The cover design is based on imagery from "A Path Through the Roots," page 1143.
TO ALL AMS MEMBERS: PLEASE VOTE

Voting Information for the 2023 AMS Election

Voting Online
AMS members who have chosen to vote online will receive an email on August 15, 2023.

The email will come from "AMS Election Coordinator" via noreply@directvote.net and the subject will be "AMS 2023 Election—login information below" (you may want to use this information to configure your spam filter to ensure delivery of this email). The body of the message will provide your unique voting login information and the web address of the voting website.

Please vote by midnight (US Eastern Time) on November 1, 2023. After midnight, the website will stop accepting votes.

Voting by Paper Ballot
AMS members who have chosen to vote by paper will receive their ballot by the middle of September. Unique voting login information will be printed on the ballot should you wish to vote online.

Paper ballots received after November 1, 2023 will not be counted.

Please note that starting with the 2024 AMS Election, paper ballots will no longer be made available. The 2024 AMS Election will move to a fully electronic model.

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Additional information regarding the 2023 AMS Election is available in Frequently Asked Questions at:
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If you have questions, please email election@ams.org or call 800.321.4267, extension 4129 (US & Canada), 401.455.4129 (worldwide).

Thank you, and please remember to vote.

Boris Hasselblatt

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The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to “advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals,” and urge all members to conduct their professional activities with this goal in mind. (as adopted by the April 2019 Council)
Leaving the AMS

In 2022, having been a member of the AMS for more than 30 years, I decided not to renew my membership for another year.

Here are my reasons:

With grave concern, I see the growing use of DEI statements as a required component for job applications, in particular in mathematical sciences. In my opinion, it has an enormous corrosive effect on the math community and education in this country. Even if one is required to say “I passionately believe that water would certainly wet us, as fire would certainly burn”, the routine affirmation of one’s beliefs as a precondition of making a living constitutes compelled speech and corrupts everyone who participates in the performance.

I grew up in the Soviet Union, where people had to affirm their fealty to ideals and the leaders embodying those ideals, on a daily basis. As years went by, I observed the remarkable ease with which passionate communists turned first into passionate pro-Western liberals and then into passionate nationalists. This lived experience and also common sense convince me that only true conformists excel in this game. Do we really want our math departments to be populated by conformists?

Currently, the compelled speech of the compulsory DEI statements affects mostly people at the beginning of their careers, that is when they are most vulnerable. The sheer logic of bureaucratic expansion suggests that those who position themselves as experts in evaluating the merits and judging the sincerity of the DEI statements will find new venues to apply their skills, affecting other demographics.

The AMS does nothing to investigate these developments. About 25 years ago, when one of the universities decided to close its PhD program in math, the AMS saw it urgent enough to dispatch a fact-finding mission. Now, as we have a social experiment on a national scale, with potentially devastating consequences, the AMS demonstrates a remarkable lack of curiosity.

I can think of several reasons for this detachment.

First, it can be that the majority of members see nothing wrong in the DEI statements, or consider them a welcome development. I, for one, would be interested to find out if this is indeed the case. A couple of years ago, several letters were circulated and published that painted an inconclusive, to say the least, picture. As far as I can tell, the discussion ended having barely started. Wouldn’t it be useful to have it restarted, now that we have seen more results of the DEI proliferation?

Second, I anticipate an argument that the AMS is “not involved in politics”. But this is the kind of “politics” that, rephrasing Pericles, will get involved with you, whether you like it or not, and hence inaction is just as political as action.

Third, people can be simply afraid to voice their opinions (admittedly, the line between the second and third reasons is blurred). The fears of being accused of having certain pernicious attitudes and creating an unsafe environment, as well as the fear of losing one’s livelihood are not without merit. However, compared to the standards set by the totalitarian movements of the past these repercussions may not seem like such a big deal. The more we are afraid to talk and act now, the more debilitating the fear becomes, and the more devastating will be the effect of our inaction.

Sincerely,

Alexander Barvinok
POSITION AVAILABLE

Executive Director

AMERICAN MATHEMATICAL SOCIETY

POSITION

The Trustees of the American Mathematical Society invite applications for the position of Executive Director of the Society. The Executive Director has the opportunity to strongly influence all activities of the Society, as well as the responsibility of overseeing a large and diverse spectrum of people, programs, and publications. The desired starting date is February 1, 2024.

DUTIES AND TERMS OF APPOINTMENT

The American Mathematical Society, founded in 1888 to further the interests of mathematical research and scholarship, serves the national and international community through its publications, meetings, advocacy, and other programs. The AMS promotes mathematical research and its communication and uses; encourages and promotes the transmission of mathematical understanding and skills; supports mathematical education at all levels; advances the status of the profession of mathematics, encouraging and facilitating the full participation of all individuals; and fosters an awareness and appreciation of mathematics and its connections to other disciplines and everyday life.

These aims are pursued mainly through an active portfolio of programs, publications, meetings, conferences, and advocacy. The Society is a major publisher of mathematical books and journals, including MathSciNet®, an organizer of numerous meetings and conferences each year, and a sponsor of grants and training programs. The Society’s headquarters are located in Providence, Rhode Island, and the Executive Director is based there. The society also maintains a print shop in Pawtucket, Rhode Island; an office in Washington, DC, that houses the Office of Government Relations and the Office of Equity, Diversity, and Inclusion; and an office in Ann Arbor, Michigan, that publishes MathSciNet.

The Society’s operations are organized into four divisions: the Public and Informational Programs Division, the Meetings and Conference Division, the Publications Division, and the Office of the Executive Director. Each division reports to the Executive Director, who is a full-time employee of the Society and is responsible for the execution and administration of the policies of the Society as approved by the Board of Trustees and by the Council. The Executive Director is a full-time employee of the Society and is responsible for the operation of the Society's offices in Providence, Pawtucket, RI; Ann Arbor, MI; and Washington, DC. The Executive Director attends meetings of the Board of Trustees, the Council, and the Executive Committee, is an ex-officio (nonvoting) member of the policy committees of the Society, and is often called upon to represent the Society in its dealings with other scientific and scholarly bodies.

The Society employs a staff of over 200 in the four offices. The directors of the various divisions report directly to the Executive Director. Information about the operations and finances of the Society can be found in its Annual Reports, available at www.ams.org/annual-reports.

The Executive Director is appointed by and serves at the pleasure of the Trustees. The terms of appointment, salary, and benefits will be consistent with the nature and responsibilities of the position and will be determined by mutual agreement between the Trustees and the prospective appointee.

DESIRED QUALIFICATIONS

The successful candidate must be a leader, and we seek candidates who additionally have as many as possible of the following:

- A doctoral degree (or equivalent) in mathematics or a closely related field.
- Substantial experience and demonstrated visibility as a professional mathematician in academic, industrial, or governmental employment, with success in obtaining and administering grants.
- Extensive knowledge of the Society, the mathematics profession, and related disciplines and organizations, with a thorough understanding of the mission that guides the Society.
- Excellent communication skills, both written and oral, and an enthusiasm for public outreach.
- Demonstrated sustained commitment to diverse, inclusive, and equitable organizational environments and substantial experience in advancing equity, diversity, and inclusion priorities in the mathematical community.
- Demonstrated leadership ability supported by strong organizational and managerial skills.
- Familiarity with the mathematical community and its needs, and an ability to work effectively with mathematicians and nonmathematicians.
- Strong interest in engaging in fundraising and enjoyment of social interactions.

APPLICATIONS PROCESS

A search committee co-chaired by Joseph Silverman (joseph.silverman@brown.edu) and Bryna Kra (kra@math.northwestern.edu) has been formed to seek and review applications. All communication with the committee will be held in confidence. Suggestions of suitable candidates are most welcome.

Applicants should submit a CV and a letter of interest on MathJobs. The letter should be at most four pages, explaining your interest in being the Executive Director of the AMS and why you consider yourself to be a compelling candidate. The majority of the letter should discuss your major accomplishments and experiences that illustrate your leadership philosophy and address the desired qualifications for the position. Applications received by September 15, 2023 will receive full consideration.
The opinions expressed here are not necessarily those of the Notices or the AMS.

Dear AMS Community,

Earlier this year, it was announced that I will become the chief executive officer of the Consortium for Mathematics and Its Applications (COMAP) in the fall. It has been an honor and a privilege to have served as your executive director these past seven years.

Lucy Maddock, the AMS chief financial officer and associate executive director of administration, has been appointed by the AMS Board of Trustees to serve as the AMS interim executive director, effective in July. I couldn’t be happier—Ms. Maddock is a strong and thoughtful leader who is deeply familiar with AMS operations. She will ably oversee our professional society’s multifaceted work. Moreover, to help maintain continuity, I will become Lucy’s special advisor in July—full time through the summer and as needed going forward.

In the coming months, you will hear more from AMS president Bryna Kra and the chair of the AMS Board of Trustees, Joe Silverman. Together with other members of AMS leadership, they will identify our next executive director. The call for applications is already out, and we encourage you to consider whether serving the AMS in this way resonates for you! Ms. Maddock will resume her role as chief financial officer once a permanent executive director is hired.

I have had the pleasure of working with so many incredible people, from staff and volunteers to many members of our community. Your passion and commitment to helping the AMS advance mathematics have made such a difference, and I thank you. As I look back at my time here, I am filled with gratitude. I am proud of what we have accomplished together and I know the AMS will continue to thrive and make a positive impact.

I’ve considered the AMS my professional home since graduate school, and being able to serve as your executive director has truly been an incredible experience. Thank you for your kindness, your generosity, and your support. It has been an honor to serve you and I will always hold a special place in my heart for the American Mathematical Society.

Please join me in welcoming Lucy Maddock as our interim executive director. I have every confidence that the AMS is in good hands, and I look forward to continuing to interact with many of you in my new role at COMAP.

Farewell, and best wishes for a bright and beautiful future! See you at the Joint Mathematics Meetings in San Francisco in 2024!

Catherine A. Roberts is the executive director of the American Mathematical Society. Her email address is exdir@ams.org.

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Tropical geometry is a relatively recent network of ideas at the intersection of geometry and combinatorics. In its purest form, tropical geometry is a combinatorial analogue of algebraic geometry and can be studied logically independently from it. The basic objects in tropical geometry are things like graphs or simplicial complexes. They are decorated with some additional data that allows them to coarsely mimic objects from algebraic geometry. The properties of these objects are constrained by combinatorial analogues of the basic theorems in geometry. For example, the tropical analogue of a Riemann surface is a graph, and there are purely combinatorial analogues of the Riemann–Roch and Riemann–Hurwitz theorems for graphs. The surprising turn is often that theorem statements in tropical geometry are the same as or similar to parallel ones in algebraic geometry, even though all the words mean something completely different.

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While these tropical objects are very interesting in their own right, tropicalization is a process that turns algebraic varieties into tropical objects. This bridges the two worlds and sets up an intricate and imperfect dictionary: geometric constraints imply combinatorial ones and combinatorial constraints imply geometric ones. An enjoyable and compelling part of the subject is that even when there is not a logical implication, known theorems on each side provide predictions and conjectures for results on the other.

The present article aims to give the reader a feel for this network, some of its subtleties, and how it is used. The first topic is a quick and dirty introduction to the process of tropicalization, and some of the early observations in the subject that might have inspired people to look closer. The next topic is the "backwards direction," i.e., the tropical inverse problem. Our main goal here will be to exhibit one interesting example of a non-realizable tropical object, and give some sense for why it is non-realizable. Finally, we sketch how tropicalization, both forwards and backwards, predicts new theorems in traditional algebraic geometry.
The prototypical tropicalization. The simplest example of tropicalization applies to spaces defined by algebraic equations inside the algebraic torus, i.e., \((\mathbb{C}^*)^n\). We start with a collection of Laurent polynomials \(f_1, \ldots, f_k\) in \(n\) variables and look at their common zeroes:

\[ V = \{ f_1 = f_2 = \cdots = f_k = 0 \} \subset (\mathbb{C}^*)^n. \]

Objects \(V\) that are defined in this fashion are the local building blocks in algebraic geometry. They are examples of algebraic varieties. The primary move in tropical geometry comes from the map:

\[ \text{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n, \]

\[ (z_1, \ldots, z_n) \mapsto (\log_t |z_1|, \ldots, \log_t |z_n|), \]

where the logarithm has base \(t\), some real number. The right-hand side tracks the logarithmic size of a tuple of complex numbers. The image of \(V\) under this map, namely \(\text{Log}_t(V)\), is the set of possible sizes of solutions to the equations \(\{ f_i \}\). The motivation behind this move is to split the process of studying solutions to polynomial systems into two steps: first study the possible sizes of solutions and then study solutions of a fixed size.

It is useful to slightly repackage the information. By the logarithm rules, if \(t\) changes the set \(\text{Log}_t(V)\) is rescaled. The tropicalization is the limit of this rescaling as the base becomes small:

\[ \text{trop}(V) = \lim_{t \to 0} \text{Log}_t(V). \]

The limit procedure reveals a simple combinatorial structure. We will see some examples soon, but for now, let us note that this process is a completely elementary but strange thing to do to an algebro-geometric object. At first sight, the nice structure that \(V\) inherits from being defined by polynomials might be destroyed by applying this transcendental “limit logarithm” operation. A fundamental theorem of Bieri and Groves from the 1980s reveals that \(\text{trop}(V)\) is far from chaotic.

**Theorem 0.0.1.** Let \(V \subset (\mathbb{C}^*)^n\) be an algebraic variety of dimension \(r\). Then the tropicalization \(\text{trop}(V)\) is a polyhedral complex of dimension \(r\) in \(\mathbb{R}^n\).

Polyhedral complexes are very concrete structures, analogous to the simplicial complexes that one encounters in topology. Just as simplicial complexes are glued from simplices, polyhedral complexes are glued from (a finite number of) polyhedra. A polyhedron in \(\mathbb{R}^n\) is any locus where a fixed finite collection of affine functions is non-negative. Examples of this include the familiar polytopes in \(\mathbb{R}^2\), as well as unbounded regions such as \(\mathbb{R}^n_{\geq 0}\). We will see a few examples below, and reader will be able to find precise definitions and numerous pictures and examples in the textbook [10].

The first entry in almost everyone’s database of tropicalizations is depicted in the figure below. It shows the tropicalization of the curve \(\{ x + y + 1 = 0 \}\) in \((\mathbb{C}^*)^2\). The tropicalization is a union of three rays (which are simple examples of polyhedra), based at the origin, in \(\mathbb{R}^2\). There is one ray each in the direction of the positive \(x\)-axis, positive \(y\)-axis, and the negative diagonal direction. The picture can be obtained by an explicit calculation, but the reader may find it a pleasant exercise to use a computer to plot a large number of points and see the figure emerge.

![Figure 1. The tropicalization of the curve \(C_1\) given by \(\{ x + y + 1 = 0 \}\).](image)

**Prettier pictures.** In the literature, the term tropicalization is used for a myriad of different operations, but they are all based on the maneuver above. An important upgrade that will be relevant in our discussion is allowing the equations \(f_1, \ldots, f_k\) to depend on the base of the logarithm \(t\). Precisely, we can choose each \(f_i\) to be a polynomial whose coefficients are polynomials, or even power series, in the variable \(t\). For some of the statements later in this article to hold, we should allow the coefficients to at least be a power series in \(t\) that converges in some neighborhood of 0, but I encourage the reader to imagine that the coefficients are polynomials in \(t\). When there is a dependence on a parameter \(t\), we sometimes refer to this as a family of varieties, imagining that for each value of \(t\) there is a different variety.

The generalization to allow variable coefficients leads to much richer pictures, which better our odds of transporting meaningful information into tropical geometry. Without varying the coefficients, the tropicalization of a cubic could, and likely would, look exactly like the tropicalization of the curve \(C_1\) that we discussed above.

For example, let us take \(f\) to be a degree-3 polynomial in \(x\) and \(y\). There are 10 possible coefficients to choose when doing this, as we can write the polynomial as a sum of monomials:

\[ f(x, y) = \sum a_{ij}x^iy^j. \]
We can now choose the $a_{ij}$ to themselves be polynomials in $t$ and then tropicalize $C_2 = \{f = 0\}$. For example, we could choose the coefficients of each $x^iy^j$ to simply be different powers of $t$. I won’t do out the calculation here, but if these powers of $t$ in front of the monomials $x^iy^j$ are chosen in a sufficiently random way, the tropicalization could look like the picture below. Different choices of coefficients $a_{ij}$ will lead to different pictures, and in fact, there are over 1000 possibilities for tropicalizations of cubics! Nevertheless, the process of going from such polynomials to their tropicalizations is rather concrete, and can be found in the early chapters of the text [10]. Despite the numerous possibilities for the actual tropicalization, in most cases, the key features of the picture below will persist.

![Figure 2](image-url)  
**Figure 2.** The tropicalization of the curve $C_2$ given by the vanishing set of cubic equations, with coefficients depending on a parameter $t$.

A first look at the tropical dictionary. The Bieri–Groves theorem already asserts that the tropicalization of a variety $V$ in $\mathbb{C}^n$ sees the dimension of $V$. The calculation of the dimension of algebraic varieties that arise naturally in mathematics has been the content of several deep theorems, so this already has significant potential.

But staring at the pictures, you might start to get a sense that there’s quite a lot more going on here. Let us examine a few suspicious coincidences. First, a little differential topology and analysis tells us what the spaces $C_1$ and $C_2$ look like when they are given their Euclidean topology. It turns out that $C_1$ is homeomorphic to a sphere $\mathbb{S}^2$ minus three points, while, for each small nonzero value of $t$, the space $C_2$ is homeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$ minus 9 distinct points.

Now let us look at the tropicalizations. Looking at $C_1$, one notices the coincidence of 3: the legs running off to infinity in trop($C_1$) and the missing points in $C_1$. In the case of $C_2$, one might notice the coincidence of 9, again between the legs and the missing points. In this case, one might also notice that the topological surface $C_2$ has genus 1 while trop($C_2$) has a cycle in the middle.

Another interesting coincidence comes from placing trop($C_1$) and trop($C_2$) in the same picture. If the pictures are shifted around a little bit, one can notice that in most cases, the two tropicalizations intersect at exactly three points. But we know that in most cases, $C_1 \cap C_2$ is also exactly three points (for each value of the parameter $t$)—if we plug in the equation for $C_2$ into $C_1$ we end up with a cubic in 1 variable.

If the degree of the polynomial $f$ above was changed from 3 to $d$, the corresponding curve $V(f)$ would—if the polynomial was chosen with a little bit of care—have genus $\frac{(d-1)(d-2)}{2}$ and have $3d$ missing points, and sure enough, there would be exactly $\frac{(d-1)(d-2)}{2}$ independent cycles in trop($V(f)$) and exactly $3d$ legs running off to infinity. Now, we’ve certainly not rigorously argued that any of these geometric invariants, such as the genus or the number of intersection numbers, are equal to the corresponding tropical ones. But if you noticed this while having a think in the park, you might be sufficiently compelled to do some more experiments!

In the course of your experiments, you would occasionally be confounded. For example, the intersection of tropicalizations of two distinct lines can sometimes be infinite. If we had chosen our degree-3 polynomial defining $C_2$ in a bad way, its tropicalization would not contain a cycle at all, and in some other cases we would find fewer than nine legs running off to infinity. In fact, these “failures” illustrate...
the quirk that tropicalization is an imperfect process: even in the best of situations it forgets lots of geometry, and it is essentially never the case that trop($V$) can recover $V$. The core principle in tropical geometry is to figure out exactly how much information can be extracted from trop($V$), because when the information is actually there, it is likely to be easier to extract from the tropicalization than from $V$ itself.

How is the dictionary used? The prototypical tropicalization above sends an algebraic variety inside $(\mathbb{C}^*)^n$ to a polyhedral complex, but there are various different flavors of tropicalization that one gets from taking the basic idea we’ve just discussed and adding technology and formalism to it, ranging from Gröbner theory to classifying stacks. In fact, it is not even clear what gets to be called a tropicalization and what doesn’t, but Table 1 is a rough collection of things that people call tropicalizations in different contexts. In each instance, one starts with a family of the objects on the left, depending in a nice way on a parameter $t$, and extracts an object of the form on the right. But the prototype tropicalization is usually hiding somewhere in this extraction process.

Let me say a few words about the last three rows in the table. The right-hand side in the fourth example might be unfamiliar. Matroids are basic objects in combinatorics that encode combinatorial patterns that arise in linear algebra, such as the independence relations between subsets of a finite set of vectors. You can read about them in an earlier article in this publication [1]. The final two examples are special types of algebraic varieties, and even if you haven’t seen them before, the point to take away is there is a lot of regularity and control in tropicalization—by specifying algebrao-geometric properties on the left, we can understand the shape of the objects appearing on the right. In fact, there are natural classes of algebrao-geometric objects that tropicalize to Möbius strips, Klein bottles, and real projective planes. The tropical dictionary is used in both directions and in multiple ways. Let us briefly discuss two uses of the dictionary that have attracted a lot of interest.

Brill–Noether theory. In Brill–Noether theory, one examines the types of meromorphic functions that a compact Riemann surface or, equivalently, a smooth projective algebraic curve can admit. Despite being a central topic in algebraic geometry as far back as the 19th century, the subject is filled with compelling open questions. In a very influential paper, Baker proposed using the tropical dictionary in Brill–Noether theory [2, 7]. Later in this article, we will discuss ways in which tropical geometry has helped shed light on such questions.

For now, let us give a blueprint for what such an application of tropical geometry might look like. One of the first interesting facts in Brill–Noether theory is that a Riemann surface of genus 3 typically will not admit a meromorphic function of degree 2, i.e., a meromorphic function such that all but finitely many values in $\mathbb{C}$ are hit twice. One can argue on theoretical grounds that if this latter statement were not true, every finite graph with 3 cycles would carry a certain combinatorial structure—a special type of “chip firing” configuration. The latter is an extremely concrete assertion, and can be checked by hand on a few sheets of paper. By finding an example of a graph that violates this, one proves a theorem in algebraic geometry. Another recent article in this publication gives a friendly introduction to chip-firing and its applications [7].

