaces and curly braces even where LaTeX does not require them, and has typed out `\begin{align*}` rather than using a macro to save keystrokes.

Spaces and braces make code easier for a human to read, `\frac{x}{n}` versus `\frac xn`. Consistency in the source file makes search-and-replace easier, and has the same benefits to a human reader that visually consistent typesetting has.

Braces around macro arguments also make a source file easier for a *machine* to parse without knowing the minutiae of LaTeX syntax. Anyone who has needed to batch-edit LaTeX source files will appreciate the vexation of handling "efficient" coding. LaTeX source files for lasting works are likely to be converted to other source file formats in the future, perhaps formats that do not yet exist. LaTeX code that explicitly delimits syntactic groupings such as macro arguments is easier to parse than code that relies on idiosyncrasies of the typesetting engine.

4. File-level Factorization

The benefits of semantic habits and centralization of typographical information tend to grow with the length of the document. On the other hand, even multiple short documents, such as daily one-page handouts for a one-semester course, benefit from judicious factorizing.

Instead of placing package inclusions and macros into the preamble of every document, factor them into an external style file. Style files confer a fringe benefit when using Overleaf: The source file preamble typically shortens to

```
\documentclass[12pt]{article}
\usepackage{Mycourse}
\begin{document}
```

Students generally find this less intimidating than scrolling past the contents of `Mycourse.sty`.

Similarly, text common to several documents may be factored into an external "configuration" file, and `\input` as needed. Placing the instructor's name(s) and the course term into an external file allows handouts to be recompiled and reused without touching the source files, only the configuration.

5. Summary

The advice in this note amounts to a handful of practices.

- In the body of a source file, avoid hard-coding fonts and spacing. Generally, avoid hard-coding for visual appearance.
- Where notation is not universal, design and use macros that reflect meaning.
- Enclose macro arguments in braces even if LaTeX does not strictly require it.
- If several documents share notation, place the necessary macros in an external style file.
- If several documents depend on a small amount of shared information, such as the term and instructor's name for a set of course handouts, place those in an external configuration file.

A collection of semantic macros defines a dictionary from concepts to notations. In this sense, macros are a powerful authorial tool for achieving flexibility, clarity, and consistency. The body of a LaTeX source file must encode the semantic content, but need not and often should not specify how that content is to be typeset.

On the other hand, not every typographical choice needs to be factored out. Semantic considerations are detailed in [Hwa11] for book-length works having particular archival and typographical desiderata. These considerations are excessive for ordinary writing, but may be of interest to prospective authors of archival works.

Semantic coding, style files, and configuration files encourage thoughtful authorship, and can lead to flexible, visually-consistent, maintainable source files. Used judiciously and habitually, these basic techniques are a substantial asset to authors, editors, and other source file maintainers.

References

[AMS02] American Mathematical Society, *User's Guide for the amsmath Package*, American Mathematical Society, 2002.

[Hwa11] A. D. Hwang, *LaTeX at Distributed Proofreaders, and the electronic preservation of mathematical literature at Project Gutenberg*, TUGBoat **32** (2011), no. 1, 32–38.

Andrew D. Hwang

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In Memory of Earl Jay Taft (1931–2021)

Uma N. Iyer, Susan Montgomery, Siu-Hung Ng, and David Radford

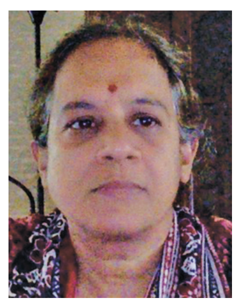
Uma N. Iyer

My first correspondence with Professor Earl J. Taft was when I was a post-doctoral visitor at the Harish-Chandra Research Institute, India, in 1999. I do not remember what exactly we corresponded about, but I do remember the thrill of receiving an answer from him for a question I had within a day. That generosity of a much renowned mathematician towards a post-doc he had never met is what I remember him the most for.

Years later, in 2006, I met him at the City University of New York (CUNY). Professor Taft regularly participated in several seminars at CUNY. In collaborative works with A. Lauve and S. Rodriguez-Romo, he constructed left Hopf algebras that are not Hopf algebras modeled after $SL_q(n)$. Our joint work [2] was a search for a left-Hopf algebra which is not a Hopf algebra containing $U_q(sl_2)$. While we were unsuccessful in identifying such an algebra, we were able to look at one non-example closely. Together with Professor Jonathan D.H. Smith, we were able to study the connections between one-sided Hopf algebras and one-sided quantum quasigroups.

Working with Professor Taft never seemed like work. He talked about various things: about his childhood in New York City, about his travels all around the world, about his amazing wife, Hessy Levinsons Taft, and her family. He could read and speak several languages fluently. He could discuss opera and literature as comfortably as he could discuss mathematics.

At the risk of repeating myself, I am grateful for his kindness and generosity. What a privilege it has been to have worked with him, to have had interacted with him and his wife. My deepest condolences to his family. *Om Shanti.*



Uma N. Iyer

Susan Montgomery

I first met Earl in the fall of 1971, when I visited Rutgers. I also met his wife Hessy and family, who were very welcoming. Of course, Earl suggested that I should look at Hopf algebras. At the time I was not interested; if an algebra had a multiplication, why did it need a comultiplication? Over the years, I would run into Earl at meetings, and again he would suggest I should look at Hopf algebras.

In 1981, I started working with Miriam Cohen on the duality between group actions on rings and rings graded

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by groups; suddenly the fact that we had two Hopf algebras and were looking at their duals made everything clear. Then I was hooked.

This began my 40 year interest in Hopf algebras. Later, I realized that one needed comultiplication in order to take the tensor product of modules, and having a suitable comultiplcation on an algebra A means that $Mod(A)$, the category of A-modules, is a tensor category. Beautiful!

So Earl was right: I should have looked at Hopf algebras earlier.

In the 70s general Hopf algebras, which are neither commutative nor cocommutative, were believed to be pathological as they did not directly arise from algebraic geometry or Lie theory. As a result, some Hopf algebra papers were published as algebraic geometry in disguise.

Drinfeld's fundamental ICM talk in 1986 on quantum groups changed this narrative; he gave solutions to the quantum Yang Baxter equation by using the representations of certain noncommutative, noncocommutative Hopf algebras. Suddenly general Hopf algebras were of wide interest (and some of the naysayers said that in retrospect they had been too harsh). But Drinfeld made one small error; he said his quantum groups were the first examples of non-commutative non-cocommutative Hopf algebras.

But in 1971 Earl constructed infinite families of such objects. He was interested in constructing examples in which the order of the antipode S could be arbitrarily large (In the commutative and cocommutative cases, it is always true that $S^2 = id$). Earl's examples have turned out to be fundamental in the classification of Hopf algebras. They are now known as Taft algebras. In a joint paper with Yevgenia Kashina and Siu-Hung Ng [3], we proved, by computing the 2nd indicators, that the Taft algebras are completely distinguished by their representation tensor categories.

As more people worked in the area, Earl came to all of our meetings and was very encouraging.

In 1988 there was an algebra meeting at the Banach Institute in Warsaw. Earl, Hessy, and I were put up at a hotel with a nice restaurant, but the menus were only in Polish. No Polish-English dictionaries were available in Warsaw. Hessy found a Polish-Spansh dictionary, which solved our problem, as she spoke Spanish fluently. So at meals, she would translate the Polish into Spanish, and then tell us what the items were. We had a lot of fun there.

I owe him a great debt, both mathematically and personally.

Susan Montgomery

Siu-Hung Ng

Some conversations with Earl from when I became his PhD student have stuck in my mind to this day. On several occasions, he expressed to me that he would continue to work as long as he could do the job mentally and physically. Earl sent me a message in November 2016 announcing his retirement in 2017. He passed away on August 9, 2021.

Figure 1. Taft at St. John's, Newfoundland, 2001.

Earl joined Rutgers University in 1959 after his three-year instructorship at Columbia University. I was admitted to Rutgers in 1992 and became Earl's PhD student in 1994. Before asking Earl to be my advisor, I took a graduate abstract algebra course and a course on quantum groups that he taught. He was a serious lecturer, and I liked his style perhaps because of my cultural background. Meanwhile, Earl was a pioneer of Hopf algebras. He discovered a groundbreaking family of Hopf algebras, now called the Taft algebras. I decided to ask Earl to be my PhD supervisor after understanding some of the interplays among Hopf algebras and many other areas of mathematics and physics.

During the time when I was his graduate student, he was working on Lie bialgebras and the Hopf algebra structure of linearly recursive sequences. Since I had some background in mathematics from my master's degree from Hong Kong, Earl showed me what he had been thinking

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in some of his research articles, and I began to work on his proposed problems. We met weekly and I reported what I discovered, even when he was on sabbatical at the Institute for Advanced Study.

In [12], Earl proved that for any Lie algebra L, if $r \in L \wedge L$ satisfies the classical Yang Baxter Equation (CYBE), then the map $\delta_r : L \to L \wedge L$, $\delta_r(x) = [r, x]$ defines a Lie bialgebra structure on L. The pair (L, δ_r) is called a *coboundary* Lie bialgebra. If L is the Witt or Virasoro algebra, then the solutions of the CYBE of the form $a \wedge b \in L \wedge L$ were classified in [12]. With Earl's guidance, I proved in our joint paper [7] that if L is the one-sided Witt algebra (the Lie algebra of derivations of $\mathbb{C}[t]$), then every Lie bialgebra structure on L is a coboundary Lie bialgebra given by a solution of the CYBE r of the form $a \wedge b$. I was thrilled by the result and grateful for Earl's encouragement to complete the paper.

In another paper [8] of Earl's, the set of linearly recursive sequences over a field $\Bbbk$ was proven to be a Hopf algebra isomorphic to $(\Bbbk[x])^\circ$, the dual Hopf algebra of $\Bbbk[x]$. In particular, they are closed under the convolution product; if $(f_i)_{i \geq 0}$ and $(g_i)_{i \geq 0}$ are $\Bbbk$-linearly recursive sequences, then so is $\left(\sum_i \binom{n}{i} f_i g_{n-i}\right)_{n \geq 0}$. He asked whether they are also closed under the q-convolution product, which is similar to the convolution product but using the q-binomial coefficients $\binom{n}{i}_q$ where $q \in \Bbbk$ is a given root of unity. We ended up answering the question affirmatively with two different approaches in [4]: the first one is from the view point of dual Hopf algebras in the category of representations of a finite abelian group and the second one is a direct combinatorial method. This joint work with Earl also brought me to the representation categories of Hopf algebras, which are fundamental examples of finite tensor categories.

In the course of working on these joint articles, Earl demonstrated how I should carry on my career as a mathematician. We tested and discussed some ideas, wrote up notes on our discussions, and organized manuscripts for publication. He carefully edited all the drafts of these articles and meticulously verified the mathematics written on the drafts. In my PhD thesis, I extended the result of [6] on Lie bialgebras to those base fields of finite characteristic. I am deeply grateful for his guidance on my graduate research projects.

After finishing these collaborations, I returned to my focus on Hopf algebras. Earl recommended that I communicate with Susan Montgomery, who first coined the term "Taft algebra." I had a few email communications with Montgomery soon after I graduated from Rutgers, and I attended some seminars at MSRI in 1999, where I was excited by an open question presented by Montgomery (cf. [4]): If p is a prime number, must a nonsemisimple Hopf algebra over $\mathbb{C}$ of dimension p^2 be a Taft algebra? An affirmative answer to the question is of fundamental importance to the theory of Hopf algebras.

Figure 2. Ng, Hoffman, Witherspoon, Richmond, Heckenberger, and Taft at Bowling Green, Kentucky, 2005.

In Earl's 1971 paper [11], he constructed the Hopf algebra $T(q)$, later called the Taft algebra, for any primitive nth root of unity $q \in \Bbbk$. The underlying $\Bbbk$-algebra of $T(q)$ is generated by a, x subjected to the relations $xa = qax$, $x^n = 0$ and $a^n = 1$, and its coalgebra structure (Δ, ϵ) is given by $\Delta(a) = a \otimes a$, $\Delta(x) = a \otimes x + x \otimes 1$, $\epsilon(a) = 1$ and $\epsilon(x) = 0$. In 1998, Andruskiewitsch and Schneider [1] proved Susan's open question for *pointed* Hopf algebras of dimension p^2. Under Earl's encouragement in 2002, long after my graduation, I finally settled this open question in [5] over $\mathbb{C}$ by proving that p^2-dimensional Hopf algebras over $\mathbb{C}$ are pointed. I could not have done this without the continuous support of Earl. The result is not only a mathematical achievement, but also a reminder of Earl's unique place in my heart.

Beside mathematics, Earl had many other interests, particularly in languages. He was fluent in Spanish and French. I still remember that he studied Chinese before his trip to Hong Kong and mainland China in 1997. It was apparent to me that he was entertained by learning new languages. He also immensely enjoyed the city life in New York because of the proximity to good food and Broadway theatre. He took great pleasure in traveling. Before 2010, I met Earl quite often in conferences. He appreciated wine with meals. I usually joined him for a bottle because he was always more relaxed and in good humor after a couple of glasses of wine. Accompanying Earl to a conference was always a delight for me.

In addition to doing mathematics research, Earl promoted mathematics in different communities. He was the founding editor of *Communications in Algebra* in 1974. In his condolence message, Professor Kar-ping Shum, former president of the Southeast Asian Mathematical Society, described Earl's contributions to the Southeast Asian mathematics community. Earl initiated the first *International*

Congress in Algebra and Combinatorics (ICAC) in Hong Kong in 1997 [10], and proposed organizing the ICAC every ten years. He was a keynote speaker at the ICAC in 1997 and 2007, and helped edit the proceedings of the ICAC in 1997, 2007, and 2017, which were all published by World Scientific Publishing. Earl also helped develop the *Southeast Asian Bulletin of Mathematics*, which now publishes six issues per year. Professor Shum emphasized that ICAC and its proceedings have had a great impact on the mathematics community.

After I graduated from Rutgers, Earl provided his down-to-earth advice on every major decision of my career. In retrospect, Earl was my thesis supervisor, my life-long career advisor, and a great friend. I am wholeheartedly thankful to him for his unfettered support all these years and I am proud to have been his student.

Siu-Hung Ng

David Radford

I am forever grateful for the role Earl played in shaping my career. I have very fond memories of visits to Rutgers University and of my association with Earl there, as well as at numerous mathematical meetings. Earl was a gracious host, funny, thoughtful, and always considerate.

He invited me to give a talk in the Algebra Seminar at Rutgers in 1974 and invited me to spend the year there as a visitor in 1975–1976, which turned out to be mathematically very stimulating and productive. Among other things, Earl led the seminar discussion of Warren Nichols's PhD thesis, the origin of the Nichols algebras. Together with Robert Wilson, we wrote a paper on the antipode of a Hopf algebra [9].

That year also provided a great opportunity for me to look for another job with very little stress. It was a fruitful transition. From Rutgers, I went to the University of Illinois Chicago in 1976 where I remained on the faculty for the rest of my academic career. I was a visitor again at Rutgers in 1979–1980.

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Earl was the first to construct finite-dimensional Hopf algebras with antipode of a given even order [11], later called the Taft algebras. In this 1971 paper, he describes the important variation of the commutative law, namely $yx = qxy$, where q is a root of unity. This variation is fundamental in his construction and the theory of quantum groups, which are important mathematical structures discovered in the 1980s.

Taft algebras are basic examples of Hopf algebras that have been studied over the years in many contexts. Quantum groups are replete with them.

David Radford

References

[1] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, *Hopf algebras of order p^2 and braided Hopf algebras of order p*, J. Algebra **199** (1998), no. 2, 430–454, DOI 10.1006/jabr.1997.7175. MR1489920

[2] Uma N. Iyer and Earl J. Taft, *The dual of a certain left quantum group*, J. Algebra Appl. **15** (2016), no. 4, 1650072, 6, DOI 10.1142/S0219498816500729. MR3462203

[3] Yevgenia Kashina, Susan Montgomery, and Siu-Hung Ng, *On the trace of the antipode and higher indicators*, Israel J. Math. **188** (2012), 57–89, DOI 10.1007/s11856-011-0092-7. MR2897723

[4] Susan Montgomery, *Classifying finite-dimensional semisimple Hopf algebras*, Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), Contemp. Math., vol. 229, Amer. Math. Soc., Providence, RI, 1998, pp. 265–279, DOI 10.1090/conm/229/03323. MR1676225

[5] Siu-Hung Ng, *Non-semisimple Hopf algebras of dimension p^2*, J. Algebra **255** (2002), no. 1, 182–197. MR1935042

[6] Siu-Hung Ng and Earl J. Taft, *Quantum convolution of linearly recursive sequences*, J. Algebra **198** (1997), no. 1, 101–119. MR1482977

[7] Siu-Hung Ng and Earl J. Taft, *Classification of the Lie bialgebra structures on the Witt and Virasoro algebras*, J. Pure Appl. Algebra **151** (2000), no. 1, 67–88. MR1770644

[8] Brian Peterson and Earl J. Taft, *The Hopf algebra of linearly recursive sequences*, Aequationes Math. **20** (1980), no. 1, 1–17. MR569945

[9] David E. Radford, Earl J. Taft, and Robert Lee Wilson, *Forms of certain Hopf algebras*, Manuscripta Math. **17** (1975), no. 4, 333–338. MR396651

[10] Kar-Ping Shum, Earl J. Taft, and Zhe-Xian Wan (eds.), *Algebras and combinatorics*, Springer-Verlag Singapore, Singapore, 1999. Papers from the International Congress (ICAC'97) held in Hong Kong, August 1997. MR1733162

[11] Earl J. Taft, *The order of the antipode of finite-dimensional Hopf algebra*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 2631–2633. MR286868

[12] Earl J. Taft, *Witt and Virasoro algebras as Lie bialgebras*, J. Pure Appl. Algebra **87** (1993), no. 3, 301–312. MR1228159

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Jerry Tunnell (1950–2022)

Joe Buhler, Alex Kontorovich, and Stephen D. Miller

Figure 1. Jerry in Paris in 1980.

Jerrold (Jerry) Tunnell was born in Dallas, Texas, in 1950. His family moved to New Mexico when he was very young, and he attended high school in Farmington, NM. His interest in mathematics led him to choose to do his undergraduate work at Harvey Mudd College. His peers noticed (as described below by Ambassador Richard Jones[1]) that he tested out of some lower-level mathematics classes, and they soon learned to ask Jerry for help on their math homework. Jerry then went to Harvard University for graduate studies, starting in 1972 and finishing with a PhD in 1977 under John Tate. After six years in Princeton (at the Institute for Advanced Study and Princeton University) he joined the faculty at Rutgers University, where he remained until his untimely death in a bicycle accident in April, 2022.

Jerry's primary research area was algebraic number theory, much of it related in one way or another to Robert Langlands's seminal ideas and conjectures connecting algebraic number theory and automorphic representations. As with other overarching "one-to-one correspondences" in mathematics, the beauty of the ideas can be stunning, and knowledge on either side often illuminates the other.

After Jerry started his graduate work in the early 1970s many of Tate's students began to work on aspects of the Langlands program. One interesting case was a conjectural bijection between certain modular forms f of weight 1 and suitable two-dimensional Galois representations ρ : $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}(2, \mathbf{C})$. P. Deligne and J-P. Serre had just written a paper that proved that if the modular form f existed, then there was a corresponding Galois representation ρ. If ρ was constructed from one-dimensional representations (by taking a direct sum or inducing), it was known how to find the modular form f. But no one had any idea about how to do this in the genuinely two-dimensional tetrahedral, octahedral, or icosahedral cases.[2] Some of Tate's students were given the rather elaborate "exercise" of finding f for some specific tetrahedral and octahedral ρ. Jerry, and three other students of Tate (Dan Flath, Bob Kottwitz, and Jim Weissinger) were able to do a few examples by starting with a quartic polynomial, finding a Galois extension of $\mathbf{Q}$ whose Galois group G was a subgroup of $\mathbf{GL}(2, \mathbf{C})$ (equipped with the tautological representation ρ), and using this to find out enough about the putative modular form f to prove its existence. One pleasant corollary was that this proved Artin's conjecture for such ρ, i.e., that the corresponding L-series, $L(\rho, s)$, was a holomorphic function of s. This vivid experience was a strong motivation for Jerry's thesis work, which proved a local version of the Langlands correspondence for (many) local fields F, relating representations of Weil–Deligne groups of F to representations of $\mathbf{GL}(2, F)$. This was an impressive piece of work, e.g., requiring a careful examination of supercuspidal representations, and the results were used in subsequent research by other mathematicians.

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[1] *A classmate, roommate, and lifelong friend of Jerry's.*

[2] *For definitions and details, see Guy Henniart's contribution.*

Jerry was outgoing and socially active, joining in whatever activities were popular with the graduate students at the time (playing table tennis in the common room, learning to juggle, riding a bicycle to Cape Cod for a weekend excursion, etc.). One of his notable traits was that he enjoyed debate. This was not antagonistic or confrontational in the slightest, but instead reflected a real joy in (a) discovering disagreement, and (b) attempting to get at the truth via something like the Socratic method. His goal was not so much to win as to enjoy the interchange and see where it led. This was surely useful for his teaching, and the occasional tenacity (a.k.a. stubbornness) that he displayed was surely useful for his research.

One of the first applications of Jerry's thesis was in his own work. While finishing his thesis, he realized that he could extend Langlands's proof of the "Deligne–Serre" converse in the tetrahedral (and some octahedral) cases to all octahedral cases. This required using his thesis, Langlands's "base change" for automorphic representations, and results of H. Jacquet, I. Piatetski-Shapiro, and J. Shalika. This result became known as the Langlands–Tunnell theorem.

Jerry's thesis and the Langlands–Tunnell theorem marked the beginning of a decade of major contributions to mathematics.

One of his seminal papers looked at the ancient problem of finding "congruent numbers." A positive integer n is a congruent number if it is the area of a right triangle whose sides are rational numbers; this turns out to be equivalent to saying that the elliptic curve $y^2 = x^3 - n^2 x$ has infinitely many rational solutions. Surprisingly, this question is also related to modular forms with weight 3/2 and this led Jerry to an efficient algorithm which determines, in many cases, whether or not n is a congruent number.[3] This idea was pursued in several directions by many others in subsequent years.

Another major paper resulted from a collaboration with Jonathan Rogawski that showed that much of the Deligne–Serre theorem could be extended to any totally real ground field.

Throughout his career, Jerry would visit New Mexico regularly. Some of his most intense nonacademic interests, not very well known even to his academic and East Coast friends and colleagues, were rooted in his long and deep connections to New Mexico and its history. This led him to acquire art and artifacts from Navajo (and other Native American) and Spanish sources. About 15 years ago, Jerry and his wife Marlene purchased an abandoned church in a small town in New Mexico, with the intent of remodeling it and the adjacent parsonage. They spent many of

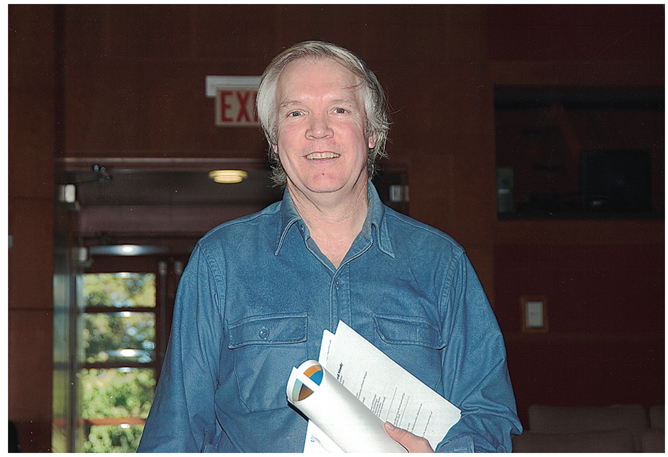

Figure 2. Jerry at IAS, Princeton.

their summers there, during which Jerry was able to spend even more time on his hobbies; Marlene is quite sure some items in his truly extensive collection were snuck into their house with the connivance of their son, Matt.

Jerry developed a real expertise in, and understanding of, these artifacts, and he became well known to artists, collectors, and museum directors. One of the jewelry makers that Jerry had "discovered" at an outdoor market learned only recently that Jerry had died in a bicycle accident. On hearing this, he burst into tears, saying that he owed his recognition and success to Jerry for having promoted his work to galleries all over the state.

Jerry's publication record later in his career does not accurately reflect his continued interest in learning and researching new mathematics, nor his impact on students and colleagues at Rutgers. He was a dedicated mentor and teacher who took the time to work closely with his students, providing guidance and support throughout their studies. He also participated actively in seminars, and was often the person who would ask the most penetrating question at the end of a talk. Jerry attended and hosted dinners, parties, and other events where he could connect with students on a personal level and foster a sense of community among them. And, of course, he rarely shied away from amiable debates with students or colleagues.

One the most vivid applications of Jerry's work was its role in Andrew Wiles's proof of Fermat's last theorem, arguably the most famous proof of the twentieth century.[4] Wiles, with Richard Taylor, showed that certain semistable elliptic curves were modular, which implied that no nontrivial integral solutions to $x^n + y^n = z^n$ existed for $n > 2$. At a certain crucial juncture, the Langlands–Tunnell theorem was a key ingredient in the argument.

[3] *See David Rohrlich's contribution.*

[4] *See Guy Henniart's contribution.*

David Rohrlich

Although I had been out of touch with Jerry Tunnell for many years, the news of his death was a brutal shock. In an earlier period of my life I had seen him frequently, first when he was finishing up his thesis at Harvard and I was a postdoc there, then when he was an assistant professor at Princeton and I was a faculty member at Rutgers, and later when he joined me at Rutgers, where we were colleagues for several years. He had an imperturbable manner and a tendency toward understatement, and he radiated a calm self-confidence without coming across as conceited. He was not one to divulge personal information gratuitously, and we never became close friends, but he was always fun to be with, and once you became involved in a conversation with him, he was an engaging interlocutor, freely revealing his knowledge and experiences and opinions in realms that you had never heard him talk about before. He was truly a man of the world, the epitome of sophistication without ostentation.

Jerry's list of mathematical publications is short but studded with diamonds. Of particular note are his work on the local Langlands conjecture for GL(2) [Tun78], the octahedral case of the Artin conjecture [Tun81], the Deligne–Serre theorem for Hilbert modular forms (joint with Rogawski) [RT83], and the connection between the congruent number problem and half-integal weight modular forms [Tun83]. In this last paper, Jerry managed to do something that very few mathematicians can ever hope to do: He used the tools of contemporary number theory and automorphic forms to make a major advance toward the solution of a problem that had remained open for a thousand years.

Let us call a right triangle *rational* if the length of each of its three sides is a rational number. Of course the area of such a triangle is also a rational number, and the congruent number problem asks for a characterization of the rational numbers that arise in this way. Since the set of rational right triangles is closed under scaling by rational numbers, one may assume that the three sides of the triangle have lengths $x^2 - 1$, $2x$, and $x^2 + 1$ for some rational number $x > 1$, whence the area is $x^3 - x$. Write this area in the form dy^2, where d is a square-free integer and y is rational. Then d is called a "congruent number," and the problem is to determine the set of positive square-free integers so obtained. Equivalently (after an argument to remove the condition $x > 1$), the problem is to characterize the set of positive square-free integers d for which there exists a rational point (x, y) on the elliptic curve $E_d : dy^2 = x^3 - x$ with $y \neq 0$. Such a point is necessarily a point of infinite order, so the conjecture of Birch and Swinnerton-Dyer predicts that d is a congruent number if and only if the L-function $L(E_d, s)$ vanishes at $s = 1$. Now $L(E_d, s)$ is just $L(E_1, \chi_d, s)$, the twist of $L(E_1, s)$ by the quadratic character χ_d corresponding to d. It follows that the vanishing and nonvanishing of $L(E_d, s)$ as d varies is controlled by a modular form of weight 3/2—a modular form which corresponds via the work of Shimura and Waldspurger to the cusp form of weight 2 underlying E_1. Identifying the relevant form of weight 3/2, Jerry was able to give a necessary condition for d to be a congruent number, a condition which is also sufficient if one grants the conjecture of Birch and Swinnerton-Dyer.

It is worth pointing out that "publishing" means "making public, disseminating," and quite apart from his published journal articles, Jerry was good at disseminating mathematical ideas and information in the midst of conversations. I certainly learned a lot from conversations with him, and I imagine others did as well. A case in point: my first encounter with the notion that there should be a version of the conjecture of Birch and Swinnerton-Dyer for twists of L-functions of elliptic curves by Artin representations was a casual remark made by Jerry at a seminar dinner in Princeton around 1980. The conversations that led to our only joint paper [RT97] are perhaps in a different category, since they were one-on-one and probably quite focused from the start, but for the record, the initiative to discuss Serre's conjecture and then to think about the case $p = 2$ was entirely Jerry's. The resulting paper appeared long after it was written, and it is only thanks to Dinakar Ramakrishnan's urging that it appeared at all. But the amusing point to emphasize here is how Jerry characterized our joint project: He cited the British novelist Graham Greene, who distinguished between his serious works and his "entertainments." Jerry proposed [RT97] as an "entertainment," a characterization which I find both delightful and very apt, given the sharp contrast with Jerry's more serious works.

Jerry figures in many of my memories of mathematical gatherings from long ago, and often an amusing or enlightening comment that Jerry made at an event is one of the things that I remember best about it. Most of the people in those memories are still alive, and it is devastating that Jerry is not. He left us much too soon.

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Figure 3. Jerry and Andrew Wiles.

Guy Henniart

In 1981 Jerry published a striking three-page research announcement [Tun81], with proofs, entitled *Artin's conjecture for representations of octahedral type*. The wonderfully concise introduction was:

> Let L/F be a finite Galois extension of number fields. E. Artin conjectured that the L-series of a non-trivial irreducible representation of $\mathrm{Gal}(L/F)$ is entire, and proved this for monomial representations. The non-monomial two-dimensional representations are those whose image in $\mathbf{PGL}(2,\mathbf{C})$ is isomorphic to the group of rigid motions of the tetrahedron, octahedron or icosahedron... Langlands proved Artin's conjecture for all representations of tetrahedral type, and certain octahedral representations when $F = \mathbf{Q}$. The purpose of this note is to prove the conjecture for all octahedral representations by using the methods of Langlands and an analytic result of Jacquet, Piatetskii-Shapiro and Shalika.

Let us explain this a bit more.

The L-series $L(\rho, s)$ of a Galois representation

$$\rho : \mathrm{Gal}(L/F) \to \mathbf{GL}(n,\mathbf{C})$$

of dimension n is a function of a complex variable s which generalizes classical examples of the Riemann zeta function, Dirichlet L-series, and the Dedekind zeta function of a number field. The L-series is initially defined for $\mathbf{Re}(s) > 1$ by an Euler product, but Artin proved that it has a meromorphic continuation to the whole complex plane, and has a nice functional equation relating $L(\rho, s)$ and $L(\rho', 1-s)$ where ρ' is the dual representation. When $n = 1$ and ρ is nontrivial, the holomorphy of $L(\rho, s)$ follows from class field theory. If ρ is *monomial*, in the sense of being induced from a one-dimensional representation, then the induced representation has the same L-function as the original one, and Artin's conjecture is true. Beyond these cases, the holomorphy of $L(\rho, s)$ for non-monomial and irreducible ρ was a mystery.

In a letter to Weil dated 1967, Langlands proposed, among other things, a very strong heuristic reason for the Artin conjecture: $L(\rho, s)$ should be **equal** to the L-series $L(\pi, s)$ of a cuspidal automorphic representation π of $\mathbf{GL}(n)$ over F, and the L-series $L(\pi, s)$ of such an automorphic representation is known to be entire. In the monomial case, an idele class character is tantamount to an automorphic representation for $\mathbf{GL}(1)$ and H. Jacquet and Langlands (following Hecke) produced such a π for $n = 2$ and monomial ρ. The general conjecture of Langlands came to be known as the strong Artin conjecture.

Irreducible non-monomial ρ of dimension 2 have image in $\mathbf{PGL}(2,\mathbf{C})$ isomorphic to a permutation group that is also the symmetry group of a regular polyhedron: A_4 (tetrahedral), S_4 (octahedral), or A_5 (icosahedral). The icosahedral case seemed out of reach (and in fact, the case of odd Galois representations over $\mathbf{Q}$ wasn't resolved until 30 years later), but the cases with solvable galois groups seemed approachable. Indeed, in those cases the Galois group has a subgroup of index 3 corresponding to a cubic extension K of the ground field F, and the restriction of ρ to K becomes monomial, so there is a unique cuspidal automorphic representation Π of $\mathbf{GL}(2)$ over K with $L(\Pi, s) = L(\rho_K, s)$.

Thus one looks for a process—called base change—associating to an automorphic representation Π of $\mathbf{GL}(2)$ over F an automorphic representation π_K of $\mathbf{GL}(2)$ over K, which would correspond to restricting ρ to $\mathrm{Gal}(L/K)$. It was hoped that this would allow π to be constructed from Π.