The central pillar in the subject is the Brill–Noether theorem, which was proved by Griffiths and Harris in the 1980s [6]. It simultaneously captures numerous results of the above form, concerning how many meromorphic functions a Riemann surface can admit of a given type, provided the Riemann surface is “generic.” We will explain the statement a bit later on. About a decade ago, Cool, Draisma, Payne, and Robeva gave an ingenious combinatorial argument proving the Brill–Noether theorem using the method that Baker had envisioned [5]. The logic of the proof is exactly the toy example outlined above—find one example (in each genus) of a graph where one can classify all the appropriate chip-firing configurations. The Brill–Noether theorem in algebraic geometry is a consequence.

The type of argument above is an “impossibility constraint”—we preclude the existence of structures in algebraic geometry by showing that they would produce impossible structures in combinatorics. It is a trickier business to turn this logic on its head, and prove the existence of geometric structures using the existence of tropical ones. It is for this reason that the tropical inverse problem is difficult and interesting.

Enumerative geometry. The tropical dictionary can also be used to “count” geometric objects, and in fact, this spurred heavy interest in the subject in the early 2000s. An instance of such a counting problem is as follows. If we fix a positive integer $d$ and choose $3d – 1$ points in the complex projective plane $\mathbb{P}^2$, one expects that there are a finite number of ways to build polynomial maps

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^2,$$

that pass through the $3d – 1$ points, and such that $\varphi$ has “degree $d,” up to changing coordinates on the source. The map $\varphi$ is given by two polynomials, and their degrees must be $d$. The count, which we denote $N_d$ is a curve counting.

1In this article, we will move back and forth between using the terminology of (smooth and projective) algebraic curve and (compact) Riemann surface, depending on which aspects of the structure we want to emphasize. They are completely equivalent, so the reader should feel free to do a “find and replace” in their minds if they have a preference.
invariant. As long as our $3d - 1$ points are chosen to be in general position, the count of curves is the same.

If $d$ is 1 then we’re trying to pass a line through two points, and when $d$ is 2 we’re trying to pass a conic through five points. In both instances, there is exactly one way to do it (though $N_3$ and $N_4$ are 12 and 620, so we’re not just repeatedly computing the number 1). The figure below shows tropicalizations of lines and conics, passing through two and five points—they turn out to be the unique such curves. In celebrated work, Mikhalkin showed that these numbers $N_d$ can be computed from an analogous tropical problem: one examines tropical curves, similar to the ones we saw in the early sections of this article, through $3d - 1$ points in $\mathbb{R}^2$, see [11]. An appropriately weighted count of these tropical curves is equal to the number $N_d$.

The engine behind Mikhalkin’s results is a solution to an appropriate tropical inverse problem: if we want to count curves, we need to know that we are not counting tropical objects that have nothing to do with algebraic geometry. Twenty years on, the theorem is well-understood from numerous different perspectives, though none of them is by any means elementary!

The two uses of the tropical dictionary above reflect what this author personally knows and loves, but there are a number of other interesting ways in which tropical geometry can be used. For example, it has number theoretic applications to the structure of solutions to Diophantine problems and applications to studying the geometry behind mirror symmetry, a deep mathematical phenomenon arising from theoretical physics. It can also be used to import tools from algebraic geometry, such as the structure of cohomology rings, to purely combinatorial questions about graphs and matroids [1].

Tropicalizing abstract curves. When we discussed the prototype for tropicalization, we started with a variety $V$ together with an embedding of it inside $(C^*)^n$. However, if one is given an “abstract” variety $V$, it doesn’t really make sense to tropicalize it. By abstract variety here, we mean a topological space that is locally isomorphic to an affine variety, together with the appropriate theory of functions. In simple terms, the space $(C^*)^n$ and anything inside it, has a canonical system of coordinates. But if $V$ does not have a distinguished set of coordinates, there is nothing to which we can apply the above limit logarithm trick.

However, for abstract algebraic curves there is a canonical tropicalization procedure. The procedure is closely related to some deep theorems in algebraic geometry, concerning the existence of a canonical compactification of the moduli space of curves, which were proved by Deligne and Mumford in the 1960s.

Stable reduction. We now contemplate a smooth projective algebraic curve. In order to get an object like this, one would have to take a finite set of equations $f_1, \ldots, f_k$ in a finite set of variables and look at their common vanishing locus. The word “curve” means that the solution set has dimension 1. The word “smooth” means that the solution set is a manifold (note that this is something you can easily check using the implicit function theorem). The word “projective” means that the ambient space in which we solve these polynomials is not $C^n$ or $(C^*)^n$, but rather complex projective space $\mathbb{CP}^n$. If this seems unfamiliar, you can also have in mind a compact Riemann surface, or even just an orientable compact topological surface that arises from a system of polynomial equations.

The key thing to take from the above paragraph is that although the curve $C$ will be defined by polynomials in some ambient space, we are not actually going to remember the way in which $C$ was defined.

In order to construct nontrivial tropical objects, we will once again assume that there is some base parameter $t$, and that the defining equations of $C$ depend on $t$ in a nice fashion. The smoothness condition tells us that for all sufficiently small nonzero values $t_0$ of $t$, the curve $C_{t_0}$ is
homeomorphic to some orientable surface of genus \( g \), with \( g \) independent of \( t_0 \).

The stable reduction theorem is a deep theorem of Deligne–Mumford on the geometry of families of algebraic curves. It is a kind of algebraic geometer's analogue of decomposing an orientable surface into pairs of pants, by cutting along real curves, and completely controls what happens to our family of curves \( C_t \) of curves when \( t \) becomes 0.

The geometric intuition behind the stable reduction theorem is the following. For all small values of \( t \), the curve \( C_t \) is a one-dimensional complex manifold (i.e., Riemann surface) whose underlying space is homeomorphic to an orientable compact topological surface of genus \( g \). We can think of the variation of the \( t \) parameter as changing the complex manifold structure on this fixed topological surface \( S \).

The theorem says that there are a finite number of disjoint closed real curves whose lengths depend on \( t \), but shrink to 0 as \( t \) goes to 0. It may be instructive to look at Figure 5. When \( t \) changes, these lengths change, and the complex structure on the \( S \) changes. However, when \( t \) becomes 0, these real curves shrink to 0. If we imagine contracting these, there is a change in topology. The resulting object, which we call \( C_0 \), might no longer be a smooth curve. But, with a few pictures, you might convince yourself that \( C_0 \) is obtained from a collection of Riemann surfaces \( D_1, ..., D_\ell \) by attaching them at points. Curves of this form are called semistable; they are not necessarily Riemann surfaces but they are always formed from Riemann surfaces by gluing them together at points.

The stable reduction theorem says that the set of all smooth curves \( C_t \) for nonzero values of \( t \) fit in, together with this new broken Riemann surface \( C_0 \), to form a nice variety \( C \), over a small disk \( D \subset C \). Part of the theorem is that this can always be done in a canonical fashion.

We now explain what the tropicalization of \( C_t \) is. First, look at \( C_0 \), which, as we have noted already, is obtained by gluing a collection of topological surfaces \( D_1, ..., D_\ell \) to each other or to themselves. We form a graph \( \Gamma \) whose vertices are in bijection with these surfaces \( V_1, ..., V_\ell \). For each point at which \( D_i \) and \( D_j \) are glued, we add an edge between \( V_i \) and \( V_j \). This is probably best illustrated graphically; see Figure 5.

This graph \( G \) is still not the tropicalization. We record two more pieces of information. First, at each vertex \( V_j \), we record the genus of the corresponding topological surface \( D_j \) in the description above.

Second, we enhance the data on the edges. Recall that for each edge \( E \) of \( G \), there was a real closed curve in the surface whose length is vanishing. Associated to this curve is a parameter, which contains information such as the rate at which the length of the curve is shrinking. We can write out this function as

\[
    r_E(t) = at^\ell_E + \text{higher-order terms in } t.
\]

We now build a new space based on \( G \) and these numbers \( \ell_E \). We simply declare that the length of the edge \( E \) is \( \ell_E \) and the result is now a metric space that naturally enhances the finite graph \( G \). Each edge is now naturally identified with an interval.

The result of this process is denoted \( \Gamma(C_t) \) and is referred to as the tropicalization of \( C_t \). Summing up, the tropicalization of a family of abstract smooth projective curves \( C_t \) is a metric space \( \Gamma\Gamma(C_t) \) that enhances a finite graph. If we keep track of the genus of the Riemann surfaces \( D_j \) above, then the sum of these genera, plus the number of independent cycles in the graph, is equal to the genus \( g \) of the curve \( C_t \).

Note that the number of independent cycles in a graph is the number of edges lying in the complement of any chosen spanning tree.

In ideal situations, the two tropicalizations we have seen so far—the first via logarithmic limit sets and the second via semistable reductions—coincide, and it does make sense to call them both tropicalizations. The precise relationship is, however, a highly nontrivial line of inquiry.

A perfect tropical inverse problem. In what follows, an abstract tropical curve will be a finite graph \( G \) together with an enhancement of it to a metric space, by the specification of a positive integer length for every edge, and a genus attached to each vertex—exactly the data structure that axiomatizes the metric graphs that arise in the above sense. The tropical inverse problem asks: does every abstract tropical curve arise as the tropicalization \( \Gamma(C_t) \) of a family of smooth projective algebraic curves? The question has a very satisfying answer.

Theorem 0.0.2. Let \( \Gamma \) be an abstract tropical curve. There exists a family of abstract tropical curves \( C_t \) whose tropicalization \( \Gamma(C_t) \) coincides with \( \Gamma \).
The theorem is a combinatorial manifestation of a beautiful fact in algebraic geometry: the moduli space of all stable curves of a given genus $g$ itself forms a smooth and compact space. If you’d like to learn more about the moduli space of curves, you can start with two earlier articles in this publication [3, 15].

Curves with a function. I now want to discuss a tropicalization process where the forwards and backwards directions of tropical geometry are not so perfectly synced up. We begin by recalling a little bit of complex analysis. Let $C$ be a compact Riemann surface (or projective algebraic curve over $\mathbb{C}$). A meromorphic function is a partially defined map

$$\varphi : C \to \mathbb{C},$$

which fails to be defined at finitely many bad points $B \subset C$. At the bad points, the function takes on the value infinity, and locally looks like $\frac{1}{z^k}$ for some $k > 0$.

Let us organize the functions above by “shape.” Precisely, on the subset where it is defined, the function is holomorphic: recalling that a Riemann surface is a space where a neighborhood of every point looks like a disk in $\mathbb{C}$, we’re saying that away from $B$ the function is given by a power series in this neighborhood. Near the bad points, the function is given by a Laurent series. As a result, at each “bad point” there is a well-defined pole order: the number $k$ in the paragraph above, or more intrinsically, the most negative power that appears in the Laurent expansion. At finitely many points $Z \subset C$, the function takes on the value 0, and there is a zero order: the first positive power that appears in the power series expansion.

In order to track these data, fix points $p_1, \ldots, p_n$ on $C$ and integers $a_1, \ldots, a_n$. We now consider pairs $(C, \varphi)$ of a compact Riemann surface and a meromorphic function $\varphi$ such that the zeroes and poles of $\varphi$ are exactly the points $p_1, \ldots, p_n$. If $a_i$ is positive (resp. negative) the point $p_i$ is a zero of $\varphi$ (resp. a pole of $\varphi$) of this order. It is a consequence of basic Riemann surface theory that the sum of all the $a_i$ has to be 0. Figure 6 depicts a meromorphic function on a genus 1 curve.

Tropicalizing the curve. Just as we’ve done on multiple occasions in this article, we can consider a situation where the curve and the function $\varphi$ depend on a parameter $t$. The abstract tropicalization discussed above gives us a metric graph $\Gamma(C_t)$. By remembering the fact that the points $p_i$ played a special role, the tropicalization procedure can be tweaked to produce a slightly enhanced version of $\Gamma(C_t)$, where for each of the points $p_1, \ldots, p_n$ we attach to $\Gamma(C_t)$ a ray $\mathbb{R}_{\geq 0}$. These “legs” are very similar to legs we saw when tropicalizing plane curves at the start of our journey. I will call this $\Gamma$.

Tropicalizing the function. By blending the abstract tropicalization and the prototypical one we started with, the data $(C_t, \varphi_t)$ produces a piecewise linear function on $\Gamma$, i.e., a map

$$f : \Gamma \to \mathbb{R},$$

that is continuous and is linear on the interior of each edge. A more refined analysis tells us two more things. First, along the $j$th leg, the slope of $f$ is equal to $a_j$. If $a_j$ is positive, the value of $f$ increases to $+\infty$ as we move along the edge. If it is negative, it decreases to $-\infty$. Second, at each vertex $V$, the sum of the slopes along the edges leaving $V$ is zero. This is called the balancing condition, and comes from the fact that the weighted count of zeroes and poles meromorphic function on a Riemann surface is 0.

Figure 7 is another picture that captures what is happening.

The non-realizables exist! Noticing the tropicalizations of pairs $(C_t, \varphi_t)$ are always abstract tropical curves together with balanced piecewise linear functions, we can now contemplate such pairs $(\Gamma, f)$ without knowing beforehand that they come from algebraic geometry. I am now going to sketch an argument that some pairs $(\Gamma, f)$ of piecewise linear functions cannot come from algebraic geometry. The argument can be made fully rigorous with the right geometric formalism.
Fix a genus $g$ for the curves that we’re interested in, and the integers $a_1, \ldots, a_n$, which together must sum to 0. We’re looking for pairs $(C, p_1, \ldots, p_n, \varphi)$, where $\varphi$ is a meromorphic function with the zero and pole data at the $p_i$ given by the $a_i$. It follows from the theory of Riemann surfaces that space of possible choices for $(C, p_1, \ldots, p_n)$ has dimension $3g - 3 + n$. In fact, this was already known to Riemann; it is recorded in a very illuminating manner in [14, Section 2.3]. However, among these, only a $2g - 3 + n$ dimensional subspace admit a function $\varphi$ of the given type. One can deduce this from the Riemann–Hurwitz theorem. It turns out that the function $\varphi$, if it exists at all, is unique up to scalar—this follows from the Riemann–Roch theorem. Putting it all together, the possible set of choices has dimension $2g - 2 + n$.

Let us specialize these numbers: take the genus $g$ to be 1, the number $n$ to be 4, and let $(a_1, a_2, a_3, a_4)$ be $(2, 2, -2, -2)$. The dimension $2g - 2 + n$ now works out to be 4.

Let us now inspect the two types of graphs of piecewise linear functions in Figure 9. In each picture, the graph is fixed but we think of the lengths as being variable; the numbers indicate the slopes of the piecewise linear function. In both cases, there are ways to change the edge lengths while maintaining the conditions on the functions: they should be continuous, piecewise linear, and balanced.

In the first case, there is an interesting constraint: the top edge of the cycle needs to have length equal to a third of the length of the borrom edge for the map to be continuous. However, in the second case, the edge lengths on the two sides of the loop are completely unconstrained by the existence of the function $f$—the existence of the function does not impose conditions on the loop. Therefore in the second case, we can move the edge lengths around to find five free parameters: four from the edge lengths and one coming from translating the piecewise linear function by a constant.

The Bieri–Groves theorem told us that, for subvarieties of $(\mathbb{C}^{\ast})^n$, the tropicalization detects the dimension. The set of $(C, p_1, \ldots, p_n, \varphi)$ with the data $g$ and $(a_1, \ldots, a_n)$ fixed, together forms a natural space, which is at least to first approximation, an algebraic variety. We will call it $M$. Similarly, the set of tropical curves equipped with balanced, piecewise linear functions with matching data, forms a new subset which I will call $T$.

Let us highlight something before carrying on. To first approximation, the space $M$ can be embedded into an algebraic torus in a canonical manner and it can be tropicalized. However, the space $T$ is a purely tropical object and we have made no assertion that the latter is the tropicalization of the former! The space $T$ plays the role of the ambient vector space $\mathbb{R}^n$ in our first theorem, which contained the tropicalization. By using our prototype construction, with some technological upgrades, one finds that there is a continuous map
trop : M \rightarrow T.
It is not too hard to show that this map corresponds precisely to taking the data $(C_i, \varphi_i)$ and sending it the corresponding tropical pair.

If we believe that tropicalization preserves dimension, even in this more general context, then we have just argued that the codomain has larger dimension than the domain. Therefore the map cannot possibly be surjective. It follows that if one chooses the edge lengths of $E_1$ and $E_2$ generically, the data $(\Gamma_2, f_2)$ cannot come from algebraic geometry.

**Harder facts.** The dimension constraint above is useful for proving negatives, i.e., that there exist non-realizable tropical curves. In our argument, it’s clear where the issue comes from. A function that is linear puts a constraint on each cycle: because there are two ways to go around the cycle, if the function $\varphi$ is non-constant on the cycle, it says that the lengths of the two paths depend on each other. If the function is constant, this constraint goes away.

The following theorem says that the converse is true, and is much harder to prove.

**Theorem 0.0.3.** Let $\Gamma$ be a tropical curve, and let $\varphi$ be a piecewise linear function. Assume that every vertex of $\Gamma$ has exactly three outgoing edges. If $\varphi$ is non-constant on all cycles of $\Gamma$, then $\varphi$ is realizable.

The result above was proved, in a much more general context, by Cheung, Fantini, Park, and Ulirsch [4]. Ignoring the condition about the three outgoing edges at every vertex, it basically says that if the dimension argument in the previous section does not apply, then realizability
Two examples of abstract tropical curves $\Gamma_1$ and $\Gamma_2$ (with legs) and piecewise linear functions $f_1$ and $f_2$. The edges are labelled with the slopes of the appropriate piecewise linear function. In the picture on the right, the cycle and its two adjacent edges are collapsed, i.e., $f_2$ is constant on this subgraph.

holds. This is not at all obvious: why does the dimension argument see all the possible ways in which the realizability problem might fail? Nevertheless, the result is true, and the proof is a nice application of some abstract machinery: logarithmic deformation theory.

In a different direction, Speyer proved a beautiful result about the realizability of tropical curves of genus 1 [13]. If we specialize this theorem to our non-realizable example from before, one finds the following simple consequence. There are two contracted edges in the figure that are labelled as $E_1$ and $E_2$, adjacent to the cycle in the graph. As a result of Speyer’s theorem, we have the following fact:

The pair $(\Gamma_2, f_2)$ above is realizable if and only if $E_1$ and $E_2$ have the same length.

Speyer’s proof of this theorem required a sophisticated piece of algebraic geometry called non-archimedean uniformization for elliptic curves, developed by Tate. Many different proofs have been found in the 15 year since then, but the result does not appear to be elementary.

Brill–Noether theory for curves. We now discuss an application of these ideas involving tropicalization and its inversion to Brill–Noether theory, which we mentioned in passing early in this article. Let us first state the basic goal of Brill–Noether theory.

Understand the geometry of spaces parameterizing of maps from curves to projective space of a given type.

The word “understand” above includes things like knowing when such spaces are nonempty, calculating their basic invariants such as dimension or Euler characteristic, or determining topological properties such as connectedness or smoothness. The phrase “of a given type” means fixing numerical parameters. The questions in Brill–Noether theory involve three simple numerical parameters: the genus $g$ of the curve $C$, the dimension $r$ of the projective space $\mathbb{P}^r$, and the degree $d$ of the maps. The numbers combine to give the Brill–Noether number:

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

If we fix a curve $C$, there is a space $W^r_d(C)$ that can be thought of as a parameter space for maps from $C$ to $\mathbb{P}^r$ of degree $d$, whose image is not contained in a hyperplane. I have suppressed a few fudge factors here, related to changes of coordinates and “base points.”

We can now state the fundamental theorem in Brill–Noether theory. Recall that the space of abstract curves $\mathcal{M}_g$ of genus $g$ has dimension $3g - 3$. In what follows, we will say that a statement holds “for the general curve” if it holds for a dense subset of curves in $\mathcal{M}_g$.

The Brill–Noether theorem. If $C$ is a general curve, the space $W^r_d(C)$ has dimension exactly $\rho(g, r, d)$.

Part of the assertion is that if $\rho(g, r, d)$ is negative, the space $W^r_d(C)$ is empty. The statement has roots in the 19th century, but was first rigorously proved in the late 20th century. As we mentioned earlier, part of this statement is accessible via an impossibility constraint—when $\rho(g, r, d)$ is negative, one has to find a graph that does not admit a certain combinatorial structure.

When $\rho(g, r, d)$ is positive, one can actually use tropical inverse problems to calculate the dimension of $W^r_d(C)$, and this was done in [9]. In order to do this, one has to prove that “sufficiently many” abstract tropical curves of genus $g$ admit a map to $\mathbb{R}^r$ of genus $g$, and that the choices for maps have $\rho(g, r, d)$ dimensions worth of choices. Extracting this statement is subtle—one needs to choose the right family of graphs and check various conditions on the maps, but once the dust has settled, the task is to analyze maps

$$\Gamma \rightarrow \mathbb{R}^r$$

from genus $g$ abstract tropical curves, that are built up from the balanced piecewise linear functions we saw earlier. The key question is then: do all of these maps come from algebraic geometry?
The strategy we saw in the previous discussion for curves with a piecewise linear function works perfectly well in this multivariate setting, and one can show that Cheung, Fantini, Park, and Ulirsch’s work applies perfectly, and allows us to deduce the complete Brill–Noether theorem—proving both the non-existence of maps when \( \rho(g, r, d) \) is negative, but also proving the existence of maps when it is positive.