At a conference in Michigan in 1975, H. Saito and T. Shintani proposed such a construction for cyclic F'/F, using a so-called twisted trace formula for $\mathbf{GL}(2)$. Langlands then "caught fire" (in Jacquet's words) and soon a preprint *Base change for* $\mathbf{GL}(2)$ appeared. For a cyclic extension F' of F, the base change π'_F of an automorphic representation π of $\mathbf{GL}(2)$ over F is an automorphic representation of $\mathbf{GL}(2)$ over F' which is invariant under the natural action of the cyclic group $\Gamma = \mathrm{Gal}(F'/F)$. Any Γ-invariant cuspidal automorphic representation π of $\mathbf{GL}(2)$ over F' is a base change of some cuspidal π, and the other possibilities are the twists of π by idele class characters corresponding to characters of Γ. The proof of Artin's conjecture for tetrahedral ρ follows from this and a clever construction of S. Gelbart and Jacquet.

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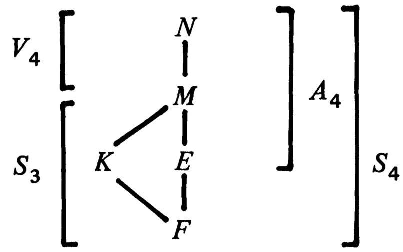

Figure 4. Field diagram for the octahedral case.

For octahedral ρ over an arbitrary number field F, one can start with the tetrahedral representation obtained by restricting Π to the quadratic field E over F. However, the techniques used in the tetrahedral case are not sufficient to determine a corresponding representation over F.

Jerry solved this puzzle by using both the cubic extension K/F (which is not Galois in the octahedral case) and the quadratic extension E/F. He used work of H. Jacquet, I. Piatetskii-Shapiro, and J. Shalika that had shown how to construct a base change associating to an automorphic representation τ of $\mathbf{GL}(2)$ over F an automorphic representation τ_K of $\mathbf{GL}(2)$ over the non-Galois cubic field K. They did not use the techniques of Gelbart and Jacquet, but rather analytic properties of the so-called Rankin-Selberg L-series.

Jerry's crucial "lemma" is that there is only one choice of $\pi = \pi_1$ or π_2 whose base change to K is the automorphic representation corresponding to ρ_K. His next main theorem is that $L(\pi, s) = L(\rho, s)$, requiring the the Euler factors in the definitions be compared term by term. So $L(\rho, s)$ is indeed entire! The idea is elegant, and the details are in Jerry's very readable note.

It is still not known how to go significantly beyond dimensions 2 or 3 for Artin's conjecture using only automorphic techniques.

Another conjecture which can be viewed as part of the broad Langlands program, but had independent and earlier origins variously attributed to G. Shimura, Y. Taniyama, and A. Weil, is often called the modularity conjecture. Roughly, it asserts that elliptic curves over the rational numbers are in bijection with certain modular forms of weight 2.

In the 1980s, it became gradually clear, through the influence of Serre and Mazur, that it might be useful to approach modularity by looking at modular representations of Galois groups (i.e., with coefficients in finite fields). The idea was to associate automorphic representations to their deformations to characteristic 0 (an ℓ-adic representation rather than a complex one). The goal was then to apply this to the modularity conjecture.

In 1985, Frey, using a curve of Hellegouarch, showed that if there was a nontrivial solution to the equation $a^p + b^p = c^p$, for $p > 3$, then modularity was likely to be false for a specific elliptic curve, built out of a, b, c. In 1985 J-P. Serre showed that modularity would indeed be false in that context if a certain precise statement about modular forms, arising out of his general conjectures on modular Galois representations, was true. This so-called "epsilon-conjecture" was proved in 1986 by K. Ribet. All of a sudden, modularity implied Fermat's last theorem!

Figure 5. Karl Rudnick on the left and Jerry on the right, in Texas in 2022.

Much of this emerged at, or near the time of, a year-long program at MSRI in algebraic number theory. This was a hot topic for the participants, of which Jerry was one. The recent successes of Langlands, Tunnell, and others were in the back of people's minds, and at first this made this attack on modularity seem promising. People knew that one had to start with modular representations for which at least one deformation had an associated automorphic representation. The group $\mathbf{PGL}(2, \mathbf{F}_2)$ is S_3, but this isn't usable in this context. However, $\mathbf{PGL}(2, \mathbf{F}_3)$ is isomorphic to S_4, and Langlands–Tunnell implies that for suitable Galois representations into $\mathbf{GL}(2, \mathbf{F}_3)$, one can indeed get an associated automorphic representation. This seemed to be exciting for a while, but no one got anywhere, and many began to feel that modularity was still far off in the future.

Fortunately, Andrew Wiles did not share that opinion! It took seven years of his hard work (and help from Richard Taylor), to prove modularity for the needed curves. One of the curious features of that proof was that at one point it required a delicate dance between the primes $p = 3$ and 5. By a strange twist of fate, the key fact needed for $p = 3$ was ... the Langlands–Tunnell theorem.

MEMORIAL TRIBUTE

Richard Jones

The squeal of brakes split the air followed by a loud "whomp" and then silence. It was day 15 of our planned cross-country bike trip from the Atlantic coast at Saint Augustine, Florida, to our alma mater in Claremont, California, where we planned to celebrate the 50th anniversary of our graduation from Harvey Mudd College. After our lunch stop, the five riders in our jaunt had become stretched out along our planned route for the afternoon. Jerry Tunnell, my longtime friend and former college roommate, was in the lead. I was trailing him by 100 yards or so and the other group members were somewhere behind me.

I quickly focused my attention on the road in front of me and was astonished to see a semi-truck trailer jackknifed across both lanes of traffic stopped a few yards in front of me. I immediately slammed on my brakes thinking that the truck must have somehow collided with a vehicle in the oncoming lane.

Wondering if Jerry had seen the accident, I dismounted and hurried around the truck to see the collision.

However, I was surprised not to see any signs of another vehicle.

As I pondered this the driver of the semi appeared on the tarmac next to me.

"I saw it," she exclaimed, pointing down the road. "That other truck hit that guy."

I immediately focused down the highway and for the first time saw Jerry. He was lying still on the tarmac a few hundred feet away.

As I hurried to him, I was heartened to see that he was lying on his side, as if resting, and that there were almost no visible signs of injuries, just a few scratches on his limbs. His bike helmet was still on and looked fully intact. This raised my hopes that perhaps he was only shaken up.

However, when I reached him he was unconscious. After calling 911, I knelt next to him and tried to rouse him. I began rubbing his back and calling his name. Another person soon joined me and began searching for a pulse while also saying his name. After a few minutes, however, the other man shook his head and stood up. Slowly it began to dawn on me that my earlier optimism had been misplaced. By the time the other members of our group began arriving on the scene, I knew that it didn't look good for Jerry.

Richard Jones had a long and distinguished career at the State Department, serving as ambassador to Israel, Kuwait, Lebanon, and Kazakhstan, and held a variety of positions (including deputy executive director of the International Energy Agency) under Presidents Clinton, Bush, and Obama.

An EMT soon arrived and began assessing Jerry's condition. After a few minutes, I was disheartened, but not surprised, to hear him calling the hospital and saying that Jerry was a "probable DOA." A short time later an ambulance arrived, but all that they could do was confirm the EMT's initial assessment with an EEG. I touched Jerry one more time, but this time his body was cold. I knew that our friend and colleague for more than 50 years, the inspiration for our cross-country expedition, was dead.

In the days and now weeks since that tragic afternoon, I have struggled to come to terms with Jerry's loss.

I have found that the best way to short circuit these bouts of self flagellation is to recall the good times Jerry and I had together over the years starting with our undergraduate careers at Harvey Mudd College (HMC) where we met on the first day of the Claremont Colleges' orientation week for new students in September 1968. We quickly found that we had something in common. Jerry was from a small town in New Mexico, and I had an older brother working in another small town in New Mexico. This was the beginning of our friendship that lasted more than 50 years.

One night during the orientation week Jerry and I went down to a mixer event at one of the other Claremont colleges with some other HMC freshmen. Jerry and another HMCer were energetically discussing a problem on a test they had taken that day to skip first semester freshman calculus. Jerry was sure of his solution and laid it out on a napkin for the other student who had not been able to solve it.

Evidently, Jerry's solution was correct; he passed the test. His reward was to be able to take an introductory course on complex variables a year earlier than most of the rest of our class, including me. Considering that Jerry's small high school in New Mexico had not offered a calculus course and he had studied it on his own over the summer, this was an impressive achievement.

As that first year progressed students' reputations gradually began to be solidified based on their performance in various areas. (Freshmen at HMC all took the same curriculum. The possibility of replacing first semester calculus with complex variables being the notable exception.) Jerry gained a reputation for being a good problem solver whose solutions were invariably correct. Other students began coming to him for help on their homework and he seemed to enjoy explaining the material to them, although once he complained to me that he had a hard time empathizing with those who could not understand points that were obvious to him.

Once, in response to my question as to why he was so good at problem solving, Jerry directed my attention to a small book on his bookshelf. It was *How to Solve It* by

G. Polya, which he evidently had acquired in high school, or perhaps when he had attended an NSF-sponsored summer program in chemistry. Among the many lessons in Polya's book is that success in solving one challenging problem builds confidence and leads to more problem-solving success. Another is that to really understand a subject area students should create and solve their own problems.

Jerry certainly took these nuggets to heart. He returned from summer vacation for sophomore year having completely worked through the more than 700 pages of Kreider, Kuller, Ostberg, and Perkins's *An Introduction to Linear Analysis*. This allowed him to skip another course and get further ahead of the rest of us. At some point that year, he proudly displayed his understanding of boundary value problems to me by devising (and solving) a problem involving the heating of a pepperoni pizza, our snack of choice for watching late night TV. His growing devotion to the study of mathematics was also evinced by his habit of bringing whatever math text he was reading with him when invited to go to the movies or some other social event, just in case things got boring.

Although we roomed with each other starting in sophomore year, we didn't actually spend much time discussing math, since we shared only one math class while at HMC. I rarely needed his help with my problems and he never needed my help with his! In fact, our academic discussions focused on Russian (which we took together for three years), or on some of the common core classes.

After graduation, we went our separate ways for graduate study. He started at Harvard in the fall of 1972, and after a gap year I entered a PhD program at UW Madison in 1973. Despite the lack of e-mail in those days we managed to stay in touch. In summer 1974, I finished a cross-country bike trip in Boston and Jerry rode his bike out from Somerville, where he was living, to meet me on the last day of my ride and guide me to his apartment. Although we had frequently ridden bikes together to and from Russian class, this was the first time that I rode with Jerry on the open road for recreation. I was pleasantly surprised that he had no difficulties in keeping pace with me on the ride into Boston.

Jerry also came to visit me and my wife in Madison on more than one occasion in the years that followed. A few years later, he visited us in northern Virginia by bike from Princeton. At that point I was already working at the US State Department. Later, when my wife and I were posted to Paris in the early 1980s, he came more than once to visit us during our three years there. On one of these visits I remember asking him about his proof of a 2000-year-old conjecture. [See David Rohrlich's contribution.] However, he said he really didn't want to discuss it, explaining that every time he was invited to a conference or to visit another mathematician he was always asked to talk about this proof and he was just sick of it. He found his subsequent work much more interesting but nobody wanted to hear about it!

After Paris we were transferred to Tunis in North Africa and, recently married, Jerry and Marlene visited us there, where we thoroughly enjoyed getting lost together on a desert road trip when our newly published map directed us to a road that had yet to be completed! Such good times continued almost annually, usually either in New Mexico or in New Jersey, for the next 40 years. During this period our friendship grew even deeper as we matured and faced life's many challenges of career and family.

Jerry was always good-natured and affable, even though he could be stubborn when he wanted to be. Sometimes I would badger him about not retiring or attending HMC's quinquennial class reunions. He quipped that he would work until he died as it was easy for him, and he was good at teaching. He promised that he would attend a reunion but not until the 50th, and it had to be by bicycle from the East Coast. This was the genesis of our ill-fated cross-country trip.

He was probably my closest friend in college, as close as a brother. He was exasperating at times, but I loved him all the same and knew we would always be close. The world is poorer for his departure from it. He will certainly be missed as a great mathematician, but he will be missed more by those of us who knew him as a true friend.

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References

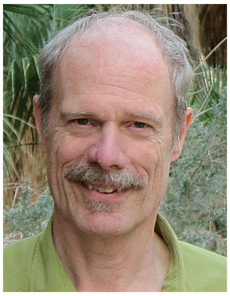
[RT83] J. D. Rogawski and J. B. Tunnell, *On Artin L-functions associated to Hilbert modular forms of weight one*, Invent. Math. **74** (1983), no. 1, 1–42, DOI 10.1007/BF01388529. MR722724

[RT97] David E. Rohrlich and Jerrold B. Tunnell, *An elementary case of Serre's conjecture*, Pacific J. Math. **Special Issue** (1997), 299–309, DOI 10.2140/pjm.1997.181.299. Olga Taussky-Todd: in memoriam. MR1610808

[Tun78] Jerrold B. Tunnell, *On the local Langlands conjecture for GL*(2), Invent. Math. **46** (1978), no. 2, 179–200, DOI 10.1007/BF01393255. MR476703

[Tun81] Jerrold Tunnell, *Artin's conjecture for representations of octahedral type*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), no. 2, 173–175, DOI 10.1090/S0273-0979-1981-14936-3. MR621884

[Tun83] J. B. Tunnell, *A classical Diophantine problem and modular forms of weight* 3/2, Invent. Math. **72** (1983), no. 2, 323–334, DOI 10.1007/BF01389327. MR700775



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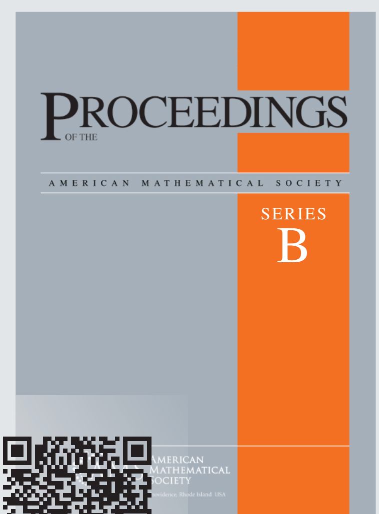
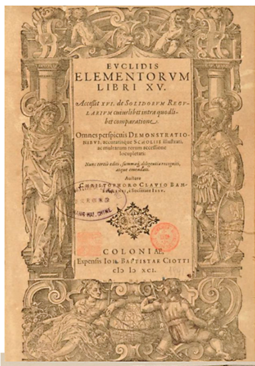
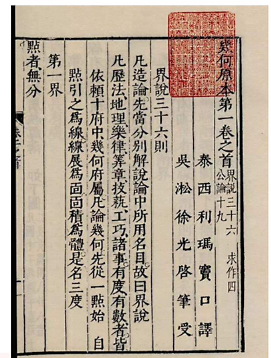
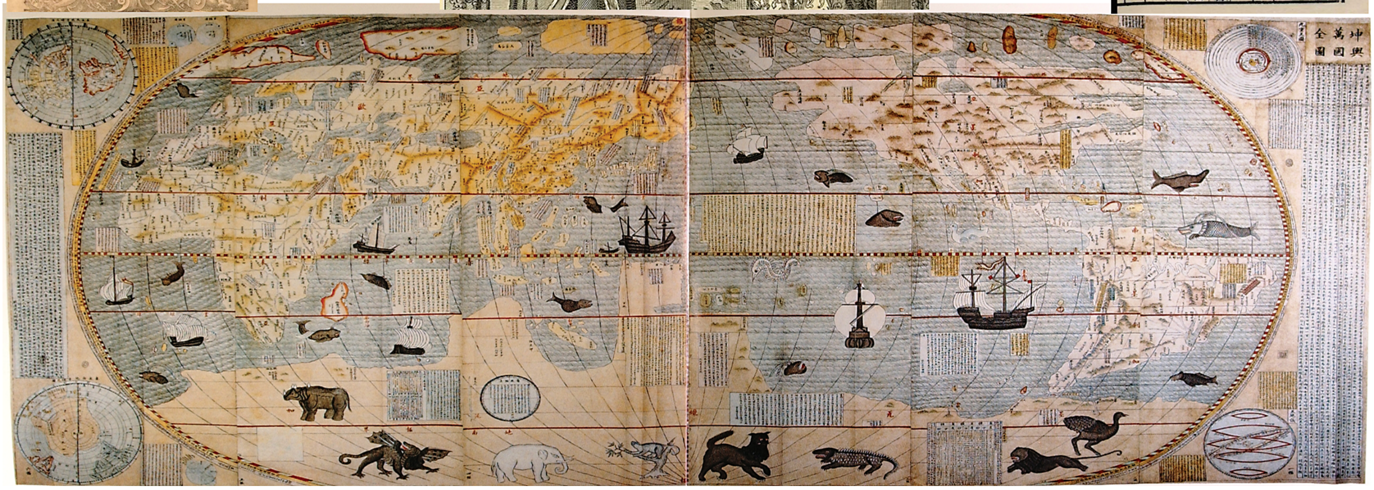


The Journey of Euclid's *Elements* to China







Chuanming Zong

In 1582, the Italian priest Matteo Ricci arrived in China with a copy of Euclid's *Elements* in his luggage. In 1606, he and the Chinese scholar Xu Guangqi started to translate it into Chinese. Unfortunately, Ricci died in 1610 in Beijing. At that time they had only finished the first six Books. The remaining seven Books were not translated into Chinese until 250 years later, in 1857, by the British missionary Alexander Wylie and the Chinese mathematician Li Shanlan. This paper tells the dramatic story.

1. Introduction

About 2,300 years ago, Euclid (330 BC–275 BC) completed his Στοιχεία, commonly known in English as *Elements*, one of the most important mathematical works to

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date. It contained thirteen Books (see [4, p. 79]):[1] I. Foundations of plane geometry. II. The geometry of rectangles. III. The geometry of circles. IV. Regular polygons in circles. V. Magnitudes in proportion. VI. Geometry of similar figures. VII. Basic arithmetic. VIII. Numbers in continued proportion. IX. Prime numbers. X. Incommensurable line segments. XI. Foundations of solid geometry. XII. Areas and volumes. XIII. The Platonic solids. In particular, it established the mathematical system of definitions, axioms, and propositions, which has served as a model of rigorous mathematical reasoning for more than two thousand years, and perhaps will last forever.

In the following centuries, Euclid's original copy was nowhere to be found. Instead, its contents were scattered in different manuscripts. At the end of the fourth century AD, Theon of Alexandria collected the manuscripts and edited a complete Greek copy. As one of the most important books in human civilization, Euclid's *Elements* has a complicated and dramatic history (see [3, 17]). It has been translated into almost every language, in more than one thousand different editions.

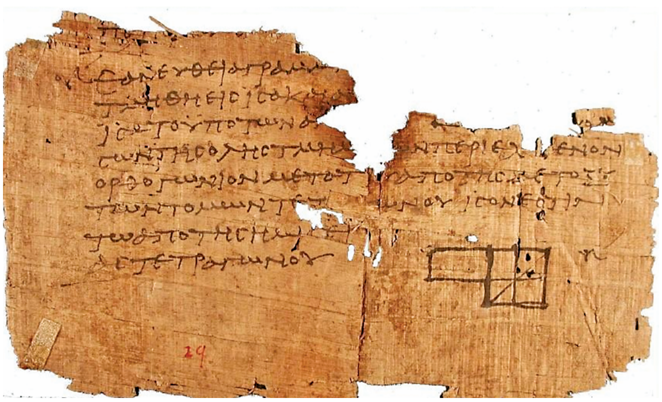

Figure 1. The earliest discovered *Elements* on papyrus.

In ancient times, China had few connections with other civilizations. More or less at the time of Euclid, the Chinese developed their own arithmetic and elementary geometry. Their earliest mathematical achievements were collected in *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art*,[2]) which was compiled during the early Han Dynasty (202 BC–220 AD). No author or editor is known (Zhang Cang (256 BC–152 BC) and Geng Shouchang in Han Dynasty were the earliest recorded compilers). The book discussed 246 practical problems, such as how to do basic arithmetic, how to compute the area of a rectangle, an isosceles triangle, or a circle, how to calculate the volume of a pyramid, a cylinder, or a ball, and how to solve linear equations, in nine chapters (see [7, 13, 14]). Of course, there were mistakes and inaccuracies in the early versions. In fact, many important later Chinese contributions to mathematics were achieved as commentaries to revisions of the *Nine Chapters*. Needless to say, it is the founding work of the Chinese mathematical tradition.

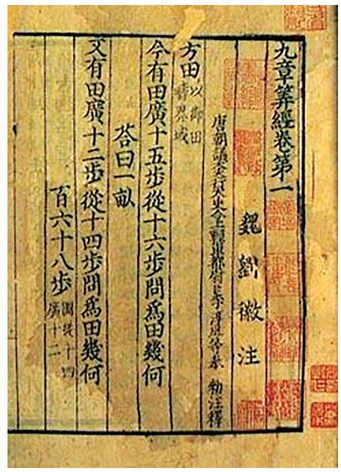

Figure 2. The title page of the *Nine Chapters*, commented by Liu Hui (c. 225–295).

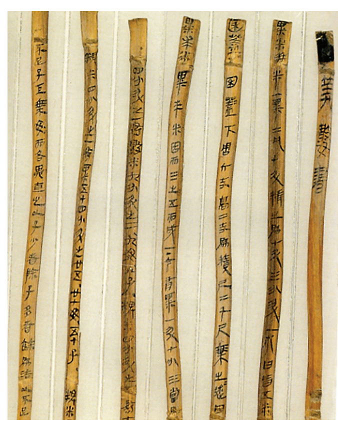

Figure 3. Bamboo strips of the *Suanshu Shu*.

In 1983–1984, Chinese archaeologists excavated an ancient tomb at Zhangjiashan in Hubei Province, which can be dated to around 186 BC in the Han Dynasty, and discovered a mathematics book, *Suanshu Shu* (*Book of Numbers and Computations*), written on some 200 bamboo strips. It dealt with 68 practical problems: 9 about multiplication and manipulating fractions, 44 on mercantile arithmetic such as collecting taxes and dividing coins, and 15 on geometric issues such as areas and volumes (see [4, 6, 7]). For example, it presented a method to determine the volumes of a cone and a frustum, under the assumption that $\pi = 3$. It is interesting to see that some problems of the *Suanshu Shu* are similar to those of the *Nine Chapters*, since the first is an original source and the second has been reedited many times.

In 221 BC, a little later than Euclid, the Qin Emperor (259 BC–210 BC) unified China. From 221 BC to 1912, China was ruled by more than four hundred emperors and Chinese society was rather closed. The earliest foreign influences to China were recorded in the Han Dynasty (202 BC–220 AD), Tang Dynasty (618–907), and Yuan Dynasty (1271–1368). Buddhism was introduced to China in the Han Dynasty from India. Islam was introduced to China in the Tang Dynasty from West Asia. Along with culture and religion, foreign knowledge such as astronomy and mathematics was also introduced to China. For example, the Persian astronomer Jamal ad-Din Bukhari worked

[1] *In some later editions, it was extended to fifteen Books, but Books XIV and XV are spurious.*

[2] *In some literature it was translated as Nine Chapters on Mathematical Procedures.*

in the Imperial Astronomical Bureau of the Yuan Dynasty and introduced Islamic astronomy into China. Marco Polo (1254–1324) also visited China in the Yuan Dynasty. However, at that time, there was no printed *Elements* yet.[3]

In 1552, the first European missionary St. Francis Xavier (1506–1552) arrived in China. In fact, he did not set foot on the Chinese mainland, but rather an island off the south coast. He died there. The early missionaries mainly engaged in religion dissemination. The Chinese did not know the systematic work of the ancient Greeks until Matteo Ricci's visit at the end of the 16th century.

For more about ancient Chinese mathematics, we refer to [4, 6, 7, 13, 14, 20].

2. Rome

In 1552, Matteo Ricci was born into a wealthy family in Macerata, Italy. He first attended a school run by the Jesuit priests in Macerata for seven years. Afterward, according to his father's wishes, he went to Rome to study law at Sapienza University. At that time, his father was the governor of his home province. However, Matteo had no interest in law, but was attracted to religious life. In 1571, he joined the Society of Jesus and went to the Jesuit College in Rome (Collegio Romano).

The college (today the Gregorian University), founded in 1551, was one of the first and most important Jesuit colleges. It offered courses not only in theology and religion, but also in mathematics, astronomy, and other subjects. When Ricci entered the college, Christopher Clavius (1537–1612) was a professor of mathematics there. In fact, it was Clavius who initiated an important tradition of Jesuit research by emphasizing applied mathematics and insisting on the need for the serious study of mathematics in the humanities.

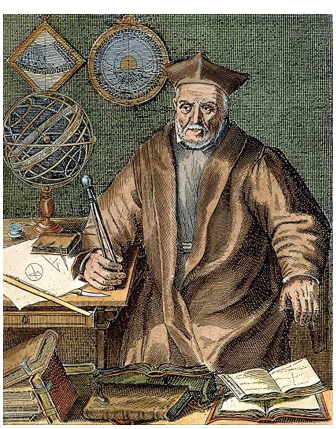

Figure 4. A portrait of Christopher Clavius.

Clavius was born near Bamberg, Germany. He entered the Jesuit order at Rome in 1555. The next year, he was admitted to the University of Coimbra in Portugal, where he learned mathematics and astronomy. In 1561, Clavius returned to Rome and studied theology at the Jesuit College. He graduated in 1564 and began teaching mathematics there. In fact, apart from a two-year stay in Naples, he remained in

Figure 5. The cover of Euclid's *Elements*, reedited by Christopher Clavius.

Rome as professor of mathematics at the Jesuit College for the rest of his life (see [9]).

In 1570, Clavius published an astronomy textbook, *Commentarius in Sphaeram Joannis de Sacro Bosco*, which influenced astronomers including Tycho Brahe, Johannes Kepler, and Galileo Galilei. His astronomical work made him highly instrumental in the calendar reform of 1582 that resulted in the introduction of the Gregorian Calendar (see [1, 9]).

At the Jesuit College in Rome, Ricci studied Latin, Greek, theology, mathematics, and astronomy. He attended Clavius's classes in mathematics and astronomy, where he learned Euclid's *Elements*, Apollonius's *Conics*, Archimedes's *Measurement of a Circle*, and Sacrobosco's *On the Sphere of the World* (see [8]). In particular, when Clavius published a Latin version of the *Elements* with his commentaries in 1574, Ricci got a copy and brought it to China later. The master and the pupil kept up a good friendship. When Ricci was in India and China, they corresponded by letters.

[3] *The Elements was first printed by the Italian publisher Erhard Ratbolt in 1482, based on the Latin version of Campanus of Novara (c. 1220–1296).*

In 1577, Ricci and three other students of the Jesuit College offered themselves for the East Indian missions. Before their departure, they were received by Pope Gregory XIII. They first went to Portugal, where Jesuit missionaries received formal instruction before going to the East, then took a six-month voyage from Lisbon to Goa, a Portuguese colony on the west coast of India. At that time, Portugal had colonies on the coasts of Africa and India and dominated this sea route. In fact, it was the only route for the European missionaries to the East. Ricci continued his studies for the priesthood in Goa and was ordained priest in 1580.

3. Peking (Beijing)

In 1557, Macao became a Portuguese colony on the south coast of China. In 1571, a house of the Jesuits had been set up at Macao. In 1582, to extend the China mission, Matteo Ricci was called from Goa to Macao by Alessandro Valignano (1539–1606), the superior of all Jesuits in the Far East and a former teacher of Ricci when he studied at the Jesuit College in Rome.

When Ricci arrived at Macao, China was at the end of the Ming Dynasty (1368–1644). The dominant hierarchy was conservative and corrupt, and establishing the Jesuit mission was very difficult. In particular, obtaining residence permissions for foreign priests was very hard. Thus, Ricci quickly learned Chinese language and culture, dressed as a Chinese scholar, presented himself as a Buddhist monk rather than as an European priest, and tried to make friends with every Chinese person he met. There are many books about Matteo Ricci (see [10, 19], for examples). For more on Ricci and Chinese mathematics, we refer to [11].

Figure 6. A portrait of Matteo Ricci.

In 1583, priests Michele Ruggieri (1543–1607) and Ricci visited the governor of Zhaoqing, a Chinese state close to Macao. The governor was deeply impressed by their demonstrations: a mechanical clock which can ring the hours, a prism which can produce colorful lights in sunshine, and a harpsichord which can make music. In particular, the governor was suffering with some disease and the priests were able to cure him quickly. This made them good friends of the governor and they obtained permission to live in Zhaoqing. In twenty years, Ricci moved from Macao in the deep south to the Chinese capital Peking step by step, from Macao (1582–1583) to Zhaoqing (1583–1589), Shaozhou (1589–1595), Nanchang (1595–1598), Nanking (1598–1600), and finally, Peking (1601–1610). Of course, he traveled with his precious books, including Euclid's *Elements*.

While Ricci was in Shaozhou, he met Qu Rukui (1548–1610), a bright young scholar and son of a top official of the Ming government. Ricci taught Qu mathematics and astronomy, and Qu introduced many important friends to Ricci. In fact, they once had the idea to translate the *Elements* into Chinese together and even finished part of the first Book. Unfortunately, the project was terminated at an early stage without known reason (see [13, p. 21]).

Having been prepared for almost twenty years, with the help of his important Chinese friends including princes, ministers, governors, and scholars, Ricci arrived in Peking in 1601 with his assistant Diego de Pantoja (1571–1618). Upon arriving in Peking, they presented the Emperor a set of well-prepared European gifts, including two magnificent mechanical clocks, a decorated harpsichord, a portrait of the Virgin Mary, and a beautiful world map in Chinese which was made by Ricci himself. They were invited to the Forbidden City, but were not received by the Emperor. Nevertheless, for his knowledge of science and technology, Ricci was appointed a royal position maintaining the clocks and other mechanisms in the Forbidden City. In this way, he obtained not only permission to reside in Peking, but also a royal salary.

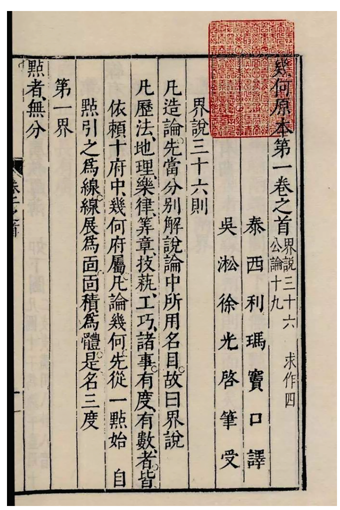

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Figure 7. The title page of the Chinese version of Euclid's *Elements*, translated by Matteo Ricci and Xu Guangqi.

In 1604, Xu Guangqi (1562–1633) succeeded in the Imperial Examination and was assigned a position in the Hanlin Academy.[4] From that point, he often visited Ricci and they became close friends. In fact, Xu first met Ricci in 1600 in Nanking, where he was baptized by Jean de Rocha (1566–1623) in 1603. Through this friendship Xu learned a lot from Ricci, not only in theology and religion, but also in astronomy and mathematics. In 1606, Ricci and Xu started to translate Euclid's *Elements* from

[4] *In the Ming Dynasty, the Hanlin Academy was somehow a secretariat of the court, where the emperor and the government could obtain advice about agriculture, astronomy, weather, natural phenomena, etc.*

Clavius's Latin version into Chinese: Ricci translated it from Latin to oral Chinese, after which Xu formulated it into classical Chinese. In 1607, they finished the first six Books and published them under the Chinese name *Jihe Yuanben* in Peking (see [13], for example). Perhaps, they planned to translate the remaining Books of the *Elements* later. Unfortunately, Xu's father died in 1607. In ancient China, a ranked official was expected to stay at home for three years if his father or mother died. Even more regrettably, when Xu returned to Peking three years later, Ricci had died.

The Chinese version of the *Elements* formally introduced "logical thinking" into Chinese mathematics. The traditional Chinese induction was mainly based on concrete examples, instead of logic and assumptions. Euclid's mathematics is different from the ancient Chinese mathematics in nature. Unfortunately, at that time the Chinese education system paid no attention to mathematics and therefore only a handful of people knew *Jihe Yuanben*. Nevertheless, many Chinese mathematical terms (such as *jihe* for geometry, *pingmian* for plane, *sanjiaoxing* for triangle, *lifangti* for cube, and *tiji* for volume) created by Ricci and Xu are still in usage today (see [8]).

Besides Euclid's *Elements*, Ricci and his collaborators also introduced several other Western books into China (see [13, p. 22]). For example, in 1607 he and Xu translated *Explanations of the Methods of Measurement* (the author is unknown) into Chinese which was published in 1617 in Beijing; in 1608 he and Li Zhizao (1564–1630) translated Clavius's *Epitome Arithmeticae Practicae* into Chinese which was recompiled and published in 1613 in Beijing.

Xu Guangqi was a very important politician and scholar in Chinese history. He made great contributions in agriculture, astronomy, and mathematics. At the end of the Ming Dynasty, the court was dominated by factions. Xu had a difficult political career in which he was dismissed or resigned several times. Nevertheless, he was appointed the minister of rites in 1630 and a grand secretary (more or less a vice prime minister) in 1632. He died at the height of his political career. There are many Chinese books about Xu (see [15], for example). In China, few historical figures have museums. There is one in Shanghai devoted to Xu Guangqi, which has an original copy of Clavius's *Elements*.

Marco Polo is often regarded as the first foreign person who introduced China to the West. Matteo Ricci certainly is the most important person who introduced the West (its religion, science and culture) into China. He drew the first world map in China, which created sensational interest among the learned Chinese. Even the Emperor ordered ten copies from him. For his great contribution to China, when Ricci died in 1610, his body was permitted by the Emperor himself to be buried in Peking. Before him, no foreign person was allowed to be buried on Chinese soil.

Figure 8. An oil painting of Matteo Ricci and Xu Guangqi.

Figure 9. The tomb of Matteo Ricci in Beijing.

There was one emperor who was enthusiastic about Euclid's *Elements*. Emperor Kangxi of the Qing Dynasty ascended the throne in 1661 when he was seven and reigned for 61 years. In 1689, he decided to learn mathematics and astronomy (see [11]). For this purpose, two Frenchmen, Joachim Bouvet (1656–1730) and Jean-Francois Gerbillon (1654–1707), were appointed royal teachers. They were missionaries sent to China by King Louis XIV of France in 1687. Since the court language in the Qing Dynasty was Manchu rather than Chinese, the teachers had to learn Manchu first and then translate the selected *Elements* into Manchu. Of course, they taught the emperor in the Forbidden City.

4. London

The first English translation of Euclid's *Elements* was published by Sir Henry Billingsley in 1570. Billingsley was born in London and attended both Oxford University and Cambridge University. He was a successful merchant. In

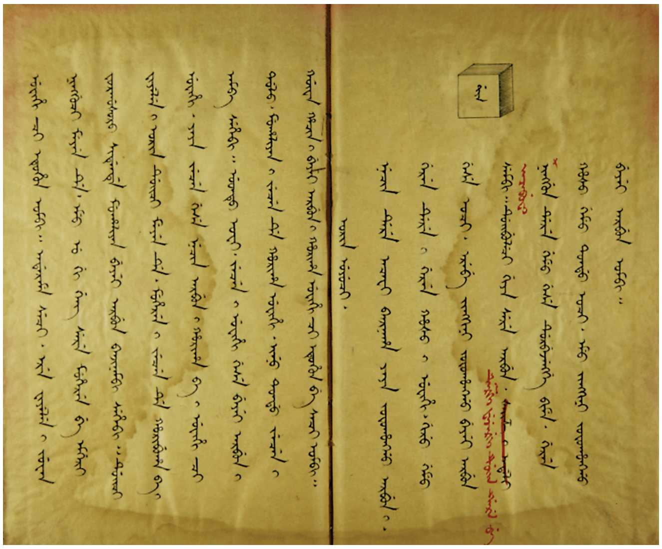

Figure 10. The Manchu translation of the *Elements* used by Emperor Kangxi. The red notes were made by the emperor.

1596, he became Lord Mayor of London. In fact, the history of the English version of *Elements* is also dramatic (see [3]).

For many years, Billingsley's translation was celebrated for its exquisite cover. However, it mistakenly wrote "EVCLIDE of Megara" instead of "EVCLIDE of Alexandria." In fact, many editions of the *Elements* had this mistake (see [2]). As indicated on the cover, this edition contained a preface by John Dee, an advisor to Queen Elizabeth I.

In the 16th, 17th, and 18th centuries, the *Elements* were translated into English editions by many scholars. In particular, Isaac Barrow (1630–1677) produced one in 1660. He completed his education at Trinity College, Cambridge, in 1648. Afterward, he was a fellow of the college and a visitor to Europe for some years. In 1660, he was appointed to the professorship of Greek at Cambridge. In 1662 he was made professor of geometry at Gresham College, and in 1663 became the first Lucasian Professor of Mathematics at Cambridge. He resigned the latter to his pupil Isaac Newton in 1669. He was appointed master of Trinity College in 1672, and held the post until his death.

When more and more English editions appeared, the *Elements* gradually entered high school classes in the UK. In 1815, Alexander Wylie was born in London. He attended a grammar school in Chelsea, where he studied Latin and mathematics including Barrow's *Elements*. After school, he became a carpenter working in Covent Garden and joined the London Missionary Society. In 1845, he started to learn Chinese by reading the Chinese Bible translated by the British and Foreign Bible Society. These preparations coincidentally created conditions for completing the Chinese translation of the *Elements*.

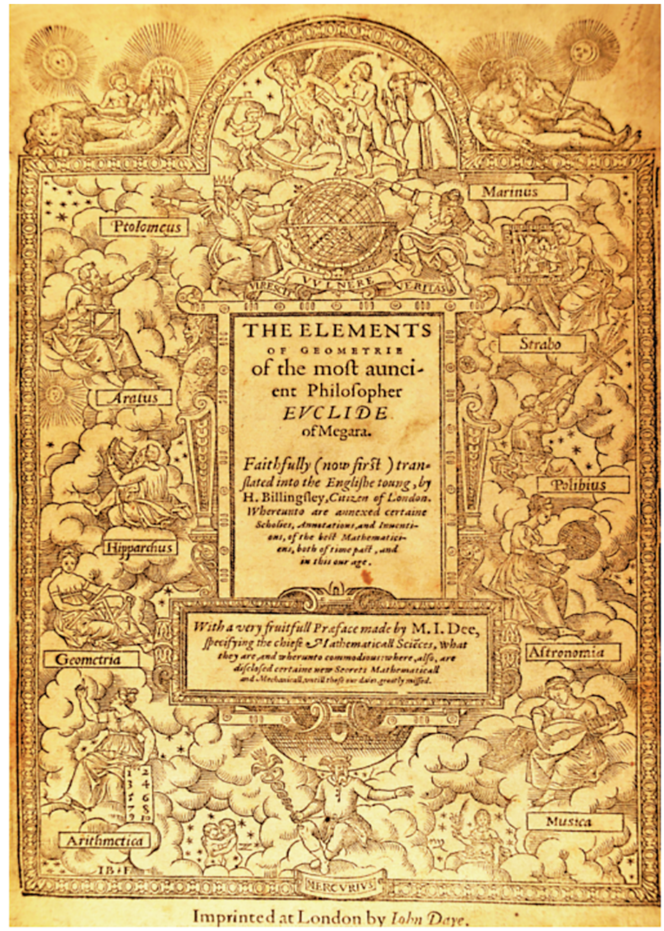

Figure 11. The cover of Billingsley's English translation of Euclid's *Elements*.

5. Shanghai

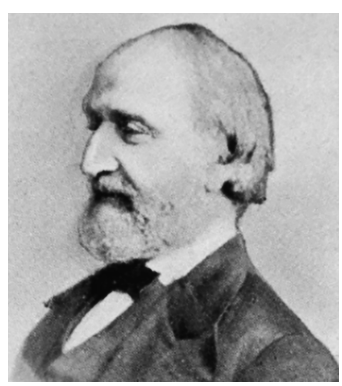

Figure 12. A portrait of Alexander Wylie.

In 1843, for the purpose of printing Chinese Bibles and other religious materials, the British missionary Walter Henry Medhurst (1796–1857) created a publishing house in Shanghai, the London Missionary Society Press. It was one of the earliest modern publishers in China. In 1847, for his Chinese language knowledge and religious background, Wylie was chosen to be an assistant manager of the publishing house. He arrived in Shanghai in August of 1847.

In the early years, the publisher indeed focused on Bibles and other religious books. Gradually, it also started to introduce culture and science. In 1852, Li Shanlan (1811–1882) joined the publishing house. At that time, he had already made a name as a mathematician. Wylie was impressed by his achievements and his mathematics knowledge.

Figure 13. A portrait of Li Shanlan.

When Wylie and Li discussed Matteo Ricci, Xu Guangqi, and their Chinese translation of the *Elements*, they decided to complete the mission of translating Euclid's *Elements* into Chinese. Unlike Ricci, Wylie did not bring the *Elements* with him when he arrived in Shanghai. In fact, they were not able to find such a book anywhere in Shanghai. So, Wylie had to ask his London colleagues to buy a copy for them. For years, it was believed that they obtained a Barrow edition. However, Yibao Xu [18] provided strong evidence in 2005 that it was a Billingsley translation.

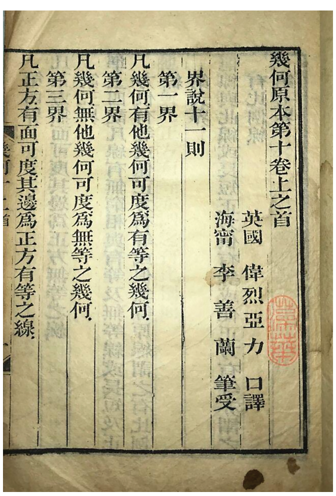
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Figure 14. The title page of the Wylie–Li translation of the *Elements*.

Wylie was not as knowledgable as Ricci in geometry, though he did learn some arithmetic and geometry in school. Luckily, Li was an established mathematician and the two complemented each other. Wylie did the oral translation and discussed with Li until the latter understood the mathematics. Then Li formulated it in standard Chinese. In 1857, Book VII to Book XV[5] of the *Elements* were completely translated into Chinese and were printed under the patronage of Han Yingbi.

Besides the *Elements*, Wylie and Li translated several other important volumes as well. In 1859, they translated Augustus De Morgan's *Algebra* and Elias Loomis' *Calculus* into Chinese. It was the first time that calculus was formally introduced into China. They started a project to translate Newton's *Philosophiae Naturalis Principia Mathematica* into Chinese. Unfortunately, it was interrupted by the Taiping Rebellion[6] and was never finished. Newton's *Principia* was translated into Chinese in 1931, by Taipu Zheng.

Like Ricci, Wylie made great contributions to the China-West exchange. Besides translating classics, he published many books and papers introducing the West to China and China to the West. In 1852, he introduced the Sun Tsu theorem and related results to Europe. In 1874, L. Mathiesen discovered the similarity between the Sun Tsu theorem and Gauss's theorem on linear congruences. Thereafter it is usually referred as the Chinese remainder theorem. For more about Wylie, we refer to [5, 13].

In 1848, Wylie married Mary Hanson in Shanghai. Unfortunately, the next year his wife died in childbirth, although their daughter survived. Wylie returned to the UK in 1877. At that time, he was weak and almost blind. He died on February 6, 1887, in London.

6. Nanking (Nanjing)

In 1860, when the Taiping rebels attacked Shanghai, Alexander Wylie returned to the UK for a short time. Li Shanlan also left the publishing house. During the next years, most of Li's property including books and papers was destroyed by the rebels.

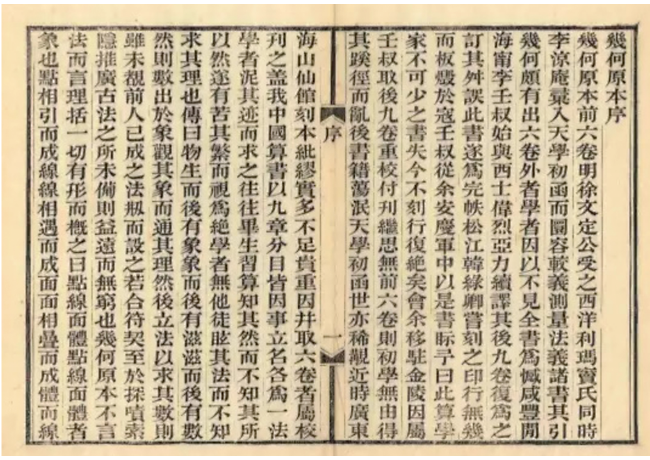
In 1862, Zeng Guofan (1811–1872) invited Li to be one of his aides, in charge of books and references. Zeng was one of the most important figures in the later Qing Dynasty. He was a key politician, a great general, and a leader of the Westernization Movement.[7] At that time, Zeng was the Liangjiang governor in charge of Jiangsu, Anhui, and Jiangxi provinces. On this occasion, Li suggested that Zeng print some important books including the Chinese version of the *Elements*. In 1864, Zeng's army suppressed the Taiping Uprising and took over Nanking again as the residence city of the Liangjiang governor.

As a leader of the Westernization Movement, Zeng strongly supported the effort to introduce Western knowledge into China. In 1865, financially supported by governor Zeng, the complete Chinese translation of Euclid's *Elements* (Book I to Book VI by Ricci and Xu, Book VII to Book XV by Wylie and Li) was published for the first time by Jinling Publishing House in Nanking. Since then, more than two thousand years after it was written by Euclid, China has had the complete *Elements*. During the past one-and-a-half centuries, more than ten different Chinese

[5] *Both Clavius's Latin version and Billingsley's English version contain the two spurious Books XIV and XV. Therefore, Wylie and Li's Chinese translation also contains the two spurious Books.*

[6] *The Taiping Rebellion (1851–1864) was a well-known peasant uprising against the rule of the Qing Dynasty led by Hong Xiuquan (1814–1864). They once took over more than half of China and built a government.*

[7] *From 1861 to 1894, for the purpose of saving the Qing empire, the ruling class pushed hard to learn science and technology from the West.*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序 一

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Figure 15. The preface of the complete Chinese translation of the *Elements*.

translations of the *Elements* (based on different editions) have been published.

Li Shanlan was the most important Chinese mathematician in the Qing Dynasty (1636–1912). In 1862, as a result of the Westernization Movement and for the purpose of training foreign language experts, the Qing government founded the School of Combined Learning in Peking. It was the most advanced institute in China at that time. Four years later, the school created a chair for astronomy and mathematics and Li was recommended. In 1868, he took the chair and held it until his death in 1882. During this time, he taught mathematics, in particular Euclid's *Elements*. For more about Li Shanlan, we refer to [12, 13, 16]. In mathematics, among other things, he is remembered for the Li Shanlan identity: *If* $n \geq m \geq 0$, *then*

$$\sum_{j=0}^{m} \binom{m}{j}^2 \binom{n+2m-j}{2m} = \binom{n+m}{m}^2.$$

Note. In Western literature, the names of ancient Chinese are often written as family name before given name. In this paper, to avoid confusion with literature, the names of Geng Shouchang, Han Yingbi, Hong Xiuqan, Li Shanlan, Li Zhizao, Liu Hui, Qu Rukui, Xu Guangqi, Zeng Guofan, and Zhang Cang follow this rule.

ACKNOWLEDGMENTS. This work is supported by the National Natural Science Foundation of China (NSFC12226006, NSFC11921001) and the National Key Research and Development Program of China (2018YFA0704701). The author is grateful to Leoni Dalla, Joseph Dauben, Richard J. Gardner, the referees, and the handling editor Laura Turner for their comments and suggestions, which essentially improved the quality of this paper. All photos in this paper are in the public domain. The map in the heading image is a copy of the first world map in China, which was created by Matteo Ricci in 1584.

References

[1] Gerald L. Alexanderson, *About the cover: Christopher Clavius, astronomer and mathematician*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 669–670, DOI 10.1090/S0273-0979-09-01269-5. MR2525739

[2] Gerald L. Alexanderson and William S. Greenwalt, *About the cover: Billingsley's Euclid in English*, Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 1, 163–167, DOI 10.1090/S0273-0979-2011-01365-9. MR2869011

[3] R. C. Archibald, *The first translation of Euclid's elements into English and its source*, Amer. Math. Monthly **57** (1950), 443–452, DOI 10.2307/2308296. MR37770

[4] June Barrow-Green, Jeremy Gray, and Robin Wilson, *The history of mathematics: a source-based approach. Vol. 1*, AMS/MAA Textbooks, vol. 45, MAA Press, Providence, RI, [2019] ©2019. MR4548178

[5] H. Cordier, *The life and labours of Alexander Wylie, agent of the British and Foreign Bible Society in China*, J. Royal Asiatic Soc. **19** (1887), 351–368.

[6] Joseph W. Dauben, *Erratum: "Suan shu shu. A book on numbers and computations" [Arch. Hist. Exact Sci.* **62** *(2008), no. 2, 91–178; MR2377455]*, Arch. Hist. Exact Sci. **62** (2008), no. 3, 347, DOI 10.1007/s00407-008-0023-0. MR2395386

[7] Joseph W. Dauben, *The evolution of mathematics in ancient China: from the newly discovered Shu and Suan shu shu bamboo texts to the Nine chapter on the art of mathematics* (English, with English and Portuguese summaries), Rev. Bras. Hist. Mat. **19** (2019), no. 37, 25–78, DOI 10.1080/14697688.2018.1538692. MR4024869

[8] P. M. Engelfriet, *Euclid in China: the Genesis of the First Chinese Translation of Euclid's Elements, Books I–VI (Jihe Yuanben, Beijing,* 1607*) and Its Reception up to* 1723, Brill, Leiden, Boston, 1998.

[9] F. A. Homann, *Christopher Clavius and the renaissance of Euclidean geometry*, Arch. Hist. Soc. Iesu **52** (1983), 233–246.

[10] P. C. Hsia, *A Jesuit in the Forbidden City: Matteo Ricci* 1552–1610, Oxford University Press, Oxford, 2012.

[11] Catherine Jami, *The emperor's new mathematics*: *Western learning and imperial authority during the Kangxi reign (1662–1722)*, Oxford University Press, Oxford, 2012. MR3711755

[12] Jean-Claude Martzloff, *Li Shanlan (1811–1882) and Chinese traditional mathematics*, Math. Intelligencer **14** (1992), no. 4, 32–37, DOI 10.1007/BF03024470. MR1188141

[13] Jean-Claude Martzloff, *A history of Chinese mathematics*, Springer-Verlag, Berlin, 1997. With forewords by Jacques Gernet and Jean Dhombres; Translated from the 1987 French original by Stephen S. Wilson and revised by the author, DOI 10.1007/978-3-540-33783-6. MR1421776

[14] Kangshen Shen, John N. Crossley, and Anthony W.-C. Lun, *The nine chapters on the mathematical art*, Oxford University Press, New York; Science Press Beijing, Beijing, 1999. Companion and commentary; With forewords by Wentsün Wu and Ho Peng Yoke. MR1740507

[15] X. Z. Wang, *Xu Guangqi, a Great Scientist in Ming Dynasty* (in Chinese), Shanghai People Publisher, Shanghai, 1985.

[16] Yusheng Wang, *Li Shanlan: forerunner of modern science in China*, Chinese studies in the history and philosophy of science and technology, Boston Stud. Philos. Sci., vol. 179, Kluwer Acad. Publ., Dordrecht, 1996, pp. 345–368, DOI 10.1007/978-94-015-8717-4_28. MR1796777

[17] B. Wardhaugh, *The Book of Wonders: The Many Lives of Euclid's Elements*. Harper Collins Publishers, 2020. MR4422087

[18] Yibao Xu, *The first Chinese translation of the last nine books of Euclid's Elements and its source* (English, with English and Chinese summaries), Historia Math. **32** (2005), no. 1, 4–32, DOI 10.1016/j.hm.2003.12.002. MR2122482

[19] Z. Zeng and Y. F. Sun, *Matteo Ricci, a Messenger of Chinese and Western Mathematical Cultural Exchange* (in Chinese), Jinan University Press, Guangzhou, 2015.

[20] C. Zong, *Modern Mathematics in China*, in preparation.

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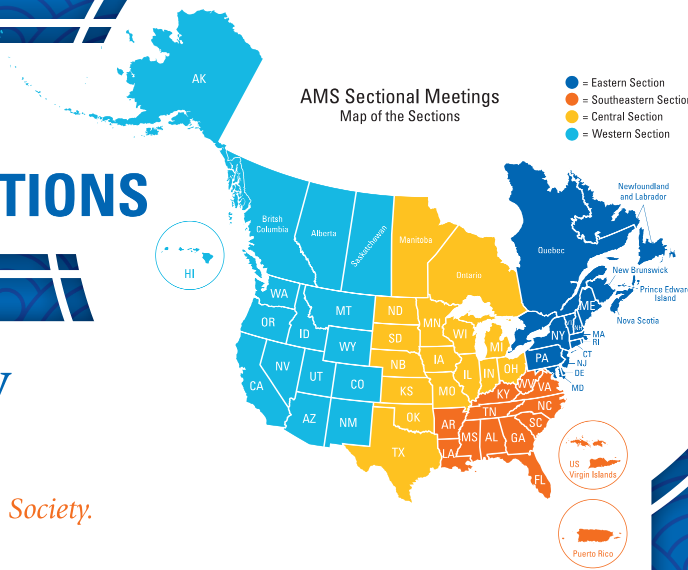

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Merging Inquiry and Math Teachers' Circles: The Math Circles of Inquiry Project

Jane Cushman, Brianna Donaldson, Keiko Dow, Ryan Gantner, C. Yousuf George, and William Jaco

The Math Circles of Inquiry project was a grant-funded project involving inquiry-based instruction, Math Teachers' Circles, college faculty, and K–12 teachers. Throughout the two-year project, college faculty members worked with K–12 teachers to create inquiry-based learning materials for classroom use. These materials were tested, refined, and disseminated using the Math Teachers' Circle model.

Inquiry-Based Learning (IBL) is a collection of student-centered instruction methods that share common themes expressed through the "four pillars" of Laursen and Rasmussen:

- Students engage in deep mathematical tasks (not just busy work or parroting what the teacher did)
- Students are given opportunities to meaningfully collaborate with their peers
- The instructor is focused on student thinking and designs tasks and organizes class time to bring this out
- Instructors foster equity in design and facilitation choices [LR]

IBL instruction has been documented to show improvements in learning and student success, and especially to close gaps between traditionally under-achieving groups of students and their counterparts (see [LR] and the references therein). It should be noted that IBL teaching is not a magic bullet: designing IBL tasks and implementing IBL instruction is demanding and takes a different mindset than other styles of teaching. The goal of the Math Circles of Inquiry project is to empower local middle and high school teachers to experiment with incorporating IBL and active learning in their classrooms.

Math Teachers' Circles (MTCs) are professional learning communities in which K–12 teachers and mathematicians meet regularly to investigate rich mathematics together [DNUW14]. Evidence suggests that MTC participation supports healthy mathematical mindsets [MGD], increases mathematical knowledge for teaching [WDAR], and enhances professional engagement [DNUW22]. MTC participation is also associated with classroom practices that promote student learning [MDK]. However, many teachers report needing additional support to translate their MTC experiences into classroom practices that actively engage students as mathematicians. That's an issue that our Math Circles of Inquiry project sought to rectify.

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Communicated by Notices *Associate Editor William McCallum.*

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Briefly, the Math Circles of Inquiry project began with bringing together middle and high school teachers interested in IBL. This project was housed at two sites: Buffalo and Rochester, NY. College faculty from the Greater Upstate New York IBL Consortium (GUNYIBL), which supports college math instructors through professional development and learning communities to effectively and equitably implement IBL methods in undergraduate courses, offered summer workshops for these K–12 teachers where they worked together to understand what IBL is and how it can be implemented in their classrooms. These teachers then worked with college faculty to develop teacher-designed, inquiry-based materials at the workshops.

For instance, one of the teachers expressed that students struggle to take advantage of different forms of linear equations: standard form, intercept form, point-slope form, and their favorite slope-intercept form. Students tend to favor the slope-intercept form even when it causes them unnecessary acrobatic mathematical maneuvers to solve a problem. We turned this frustration into a series of inquiry-based exercises which help students to experience and describe the differences by themselves through the activities. Our teacher-designed, inquiry-based materials were based on actual teachers' needs and facilitation choices throughout a series of inquiry-based lessons that were carefully built with their own students and classroom experiences in mind.

During the workshop, an exciting and collaborative environment was developed by exchanging ideas and classroom experiences, as well as educating each other with new technologies (such as creating activities using Desmos for their classrooms). This became the root of our IBL community that we were able to grow for the next two years. After using their IBL course materials in their own classrooms, the teachers then showcased them at the MTCs meetings, which were held approximately three times per semester at a college campus. Part of the course material was selected for the audience to try during the meeting just as their students would have in their classrooms. We often had very lively conversations with the participants about the materials themselves, how their students would solve them, common pitfalls, and new ideas of activities. After a year of doing this, many of the regular attendees at the MTC meetings were invited into the second year of our project to develop IBL course materials of their own. We continued with this cycle of developing and testing, having the newly onboarded teachers present their materials at MTC events for understanding, refinement, and dissemination.

Benefits for K–12 and Higher Education Faculty

First and foremost, we all seek to promote better teaching of mathematics, especially in K–12 schools. College faculty benefit from a deeper understanding of the mathematics their students have been exposed to before coming to college. IBL teaching, through active learning and student-centered reasoning, has been shown to broaden interest in mathematics [BP]. And it is clear that every college course—mathematics or not—benefits from having students who are intellectually curious, engage deeply with the material, search for evidence to draw logical conclusions, and communicate clearly with others. IBL has been shown to do all of these [BP, LR]. Furthermore, by engaging college faculty with innovative K–12 educators, mathematics instruction at the collegiate level can also be improved. Indeed, when reflective teachers are exposed to new ideas, everyone can flourish.

Not only does the experience help the students, but the K–12 teachers who participated in our project with IBL teaching were able to deepen their understanding of the topic, critically evaluate their students' needs and design material needed for the classroom, collaborate with their peers, and work on teaching skills that promote active learning. Furthermore, during the MTC meetings, teachers were able to engage in genuine math inquiry themselves, improving their ability to teach in this style. Our project provided a time and place for local K–12 teachers to meet, share, and work together with people with common interests outside of their own schools. All of the participants in the Math Circles of Inquiry Project were themselves teachers. This was critical to gain authentic feedback on their IBL course material, gain acceptance into the community, and develop the materials in the first place. Course material presentations at MTC series often started with sharing any difficulties the author had with the existing materials and motivations behind their creation. This approach grabbed attention right away and made the rest of the presentation approachable and authentic, similar to how students' presentations in inquiry-based classrooms invite more open communication among the students compared to a traditional lecture. By the end of the presentation, we often received ideas for extension activities and new ideas from the participants, which indicates that they were starting to envision themselves as authors of new and exciting inquiry-based course materials and becoming a part of our community.

Finally, college faculty can derive indirect benefits from participating in such a project. The K–12 teachers get to know the college campus and faculty and associate them with fun, interesting mathematics activities. The college faculty members get to know the K–12 teachers on a more personal level and gain deeper insight into how and what they teach in their classroom. This can have an impact on college teaching and curricula, especially for first-year college students. This enhanced collaboration and relationship building between college faculty and teachers from across the region can be seen as a potential recruiting opportunity for the host college.

How Do Typical MTCs Work?

Although the specifics of implementation vary, all MTCs share several core features: 1) regular meetings (6–8 per year); 2) involvement of both K–12 teachers and mathematicians; 3) a problem-solving component with significant time (1.5 hours or more per meeting) devoted to collaborative investigation of nonroutine, rich, low-threshold, high-ceiling problems; and 4) additional time for reflection and social interactions to build community. MTCs have been implemented at more than 100 sites nationwide (see `https://mathcircles.org/`). Sites typically hold academic-year meetings for multiple years, with intensive workshops as needed to sustain or increase membership.

Teachers are treated as mathematician colleagues in MTC sessions, working through open-ended and inquiry-driven problems. The teachers in this project did this while also collaboratively developing, testing, and refining curriculum-relevant classroom lessons in IBL style.

What's Different About the Math Circles of Inquiry Project?

Math Circles of Inquiry allows for transference of MTC virtues to the classroom by setting up the MTC sessions in a deliberate way. Most importantly, the K–12 teachers serve as "teacher-leaders" in this model. The teacher-leaders develop their own materials with their students and resources in mind. Then they facilitate much of the MTC meetings. College faculty are involved in both the creation and dissemination steps, but the teacher-leaders are invited to set the agenda for the meetings. Often the college faculty contribute by adding another inquiry activity at the end of the MTC meetings—usually related to and extending the course materials already presented—to deepen the engagement of the teachers as mathematicians.

During the MTC meetings, IBL course materials are presented. But they are not simply shown off and left for comment; they are presented using the method of pedagogy intended for the lesson. Thus, this model demonstrates not only materials that can be used in the classroom, but also the pedagogical methods to use them. Both while the materials are worked through and after completion, participants actively give feedback and hold discussions about possible adaptations that could be made for various classroom circumstances. And for some participants at MTC meetings, classroom materials with new approaches developed by their peers are more relatable and exciting to think about and use in their classroom than those offered by traditional textbooks and publishers.

Finally, the Math Circles of Inquiry project benefits from the growing number of people involved. As more and more teachers are onboarded, the level of discourse encompasses broader perspectives. With more colleagues involved in the project, teachers are able to get more insight into how other schools teach, build the support system outside of their own schools, and promote inquiry, what types of materials are used, and how those materials are made available to the students.

Details About Establishing and Running a Circle

In this section we give some details about establishing and running the Math Circles of Inquiry which we believe were valuable to its success. From the onset, we required a partnership between K–12 educators and higher-education faculty who were interested in working together to implement mathematics learning by inquiry in their classrooms. We were able to lean on an existing inquiry community of faculty (GUNYIBL), but support from a few committed faculty who are well versed in inquiry-oriented methods would also suffice. The MTCs we used were largely new creations, though building on existing frameworks has advantages. However they are designed, the following blueprint is helpful:

Inception: A core of enthusiastic educators. A core group of middle and high school teacher-leaders and higher-education faculty developed, adapted, tested, and refined curriculum-relevant, inquiry-oriented classroom materials. Teacher-leaders were trained in multi-day sessions in the summer. These sessions involved aligning our understanding of IBL, generating ideas for content areas that need curricular development, discussing best practices of writing IBL activities, and working in teams to begin curricular development.

Establishing a collaborative network: Workshops and mentoring. Teacher-faculty teams co-led meetings and workshops for other local teachers, focused on introducing the classroom materials, deeply investigating the mathematics in the materials, and mentoring teachers as they implemented the materials. We held MTC sessions approximately three times per semester during the academic year at a college campus. Dinner was served. The sessions were led by teacher-leaders and often supplemented with problems facilitated by college faculty members. These sessions were open to anyone who wished to attend, and professional development credit was awarded. These sessions were advertised through various methods, including mailing lists, social media, our local teacher association, and conferences.

Broadening awareness: Deliver the message, tell the story. Teacher-leaders and faculty shared their work at local and regional professional meetings, and with administrators from K–12 and higher education. Bringing more people into the program helps it flourish, prevents burnout, and reaches a wider audience.

Conclusion

The authors have been involved in Math Circles and MTCs for many years. What was unique in this project was the intentional development of teacher leaders and classroom-ready IBL materials written by the teachers for wider distribution and use. We observed a true sense of agency and autonomy among the teacher participants. The teacher leaders were able to drive the discussions, which allowed for genuine teacher-to-teacher collaboration in the creation, critique, and pedagogical aspects of the materials. Using the MTCs format, we were able to engage in robust discussion, but in this project the discussion was driven by the teachers themselves, rather than college faculty. Furthermore, not only was the content brought out but the pedagogy surrounding the content was thoughtfully modeled and critiqued. Thus, the discussions were able to reflect many angles: mathematical content and curiosity, pedagogical methods, teacher thinking about mathematics, student thinking about mathematics, and the perspective of college faculty. In the end, the teacher participants were able to leave every MTC event with a deeper understanding of mathematical material *and* actionable items related to teaching mathematics, either content or pedagogy (or both). And teacher leaders left the project with the skills, materials, and confidence to drive positive change in their own departments and schools. Finally, the college faculty involved were able to learn quite a bit from the teachers in the program. Seeing courses taught from a different perspective (college vs. high school) allowed us to see a wider range of pedagogical techniques and approaches as well as have discussions about teaching that are often left alone in the college math department.

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References

[DNUW14] B. Donaldson, M. Nakamaye, K. Umland, and D. White, *Math Teachers' Circles: Partnerships between mathematicians and teachers*, Notices of the American Mathematical Society **61** (2014), no. 11, 1335–1341, available online at `http://www.ams.org/notices/201411/rnoti-p1335.pdf`.

[MGD] K. Milless, C. Good, and B. Donaldson, *K–12 Math Instructors' Attitudes toward and Identification with Math Following Participation in Math Teachers' Circles*, poster presented at Society for Personality and Social Psychology Annual Convention (2018), Atlanta, GA.

[WDAR] D. White, B. Donaldson, A. Hodge, and A. Ruff, *Examining the effects of Math Teachers' Circles on aspects of teachers' mathematical knowledge for teaching*, International Journal for Mathematics Teaching and Learning, published online September 26, 2013, at `https://www.cimt.org.uk/journal/white.pdf`.

[DNUW22] B. Donaldson, M. Nakamaye, K. Umland, and D. White (in press), *The impact of Math Teachers' Circles: Case studies*, The Mathematics Educator.

[MDK] P. D. Marle, L. L. Decker, and D. H. Khaliqi, *An inquiry into Math Teachers' Circle: Findings from two year-long cohorts*, paper presented at the School Science and Mathematics Association Annual Convention (2012), Birmingham, AL.

[LR] S. L. Laursen and C. Rasmussen, *I on the prize: Inquiry approaches in undergraduate mathematics*, International Journal of Research in Undergraduate Mathematics Education **5** (2019), 129–146, `https://eric.ed.gov/?id=EJ1210552`.

[BP] R. Bruder and A. Prescott, *Research Evidence on the Benefits of IBL*, ZDM: the international journal on mathematics education **45** (2013), no. 6, 811–822, `https://www.researchgate.net/profile/Regina-Bruder/publication/271660668_Research_evidence_on_the_benefits_of_IBL/links/58b1b0f3aca2725b5416e747/Research-evidence-on-the-benefits-of-IBL.pdf`.