**Beyond general curves.** While the Brill–Noether theorem tells us precisely when maps to \( \mathbb{P}^r \) do and do not exist from general curves of a given genus, it does not settle the story. For example, for every value of the genus \( g \), there exist curves of genus \( g \) that admit a map to \( \mathbb{P}^1 \) of degree 2. Since \( \rho(g, r, d) \) is negative, this is “unexpected.” In order to get a more complete picture we must go beyond the general curve, and to do this we need a way to measure the failure of a curve to be general.

One classical way to measure that failure of a curve to be general is the gonality. The gonality of a curve \( C \) is the smallest degree of a map from that curve to \( \mathbb{P}^1 \). The gonality takes values between 2 and \( \lceil \frac{g+1}{2} \rceil \). The curves of gonality 2 are called hyperelliptic and appear in numerous contexts. Classical geometry had given us an understanding of curves of gonality 2, 3, and 4 in all genus, but the general case remained open.

In 2016, Pflueger revisited this problem from the point of view of tropical geometry [12]. For each value \( k \) of the gonality, Pflueger studied carefully-chosen tropical curves \( \Gamma^k \), one in each genus. These were known to be tropicalizations of algebraic curves of gonality \( k \). He then performed a beautiful combinatorial analysis to calculate the dimension of a space \( W^r_d(\Gamma^k) \)—a purely tropical geometric object that plays the role of \( W^r_d(C) \) above. The combinatorial analysis was done via the combinatorics of chip-firing and Young tableaux.

The result of the combinatorial analysis was a formula for the dimension of \( W^r_d(\Gamma^k) \), which we will call \( \rho_k(g, r, d) \). The definition of this number is a little complicated but similar to the \( \rho(g, r, d) \) above. Together with Jensen, we proved a \( k \)-gonal refinement of the classical Brill–Noether theorem [9].

The Brill–Noether theorem. *If \( C \) is a general curve of gonality \( k \), the space \( W^r_d(C) \) has dimension exactly \( \rho_k(g, r, d) \).*

The proof of this theorem involves a solution to a tropical inverse problem. Points in the space \( W^r_d(\Gamma^k) \) for Pflueger’s graphs \( \Gamma^k \) turn out to give rise to maps

\[
\Gamma^k \rightarrow \mathbb{R}^r,
\]

which are balanced piecewise linear functions in each coordinate—just as we saw when discussing the tropical inverse problem. However, in each coordinate, they behave more like the badly behaved examples with contracted cycles than the nicer examples where the earlier results of Cheung, Fantini, Park, and Ulirsch apply.

By combining Speyer’s theorem on realizability with results in logarithmic geometry and non-archimedean analysis that had been developed in the preceding years, we solved the relevant tropical inverse problem to show Pflueger’s analysis on the tropical side correctly predicted the dimensions of the corresponding algebraic varieties.

My goal was to convey a sense of what these tropical inverse problems look like, and what kind of information their answers can reveal. However, I should say that in the years since these results were first proved, the subject of Brill–Noether theory beyond the general curve has flourished, with many new beautiful results using a variety of different and new methods. See the survey [8] and references therein.

**References**


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Normalizing Flows Aided Variational Inference: A Useful Alternative to MCMC?

Sumegha Premchandar, Bhattacharya Shrijita, and Maiti Tapabrata

1. Introduction
A major area of contemporary statistics research is learning to model probability distributions of varying complexity. The problem of learning to characterize probability distributions broadly takes two forms: estimating a probability density given samples from it and approximating densities that are known only up to a normalizing constant. The latter avenue of research has applications in Bayesian inference, where we wish to generate samples from the posterior distribution of model parameters given observed data.

This review aims to discuss the use of normalizing flows for variational inference (VI), a method wherein we can approximate and sample from complex probability densities [RM15]. This type of probabilistic modeling lies in the second avenue of research, where we do not have a normalizing constant for probability densities of interest. VI is a tool that emerged in machine learning to...
approximate probability densities. It is often applied in Bayesian statistics as a more scalable alternative to Markov Chain Monte Carlo (MCMC) methods for large datasets. Although scalable, earlier works such as mean-field or structured VI are limited when approximating more complex, multimodal probability distributions. Normalizing flows are mappings from a simple base distribution to a more complex probability distribution. They are primarily used for modeling continuous distributions and can be used to specify very flexible probability models, thus improving the accuracy of VI algorithms.

There already exist comprehensive reviews for normalizing flow methods in general. An overview of different normalizing flow families is provided in [KPB2005], while [PNR+21] goes into depth on each family of flow models and extends this discussion to newer areas, such as flows for discrete variables. These reviews are an overarching look at flows for probabilistic modelling and are focussed on applications in the machine learning literature. Discussion of applications of a more classical statistical nature is limited. An excellent exposition and survey of VI from a probabilistic perspective is given in [BKM17]. However, they only cover variational families of a parametric nature, such as mean-field and structured VI. We extend the discussion to variational families specified by normalizing flow models. Further, our review is written for readers entirely new to the area.

In latent variable modeling, we aim to learn the conditional distribution of latent variables \( \mathbf{z} = (z_1, z_2, \ldots, z_d) \) given observed data \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \), that is, \( \pi(\mathbf{z}|\mathbf{x}) \). We explain how solving this problem is useful in Bayesian statistics. In parametric statistics, stochasticity in the observed data is often described using a specific probability distribution \( p(\mathbf{x}|\mathbf{z}) \), where \( \mathbf{z} \) needs to be estimated from the data \( \mathbf{x} \). In Bayesian inference, we assume a prior distribution \( \pi(\mathbf{z}) \) on \( \mathbf{z} \) representing our beliefs about the model parameter prior to observing the data. Based on the data, we update our beliefs via the posterior distribution \( \pi(\mathbf{z}|\mathbf{x}) \). The posterior can be calculated by Bayes theorem:

\[
\pi(\mathbf{z}|\mathbf{x}) = \frac{p(\mathbf{x}|\mathbf{z})\pi(\mathbf{z})}{\int_{\mathbf{z}} p(\mathbf{x}|\mathbf{z})\pi(\mathbf{z}) d\mathbf{z}}.
\]

For cases where the marginal likelihood \( m(\mathbf{x}) = \int_{\mathbf{z}} p(\mathbf{x}|\mathbf{z})\pi(\mathbf{z}) d\mathbf{z} \) is intractable we resort to approximate inference. MCMC methods have long been the go-to for sampling from posterior distributions when \( m(\mathbf{x}) \) cannot be computed. MCMC algorithms generate samples from a Markov Chain whose stationary distribution converges to the target distribution of interest. One prominent example is the Metropolis-Hastings method [CG95], of which the

\[\text{Gibbs sampling algorithm [CG92] is a special case. However, these methods may not always scale well to high-dimensional models and can be slow to converge for multimodal distributions. VI has shown promise as a scalable alternative to MCMC. In VI, the target distribution is approximated by a family of distributions } Q \text{ among which we choose the optimal distribution } q^* \text{ to be “closest” to the target. To determine “closeness,” KL-divergence is often used. Intuitively, KL-divergence is something akin to a distance between 2 probability distributions. Thus, probabilistic modeling with VI becomes an optimization problem:}

\[q^* \in \arg \min_{q \in Q} KL(q||\pi(.|\mathbf{x})).\]

\[\text{Mean-field VI (MF-VI) is a popular approach in which the variational family } Q \text{ is defined based on the assumption that latent variables are independent. The mean-field assumption is useful for faster computations during optimization but is restricted in the complexity of densities we can approximate. Structured VI takes this one step further by allowing dependencies across latent variables. However, even with Structured VI we cannot guarantee that we can approximate any density arbitrarily well. This is where normalizing flows come in.}

When should we use normalizing flows VI? In [BKM17], the authors observe that “VI is suited to large data sets and scenarios where we want to quickly explore many models; MCMC is suited to smaller data sets and scenarios where we happily pay a higher computational cost for more precise samples.” While this is generally true of MF-VI, normalizing flows VI lies somewhere between MCMC and other variational approximation approaches in terms of computational efficiency and accuracy. To shed some light on how normalizing flows VI compares to other sampling methods such as MCMC and MF-VI, we implement variational inference with neural autoregressive flows [HLC18] for several examples. These examples cover classical Bayesian statistical applications in exponential family models, Gaussian linear regression and logistic regression. We cover scenarios of varying dimensions and complexity of the target distribution. This gives us a high-level idea of scalability vs. accuracy for these methods but is by no means a rigorous treatment of the topic.

We begin the following section by introducing normalizing flows and elaborate on how to use them for VI. We then proceed to examples in Section 3. Finally, we discuss some important takeaways & challenges remaining in the area in Section 4.

2. Normalizing Flows

The main idea behind normalizing flows is to transform some simple base distribution on a continuous support into a “target” distribution that is usually more complex, via a series of bijective, differentiable transformations (diffeomorphisms).
Let \( Z \in \mathbb{R}^d \) be a random variable whose density we wish to model. We begin with a random variable \( U \) sampled from some base distribution \( p_U(u) \) also defined on support \( \mathbb{R}^d \) and apply a diffeomorphism \( T : \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that \( Z = T(U) \). The density of \( Z \) is then given by the change of variable formula:

\[
p_Z(z) = p_U(u)|J_T(u)|^{-1}.
\]

\( |J_T(u)| \) denotes determinant of Jacobian of \( T \) w.r.t \( u \). Thus, the function \( T \) transforms the density \( p_U(u) \) into \( p_Z(z) \). This process, wherein samples from one probability den-

viates to be diffeomorphisms, the change of variables frame “flow” through a mapping to obtain another density is called a normalizing flow. A natural question to ask is whether normalizing flows can be used to transform a simple base distribution (e.g., uniform or standard normal distribution) into any target distribution. [PNR+21] contains a constructive argument to show that normalizing flows can indeed recover any target density under rather general conditions. In practice, this is heavily dependent on the transformations \( T \) that we employ.

Discrete and continuous-time flows. Normalizing flows are mainly of two types—discrete time (finite flows) and continuous time (infinitesimal flows) [PNR+21]. Discrete-time normalizing flows are constructed by choosing a finite sequence of transformations \( T_1, T_2, \ldots, T_K \) and applying them successively to some base distribution \( p_U(u) \) such that \( z_K = T_K \circ T_{K-1} \circ \cdots \circ T_1(u) \). Since we choose all transformations to be diffeomorphisms, the change of variables formula applies and we have:

\[
p_Z(z_K) = p_U(u) \times |J_{T_K}(z_{K-1})|^{-1} \times |J_{T_{K-1}}(z_{K-2})|^{-1} \cdots |J_{T_1}(u)|^{-1}.
\]

The number of transformations \( K \), is often called the flow depth. Increasing flow depth can help us model progressively more complex densities at the expense of increased computational cost due to the calculation of the determinant for Jacobian matrices \( J_{T_k}(.) \).

We can think of discrete time flows as modelling the evolution of a probability density at \( K \)-many time points. In contrast, continuous-time normalizing flows model this evolution continuously from some time \( t = 0 \) to \( T \) as an ordinary differential equation \( \frac{dz}{dt} = f(t, z) \). A well-known example of a continuous time flow is the Hamiltonian flow, which is used for MCMC sampling [Nea11].

2.1. Normalizing flows for variational inference. We now expand on how normalizing flows are used to aid VI. As before, let \( z = z_{1:d} \) be the latent variables, \( x = x_{1:n} \) be the observed data and \( \pi(z|x) \) be the conditional distribution we wish to sample from:

\[
\pi(z|x) = \frac{p(x|z)\pi(z)}{m(x)}.
\]

VI approximates the target density by choosing a family of distributions \( Q = \{q_\phi | \phi \in \Phi \} \) and selecting the optimal distribution in this family \( q_\phi \) “closest” to the target density in terms of KL-divergence:

\[
q_\phi^* = \arg\min_{q_\phi \in Q} KL(q_\phi||\pi(\cdot|x)).
\] (1)

Other metrics such as more generalized \( \alpha \)-divergence measures [LT16] can be used in place of KL-divergence. However, KL-divergence is popular due to its versatility and relative ease of implementation. The optimization in (1) is difficult to work with due to the presence of the intractable marginal likelihood \( m(x) \). In practice, we maximize the evidence lower bound (ELBO) with respect to the variational parameters \( \phi \) due to its equivalence to (1). The ELBO is the negative KL-divergence between the variational distribution \( q \) and the joint distribution \( p(x, z) \) of latent variables and observed data:

\[
\max_{q_\phi \in Q} ELBO(q_\phi, \pi(\cdot|x))
\]

\[
E_{q_\phi(z)} h(z) = E_{q_\phi(z)} h(T_K \circ T_{K-1} \circ \cdots \circ T_1(z_0)).
\]

(4)

(3) follows from the change of variable formula and (4) is a well-known property of expectation. We simplify the maximization of the ELBO in (2):

\[
\max_{q_\phi \in Q} E_{q_\phi(z)} \left[ \ln p(x, z, \pi(z) - \ln q_\phi(z) \right] \]

\[
= \max_{q_\phi \in Q} \left[ E_{q_\phi(z)} \left[ \ln p(x|z)\pi(z) - \ln q_\phi(z) \right] \right] \]

\[
+ E_{q_\phi(z)} \left[ \sum_{k=1}^{K} \ln \left| \frac{\partial T_k}{\partial z_{k-1}} \right| - E_{q_\phi(z)} \left[ \ln q_0(z_0) \right] \right] \]

\[
= \max_{q_\phi \in Q} \left[ E_{q_\phi(z)} \left[ \ln p(x|z)\pi(z) - \ln q_\phi(z) \right] \right] \]

\[
+ E_{q_\phi(z)} \left[ \sum_{k=1}^{K} \ln \left| \frac{\partial T_k}{\partial z_{k-1}} \right| \right].
\] (6)

Equations (3) and (4) jointly imply (5). We are essentially re-parametrizing the expectation in terms of the base distribution \( q_0 \). In (6), we are able to drop \( E_{q_\phi(z)} \left[ \ln q_0(z_0) \right] \) because it is free of the parameter \( \phi \). In practice, optimizing over \( q_\phi \in Q \) effectively becomes optimizing over the parameters \( \phi \) of transformations \( (T_k)_{k=1}^{K} \). \( \phi \) are referred
to as flow parameters. In general, for $d$-dimensional latent variables $z$, calculating the determinant of Jacobian $J_{T_k}(z_{k-1})$ takes $O(d^3)$ time [PNR+21]. Therefore, in addition to $T_1, T_2, ... , T_k$ being diffeomorphisms, they are often selected such that computational complexity of calculating $J_{T_k}(z_{k-1})$ is $O(d)$.

There are a myriad of ways in which we can choose the normalizing flow transformations. Intuitively, if we choose $T_k$ to be deep neural networks we should be able to approximate almost any well-behaved function. But how do we ensure computational feasibility? Neural autoregressive flows (NAF) [HLCK18] manage to achieve this balance. NAF satisfy the “Universal approximation property.” This means that they can approximate any probability distribution within an arbitrarily small error margin provided the flow depth $K$ is large enough. Further, the autoregressive structure of these flows ensures the Jacobian determinants can be computed in $O(d)$ time. Note that this is just one among many families of normalizing flows. Given these nice properties we choose to use NAF for our examples in Section 3.

2.1.1. Neural autoregressive flows. Autoregressive flows are among the most popular normalizing flows discussed in the literature. We discuss some of the principals behind autoregressive normalizing flows. We concentrate on describing NAF since we use these for the examples in which we contrast normalizing flows, MCMC and MF-VI.

Continuing with the same notation, we denote the input from the base distribution by $u = u_{1:d}$ and transformed latent variable by $z = z_{1:d}$. Autoregressive flows are constructed such that each transformed variable $z_i, 1 \leq i \leq d$ is dependent only on the first $i$ inputs $u_{1:i}$. More specifically, the transformer $T = (t_1, t_2, ... , t_d)$ is made up of $d$ many diffeomorphisms such that:

$$z_i = t_i(u_i, c_i(u_{1:i-1})) \quad 2 \leq i \leq d.$$

$t_i$ is parameterized by the vector $c_i(u_{1:i-1})$. The function $c_i : \mathbb{R}^{i-1} \rightarrow \mathbb{R}^m$ is referred to as conditioner and it enforces the autoregressive property for the normalizing flow (see Figure 1). As the name suggests, NAF uses a neural network for $t_i$. The 2 types of transformations used are:

1. Deep Sigmoidal Flow (DSF) - This neural network uses a single hidden layer.
2. Dense Deep Sigmoidal Flow (DDSF) - This uses a deep neural network.

For readers new to the topic, think of a neural network as a somewhat complex function that takes some inputs and applies a series of operations and transformations to them. They generally involve multiplication of inputs with weight matrices, translation, and the application of certain “activation” functions. The DSF network is formally defined as:

$$z_i = \sigma^{-1}(w_i^\top \sigma(a_i,u_i + b_i)) \quad a_i, w_i, b_i \in \mathbb{R}^k \quad 1 \leq i \leq d.$$ 

Here $k$ is the number of nodes in the hidden layer and $\sigma(x) = 1/(1 + e^{-x})$ is an activation function. $a_i$ and $w_i$ are constrained as $a_{i,j} > 0 \forall i, j$, $0 < w_{i,j} < 1$, $\sum_j w_{i,j} = 1$. This ensures invertibility of $t_i$ [HLCK18]. The DDSF transformation has the capacity to be more expressive than DSF due to the universal approximation properties of deep neural networks, albeit at an increased computational cost.

Until now we have discussed the choice of the transformers $t_i$ for NAF. To construct the conditioner there are no constraints such as invertibility on the functions $c_i, 1 \leq i \leq d$. A natural choice is to use a neural network for the conditioner as well. However, using a distinct neural network for each $c_i$ is computationally infeasible as $d$ increases. This is because we have to store and optimize over $d$ networks each with different parameters. [PNR+21] discusses a range of conditioners that leverage parameter sharing across $c_i$. Following [HLCK18], we adopt the popular masked conditioner approach. Masked conditioners take $u_{1:d}$ as inputs to a neural network and calculate all the parameters $c_i(c_i(u_{1:i-1}))^d_{i=2}$ for the transformers in a single forward pass. For a network with a single hidden layer, the autoregressive dependency structure is enforced by multiplying the weight matrices $\mathcal{W}_1$ & $\mathcal{W}_2$ by masking matrices $\mathcal{M}_1, \mathcal{M}_2$ of the same dimension. $\mathcal{M}_1, \mathcal{M}_2$ consist of binary $1 - 0$ entries such that a 0 entry in $\mathcal{M}_i$ implies the corresponding weighted connection is dropped from the network. Therefore entries in $\mathcal{M}_1, \mathcal{M}_2$ are chosen such that there is no connection between the $i$th input $u_i$ and 1, 2..., $(mi)$ outputs of the network. Here $m$ is a multiplier which tells us how many parameters are required for each $t_i$. The weight matrices $\mathcal{W}_1 \in \mathbb{R}^{d \times k}$ and $\mathcal{W}_2 \in \mathbb{R}^{k \times md}$ correspond to the hidden and output layer respectively for the conditioner network. See Figure 1 in [GGML15].
2.1.2. Implementation. Recall that for normalizing flows aided variational inference (FAVI) we maximize the ELBO:

$$
\mathcal{L}(q_\phi) = E_{q_0(\mathbf{z}_0)}[\ln p(\mathbf{x}, \mathbf{z}_K)] + E_{q_0(\mathbf{z}_0)}[\sum_{k=1}^K \ln |\frac{\partial T_k}{\partial \mathbf{z}_{k-1}}|].
$$

$(T_K)_k=1^K$ and $\mathbf{z}_K$ depend on $\phi$ in the equation above. In general, $\mathcal{L}(q_\phi)$ will not have a closed form expression. Additionally, standard coordinate-wise gradient ascent algorithms are computationally inefficient for large datasets. As a result, the Stochastic Gradient Ascent (SGA) algorithm is often used for optimizing the ELBO.

SGA is an iterative method that uses the following update for the flow parameters $\phi$ at step $t$:

$$
\phi_{t+1} = \phi_t + \alpha_t l(\phi_t).
$$

Here $l(\phi)$ is a realization for an unbiased estimator of $\nabla_\phi \mathcal{L}(q_\phi)$, the gradient for the ELBO. We can calculate it by sampling $\mathbf{z}_0^{(1)}, \mathbf{z}_0^{(2)}, \ldots, \mathbf{z}_0^{(S)}$ from $q_0(.)$ and passing them through the transformations $T_1, T_2, \ldots, T_K$ to get $\mathbf{z}_K^{(1)}, \mathbf{z}_K^{(2)}, \ldots, \mathbf{z}_K^{(S)}$.

$$
l(\phi) = \frac{1}{S} \sum_{s=1}^S \left[ \nabla_\phi \ln p(\mathbf{x}, \mathbf{z}_K^{(s)}) + \sum_{k=1}^K \nabla_\phi \ln |\frac{\partial T_k}{\partial \mathbf{z}_{k-1}}| \right].
$$

SGA almost surely converges to a local minimum for non-convex functions and global minimum for pseudo-convex functions when learning rates satisfy $\sum_{t=1}^\infty \alpha_t = \infty$ & $\sum_{t=1}^\infty \alpha_t^2 < \infty$. ([RM51], [Bot98]).

In practice, choosing the learning rate $\alpha_t$ is nontrivial. When $\alpha_t$ is too large we may overshoot the maxima and when $\alpha_t$ is too small then SGA will learn too slowly. For our experiments, we use the Adam algorithm [KB14] which uses an adaptive learning rate that incorporates information about the scale of different components in the parameter vector $\phi$. We use a standard normal distribution for $q_0(\mathbf{z}_0)$.