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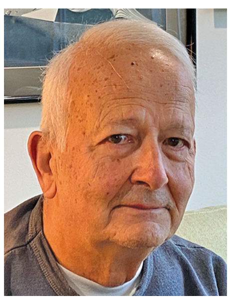

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OPINION

SocialOffset: Making a Difference, One Conference at a Time

Elena Gerstmann

Note: The opinions expressed here are not necessarily those of Notices.

I spend a lot of my time around engineers and mathematicians. I'm the executive director of INFORMS (the Institute for Operations Research and the Management Sciences). Previously I was on the executive teams of IEEE (The Institute of Electrical and Electronics Engineers) and ASME (The American Society of Mechanical Engineers). My doctorate is in social psychology. But I'm not writing about these experiences. I'm writing about a new non-profit organization my wife and I recently started. I believe this organization, *SocialOffset* (`https://socialoffset.org`) will resonate with many AMS members in all disciplines who attend conferences.

Origin Story

Let me start at the beginning. In 2019, I was in the Dallas airport about to fly home to New Jersey, and I was looking for something relatively healthy to eat. The only decent choice was Chick-fil-A. I hadn't been to a Chick-fil-A for years because, as a lesbian, I don't like their anti-gay track record.[1] I reluctantly purchased a chicken sandwich. I then proceeded, with my sandwich, to sit in an uncomfortable airport chair and attempt to calculate the percent of the purchase price that was profit that could flow into the wallets of people who might give this money to an anti-gay cause. I thought, "I should create an app to help people assuage their guilt related to their reluctant purchases." I imagined this app would have two key components. The first would be a calculator that would determine how much profit is made from a purchase—for example, a hypothetical \$8.21 sandwich might create a hypothetical profit of \$2.44. The second component would be the ability to easily donate the identical profit amount to a charity that does align with one's values. I talked to a few friends about this idea, but it never got traction. (Remember I'm a social psychologist who works for associations; I'm not an app developer or an entrepreneur.)

Fast forward to March 2022, when my wife, who is also in the association and events business, was planning to travel to Austin, Texas to attend the SXSW (South by Southwest) conference. Texas Governor Abbott had recently issued orders to investigate parents and physicians who provide trans children with gender-affirming care. My wife, who identifies as non-binary, was upset and considered not attending the conference. She wound up attending and was pleased that SXSW quickly assembled

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[1]*While some may argue whether Chick-fil-A is currently anti-LGBTQ+, few will disagree that they have been in the past. My belief is that by buying their food now, I put money into the pockets of the family-owned corporation who may use profits to personally give to anti-LGBTQ+ groups who are working against my rights. For family-owned corporations it is near impossible to separate the profits of the business from the riches of the owners.* `https://www.thetaskforce.org/from-chick-fil-a-to-enda/`

supportive trans-related programming and opportunities for donations.

Fast forward to July 2022, post-Dobbs, when my wife and I were planning to attend an association conference in Nashville, Tennessee. Tennessee's anti-abortion trigger law was going to go into effect shortly after the conference. A friend started an email chain with a number of mutual work friends who were going to the conference and were pro-choice, to discuss some type of action we might all take. I was on this email chain, and it got me thinking.

My wife and I then had a long car ride and this was a topic of discussion. We went back and forth about what we should do when we needed to travel to locations that don't align with our personal social values. As an event strategist, she sees the world through events. She believes in the transformative power and beauty of in-person events to bring people together to share ideas, collaborate, and solve problems. I was raised to see the world through the lens of social justice and the need to take a stand to support my values. As we went back and forth on the pros and cons of attending events and spending money in places where we know our tax dollars will be spent in ways we disagree with, one of us (we don't know who said it first) said "social offset, like a carbon offset."

And that's how SocialOffset came to be. We spent the next several weeks buying the rights to the web domain and building a minimum viable product to test the market at the conference in Nashville—the ASAE[2] Annual Meeting, a 5,000-person conference for the association and event industry. It was amazing. Not only did we raise over $3,500 for a local reproductive health organization but we raised $15,000 from three city destination management organizations (Visit Salt Lake, Visit Austin, and Visit Seattle) to help us build a fully functioning website. We also received a lot of positive feedback including quotes like, "That is such a simple but brilliant idea. Please build it."

Toddler Stage

In the fall of 2022, we incorporated, populated a board of directors, hired a design firm to design a logo[3] and a website, and hired a firm to build the website. We purposely invited six association and industry rock stars to be on our board of directors so we could move quickly and with experience and expertise. In January of 2023, we launched a fully functioning site. (The website is `https://SocialOffset.org`.) In February of 2023, we learned that the IRS granted us non-profit, 501(c)(3) status.

Given the board's background in association events, the first phase is focused on conferences sponsored by professional and trade associations. This also aligns with the post-Dobbs press related to boycotts and events. *The Chronicle of Higher Education* had an early article on July 12, 2022 called *"To Boycott or Not? Academic Conferences Face Pressure to Avoid Abortion-Hostile States"* by Sylvia Goodman. Since then, there have been plenty of other articles in the academic space about scholarly conferences in politically contentious destinations.

We know that nonprofit and association boards and conference organizing committees are energized by the opportunity to relaunch events in a post-Covid environment, but we also know some conference locations risk alienating members and attendees and may cause reputational harm to the hosting associations. Boycotts of cities, especially in a post-Trump world, are not effective and are most likely to hurt the very people many of us want to help. For example, many frontline hospitality workers are women and of lower socio-economic status, and are only paid when there is work for them to do. Removing an event is effectively removing their paycheck for something they have no control over.

I've paid close attention to Stacy Abrams. Abrams, during her Atlanta gubernatorial race, was very vocal about resisting boycotts.[4] "My deep concern is that if we call for a boycott, the very people who are helping change the nature of economic opportunity and political opportunity will leave us behind," she said. "So my message is stay and fight. Come and lift up your voices and join us."

Nonetheless, we can't ignore the distress some attendees feel when asked to attend a conference in a state with policies that go against their core values. In fact, their calls to boycott come from a desire to do something. They are conflicted between needing to engage with their professional and industry networks and not wanting to support certain states. We can't sweep this personal conflict and desire to do good under the rug.

We hope SocialOffset gives everyone involved a way to balance these needs. **At its most basic level, SocialOffset says "no" to event boycotts and "yes" to making a difference through local impact.**

The product of SocialOffset is a page for each conference with a dedicated URL like `SocialOffset.org/eventname`. This dedicated page a) provides guidance on the amount of money that is needed to offset the taxes being paid by an attendee or exhibitor and b) lists 3–6 local charities that have been vetted. Individuals can easily and seamlessly give directly to the charities. One hundred percent of the funds donated by meeting attendees go to the charities. Our 2023 causes are racial justice, LGBTQ+ equality, hunger relief, housing security, environmental

[2] *ASAE is the leading organization for those who work for associations. While ASAE prefers its acronym, it stands for American Society of Association Executives.*

[3] *I love our logo. It is purposely purple so it is not red or blue.*

[4] `https://www.forbes.com/sites/alisondurkee/2021/04/13/companies-in-battleground-states-warn-gop-not-to-follow-georgia-and-pass-voting-restrictions/?sh=c5fe29028394`

sustainability, and reproductive freedom. After the conference, the sponsoring association receives a report stating how much was raised. For privacy reasons, the individual donor names are not shared. Because SocialOffset is a nonprofit, donations by attendees are tax-deductible.

To help cover the cost of operating, and to ensure that 100% of funds raised always go to designated charities local to the event, SocialOffset charges event organizers a very modest fee. We also rely on sponsors like destinations, convention centers, organizations in the association or event spaces, and other supplier companies to make annual investments in this venture.

Future Stages

I dream of having pages built for musicians (imagine `SocialOffset.org/TaylorSwift`, `SocialOffset.org/P!nk`, `SocialOffset.org/Lizzo`, `SocialOffset.org/BrandiCarlile`[5]) so as they travel to different cities and reluctantly give a lot of money to the state governments they disagree with, SocialOffset can provide an easy and streamlined way for them and their fans to offset their generated tax dollars.

And... of course, one day we plan to create my original idea from 2019, which was to build a simple app that allows anyone to bring their values wherever they go, quickly calculate an amount that needs to be offset, and within a few seconds donate to a charity that aligns with their beliefs.[6]

Elena Gerstmann

Credits

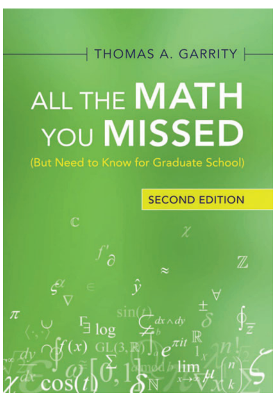
Photo of Elena Gerstmann is courtesy of INFORMS/Max Liberatore-Resnick.

[5] *Yes, these are my four favorite musicians.*

[6] *We are always looking for volunteers to assist with our vision. Please contact us if you are interested.*

All the Math You Missed (But Need to Know for Graduate School)

Reviewed by Sarah Cannon



All the Math You Missed (But Need to Know for Graduate School)
By Thomas A. Garrity.
Cambridge University Press, 2021, 416 pp.

Students entering graduate school in mathematics come from a diversity of backgrounds and experiences. It's rare for a beginning graduate student to already have all the mathematical knowledge they need to be successful; students frequently find themselves needing to quickly learn new areas of mathematics they had not been exposed to before. The new edition of *All the Math You Missed (But Need to Know for Graduate School)* by Thomas A. Garrity is intended to help students do exactly this. It covers twenty topics but also highlights the commonalities between them, such as the fundamental nature of equivalence problems and the broad use of functions. It seeks to provide a beginning graduate student in mathematics, or other readers of a similar background, with the key definitions, problems, results, and approaches in each of these areas. Throughout, the book has an informal, friendly tone, focusing on motivation and the big picture rather than rigor. It puts things in an understandable order intended to promote learning and intuition—for example, at times it uses a lemma before providing a proof, which gives the reader an understanding of the lemma's usefulness and a motivation to learn more about it before diving into the details of its possibly quite technical proof.

Each chapter is intended to be a first touchpoint for students to begin to explore that topic. Research shows that when people learn, they build knowledge structures in their brain, and seek to attach new information onto their existing knowledge structures. A novice in an area will frequently have sparse, disconnected knowledge structures, while an expert's knowledge structures will be dense, well connected, and well-organized [1]. This book serves the purpose of helping students build an initial knowledge structure for a new area: while not covering every detail, it highlights the key ideas and connections between them and gives students an idea of the important features of a field. Once students have begun to understand an area, they are then well-equipped to seek out additional sources to make their knowledge more robust; having the initial knowledge structure provided by this book will make these future explorations easier. Towards this end, each chapter features pointers to additional references students can seek out to learn more as well as some exercises students can attempt to expand their understanding. These range from looking something up in a different textbook to proving

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one of the results in the chapter to diving deeper into some aspect of the topic.

For example, the book begins with a chapter on Linear Algebra and the sentence, "Though a bit of an exaggeration, it can be said that a mathematical problem can be solved only if it can be reduced to a calculation in linear algebra." The chapter highlights the basic important points of linear algebra (vector spaces, linear transformations, determinants, eigenvalues), beginning with the concrete example of $\mathbb{R}^n$ before moving on to the general definition of a vector space. In addition to definitions and examples, a big emphasis is placed on intuition and conceptual understanding. For example, the author gives three different ways of understanding the determinant: with an inductive definition, as the unique function on matrices satisfying three important properties, and in terms of volumes of transformed unit cubes. The author also doesn't shy away from going slightly out-of-order when doing so makes the most sense for the conceptual understanding of the novice reader. For example, when he states a key theorem of linear algebra—a list of equivalent conditions for a matrix to be invertible—he includes a condition about eigenvalues even though they haven't been discussed yet, along with a clear caveat that definitions of eigenvalues are coming. This allows the complete theorem to be stated at a logical point in the narrative (after discussion linear transformations and determinants) while also motivating the later discussion of similar matrices and eigenvalues. I also particularly like that the author makes it clear when his examples are carefully engineered to work out nicely: "I did not just suddenly 'see' that A and B are similar. No, I rigged it to be so."

The chapter on Probability would have been a particularly good resource for me, had I known about it when I needed it: as a first year graduate student, I found myself taking a class in probabilistic combinatorics and beginning research in randomized algorithms, despite never having studied probability before. The chapter includes basic, clear explanations of sample spaces and variance, including the two formulas for variance, which would have been extremely useful for me at the time. I particularly liked that the chapter focused on the central limit theorem just for Bernoulli random variables – the simple case encapsulates all the concepts and ideas of the more general central limit theorem while also being more straightforward for a beginner to comprehend. The chapter includes the clever proof that the integral of a normal random variable is one, a straightforward proof of the central limit theorem, and a proof I was unfamiliar with for Stirling's approximation for $n!$. While I've used Stirling's formula many times, I'd actually never seen this proof, which uses the ideas of the central limit theorem within the proof. I like that the chapter highlighted the connection between what I typically understand as two entirely different tools, the central limit theorem and Stirling's formula; it certainly helped me connect these two concepts in a way I hadn't before. Learning about such connections will be extremely useful to beginning graduate students as they build their knowledge in these fields.

In a departure from many math textbooks, there is also a chapter about theoretical computer science. Though the chapter is titled "Algorithms," it discusses a bit of complexity as well. As someone who works on the boundary of math and theoretical computer science, it was nice to see this included. This chapter is motivated by the distinction between existence proofs and constructive proofs: sometimes one can prove something exists without knowing how to find it, while sometimes the main motivation is being able to find the desired object. The chapter certainly takes a much more mathematical approach to algorithms and complexity than computer science textbooks usually do, but this makes sense for the intended audience, which is math graduate students. The selection of topics also reflects the interests of the intended audience of mathematicians, covering graphs, sorting lower bounds, P vs. NP, and Newton's method for finding zeros of functions.

While I've highlighted the chapters that are most relevant to my interests and experiences, the book is comprehensive, including twenty chapters on topics ranging from Fourier Analysis to Category Theory:

- Linear Algebra
- ε and δ Real Analysis
- Calculus for Vector-Valued Functions
- Point Set Topology
- Classical Stokes' Theorem
- Differential Forms and Stokes' Theorem
- Curvature for Curves and Surfaces
- Geometry
- Countability and the Axiom of Choice
- Elementary Number Theory
- Algebra
- Algebraic Number Theory
- Complex Analysis
- Analytic Number Theory
- Lebesgue Integration
- Fourier Analysis
- Differential Equations
- Combinatorics and Probability Theory
- Algorithms
- Category Theory

Whatever topic a math graduate student wants to learn about, they'll be able to find some relevant sections in this book. Beyond graduate students, it can also be a useful resource for experts in other areas who want to learn about various mathematical topics; if I found myself needing to learn about a new area of mathematics, this book is where I'd start.

The field of mathematics is often beset by the "myth of genius," a belief that big advances in the field come from a select few who have some innate talents the rest of us cannot hope to learn [4]. However, that's rarely the case. As the author says in the preface, "I know of no serious mathematician who finds math easy. In fact, most, after a few beers, will confess how slow and stupid they are." Math is not easy, but it can be learned. Pedagogical research at the university level shows that students are more likely to succeed when they have positive expectancies, that is, when they believe they can successfully achieve their desired goals and outcomes [1]. This book is an extraordinarily helpful tool that can make it possible for students to believe they are capable of learning a new topic and exploring a new area, by explaining key ideas and approaches in a friendly, accessible way.

Work has shown that in scientific fields where the "myth of genius" is more pervasive, academic departments tend to have fewer numbers of women [5]. Counteracting this myth and empowering students to believe they are capable of success is thus critical for improving gender diversity across mathematics. For example, seminal work of Jane Margolis and Allan Fisher studying the experiences of female students in a male-dominated undergraduate computer science program connected female students' success to having a growth mindset, a belief that even if they don't immediately understand something now, with hard work and effort they'll be able to in the future [6]. Related work looking at high school math students had similar findings: students with a growth mindset were on average more successful, and this effect was stronger for female students [3]. Though not specific to math, interventions focused on encouraging students to see intelligence as malleable rather than fixed also led to improved outcomes among African American college students, with more improvements seen than in white students given similar interventions [2]. According to these studies, a growth mindset is a critical tool for fostering student success. By helping all kinds of students believe they are capable of learning, this book has the potential to increase diversity in mathematics.

Overall, this book is a valuable resource for beginning graduate students. By introducing students to new areas of mathematics in an intuitive way with a friendly, accessible tone, it enables them to build on their mathematical knowledge and gain confidence in their abilities in new areas. This makes it more possible for a student who has gaps in their mathematical knowledge to be successful in graduate school, an admirable goal that helps us bring more people into our mathematical community.

References

[1] Susan A. Ambrose, Michael W. Bridges, Michele DiPietro, Marsha C. Lovett, and Marie K. Norman, *How Learning Works: 7 Research-Based Principles for Smart Teaching*, Jossey-Bass, 2010. Foreword by Richard E. Mayer.

[2] Joshua Aronson, Carrie B. Fried, and Catherine Good, *Reducing the Effects of Stereotype Threat on African American College Students by Shaping Theories of Intelligence*, Journal of Experimental Social Psychology **38** (2002), no. 2, 113–125.

[3] Jessica L. Degol, Ming-Te Wang, Ya Zhang, and Julie Allerton, *Do Growth Mindsets in Math Benefit Females? Identifying Pathways between Gender, Mindset, and Motivation*, Journal of Youth Adolescence **47** (2018), 976–990.

[4] Evelyn Lamb, *The Media and the Genius Myth*, Roots of Unity Blog, Scientific American, February 15, 2015. `https://blogs.scientificamerican.com/roots-of-unity/the-media-and-the-genius-myth/`.

[5] Sarah-Jane Leslie, Andrei Cimpian, Meredith Meyer, and Edward Freeland, *Expectations of brilliance underlie gender distributions across academic disciplines*, Science **347** (2015), no. 6219, 262–265.

[6] Jane Margolis and Allan Fisher, *Unlocking the Clubhouse*, MIT Press, 2002.

Sarah Cannon

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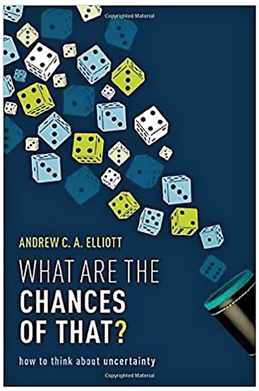

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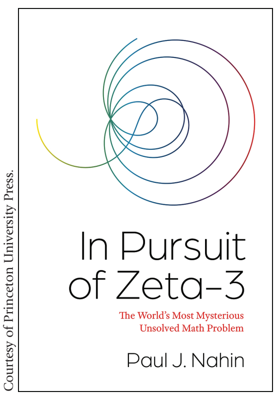
What are the Chances of That?
How to Think About Uncertainty
By Andrew C. A. Elliott. Oxford University Press, 2021, 384 pp.

The probability of random events can go against our intuition and have surprising outcomes. Two famous examples of this are the Monty Hall problem and the birthday problem, although examples can be found in all aspects of life. A better understanding of probability can help us reconcile the counterintuitive nature of the randomness we encounter.

One inherent obstacle when thinking about random outcomes is that while probability is a statement about the long run, we experience the outcomes of randomness as individual events. How can we relate the long run with these single occurrences? Elliott terms this complicated duality as the individual versus the collective. He identifies this as one of the five sources of tension that are fundamental to probability which create a barrier to understanding it. Each of the five dualities is clarified through varied examples. For instance, the individual versus collective dichotomy is explained in terms of vaccines, crime statistics, and music, to name a few.

Written for general audiences, this book is extremely engaging and filled with relevant examples of the misuse of probability that immediately draw the reader in. Each section starts with easy to state probability questions that challenge our intuition, after which, various probability concepts are introduced and discussed. The end of each section gives the solution to the problem stated at the beginning, making use of the ideas developed throughout the section. The book emphasizes intuition over technical reasoning, although where technical arguments would enhance the reader's understanding, the explanations are boxed off and clearly marked so that the reader knows that what follows will be more rigorous. This is an excellent and inviting book for anyone looking to hone their knowledge of probability.



Courtesy of Princeton University Press.

In Pursuit of Zeta-3
The World's Most Mysterious Unsolved Math Problem
By Paul J. Nahin. Princeton University Press, 2021, 344 pp.

Zeta-3 is the value of the Riemann–Euler zeta function at 3, $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$. While the zeta function is most famous for its role in the Riemann hypothesis, there are many open questions surrounding various aspects of it. For example, there is no known symbolic representation for $\zeta(k)$ where $k \geq 3$ is a positive, odd integer (in contrast with when k is a positive even integer). Nahin focuses specifically on the attempts to find this representation for $k = 3$. Zeta-3 has a number of applications in physics and engineering, which appeals to the author as an electrical engineer.

In its treatment of $\zeta(3)$, this book draws on background covered in the calculus sequence. In particular, infinite series, integration techniques, and multivariable integration feature prominently. The preface contains a brief introduction to mathematical induction which is utilized throughout the book and is also there to ensure that the reader's mathematical maturity is at a level to engage with the mathematics in the book. Each section contains multiple challenge questions, the solutions to which are included. Additionally, MATLAB code that accompanies the calculation being discussed is often provided.

Nahin's writing style is welcoming and approachable; he easily draws the reader in with his storytelling ability and sense of humor. This book is accessible to anyone with a good grasp of second semester calculus and may be of particular interest to math, physics, or engineering majors. Mathematicians looking for a light math read will also find this engaging and worth picking up!

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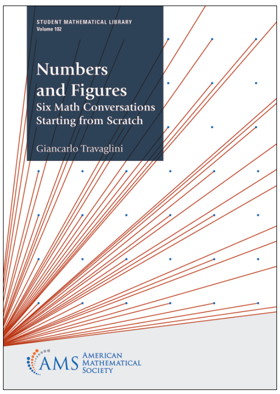
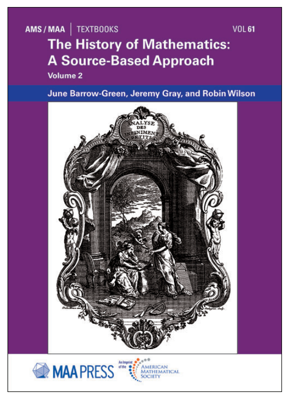


Numbers and Figures
Six Math Conversations Starting from Scratch
By Giancarlo Travaglini.
STML/102.

Numbers and Figures contains six chapters illuminating topics in number theory by providing a connection to geometry. Part A of each chapter provides a very gentle introduction to the topic that should be accessible to any mathematically interested undergraduate. In Part B the topic is followed up in considerable depth. Any of the chapters could form the basis for a rewarding project in a number theory or combinatorics class; reading the whole book would make a beguiling independent study or seminar class.

Chapter Two, for example, opens with several instances of Simpson paradox. There is a lovely two-dimensional geometric illustration for the 2×2 case of the paradox using diagonals of parallelograms. Simple cases of the paradox lead directly to consideration of Farey addition. Part B opens by asking about rational approximations to irrational numbers. We get a quick proof of Dirichlet's theorem: any irrational can be approximated by a rational $\frac{p}{n}$ to within $1/n^2$ for infinitely many different values of n. Conversely, algebraic irrationals cannot be too closely approximated by rationals (Liouville's theorem), which gives a simple proof of the transcendence of Liouville's number. After an interlude on Farey sequences and Ford circles, we get a proof of Hurwitz's, best possible, improvement to Dirichlet's theorem which puts a factor of $\sqrt{5}$ in the denominator. Farey sequences are used to generate rational approximations, connections are drawn to the Basel problem and the Fibonacci numbers.

Every chapter is similarly rich with deep and beautiful ideas not usually explored in the standard undergraduate curriculum. The book is self-contained, everything is proven, and each chapter concludes with a list of suggested readings and a small number of challenging exercises.



The History of Mathematics
A Source-Based Approach, Volume 2
By June Barrow-Green, Jeremy Gray, and Robin Wilson.
TEXT/61, 2022, 687 pp.

The History of Mathematics: A Source-Based Approach describes the history of mathematics with an incredible richness of detail. But, and this is the first of two defining and exceptional features of the book, the authors explicitly and constantly keep in the foreground historical questions: How do we know this? And, How confident can we be that we are correct? Of course, the answer to the first question is that our predecessors left documents and we can read them. And that brings us to the second exceptional feature: the book includes excerpts of primary sources with the expectation that we will read them as a historian would. The authors are making the point that to understand history one must struggle with source material and try to understand what the author of it is saying in his or her own terms.

This, the second volume of a two-volume set, takes the reader from the invention of calculus to the beginning of the twentieth century. The initial discoverers of calculus are thoroughly investigated, and special attention is paid to Newton's *Principia*. The eighteenth century is presented primarily in terms of the development of calculus and its applications. Mathematics blossomed in the nineteenth century and the book explores progress in geometry, analysis, foundations, algebra, and applied mathematics, especially celestial mechanics. The reader learns not only the history of mathematical ideas, but also how to think like a historian. These volumes were designed as textbooks for the authors' Open University course, but they can also be profitably read by mathematicians.

This AMS Bookshelf is by AMS retired Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is `skennedy@carleton.edu`.

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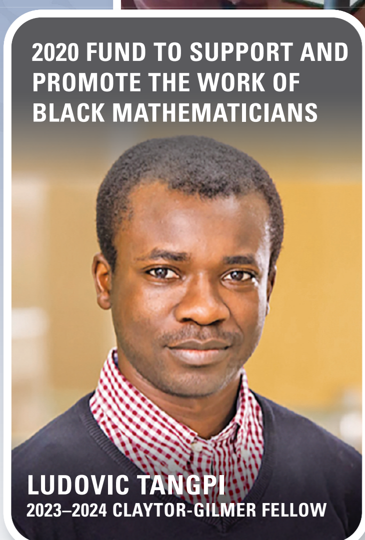

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On Best Practices for the Recruitment, Retention, and Flourishing of LGBTQ+ Mathematicians

Ron Buckmire, Amanda Folsom, Christopher Goff, Alexander Hoover, Joseph Nakao, and Keri Ann Sather-Wagstaff

Introduction

Recently, more people in the mathematics community have been acknowledging something, which to many of us, is an obvious fact: mathematics is a human endeavor. That is, mathematics is done, taught, learned, researched and discovered by people. A corollary of this fact is the identities of the people who are "doing the math" is important; this idea undergirds much of the recent increased activity of diversity, equity, and inclusion (DEI) initiatives in the mathematics community.

Included in this burgeoning awareness that the identity of who is allowed to participate fully in the mathematics community is important is the idea that lesbian, gay, bisexual, transgender, and queer inclusive (LGBTQ+) people can be mathematicians also. So, as institutions and individuals start to take their commitment to DEI more seriously, in addition to considering how to broaden participation by people whose race, ethnicity, or gender are currently underrepresented in mathematics, we also need to consider the sexual orientation and gender identity of mathematicians. The question arises: how can we in the mathematics community best support LGBTQ+ mathematicians in various aspects of their professional lives?

A first step someone can take to show their support is to familiarize themselves with current and outdated terminology. For instance, the difference between "LGBTQ" and "LGBTQ+" is the "+" signifying all of the gender identities and sexual orientations not specifically covered by the other five letters. Other current terms include "nonbinary," "deadname," and "cisgender." On the contrary, several terms such as "homosexual" and "sex change" are outdated and possibly offensive. We refer the reader to three excellent glossaries for more information [4, 10, 11].

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At the Joint Mathematics Meetings (JMM) in 2022, there was a panel discussion to address the question of what are the best practices for the recruitment, retention, and flourishing of LGBTQ+ mathematicians. Organized by members of Spectra, the Association for LGBTQ+ Mathematicians, the panel featured mathematicians sharing their personal recommendations for how to support transgender and nonbinary mathematicians at work, how to recruit LGBTQ+ mathematicians to your department, and how to support graduate students who identify as members of the LGBTQ+ community. This article grew out of the organization of that panel and the ensuing discussions and feedback that followed. We hope that by sharing possible answers to the questions raised by the panel in this venue, we will reach a wider audience than just those who were able to attend the JMM in 2022. After each panelist has shared their detailed contribution, brief concluding remarks will follow.

Supporting Transgender and Nonbinary Mathematicians in the Workplace

Keri Ann Sather-Wagstaff

I need to start with a few disclaimers.

First, I carry a huge amount of privilege. I am a White full professor at an R1 institution. Trans people who are racially minoritized and who live on other axes of marginalization generally experience significantly more/different bias and discrimination due to their transness and other factors. And those at other types of institutions will have different experiences as well. This is not to suggest that R1 institutions are free from transphobia, far from it. But I have some privilege here within the research community.

Next, the trans and nonbinary community, like all marginalized communities, is not monolithic. This piece consists of my opinions and my experiences, and the opinions and experiences of other trans people will vary. On the other hand, while I write this from the perspective of a faculty member, much of it also applies to staff members, grad and undergrad students, and others. Overall, my point is that you should take the following suggestions in the appropriate context and not as a rulebook. In particular, none of what I write here should be construed in any way as legal advice.

The single most important thing you can do to support and affirm your trans and nonbinary colleagues is to use their correct name and pronouns without question or hesitation. Do not deadname or misgender us; do correct anyone who does so. Do not ask us or anyone else for our deadnames; it is not information that you or anyone else needs to know unless there is a valid, legal requirement for it. Do not ask a person for information about their sex or the sex of another person.

To be clear, I use the words "name" here as others might use "lived name" or "chosen name," and similarly for "pronouns." It is common to call these "preferred name" and "preferred pronouns," but that suggests that they are optional for you to use, which they are not. To deadname or misgender a trans person is transphobic, unprofessional, and inappropriate.

It can be hard for some cis people (not all cis people, of course) to understand how harmful it can be to deadname or misgender a trans/nonbinary person. It may be helpful to think about it like a professor who is addressed as Mr. or Mrs. instead of Dr. or Prof., though I recognize that not everyone understands or can fully appreciate the myriad ways that this can be harmful, especially to members of our community with other marginalized identities.

As part of using your colleagues' names and pronouns correctly, you can work to ensure that their nametags, doorplates, email aliases and signatures, video conferencing identities (e.g., Zoom windows), and other identifying markers use their correct names and pronouns. This helps to ensure that your colleagues are not being deadnamed and/or misgendered. This includes making sure that they are able and allowed to include their pronouns in these environments. You can also share your own pronouns in your email signature, your video conferencing window, and whenever you introduce yourself. This helps to normalize the practice, so trans and nonbinary people aren't the only ones doing it.

If you are a journal editor or involved in other aspects of publishing or reviewing scholarly texts, help your trans/nonbinary authors and editors to correct their deadnames in their past publications and publicly displayed reviews. This is becoming common practice. Many journals and publishers already have policies for this. Others that do not might be convinced to do so when they learn they might not have to reinvent the wheel.

Bathrooms. This topic is particularly challenging. Let me be absolutely clear here: trans and nonbinary people need to have access to and be able to use restrooms that match their correct gender. This is a matter of fundamental human dignity. This includes having at least one gender neutral restroom in any building where you have office or classroom space.

Have our backs if you hear someone deadnaming us, misgendering us, using transphobic slurs, or talking about whether we pass or whether we should use another restroom. Be an ally whether we are or are not present. Tell your colleagues how inappropriate their comments and behavior are. Consider discussing the behavior with your chair, Office of Access and Equity, or other appropriate individual/office. Be mindful that your trans or nonbinary colleague may or may not want you to make a big deal about it. Consider making a plan with them before any such incident occurs.

If you accidentally misstep with your trans or nonbinary colleague, apologize quickly and move on. Do this if you deadname or misgender them or do anything flagged in this article as inappropriate or insensitive. Be careful about making a big deal about it or placing another emotional burden on the injured party when your action already caused harm.

Educate yourself. Learn about relevant institutional policies and procedures. Learn about experiences of trans and nonbinary people. Google is your friend, so you don't have to ask us to commit the extra emotional labor required to educate you ourselves. And remember that the trans and nonbinary community is not monolithic, so not everything you read will apply to each of us.

Listen to us. Don't make assumptions about us or our experiences. This includes not assuming that someone is trans or nonbinary if they have not told you they are.

If we tell you something is harming us, believe us. If we tell you that we are being harmed, we are giving you an opportunity to help retain us as colleagues and to help us flourish in the department. If you want to be an ally, tell us that you hear us and ask us how you can help. Don't tell us that our experience doesn't sound so bad, or that you or someone else had it way worse. Don't suggest that we imagined it or that we're being too sensitive. Don't try to justify or downplay the harm. Don't tell us not to take it personally. Listen actively. And don't tell us to suck it up or to just get a job somewhere else.

Respect our privacy and our boundaries. Do not ask us if we are trans or nonbinary if we have not come out to you. If we have come out to you, don't ask us about medical matters. If a trans or nonbinary person shares personal information with you about their transition (including coming out) that they're not sharing with the whole department, listen to them actively, thank them for sharing and for trusting you, and assure them that you will keep everything they tell you in the strictest confidence. Be extremely careful discussing any medical aspect of their transition with them, as such discussions can involve topics that are probably not appropriate to discuss in a professional environment or with a professional colleague. Similarly for any aspect of your colleague's sex life. Don't share any personal information about your trans or nonbinary colleague with others without their explicit permission. Do not ask anyone else about your trans or nonbinary colleague's personal information.

Best Practices for Recruitment of LGBTQ+ Faculty

Amanda Folsom

According to a 2022 Gallup poll, 7.1% of adults in the US self-identify as LGBTQ+, an all-time high, doubling the percentage recorded a decade earlier in 2012, and up from 5.6% in 2020 [7]. Among the "zoomers," the Generation Z adult population (with birth dates between ~ 1997–2012), a demographic cohort that encompasses most current college students, the same poll reports that one in five self-identify as LGBTQ+. The uptick in these reported statistics are likely in part reflective of increased societal acceptance of the LGBTQ+ population, but they should still be read with a grain of salt: the LGBTQ+ community is still marginalized. Moreover, these numbers almost certainly do not reflect the actual size of the LGBTQ+ population, as many in the community are not "out," meaning they choose not to publicly disclose their sexual orientation or gender identity.

What does all of this mean in terms of the work force, and in mathematics academia in particular? For one, the new generation entering the work force does so with more expectations of equality, fair treatment, and support. Current-day college students approach their undergraduate years with similar expectations (as do current graduate students). More and more employers are building explicit policy and workplace support pertaining to LGBTQ+ employees under the larger umbrella of diversity, equity, and inclusion initiatives. Below we consider several best practices for the recruitment and retention of LGBTQ+ faculty, with mathematics academia in mind. Given our publication outlet, most of our discussion is US-focused; for further international resources and rights, see for example [8, 9, 15].

Know the Law. Under Title VII of the Federal Civil Rights Act of 1964 in the US, federal law prohibits discrimination based on sex including sexual orientation and gender identity; as a result, LGBTQ+ people may file complaints with the Equal Employment Opportunity Commission [13]. Approximately half of US states also uphold certain additional state laws against employment discrimination based on sexual orientation and/or gender identity, though these vary in scope and protections offered [2]. Some universities and colleges also publish their own policy regarding discrimination based on sexual orientation or gender identity (these may be found in a faculty handbook, Dean's Office website, or Office of Diversity, Equity, and Inclusion website, etc.), and many now offer explicit trainings for faculty and search committees on inclusive hiring practices, legality, DEI, implicit bias, and more. As a faculty member in a department, member of a hiring committee, or as a job seeker, it is important to approach any hiring situation with the law and explicit state and university policy in mind.

Job Postings. A scan of job postings for math faculty positions on `https://www.mathjobs.org` will quickly reveal many non-discrimination clauses included, such as *"We are an equal opportunity employer; all qualified applicants will receive consideration for employment."* Some current-day ads go further to include explicit phrasing such as *"...without regard to race, color, religion, sex, sexual orientation, gender identity, national origin, disability,...,"* and some actively encourage such candidates to apply through wording such as *"we encourage potential candidates from underrepresented and/or historically excluded groups to apply"* or *"...including, but not limited to, racial and ethnic minorities, women, members of the LGBTQ+ community, persons with disabilities, persons from lower-income background and first-generation college graduates."* Even a single sentence in a job ad making explicit reference to anti-discrimination, tolerance, and support for LGBTQ+ candidates can go a long way, and send signals of inclusivity to the candidate at the earliest stage of the math job application process. Conversely, with more and more universities making such explicit references in jobs ads these days, omitting such a sentence may send a negative signal to math-job-seeking members of the LGBTQ+ community.

A job ad can also be an opportunity for departments to make mention to a diverse student population, including LGBTQ+ students, and a new faculty member's role in this regard, e.g. *"we serve a diverse undergraduate and graduate student population, and are interested in applicants with a record of successful teaching and mentoring of students from all backgrounds, including first-generation college students, low-income students, racial and ethnic minorities, women, LGBTQ+, etc., and who have a demonstrated ability to contribute to undergraduate diversity initiatives in STEM."*

While `https://www.mathjobs.org` is a major outlet for academic jobs in mathematics, it is not the be-all and end-all. Departments should consider advertising open positions more broadly, for reasons including more potential diversity in their recruitment efforts. To this end, some additional job ad outlets to consider include Spectra, the Association for LGBTQ+ Mathematicians, the Association for Women in Mathematics (AWM), and the National Organization of Mathematicians (NAM), which promotes and supports research in the mathematical sciences, especially for underrepresented minorities.

The Interview. At any point during the recruitment process, it is currently illegal (in violation of Title VII) for those doing the hiring to ask questions that force a candidate to reveal personal information such as sexual orientation, marital status, age, citizenship, and more. This is important for both the interviewing department and job candidate to know. Interviewers should not assume a person's pronouns (e.g., they/their, she/her, he/his) and should in general be mindful of what is said during casual conversations; the candidate is interviewing the departments in a sense, too. Interviewers should let the candidate lead any conversation about their gender identity or sexual orientation, which they may choose not to discuss at all, and interviewers should not pry if the candidate voluntarily opens up. In case a faculty interviewee chooses to come out at any point before, during, or after the interview, it could be helpful for the search committee members to have thought through in advance how they would respond in the moment, to show support and community—and even better, to actively indicate interest in recruiting and retaining diverse faculty. When setting up an interview schedule, if time permits, hiring departments may consider asking candidates if they would like to request optional appointments with a campus group or person of their choosing; this open-ended invitation could allow a candidate to talk to others on campus about LGBTQ+ culture or issues, without any presumption on the part of the search committee.

In the Department and on Campus. In addition to a job ad potentially sending signals of tolerance, support, and welcoming, department-specific websites can do the same. For example, a department may choose to include a DEI sub-page which makes specific reference to the LGBTQ+ community in some way, or links to relevant campus resources such as a Queer Resource Center, or LGBTQ+ faculty groups. Physical signage on campus and in departments can go a long way, too. *"One of the first initiatives of these groups was to hang a permanent pride banner outside the university center... an openly gay faculty member told the group that this is what convinced them to come to Adelphi* [6]*."* The AMS and Spectra have recently partnered to produce posters highlighting LGBTQ+ mathematicians, and campus Queer Resource Centers typically have free "swag" to hand out such as "LGBTQ+ Safe Space" or "Ally" stickers – a gesture as small as getting a hold of these free materials and hanging them around the department or outside your office is meaningful.

Retention and Recruitment. Kerry Ann Rockquemore, author, speaker, and academic in the field of faculty development and leadership, bluntly tells us *"For a Diverse Faculty, Start With Retention"* [3]. We leave you, reader, with Kerry's five important questions to consider regarding retention and recruitment of LGBTQ+ faculty and diversity in hiring more broadly:

1. Do you really know why faculty members have left your department?
2. Have you asked current faculty members if they have what they need to succeed?
3. Is a structure in place to support newly recruited faculty members?
4. Is there an ugly reality that nobody wants to face?
5. Are you actually behaving like an ally in your department?

Supporting LGBTQ+ Graduate Students

Joseph Nakao

A student's success in graduate school can heavily depend on whether or not they feel safe, supported, and part of a community. There are already countless stressors that cause anxiety throughout graduate school such as coursework, research, finances, and one's personal life. Yet, departments seldom consider the struggles of being LGBTQ+. The challenges LGBTQ+ students face are constant and everywhere. From not being addressed by one's pronouns, to being blatantly discriminated on the street, to being one of a few out LGBTQ+ individuals in a department, all these instances take an immense toll on a student's mental health. Before moving forward, I acknowledge that facing challenges because of one's identity is not unique to LGBTQ+ individuals and applies to all marginalized groups.

What can departments do? Many of the topics addressed in the other sections of this article also apply to graduate students. The purpose of this section is to focus specifically on graduate students. As mathematicians, we understand the importance of stating assumptions before stating results. A department's desire to increase their commitment to diversity, equity, and inclusion follows a similar structure. Before discussing ways to improve DEI, a department should first assume a position of humility and an honest willingness to change. Talking about improving DEI is the "easy" part; taking action and following through on your word is the "hard" part. Without this mindset, a department will be unable to fully support their graduate students.

I believe there are two indicators of a department's support of LGBTQ+ graduate students: (1) the level of friendliness and thoughtfulness within a department, and (2) the promotion and representation of LGBTQ+ mathematicians. I have found that building a departmental atmosphere of thoughtfulness and friendliness allows LGBTQ+ inclusivity to flourish. A graduate student's department should serve as an oasis within a university and town/city. Their department should make them feel welcome, respected, and safe. LGBTQ+ students want to feel included without fear of judgment. Simply interacting with faculty and students who are thoughtful and express interest in knowing you can significantly reduce anxiety. I encourage faculty and other students to occasionally take a stroll past student offices/cubicles and check in on how their classes and research are going. This is a small yet personal example that can help with the first indicator.

As with most things in life, communication is key. More importantly, it's the students' opinions that matter! This cannot be overstated. Students need to feel comfortable sharing their concerns with the department, and they need to have confidence that their concerns will be addressed. Hence my emphasis on working to build a feeling of openness and camaraderie between faculty and students. I have found that when students feel more connected with their department, conversations on DEI come naturally. However, I want to note that departments should clearly preface any communication with a disclaimer that they might be required to report certain conversations. Another idea to help spur communication and trust between faculty and students is to have a graduate student representative on the graduate studies committee. Having a graduate student elaborate and point out concerns students have can also develop trust between both bodies. But, it is imperative that the faculty take the concerns raised with seriousness and are willing to work with students to resolve the issues.

Another idea is to host weekly meetings between all instructors and TAs to discuss the previous week's discussions/labs. (This could be broken down by course, e.g., all instructors and TAs for Calculus III meet). In general, the TAs take turns talking about their sessions: what went well, what went poorly, and any situations that they need guidance on. For example, perhaps a student started getting visibly upset because they struggled to understand a problem that everybody else seemingly understood. Dealing with these sensitive student situations requires experience, patience, and no small amount of empathy. The instructors and TAs then bounce ideas in a Socratic manner. Although topics of DEI might not be a talking point each week, these weekly conversations increase trust. The level of friendliness and thoughtfulness within a department between faculty and students will follow naturally from these more personal situations.

Ideally, a department should have widespread minority representation, including LGBTQ+ faculty and graduate students. This is not always the case. Nonetheless, a department can still stay current on articles, conferences, and workshops that highlight LGBTQ+ mathematicians. Places to look include: LGBTQ+ specific minisymposia and panels, conference/workshops listings on the websites of the major mathematical societies, and the Spectra newsletter. Encouraging students to read these articles and attend these events, even if you and the student do not identify as LGBTQ+, goes a long way in showing your support for LGBTQ+ mathematicians. I believe it is incredibly beneficial for graduate students to see other successful LGBTQ+ mathematicians. This exposes students to similar and relatable mathematicians that they can look up to.

LGBTQ+ graduate students face a wide range of challenges, and mathematics departments have a role to play in providing a safe and supportive atmosphere. By encouraging a thoughtful departmental atmosphere and promoting LGBTQ+ mathematicians, departments can create a more supportive environment for their LGBTQ+ graduate students. Ultimately, the graduate students are the people to listen to, and I encourage departments to facilitate discussion between faculty and students to open up communication.

Concluding Remarks

These three contributions have described several actions that departments and faculty members can take (and avoid) to support transgender and nonbinary colleagues, as well as to recruit and retain LGBTQ+ faculty and graduate students. The suggestions contained herein will go a long way toward building a more inclusive department, and a more inclusive mathematics community as a whole.

But it is important to remember that these are just three contributions from a large and diverse community. As these panelists mention, mathematicians from underrepresented groups who also identify as LGBTQ+ face intersecting layers of prejudice and discrimination. Fortunately, treating all individuals with respect, recognizing and celebrating everyone's diverse backgrounds, and reflecting on the ideas and suggestions in this article can go a long way to creating a professional mathematics community welcoming to all, especially those of us in the LGBTQ+ community.

If you are interested in supporting the LGBTQ+ mathematical community, please consider joining Spectra, the Association for LGBTQ+ Mathematicians. Spectra regularly provides programming at the Joint Math Meetings, including the plenary Lavender Lecture, as well as panels like the one detailed here, and has held a social reception at every JMM since 1996. We also organized an online research conference hosted by The Institute for Computational and Experimental Research in Mathematics (ICERM) for LGBTQ+ mathematicians in 2021 [14], and are planning the next one as of this writing. For more information, please consult other articles in the *Notices* such as [1], chapter contributions such as [5], or Spectra's website [12].

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References

[1] Robert Bryant, Ron Buckmire, Lily Khadjavi, and Douglas Lind, *The origins of spectra, an organization for LGBT mathematicians*, Notices Amer. Math. Soc. **66** (2019), no. 6, 875–882. MR3929579

[2] Employment Nondiscrimination, `https://www.lgbtmap.org/equality-maps/employment_non_discrimination_laws`.

[3] *For a Diverse Faculty, Start With Retention*, Jan. 26, 2016, `https://www.insidehighered.com/advice/2016/01/06/how-retain-diverse-faculty-essay`.

[4] Glossary of Terms, `https://www.hrc.org/resources/glossary-of-terms`.

[5] Christopher Goff, George Bradley, Alexander Hoover, and Aubrey Kemp, *Fostering a Community of LGBTQ+ Mathematicians*, in Della Dumbaugh and Deanna Haunsperger (eds.), Count Me In: Community and Belonging in Mathematics, AMS/MAA Classroom Resource Materials, 2022, 191–200. Available at `https://www.maa.org/sites/default/files/pdf/pubs/books/members/CountMeIn.pdf`.

[6] Perry Green, *Actively Seeking Diverse Faculty*, Dec. 12, 2018, `https://www.insidehighered.com/advice/2018/12/12/advice-attracting-and-retaining-diverse-faculty-members-opinion`.

[7] Jeffrey M. Jones, *LGBT Identification in U.S. Ticks Up to 7.1%*, Feb. 17, 2022, `https://news.gallup.com/poll/389792/lgbt-identification-ticks-up.aspx`.

[8] LGBT Rights, `https://www.hrw.org/topic/lgbt-rights`.

[9] LGBTI Rights, `https://www.amnesty.org/en/what-we-do/discrimination/lgbti-rights`.

[10] LGBTQ Glossary, `https://studentaffairs.jhu.edu/lgbtq/education/glossary/`.

[11] LGBTQ+ Glossary, `https://health.ucdavis.edu/diversity-inclusion/LGBTQI/LGBTQ-Plus.html`.

[12] Spectra: The Association for LGBT Mathematicians, `http://lgbtmath.org`.

[13] Title VII of the Civil Rights Act of 1964, https://www.eeoc.gov/statutes/title-vii-civil-rights-act-1964.
[14] VIRTUAL ONLY: Spectra LGBTQ+ in Mathematics Conference, https://icerm.brown.edu/topical_workshops/tw-21-smc.
[15] #OUTLAWED "The love that dare not speak its name," http://internap.hrw.org/features/features/lgbt_laws.


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Credits

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Національна Академія Наук України
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Voices from the Bombed Universities of Ukraine

Masha Vlasenko and Efim Zelmanov

The National Universities of Kyiv and Kharkiv have been severely damaged by Russian missiles. We have interviewed several mathematicians working at these two universities. **Oksana Bezushchak** is the dean of the Faculty of Mathematics and Mechanics at the Taras Shevchenko National University of Kyiv. **Sergey Gefter** is the head of the Department of Pure Mathematics at the Vasyl Karazin Kharkiv National University. **Volodymyr Kadets** is a professor in the same Department of Pure Mathematics in Kharkiv, who was wounded during a shelling of the city in March of 2022.

Figure 1. Faculty of Economics and Karazin School of Business, Karazin University.

Masha Vlasenko is an associate professor at the Institute of Mathematics of the Polish Academy of Sciences. Her email address is mvlasenko@impan.pl.

Efim Zelmanov is a professor at the University of California, San Diego. His email address is ezelmanov@ucsd.edu.

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i) Could you please tell us about the destruction of your university. Is there any reconstruction going on? Are students and faculty involved in this process? How long will reconstruction take, in your opinion?

Bezushchak: The buildings of the Kyiv University have been attacked by Russian missiles several times. The most terrible attack so far happened on December 31, where 13 buildings were heavily damaged. Later professors, students, and volunteers removed tons of damaged glass, scrap, and wreckage. We live in hard times for our country and economy, so the rebuilding of our university depends on no one but ourselves. Many students help with the dirty work of cleaning and getting rid of the mess. Our professors used electric and circular saws during their vacations, and covered the damage with plywood shields. Students, alumni, professors, patrons, scientists, and friends are the ones we can count on. Nevertheless, we are proud of our university, because our buildings took the main damage, sheltering and securing the residential buildings which saved many innocent lives. We have already raised donations from the whole university community to prepare and launch the first stage of the repair. We are trying to raise more money to be able to cover as much of the damage as possible, because it is crucial to repair the damage quickly to avoid subsequent destruction and more expense. We are grateful to many foreign partners that have supported our efforts. Right now, we cannot state any certain dates. It is a matter of time.

Gefter: Already in the early morning of February 24, 2022, Russian Federation launched a missile attack on the buildings of the Faculty of Physics and Technology of Kharkiv University, which caused significant destruction. The buildings of the Faculty of Economics, the Institute of Public Administration, the sports complex, the Institute of Banking, and the university clinic were destroyed. Other buildings and structures were significantly damaged,

Figure 2. Mathematicians doing reconstruction work, with Oksana Bezushchak sawing, Kyiv University.

including: the library building, the nature museum, and the dormitories. In the main building of the university, there is a great deal of damage and almost half of the windows have been lost. According to Kharkiv University vice-rector Anatoly Babichev, who is an alumni of the Faculty of Mathematics and Mechanics, the losses to the university infrastructure are preliminarily estimated at more than 100 million dollars. Of course it is impossible to pay this with the University's budget. In addition, complete reconstruction is impractical because missile strikes on Kharkiv continue. Students, professors, and other employees of the university have worked on bringing the premises to a more or less normal state: they cleaned the glass, repaired what they could as nonprofessionals, blocked the windows, dismantled the rubble, and removed the equipment that was salvageable. The rebuilding process will be long and difficult, but all this will be after the victory.

ii) How has the war affected mathematical life in Ukraine? Are you able to teach under bombs?

Kadets: The war has affected the lives of mathematicians in the same ways that it affected the lives of other Ukrainians: by losses, danger, psychological trauma, forced relocations to other cities and even other countries. Mathematicians, and especially "pure" mathematicians, by the very essence of their work are in a privileged position compared to other scientists, because they do not need any equipment for their research. The main thing is the presence of an interesting problem and an inner urge to do research. For me personally, during the war, this inner drive has decreased. I can hardly concentrate and distract myself from current news and everyday problems.

Regarding teaching, on the day of the attack, the administration suspended the work of the university, and we did not work for about a month. It was a very smart decision, because since the beginning of the full scale war, Kharkiv was under heavy shelling. Battles took place on the outskirts, and in the first days, small groups of the enemy even broke into the city itself. I witnessed a combat clash right under the windows of my house. Some university buildings were actually destroyed (the Faculty of Physics and Technology suffered the worst). The central building was damaged, which led to the shutdown of the university servers and, as a result, the university website.

Classes resumed at the end of March. By that time, many teachers and students had evacuated to other, safer places, but in those new places they often had limited access to the internet and difficult living conditions. I missed two classes because I was in the hospital due to an injury. In September we returned to teaching by Zoom, which we had gotten used to during the pandemic. At that time, I was no longer "under bombs", because my wife and I left for Israel. The students were much worse off, not only because of the immediate danger, but also because of the constant blackouts caused by Russian shelling, problems with heating, etc. I feel uncomfortable teaching in comfortable conditions and safety, when many students have a completely different situation.

Bezushchak: Imagine you are giving a lecture, no matter online or offline. And you hear beating and hitting noises, that are so loud, they cannot be ignored. Is it easy to concentrate and keep working? Now imagine that any of those noises could suddenly terminate your life or the life of someone you love or care. What about your concentration now? And apart from that, after the next hit your electricity disappears for an hour, a day, or a week, who knows. Will you be able to carry out any activity as usual? You decide.

Regardless of the situation, both professors and students keep doing their best to handle it, to teach our future specialists as best we can.

Gefter: The war changed all aspects of life in Ukraine, and, in particular, it affected mathematical life in a fundamental way. During the first months of the invasion, many Ukrainian mathematicians had almost no opportunity to do research, their life was often taken up by their and their family's physical survival. Some mathematicians

Figure 3. Faculty of Physics and Technology, Karazin University.

Figure 4. Karazin University.

ended up in the occupied territories. A significant number of mathematicians from Kharkiv were forced to move to safer places in Ukraine, or even outside its borders. Of course, many scientific conferences were canceled or postponed. Currently, in the unusual conditions of martial law, research continues, and online scientific seminars are held. The Faculty of Mathematics and Informatics continues to actively work with schoolchildren. In January 2023, our Faculty organized the All-Ukrainian Internet Mathematics Olympiad. Regional stages of student olympiads and of the Junior Academy of Sciences in mathematics will be held in the spring.

At the end of March 2022, classes at the Kharkiv University resumed in remote form. Power and internet outages are not an insurmountable obstacle for those who want to work and study. It is not easy to teach and study under rocket shellings, artillery, and mortar fire. Some students who remained in Kharkiv joined online classes from bomb shelters. Sometimes, during air raids, classes had to be stopped and postponed. After heavy rocket attacks, electricity and the internet were often cut off for a long time. In this case, classes and consultations were postponed. Thus, the educational process actually does not stop.

iii) Is there anything the mathematical community could do to help?

Kadets: From the first days of the full-scale war, I personally felt the support of the mathematical community. I received letters from my colleagues from many countries (Bulgaria, China, Czechia, Germany, Great Britain, Hungary, India, Israel, Italy, Poland, Spain, the USA, I apologize if I did not remember someone) with words of support, as well as with touching offers of shelter, temporary work, etc. This moral support was important and I am grateful to everyone for it. By the way, none of my colleagues working in Russia even asked "how are you"... Then there began to appear numerous initiatives providing grants to Ukrainian scientists who had gone abroad, and opportunities for students to study at foreign universities. What we lack is support for mathematicians and university teachers who remain in Ukraine. It would be good if not only famous mathematicians could count on such support, but also ordinary teachers, because many of them are now in a difficult situation, and without them the collapse of the education system is inevitable.

Bezushchak: Not only could help. The help has already started! We express our deep gratitude to mathematicians from many countries who work hard to help Ukrainian refugees. The International Mathematical Union moved the ICM-2022 out of St. Petersburg in protest against the war. This year four well-known mathematicians—Pavel Etingof (MIT), Roman Vershynin (University of Texas), Maryna Viazovska (EPFL), and Efim Zelmanov (University of California)—volunteered to do what they do best: teach the young generation of Ukrainian mathematicians. Maybe for the first time ever our freshmen were taught simultaneously by two Fields medalists. Exams and discussions were handled by our faculty.

We had a bright and extremely talented young mathematician, Yulia Zdanovska. She taught math to talented teenagers. One of the projects she played a huge role in was called "Teach for Ukraine". Unfortunately, you can never entirely appreciate one's impact until you lose it. Yulia Zdanovska was brutally killed by Russians in her home city, Kharkiv. Now the address of our Department is: Yulia Zdanovska street.

When Pavel Etingof learned about our Yulia and her project, he initiated an MIT-sponsored project for talented high school students in Ukraine. Some time ago he was a talented high school student in Ukraine himself.

Gefter: These days the mathematicians at the university, as well as the entire university community, need support.

Figure 5. Karazin University.

Under our conditions of long-term power outages, additional equipment is needed to conduct classes. We have an urgent need for 15–20 laptops and corresponding power banks for teachers and students. We also would be grateful if we could get a license package for one of the latest versions of Maple (online and desktop versions).

iv) There is an opinion that science should not be mixed with politics and that mathematical activities should go on as usual. What do you think about it?

Kadets: The word "politics" somehow bothers me in this context. Politics is about differences of opinion. Shelling of residential areas, killing, robbing, and abusing people are crimes. The attitude towards this is also not a political choice, but a choice between complicity in crime and taming criminals.

As for the essence of the question, what "as usual" are we talking about? Is it possible to imagine, for example, organizing an international scientific conference in Russia now? Any event of this kind would be de facto support of the Russian government. Funding of the leading Russian journals comes from the government. How could we publish there in these circumstances?

Bezushchak: There is a common belief that science as well as sports, art, etc. should be separated from politics. That is to say that these activities should unite nations, and be the main driver to spread love and unity. But our mathematicians take up arms to defend our country. Our young students die as a result of missile launches. And those that sit in shelters are afraid of being buried alive. When things go in such a way, no normal human being can stand aside pretending that nothing is happening. Regardless of your personal opinion about the war, Russia keeps killing us. The rest is a matter of one's personal conscience. To sum up, we keep carrying on our mission to bring up the next generation of Ukrainians to form a prosperous European future of Ukraine.

Gefter: I agree with this opinion, but one can hardly call the war in Ukraine politics. Therefore, unfortunately, mathematical activities in Ukraine will not be able to go as usual for a long time.

Figure 6. Anton Ryzhov, at the border of Donbas.

The last question is addressed to scientists who defend their country on the battle field. We received answers from two professors of mathematics at the Kyiv University. **Oleksiy Kapustyan** is the winner of the award of the President of Ukraine for young researchers and the award of the National Academy of Sciences of Ukraine. His research interests are in the fields of dynamical systems and differential equations. Oleksiy joined the Armed Forces of Ukraine on March 3, 2022. In April–May 2022, he participated in fighting near Popasna in the Luhansk region. In August 2022, after being in the hospital, Oleksiy was discharged due to health conditions. In September he resumed his work at the university. In December 2022 Oleksiy was elected as a head of the Department of Integral and Differential Equations. **Anton Ryzhov**'s fields of research are biostatistics, data analysis, cancer epidemiology, and optimal control. As of February 25, 2022, Anton is an officer of the Territorial Defence Forces, the military reserve component of the Armed Forces of Ukraine. He served in

Kyiv until November 2022. As of December 2022 Anton fights near Bakhmut in Donetsk region.

v) You are a mathematician. What made you take arms and risk your life?

Kapustyan: A professional mathematician, a classical university career—and suddenly a commander of a rifle platoon. What made me to take this step? It is difficult to name one reason. On that day, February 24, 2022, I was confused just as everyone else was. Something had happened that simply could not happen—the war. Probably, the habit of analyzing prevailed over my emotions and helped to highlight the important idea, or strategy: if you don't know what to do, do what you have to do. And then everything became simple. I realized that protecting my family, my beloved ones, defending my home with weapons, would be the right thing to do, the only solution under these initial conditions.

Ryzhov: Yes, I am a mathematician. But also, I am a father, a son, and a Ukrainian citizen. When a powerful and enormously cruel enemy is going to destroy everything you love, there's not much time to decide. Being at the front almost from the first day of the war, I make use of my mathematical training as much as possible. I am grateful for all the support my colleagues from the university have provided to me and my unit, we believe that only united we will win. But I also see all the equipment and machinery and other supplies our friends from the West are sending to Ukraine to help us win this war. Then, I can get back to my room and continue all my suspended projects and start teaching new classes, which I am missing the most.

Masha Vlasenko

Efim Zelmanov

Credits

Figures 1, 3, 4, and 5 are courtesy of Karazin University/Viktoriia Yakymenko.
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What's it Like for a Mathematician to Run for Congress?

Jerry McNerney

My journey in Congress started when my son, Michael, urged me to run in 2004. He was in the Air Force, and when he received his absentee ballot for the primary election in the mail, he noticed that our political party did not have a candidate for our congressional seat. He called and urged me to run for the seat. He was serving our country, he said, and wanted me to serve our country by running for Congress. It was late in the election cycle since absentee ballots had already gone out, but still time to run as a write-in candidate. After some deliberation, I decided to take on the challenge, but had a lot to learn.

I was not the first mathematician in Congress. Dalip Singh Saund preceded me in the 1950s. From the Fresno, CA area, Dr Saund was also the first Asian American elected to Congress. He clearly climbed some very steep mountains in his career. Today, a large portrait of Dr. Saund hangs in the Capitol building in the stairwell just east of the House floor in tribute to his achievements. Ironically, both Dr Saund and I, the only two mathematicians ever in Congress, are from the Central Valley of California, which makes it seem like the Central Valley is a hotbed of mathematicians in politics.

My academic math background is differential geometry. I earned a PhD from the University of New Mexico in 1981. Studying math as a graduate student while starting a family was one of the most joyous periods of my life. Any mathematician loves an intellectual challenge that involves understanding and unlocking secrets of our universe.

Politics, on the other hand, involves a different set of interests, skills, and ambitions than mathematics. It's true

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that politicians must learn a great deal about the world around them, but being a successful politician in a democracy is more about communicating, relationship building, and projecting a self-image that appeals to a broad spectrum of the electorate. You need to be sociable, and you have to be obsessed about the job to be successful.

After getting my PhD, instead of going into academia, I opted for industry and spent 20+ years developing wind energy and smart grid technologies. This was a very rewarding and sometimes exciting career path. I was on the ground level with the development of some very successful and impactful technologies. I used math skills in developing wind turbine dynamics models and control systems simulations as well as statistics and a general ability to think clearly about field data and wind energy economics. However, the company I worked for during most of my years went bankrupt because of bad engineering management decisions. This made me realize that in industry, I needed to be more than an engineer, I needed to delve into management.

Making the transition to politics from mathematics and engineering was incredibly difficult. Public speaking, raising campaign money, communicating effectively on any subject, listening carefully, empathizing with constituents and donors, being subject to close scrutiny, being on display at all times, being ready for a political confrontation at all times, withstanding public personal criticism, and other experiences in politics are well outside the comfort zone of some mathematicians. In politics, I had to learn to accept that people will outright lie, and I had to be prepared to react in an appropriate way. Being elected to public office also has significant rewards, mainly being able to make a difference in people's lives, but it's also an ego boost. Strangers recognized me, often with applause, when I entered a room. Some people were awed to meet me and wanted to take their picture with me and hear my thoughts on any number of issues. As a member of Congress, I had a great staff to manage my schedule, help prevent me from making mistakes, protect me when I did make mistakes, and brief me on issues in front of me. I had a say on many issues of the day and got to participate in committee hearings with some of the biggest experts in the country. I also had the opportunity to meet interesting and prominent people and go on incredible trips overseas.

Let's take a look at the challenges I faced starting with public speaking. I had taught undergraduate math courses as a graduate student and had given many technical presentations during my years in industry. But public speaking in a political setting is a different experience. For me, being in front of a class of bright students for the first few classes in a semester was always anxiety producing. Professional presentations could also be a challenge. Strong preparation was essential. My consciousness changes when I get in front of people. I would freeze up and forget what I was going to say. When I started running for Congress in 2004, my speeches were terrible. Memorization was impossible. The best thing was to be able to read a speech from a podium. Next best was to have some note sheets with key words. Ad libbing was out of the question. I met with a speech coach, but it didn't seem to help much. I met with a psychologist, with no improvement. The thing that made a difference was a lesson with an acting coach. It helped me open up. I still stumbled, many times. After one disaster, my biggest donor emailed me that he'd had it with me. I could get slightly choked up while speaking about emotional issues. But I was determined because I was so angry about the 2003 war in Iraq. Whatever it took, I was going to do it. After a while, I realized that people still stuck with me even though I had very bad speaking episodes. I had the worst, the most embarrassing experiences and survived. It would get better, and it did. Sure, I still get nervous and still forget some of what I want to say, but somehow, I get through speeches and often enjoy it. Practice makes better.

Raising campaign money is critical to political success. It is said that money is the mother's milk of politics. It is true that politicians need campaign money to be successful. Some people are self-funders. I wasn't. As a sometimes-employed wind energy consultant, I wasn't rich. I had to depend on donors to produce a campaign. That included hiring political consultants, doing polls, producing and distributing literature, having a field campaign, and getting into mass media. It was frustrating that you could spend lots of money on different campaign activities and have little idea what was effective. As a new candidate, I had no name ID and no fundraising base. Fundraising was a struggle. Some professionals, such as lawyers, have a natural fundraising base. Mathematicians aren't very big political donors and, neither are wind energy engineers. I had fundraising events that, even with a lot of work, raised less than $100. Considering that you need millions of dollars to win a congressional seat, this was not going to work. I was advised to make a list of everyone I had ever known and call each one and ask for more money than you think they could ever donate. In addition, if I couldn't raise $100,000 within a couple of weeks, then I shouldn't waste my time or anyone else's time. Calling and connecting with friends and acquaintances was fun at first. One thing I learned is that you really can't guess who will give and who won't. I spoke to acquaintances who detested my political leanings and will probably never speak to me again. Fundraising got tiresome. I only raised about $40,000 from my personal list and that money disappeared very quickly. I had to purchase lists and start cold calling. I had to schedule "call time" for hours every day. I lost the general election in 2004 but persisted and ran again in 2006. After winning the primary election, I hired a fund-raising advisor who specialized in challengers. He had a list of about 50,000 donors nationwide and told me that I needed to start calling people on the East Coast at 6 a.m. Pacific time and call straight through until 9 p.m. Pacific time, and I needed to do this every day for the next three months. He said it would be like crawling on crushed glass—it was. But it worked. After a few months, I started getting a buzz as a candidate around the country. Donations started flowing in. Fundraising persisted as a requirement well after I got elected. My district was a Republican district, and I was, and am, a Democrat, so I would be targeted until the districts were redrawn after the 2010 census. I had to spend about 20 hours per week on call time. I got a better partisan district after the 2010 election, but then the party demanded that I pay dues to the party's campaign arm, hundreds of thousands of dollars every two years. It took the enthusiasm from me. It compounds the frustration for me to know that much of the campaign money is used to attack the opponent, and since almost every campaign for Congress does this, it reinforces the general belief that most politicians in Washington are crooks. Everyone hates call time, although some people slog through it better than others.