Note that, the outputs of the normalizing flow transformations $\mathbf{z}_k$ are unconstrained, i.e., they belong to $\mathbb{R}^d$, since they are outputs of a neural network. Sometimes the latent variable space is restricted, for example, our model may have a variance parameter $\sigma^2 > 0$. More formally, when $z_i \in S \subset \mathbb{R}$ we apply a final transformation $T_{K+1}$ to constrain $\mathbf{z}_{K,i}$. For instance, if $z_i > 0$ then we set $\mathbf{z}_{K,i} = \ln(1 + e^{2K,i})$.

3. Illustrative Examples

Here, we implement the FAVI algorithm on some examples. We also provide comparisons to MCMC and MF-VI where applicable. Note that MCMC comprises a wide class of algorithms ranging from the more basic Random Walk Metropolis Hastings (RW-MH) and Gibbs sampling methods to approaches that make use of gradient information such as the Hamiltonian Monte-Carlo (HMC) [Neal11].

We use either the RW-MH or Gibbs sampling methods as a baseline since these are widely used in classical applications of Bayesian inference. See Section 4 for a detailed discussion of contemporary MCMC literature. Section 3.1 discusses FAVI for exponential family models, followed by 3.2 in which we sample from un-normalized energy density functions. We then move onto Bayesian linear and logistic regression in 3.3 and 3.4. Through these examples we hope to elucidate how FAVI works in different contexts.

3.1. The exponential family. In many applications of Bayesian inference the complete conditionals $p(z_i|\mathbf{z}_{-i}, \mathbf{x})$ $1 \leq i \leq d$ of latent variables belong to the exponential family $p = \frac{h(\mathbf{z})}{A(\eta)} \exp \eta l(\mathbf{z})$. This class of models is known as conditionally conjugate exponential family models and its broad applicability makes it of interest to statistical
practitioners. [BKM17] discusses the derivation for coordinate ascent variational inference (CAVI), an MF-VI method, for this class of models. It is natural to extend this discussion to the FAVI algorithm for exponential family models.

FAVI performing well on low-dimensional examples is a necessary but not sufficient condition for them to be reliable for high-dimensional VI problems. This motivates our choice of examples:

1. \( (y_i)_{i=1}^n \mid p \sim \text{Bernoulli}(p) \) \( p \sim \text{Beta}(a, b) \)
2. \( (y_i)_{i=1}^n \mid \mu, \sigma^2 \sim N(\mu, \sigma^2) \), \( p(\mu, \sigma^2) : \mu \sim N(0, \tau^2) \) \( \perp \perp \sigma^2 \sim \text{Inv-Gamma}(v_1,v_2) \)

In the first case the posterior has a closed form with which we can compare the density obtained by flows.

\[ p(y_{1:n} \mid \theta) \sim \text{Beta}(a + n\bar{y}, b + (n - n\bar{y})) \]

In the second example, we compare with results obtained by Gibbs sampling. We also include results from MF-VI.

We see from Figure 3 that FAVI, MF-VI, and Gibbs produce similar results. Figure 4 is a demonstration of how the density obtained from FAVI converges to the true distribution over the training epochs. From the plot of ELBO against epochs we see that at around the 20th epoch there is a plateau. This indicates the density from FAVI has changed shape and is approaching the true distribution.

3.2. Sampling from multimodal densities. The primary advantage of normalizing flows is their ability to recover highly multimodal target distributions with complex dependencies. In [HLCK18], authors use NAF to sample from multimodal energy density functions, for which the normalizing constant is unknown. They do not however, provide a comparison to MCMC methods. Given that MCMC methods are theoretically guaranteed to converge to the target distribution of interest, we believe it would be useful to include this comparison. We contrast both accuracy and computational time for NAF and the RW-MH Algorithm, for sampling from the energy density functions \( U_1 - U_9 \) (see Figure 5).

We compare the two methods based on run-time and kernel density estimates (k.d.e). Run-time is measured from the first iteration for the algorithm till convergence. For comparing the densities generated by both methods we calculate kernel density estimates (k.d.e) from the samples. We use a Gaussian kernel, on a grid of size 200 x 200. We then calculate square root of sum of squares error (\( \sqrt{\text{SSE}} \)) for k.d.e over the grid as

\[ \sqrt{\sum_{i=1}^{N=40,000} (\hat{f}(x_i) - f_{\text{True}}(x_i))^2} \]

Here, \( \hat{f}(\cdot) \) is the kernel density estimator obtained from either the FAVI/RW-MH algorithm and \( f_{\text{True}}(\cdot) \) is the true density.\(^2\) This is equivalent to the Frobenius norm of errors between true and estimated density on our grid.

Determining convergence. For assessing convergence of the random-walk Metropolis Hastings we visually inspect the autocorrelation and trace plots. The plots for many energy functions display non-negligible autocorrelation upto lag 40, therefore we thin the samples by 40 and run the chain for 400,000 samples. We choose this run-time in order to obtain a sufficient sample size of 10k to get richer kernel density plots. We run FAVI for 15k epochs based on stabilization of the loss function and also generate 10k samples after training. For both, the RW-MH and FAVI algorithms there is a degree of subjectivity to determining convergence since we use visual inspection. Empirical convergence criteria such as trace plots, \( \hat{R} \) [GR92] and zero autocorrelation in the samples does not guarantee convergence of the Markov chain. Although satisfying these criteria is not sufficient for convergence, it is necessary for us to gain confidence that the Markov chain is approaching the stationary distribution.

\(^2\)Since we do not have the closed form for the true density we normalize the energy functions using numerical integration from Scipy’s integrate module.
Results. Table 1 reports the average $\sqrt{\text{SSE}}$ of k.d.e for both FAVI and RW-MH algorithms across the best three of five trials (based on loss) ± standard deviation. We chose the best three because the loss does not converge in all cases for the FAVI algorithm, due to sensitivity to choice of initialization and the presence of local minima. We observe that the RW-MH algorithm outperforms FAVI in terms of k.d.e metrics by a small–medium margin with one exception, $U_6$. We also see from the standard deviations that the RW-MH algorithm is more stable than FAVI, which is highly sensitive to the initialization of flow parameters $\phi$. For instance, $U_5$ has a standard deviation of $\sqrt{\text{SSE}}$ 0.95 for FAVI and only 0.01 for RW-MH. In terms of computational time, approximately half of the energy functions have run-time of a similar order for the FAVI and RW-MH algorithms. For the remaining functions we observe:

1. For $U_3$ and $U_5$ the RW-MH algorithm takes approximately 60% of the run-time that FAVI does.
2. For $U_4$ and $U_9$ the trend is reversed and FAVI takes only 30% of the RW-MH algorithm run-time.

Upon closer examination of the function forms we see that $U_3$ and $U_5$ are relatively simple functions to evaluate over a particular sample whereas $U_4$ and $U_9$ are complex functions, being a mixture of multiple densities. $U_4$ is a mixture Gaussian density with four components and $U_9$ is a mixture of $U_3$ and part of $U_8$. Thus, unlike FAVI, the RW-MH algorithm does not scale as complexity of the target density increases.

![Figure 5. True density.](image)

3.3. Linear regression. In this section we implement the FAVI algorithm on a Bayesian linear regression example to sample from the posterior of regression parameters given the data, $\pi(\beta, \sigma^2 | D)$. We use the framework below:

$$y_i \sim x_i^T \beta + \epsilon_i, \quad \beta \in \mathbb{R}^p, \quad \epsilon_i \overset{i.i.d}{\sim} N(0, \sigma^2) \quad 1 \leq i \leq n$$

$$\pi(\beta, \sigma^2) : \beta \sim N(0, \tau^2 I_p) \quad \perp \quad \sigma^2 \sim \text{Inv-Gamma}(a, b).$$

![Figure 6. Energy density functions $U_1 - U_9$.](image)

| Table 1. Avg. $\sqrt{\text{SSE}}$ ± s.d. of k.d.e. for $U_1 - U_9$ (smaller values are better). Avg. algorithm run-time in seconds ± s.d. for $U_1 - U_9$ (smaller values are better). |
|------------------------|------------------------|------------------------|------------------------|------------------------|
|                        | FAVI   | RW-MH   | FAVI   | RW-MH   |
| $U_1$                  | 0.98 ± 0.05 | 0.94 ± 0.01 | 114 ± 1 | 119 ± 4 |
| $U_2$                  | 0.46 ± 0.03 | 0.42 ± 0.01 | 118 ± 7 | 173 ± 31 |
| $U_3$                  | 0.20 ± 0.01 | 0.18 ± 0.01 | 109 ± 3 | 68 ± 8 |
| $U_4$                  | 0.62 ± 0.04 | 0.56 ± 0.02 | 122 ± 1 | 465 ± 84 |
| $U_5$                  | 1.72 ± 0.95 | 1.17 ± 0.01 | 111 ± 2 | 69 ± 4 |
| $U_6$                  | 1.21 ± 0.03 | 1.25 ± 0.00 | 145 ± 15 | 212 ± 40 |
| $U_7$                  | 1.07 ± 0.01 | 1.07 ± 0.02 | 122 ± 7 | 166 ± 8 |
| $U_8$                  | 1.11 ± 0.04 | 1.10 ± 0.00 | 127 ± 12 | 251 ± 13 |
| $U_9$                  | 1.25 ± 0.04 | 1.15 ± 0.01 | 131 ± 2 | 303 ± 45 |

Although not standard practice, we report results across different initializations to contrast the stability across FAVI and MCMC. Further, run-time varies across trials and averaging gives us a better idea of the true run-time.
We compare FAVI to both MF-VI and the Gibbs sampling algorithm. We use Gibbs sampling because it relies on the complete conditionals for latent variables $\pi(\beta|y_{1:n})$ and $\pi(\sigma^2|y_{1:n},\beta)$ which are easily available in this case. Through this example, we can gain some insight on the scalability and accuracy contrast between the three methods in a classical statistical setup. To assess effect of both sample size and dimensionality on the performance of these methods we use a grid of $(n, p)$ combinations. We allow $n$ (sample size) to take values 50, 100, and 200, while $p$ ($\beta$ dimension) takes values 2, 20, 50, and 100.

For our experiments, we simulate the true data generating $\beta_0$ from the Uniform($\frac{1}{2}, 2$) distribution. We assume $\sigma_0 = \tau = 1$ where $\sigma_0$ is the true value of model parameter $\sigma$.

The $p$-dimensional predictor variables $x_1, x_2, \ldots, x_n \in \mathbb{R}^p$ are simulated from a multivariate normal distribution $N(0, I_p)$. For the cases where $p \geq 20$, we set only 20% of the variables to be non-zero in order to ensure the latent variable space is sparse, that is, $K \ll p$ where $K$ is the number of non-zero components in $\beta_0$.

Similar to Section 3.2, convergence of the Gibbs sampling algorithm is determined by a combination of trace and autocorrelation plots. We thin the samples by a factor of 10 to ensure 0 autocorrelation. We initialize $\beta$ with its O.L.S estimate for faster convergence. Convergence of FAVI and MF-VI is ascertained via stabilization of the loss function.

Results. In order to visualize the difference between densities approximated by the three approaches (FAVI, MF-VI and Gibbs) we use kernel density plots. For the case where $p = 2$, we can easily visualize the posterior distributions of $\beta$ and $\sigma^2$. For higher-dimensional examples we use the kernel density plots for SSE of $\beta$; $g(\beta) = ||\beta - \beta_0||^2_2$ where $\beta$ is sampled from the posterior $\pi(\beta|y_{1:n})$. We present density plots for $n = 100$ and varying $p$ in Figure 8. We report the model predictive root mean squared error ($\sqrt{\text{MSE}}$) on test data $\sqrt{\sum_{i=1}^{n_{test}}(y_i - \hat{y}_i)^2/n_{test}}$. Here $\hat{y}_i = x_i^T \hat{\beta}$ is the predicted value for the $i^{th}$ sample based on mean of the posterior samples $\hat{\beta} = 1/N \sum_{n=1}^{N} \hat{\beta}_n$. Here $N$ is the number of $\beta$ samples generated and is set to be 10k. To get a sense of variance of the posterior distribution for $\beta$ we also report $\overline{s_\beta}$. This is obtained by first computing sample standard deviation from posterior samples for each $\pi(\beta_i|y_{1:n})$ as $s_{\beta_i} = (1/(N - 1)) \sum_{n=1}^{N} (\beta_i^n - \hat{\beta}_i)^2$ for $1 \leq i \leq p$. We then aggregate these by averaging across dimensions as $\overline{s_\beta} = (1/p) \sum_{i=1}^{p} s_{\beta_i}$. By reporting $\sqrt{\text{MSE}}$ and $\overline{s_\beta}$ we are able to capture model predictive performance and uncertainty quantification for the three methods.

We observe from density plots that for cases where $p = 2$ the difference across 3 algorithms is insubstantial (Figure 7). As dimension $p$ increases we see that the FAVI results are reasonably close to Gibbs sampling (the gold standard) but MF-VI produces a peaked distribution, indicating it under estimates uncertainty in the samples (Figure 8). Performance metrics in Table 2 show that all 3 methods have similar predictive performance. For uncertainty quantification, $\overline{s_\beta}$ shows that MF-VI has lower posterior variance in general than the other two methods, while FAVI is comparable to Gibbs. There are 2 notable exceptions when $n = 100, p = 100$ and $n = 50, p = 100$. In these cases MF-VI has larger $\overline{s_\beta}$ by 0.01 points than FAVI but this difference is insubstantial.

For the 3 cases where $p \geq n$, the Gibbs sampling algorithm breaks down due to the instability of the following matrix inversion: $(I_p/\tau^2 + X^TX/\sigma^2)^{-1}$. Since FAVI, like MF-VI, does not require the matrix inversion step and can specify an arbitrarily flexible family of densities, it could be a promising alternative to Gibbs sampling in such a set-up.

Table 3 reports average algorithm run-times ± s.d. across 5 trials. In general, MF-VI runs the fastest, followed by Gibbs sampling and then FAVI. However, for the highlighted cases of $p = 50$ and $n \geq 100$ this trend switches and Gibbs sampling becomes slower than FAVI. Given that Gibbs sampling breaks down in high dimension, it is difficult to discern a pattern and contrast scalability of the two methods.

---

4 There are modifications of Gibbs sampling that can circumvent this and a more thorough exploration is required.
We consider the following model:

\[ y_i \sim \text{Bernoulli}(p_i), \quad p_i = \frac{\exp(x_i^\top \beta)}{1 + \exp(x_i^\top \beta)} \]

\[ \pi(\beta) : \beta \sim N(0, \tau^2 I_p). \]

We use the same simulation set-up for \( \beta \) and \( x \) as of Section 3.3 on linear regression. Here we use RW-MH instead of Gibbs sampling because we no longer have closed form complete conditionals. Maintaining consistency with previous experiments, we use trace and autocorrelation plots to decide on convergence.\(^5\) We initialize the RW-MH algorithm with the maximum likelihood estimates for \( \beta \). We use plots and metrics as in 3.3, replacing \( \sqrt{\text{MSE}} \) by Accuracy.

Results. Density plots for \( \beta \) when \( n = 100, p = 2 \) are presented in Figure 9. From the contour plot we see that MF-VI does not capture the elliptical structure of the joint distribution of \( \beta \) which the other two methods display. For higher dimensions, kernel density plots of \( \beta \) SSE, \( \|\beta - \beta_0\|^2_2 \) when \( \beta \sim \pi(\beta|y_{1:n}) \) are presented in Figure 10. Similar to the trend displayed by Gaussian linear regression, we see that kernel density plots for MF-VI seem to center on a lower SSE. The FAVI and RW-MH algorithms perform similarly with respect to uncertainty quantification as dimension \( p \) increases but MF-VI has lower aggregate posterior variance for \( \beta \) (See Table 4).

All three methods display identical model predictive Accuracy given by

\[ \text{Accuracy} = \frac{\text{Correct Predictions}}{\text{Total Predictions}} \]

\( \sqrt{\text{MSE}} \) is sampled from \( \pi(\beta|y_{1:n}) \).

Table 2. Linear regression: Model predicted \( \sqrt{\text{MSE}} \) (smaller values are better). \( \text{Avg. s.d. of } \beta \) samples. We use * when the result can’t be computed.

<table>
<thead>
<tr>
<th>(n, p)</th>
<th>Pred ( \sqrt{\text{MSE}} )</th>
<th>Avg. ( \beta ) s.d. ( \bar{\beta} )</th>
<th>Gibbs</th>
<th>FAVI</th>
<th>MF-VI</th>
<th>Gibbs</th>
<th>FAVI</th>
<th>MF-VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 2)</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>0.17</td>
<td>0.17</td>
<td>0.16</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>(50, 20)</td>
<td>0.88</td>
<td>0.92</td>
<td>0.88</td>
<td>0.24</td>
<td>0.23</td>
<td>0.17</td>
<td>0.24</td>
<td>0.23</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>*</td>
<td>3.06</td>
<td>3.41</td>
<td>*</td>
<td>0.47</td>
<td>0.46</td>
<td>*</td>
<td>0.47</td>
</tr>
<tr>
<td>(50, 100)</td>
<td>2.93</td>
<td>2.93</td>
<td>2.64</td>
<td>*</td>
<td>0.77</td>
<td>0.78</td>
<td>*</td>
<td>0.77</td>
</tr>
<tr>
<td>(100, 2)</td>
<td>1.04</td>
<td>1.04</td>
<td>1.04</td>
<td>0.11</td>
<td>0.11</td>
<td>0.11</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td>(100, 20)</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
<td>0.14</td>
<td>0.14</td>
<td>0.12</td>
<td>0.14</td>
<td>0.12</td>
</tr>
<tr>
<td>(100, 50)</td>
<td>1.93</td>
<td>1.90</td>
<td>1.93</td>
<td>0.17</td>
<td>0.16</td>
<td>0.11</td>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>(100, 100)</td>
<td>*</td>
<td>2.64</td>
<td>2.92</td>
<td>*</td>
<td>0.46</td>
<td>0.47</td>
<td>*</td>
<td>0.46</td>
</tr>
<tr>
<td>(200, 2)</td>
<td>1.07</td>
<td>1.07</td>
<td>1.07</td>
<td>0.08</td>
<td>0.09</td>
<td>0.08</td>
<td>0.08</td>
<td>0.09</td>
</tr>
<tr>
<td>(200, 20)</td>
<td>1.10</td>
<td>1.09</td>
<td>1.10</td>
<td>0.09</td>
<td>0.09</td>
<td>0.08</td>
<td>0.09</td>
<td>0.09</td>
</tr>
<tr>
<td>(200, 50)</td>
<td>1.26</td>
<td>1.26</td>
<td>1.26</td>
<td>0.10</td>
<td>0.10</td>
<td>0.08</td>
<td>0.10</td>
<td>0.08</td>
</tr>
<tr>
<td>(200, 100)</td>
<td>1.86</td>
<td>1.87</td>
<td>1.86</td>
<td>0.14</td>
<td>0.13</td>
<td>0.09</td>
<td>0.14</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Table 3. Linear regression: Avg. algorithm run-time ± s.d. over five trials (smaller values are better). We use * when the result can’t be computed.

<table>
<thead>
<tr>
<th>(n, p)</th>
<th>Gibbs</th>
<th>FAVI</th>
<th>MF-VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 2)</td>
<td>55 ± 0</td>
<td>252 ± 8</td>
<td>22 ± 1</td>
</tr>
<tr>
<td>(50, 20)</td>
<td>79 ± 5</td>
<td>374 ± 21</td>
<td>23 ± 1</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>*</td>
<td>606 ± 41</td>
<td>122 ± 10</td>
</tr>
<tr>
<td>(50, 100)</td>
<td>*</td>
<td>1055 ± 84</td>
<td>922 ± 69</td>
</tr>
<tr>
<td>(100, 2)</td>
<td>59 ± 4</td>
<td>267 ± 10</td>
<td>24 ± 2</td>
</tr>
<tr>
<td>(100, 20)</td>
<td>85 ± 5</td>
<td>383 ± 24</td>
<td>24 ± 1</td>
</tr>
<tr>
<td>(100, 50)</td>
<td>953 ± 54</td>
<td>549 ± 23</td>
<td>36 ± 2</td>
</tr>
<tr>
<td>(100, 100)</td>
<td>*</td>
<td>1087 ± 56</td>
<td>778 ± 80</td>
</tr>
<tr>
<td>(200, 2)</td>
<td>57 ± 3</td>
<td>254 ± 13</td>
<td>23 ± 1</td>
</tr>
<tr>
<td>(200, 20)</td>
<td>80 ± 0</td>
<td>568 ± 10</td>
<td>23 ± 0</td>
</tr>
<tr>
<td>(200, 50)</td>
<td>997 ± 110</td>
<td>567 ± 31</td>
<td>24 ± 2</td>
</tr>
<tr>
<td>(200, 100)</td>
<td>833 ± 37</td>
<td>101 ± 28</td>
<td>35 ± 1</td>
</tr>
</tbody>
</table>

\( ^5 \)For most cases the autocorrelation after thinning is between 0.0 – 0.2, however when \( p = 100 \) we allow autocorrelation of 0.4 given computational considerations.
Figure 9. Logistic regression: \( \pi(\beta_1, \beta_2 | y_1:n) \) when \( n = 100, p = 2 \).

Figure 10. Logistic regression: Density plots of \( \| \beta - \beta_0 \|_2^2 \) where \( \beta \) is sampled from \( \sim \pi(\beta | y_1:n) \).

Table 4. Logistic regression: Model accuracy (larger values are better). | Avg. s.d. of \( \beta \) samples. Here RW-MH is abbreviated as MH.