Communicating and listening are key. I couldn't be an effective politician if I couldn't communicate, and I couldn't communicate if I didn't listen. Listening, especially to personal issues, takes patience. But when I focused on listening and turned off outside distractions while listening, I began to get something out of it. People brought emotional issues to me, and it energized me to empathize with them.

People will lie to you. This wasn't something I was used to as a mathematician. I suppose some mathematicians lie, but there's really no benefit to it because you'll get caught. The first time someone lied in response to a question I asked in committee, I was so taken aback that I didn't know how to respond. I blanked out. I learned that I had to prepare for that kind of behavior by being ready to counter or cross-examine in lawyer speak.

Then there's the issue of being a public figure. My personal life went under a microscope. Anything I said or did was liable to end up in the media. I had to be ready to respond to a personal attack. When someone attacks me unexpectedly, I go into a stunned mode and freeze up. When there was any possibility of a personal attack, I prepared for that. In politics, your family may or may not be attacked in the media, which is even more difficult. I was lucky, no one went after my family.

Walking into Congress for the first time was euphoric. We had Marine guards when we went to the White House and back. It was deeply moving to me that these young men and women would take a bullet for me. I got to meet the president, the first lady, and their staff. I got to choose an office and hire a staff. Hiring my first staff was critically important because they were essential in helping set my congressional direction. Decisions made the first months impacted my entire career.

Committees are the work of Congress. There are hearings in which invited witnesses, sometimes under subpoena, testify. They are usually experts, and every member on the committee has a turn to question the witnesses five minutes in hearings. I had to prepare. If I fumbled, it would end up in the media. I thrived in that setting. I could ask anything, and I developed a reputation for asking technical questions about science or tech issues. Committees also held markups in which bills were discussed and amended. These could be contentious and could last a long time. One markup lasted 27 hours straight. The markup on the Affordable Care Act (Obamacare) lasted three weeks, although we stopped every night at about 9 or 10 p.m.

Activities on the House floor, votes, debates, and speeches were a big part of the experience. Members are allowed to give one-minute speeches at the beginning of each day on any topic. I gave a one-minute speech on the twin prime problem. It was one of the more memorable speeches I gave since it's easy to explain and understand in one minute. I got a lot of joking and positive feedback from that one. During House floor votes some members mill around the floor and some stay seated. It seems chaotic, but it's a good time to connect with colleagues. Voting could run for hours, not ending until the early morning. Some debates were interesting. Most of the time debates were about some arcane subject of interest to a narrow interest group. Sometimes debates were very emotional and contentious. I gave my share of boring speeches, but also gave a few hot speeches that got my blood boiling. Most speeches in the House are read, and that was a good approach for me except that I often make minor stumbles that only I seemed to notice.

Congressional trips, called COngressional DELegations, or CODELs were some of the most rewarding and enjoyable parts of the job. CODELs were usually international, sometimes on military aircraft. We almost always got to meet the head-of-state and other dignitaries in any country we visited, and we had the opportunity to visit historical sights. The staff took care of the planning and execution of the trip details. I took trips with scientific goals in the polar regions, twice to Greenland, once to northern Norway and Alaska, and once to Antarctica. The science in these locations was absolutely stunning, mostly funded by the National Science Foundation, and some by the Department of Energy and the National Oceanic and Atmospheric Administration. The South Pole scientists were measuring resonances in the early universe! I went on a domestic CODEL to the Boulder, CO, area in September of 2022 to visit the National Center for Atmospheric Research, the National Renewable Energy Laboratory, the National Oceanic and Atmospheric Administration, and the University of Colorado. It was a very impressive science community. My last CODEL was to the International Thermonuclear Experimental Reactor (ITER) fusion facility in France in October of 2022, where I urged the directors of the facility to publicly explain the reasons for their delays and cost overruns. They did, which helped clear the air.

Another word about the congressional staff. They are mostly bright, hardworking, idealistic, and committed individuals, mostly in their 20s. I depended on their work. Since I was interested in science issues, they helped me organize caucuses, one of which was the AI Caucus. The young woman who organized caucus meetings with AI experts from around the world was incredibly energetic. The purpose of the caucus was to inform my colleagues and their staffers about AI so they could make good decisions on the subject. In general, the staff kept me in the loop on issues I cared about. Staff who worked directly for a committee were more specialized and were just as helpful on issues related to committee jurisdiction.

People often ask me if I used mathematical thinking in Congress. The answer is somewhat, but not as much as I hoped. My colleagues clearly respected my math PhD background. For example, my colleague and cochair of the AI Caucus would always introduce me by announcing the title of my dissertation. But effectiveness in Congress depends on your jurisdiction and seniority, how much money you raise and use to advance the party goals, how much you get in the news, and how often and authoritatively you speak up in the party caucus and on the House floor. Even on the budget issues and projections, math didn't play much of a part. My colleagues sometimes looked to my opinions on science/tech issues. I raised my background in committee hearings and markups to illustrate a point and that sometimes made an impression. My background was helpful in understanding technical issues and it was usually very satisfying to be able to participate in discussions about science or technical issues for similar reasons that I might have enjoyed a college course in math or science—the opportunity to learn and help advance the state of the art or at least funding for important science issues.

In appropriations or raising federal dollars, members of Congress can help direct money for their pet causes. It can be an arduous and arcane process, but with persistence and good legislative strategy members can direct federal money to specific projects. In addition to the needs of people in my district in education, transportation, public safety, and so on, I was able to make sure some science and math organizations got federal money. I helped to make sure that the NSF got at least as much money each year as it did in prior years. I made sure the DoE also got as much money as I could for its research projects. In one case, my appropriations for research in solar radiation management was the first government money in any country to do so.

As a mathematician, Congress was frustrating. In the House, I was one of 435 members, each with his or her own priorities and egos. Since you need more than half of the members to get things passed in the House, getting anything done can be very difficult and time consuming. It can take a decade or more to do some things. I started working to get a veterans health center in my district and it took all 16 of my years in Congress to get it done. It takes years or even decades to become a chair of a committee, where most of the power lies in getting legislation passed through the House. After the House passes legislation, the Senate needs to pass the same exact bill before it can go to the Oval Office. The vitriol and shrill attacks in Washington were difficult for me to see and deal with and it was usually counterproductive. I avoided partisan fighting as much as possible, but there were times when it was necessary.

I met with mathematicians frequently in Washington. Sometimes I would be invited to math events, lectures, dinners, or parties. It was reassuring to see more prominent women in mathematics today than when I graduated. I always looked forward to being around other mathematicians. It lifted me up and helped generate enthusiasm for the job.

Being in Congress requires quite a bit of time away from home. Members of Congress either live in the DC area or commute back and forth to their districts every week when Congress is in session, about 40 weeks per year, usually three nights per week. That can be hard on the family, especially if you have kids still at home. My kids are all gone from our home, but it was still difficult on my wife.

I decided to run for Congress for the right reasons, my son's request, and my determination to help stop an unjust war of choice. I decided to leave for the right reasons as well. Too much money involved in campaigning, hardship on my wife, and a new district after the 2020 census that cut me entirely out of what had been my home in the San Francisco Bay Area. I accomplished a lot for my district, for veterans, and for the issues I focused on. Would I do it again, and was it worth the sacrifice of my personal life? The answer is yes, I feel that I made a difference. But

I'm now greatly relieved to get back into activities that I love, such as fusion and AI. I do miss many of my constituents and colleagues and the attention and prestige of being in Congress. But it was time for me to leave Congress, and I very much look forward to re-establishing myself in the mathematics community doing research and trying to make a difference in our world in a way that better suits my personality.

For me, mathematics required a commitment to a rather narrow lane. This can be limiting and reduces the usefulness of mathematics to the more general society at a time when society needs to pull all of its resources together to address its most difficult challenges, especially since this requires some of the most highly talented and driven members of society. I hope this article will encourage mathematicians to be more open to professional activities that may be outside of their academic training, and math departments to be open to inviting mathematicians that have nonacademic experience into their faculty to collaborate and share their knowledge.

My last day as a US congressman was January 3, 2023.

Jerry McNerney

Credits

All photos are courtesy of the author.

The Mathematics of Digital Signatures

Angela Robinson

Handwritten signatures have been used to verify the authenticity of documents for centuries. In the late 1970s, mathematicians discovered pivotal techniques to construct digital signatures for authenticating any message or data that can be represented as bit strings [DH76, RSA78]. Digital signatures quickly evolved from theoretical tools to essential components of our global digital world. Software patches and updates are digitally signed by the provider so that before a device installs the update, the device can verify the origin of the software and that the software package was not modified after the signature was applied. Transactions on public blockchains like Bitcoin and Ethereum are digitally signed by the coin-sender to authenticate the details of the transaction (coin amount to be sent, recipient, etc.) and to prevent unauthorized changes to the transaction details.

Components of a digital signature. Digital signature algorithms (DSAs) are used to establish authenticity and integrity. The former assures that the signed message originated from the claimed sender and the latter assures that the message has not been changed during transit. A typical example is illustrated in Figure 1.

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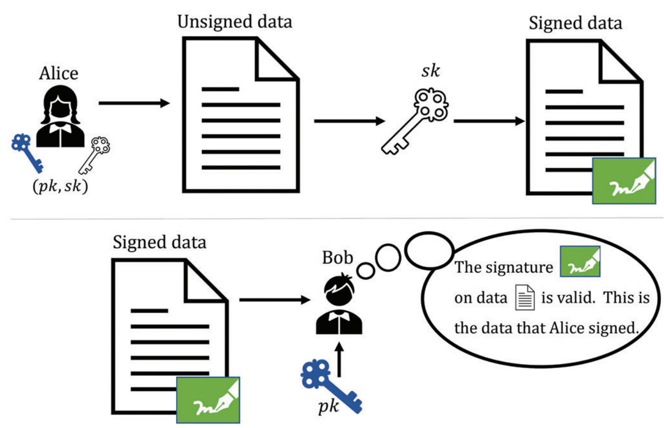

Figure 1. Alice generates a pair of keys and sends a signed message to Bob. Bob uses Alice's public key to verify the signature.

There are three components of a digital signature scheme: `key generation`, `sign`, `verify`.

During the first phase of the protocol, `key generation` generates a pair of keys: one public key *pk* for signature verification and one secret key *sk* for signing messages. To prevent forgeries, it is critical that the signer does not share *sk* with anyone.

The `sign` algorithm can use *sk* to sign any message or data. Unlike handwritten signatures, a digital signature depends on the message and will thus vary. Upon receipt of a signed message, the message recipient uses public knowledge of *pk* to run the `verify` algorithm to determine whether the signature is valid with respect to the message.

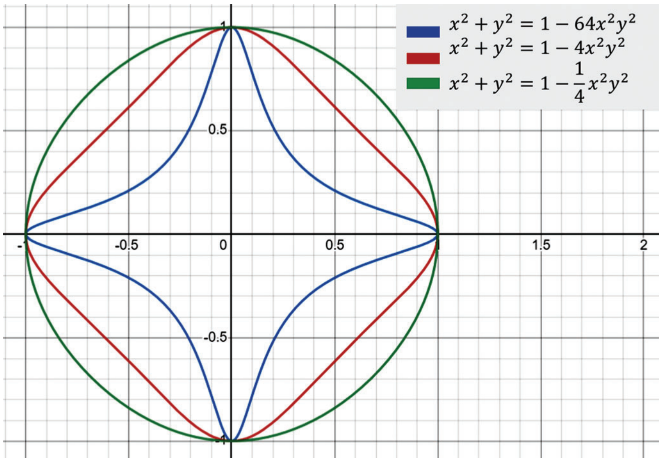

Figure 2. Edwards curve examples.

Constructing digital signatures. Many digital signature schemes are constructed using *trapdoor* functions. That is, functions that are easy to compute but difficult to invert without some knowledge of the trapdoor. One such digital signature scheme is the Edwards-curve Digital Signature Algorithm (EdDSA).

EdDSA belongs to a family of digital signature algorithms that use elliptic curves over finite fields. It is well known that for an elliptic curve $\mathcal{E}$ defined over a finite field $\mathbb{F}_q$, $\mathcal{E}(\mathbb{F}_q)$ forms an abelian group under addition with respect to a particular addition law. At a high level, an Edwards curve $\mathcal{E}$ is defined by $x^2+y^2=1+dx^2y^2$ over $\mathbb{F}_q$, $d \neq 0,1$ [Edw07]. Some examples of Edwards curves over $\mathbb{R}$ with various d values are given in Figure 2. The Edwards addition law is, for two points $(x_1,y_1),(x_2,y_2) \in \mathcal{E}(\mathbb{F}_q)$,

$$(x_1,y_1)+(x_2,y_2)=\left(\frac{x_1y_2+x_2y_1}{1+dx_1x_2y_1y_2}, \frac{y_1y_2-x_1x_2}{1-dx_1x_2y_1y_2}\right).$$

The security of EdDSA is based on the difficulty of solving the Elliptic Curve Discrete Logarithm Problem (ECDLP). Let $P \in \mathcal{E}(\mathbb{F}_q)$ be a point of prime order p, and let $\langle P\rangle$ be the subgroup generated by P. For any $Q \in \langle P\rangle$, $Q=a\cdot P$ for some $a \in [0,p-1]$. The ECDLP is, given P,Q, and $\mathcal{E}$, find a. Computing $a\cdot P$ is computationally easy, but solving the ECDLP requires significantly more computation time. In fact, for appropriately chosen parameters, solving the ECDLP is computationally infeasible.

EdDSA. The following is a highly simplified version of how the components of EdDSA are used to sign a message m using an Edwards curve $\mathcal{E}(\mathbb{F}_q)$. Let H be a function that maps messages of arbitrary length to bit strings of a fixed size in a manner that is difficult to invert and difficult to find a pair of distinct input strings that map to the same output.

`key generation`: Selects a point $P \in \mathcal{E}(\mathbb{F}_q)$ of prime order p, and a random integer $a \in [1,p-1]$. The point P is a public parameter, *pk* is the point $Q=a\cdot P=(x,y)$, and *sk* is the scalar a.

`sign`: Given message m and a, computes integer $r=H(H(a)+m) \mod q$ and point $R=r\cdot P$. For simplicity, assume that m and points R,Q are encoded in such a way that enables addition modulo q. Let $h=H(R+Q+m) \mod q$. The signature on m is the pair $(R,s)=(R,r+h\cdot a \mod q)$.

`verify`: Computes $h=H(R+Q+m) \mod q$ and two points: $P_1=s\cdot P$, and $P_2=R+h\cdot Q$. If $P_1=P_2$, the signature (R,s) is accepted. Otherwise, rejected. The curious reader is encouraged to check that `verify` accepts well-formed signatures.

Integrity is broken if an attacker manages to forge a signature on a new message m'. It is not known how to forge EdDSA (at cryptographic sizes) signatures without knowledge of *sk*, and recovering *sk* from *pk* requires one to solve the ECDLP. The security of EdDSA depends on the difficulty of solving the ECDLP.

The predecessor of EdDSA, the elliptic curve digital signature algorithm (ECDSA), emerged in the 1990s and performed remarkably better than the DSAs of the 1970s. EdDSA varies a bit from ECDSA but can provide even more efficiency with additional security protections. EdDSA has recently been included in US cryptographic standards [Nat23].

References

[DH76] Whitfield Diffie and Martin E. Hellman, *New directions in cryptography*, IEEE Trans. Inform. Theory **IT-22** (1976), no. 6, 644–654, DOI 10.1109/tit.1976.1055638. MR437208

[Edw07] Harold M. Edwards, *A normal form for elliptic curves*, Bull. Amer. Math. Soc. (N.S.) **44** (2007), no. 3, 393–422, DOI 10.1090/S0273-0979-07-01153-6. MR2318157

[Nat23] National Institute of Standards and Technology, *Digital signature standard (DSS)*, Technical Report Federal Information Processing Standards (FIPS) Publication 186-5, U.S. Department of Commerce, Washington, D.C., 2023.

[RSA78] R. L. Rivest, A. Shamir, and L. Adleman, *A method for obtaining digital signatures and public-key cryptosystems*, Comm. ACM **21** (1978), no. 2, 120–126, DOI 10.1145/359340.359342. MR700103

Angela Robinson

Credits

Figure 1 and Figure 2 are courtesy of Angela Robinson.
Photo of Angela Robinson is courtesy of Keyi Liu.

AMS Prizes & Awards

NEW! Ivo and Renata Babuška Thesis Prize

The Ivo and Renata Babuška Thesis Prize is awarded annually ($3,000) to the author of an outstanding PhD thesis in mathematics, interdisciplinary in nature, possibly with applications to other fields.

About this Prize

Ivo Babuška is a Czech-American mathematician whose honors include five doctorates honoris causa, the Czechoslovak State prize for Mathematics, the Leroy P. Steele Prize, the Birkhoff Prize, the Humboldt Award of Federal Republic of Germany, the John von Neumann Medal, the Neuron Prize Czech Republic, the ICAM Congress Medal (Newton Gauss), the Bolzano Medal, and the Honorary Medal De Scientia Et Humanitate Optime Meritis. Asteroid 36060 Babuška was named in his honor by the International Astronomical Union.

Renata Babuška (nee Mikulášek) was Ivo's wife and partner for 63 years. Renata grew up in Prague, Czechoslovakia and graduated from Charles University in 1953 with a degree in Mathematical Statistical Engineering. Upon graduation, she was assigned to the Education Department as an administrator evaluating universities and technical schools. Two years later she became an Assistant Professor of Mathematics at the Czech Technical University. After moving to the US, Renata worked as a data and computing management consultant for different government agencies in Washington, DC. She liked to point out that behind every successful man is a strong woman and he often said that without Renata, he would not have accomplished all that he did.

Babuška was a Distinguished Professor at the University of Maryland at College Park and then the Robert B. Trull Chair in Engineering, TICAM Senior Research Scientist, Professor of Aerospace Engineering and Engineering Mechanics, and Professor of Mathematics at the University of Texas, Austin. He is a Fellow of SIAM, ACM, and ICAM, a member of the US National Academy of Engineering, the Academy of Medicine, Engineering, and Sciences of Texas, the European Academy of Sciences, and an honorary Foreign Member of the Czech Learned Society.

Babuška's work spans the fields of theoretical and applied mathematics with emphasis on numerical methods, finite element methods, and computational mechanics. His interest in fostering collaboration among mathematicians, engineers, and physicists led him to establish this prize to encourage and recognize interdisciplinary work with practical applications.

The Ivo and Renata Babuška Thesis Prize is awarded in line with other AMS Prizes and Awards, according to governance rules and practice in effect at that time.

Next Prize: Inaugural Prize January 2024

Nomination Period: 1 February – 30 June 2023

Nomination Procedure:

1. The prize will recognize a thesis for a PhD granted between July 1 of year -1 and June 30 of year 0 and will be presented at the Joint Mathematics Meetings in January of year +1.
2. The nominating institution will be a PhD-granting institution that is either a. located in the United States of America (USA), or b. located outside the USA and an institutional AMS member at the time of the nomination.
3. One PhD thesis may be nominated by a nominating institution.
4. The nominating institution will submit a copy of the thesis along with a letter in support of the nomination, and both will be written in English.
5. A selection committee will be appointed by the AMS President.

To make a nomination go to `https://www.ams.org/babuska-prize`.

NEW! The Elias M. Stein Prize for New Perspectives in Analysis

The Elias M. Stein Prize for New Perspectives in Analysis is awarded for the development of groundbreaking methods in analysis which demonstrate promise to revitalize established areas or create new opportunities for mathematical discovery. The current prize amount is US$5,000 and the prize is awarded every three years for work published in the preceding six years.

About this Prize

This prize was endowed in 2022 by students, colleagues, and friends of Elias M. Stein to honor his remarkable legacy in the area of mathematical analysis. Stein is remembered for identifying many deep principles and methods which transcend their original context, and for opening entirely new areas of research which captivated the attention and imagination of generations of analysts. This prize seeks to recognize mathematicians at any career stage who, like Stein, have found exciting new avenues for mathematical exploration in subjects old or new or made deep insights which demonstrate promise to reshape thinking across areas.

Next Prize: Inaugural Prize January 2024

Nomination Period: 1 February – 30 June 2023

Nomination Procedure: Nominations can be submitted between February 1 and June 30. Nominations should include a letter of nomination and a brief citation to be used in the event that the nomination is successful.

To make a nomination go to `https://www.ams.org/stein-prize`.

Joint Prizes & Awards

JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

About this Award

This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.

Next Prize: January 2024

Nomination Period: open

Nomination Procedure: Nominations should be submitted on `mathprograms.org`. Note: Nominations collected before September 15th in year N will be considered for an award in year $N+2$.

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Bulletin of the AMS	**Susan J. Friedlander** (1 Jul 2005 – 31 Jan 2024)
Colloquium Publications	**Mark Kisin** (1 Feb 2021 – 31 Jan 2025)
Communications of the AMS	**Qiang Du** (20 Dec 2022 – 31 Jan 2025)
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Proceedings of the AMS	**David Futer** (1 Feb 2022 – 31 Jan 2026)
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2023 Executive Committee

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Society Governance

The American Mathematical Society has a bicameral governance structure consisting of the Council (created when the Society's constitution was ratified in December 1889) and the Board of Trustees (created when the Society was incorporated in May 1923). These bodies have the ultimate responsibility and authority for representing the AMS membership and the broader mathematical community, determining how the AMS can best serve their collective needs, and formulating and approving policies to address these needs. The governing bodies determine what the Society does and the general framework for how it utilizes its volunteer, staff, and financial resources.

The Governance Leadership consists of the Officers (President, President Elect or Immediate Past President, three Vice Presidents, Secretary, four Associate Secretaries, Treasurer, and Associate Treasurer), the Council, Executive Committee of the Council, and Board of Trustees.

The Council[1] formulates and administers the scientific policies of the Society and acts in an advisory capacity to the Board of Trustees. Council Meetings are held twice a year (January and the spring).

The Board of Trustees receives and administers the funds of the Society, has full legal control of its investments and properties, and conducts all business affairs of the Society. The Trustees meet jointly with the Executive Committee of the Council twice a year (May and November) at ECBT Meetings.

The Council and Board of Trustees are advised by over 100 Committees, including six Policy Committees (Education; Equity, Diversity, and Inclusion; Meetings and Conferences; Profession; Publications; and Science Policy) and over 20 Editorial Committees for the various journals and books it publishes.

The Council and Board of Trustees are also advised by the Executive Director and the Executive Staff, who are responsible for seeing that governance decisions are implemented by over 200 staff members.

Learn more at `www.ams.org/about-us/governance`.

[1] *Bylaws, Article IV, Section 1. The Council shall consist of fifteen members at large and the following* ex officio *members: the officers of the Society specified in Article I, the chair of each of the editorial committees specified in Article III (committees for the* Bulletin, *for the* Proceedings, *for the* Colloquium Publications, *for the* Journal, *for* Mathematical Surveys and Monographs, *for* Mathematical Reviews; *a joint committee for the* Transactions *and the* Memoirs; *and a committee for* Mathematics of Computation), *any former secretary for a period of two years following the terms of office, and members of the Executive Committee (Article V) who remain on the Council by the operation of Article VII, Section 4.*

2022 Contributors

Credit: Mike Nyman Photography

Catherine A. Roberts
AMS Executive Director

Dear AMS Members and Friends,

Throughout 2022 we felt the pent-up demand and enthusiasm for connection, for collaboration, and for many of the traditions that mark the mathematical calendar. It was inspiring and affirming to see mathematicians from across the country and at all stages of their careers working together to strengthen the fabric of our community—from AMS sectional meetings, to our Mathematics Research Communities, to preparing for the 2023 Joint Mathematics Meetings. Donors to the AMS expressed their ongoing confidence in the people, the programs, and the mission of our Society this year with their support of new prizes and awards, ongoing initiatives, and new programs. This report highlights some of these wonderful gifts. Please join me in thanking our donors for their generosity and vision.

AMS donor gifts created an extraordinary number of exciting new prizes, fellowships, sponsorships, and awards last year that celebrate and support mathematicians: the Ivo and Renata Babuška Thesis Prize, awarded to the author of an outstanding PhD thesis in mathematics; the Stefan Bergman Fellowship, awarded to a research mathematician conducting research in real analysis, complex analysis, or partial differential equations; the Bose, Datta, Mukhopadhyay and Sarkar Fund, endowing a JMM Current Events Bulletin lecture; the I. Martin Isaacs Prize for Excellence in Mathematical Writing, awarded for excellence in writing of a research article; and three rotating Elias M. Stein Prizes to be awarded for New Perspectives in Analysis, for Transformative Exposition, and for Commitment to Mentoring.

In addition, donors continue to strengthen the programs the AMS provides at the Joint Mathematics Meetings to support connection and collaboration. The Next Generation Fund provided travel grants for graduate students, donor support underwrote travel grants for faculty at Primarily Undergraduate Institutions (PUIs), and an anonymous donor created undergraduate opportunities at the JMM for presentations, poster sessions, networking, and socializing.

Finally, the AMS was extraordinarily fortunate last year to receive news of additional donor bequests. These gifts demonstrate a shared commitment to the enduring values of our Society, since they will support the AMS long after the donor's lifetime. The AMS celebrates and honors these donors by including them in the Thomas A. Fiske Society. This Contributor Report lists all Fiske Society members, highlighting both those who are new this past year, as well as those whose bequests to the AMS have been received. We continue to be grateful for their legacy of support for all areas of the AMS.

The mathematical community benefits from contributions from mathematicians of all ages, in all professions, at all institutions, and from all backgrounds. On behalf of the Society, thank you for your generous and thoughtful gifts to our mathematical community: your support encourages and inspires us all.

Catherine A. Roberts
Executive Director

AMS Donors Create New Prizes


Ivo and Renata Babuška

Photo courtesy of the Babuška family

We are grateful to Czech-American mathematician Ivo Babuška for creating the **Ivo and Renata Babuška Thesis Prize**, which will be awarded annually to the author of an outstanding PhD thesis in mathematics, interdisciplinary in nature, possibly with applications to other fields. The prize honors Renata Babuška (nee Mikulášek), Professor Babuška's wife and partner for 63 years, who passed away in 2020. Babuška's interest in fostering collaboration among mathematicians, engineers, and physicists led him to establish this prize to encourage and recognize interdisciplinary work with practical applications.

The inaugural Ivo and Renata Babuška Thesis Prize will be awarded in 2024.


I. M. (Marty) Isaacs

Photo was taken by Yvonne M. J. Nagel

The **I. Martin Isaacs Prize for Excellence in Mathematical Writing** was established in 2022 by a generous gift from Professor I. M. (Marty) Isaacs of the University of Wisconsin. The prize is to be awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years. Inspired by his experience advising twenty-nine PhD students and helping them improve their mathematical writing, Isaacs's motivation for creating the prize was to encourage and recognize the highest standard of mathematical exposition in AMS journals.

At the time of writing, the date of the inaugural I. Martin Isaacs Prize for Excellence in Mathematical Writing is yet to be announced.

In Tribute

The following friends, colleagues, and family members are being specially honored by commemorative gifts. The AMS is pleased to be the steward of donors' generosity in their name.

Gifts made in honor of the following individuals:

Pushpa R. Agrawal *by Jagdish C. Agrawal*
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Nicole Simkins *by Stephen J. Tierney*
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Gifts made in memory of the following individuals:

Steve Armentrout *by Arnold Selnick*
Georgia Benkart *by Francis Bonahon, Erica Flapan, Karen Saxe, and Anonymous*
Chun-Ting Chen *by Jungkai Alfred Chen*
Frank Nelson Cole, Charles A. Cole, Charles A. Cole, Jr., and Richard K. Cole *by Carolyn Cole*
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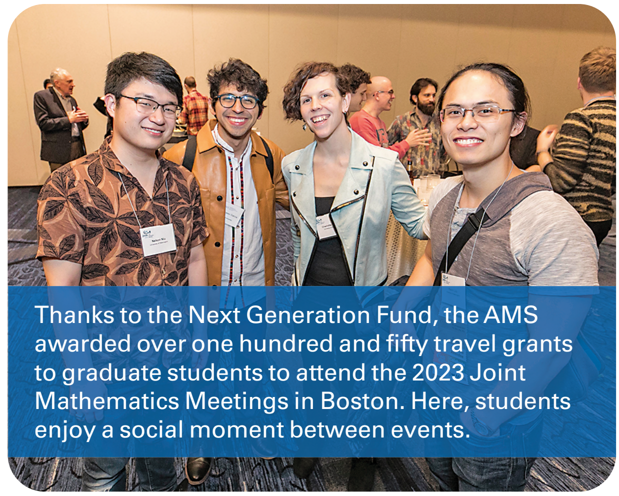
Highlight on the Next Generation Fund

With the generous help of an anonymous benefactor and many other AMS donors, the Campaign for the Next Generation raised $3 million to create an endowment to help graduate students and early career mathematicians as they establish their professional lives. The Next Generation Fund provides funding for the AMS Graduate Student Travel Grant program and as it grows, will provide secure, dedicated support to influential career-building programs including Mathematics Research Communities, AMS Employment Center, JMM Child Care Grants, and AMS Graduate Student Chapters. A top AMS priority, the Fund is designed to adapt to future needs and will support many individuals each year at modest but impactful levels.

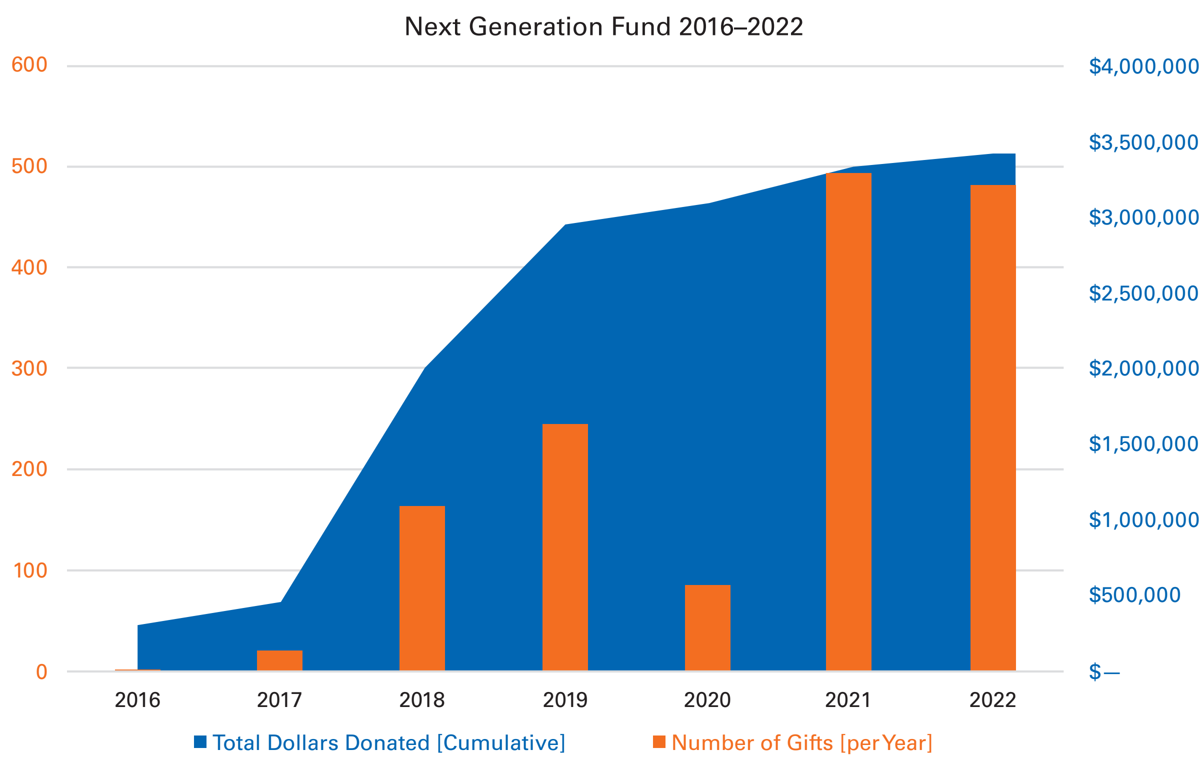


Thanks to the Next Generation Fund, the AMS awarded over one hundred and fifty travel grants to graduate students to attend the 2023 Joint Mathematics Meetings in Boston. Here, students enjoy a social moment between events.

The *Next Generation Fund* 2016–2022 by the numbers:

Total Gifts: **$3,424,603**

Donors:	Number of gifts:	Median gift:	AMS Graduate Student Travel Grants awarded for 2023 JMM:	Named Endowed Funds in NextGen:
948	1,495	$40	155	13



Create Your Legacy: The Thomas S. Fiske Society

Members of the Thomas S. Fiske Society create a personal legacy supporting mathematics by naming the AMS in their will, retirement plan, or other gift vehicle. The AMS celebrates the following people for their thoughtful vision. **Fiske Society members inducted in 2022 are indicated in bold.** *Asterisk indicates deceased.*

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The American Mathematical Society offers donors the opportunity to establish named endowed funds at the AMS. These endowed funds support research and the mathematical community for future generations while commemorating a family member, colleague, or mentor. (The number in brackets indicates the year the fund was created.)