<table>
<thead>
<tr>
<th>( n, p )</th>
<th>Accuracy</th>
<th>Avg. ( \beta ) s.d.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(50, 2)</td>
<td>0.80</td>
<td>0.41</td>
</tr>
<tr>
<td>(50, 20)</td>
<td>0.80</td>
<td>0.56</td>
</tr>
<tr>
<td>(50, 50)</td>
<td>0.50</td>
<td>0.88</td>
</tr>
<tr>
<td>(50, 100)</td>
<td>0.60</td>
<td>0.77</td>
</tr>
<tr>
<td>(100, 2)</td>
<td>0.70</td>
<td>0.77</td>
</tr>
<tr>
<td>(100, 20)</td>
<td>0.70</td>
<td>0.71</td>
</tr>
<tr>
<td>(100, 50)</td>
<td>0.50</td>
<td>0.40</td>
</tr>
<tr>
<td>(100, 100)</td>
<td>0.50</td>
<td>0.36</td>
</tr>
<tr>
<td>(200, 2)</td>
<td>0.85</td>
<td>0.25</td>
</tr>
<tr>
<td>(200, 20)</td>
<td>0.85</td>
<td>0.25</td>
</tr>
<tr>
<td>(200, 50)</td>
<td>0.55</td>
<td>0.25</td>
</tr>
<tr>
<td>(200, 100)</td>
<td>0.55</td>
<td>0.25</td>
</tr>
<tr>
<td>(200, 200)</td>
<td>0.78</td>
<td>0.25</td>
</tr>
</tbody>
</table>

4. Looking Ahead

This article discusses how normalizing flows are a useful tool in probabilistic modeling. The examples we covered...
in Section 3 confirm that FAVI lies somewhere between basic MCMC methods (RW-MH, Gibbs sampling) and MF-VI with respect to accuracy and scalability. They can approximate multimodal densities and come reasonably close to MCMC for uncertainty quantification while retaining some scalability. An exciting feature of FAVI is a high degree of flexibility over the desired levels of expressivity and scalability for the probabilistic model. This is due to our ability to select the flow depth, type of transformation, and the number of flow parameters \( \phi \).

There are still many challenges remaining in this area. There is no rigorous study on the scalability vs. expressivity trade-off in the literature. Many normalizing flow families such as NAF have universal approximation properties which allow them to approximate any distribution arbitrarily well, given enough flow depth. This does not, however, provide answers on the flow depth required to achieve desired levels of accuracy. In contrast, MCMC methods have effective sample size measures, which indicate the minimum amount of samples needed to obtain the required quality in posterior samples.

Another research direction would be improving FAVI’s scalability without a significant accuracy loss. We observe that FAVI can be computationally expensive in Bayesian inference if the likelihood \( p(x|z) \) is difficult to evaluate. There are already efforts in this direction. [WLS22] replaces the likelihood with a surrogate likelihood that we can learn while training the flow transformations. In addition, we see in Section 3.2, that FAVI can be highly sensitive to the initialization of flow parameters \( \phi \). Thus FAVI can take longer to converge to the minima with bad initializations. It follows that a beneficial avenue of research would be going beyond naive initializations for FAVI.

As discussed in Section 3, we have used the RW-MH and Gibbs sampling algorithms for our experiments. However, there exists a range of other MCMC algorithms in the literature. Among the most popular is the HMC algorithm [Nea11], and its self tuning variant, the No-U-Turn sampler (NUTS) [HG14]. These methods have achieved considerable success by leveraging gradient information to make jumps through the state space for the target distributions. In fact, HMC scales at \( O(d^{1/4}) \) iterations to achieve 2 nearly independent samples in comparison to \( O(d) \) time for RW-MH. Thus, HMC is considered to be the gold standard for unimodal high-dimensional regimes, if computationally feasible. [WJC19] contains a comparison of FAVI and HMC on 13 Bayesian linear regression models. Their results indicate that FAVI is competitive with HMC, having a lower MSE in 5 of 13 models. [MPS18] show that for highly multimodal distributions the above scaling regime need not hold. Specifically, HMC and the RW-MH algorithm behave the same way, with their spectral gaps decaying at the same rate. Thus, FAVI has the potential to compete with HMC for multimodal densities. A more rigorous, wide scale exploration of how FAVI compares to gradient based MCMC methods is essential.

Normalizing flows can also be used to aid MCMC sampling. We expand on some existing ideas to do this. For multimodal target distributions, the MCMC chains may converge slowly, and samples are highly correlated. The MH algorithm, uses a “proposal density” \( p(z) \) from which we generate candidates for posterior samples. The proposal density is often selected to be easy to sample from, for example, the Gaussian distribution. Unfortunately, when the target density has a complicated geometry, this results in slow exploration of the sample space. To address this, we can use normalizing flows to shift the proposal density space to a “distorted” space. This is possible because normalizing flows are nothing but a reshaping of one density into another. The MH algorithm is then able to move faster through this “distorted” space. Recently, inverse autoregressive flows have been used to aid HMC sampling [HSD+19].

Until now, we have discussed how normalizing flows can be used for learning densities on continuous support. What happens for discrete probability distributions? There is an equivalent change of variable formula for flows on discrete distributions:

\[
  p_\mathcal{Z}(z) = p_\mathcal{U}(T^{-1}(z)).
\]

\( T : \mathcal{U} \rightarrow \mathcal{Z} \) is a bijection between two discrete spaces \( \mathcal{U}, \mathcal{Z} \). However, some issues exist with using the above for learning discrete distributions. For one, we rely heavily on the base distribution for expressivity in discrete flow models. We need to incorporate dependencies across variables into the base distribution itself. Further, there is no research on modeling joint discrete-continuous distributions. We expect this to be a popular avenue of research in the near future.

Normalizing flows are a significant advancement for probabilistic modeling, particularly for VI. This area is in the nascent stages, and many issues need to be tackled. We hope to see future collaborations between computer scientists and statisticians to address some of these issues, thus enabling wider adoption of normalizing flows, especially for Bayesian inference.

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\[\text{We see this occur for the mixture Gaussian density U4 in section 3.2}\]
References


Credits

Opening image is courtesy of NicoElNino via Getty.
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Photo of Bhattacharya Shrijita is courtesy of Bhattacharya Shrijita.
Photo of Maiti Tapabrata is courtesy of Michigan State University/Harley Seeley.
1. What Are Theories?

In mathematics one encounters theories of groups, fields, modules, vector spaces, Hilbert spaces, Banach algebras, probability spaces etc. Here theory means an assemblage of axioms that single out structures sharing certain features of interest to mathematicians in various contexts. There is usually no intention that these axioms ought to characterize a single structure—usually quite the opposite. Obviously, the word "axiom" is not meant to convey that one deals with indisputable self-evident truth as in the original meaning of that word since ancient times and until the 19th century. But by adopting a larger perspective, one can also ask what underpins the whole enterprise of mathematics. Surely, it draws on logic, but in addition one needs a plethora of mathematical objects either given to us from the start or constructed in some way. In other words, what are the axioms of mathematics (in the old sense)? Euclid intended to answer this question for geometry.

Axiomatizations sufficient unto the task of undergirding the entire edifice of mathematics came rather late. Rigorous accounts of the laws of logic, especially the logic of the quantifiers, had to await the late 19th century. Frege achieved this for logic in his Begriffsschrift (concept script) from 1879 and then attempted an axiomatization of mathematics in Grundgesetze der Arithmetik (1893). These first steps were followed by the massive Principia Mathematica (1910–1913) of Whitehead and Russell, and, springing forth from Cantor’s set theory, the axiomatization of set theory by Zermelo (1908), culminating in the axiomatic system of Zermelo–Fraenkel set theory, ZFC. In this article we shall mean by theory an axiomatic system in the sense of the foregoing sentence. The intent of such a theory is to provide an axiomatic foundation for the whole of mathematics or at least substantial chunks of it. To distinguish them from theories of groups, Hilbert spaces etc., one should perhaps call them metamathematical theories. But as this article will be concerned exclusively with the latter kind, I shall address them just as theories. In the wake of Hilbert’s program and Gödel’s incompleteness theorems, it became clear that theories can have very different consistency strengths, also called proof strengths. The purpose of this article is to describe and explain how the strength of such theories can be measured by transfinite...
ordinals, and how this ordinal-theoretic characterization can be milked to extract information about the provably computable functions of the theory and, moreover, yield unprovability and combinatorial independence results.

2. The Theory of Natural Numbers

The natural numbers equipped with the usual arithmetic functions arguably constitute the most important structure of mathematics.

The laws that govern this structure were explicated by Dedekind in his famous essay *Was sind und was sollen die Zahlen?* They gave rise to a system of axioms, the Dedekind–Peano axioms, which collectively are known as elementary number theory or first order arithmetic, but nowadays mainly called Peano arithmetic, PA.

Definition 2.1. A theory designed with the intent of axiomatizing the structure

\[ \mathcal{R} = (\mathbb{N}; 0, 1, +, \cdot, \text{exp}, <) \]

of the naturals is *Peano arithmetic*, PA. The language of PA has relation symbols =, < for the equality and the less-than relation, respectively, the function symbols +, ·, \text{exp} (for addition, multiplication, exponentiation) and the constant symbols 0 and 1. The Axioms of PA comprise the usual equations and laws for addition, multiplication, exponentiation, and the less-than relation. In addition, PA has the Induction Scheme

\[
(\text{IND}) \quad \varphi(0) \land \forall x [\varphi(x) \rightarrow \varphi(x + 1)] \rightarrow \forall x \varphi(x)
\]

for all formulae \( \varphi \) of the language of PA.

By the usual laws of \( \mathcal{R} \) we mean the following: 0 \( \neq 1 \), \( \forall x x + 1 \neq 0 \), \( \forall x \forall y [x + y = y + x] \), \( \forall x x + 0 = x \), \( \forall x \forall y [x + (y + 1) = (x + y) + 1] \), \( \forall x \forall y [x \cdot 0 = 0] \), \( \forall x \forall y [x \cdot (y + 1) = x \cdot y + x] \), \( \forall x \forall y (x < y + 1 \leftrightarrow (x < y \lor y = x)) \), \( \forall x x^0 = 1 \), and \( \forall x \forall y (x^{y+1} = x^y \cdot x) \), writing \( x^y \) for \( \text{exp}(x, y) \).

As the axioms of PA enable one to do coding, its language is rather expressive. Several famous conjectures such as the twin prime conjecture and those of Goldbach and Riemann\(^1\) can be expressed in it. Moreover, many theorems about numbers such as the prime number theorem (and, conjecturally, Fermat’s last theorem, i.e., Wiles’ theorem) can be proved in PA.

Gerhard Gentzen gave two different consistency proofs for PA in 1936 and 1938, that is, that no contradiction (inconsistency), such as \( 0 = 1 \), can be inferred from the axioms of PA. He had developed a theory of proofs as suggested by Hilbert in 1917 [Hil17].\(^2\)

To conquer this field [concerning the foundations of mathematics] we must turn the concept of a specifically mathematical proof itself into an object of investigation, just as the astronomer considers the movement of his position, the physicist studies the theory of his apparatus, and the philosopher criticizes reason itself.

Gentzen [Gen38] presented an ingenious procedure \( \mathcal{R} \) whereby any alleged proof \( P \) of \( 0 = 1 \) in PA gets reduced to another proof \( \mathcal{R}(P) \) of \( 0 = 1 \). The proof \( \mathcal{R}(P) \) may not be less complex than \( P \) in any ordinary sense, where one just counts the number of symbols or lines, however, Gentzen assigned transfinite ordinals to proofs to the effect that \( \mathcal{R}(P) \) receives a smaller ordinal than \( P \). As a result, an inconsistency will lead to an infinite descending sequence of ordinals. However, as an infinite descent is impossible in the ordinals, no contradiction can be deduced from PA. It is of note that all the steps in his argument concerning the manipulation of proofs are very concrete and elementary. For instance, \( \mathcal{R} \) and the ordinal assignment are given by basic functions.\(^3\) It is only the invocation of the principle of no-infinite-descent that transcends elementary means.

Gentzen’s result was also optimal in that he used ordinals below the first ordinal \( \rho > 0 \) such that \( \omega^\rho = \rho \) and showed that no smaller ordinal segment sufficed. What is actually meant by *ordinal* will be explained next.

3. Ordinals, Wellorderings, Ordinal Representation Systems

A set \( A \) is *transitive* if \( y \in A \) and \( x \in y \) entails \( x \in A \). In set theory, ordinals are introduced as rather abstract, mostly transfinite objects, namely transitive sets \( A \) whose elements happen to be linearly ordered by the elementhood relation \( \in \), that is, for all \( x, y \in A \), \( x \in y \) or \( y \in x \) or \( x = y \). However, many interesting ordinals \( \alpha \) have concrete representations as term systems and can be defined as orderings of subsets of \( \mathbb{N} \).

Definition 3.1. A set \( A \) equipped with a total ordering \( < \) (i.e., \( < \) is transitive, irreflexive, and \( \forall x, y \in A [x < y \lor x = y \lor y < x] \)) is a wellordering if every nonempty subset \( X \) of \( A \) contains a \( < \)-least element, i.e., \( \exists u \in X ) \forall y \in X [u < y \lor u = y] \).

An *ordinal* is a transitive set wellordered by the elementhood relation \( \in \).

Fact 3.2. Every wellordering \((A, <)\) is order isomorphic to an ordinal \((\alpha, \in)\).

The crucial property of a wellordering \(< \) can be expressed equivalently in a positive manner via the pertinent

\[ \text{Those basic functions can be taken to be the primitive recursive functions. Primitive recursion is given by the equations } f(0, \vec{x}) = g(\vec{x}) \text{ and } f(s, \vec{x} + 1) = h(s, f(s, \vec{x}), \vec{x}), \text{ where } \vec{y} = y_1, \ldots, y_n \text{ and } g, h \text{ are previously defined primitive recursive functions. The collection of primitive recursive functions is obtained from the constant 0 function, } S(x) = x + 1 \text{ and the projection functions } \pi_k(x_1, \ldots, x_n) = x_k \text{ by closure under composition and the above recursion schema.} \]

\(^1\)This was first observed by Turing.

\(^2\)Gentzen published a beautiful proof system in 1935, known as the sequent calculus, in which any proof can be transformed into one without detours. The latter result is called Gentzen’s Hauptsatz.
transfinite induction principle:

If \( \forall u < x \ P(u) \) implies \( P(x) \) for every \( x \in A \),
then \( \forall y \in A \ P(y) \),

where \( P \) is an arbitrary property. Note that \((\ast)\) is similar to
the strong induction principle for the standard ordering of
the naturals.

Ordinals are traditionally denoted by lowercase Greek
letters \( \alpha, \beta, \gamma, \delta, \ldots \), and the relation \( \in \) on ordinals is
notated simply by \(<\). The operations of addition, multi-
plication, and exponentiation can be defined on all ordinals,
however, addition and multiplication are in general not
commutative.

The ordinals that Gentzen used in his consistency proof
of PA can be nicely explained in terms of a normal form
theorem due to Cantor.

**Theorem 3.3 (Cantor, 1897).** For every ordinal \( \beta > 0 \) there
exist unique ordinals \( \beta_0 > \beta_1 > \cdots > \beta_n \) and nonzero natural
terms \( k_0, \ldots, k_n \) such that
\[
\beta = \omega^{\beta_0} \cdot k_0 + \cdots + \omega^{\beta_n} \cdot k_n. \tag{1}
\]

The representation of \( \beta \) in \((1)\) is called the Cantor normal
form. We shall write \( \beta = \text{CNF} \omega^{\beta_1} \cdot k_0 + \cdots + \omega^{\beta_n} \cdot k_n \)
to convey that the right-hand side exhibits \( \beta \)'s normal form.

\( \varepsilon_0 \) denotes the least ordinal \( \alpha > 0 \) such that \((\forall \beta < \alpha) \omega^\beta < \alpha\). \( \varepsilon_0 \) can also be described as the least ordinal \( \alpha \) such that \( \omega^\alpha = \alpha \).

Ordinals \( \beta < \varepsilon_0 \) have a Cantor normal form with ex-
ponents \( \beta_1 < \beta \) and these exponents have Cantor normal
forms with yet again smaller exponents. As this process
must terminate, ordinals \( < \varepsilon_0 \) can be coded by natural
numbers. Indeed, such ordinals need not be conceived as
denizens of a lofty set-theoretic realm. They can be iden-
nified simply by \( \varepsilon_0 \). Indeed, such ordinals need not be
conceived as \( < \varepsilon_0 \).

\( \varepsilon_0 \) is an arbitrary property. Note that \((\ast)\) is similar to
the strong induction principle for the standard ordering
of the naturals. Indeed, such a statement was un-
proved in PA and in 1938 that PA
does not prove transfinite induction over all ordinal terms
(whose set represents the ordinal \( \varepsilon_0 \)). So the idea was born
that \( \varepsilon_0 \) characterizes the proof-theoretic strength of PA.

4. **Combinatorial Independence**

The unprovability of the principle of transfinite induction
up to \( \varepsilon_0 \) from PA is an important result. However, it has
a rather logical or set-theoretic flavor. So one might ask
whether a purely number-theoretic statement could be dis-
tilled from it, that is, one using solely concepts familiar
to any number theorist. Indeed, such a statement was un-
earthed by Goodstein in 1944. He realized that there exists a
similarity between the existence of Cantor normal forms
and the fact that any positive integer \( m \) has a unique
representation with regards to a base \( b \geq 2 \), that is, \( m \) can be
uniquely expressed in the form
\[
m = b^{n_1} \cdot k_1 + \cdots + b^{n_r} \cdot k_r, \tag{2}
\]

where \( m > n_1 > \cdots > n_r \geq 0 \) and \( 0 < k_1, \ldots, k_r < b \). As each \( n_i > 0 \) is itself of this form we can repeat
this procedure, arriving at what is called the complete b-
representation of \( m \). In this way we get a unique representation
of \( m \) over the alphabet \( 0, 1, \ldots, b, +, \cdot \). For example, with \( b = 3 \) one has
\[
7625597485157 = 3^{37} \cdot 1 + 3^4 \cdot 2 + 3^1 \cdot 2 + 3^0 \cdot 2 = 3^{31} \cdot 2 + 2 + 2.2 + 2.
\]

Goodstein [Goo44], then, proceeded to define wond-
erous sequences of naturals, nowadays of course called
Goodstein sequences.

**Definition 4.1.** For naturals \( m > 0 \) and \( c \geq b \geq 2 \) let
\( S_c^b(m) \) be the integer resulting from \( m \) by replacing the base
\( b \) in the complete b-representation of \( m \) everywhere by \( c \).

\[\text{[It is often stressed that ordinal representation systems are computable structures, which is true and allows for the treatment of their order-theoretic and algebraic aspects in very weak systems of arithmetic, but it is only one of their distinguishing features. Overstating the computability aspect tends to give the impression that their study is part of the venerable research area of computable orderings. In actual fact, the two subjects have very little in common.]}\]
For example $S_3^2(34) = 265$, since $34 = 3^3 + 3 \cdot 2 + 1$ and $4^4 + 4 \cdot 2 + 1 = 265$.

Given any natural number $m$ and nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) \geq 2$ define

$$m_0^f = m, \quad \ldots, \quad m_{i+1}^f = S_{f(i+1)}^f(m_i^f) \sim 1,$$

where $k \sim 1$ is the predecessor of $k$ if $k > 0$, and $k \sim 1 = 0$ if $k = 0$.

$(m_i^f)_{i \in \mathbb{N}}$ is said to be a Goodstein sequence. Note that $(m_i^f)_{i \in \mathbb{N}}$ is uniquely determined by $f$ and its starting point $m = m_0^f$.

Goodstein sequences, even for a modest starting point such as 4, quickly climb up to gigantic numbers, but then miraculously begin their descent after reaching an outlandishly large stage in the sequence.

Theorem 4.2 (Goodstein 1944). Every Goodstein sequence terminates, i.e., there exists $k$ such that $m_i^f = 0$ for all $i \geq k$.

Proof. This is seen by assigning ordinals to the numbers $m_i^f$, effected by replacing the base $f(i)$ in the complete $f(i)$-representation of $m_i^f$ by $\omega$. One then sees that the resulting ordinal sequence decreases until it hits 0. Hence $m_i^f = 0$ for a sufficiently large $i$ as there are no infinite descending sequences of ordinals.

Conversely, the principle of termination of Goodstein sequences entails constructively that an infinite decent among the ordinal (representations) below $\varepsilon_0$ is impossible. This insight gives rise to an independence result. The language of PA, however, is not sufficiently rich to talk about arbitrary sequences of numbers, but it is capacious enough for formalizing the notion of Turing computable sequences or primitive recursive sequences (as defined in footnote (3)). Thus, Goodstein’s results from 1944 yield an equivalence provable in PA.

Corollary 4.3. Over PA the following are equivalent:

(i) Every primitive recursive Goodstein sequence terminates.

(ii) There are no infinitely descending primitive recursive sequences of ordinal representations $\varepsilon_0 > \alpha_0 > \alpha_1 > \alpha_2 > \ldots$.

In light of Gentzen’s consistency proof for PA, which uses the primitive recursive reduction operator $\mathcal{R}$ and a primitive recursive assignment of ordinals to proofs, the following number-theoretic independence result emerges from Gentzen’s and Goodstein’s insights.

Theorem 4.4 (Gentzen and Goodstein). Termination of primitive recursive Goodstein sequences is not provable in PA.

The case when $f$ is just a shift function has received special attention. Given any $m$ we define $m_0 = m$ and $m_{i+1} = S_{i+3}^2(m_i) \sim 1$ and call $(m_i)_{i \in \mathbb{N}}$ a special Goodstein sequence. Thus $(m_i)_{i \in \mathbb{N}} = (m_i^d)_{i \in \mathbb{N}}$, where $id_2(x) = x + 2$. Special Goodstein sequences can differ only with respect to their starting points.

As shown much later by Kirby and Paris in 1982 [KP82], already the termination of special Goodstein sequences is unprovable in PA. They used model-theoretic tools. Another famous Ramsey-type independence result is the Paris–Harrington theorem from 1977.

4.1. The general form of ordinal analysis of a theory. Gentzen’s analysis of PA gave rise to the idea that ordinals can measure the strength of theories. Given a theory $T$, in which at least some basic parts of mathematics can be developed, and an ordinal representation system (ORS) one would like to associate an ordinal in the ORS to $T$. First one should assume that $T$ contains means to develop some basic arithmetic. In practice this means that $T$ contains the system of primitive recursive arithmetic, PRA. The latter theory has often been equated with Hilbert’s finitism. Let $F$ be such a basic theory.

Definition 4.5. Suppose that there is an ordinal $\rho$ in the ORS such that $F$ together with the statement

There are no infinitely descending primitive recursive sequences of ordinals below $\rho$.  

proves the consistency of $T$, and, moreover, $\rho$ is the least such ordinal. Then $\rho$ is said to be the proof-theoretic ordinal of $T$.

We will treat the above as a working definition.\footnote{A much more thorough discussion of this notion can be found in [Rat99, Section 2]. In practice, this also entails that $F + (\star)$ proves all theorems of $T$ of the complexity of the twin prime conjecture.}

5. The Birth of Second-order Proof Theory

Although a fair chunk of mathematics can be developed in PA, its expressiveness is rather restricted as it doesn’t accommodate a reasonable theory of the reals. Hilbert’s work on axiomatic geometry marked the beginning of his lifelong interest in the axiomatic method. For geometry, he solved the problem of consistency by furnishing arithmetical-analytical interpretations of the axioms, thereby reducing the question of consistency to the consistency of the theory of the real numbers. The consistency of the latter system of axioms is therefore the ultimate problem for the foundations of mathematics. It became the second problem on Hilbert’s famous list of problems from 1900.

The language of second-order arithmetic, $\mathcal{L}_2$, is an augmentation of that of PA in that it has variables $X, Y, Z, \ldots$ ranging over sets of natural numbers which can be quantified over and identified with the real numbers. The full theory $Z_2$ has a comprehension axiom to the effect that

$$\{n \in \mathbb{N} \mid \varphi(n)\}$$

\footnote{A much more thorough discussion of this notion can be found in [Rat99, Section 2]. In practice, this also entails that $F + (\star)$ proves all theorems of $T$ of the complexity of the twin prime conjecture.}
is a set for any formula \( \varphi(n) \) of \( \mathbb{Z}_2 \). (3) is a relative of the set-theoretic comprehension axiom that is responsible for the paradoxes of naïve set theory (as it came to be called) found by Cantor, Russell, and others. \( \mathbb{Z}_2 \) is a rather strong theory. It can accommodate a great deal of ordinary mathematics, especially a theory of the reals. Hermann Weyl was very disturbed by the paradoxes. The expression foundational crisis was coined by him [Wey21]. As a consequence, he would only countenance arithmetical comprehension, that is, only instances of (3), where the formula \( \varphi(n) \) doesn’t contain quantifiers over sets of numbers. In his view (and that of other predicativists, including Poincaré), if \( \varphi(n) \) contains such quantifiers, then these quantifiers already range over the set asserted to exist by (3), thus constituting a circularity. An instance of the latter is said to be an impredicative comprehension axiom.

5.1. Takeuti’s fundamental conjecture. Proving the consistency of second-order arithmetic \( \mathbb{Z}_2 \) that is, solving Hilbert’s second problem, became the holy grail problem of proof theory. In the late 1940’s, Gaisi Takeuti wanted to extend Gentzen’s methodology to \( \mathbb{Z}_2 \). He formulated a sequent calculus GLC (for generalized logical calculus) which encompassed \( \mathbb{Z}_2 \). The idea was to prove a Haupt- satz for GLC à la Gentzen from which the consistency of analysis, i.e., \( \mathbb{Z}_2 \), would follow. This came to be known as Takeuti’s fundamental conjecture. Looking back at that time, Takeuti wrote:

Having proposed the fundamental conjecture, I concentrated on its proof and spent several years in an anguished struggle trying to resolve the problem day and night. [Tak03, p. 133]

It was only much later, in the 1960s, that Takeuti appreciated that he had made substantial progress. Rather than trying to prove the whole conjecture, he finally concentrated on partial results at the suggestion of Maehara. In his 1967 paper [Tak67], he gave an ordinal analysis of the subsystem of \( \mathbb{Z}_2 \) with \( \Pi^1_1 \)-comprehension (\( \Pi^1_1 \)-CA) whose main axiom scheme asserts that

\[
\{ n \in \mathbb{N} \mid \forall X \vartheta(n, X) \}
\]

is a set whenever the formula \( \vartheta(n, X) \) contains no further second-order quantifiers; so only quantifiers over natural numbers are allowed therein. For this Takeuti returned to Gentzen’s method of assigning ordinals (ordinal diagrams, to be precise) to purported derivations of the empty sequent (inconsistency). A sufficiently strong ordinal representation system that captures its strength will be described later.

The next section will be concerned with investigations as to which set existence axioms are necessary for developing various parts of mathematics, and what ordinal strengths these chunks have. From the latter vantage point, the fragment of \( \mathbb{Z}_2 \) based on \( \Pi^1_1 \)-CA turns out to be rather strong in that most of the theorems of ordinary mathematics can be proved in it.

For the sake of comparison, however, it’s perhaps worth pointing out that \( \Pi^1_1 \)-CA is very weak when compared to \( \mathbb{Z}_2 \), while \( \mathbb{Z}_2 \) is a hugely weaker theory than the axiomatic set theory ZFC.

6. Theories for the Development of Mathematics

It was already mentioned that Weyl in 1918 took the radical consequence of ditching all the mathematics that relied on impredicative set existence axioms. He accepted the infinite set \( \mathbb{N} \) as a basis but all further sets had to be obtained by arithmetical comprehension from previously introduced sets of numbers. The resulting theory from [Wey18] is a conservative extension of \( \text{PA} \), that is, it proves the same theorems about numbers as \( \text{PA} \). Amazingly, contrary to first expectation, he could salvage or rather resurrect a great deal of analysis in his theory. Weyl’s book became a blueprint for further investigations. A long list of mathematical logicians (e.g., Hilbert, Bernays, Lorenzen, Takeuti, Feferman, Friedman, and Simpson to name a few) showed that large swathes of ordinary mathematics can be undergirded by theories of fairly modest consistency strength. To some extent, this confirms what Hilbert surmised in his conservativity program, namely that number-theoretic results (i.e., those expressible in the language of number theory) proved in abstract, nonconstructive mathematics can often be proved by much more elementary means. To obtain such results, logicians have developed elaborate theories for the formalization of mathematics, and shown, by a plethora of elaborate techniques from mathematical logic, that they are conservative over various elementary theories.

It is known that virtually all of ordinary mathematics can be formalized in Zermelo–Fraenkel set theory with the axiom of choice, \( \text{ZFC} \). Hilbert and Bernays [HB39] verified that large swathes of mathematics can be formalized in second-order arithmetic. Owing to these observations, proof theory initially focussed on subsystems of second-order arithmetic. Further scrutiny revealed that small fragments are sufficient. Continuing in the wake of Hilbert and Bernays, a research program, dubbed Reverse Mathematics, was founded by H. Friedman [Fri75] some fifty years ago and then extensively developed by S. Simpson (see [Sim09]). The idea is to ask whether, given a theorem, one can prove its equivalence to some axiomatic system, with the aim of determining what proof-theoretical resources are necessary for the theorems of mathematics. More precisely, the objective of reverse mathematics is to investigate the role of set existence axioms in ordinary mathematics. The main question can be stated as follows:
Given a specific theorem \( \tau \) of ordinary mathematics, which set existence axioms are needed in order to prove \( \tau \)?

Central to the above is the reference to what is called “ordinary mathematics.” This concept, of course, doesn’t have a precise definition. Roughly speaking, by ordinary mathematics we mean main-stream, non-set-theoretic mathematics, i.e., the core areas of mathematics which make no essential use of the concepts and methods of set theory and do not essentially depend on the theory of uncountable cardinal numbers.

6.1. Subsystems of second-order arithmetic. The framework chosen for studying set existence in reverse mathematics, though, is second-order arithmetic rather than set theory. Second-order arithmetic, \( \mathbb{Z}_2 \), is a two-sorted formal system with one sort of variables \( x, y, z, \ldots \) ranging over natural numbers and the other sort \( X, Y, Z, \ldots \) ranging over sets of natural numbers. The language \( \mathcal{L}_2 \) of second-order arithmetic also contains the symbols of \( \text{PA} \), and in addition has a binary relation symbol \( \in \) for elementhood. Formulae are built from the prime formulae \( s = t, s < t, \) and \( s \in X \) (where \( s, t \) are numerical terms, i.e., terms of \( \text{PA} \)) by closing off under the connectives \( \land, \lor, \to, \neg \), numerical quantifiers \( \forall x, \exists x \), and set quantifiers \( \forall X, \exists X \).

The basic arithmetical axioms in all theories of second-order arithmetic are the defining axioms for \( 0, 1, +, \cdot, \exp, < \) (as for \( \text{PA} \)) and the induction axiom
\[
\forall X \left( 0 \in X \land \forall x \left( x \in X \to x + 1 \in X \right) \right) \to \forall x (x \in X).
\]

We consider the axiom schema of \( \mathcal{C} \) comprehension for formula classes \( \mathcal{C} \) which is given by
\[
\mathcal{C} \rightarrow \text{CA} \quad \exists X \forall u \left( u \in X \leftrightarrow F(u) \right)
\]
for all formulae \( F \in \mathcal{C} \) in which \( X \) does not occur. Natural formula classes are the arithmetical formulae, consisting of all formulae without second-order quantifiers \( \forall X \) and \( \exists X \), and the \( \Pi^n_1 \) formulae, where a \( \Pi^n_1 \) formula is a formula of the form \( \forall X_1 \ldots \forall X_n A(X_1, \ldots, X_n) \) with \( \forall X_1 \ldots \forall X_n \) being a string of \( n \) alternating set quantifiers, commencing with a universal one, followed by an arithmetical formula \( A(X_1, \ldots, X_n) \).

For each axiom scheme \( \text{Ax} \) we denote by \( (\text{Ax})_0 \) the theory consisting of the basic arithmetical axioms plus the scheme \( \text{Ax} \). By contrast, \( (\text{Ax}) \) stands for the theory \( (\text{Ax})_0 \) augmented by the scheme of induction for all \( \mathcal{L}_2 \) formulae.

An example for these notations is the theory \( (\Pi^1_1 \rightarrow \text{CA})_0 \) which has the comprehension schema for \( \Pi^1_1 \) formulae.

For many mathematical theorems \( \tau \), there is a weakest natural subsystem \( S(\tau) \) of \( \mathbb{Z}_2 \) such that \( S(\tau) \) proves \( \tau \). Very often, if a theorem of ordinary mathematics is proved from the weakest possible set existence axioms, the statement of that theorem will turn out to reversible in that it implies those axioms over a still weaker base theory, giving rise to

the name Reverse Mathematics for this theme. Moreover, it has turned out that \( S(\tau) \) often belongs to a small list of specific subsystems of \( \mathbb{Z}_2 \) dubbed \( \text{RCA}_0, \text{WKL}_0, \text{ACA}_0, \text{ATR}_0 \), and \( (\Pi^1_1 \rightarrow \text{CA})_0 \), respectively. The systems are enumerated in increasing strength. The main set existence axioms of \( \text{RCA}_0, \text{WKL}_0, \text{ACA}_0, \text{ACA}_0^\#, \text{ATR}_0 \), and \( (\Pi^1_1 \rightarrow \text{CA})_0 \) are comprehension for computable sets, König’s lemma for binary branching trees, arithmetical comprehension, existence of \( \omega \)-fold Turing jumps, arithmetical transfinite recursion, and \( \Pi^1_1 \) comprehension, respectively. For exact definitions of all these systems and their role in reverse mathematics see [Sim09].

The below list is just meant to give an idea of what kind of mathematics can be developed in the various theories of RM and how they can be located on the theory scale of RM.

- \( \text{RCA}_0 \): “Every countable field has an algebraic closure”;
- “Every countable ordered field has a real closure.”
- \( \text{WKL}_0 \): “Cauchy–Peano existence theorem for solutions of ordinary differential equations”;
- “Hahn–Banach theorem for separable Banach spaces.”
- \( \text{ACA}_0 \): “Bolzano–Weierstrass theorem”;
- “Every countable commutative ring with a unit has a maximal ideal.”
- \( \text{ACA}_0^\# \): Proves the “Auslander–Ellis theorem” of topological dynamics.
- \( \text{ATR}_0 \): “Every countable reduced abelian \( p \)-group has an Ulm resolution.”
- \( (\Pi^1_1 \rightarrow \text{CA})_0 \): “Every uncountable closed set of real numbers is the union of a perfect set and a countable set”; “Every countable abelian group is a direct sum of a divisible group and a reduced group.”

The proof-theoretic strength of both, \( \text{RCA}_0 \) and \( \text{WKL}_0 \) is the same and considerably weaker than that of \( \text{PA} \) while \( \text{ACA}_0 \) has the same strength as \( \text{PA} \). In terms of proof-theoretic ordinals, one has \( |\text{RCA}_0| = |\text{WKL}_0| = \varepsilon_0 \) and \( |\text{ACA}_0| = \varepsilon_0 \). The proof-theoretic ordinals of \( \text{ACA}_0^\#, \text{ATR}_0 \), and \( (\Pi^1_1 \rightarrow \text{CA})_0 \), however, elude expression in the ORS introduced so far.

7. Ordinal Representation Systems for the 1960s

In the case of \( \text{PA} \), Gentzen could rely on Cantor’s normal form for a supply of ordinal representations. For stronger theories, though, segments larger than \( \varepsilon_0 \) have to be employed. Ordinal representation systems utilized by proof theorists in the 1960s arose in a purely set-theoretic context. This subsection will present some of the underlying ideas as progress in ordinal-theoretic proof theory also hinges on the development of sufficiently strong and transparent ordinal representation systems.

In 1904, Hardy wanted to “construct” a subset of \( \mathbb{R} \) of size \( \aleph_1 \). His method was to represent countable ordinals via increasing sequence of natural numbers and then to correlate a decimal expansion with each such sequence.
Hardy used two processes on sequences: (i) Removing the first element to represent the successor; (ii) Diagonalizing at limits.

Hardy’s two operations give explicit representations for all ordinals $< \omega^2$. Veblen [Vebl08], then, extended the initial segment of the countable for which fundamental sequences can be given effectively. The new tools he devised were the operations of derivation and transfinite iteration applied to continuous increasing functions on ordinals.

**Definition 7.1.** Let $ON$ be the class of ordinals. A (class) function $f : ON \to ON$ is said to be increasing if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and continuous (in the order topology on $ON$) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal $\lambda$ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$. $f$ is called normal if it is increasing and continuous.

The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at $\omega$ since $\lim_{\xi<\omega} (\xi + \omega) = \omega$ but $\lim_{\xi<\omega} \xi + \omega = \omega + \omega$.

**Definition 7.2.** The derivative $f'$ of a function $f : ON \to ON$ is the function which enumerates in increasing order the solutions of the equation $f(\alpha) = \alpha$, also called the fixed points of $f$.

If $f$ is a normal function, $\{ \alpha : f(\alpha) = \alpha \}$ is a proper class and $f'$ will be a normal function, too.

**Definition 7.3.** Now, given a normal function $f : ON \to ON$, define a hierarchy of normal functions as follows:

$$f_0 = f \quad \text{and} \quad f_{\alpha+1} = f'_{f_\alpha}$$

for and for limits $\lambda$:

$$f_\lambda(\xi) = \xi^{\text{th}} \text{ element of} \bigcap_{\alpha<\lambda} (\text{Range of } f_\alpha).$$

In this way, from the normal function $f$ we get a two-place function, $\varphi_f(\alpha, \beta) := f_\alpha(\beta)$. Veblen then discusses the hierarchy $\varphi_\alpha := \varphi_f$, where $f(\beta) = \omega^\beta$.

**Theorem 7.4** (Veblen normal form). For every ordinal $\alpha > 0$ there exist uniquely determined ordinals $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ such that:

1. $\alpha = \varphi_{\xi_1}(\eta_1) + \cdots + \varphi_{\xi_n}(\eta_n)$
2. $\varphi_{\xi_i}(\eta_1) \geq \cdots \geq \varphi_{\xi_n}(\eta_n)$
3. $\eta_1 < \varphi_{\xi_i}(\eta_i)$ for $i = 1, \ldots, n$.

The least ordinal $\rho > 0$, such that $\varphi_{\rho}(\eta) < \rho$ whenever $\xi, \eta < \rho$, is traditionally called $\Gamma_0$. As the ordering of representations in Veblen normal form can be determined by a recursive procedure, similarly as for the Cantor normal, one arrives at an ordinal representation system for $\Gamma_0$. With its help, the proof-theoretic ordinals of some further systems of RM can be exhibited:

$$|ACA_0| = \varphi_2(0)$$

and

$$|\text{ATR}_0| = \Gamma_0.$$  The one for $(\Pi_1^1-\text{CA})_0$, however, still remains elusive.

8. **Collapsing Functions Beyond Veblen**

Veblen extended his approach, first to functions having a finite number of ordinal arguments, but then also to a transfinite number of arguments, with the proviso that in, for example $\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$, only a finite number of the arguments $\alpha_\eta$ may be nonzero. Finally, Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions $\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$ with $\eta < \delta$, and each $\alpha_\eta < \delta$.

Though the “great Veblen number” (as $E(0)$ is sometimes called) is quite an impressive ordinal, it does not furnish an ordinal representation sufficient for the task of analyzing a theory as strong as $\Pi^1_1$ comprehension. Of course, it is possible to go beyond $E(0)$ and initiate a new hierarchy based on the function $\xi \mapsto E(\xi)$ or even consider hierarchies utilizing finite-type functionals over the ordinals. Still all these further steps amount to rather mundane progress over Veblen’s methods. In 1950 Bachmann [Bac50] presented a new kind of operation on ordinals which dwarfs all hierarchies obtained by iterating Veblen’s methods. Bachmann built on Veblen’s work, but his novel idea was the systematic use of a regular uncountable cardinal to keep track of the functions defined by diagonalization.

Let $\mathcal{N}_1$ be the first uncountable ordinal. Bachmann defines a set of ordinals $\mathcal{B}$ closed under successor such that with each limit $\lambda \in \mathcal{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathcal{B}$ of length $\tau_\lambda \leq \mathcal{N}_1$ and $\lim_{\xi<\lambda} \lambda[\xi] = \lambda$. A hierarchy of functions $(\varphi_\alpha)_{\alpha \in \mathcal{B}}$ is then obtained as follows:

$$\varphi_0(\beta) = \omega^\beta,$$

$$\varphi_{\alpha+1}(\beta) = (\varphi_\alpha)^{\varphi_\alpha}(\beta),$$

$$\varphi_\alpha = \text{En}\left(\bigcap_{\xi<\tau_\lambda} (\text{Range of } \varphi_{\lambda[\xi]})\right) \quad \text{if } \tau_\lambda < \mathcal{N}_1,$$

$$\varphi_\lambda = \text{En}(\{\beta < \mathcal{N}_1 : \varphi_{\lambda[\xi]}(\beta) = \beta\}) \quad \text{if } \tau_\lambda = \mathcal{N}_1,$$

where $f = \text{En}(X)$ means that $f$ enumerates the ordinals of $X$.

Modern approaches are much simpler and more transparent. We will briefly look at this.

8.0.1. The Bachmann–Howard ordinal.

**Definition 8.1.** Let $\Omega_1$ be a sufficiently “big” ordinal. We define the sets $B_{\Omega_1}(\alpha)$ and ordinals $\varphi_{\Omega_1}(\alpha)$ by transfinite
recursion on \( \alpha \) as follows

\[
B_{\Omega_1}(\alpha) = \left\{ \begin{array}{ll}
\text{closure of } [0, \Omega] \\
\text{under:}
\end{array} \right.
\]

\[
\begin{array}{l}
+ \xrightarrow{\xi \mapsto \omega^\xi} \\
\xi \xrightarrow{\psi_{\Omega_1}(\xi)} \xi < \alpha
\end{array}
\]

\[
\psi_{\Omega_1}(\alpha) = \min \{ \rho < \Omega_1 : \rho \notin B_{\Omega_1}(\alpha) \}.
\]  

(5)

Now, the foregoing definition is vague in some important respect. What does it mean for \( \Omega_1 \) to be sufficiently big? This can be defined implicitly by requiring that \( \psi_{\Omega_1}(\alpha) \) is defined for all \( \alpha \), which is implicitly assumed in (5), meaning that there always exists an ordinal \( \rho < \Omega_1 \) with \( \rho \notin B_{\Omega_1}(\alpha) \). One can see, via a simple cardinality argument, that equating \( \Omega_1 \) with the first uncountable ordinal, that is, the cardinal \( \kappa_1 \), will work. But this is surely overkill: much smaller countable ordinals can be substituted for \( \Omega_1 \). The smallest for which it works is called the Bachmann–Howard ordinal. It is usually denoted by \( \psi_{\Omega_1}(\varepsilon_{\Omega_1+1}) \), where \( \varepsilon_{\Omega_1+1} \) stands for the least ordinal \( \eta > \Omega_1 \) such that \( \omega^\eta = \eta \). And, miraculously, the Bachmann–Howard ordinal can be captured by a primitive recursive ORS over the alphabet \( 0, \Omega_1, +, \omega, \varepsilon, \psi_{\Omega_1} \).