Satyendra Nath Bose, Mahadev Datta, Salilesh Mukhopadhyay and Pranab K. Sarkar Fund (2021)
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The people and organizations listed below made gifts to the AMS between January 1–December 31, 2022. The AMS thanks every donor on behalf of the beneficiaries for their generosity. Every gift helps advance mathematics. *Asterisk indicates deceased.*

Photo courtesy of Joseph Silverman

"The AMS is a wonderful organization that provides a community for mathematicians at all stages of their careers. The programs, meetings, and resources that the AMS offers create opportunities for all of us. I am proud to support the AMS, because I know that my donation makes a difference in supporting the whole of our mathematical profession."

—Joseph H. Silverman
Chair, AMS Board of Trustees

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Supporting Undergraduates

The AMS Undergraduate Opportunity Awards are one-time grants awarded to undergraduate mathematics majors, helping to ensure that financial hardship does not stand in the way of completing their degree program. Over two hundred promising math students have received Waldemar J. Trjitzinsky Memorial Awards (established by the Trjitzinsky family in 1991) and Edmund Landau Awards (established by an anonymous donor in 2020). In 2022, nine Trjitzinsky and one Landau Awards of $3,000 were made.

Photo courtesy of Kaia De Vries

2022 Landau Award winner Kaia De Vries, a sophomore at the University of Maine, Orono, is pursuing a mathematics major and a computer science minor. De Vries first became interested in math as a child and hopes to become involved in mathematics research as an undergraduate. De Vries plans to continue her math studies in graduate school.

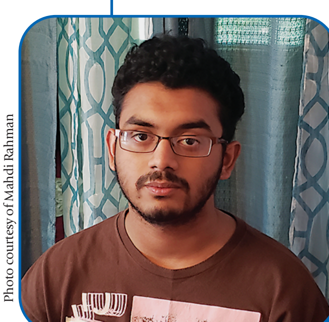

Photo courtesy of Mahdi Rahman

2022 Trjitzinsky Award winner Mahdi Rahman is a junior at the University of Buffalo-SUNY, majoring in mathematics. He had considered becoming an engineer or a doctor, but this changed when he watched math videos to prepare for the SAT and realized he actually enjoyed mathematics. Mahdi is particularly interested in algebra and analysis and aspires to dedicate himself to these fields.

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Photo courtesy of Augenstein family.

A Gift to the Mathematics of Tomorrow

An AMS member for 25 years, Walt made a generous unrestricted bequest to the AMS. Walt told us, "I am profoundly glad to be able to contribute to the progress of mathematics and the AMS throughout the world. A gift to the AMS and to the future of mathematics is an expression of my values." We are deeply grateful for Walt's dedication to mathematics.

Pictured: Walter O. Augenstein, Jr. (1951–2020)

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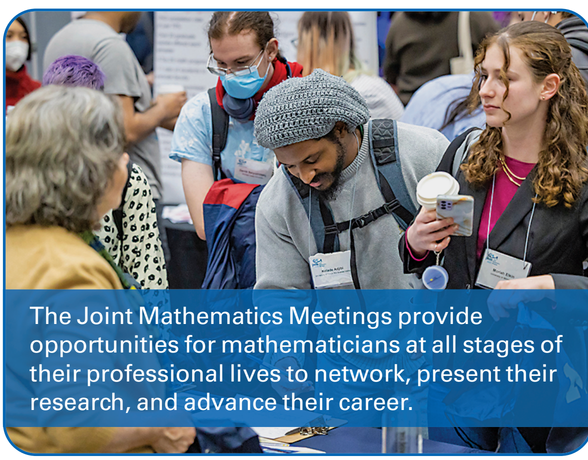

The Joint Mathematics Meetings provide opportunities for mathematicians at all stages of their professional lives to network, present their research, and advance their career.

"Attending the Joint Mathematics Meetings was vital for me to take the next step in my academic career, both by presenting my own work, as well as meeting other graduating students and more senior mathematicians.

Having this grant made it much easier to focus on the mathematics and networking of the conference instead of how I was going to afford to attend, and for that I am incredibly grateful."

—*AMS Graduate Student Travel Grant Recipient, JMM 2023*

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John A. Zweibel

Thank you!

This report reflects contributions from January 1, 2022, through December 31, 2022. Accuracy is important to us and we apologize for any errors. Please bring discrepancies to our attention by calling AMS Development at 401.455.4111 or emailing development@ams.org. Thank you.

THE

NEXT GENERATION

Early-career AMS members take a moment

Favorite memory from an AMS event: The excitement of seeing the big mathematicians whose papers I've read.

Were you inspired by a mathematician?: I was inspired to study math just in order to make it simpler for others. This is because I was the only one that passed math in my graduating high school class and I thought this could be simpler because I believed my classmates were smart too.

Favorite color: Blue

Field of research: Differential Equations

Maria Amarakristi Onyido

Assistant Professor, Northern Illinois University

What does the AMS mean to you?: The AMS is a good resource for staying up-to-date on the latest research and networking.

Describe a situation when you first fell for math: When I won the math competition at my school during the 7th grade.

What is the best piece of advice you received that helped you get through graduate school?: Think about the end product and stay positive.

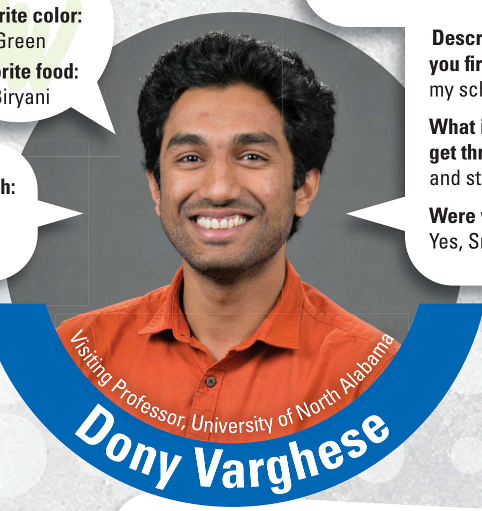
Were you inspired by a mathematician?: Yes, Srinivasa Ramanujan

Hobby: Playing Soccer

Favorite color: Green

Favorite food: Biryani

Field of research: Matroid Theory



Dony Varghese

Visiting Professor, University of North Alabama

Describe a situation when you first fell for math: I was at middle school when I first took a geometry class. I was totally fascinated with the elegance and the beauty of mathematics. Geometry helps discover patterns and understand the world around us.

Favorite memory from an AMS event: Meeting my friends and seeing old colleagues.

Field of research: Statistics and Applied Mathematics

Mariam Khachatryan

Assistant Professor, Augustana University

AMS
American Mathematical Society
Advancing research. Creating connections.

OF MATHEMATICS

to share a little about themselves:

#AMSMember

Hobby: Harpist (I've played for a former prime minister of England)

Field of research: Probability and Statistics

What does the AMS mean to you?: Math has always been a way to connect with a community of people who love what I love. The AMS has enabled that connection at both my institution and more broadly at conferences and online.

Hobby: Soccer

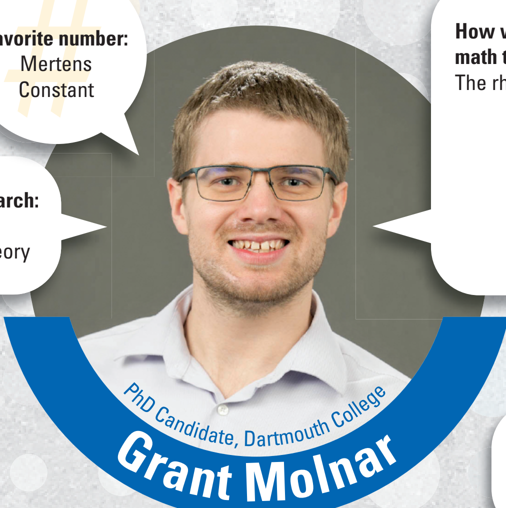
What does the AMS mean to you?: Opportunity, Growth, Success

What do you think is the most important service the AMS offers?: Conferences, Opportunities

Field of research: Applied Mathematics

Favorite number: Mertens Constant

Field of research: Analytic Number Theory



How would you describe math to a non-math person?:
The rhythm of a metered rhyme:

The fractal shape of falling snow
A contour changing over time,
both swift and slow
The logos of the wise and great
The mind unlocked, alert, and free
The dance of chaos, facing fate.

www.ams.org/membership

Catherine Roberts Resigns

Scott Turner

After seven years as executive director of the American Mathematical Society (AMS), Catherine Roberts will step down at the end of August to become chief executive officer of the Consortium for Mathematics and its Applications (COMAP). Roberts called her AMS service "the honor of a lifetime. The AMS is, and will remain, my professional home, as it has been for me since graduate school." By joining COMAP, Roberts said that she is returning "to my roots, promoting math modeling and education."

"We thank Catherine Roberts for her leadership and for her many accomplishments as executive director," said Bryna Kra, AMS president, and Joseph Silverman, AMS Board of Trustees chairperson. "She has been integral to the functioning of the AMS.

"Her achievements include the highly successful campaign for the Next Generation Fund, steering the AMS through both political turmoil and the COVID-19 pandemic, growing existing AMS programs, and expanding into new areas. Catherine's presence will be missed."

Launched in 2020, the Next Generation Fund supports doctoral students and recent PhD recipients as they navigate their early careers. During Roberts' tenure, the AMS also added new prizes and fellowships and increased financial and professional support to help mathematicians advance their research.

Roberts positioned the AMS for a thriving future in other ways. She implemented a strategic plan that resulted in rebranding the AMS identity and establishing offices of Membership, Communications, and Equity, Diversity, and Inclusion. Roberts reimagined the Joint Mathematics Meetings as a robust partnership that now includes more than a dozen professional societies and more research areas and aspects of the mathematical sciences than ever before. She also helped amplify the voices and interests of mathematicians in Washington, DC, expanding AMS advocacy to include education policy and opening new offices near Capitol Hill.

The AMS grew under Roberts' leadership. The AMS book program expanded with the acquisition of MAA Press; the AMS launched a new gold open access journal, *Communications of the AMS*; the workflow for MathSciNet was converted to paperless; and the AMS printing and distribution facility transformed to a digital press operation.

"Catherine Roberts repeatedly impressed me by her ability to navigate challenges," said Ruth Charney, immediate past president of the AMS. "She has been a huge asset to the AMS, and will be hard to replace."

Over the next few months, the Board of Trustees will conduct a search for a new executive director.

Credits

Photo is courtesy of Billy Durvin.

Scott Turner is the director of communications at the American Mathematical Society. His email address is `sxt@ams.org`.

DOI: https://doi.org/10.1090/noti2701

At JMM 2023, PEPs Provide Professional Development

Elaine Beebe

At JMM 2023, Professional Enhancement Programs (PEPs) provided professional development on a number of mathematical topic areas, from pedagogy to programming to artistic perspective.

The eight PEPs drew 136 participants, who registered in advance and paid a small fee to enroll.

We sat in on three PEPs to see learning in action.

* * *

It was obvious that "Visualizing Projective Geometry Through Photographs and Perspective Drawings" would be hands-on when Annalisa Crannell (Franklin and Marshall College) handed out class materials: rulers, packets of paper, pencils, and erasers.

Crannell and Fumiko Futamura (Southwestern University) had tandem-taught this course several times before, to groups of as many as several dozen students.

No artistic experience is required to attend. The class uses practical art puzzles in perspective drawing and photography to inspire the mathematics of projective geometry.

"In particular," the syllabus notes, "we use a geometrical analysis of Renaissance art and of photographs taken by students to motivate several important concepts in projective geometry, including Desargues's Theorem and the use of numerical projective invariants."

Or as Crannell said in class, "Think of a shadow as being a function," as participants diligently sketched vanishing points and 3D letters.

Discovery was crucial to the learning experience, Crannell and Futamura said.

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DOI: https://doi.org/10.1090/noti2699

"When we presented one of our favorite problems about drawing letters, one participant suggested a computation approach to solution that at first seemed improbable," Crannell said. "A second participant suggested a geometric solution that helped with a proof that the first solution was in fact correct, and the combination of those two solutions led to fascinating generalizations that we, the organizers, hadn't thought of before. So much fun! And very rewarding."

Futamura added, "I agree, what Annalisa described was an exciting moment for all of us! This is an accessible and intriguing problem with a number of creative solutions, so our participants often experience the joy of discovery.

"After years of teaching this and thinking we've seen every possible solution, it was especially wonderful that Annalisa and I got to experience that same joy of discovery this time around!"

The PEP was also a chance for a range of mathematical scientists to come together and learn in person after several years of cancelled or virtual gatherings.

"It was great to solve problems in community again," Crannell said.

* * *

Skill-building aptly described the course, "Inclusive Active Learning in Undergraduate Academics," presented by Nancy Kress (University of Colorado at Boulder), Rebecca Machen (University of Colorado at Boulder), Antonio Martinez (California State University Long Beach), Wendy Smith (University of Nebraska-Lincoln), and Matt Voigt (Clemson University).

Even the dynamic classroom methods that the instructors used with their faculty-level students at JMM would be replicable in an undergraduate setting.

"We're going to organize in visibly random groups, which is an effective strategy to use," Smith said to the

23-person class, seated at round tables throughout a hotel conference room.

"And you're going to move again, so the groups will change," she said. "Why are we doing this? To foster a sense of student belonging. This is critical for them [undergraduates] to have, to be able to stick it through. Student belonging is critical to STEM."

Another classroom tip from the instructors: On the first day, have students introduce themselves in order to hear the actual pronunciation of their names, instead of a teacher guessing and fumbling through an attendance list.

During the PEP, discussing the unconscious bias of the phrase "you guys" raised a few hackles. "But girls say that. All the time," said a class member.

Replied Smith, "Students don't receive it as meaning all of them, even if it's what's intended."

An animated discussion began with the premise, "Have you ever experienced or witnessed a microaggression?"

"Everyone has experienced this once in a while," one male class member said, slightly dismissive.

"During my career, absolutely!" countered his tablemate, also male. "The way I look, the way I sound, the Indian accent."

"Who decides?" says Classmate 1. "If you tell me my English is really good, I take it as a compliment."

"Even if it wouldn't bother you personally, it might bother students," a third classmate said.

The tablemate agreed. "A microaggression is based on their experiences in their past," he said. "How it lands is going to be at their individual level."

"I grew up in an environment of microaggressions," another classmate shared. "All. The. Time."

"Death by one thousand paper cuts," Smith said, adding that if an incident happens in front of the whole class, the instructor should address it in front of the whole class, then follow up independently with the affected student.

"Inclusive Active Learning" included the extensive use of skill-building videos and role-playing of scenarios from classroom-level to department-level. "We definitely want you to try this at home!" Kress said. "And share with your colleagues."

* * *

From gerrymandering to the Super Bowl, a wide range of current events are ripe for mathematicians' unique expertise and perspective.

This principle fueled the PEP, "Using Your Voice for Influence and Impact: Incorporating Mathematics into Public Discourse," in which math faculty members and graduate students learned how to draft an opinion piece for a newspaper or other non-academic publication.

Versions of this course have been taught since prepandemic times by Francis Su (Harvey Mudd College) and Kira Hamman (Penn State Mont Alto), each of whom has a distinguished record of publishing opinion pieces. Hamman writes a regular column for her local paper in Pennsylvania, while Su has published timely op-eds in newspapers such as the *Los Angeles Times* and the *Sacramento Bee*.

Communicating with a mainstream audience in a media outlet requires quite a different skill set than writing an academic paper. The writing process is almost entirely flipped. A newspaper column front-loads the important points, Su explained, rather than spending 25 pages to arrive at a conclusion. The news world moves much more quickly than academia, so deadlines are short. "And the news hook cannot be underestimated," said panelist Rafe Jones (Carleton College).

Panelist Audrey Malagon (Virginia Wesleyan University), described her experience of taking Su and Hamman's writing course in 2018. "I decided, I could share what I know, or I could keep it to myself." The opinion piece she crafted during that session was published in the *Virginian-Pilot* daily newspaper and led to an avalanche of opportunities for her, from writing to public-policy work.

By the end of the course, each of the 11 participants had drafted an opinion piece peer-edited by classmates and reviewed by the instructors. With more editing, some motivation, and a little luck, these columns might appear in print.

* * *

PEPs will return for JMM 2024, said Catherine Roberts, AMS executive director.

"As we reimagined JMM, the math community expressed a desire for the meetings to include more professional development opportunities," she said. "PEPs offer varied topics, suggested and presented by members of the community.

"We look forward to expanding those topics to help attendees develop professionally in wide-ranging ways."

Math, Friendship, and Travel Grants

Elaine Beebe

At 8:30 a.m. Aida Maraj, postdoctoral assistant professor at the University of Michigan, welcomed JMM 2023 attendees to "Enumerating in Algebra Using Formal Languages and Automata."

"Especially at this early time! I hope you had your coffee," she said brightly to a small conference room of mathematicians of all ages and career stages.

Part of Maraj's work, mostly conducted with her PhD advisor, Uwe Nagel (University of Kentucky), is on asymptotic phenomena for infinitely many objects such as ideals and algebras related by a group action.

"In particular, we can record detailed quantitative data for infinitely many related objects in a multivariate formal power series, called equivariant Hilbert series," she explained later. "The rational form of these series, if it exists, is of interest as it can provide additional data about the objects in study. In searching for this rational form, we often utilize theory of languages and their automata from computer science."

Seated in the back row of the conference room, paying close attention throughout the 20-minute talk, was Jane Ivy Coons of the University of Oxford, a supernumerary teaching fellow at St. John's College and an affiliate researcher at the Mathematical Institute at Oxford.

This is a story of math and friendship. And travel grants.

In 2018, Maraj was in the third year of her PhD at the University of Kentucky when she received an AMS Graduate Student Travel Grant, to attend and present at the Spring Central AMS sectionals at the Ohio State University.

Elaine Beebe is the communication and outreach content specialist at the American Mathematical Society. Her email address is exb@ams.org.

DOI: https://doi.org/10.1090/noti2700

Figure 1.

The grant was "encouraging," Maraj said. "It gave me some confidence that I was becoming a mathematician."

At the Ohio State sectionals, Maraj met Coons, then a PhD student and NSF Graduate Research Fellow at North Carolina State University. Coons became Maraj's first collaborator apart from her adviser.

Five years later, they continue to work together in their shared field of algebraic statistics. "We are analyzing semialgebraic sets of symmetric matrices that parametize certain multivariate Gaussian models," Maraj said.

"This JMM particularly contained many talks in algebraic statistics and related areas by experts and young mathematicians," Maraj observed. She and Coons presented projects of which they have been authors: Coons on Brownian motion tree models and Maraj on colored Gaussian graphical models.

In addition, Maraj and Coons each presented in the AMS Special Session on Applied Enumerative Geometry. "Most of the talks focused on applications of discrete and combinatorial geometry to areas such as statistics, computer vision, and numerical algebraic geometry," Coons said.

"Even with the busy schedule, Jane and I found some time to discuss math in person together," Maraj said. "Obviously, we hung out and had drinks; we are collaborators *and* good friends!"

The American Mathematical Society administers several travel grant programs. "We believe in the power of connecting people," said Sarah Bryant, director of programs for the AMS. "We support travel that allows mathematicians to engage in research collaboration, build their personal and professional networks in mathematics, and share their research results with the mathematical community."

Some travel grants are funded by donations to the AMS; others by outside funding from sources such as the National Science Foundation (NSF). Most grants provide partial travel support. Each individual program's web page provides more information about its grant, including deadlines, eligibility, and restrictions.

In 2022, Maraj received an AMS Simons Travel Grant, administered by the AMS with support from the Simons Foundation. Each AMS Simons Travel Grant provides an early-career mathematician with $2,500 each year for two years, to be used for research-related travel.

This grant "is significantly helping my career," Maraj said. "It has removed a lot of stress. I am not scared to accept an invitation to give a talk."

"Some conferences such as JMM are significant for a postdoc's career, but expensive," she said. "I seriously doubt that I would be able to attend these meetings if not for the support from the Simons grant."

Coons added, "In-person meetings are absolutely critical for me. They give me the opportunity to meet people in my field of varying stages in their careers and to discuss interesting new mathematics with them."

"In fact, many of my ongoing collaborations with Aida and others have stemmed from discussions that took place at conferences," she said.

"In-person meetings are crucial," Maraj agreed. "I have met many of my collaborators and mathematicians who have become my official and unofficial mentors in these meetings."

"A lot of research is done when we are in the same place."

Credits

Figure 1 is courtesy of Kyle Klein.

AMS Updates

MathSciNet Update Improves Search Tools, Design, Accessibility Features

On June 18, the American Mathematical Society will launch the new MathSciNet interface. The AMS has updated MathSciNet's features and layout consistently since its release in 1996, but the foundational search structure remained mostly the same. The new interface improves MathSciNet's design, search tools, and accessibility features.

Notably, the interface has shifted to a simpler design, replacing the stacked search fields of the past with a cleaner, more intuitive single field. Accessibility features include improved clarity through fonts and contrast, as well as better screen reader compatibility. And the entire update has been designed with mobile in mind; MathSciNet now works better than ever with mobile devices. The update also includes improved filtering and refining processes, making it easier than ever to find what you're looking for on MathSciNet.

The new interface will launch on June 18 at `mathscinet.ams.org`.

DOI: https://doi.org/10.1090/noti2702

Deaths of AMS Members

J. R. Barreca, of Canada, died on March 8, 2023. Born on May 30, 1955, he was a member of the Society for 44 years.

Howard Lee Dachslager, of Indianapolis, Indiana, died on March 30, 2023. Born on May 1, 1932, he was a member of the Society for 20 years.

Heinz Dietrich Doebner, of Germany, died on October 4, 2022. Born on May 11, 1931, he was a member of the Society for 21 years.

Elinor Evenchick Berger, of New York, New York, died on June 5, 2021. Born on October 25, 1942, she was a member of the Society for 58 years.

Christopher K. King, of Boston, Massachusetts, died on March 15, 2023. Born on January 11, 1959, he was a member of the Society for 35 years.

Julius Korbas, of Slovakia, died on August 21, 2022. Born on March 22, 1955, he was a member of the Society for 31 years.

Robert E. O'Malley, Jr., of Seattle, Washington, died on December 31, 2020. Born on May 23, 1939, he was a member of the Society for 59 years.

Terence J. Reed, of Greensboro, North Carolina, died on July 26, 2021. Born on May 24, 1935, he was a member of the Society for 62 years.

C. Rousseau, of Memphis, Tennessee, died on April 10, 2020. Born on January 13, 1938, he was a member of the Society for 49 years.

Bertram M. Schreiber, of Detroit, Michigan, died on August 12, 2019. Born on November 4, 1940, he was a member of the Society for 54 years.

Monty J. Strauss, of Plano, Texas, died on September 9, 2022. Born on August 26, 1945, he was a member of the Society for 56 years.

Robert Tubbs, of Boulder, Colorado, died on April 11, 2023. Born on January 29, 1954, he was a member of the Society for 46 years.

Lieven Vanhecke, of Belgium, died on March 11, 2023. Born on May 26, 1939, he was a member of the Society for 50 years.

Mathematics People

Nominations Open for 2023 SASTRA Ramanujan Prize

The Shanmugha Arts, Science, Technology, Research Academy (SASTRA), based in the state of Tamil Nadu in South India, has called for nominations for the 2023 SASTRA Ramanujan Prize of $10,000.

The deadline for nominations is July 31, 2023, for a mathematician who has contributed outstanding work in an area of mathematics influenced by the late Indian mathematician Srinivasa Ramanujan. The winner will be a mathematician not exceeding the age of 32: "set at 32 because Ramanujan achieved so much in his brief life of 32 years," according to the prize organizers.

Each year, the winner is invited to give a talk and receive the prize at an international conference conducted by SASTRA in Kumbakonam, Ramanujan's hometown, around Ramanujan's birthday, December 22: held this year December 21–22, 2023.

For more information about the prize and the nomination process, see `https://qseries.org/sastra-prize/nominations-2023.html`.

—SASTRA Ramanujan Prize

Childs Receives Michler Prize

The Association for Women in Mathematics has awarded the 2023–2024 Ruth I. Michler Memorial Prize to **Lauren M. Childs**, associate professor, Department of Mathematics, Virginia Tech.

Childs has been selected to receive the Michler Prize for her research accomplishments in mathematical biology. She will pursue a research project advancing mathematical theory and methods for trait-based models of infectious disease, including integral projection models. Such models also will be used to study the spread of infectious disease—in particular, malaria—and associated population dynamics.

Childs will spend an upcoming semester visiting Cornell University, where she earned her PhD in 2010. She is a member of the American Mathematical Society.

—Association for Women in Mathematics

Wang Awarded 2023 Clifford Prize

Xieping Wang, an associate research fellow at the University of Science and Technology of China (USTC) in Hefei, has been selected as the recipient of the 2023 W. K. Clifford Prize for his outstanding contributions to the field of Clifford analysis.

Wang completed his PhD from USTC in 2017. He already has made significant contributions to the field of Clifford analysis, particularly in the area of slice hyperholomorphic functions. His achievements include establishing geometric function theory for slice regular functions over octonions; proving a boundary Schwarz lemma for slice regular self-mappings of the unit ball in the quaternionic space; proving uniqueness of complex geodesics with prescribed boundary value and direction in strongly linearly convex domains; and establishing a Hartog's type extension theorem for pluriharmonic functions on $(n-1)$-complete complex manifolds of dimension $n > 1$. Wang's work has been published in such journals as *Transactions of the AMS*, *Mathematische Annalen*, *Pacific Journal of Mathematics*, and *Journal of Geometric Analysis*.

The W. K. Clifford Prize is an international scientific prize intended to encourage young researchers to compete for excellence in research in theoretical and applied Clifford algebras and their analysis and geometry. Awarded every three years at the International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA), the W. K. Clifford Prize will be presented to Wang at the 13th ICCA in Holon, Israel, in June 2023.

—ICCA

DOI: https://doi.org/10.1090/noti2725

Association for Symbolic Logic Presents Prizes

The Association for Symbolic Logic (ASL) presented the Gerald Sacks Prize and the Shoenfield Prize for 2022.

Francesco Gallinaro, Leeds University, and **Patrick Lutz**, University of California, Berkeley, were awarded the Sacks Prize, which is presented annually for the most outstanding doctoral dissertation in mathematical logic. The prize was established to honor the late Professor Gerald Sacks of MIT and Harvard for his unique contribution to mathematical logic, particularly as an adviser to a large number of excellent PhD students. The Sacks Prize became an ASL Prize in 1999; it consists of a cash award plus five years' free membership in the ASL.

Francesco Gallinaro received his PhD in 2022 from Leeds University under the joint supervision of Vincenzo Mantova and Dugald Macpherson. His thesis, "Around exponential-algebraic closedness," provides further evidence towards the quasi-minimality property of the field of complexes enriched with the exponential map, conjectured by Boris Zilber. For a family of varieties (defined by equations that are dimensionally likely to have solutions), Gallinaro shows that the exponential-algebraic closedness property holds in the field of complexes, then considers the analogous problem for abelian varieties with their associated exponential maps and finally in the upper half plane endowed with other analytic functions such as the elliptic modular function. His novel approaches also demonstrate a mastery of quite different techniques and are strongly expected to enable further progress.

Patrick Lutz received his PhD in 2021 from the University of California, Berkeley, under the supervision of Theodore A. Slaman. His dissertation, "Results on Martin's Conjecture," contains some of the most substantial progress in decades on Martin's Conjecture, including a proof of the first part of Martin's Conjecture for order-preserving functions and of its analog for regressive functions on the hyperarithmetic degrees. Its methods also open up new possibilities in the study of Martin's Conjecture. The proofs involve several novel ideas as well as a powerful combination of methods from set theory and computability theory, with applications beyond Martin's Conjecture, including the most significant advance in decades on a question of Sacks about embeddability of continuum-sized partial orders into the Turing degrees.

The Shoenfield Prize, a cash award, is awarded every three years to a book and an expository article for outstanding expository writing in the field of logic. Any new book published during the nine years prior to the award year is eligible; any article published during the six years prior to the award year is eligible.

Paolo Mancosu, Sergio Galvan, and **Richard Zach** received the Shoenfield Prize for the book *An introduction to proof theory—normalization, cut-elimination, and consistency proofs* (Oxford University Press, Oxford, 2021. xii+418 pp.)

Vasco Brattka received the Shoenfield Prize for the article "A Galois connection between Turing jumps and limits," *Log. Methods Comput. Sci.* **14** (2018), no. 3, Paper No. 13, 37 pp.

—*Association for Symbolic Logic*

Ames Awards Winners Announced

The editors and publisher of the *Journal of Mathematical Analysis and Applications* (JMAA) announced the winners of the 2021 Ames Awards last summer.

Michal Goliński and **Adam Przestacki** (Adam Mickiewicz University, Poland) were honored for their article "The invariant subspace problem for the space of smooth functions on the real line." **Alexander Keimer** (University of California, USA) and **Lukas Pflug** (Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany) were honored for their article "On approximation of local conservation laws by nonlocal conservation laws."

The annual JMAA Ames Awards are in memory of Dr. William F. Ames, editor-in-chief of JMAA from 1991–2006. After Ames's death in 2008, awards in pure and applied mathematics were established by his relatives to recognize Ames's many years of outstanding service to the Journal and contributions to the field of applied mathematics. Each award consists of a certificate of merit and a monetary prize of USD $2,500 donated by Elsevier and the AMS.

—*Elsevier*

Classified Advertising

Employment Opportunities

RHODE ISLAND

ICERM Partnerships for Research Innovation in the Mathematical Sciences

The Institute for Computational and Experimental Research in Mathematics (`https://icerm.brown.edu/`) is proud to be an eligible partner in the NSF Division of Mathematical Sciences' Partnerships for Research Innovation in the Mathematical Sciences (`https://www.nsf.gov/pubs/2023/nsf23560/nsf23560.htm?WT.mc_ev=click&WT.mc_id=&utm_medium=email&utm_source=govdelivery`)! PRIMES is a new funding opportunity to build lasting ties between DMS-supported math research institutes and minority-serving institutions. Eligible Historically Black Colleges and Universities, Hispanic-Serving Institutions, Tribal Colleges and Universities, and Asian American and Pacific Islander-Serving Institutions are invited to submit a proposal nominating a faculty member to serve as a co-Principal Investigator alongside a Director, Associate Director, or equivalent, at one of the eligible partner institutes.

We would like to invite faculty at eligible minority-serving institutions to start the PRIMES process by applying to participate in one of the following ICERM semester programs: 'Topology and Geometry in Neuroscience' (Fall '23), where leading researchers at the interfaces of topology, geometry and neuroscience will take stock of recent work and outline future directions; 'Numerical PDEs: Analysis, Algorithms, and Data Challenges' (Spring '24), where researchers will discuss the current state-of-the-art and emerging trends in computational PDEs; 'Theory, Methods, and Applications of Quantitative Phylogenomics' (Fall '24), where mathematicians, statisticians, computer scientists, and experimental biologists will come together to address the challenges involved in genome-scale phylogenetic inference; and 'Geometry of Materials, Packings, and Rigid Frameworks' (Spring '25), where researchers will integrate diverse fields of discrete mathematics, geometry, theoretical computer science, mathematical biology, and statistical and soft matter physics.

3

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The *Notices* Classified Advertising section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

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Upcoming deadlines for classified advertising are as follows: September 2023—June 23, 2023; October 2023—July 21, 2023; November 2023—August 25, 2023; December 2023—September 22, 2023.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to `classads@ams.org`.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Mr. Albert Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

2

SOUTH KOREA

Korea Institute for Advanced Study (KIAS)
Call for Applications: Positions
in Pure and Applied Mathematics

Founded in 1996, KIAS is committed to the excellence of research in basic sciences, namely mathematics, theoretical physics, and computational sciences, through high-quality research programs and a strong faculty body consisting of distinguished scientists and visiting scholars.

The School of Mathematics and the Center for Mathematical Challenges boast internationally-renowned faculty members whose research bring prestigious visitors from diverse research areas, nurturing a research environment that encourages interaction and collaboration not only on-campus but beyond.

Qualified, outstanding candidates in the field are encouraged to frequently check Mathjobs.org and KIAS Jobs website (`https://jobs.kias.re.kr`), where detailed information is updated when faculty and postdoctoral research fellow positions become available.

4

APPLY FOR THE

AMS *Claytor-Gilmer* FELLOWSHIP

Pursue your mathematics research more intensely with support from the AMS Claytor-Gilmer Fellowship.

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Further information and instructions for submitting an application can be found at the fellowship website: **www.ams.org/claytor-gilmer**

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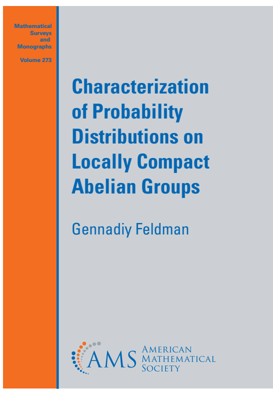
New Books Offered by the AMS

Analysis



Characterization of Probability Distributions on Locally Compact Abelian Groups

Gennadiy Feldman, *B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences, Kharkiv, Ukraine*

It is well known that if two independent identically distributed random variables are Gaussian, then their sum and difference are also independent. It turns out that only Gaussian random variables have such property. This statement, known as the famous Kac-Bernstein theorem, is a typical example of a so-called characterization theorem. Characterization theorems in mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions of these random variables. The first results in this area are associated with famous 20th century mathematicians such as G. Pólya, M. Kac, S. N. Bernstein, and Yu. V. Linnik.