Note that the function \( \psi_{\Omega_1} \) differs significantly from previous proof-theoretic functions, such as \( \beta \mapsto \omega^\beta \) and \( \varphi_\beta \), in that \( \psi_{\Omega_1}(\alpha) \) can be (and in the most interesting cases is) a smaller ordinal than \( \alpha \). Such proof-theoretic functions have been called collapsing functions.

Now, \( \psi_{\Omega_1}(\varepsilon_{\Omega_1+1}) \) is still much smaller than the proof-theoretic ordinal of \( \Pi_1^1-CA_0 \). It is, however, the proof-theoretic ordinal of an important set theory, called Kripke–Platek set theory (see [Jäg86]). To climb up to the strength of \( \Pi_1^1-CA_0 \) one needs infinitely many sufficiently large ordinals \( \Omega_1 < \Omega_2 < \cdots < \Omega_n < \Omega_{n+1} < \cdots \), each equipped with their own collapsing function \( \psi_{\Omega_n} \). Again one can use the infinitely many uncountable \( \kappa_n \)’s to play the role of the \( \Omega_n \)’s. But that again amounts to an enormous overkill. Countable avatars for the \( \Omega_n \)’s suffice. The proof-theoretic ordinal of \( \Pi_1^1-CA_0 \) is usually notated by \( \psi_{\Omega_1}(\omega) \), where \( \omega = \sup_{n \in \mathbb{N}} \Omega_n \).

By now we have become acquainted with (an idea of) all proof-theoretic ordinals of theories used in RM. Notwithstanding that \( \Pi_1^1-CA_0 \) is rather capacious as a framework for ordinary mathematics, there are still very interesting results from graph theory which it cannot prove and to which we turn next. This will provide another example of an independence result obtained via ordinal analysis.

9. The Graph Minor Theorem

If a graph \( X \) is obtained from a graph \( Y \) by first deleting some vertices and edges, and then contracting some further edges, \( X \) is said to be a minor of \( Y \). The following theorem holds.

Theorem 9.1 (Robertson and Seymour 1986–2004). If \( G_0, G_1, G_2, \ldots \) is an infinite sequence of finite graphs, then there exist \( i < j \) so that \( G_i \) is isomorphic to a minor of \( G_j \).

As to the importance attributed to the graph minor theorem, I quote from a book on graph theory [Die10, p. 333.]

"Our goal [...] is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer: in every infinite set of graphs there are two such that one is a minor of the other. This minor theorem, inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages."

Theorem 9.1 (GMT hereafter) has many important consequences. Here are a few of them.

Corollary 9.2. (i) (Vázsonyi’s conjecture) If all the \( G_k \) are trivalent, then there exist \( i < j \) so that \( G_i \) is embeddable into \( G_j \).

(ii) (Wagner’s conjecture) For any 2-manifold \( M \) there are only finitely many graphs which are not embeddable in \( M \) and are minimal with this property.

A further important consequence of GMT is that any minor closed class of graphs can be characterized by finitely many forbidden minors (a vast generalization of the case of planar graphs). This has important predictive algorithmic consequences: Membership in any minor closed class of graphs can be decided in polynomial (even cubic) time. A case in point (see [Die10, p. 367]) is the class of knotless graphs, that is, finite graphs which can be embedded in \( \mathbb{R}^3 \) such that none of its cycles forms a nontrivial knot. This class is minor closed, so there is a polynomial algorithm. Currently, such an algorithm is not known, but it exists owing to GMT.

GMT is not provable in the strongest system of RM. This independence is a consequence of the ordinal analysis of \( \Pi_1^1-CA_0 \) in that GMT proves the wellorderedness of its ordinal.

Theorem 9.3 (Friedman, Robertson, Seymour [FRS87]). GMT is not provable in \( \Pi_1^1-CA_0 \).

The paper [KR20] investigated upper bounds for the proof strength of GMT. If one adds a principle of induction, called \( \Pi_2^1 \) bar induction, to \( \Pi_1^1-CA_0 \) one can prove GMT and many of its generalizations. The resulting system is well within the scope of proof theory of the 1970s.

10. Beyond \( \Pi_1^1 \)-comprehension

Proof theorists have widened the realm of theories for which ordinal analyses have been attained way beyond the level \( \Pi_1^1 \)-comprehension, especially through the work of Ari (e.g., [Ara15]) and the author (e.g., [Rat95]). The current state of the art is that subsystems of \( \mathcal{Z}_2 \) with \( \Pi_1^1 \)-comprehension can be handled, that is, comprehension
of the form

\[ \{ n \in \mathbb{N} \mid \forall X \forall Y \theta(X, Y, n) \}, \]

where \( \theta(X, Y, n) \) contains no set quantifiers. The difference between the proof power of \( \Pi^1_1 \)-comprehension and \( \Pi^1_2 \)-comprehension is almost unimaginably huge. In section 8 we saw that ideas from uncountable cardinals played a role in devising an ORS capable of encapsulating \( \Pi^1_1 \)-comprehension. Viewing the representation system as a miniaturization of some cardinal notion has become an important source of inspiration for proof theorists. Accordingly, ideas from large cardinal notions such as inaccessible, Mahlo, and weakly compact cardinals have entered the design of ORSs. A cardinal notion germane to \( \Pi^1_1 \)-comprehension is that of the much larger \( \textit{shrewd cardinals} \) defined in [Rat95]. Their existence follows from those of subtle cardinals.

The next barrier is \( \Pi^1_1 \)-comprehension. One might hope that this is somewhat the generic case that can be generalized to yield an ordinal analysis of any \( \Pi^1_n \)-comprehension, and thus of \( Z_2 \). We will see.

Note: The rules of the AMS Notices for this type of article do not allow more than 20 references. A longer version with the same title and including many more references is available on arXiv.

References


Michael Rathjen

Credits

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2023 TexPREP-Lubbock students pose proudly next to the gumball machine they designed during this hands-on math, science, and engineering summer program.
Mathematizing Human Perception

Brett Jefferson

In this exposition, I aim to encourage mathematicians to learn about and conduct research in the area of cognitive science by walking the reader through the process of utilizing tools and formalism from mathematics to address challenges in perception. First, I introduce the audience to a historically considered phenomena from psychology, called blob processing. Then, I present one approach to mathematically model the phenomena. Throughout, I introduce psychological findings to show how we can incorporate observed experimental results into our mathematical model.

A Brief Foreword on the Importance of Mathematical Psychology

I’d like to pick on the bully in the room a bit. In the last decade, the area of data science has grown massively due to advances in deep learning. It touches almost every scientific discipline and algorithms are being developed that are undeniably accurate in their predictive power and robustness. However, despite the popularity of machine learning approaches to applied problems, there are many observable phenomena that require more finesse and rigor for an appropriate explanation than deep learning can provide. Psychology is full of such examples. It is amazing how humans are able to pick out a loved one’s voice in a noisy crowd, or quickly identify a word before the actual letters are discernible. Likewise, mathematics is much broader than machine learning and provides a way of thinking and working through real-world problems. To describe and model human behaviors, psychologists and mathematicians (not always distinct people) have a long history of working together to axiomatize underlying principles, uncover the right decomposition of perceptual parts, and prove theorems about how and why the human experience doesn’t simply cease to function! While this exposition is not a review of this history, a wonderful review of how mathematics and psychology have played in the same sandbox can be found in another AMS column article by Joseph Malkevitch [Mal15].
I encourage this generation of mathematicians to continue this history of marrying psychological principles and mathematical language, as it poses rich opportunities for mathematicians to find exciting new problems and research avenues. Here, I showcase one example of how a psychological observation can benefit from mathematical concepts and the rigorous formalism of mathematics. That methodology can, in turn, formalize the space of open questions around that observation. This article serves to provide just one example of what I call “mathematizing” a psychological observation.

The Blob Phenomena

We first think about a psychology problem (usually presented as an experiment with a puzzling result). Consider three rectangles of varying height and width, \((h_1, w_1), (h_2, w_2), (h_3, w_3)\). Over multiple rounds, you present a single rectangle very quickly (just a couple of tenths of a second) and task an observer (a priori) with identifying either height only or width only by responding with the appropriate value. First, you use the three rectangles from Figure 1a and, later, you rerun the experiment with rectangles from Figure 1b. The data collected are accuracy and response time. Once the experiments are complete, you compare performance across the two stimulus sets.

One will find that on average, judgments from stimulus set 1 were less accurate and slower than judgments on stimulus set 2. This is a bit of a mystery. Both sets of stimuli have two dimensions, height and width, that can be used to discriminate the rectangles. One might even posit that since the dimensions vary in a redundant and monotonic way in stimulus set 1, that we should expect the opposite behavioral result. Understanding how parts come together to form a perceptual experience is a major theme in perception research [TG11, Tre86, PP11]. [Loc72] proposes a descriptive model for why this occurs for the rectangles. The blob processing model posits that visual stimuli are initially perceived in a holistic way with the sum being greater than the parts. Higher-order features emerge that are perceived faster and are also easier to discriminate than the part-wise (single-dimension) components. As such, Lockhead suggests that the rectangles in the second stimulus set have more shape features than the squares in the first set. The notion of blob processing has been a persistent model throughout the decades following Lockhead’s note. Namely, in the area of letter and word recognition, there’s evidence that (1) words are identified before individual letters, (2) word shape and orthography are important for word recognition, and (3) neighboring words also serve as cues for word recognition. The type of holistic processing evidenced in this research area points to words (respectively sequences of words) being perceived as a single object rather than a collection of letters (respectively words) with general shape features playing an important role. In fact, the word “blob” appears frequently in the literature when investigating early visual perception and speeded-judgement tasks and the idea of a “fuzzy” to “sharp” perception is an intuitive notion.

This phenomena is not restricted to rectangular stimuli. In the 1980s Townsend published two studies in which he created a stimulus set comprised of the factorial combinations of three line parts and a curve part (Figure 2). Participants were asked to identify which components were shown. Townsend showed that the components/parts were not perceived independent of one another. Findings included that the probability of a correct identification increased when there were more components present, the angle between line parts increased, and the stimuli was presented for longer durations [THE8408, THK8807]. These studies also highlighted the importance of time by modeling this discrimination of parts of a single stimulus as a process whereby over the time course of stimulus onset, exposure, and stimulus offset, more and more information is accumulated about the stimulus.

At this point, the reader may have started developing a few theories on why and how such processing occurs. The reader may be thinking through the technical process of how our eyes transform light into neurological signals that
our brain then sends down various pathways that lead to perceived visual experiences. However, that level of end-to-end explanation does not quite prove satisfying from the point of view of providing an understanding of mapping direct neuroscience stimulus inputs to perception ... not to mention the neuroscience community still doesn’t have the tools for a complete classification in this way. Rather, we desire a mathematical expression or theory to properly understand how the measurable visual components come together to form this blob and, for that matter, a rigorous definition of what a blob is.

In the next sections, we’ll explore a straightforward definition for a blob that is intuitive and testable. After, we’ll see how different stimuli can be used to test the subsequent hypotheses. Finally, an analysis method is presented that relates to blob processing theory.

**Mathematical Formulations**

**Mathematizing** a repeatable human behavioral phenomena is both beautiful and terrifying at once. While it is satisfying to develop a rigorous framework for study, it can be difficult to know if the mathematics adequately (or appropriately) captures the observation. In mathematics, one tends to constrain the objects and rules for objects to well-studied and agreed upon structures like topological spaces, continuous maps, or algebraic structures. By contrast, in mathematical psychology, the phenomena often are so strange that it can be difficult to relate them to each other via existing structures. It’s not uncommon for new areas of study to be derived from observed human behaviors. Fortunately, for the aforementioned phenomena, there are some ready structures that are proposed to define blob processing.

Any definition of blob processing should be compatible with observable processing properties. Those properties are:

1. **Refinement.** When a stimulus is presented, as time increases, the blob for that stimulus should decompose into a fixed set of constituent parts for sufficient viewing conditions.
2. **Complexity.** Assuming that for a given stimulus there is a fixed number of constituent parts, as stimuli become more complex (containing more parts) they become more discriminable from substituents or stimuli comprised of a subset of constituent parts.
3. **Orthogonality.** Given a set of constituent parts, we assume that when two parts are not similar then the parts are discernible and discriminable earlier in time.

**Remark.** This list of properties is not intended as an exhaustive list. In fact, property 2 is arguably not necessary as there are many examples where adding more features to a stimulus may serve to slow down discrimination rather than facilitating it. For this exposition, the properties primarily support the findings of [THE8408, THK8807].

These processing properties directly describe the observed phenomena from Townsend’s study. Lockhead’s review specifies that when there are multiple dimensions to a stimulus (like length and width) then the additional dimensions contribute to the perception and discrimination of the stimuli. This is akin to modeling stimuli in a multidimensional metric space rather than projecting to a unidimensional space. Note that there are two ends of the discussion here. The first end is at the early stage processing where stimuli can be thought of as barely detectable and coarse. Here the concept of dimensions isn’t relevant because not enough information has been processed to even guess at the dimensionality of the stimulus. On the other end of the discussion, at a later stage of processing, stimuli are modeled together in a higher-dimensional space (often the experimenter defined manipulations each being a dimension) and as being comprised of perceivable parts. Each part can be modeled in the higher-dimensional space. For example, the rectangles in Figure 1 can be embedded as points in the 2-dimensional space with width corresponding to x-coordinates and length to y-coordinates. Simultaneously, the lower line segment of each rectangle can also be embedded in this space where the y-coordinate is 0. More commonly, a set of parts is defined first, and stimuli are created from combinations of parts. For this discussion, we take this later view and treat the transition from blob to not blob as a question of when the parts become discernible.

**Definition 1 (Part).** Let $\mathcal{T}$ be a task for participants to complete, and $\mathcal{S}$ be a fixed set of visually presented stimuli for that task. Let $\{p_j\}_{1..N}$ be a set of subsets of $\mathbb{R}^2$ along with an inner product such that for each $s \in \mathcal{S}, s = \{p_j\}_{1..N}$. We will call the $p_j$ the parts of a stimulus, $s$. It is also useful to call the inner product the similarity between parts.

[Jef18] followed on [THE8408] and proposed that blob processing for a stimulus is a time sensitive function on the Fourier decomposition of visual stimuli. Look ahead to Figure 3 for example stimuli and their Fourier representations in polar coordinates. Let $F \times O$ be the Cartesian product of spatial frequencies (measured in the number of cycles subtended at the eye per degree) and orientations (measured in degrees of rotation from horizontal). For each frequency-orientation combination, $(f, o)$, of the decomposition, there is an amplitude determined by the stimulus. When the amplitude at $(f, o)$ is greater than a threshold specific to that $(f, o)$ a human can perceive...
that component. We refer to the threshold as the sensitivity of the human to a frequency-orientation component (higher sensitivity implying lower threshold). Sensitivity (as a widely used term in psychology) can refer to neuron higher sensitivity implying lower threshold. Sensitivity from stimulus onset be denoted, contrast, and previously perceived points that the stimulus becomes perceptible in a matter of a couple hundred milliseconds. At this point a blob can now be defined.

Definition 2 (Blob). For a task, \( T \), and set of stimuli, \( S \), a stimulus \( s \) is perceived as a blob at time \( t \) if for some part of the stimulus, \( p_i, A(s) \) restricted to the frequency and orientation components of \( p_i \) is less than \( B_{t,s}(p_i) \) for a sufficient amount of frequencies and orientations.

Remark. The failure of a few components of \( A(s) \) to reach threshold doesn’t prevent accurate perception of stimulus parts, much like using fewer terms in the Taylor expansion of a function doesn’t prevent understanding of the overall function shape. In fact, there is a large body of research surrounding critical Fourier components for foveal (central vision) and peripheral stimulus identification.

Each of the desired processing properties can now be stated in terms of \( B \) and \( A \).

1. Refinement. The refinement property requires that for \( s \in S \), there exists \( \hat{s} \in T \) so that whenever \( t > \hat{t}_s \), \( B_{t,s} \leq A(s) \) for sufficiently many \((f,o) \in F \times O\).

2. Complexity. There exists a minimum \( \delta_d \in \mathbb{R} \) so that for \( s_i, s_j \in S \) with \( s_i \subset s_j \), there exists \( d_{i,j} \in T \) so that whenever \( t \geq d_{i,j} \) we have \( \|B_{t,s_i} - B_{t,s_j}\| \geq \delta_d \) and humans can accurately distinguish \( s_i \) from \( s_j \). We refer to \( \delta_d \) as the discrimination threshold and \( d_{i,j} \) as the discrimination time. Note that \( \delta_d \) is not stimulus dependent and is minimum over all stimulus comparisons. Also, \( d_{i,j} \) is ideally small enough that parts are not discernible. Complexity is how blobs are distinguished from one another.

3. Orthogonality. Given four parts, \( p_1, p_2, p_3, \) and \( p_4 \), where

- \( d_{1,2} \) is the discrimination time for \( \{p_1, p_2\} \),
- \( d_{3,4} \) is the discrimination time for \( \{p_3, p_4\} \),
- \( \langle p_1, p_2 \rangle \leq \langle p_3, p_4 \rangle \),

we have \( d_{1,2} \leq d_{3,4} \).

With a working definition of a blob, blob processing, and mathematical expressions for the three processing properties, we make note of a few directions in which one could go. First, one could start to study the nature of the sensitivity function, \( B \). This includes understanding if frequency and orientation have independent effects on sensitivity, asking if sensitivity is monotonic with time, or how sensitivity changes with different and related stimulus sets. In the next sections we explore some of these questions.

Another direction that one could pursue is characterizing the similarity inner product. This is one of the more difficult directions to validate in psychology because similarity judgements and responses can be modulated by experimenter instructions, context, and experiment environment settings... all things that ideally would have no
bearing on a true similarity representation. We often desire a static similarity space to make understanding other phenomena more feasible. There are models in psychology dedicated to unveiling such a representation, but this is outside the scope of this exposition. For our purposes, we assume there is such an inner product and move on.

### Testing Framework

Testing the blob model assumption that the Fourier decomposition is the right way to describe the stimulus space faces many challenges. This assertion is one based in human performance and humans, after all, vary in behavior, learn, adapt, and evolve. In truth many psychology models are, at best, contextually true: that is, they are specific to a stimulus set, population demographic, experiment settings, and measure of behavior. Strong theories show consistent results when these conditions change. A theory of visual perception based on spatial frequency channels (or dedicated processors) has been explored for an array of simple and complex stimuli. While alternative theories exist, as in [Lup79], there still remains strong physiological and psychophysical evidence that humans are wired to detect spatial frequency and orientation information [BC69, Dev82, VSG89]. The blob processing model brings to the forefront the question of time-course of processing. Namely, for early stage processing, we can ask three questions:

1. Do distinct, yet co-occurring frequencies facilitate, inhibit, or act independently on relative sensitivity? Using our definitions, for fixed \( o \in O \), sufficiently low \( t \in T \), and \( f_1 \neq f_2 \in F \), what is the relationship between \( B_{t,s}(\{(f_1, o), (f_2, o)\}) \), \( B_{t,s}(\{(f_1, o)\}) \), and \( B_{t,s}(\{(f_2, o)\}) \)?
2. Likewise, do distinct, yet co-occurring orientations facilitate, inhibit, or act independently on relative sensitivity?
3. Lastly, is sensitivity to frequency independent of sensitivity to orientation? We desire the relationship between two frequencies (resp. orientations) to be independent of orientation (resp. frequency). Equivalently, are the following true:
   (a) Given frequencies \( f_1 \) and \( f_2 \), there exists a constant, \( c_o \) so that for any orientation, \( \hat{o} \),
   \[
   B_{t,s f_1}(\{(f_1, \hat{o})\}) = c_o \cdot B_{t,s f_2}(\{(f_2, \hat{o})\})
   \]
   for stimuli that only contain the respective frequencies and
   (b) Given orientations \( o_1 \) and \( o_2 \), there exists a constant \( c_f \) so that for any frequency, \( \hat{f} \),
   \[
   B_{t,s o_1}(\{(\hat{f}, o_1)\}) = c_f \cdot B_{t,s o_2}(\{(\hat{f}, o_2)\})
   \]
   for stimuli that only contain the respective orientations.

Note that the stimulus corresponding to a single frequency and single orientation is a 2-dimensional linear sinusoidal pattern that changes in brightness. The stimulus corresponding to a single frequency and all orientations is a radial sinusoidal pattern; and the stimulus corresponding to a single orientation and all frequencies is a line. See Figure 3.

Answering the posed questions requires determining values for \( B_{t,s} \) (human sensitivities) for different stimuli. This is done experimentally. The blob phenomena presented here were observed in identification experiments. The output of such an experiment is a confusion matrix, \( n \times n \) where \( n \) is the number of stimuli and possible responses. The matrix of integers describe response frequencies with rows corresponding to individual stimuli and columns to responses. We must tie responses to perception at this point so we make the following assumption.

**Assumption 1.** If \( A(s) \geq B_{t_{s_1}, s_1} \) at minimum time \( t_{s_1} \) for sufficiently many points for stimulus \( s_1 \), then \( s_1 \) may be a response at time \( t \geq t_{s_1} \).

We don’t assume that \( s_1 \) will be a response because several stimuli may meet this criterion. At this point, we note there are many candidate response models corresponding to choice experiments. One class of response models considers behaviorally derived weights for each potential response and chooses the maximum weight. Another class represents alternative responses in a metric space and the response closest to a behaviorally derived representation is chosen. Yet a third class of response models are race models in which the first response to reach a threshold is the chosen response. For this application we use the race model to make a second assumption regarding choice. This is made out of convenience rather than evidence.

**Assumption 2.** For \( \{s_{1_1}, ..., s_{1_M}\} \) in the response candidate set, the response corresponding to the minimum of \( \{t_{s_{1_1}}, ..., t_{s_{1_M}}\} \) is chosen.