By now, the corresponding theory on the real line has basically been constructed. The problem of extending the classical characterization theorems to various algebraic structures has been actively studied in recent decades. The purpose of this book is to provide a comprehensive and self-contained overview of the current state of the theory of characterization problems on locally compact Abelian groups. The book will be useful to everyone with some familiarity of abstract harmonic analysis who is interested in probability distributions and functional equations on groups.

This item will also be of interest to those working in probability and statistics.

Mathematical Surveys and Monographs, Volume 273
July 2023, 240 pages, Softcover, ISBN: 978-1-4704-7295-5, 2020 *Mathematics Subject Classification*: 60B15, 62E10, 43A25, 43A35; 39B52, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/273

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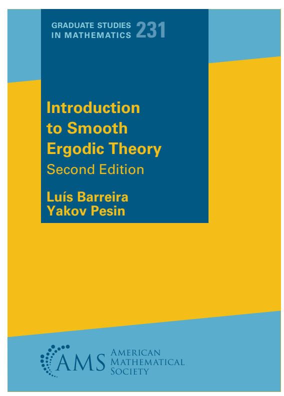
Differential Equations



Introduction to Smooth Ergodic Theory
Second Edition

Luís Barreira, *Universidade de Lisboa, Lisbon, Portugal*, and **Yakov Pesin**, *Pennsylvania State University, University Park, PA*

This book is the first comprehensive introduction to smooth ergodic theory. It consists of two parts: the first introduces the core of the theory and the second discusses more advanced topics. In particular, the book describes the general theory of Lyapunov exponents and its applications to the stability theory of differential equations, the concept of nonuniform hyperbolicity, stable manifold theory (with emphasis on absolute continuity of invariant foliations), and the ergodic theory of dynamical systems with nonzero Lyapunov exponents. A detailed description of all the basic examples of conservative systems with nonzero Lyapunov exponents, including the geodesic flows on compact surfaces of nonpositive curvature, is also presented. There are more than 80 exercises. The book is aimed at graduate students specializing in dynamical systems and ergodic theory as well as anyone who wishes to get a working knowledge of smooth ergodic theory and to learn how to use its tools. It can also be used as a source for special topics courses on nonuniform hyperbolicity. The only prerequisite for using this book is a basic knowledge of real analysis, measure theory, differential equations, and topology, although the necessary background definitions and results are provided.

In this second edition, the authors improved the exposition and added more exercises to make the book even more student-oriented. They also added new material to bring the book more in line with the current research in dynamical systems.

Graduate Studies in Mathematics, Volume 231
August 2023, approximately 340 pages, Hardcover, ISBN: 978-1-4704-7065-4, LC 2022050423, 2020 *Mathematics Subject Classification*: 37D25, 37C40, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/231

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Graduate Studies in Mathematics, Volume 231
August 2023, approximately 340 pages; Softcover, ISBN: 978-1-4704-7307-5, LC 2022050423, 2020 *Mathematics Subject Classification*: 37D25, 37C40, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/231.S

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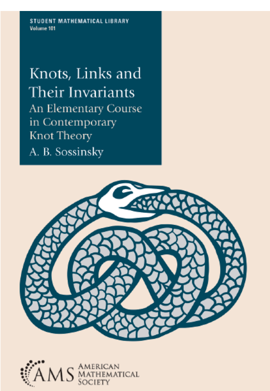
Geometry and Topology



Knots, Links and Their Invariants

An Elementary Course in Contemporary Knot Theory

A. B. Sossinsky, *Independent University of Moscow, Russia, and Poncelete Laboratory IUM-CNRS, Moscow, Russia*

This book is an elementary introduction to knot theory. Unlike many other books on knot theory, this book has practically no prerequisites; it requires only basic plane and spatial Euclidean geometry but no knowledge of topology or group theory. It contains the first elementary proof of the existence of the Alexander polynomial of a knot or a link based on the Conway axioms, particularly the Conway skein relation. The book also contains an elementary exposition of the Jones polynomial, HOMFLY polynomial and Vassiliev knot invariants constructed using the Kontsevich integral. Additionally, there is a lecture introducing the braid group and shows its connection with knots and links.

Other important features of the book are the large number of original illustrations, numerous exercises and the absence of any references in the first eleven lectures. The last two lectures differ from the first eleven: they comprise a sketch of non-elementary topics and a brief history of the subject, including many references.

Student Mathematical Library, Volume 101
July 2023, approximately 142 pages, Softcover, ISBN: 978-1-4704-7151-4, LC 2022051584, 2020 *Mathematics Subject Classification*: 55–XX, 51–XX, 20–XX, List US$59, **AMS Institutional member US$47.20**, All Individuals US$47.20, Order code STML/101

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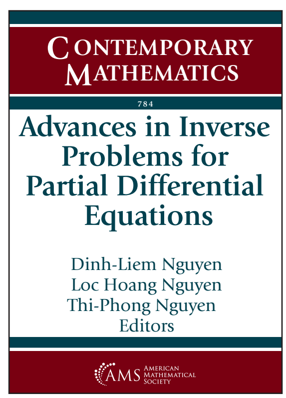
New in Contemporary Mathematics

Applications



Advances in Inverse Problems for Partial Differential Equations

Dinh-Liem Nguyen, *Kansas State University, Manhattan, KS*, **Loc Hoang Nguyen**, *University of North Carolina at Charlotte, NC*, and **Thi-Phong Nguyen**, *New Jersey Institute of Technology, Newark, NJ*, Editors

This volume contains the proceedings of two AMS Special Sessions "Recent Developments on Analysis and Computation for Inverse Problems for PDEs," virtually held on March 13–14, 2021, and "Recent Advances in Inverse Problems for Partial Differential Equations," virtually held on October 23–24, 2021.

The papers in this volume focus on new results on numerical methods for various inverse problems arising in electrical impedance tomography, inverse scattering in radar and optics problems, reconstruction of initial conditions, control of acoustic fields, and stock price forecasting. The authors studied iterative and non-iterative approaches such as optimization-based, globally convergent, sampling, and machine learning-based methods.

The volume provides an interesting source on advances in computational inverse problems for partial differential equations.

This item will also be of interest to those working in differential equations.

Contemporary Mathematics, Volume 784
June 2023, 206 pages, Softcover, ISBN: 978-1-4704-6968-9, LC 2022046723, 2020 *Mathematics Subject Classification*: 49N45, 65M32, 65N21, 65R20, 78A46, 78M22, 80A23, 86A22, List US$130, **AMS members US$104**, **MAA members US$117**, Order code CONM/784

`bookstore.ams.org/conm-784`

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Multiplicative Invariant Fields of Dimension ≤ 6

Akinari Hoshi, *Niigata University, Japan*, **Ming-chang Kang**, *National Taiwan University, Taipei, Taiwan*, and **Aiichi Yamasaki**, *Kyoto University, Japan*

Memoirs of the American Mathematical Society, Volume 283, Number 1403
March 2023, 137 pages, Softcover, ISBN: 978-1-4704-6022-8, 2020 *Mathematics Subject Classification*: 14E08, 20C10, 14F22, 20J06, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1403

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Weight Multiplicities and Young Tableaux Through Affine Crystals

Jang Soo Kim, *Sungkyunkwan University, Suwon, South Korea*, **Kyu-Hwan Lee**, *University of Connecticut, Storrs, CT*, and **Se-jin Oh**, *Ewha Womans University, Seoul, South Korea*

Memoirs of the American Mathematical Society, Volume 283, Number 1401
March 2023, 88 pages, Softcover, ISBN: 978-1-4704-5994-9, 2020 *Mathematics Subject Classification*: 05E10, 17B37, 81R50, 16T30, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1401

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Analysis

Spectral Properties of Ruelle Transfer Operators for Regular Gibbs Measures and Decay of Correlations for Contact Anosov Flows

Luchezar Stoyanov, *University of Western Australia, Crawley, Western Australia, Australia*

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 283, Number 1404
March 2023, 121 pages, Softcover, ISBN: 978-1-4704-5625-2, 2020 *Mathematics Subject Classification*: 37D20, 37C40, 37D25; 37C25, 37C30, 37D40, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1404

`bookstore.ams.org/memo-283-1404`

On Singular Vortex Patches, I: Well-Posedness Issues

Tarek M. Elgindi, *Duke University, Durham, NC*, and **In-Jee Jeong**, *Seoul National University, Republic of Korea*

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 283, Number 1400
March 2023, 89 pages, Softcover, ISBN: 978-1-4704-5682-5, 2020 *Mathematics Subject Classification*: 37B55, 76B47, 76D03, 35Q31, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1400

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Geometry and Topology

Gromov's Theory of Multicomplexes with Applications to Bounded Cohomology and Simplicial Volume

Roberto Frigerio, *Università di Pisa, Italy*, and **Marco Moraschini**, *Universität Regensburg, Bavaria, and Università di Bologna, Italy*

This item will also be of interest to those working in algebra and algebraic geometry.

Memoirs of the American Mathematical Society, Volume 283, Number 1402
March 2023, 153 pages, Softcover, ISBN: 978-1-4704-5991-8, 2020 *Mathematics Subject Classification*: 55N10, 55U10, 57N65, 57R19; 20J06, 43A07, 53C23, 55Q05, 57Q05, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1402

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Number Theory

Local Coefficients and Gamma Factors for Principal Series of Covering Groups

Fan Gao, *Zhejiang University, Hangzhou, China*, **Freydoon Shahidi**, *Purdue University, West Lafayette, Indiana*, and **Dani Szpruch**, *Open University of Israel, Raanana, Israel*

This item will also be of interest to those working in algebra and algebraic geometry.

Memoirs of the American Mathematical Society, Volume 283, Number 1399
March 2023, 135 pages, Softcover, ISBN: 978-1-4704-5681-8, 2020 *Mathematics Subject Classification*: 11F70; 22E50, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1399

bookstore.ams.org/memo-283-1399

Probability and Statistics

Dynamics of the Box-Ball System with Random Initial Conditions via Pitman's Transformation

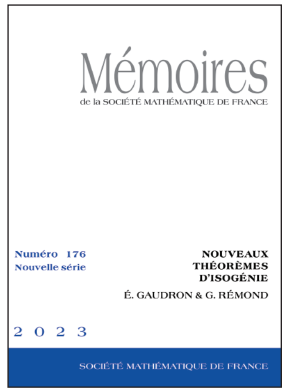
David A. Croydon, *Kyoto University, Japan*, **Tsuyoshi Kato**, *Kyoto University, Japan*, **Makiko Sasada**, *University of Tokyo, Japan*, and **Satoshi Tsujimoto**, *Kyoto University, Japan*

Memoirs of the American Mathematical Society, Volume 283, Number 1398
March 2023, 99 pages, Softcover, ISBN: 978-1-4704-5633-7, 2020 *Mathematics Subject Classification*: 37B15; 60G50, 60J10, 60J65, 82B99, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/283/1398

bookstore.ams.org/memo-283-1398

New AMS-Distributed Publications

Algebra and Algebraic Geometry



Nouveaux Théorèmes D'isogénie

É. Gaudron, *Université Clermont Auvergne, CNRS, LMBP, Clermont-Ferrand, France*, and **G. Rémond**, *Institut Fourier, Grenoble, France*

Given a finitely generated field extension K of the rational numbers and an abelian variety C over K, the authors consider the class of all abelian varieties over K which are isogenous (over K) to an abelian subvariety of a power of C.

The authors show that there is a single, naturally constructed abelian variety $C^\flat$ in the class whose ring of endomorphisms controls all isogenies in the class. Precisely, this means that if d is the discriminant of this ring then for any pair of isogenous abelian varieties in the class there exists an isogeny between them whose kernel has exponent at most d.

Furthermore, the authors prove for any element A in the class, the same number d governs several invariants attached to A such as the smallest degree of a polarisation on A, the discriminant of its ring of endomorphisms or the size of the invariant part of its geometric Brauer group. All these are bounded only in terms of d and the dimension of A.

In the case where K is a number field the authors go further and show that the period theorem applies to $C^\flat$ in a natural way and gives an explicit bound for d in terms of the degree of K, the dimension of $C^\flat$ and the stable Faltings height of C. This, in turn, yields explicit upper bounds for all the previous quantities related to isogenies, polarisations, endomorphisms, and Brauer groups which significantly improve known results.

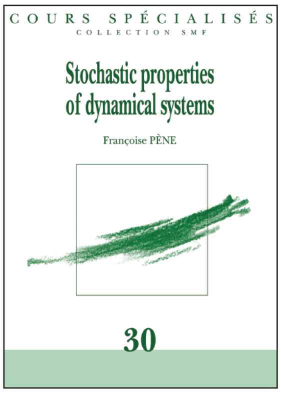
This item will also be of interest to those working in number theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 176
March 2023, 136 pages, Softcover, ISBN: 978-2-85629-948-7, 2020 *Mathematics Subject Classification*: 11G10, 14K02, 11R54, 14K15, List US$57, **AMS members US$45.60**, Order code SMFMEM/176

`bookstore.ams.org/smfmem-176`

Analysis



Stochastic Properties of Dynamical Systems

Françoise Pène, *UBO, UFR Sciences et Techniques Département de Mathématics, Brest, France*

This book provides an introduction to the study of the stochastic properties of probability preserving dynamical systems. The material is suitable for master's students who have completed their first year. The definitions and results are illustrated by examples and corrected exercises. The book presents the notions of Poincaré's recurrence, of ergodicity, of mixing and also sheds light on existing links between dynamical systems and Markov chains. The final objective of this book is to present three methods for establishing central limit theorems in the context of chaotic dynamical systems: a first method based on martingale approximations, a second method based on perturbation of quasi-compact linear operators, and a third method based on decorrelation estimates.

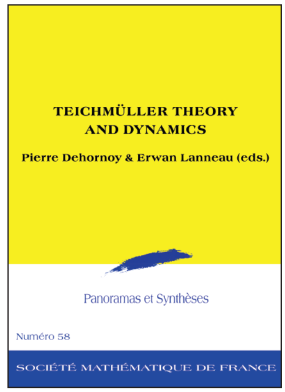
This item will also be of interest to those working in probability and statistics.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Cours Spécialisés—Collection SMF, Number 30
March 2023, 276 pages, Hardcover, ISBN: 978-2-85629-967-8, 2020 *Mathematics Subject Classification*: 37–00, 37A25, 37A30, 60F05, 60G42, 47A55, List US$81, **AMS members US$64.80**, Order code COSP/30

`bookstore.ams.org/cosp-30`

Number Theory



Teichmüller Theory and Dynamics

Pierre Dehornoy and **Erwan Lanneau**, *Université Grenoble Alpes, CNRS, Institut Fourier, Grenoble, France*, Editors

This edition of Panoramas and Synthéses follows the 27th edition of the Summer School in Mathematics on Teichmüller Dynamics, Mapping Class Groups and Applications. It took place from June 11–22, 2018 at the Institut Fourier (UMR CNRS 5582) of Grenoble. During this school, twelve specialists presented the basics of the theory of translation surfaces and their moduli spaces, as well as the recent advances in the field.

This volume brings together four texts, all based on the lecture notes of the school, and illustrates the interaction between Teichmüller theory and dynamics.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Panoramas et Synthèses, Number 58
March 2023, 136 pages, Softcover, ISBN: 978-2-85629-966-1, 2020 *Mathematics Subject Classification*: 37D40, 37D20, 37A25, 37C30, 53D30, 53C26, 32G15, 32G20, 32Q45, List US$65, **AMS members US$52**, Order code PASY/58

`bookstore.ams.org/pasy-58`

Meetings & Conferences of the AMS

June/July Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: stovall@math.wisc.edu; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: mamanes@hawaii.edu; telephone: (808) 956-4679.

Meetings in this Issue

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Buffalo, New York

University at Buffalo (SUNY)

September 9–10, 2023

Saturday – Sunday

Meeting #1188

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 3

Deadlines

For organizers: Expired
For abstracts: July 18, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jennifer Balakrishnan, Boston University, *Title to be announced.*
Sigal Gottlieb, University of Massachusetts, Dartmouth, *Title to be announced.*
Sam Payne, UT Austin, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Automorphic Forms and L Functions (Code: SS 1A), **Xiaoqing Li** and **Joseph A. Hundley**, University at Buffalo, SUNY.

Building Bridges Between $\mathbb{F}_1$-Geometry, Combinatorics and Representation Theory (Code: SS 2A), **Jaiung Jun**, SUNY New Paltz, **Chris Eppolito**, The University of the South, and **Alexander Sistko**, Manhattan College.

Combinatorial and Categorical Techniques in Representation Theory (Code: SS 3A), **Nicholas Davidson**, College of Charleston, **Robert Muth**, Duquesne University, **Tianyuan Xu**, Haverford College, and **Jieru Zhu**, Université Catholique de Louvain.

Difference and Differential Equations: Modeling, Analysis, and Applications to Mathematical Biology (Code: SS 4A), **Nhu N. Nguyen** and **Mustafa R. Kulenovic**, University of Rhode Island.

Ergodic Theory of Group Actions (Code: SS 5A), **Hanfeng Li**, SUNY at Buffalo, and **Jintao Deng**, University of Waterloo.

Financial Mathematics (Code: SS 6A), **Maxim Bichuch**, University at Buffalo, and **Zachary Feinstein**, Stevens Institute of Technology.

From Classical to Quantum Low-Dimensional Topology (Code: SS 7A), **Adam S. Sikora**, University At Buffalo, SUNY, **Roman Aranda**, Binghamton University, and **William Menasco**, University at Buffalo.

Gauge Theory and Low-Dimensional Topology (Code: SS 8A), **Cagatay Kutluhan**, University at Buffalo, **Paul M. N Feehan**, Rutgers University, New Brunswick, **Thomas Gibbs Leness**, Florida International University, and **Francesco Lin**, Columbia University.

Geometry of Groups and Spaces (Code: SS 9A), **Johanna Mangahas**, University at Buffalo, **Joel Louwsma**, Niagara University, and **Jenya Sapir**, Binghamton University.

Geometry, Physics and Representation Theory (Code: SS 10A), **Jie Ren**, SUNY Buffalo.

Homological Aspects of p-adic Groups and Automorphic Representations (Code: SS 11A), **Karol Koziol**, Baruch College, CUNY, and **Anantharam Raghuram**, Fordham University.

Inverse Problems in Science and Engineering (Code: SS 12A), **Sedar Ngoma**, SUNY Geneseo.

Nonlinear Partial Differential Equations in Fluids and Waves (Code: SS 13A), **Qingtian Zhang**, West Virginia University, and **Ming Chen**, University of Pittsburgh.

Nonlinear Wave Equations and Integrable Systems (Code: SS 14A), **Gino Biondini**, SUNY Buffalo, and **Alexander Chernyavsky**, SUNY at Buffalo.

Probability, Combinatorics, and Statistical Mechanics (Code: SS 15A), **Douglas Rizzolo**, University of Delaware, and **Noah Forman**, McMaster University.

Recent Advances in Numerical Methods for Fluid Dynamics and Their Applications (Code: SS 16A), **Daozhi Han**, The State University of New York at Buffalo, **Guosheng Fu**, University of Notre Dame, and **Jia Zhao**, Utah State University.

Recent Advances in Water Waves: Theory and Numerics (Code: SS 17A), **Sergey Dyachenko**, University At Buffalo, and **Alexander Chernyavsky**, SUNY at Buffalo.

Recent Developments in Operator Algebras and Quantum Information Theory (Code: SS 18A), **Priyanga Ganesan**, University of California San Diego, **Samuel Harris**, Northern Arizona University, and **Ivan G. Todorov**, University of Delaware.

Recent Trends in Spectral Graph Theory (Code: SS 19A), **Michael Tait**, Villanova University, and **Shahla Nasserasr** and **Brendan Rooney**, Rochester Institute of Technology.

Representation Theory and Flag Varieties (Code: SS 20A), **Yiqiang Li**, University at Buffalo, and **Changlong Zhong**, SUNY Albany.

Topics in Combinatorics and Graph Theory (Code: SS 21A), **Rong Luo** and **Kevin G. Milans**, West Virginia University, and **Guangming Jing**, University of West Virginia.

Omaha, Nebraska

Creighton University

October 7–8, 2023

Saturday – Sunday

Meeting #1189

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 4

Deadlines

For organizers: To be announced
For abstracts: August 8, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Lydia Bieri, University of Michigan, *Title To Be Announced*.

Aaron J. Pollack, University of California, San Diego, *Title To Be Announced*.
Christopher Schafhauser, University of Nebraska-Lincoln, *Title To Be Announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Graph Theory and Combinatorics I (Code: SS 7A), **Bernard Lidický** and **Steve Butler**, Iowa State University.

Advances in Operator Algebras I (Code: SS 10A), **Christopher Schafhauser**, UNIVERSITY of NEBRASKA–LINCOLN, and **Ionut Chifan**, University of Iowa.

Analytic number theory and related fields I (Code: SS 14A), **Vorrapan Chandee** and **Xiannan Li**, Kansas State University, and **Micah B. Milinovich**, University of Mississippi.

Applied knot theory I (Code: SS 15A), **Isabel K. Darcy**, University of Iowa, and **Eric Rawdon**, University of St. Thomas.

Automorphic forms, their arithmetic, and their applications I (Code: SS 8A), **Aaron J. Pollack**, UC San Diego, and **Spencer Leslie**, Boston College.

Commutative algebra, differential operators, and singularities I (Code: SS 11A), **Uli Walther**, Purdue University, **Claudia Miller**, Syracuse University, and **Vaibhav Pandey**, Purdue University.

Commutative Algebra i (Code: SS 9A), **Thomas Marley**, University of Nebraska- Lincoln, and **Eloísa Grifo**, University of Nebraska-Lincoln.

Discrete, Algebraic, and Topological Methods in Mathematical Biology I (Code: SS 6A), **Alexander B. Kunin**, Creighton University.

Enumerative Combinatorics I (Code: SS 4A), **Seok Hyun Byun**, Clemson University, **Tri Lai**, University of Nebraska, and **Svetlana Poznanovic**, Clemson University.

Foliations, flows, and groups I (Code: SS 16A), **Ying Hu**, University of Nebraska Omaha, and **Michael Landry**, Washington University in Saint Louis.

Fractal Geometry and Dynamical Systems ! (Code: SS 5A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, **Manuj Verma**, Indian Institute of Technology Delhi, New delhi, 110016, India, and **Megha Pandey**, Indian Institute Of Technology, Bhu(Varanasi)".

Harmonic Analysis in the Midwest I (Code: SS 20A), **Betsy Stovall**, University of Wisconsin-Madison, and **Terence L.. J. Harris**, Cornell University.

Homotopy theory I (Code: SS 21A), **Prasit Bhattacharya**, New Mexico State University, **Agnès Beaudry**, University of Colorado Boulder, and **Zhouli Xu**, University of California, San Diego.

Interactions of Floer Homologies, Contact Structures, and Symplectic Structures I (Code: SS 19A), **Robert DeYeso III** and **Joseph Breen**, University of Iowa.

Mathematical modeling and analysis in ecology and epidemiology I (Code: SS 17A), **Yu Jin**, University of Nebraska-Lincoln, **Shuwen Xue**, Northern Illinois University, and **Chayu Yang**, University of Nebraska-Lioncoln.

Nonlinear PDE and Free Boundary Problems I (Code: SS 3A), **William Myers Feldman**, University of Utah, and **Fernando Charro**, Wayne State University.

Progress in Nonlinear Waves I (Code: SS 2A), **David M. Ambrose**, Drexel University.

Recent Development in Advanced Numerical Methods for Partial Differential Equations I (Code: SS 13A), **Mahboub Baccouch**, University of Nebraska At Omaha.

Recent Developments in Theories and Computation of Nonlocal Models I (Code: SS 18A), **Anh Vo** and **Scott Hootman-Ng**, University of Nebraska-Lincoln, and **Animesh Biswas**, University of Nebraska Lincoln.

Topology of 3- and 4-Manifolds ! (Code: SS 12A), **Alexander Zupan**, University of Nebraska-Lincoln, **Roman Aranda**, Binghamton University, and **David Auckly**, Kansas State University.

Varieties with unexpected hypersurfaces, geproci sets and their interactions I (Code: SS 1A), **Brian Harbourne**, University of Nebraska, and **Juan C. Migliore**, University of Notre Dame.

Mobile, Alabama

University of South Alabama

October 13–15, 2023

Friday – Sunday

Meeting #1190

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 15, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Theresa Anderson, Carnegie Mellon, *Title to be announced*.
Laura Ann Miller, University of Arizona, *Title to be announced*.
Cornelius Pillen, University of South Alabama, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Extremal Combinatorics (Code: SS 17A), **Joseph Guy Briggs**, Auburn University, and **Chris Cox**, Iowa State University.

Analysis, Computation, and Applications of Stochastic Models (Code: SS 11A), **Yukun Li**, University of Central Florida, **Feng Bao**, Florida State University, **Xiaobing Henry Feng**, The University of Tennessee, and **Liet Anh Vo**, University of Illinois, Chicago.

Categorical Representations, Quantum Algebra, and Related Topics (Code: SS 7A), **Arik Wilbert**, University of Georgia, **Mee Seong Im**, United States Naval Academy, Annapolis, and **Bach Nguyen**, Xavier University of Louisiana.

Combinatorics and Geometry Related to Representation Theory (Code: SS 8A), **Markus Hunziker** and **William Erickson**, Baylor University.

Cyberinfrastructure for Mathematics Research & Instruction (Code: SS 9A), **Steven Craig Clontz**, University of South Alabama, and **Tien Chih**, Oxford College of Emory University.

Discrete Geometry and Geometric Optimization (Code: SS 13A), **Andras Bezdek**, Auburn University, Auburn AL, **Ferenc Fodor**, University of Szeged, and **Woden Kusner**, University of Georgia, Athens GA.

Dynamics and Equilibria of Energies (Code: SS 4A), **Ryan Matzke**, Technische Universität Graz, and **Liudmyla Kryvonos**, Vanderbilt University.

Dynamics of Fluids (Code: SS 20A), **I. Kukavica**, University of Southern California, **Dallas Albritton**, Princeton University, and **Wojciech S. Ozanski**, Florida State University.

Ergodic Theory and Dynamical Systems (Code: SS 5A), **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and **Joanna Furno**, University of South Alabama.

Experimental Mathematics in Number Theory and Combinatorics (Code: SS 2A), **Armin Straub**, University of South Alabama, **Brandt Kronholm**, University of Texas Rio Grande Valley, and **Luis A. Medina**, University of Puerto Rico.

Extremal and Probabilistic Combinatorics (Code: SS 14A), **Sean English**, University of North Carolina Wilmington, and **Emily Heath**, Iowa State University.

Mathematical Modeling of Problems in Biological Fluid Dynamics (Code: SS 1A), **Laura Ann Miller**, University of Arizona, and **Nick Battista**, The College of New Jersey.

New Directions in Noncommutative Algebras and Representation Theory (Code: SS 18A), **Jonas T. Hartwig**, Iowa State University, and **Erich C. Jauch**, University of Wisconsin - Eau Claire.

Number Theory and Friends (Code: SS 21A), **Robert James Lemke Oliver**, Tufts University, **Theresa Anderson**, Carnegie Mellon, **Ayla Gafni**, University of Mississippi, and **Edna Luo Jones**, Duke University.

Recent Advances in Low-dimensional and Quantum Topology (Code: SS 16A), **Christine Lee**, Texas State, and **Scott Carter**, University of South Alabama.

Recent Developments in Graph Theory (Code: SS 15A), **Andrei Bogdan Pavelescu**, University of South Alabama, and **Kenneth Roblee**, Troy University.

Recent progress in numerical methods for PDEs (Code: SS 12A), **Muhammad Mohebujjaman**, Texas A&M International University, **Leo Rebholz**, Clemson University, and **Mengying Xiao**, University of West Florida.

Representation Theory of Finite and Algebraic Groups (Code: SS 3A), **Daniel K. Nakano**, University of Georgia, **Pramod N. Achar**, Louisiana State University, and **Jonathan R. Kujawa**, University of Oklahoma.

Rings, Monoids, and Factorization (Code: SS 10A), **Jim Coykendall**, Clemson University, and **Scott Chapman**, Sam Houston State University.

Theory and Application of Parabolic PDEs (Code: SS 19A), **Wenxian Shen** and **Yuming Paul Zhang**, Auburn University.

Topics in harmonic analysis and partial differential equations (Code: SS 6A), **Jiuyi Zhu** and **Phuc Cong Nguyen**, Louisiana State University.

San Francisco, California

Moscone North/South Convention Center

January 3–6, 2024

Wednesday – Saturday

Meeting #1192

Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: Expired
For abstracts: September 12, 2023

Tallahassee, Florida

Florida State University in Tallahassee

March 23–24, 2024

Saturday – Sunday

Meeting #1193

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: August 22, 2023
For abstracts: January 23, 2024

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Meeting #1194

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: September 5, 2023
For abstracts: February 13, 2024

Milwaukee, Wisconsin

University of Wisconsin- Milwaukee

April 20–21, 2024

Saturday – Sunday

Meeting #1195

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: September 19, 2023
For abstracts: February 20, 2024

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Meeting #1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: October 4, 2023
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Recent Advances in Differential Geometry, **Zhiqin Lu**, University of California, **Shoo Seto** and **Bogdan Suceavă**, California State University, Fullerton, and **Lihan Wang**, California State University, Long Beach.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Antonio, Texas

University of Texas, San Antonio

September 14–15, 2024

Saturday – Sunday

Meeting #1198

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: February 13, 2024
For abstracts: July 23, 2024

Savannah, Georgia

Georgia Southern University, Savannah

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: To be announced
For abstracts: To be announced

Albany, New York

State University of New York at Albany

October 19–20, 2024

Saturday – Sunday

Meeting #1200

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 19, 2024
For abstracts: August 27, 2024

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Meeting #1201

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 26, 2024
For abstracts: September 3, 2024

Auckland, New Zealand

December 9–13, 2024

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

MEETINGS & CONFERENCES

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

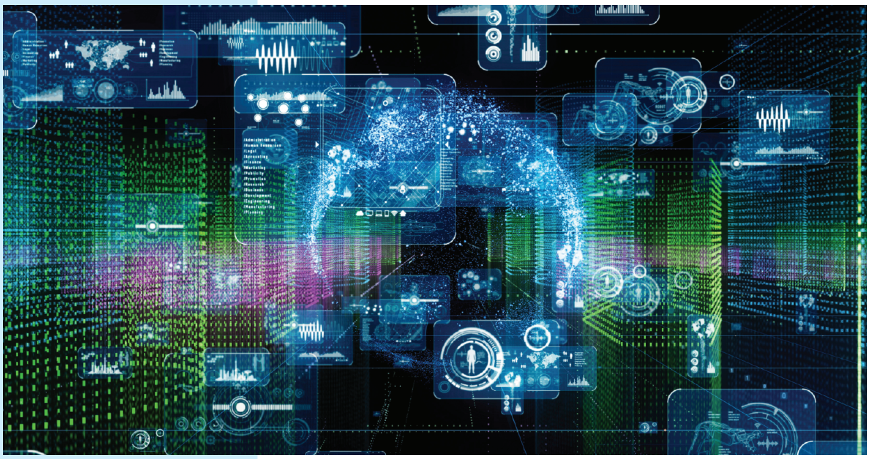
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Giving Health Care Policy a Dose of Mathematics



metamorworks iStock / Getty Images Plus via Getty Images

When a hospital or government wants to adjust their health policies — for instance, by encouraging more frequent screenings for certain diseases — how do they know whether their program will work or not?

If the service has already been implemented elsewhere, researchers can use that data to estimate its effects. But if the idea is brand-new, or has only been used in very different settings, then it's harder to predict how well the new program will work. Luckily, a tool called a microsimulation can help researchers make an educated guess.

A microsimulation predicts the effects of a policy or program by modeling individual behavior. For instance, the Urban Institute's Health Insurance Policy Simulation Model (HIPSM) simulates individual people, using publicly available data to create a realistic sample of the American population. The HIPSM simulates the reactions of insurance companies to policy change, reactions of employers to insurance companies, and finally how people will make choices about their health insurance based on the resulting options. The process is repeated until the model settles into an equilibrium. By adjusting the parameters of their model to match real-world data, the HIPSM can estimate how much people value health insurance covering the cost of medical services, and therefore how people will make decisions.

Governments around the world have adopted microsimulations to get insight into programs and policies. In Canada, for example, the OncoSim model quantifies the effects of screening programs on over 30 different types of cancer. As of 2018, microsimulation in health care had made its way to every populated continent.

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References: "Health Models." *Statistics Canada.* Retrieved 22 Dec. 2021, https://www.statcan.gc.ca/en/microsimulation/health/health.

D. J. Schofield, M. J. B. Zeppel, O. Tan, S. Lymer, M. M. Cunich, and R. N. Shrestha. "A brief, global history of microsimulation models in health: Past applications, lessons learned and future directions." *International Journal of Microsimulation.* 30 Apr 2018. **11**(1); 97–142. DOI: 10.34196/IJM.00175

M. Buettgens and J. Banthin. "The Health Insurance Policy Simulation Model for 2020." The Urban Institute. https://www.urban.org/sites/defaultfiles/publication/103412/the-health-insurance-policy-simulation-model-for-2020.pdf.

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