Lastly, since responses are not deterministic, we must introduce stochasticity. Again, there are many models for how stochasticity enters responses. One class of models considers information accumulation to be a noisy process, where information is gained (learned) and lost (forgotten or, more conservatively, inaccessible). In this model, this might be introducing Gaussian noise to \( B_{t,s} \) at each point in the decomposition. Alternatively, we may consider sampling the stimulus to be a noisy process. This would mean that \( A(s) \) is the the correct amplitude for a random sample of frequencies and orientations. Out of convenience, we model the former.
Assumption 3. There exists an experimentally determined parameter $\varepsilon$ so that for all stimuli and all $(f, o) \in F \times O$,

$$B_{t,s}(f, o) = \mathcal{N}(0, \varepsilon) + B_{t,s}$$

for deterministic function $B_{t,s}$.

Signal detection theory. After attaining judgements in a confusion matrix, the experimenter will need an analysis technique to separate signals from different sources (or stimuli). A widely used approach is signal detection theory. We will call the no-stimulus condition the noise and the stimulus-presented condition the signal due, in part, to the history of signal detection theory in psychology and to focus the reader on a particular stimulus of interest. In psychology, signal detection theory is an application of statistical decision theory to stimulus detection. It describes a perceptual process where there is no fixed threshold for detecting a stimulus (or part of a stimulus) but rather an observer-specific threshold. Briefly, we assume that while the participant views a stimulus (including the null stimulus) samples are drawn that are classified as either coming from a signal distribution or a noise distribution. We model these distributions as Gaussian $\mathcal{N}(0, \sigma_N)$ and $\mathcal{N}(\mu_S, \sigma_S)$ on a common support where a decision boundary ($x^*$ on the support) determines if participants respond that a stimulus (or part) was present or not. We fix the mean of the noise distribution to $\mu_N = 0$ and variance of the noise distribution to $\sigma_N = 1$ without loss of generality. Modellers often assume the variance of the signal distribution is equal to that of the noise distribution ($\sigma_N = \sigma_S = 1$) for computational simplicity and this assumption provides an easier interpretation later. From data, the mean of the signal distribution, $\mu_S$, and $x^*$ are fit so that

$$\int_{x^*}^{\infty} f_S(x)dx = P(\text{Respond Signal} \mid \text{Signal})$$

and

$$\int_{x^*}^{\infty} f_N(x)dx = P(\text{Respond Signal} \mid \text{No Signal}),$$

where $f_S$ and $f_N$ are density functions for the respective distributions. A picture communicates the concepts more efficiently (see Figure 4). Note, that the larger $\mu_S$ is, the
more correct identifications occur and the better sensitivity is. The standardized distance between the means is referred to as sensitivity and is denoted $d'$. Let $H$ (hit) denote $P$(Respond Signal $|$ Signal Stimulus) and $FA$ (false alarm) denote $P$(Respond Signal $|$ No Stimulus). Then

$$d' = z(H) - z(FA),$$

where $z$ denotes the z-score. Since we are already using the word sensitivity to be inversely proportional to thresholds $z$, we can call this computation SDT sensitivity.

The SDT sensitivity can provide insight on the relationships between functions $B_{t,j}$ and $B_{t,j'}$. Jef18 conducted an experiment with diagonal line stimuli (see Figure 3 for example) presented for very short durations (50 ms) with contrast level adjusted for each participant so that accuracy in the identification task was 60%. There were two line parts (line 1 at 45 degrees, referred to as $l_1$ and line 2 at 135 degrees, referred to as $l_2$) that provided for four stimuli ($l_1, l_2, l_1 + l_2$, and blank). That study showed

- the probability of responding accurately in the identification task when both diagonals were present is lower than the product of probabilities in single line presentations (inhibiting; non-independence of orientation),
- SDT sensitivity decreased for each line part with an increase in the number of parts, and
- higher correct “blank” responses than correct responses when a stimulus was presented.

The SDT decrease implies the probability of $B_{t,l_1+l_2}$ or $B_{t,l_1} \leq A(l_1)$ first is higher than the probability of $B_{t,l_1+l_2}$ or $B_{t,l_1} \leq A(l_1 + l_2)$ first. However, we've defined $A$ to be the amplitudes corresponding to the Fourier decomposition and since the study only considered simple stimuli, $A(l_1 + l_2)$ agrees with $A(l_1)$ on the corresponding components of $l_1$. This may lead us to believe that either the sensitivity functions are modulated by the exact stimulus being presented, or the other sensitivity functions (one for $l_2$ and one for “blank”) are winning the race model more frequently for the $l_1 + l_2$ stimulus. In fact, the third finding shows that the “blank” sensitivity is not negligible.

These results may appear to be at odds with the complexity blob processing requirement, but they are not. In Jef18 the $l_1 + l_2$ stimulus was not correctly identified above chance (25%), in many cases being confused with the “blank” stimulus, where nothing was shown. So the discrimination threshold for that stimulus was not reached to satisfy the complexity property. Concurrently, for higher-contrast stimuli, many authors have shown evidence that for longer display times, two oblique-angle stimuli should act independently of one another (single frequency [CK66] and line stimuli [Gil68]) and provide higher accuracy and faster response times in discrimination tasks. We conjecture the following:

**Conjecture 2.** For small enough $t$, Blob processing is highly sensitive to low-frequency stimulus content. That is, for all stimuli, $\left\| \frac{d}{dt} B_{t,s}(small f, -t) \right\|$ decreases as $t$ increases.

In another experiment from the same study, Jef18 ran the same identification task with a different set of stimuli. Two spatial frequencies were provided. For each frequency, a sinusoidal radial stimulus was produced with that frequency, one with 1 cycle ($1c$), and a second with 5 cycles ($5c$). A combination stimulus ($1c + 5c$) and a “blank” stimulus were also used (see Figure 3). These frequencies have been largely agreed upon to be independent through adaptation studies [DeV77], masking studies [KKS73], and detection studies [GN71]. Jef18 showed that SDT sensitivity increased when going from the $5c$ stimulus to the $1c + 5c$ stimulus, but not from the $1c$ stimulus. The study concluded that there was a dominance effect of the $1c$ frequency over the $5c$ stimulus in that participants were more likely to respond that a stimulus was the $1c$ stimulus when the $5c$ part was added to the $1c$ part. This study provides some support for the second conjecture. Additionally, the $5c$ stimulus was often confused for the “blank” stimulus.

In a final experiment, participants were asked to identify which of four stimuli were presented. Stimuli were Gabor patches. The lower image in Figure 3 shows a Gabor patch, a stimulus with alternating black and white linear patterns at different orientations and frequencies. This task tests the ability to simultaneous discriminate between orientation and and frequency. Jef18 found that stimulus frequency discrimination was largely unimpaired by changes in orientation, but low-frequency perception deteriorated separation of same-oriented stimuli. A model that could explain this observations is that the sensitivity functions may have a uniform continuity constraint for a given $t$. The observation that responses for a given stimulus have systematic confusions across dimensions can be explained by sensitivity functions being more or less consistent across bands of orientations and frequencies. A final conjecture on sensitivity functions follows.

**Conjecture 3.** $B_{t,s}$ is uniformly continuous in $f$ and $o$ for each $t$. Further, $B_{t,s}$ is continuous in $t$.

While there may be other models that explain the collective findings across these experiments, we have three conjectures that can be tested by other paradigms. Without using mathematics, these disparate experimental results would lack a unifying concept to organize and tie them together. Recall, that Figure 1 contained rectangular stimuli. We now have a language and framework for describing the “right” features of the stimuli for explaining varied accuracy and reaction times. In Fourier space, the distance between the stimuli in set 2 are larger than the distances between stimuli in set 1. These distances may map directly to perceptual distances and the sensitivity of rectangles is likely
higher than that of squares due to the higher-frequency content.

Open Questions

As a psychological construct, the blob processing model provides a testable model for the concept of the blob. As a mathematical construct, the simple functional approach gives an intuitive and flexible base case.

To the degree possible, I encourage readers to consider some open questions that I find interesting for this particular phenomena:

- The inner product as a similarity measure was left undefined. When judging similarity is there a more appropriate group structure or action that captures behaviors?
- Are there topological properties inherent in the surface of a stimulus sensitivity $B_{ls}$ that are indicative of deficits or changes over time? Likewise, are there topological properties applicable to similarity space?
- This discussion avoided making claims about $B$ outside of a particular range of frequencies and orientations relevant to a stimulus. What would changes in the support of $B$ do for SDT sensitivity?

There is research in this space that considers more complex stimuli [TS22], and interested readers should consider some of the citations in this notice as a good entry point to the many viewpoints on the relationship between spatial frequency and orientation (especially [VSG89]). More generally, I encourage mathematicians to learn more about the very diverse world of mathematical psychology. There’s a strong chance that a phenomena will fascinate you, and there’s a guarantee that your expertise can be valuable in helping to describe the phenomena!

References


[TS22] Hikari Takebayashi and Jun Saiki, Restriction of orientation variability and spatial frequency on the perception of average orientation, Perception 51 (2022), no. 7, 464–476, PMID: [35578553].


Brett Jefferson

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Applying for Grants and Jobs

A Practical Guide to Writing an NSF Grant Proposal

Matthew Badger

Your think that your research is going well. You have a shiny new theorem or a novel proof technique or a cool example. You also have some ideas on how to build on your recent research. You want to tell the US mathematics community about your recent successes and share the excitement that you have for your research program. Maybe your faculty advisor, postdoctoral mentor, or colleague suggests that you apply for a grant. Maybe you take the initiative on your own. Whether the reason is to obtain summer salary to support your research, fund travel to conferences, reimburse visitors, pay students, or all of the above, you decide to submit a grant proposal to the National Science Foundation (NSF). This is not something that you have done before—or at least it has been a while since you last prepared a proposal.

What exactly do you have to do?

In this guide, I will outline how to prepare and submit a proposal to the Division of Mathematical Sciences (DMS) at NSF. The goal is to demystify the process and help you with your first proposal from start to finish. There are different types of grants available at the NSF, but I will focus on how to prepare and submit a standard grant proposal to one of the current disciplinary research programs:

1. Algebra and Number Theory;
2. Analysis;
3. Applied Mathematics;
4. Combinatorics;
5. Computational Mathematics;
6. Foundations;
7. Geometric Analysis;
8. Mathematical Biology;
9. Probability;
10. Statistics;
11. Topology.

If your research bridges two or more areas, then it is likely that your proposal will be reviewed by experts from multiple fields, but you still have to choose one of the 11 programs for your initial submission.

Grants from the NSF are available to researchers and educators who are employed at US institutions of higher education. There are no citizenship requirements for investigators on most NSF grants. The NSF strongly encourages women, minorities, and persons with disabilities to participate fully in its programs. While a grant proposal is initiated by you, the final proposal will be submitted by your employer on your behalf.

Standard grants are typically awarded for periods of 36 months. When you apply within 10 years of earning your PhD, you are automatically considered an early-career researcher and this may be taken into account during the review process. You do not need to have a permanent position to apply for a grant; postdocs are eligible too and
Early Career

have received awards. (An award is a funded grant.) If you change institutions within the US, then it may be possible to transfer your grant.

Much of the advice to follow also applies for career-stage specific grants such as MSPRF ("Mouse Proof," for new PhDs) and NSF CAREER (for tenure-track assistant professors). Just be aware that proposals for those awards come with additional restrictions and requirements. More on this near the end.

Overview. In NSF jargon, the individual who is responsible for carrying out research and other proposed activities is called the Principal Investigator (PI). Throughout your proposal, you should write in the third person and refer to yourself as "the PI" instead of by your name; this helps reviewers easily identify when you are discussing your past work and what you will do if the grant is funded. As the PI, you will prepare most of the proposal, which is comprised of a series of documents. Financial parts of the proposal may be prepared with assistance from university staff. Once all required documents are uploaded to research.gov and reviewed by your university’s Sponsored Projects Office (SPO), your proposal will be submitted to the NSF by an Authorized Organizational Representative (AOR). At some institutions, the SPO is called the Sponsored Research Office.

Each of the research programs is overseen by NSF program officers, who make the final decisions about which grants to fund. If you have technical questions as you prepare your proposal ("Am I allowed to...?", "Should I...?") the best way to get answers is to ask a program officer. Names and email addresses of the current program officers for each program can be found on the NSF DMS website.

After submission, program officers will browse your proposal to determine its topic and group it with similar proposals. Next, your proposal will be sent for review by anonymous experts, who are usually mathematicians who were funded by NSF in the past. Anyone that you may have a conflict of interest with, including colleagues, collaborators, and former mentors, will not be chosen to review your proposal. This means that you should target your proposal, the NSF DMS website. With the exception of Mathematical Biology, which now accepts proposals year round, each of the other 10 programs has a single submission window per year, usually in the fall. For example, the next round of proposals in Probability are due between September 10 and September 25, 2023. But the next proposals in Topology are not due until November 7, 2023. Before you do anything else, look up the deadline for the program that you want to apply to. Make sure that you have time to complete a proposal.

Proposals are evaluated using the NSF’s two major review criteria. Intellectual Merit includes the perceived importance of the proposed research within contemporary mathematics or other areas of science; whether the proposed research explores creative, original, or potentially transformative concepts; how well qualified the PI is to carry out the proposed research; and the likelihood that the project can be carried out in the proposed time frame. Broader Impacts are more nebulous, but refer to activities in the proposal or carried out by the PI with the potential to benefit society or to achieve specific, desired societal outcomes. For an in-depth discussion, see Max Lieblich’s article "What is Broader Impact?" in the August 2021 issue of the Notices. There is no official weighting of the review criteria; the most competitive proposals will have excellent Intellectual Merit and excellent Broader Impacts. Expectations for Broader Impacts in a proposal are less for a junior investigator than for a senior investigator.

How to get started. A grant proposal is a complex document with many components. Start early! The more time that you give yourself to write and revise, the better chance you have to submit a competitive proposal. I recommend beginning work at least two months before the submission deadline. If your proposal is due in September, then you should ideally start no later than early July. (Life is not always ideal.)

At the start of a new proposal, you should:

1. Identify the NSF submission deadline. Current information about grant programs is always available on the NSF DMS website. With the exception of Mathematical Biology, which now accepts proposals year round, each of the other 10 programs has a single submission window per year, usually in the fall. For example, the next round of proposals in Probability are due between September 10 and September 25, 2023. But the next proposals in Topology are not due until November 7, 2023. Before you do anything else, look up the deadline for the program that you want to apply to. Make sure that you have time to complete a proposal.

2. Identify any internal deadlines and reach out to university staff. It is common for a university’s SPO to require an internal submission deadline anywhere between 5 days to 14 days before the NSF deadline. This is needed so that they can audit your proposal and leave time for you to fix any issues before submitting the proposal to the NSF. If you have not applied for external funding at your institution before, ask your mentor or supervisor about the local procedure and any department or university resources that are available to assist you.

3. Create a folder. Download the PAPPG. Make a separate folder for each proposal that you work on. Download a copy of the current Proposal & Award Policies & Procedures Guide (PAPPG). This is a very long document that
specifies the rules for all grant proposals to the NSF. Ignore the temptation to use the version of the PAPPG from last year, since it has probably been updated. Refer to the PAPPG as you work on each part of the proposal.

4. Get an NSF ID. Login to research.gov. Start a new proposal. If you do not already have an NSF ID, you can request one on research.gov. After logging in, find the link to create a new Full Proposal. You will be prompted to fill out basic information, starting with your organization (university). On the next page, use the search box to find the code for the program that you are applying to (Geometric Analysis, Foundations, etc.). For a standard grant, you should select Research Proposal for the proposal type and Single Proposal for the collaborative status. Finally, enter a short title for your proposal—for examples, look up recently funded awards on the NSF DMS website. If everything went right, you will be assigned a temporary proposal number and see a screen with the list of required and optional documents that make up the proposal.

5. Begin working on the proposal. Most required documents may be prepared in LaTeX, Google Docs, or Microsoft Word, and uploaded to research.gov as PDFs. Exceptions are the Budget and Cover Sheet, which are entered directly on research.gov, and a trio of Senior Personnel Documents, which are filled in using NSF-supplied templates. It is time to work on these documents.

Cover Sheet (1% effort). Most of the Cover Sheet is auto-filled with information that you entered when creating the proposal. In addition, you must enter the country codes for any planned foreign collaboration or travel. You must also specify the award period (36 months) and pick a requested start date (1st or 15th of a month). For applications made in Fall 2023, a typical start date would be 6/1/2024. This will let you be paid summer salary starting in summer 2024.

Project Description and References Cited (75% effort). This is the most important part of your proposal. In the Project Description (PD), you have up to 15 pages to say what you want to do, how you will try to do it, why it is reasonable, why you are qualified to do it, and what its broader impacts are. The PAPPG specifies 8.5 × 11-inch page size, at least 1-inch margins on all sides, either Times New Roman or Computer Modern (default LaTeX) fonts of size 11pts or higher, and no more than 6 lines of text per 1 inch of vertical space. Smaller fonts in mathematical expressions are okay. Plan to use all 15 pages as anything shorter may convey a lack of seriousness to reviewers. Figures are allowed and can be effective, but should be used sparingly, because they quickly reduce the amount of space you have to write. You may not include hyperlinks to external documents or do anything else to get around the 15-page limit.

You should use citations in the PD like in a research paper, but References Cited do not count against the 15-page limit and must be uploaded separately. The easiest way to do this is to prepare the PD with references at the end like you would for a journal article. Then, once the document is in its final form, use a free web app to split the PDF into two files and upload them separately on research.gov. Encourage you to use author-year citation labels like [Bad11] or [BG22] instead of numerical labels like [23], [42], because the names and dates help create a better narrative.

I think of the body of the PD as being organized into three differently sized “blocks.”

Block #1 Prior NSF Support (0–5 pages). If you have received support on one or more NSF grants with an end date in the last five years (e.g., you were an RA on your advisor’s NSF grant), then you must have a separate section labeled Results from Prior NSF Support. The PAPPG specifies a hard limit of 5 pages for discussing prior support. Start by listing the number of each grant, the PI of the grant, the dates of the grant (e.g., 6/2020–5/2023), and the total dollar amount of the grant. Next, under a required subsection labeled Intellectual Merit of Prior NSF Support, cite any papers or preprints from prior support. Then describe in short, titled, minimally technical paragraphs the major outcomes and scientific impact of prior support. Finally, under a required subsection labeled Broader Impacts of Prior NSF Support, summarize any broader impacts from the period of the award. It is okay if portions of this block are duplicated in other parts of the proposal. If you have not received any prior NSF support, start your PD with the second block.

Block #2 Research Problems (9–14 pages). The bulk of the PD should revolve around a series of clearly stated problems that you propose to work on. Group related problems into their own sections. Start each section with general background and describe any major advances in the last 5–10 years. While there is no exact rule, I recommend stating between 8 and 12 problems—some easier, some harder, none out of reach. Problems can take a variety of forms, including questions, conjectures, and statements like “Prove . . . or find a counterexample.” Number your problems so that you may refer to them throughout the proposal and your reviewers can refer to them in their reviews. Before and/or after each problem statement make sure to describe both the context for the problem and one or more strategies to attack the problem. Discuss how difficult each problem is and your estimation of the likelihood it will be solved. If you forget to discuss a strategy for one or more problems, it will be called out as a weakness in the reviews. Novelty or variety of problems may be viewed as a strength. Relating your narrative back to your past work can be seen as evidence that you are qualified to work on
the proposed problems. It is okay to report on work in progress.

Block #3 Broader Impacts (1/2–2 pages). You are required to have a separate section labeled Broader Impacts, in which you describe any broader impacts of the project or being carried out by the PI parallel to the project. The length of this section should scale with the number of years past your PhD.

As a final remark, although I have seen it done, there is currently no requirement in the PAPPG to have a separate section in the PD labeled Intellectual Merit. Your exposition in Block #2 already discussed Intellectual Merit at length.

Project Summary (5% effort). This is a specialized abstract of your project, which is comprised of exactly 3 paragraphs. It is used to help program officers categorize your proposal and is useful for reviewers when they are writing their reviews. Do not include any mathematical notation, equations, or references. The Project Summary (PS) must be entirely self-contained. Delay writing the PS until after you have a solid draft of the PD.

Paragraph #1. Overview. (The PAPPG says that you actually have to label this paragraph “Overview.” A similar rule applies for paragraphs #2 and #3.) Write in a few sentences—seriously, the overview should be short—about the major topics and themes of the proposal. After reading the overview for a proposal to the Analysis Program, for example, a program officer should be able to decide whether the proposal focuses on complex analysis, geometric function theory, harmonic analysis, etc.

Paragraph #2. Intellectual Merit. This paragraph should set the context for the proposed research and give a high-level description of different clusters of problems tackled in the proposal. Discuss the novelty of proposed problems and methodology, connections with other areas of mathematics or science (if any), and the potential impact of the research. This paragraph should take around 50% of the PS.

Paragraph #3. Broader Impacts. Highlight past, present, and future broader impacts in the proposal and carried out by the PI. This paragraph need not be exhaustive, since you described broader impacts in more detail in the PD.

Budget and Budget Justification (10% effort). The Budget is a line-item accounting of the money requested for proposed activities, broken up into categories that are specified in the PAPPG (Salaries, Equipment, Travel, etc.). Pay close attention to the section on Allowable and Unallowable Costs. Rates for certain expenses (Overhead, Fringe Benefits) are determined by your university in negotiation with a federal agency. The Budget is entered using a form on the research.gov website. You must also upload a Budget Justification, which is a companion document of 5 pages or less that details your planned expenditures in plain sentences. There are no simple instructions for this part of the proposal. You should ask for help from department or university staff or senior colleagues to prepare the budget. Even so, do not neglect these documents. Your university may not allow you to spend money that you receive from the NSF unless you described the expense in advance in the Budget Justification.

To start work on the budget, list everything that you might want to spend money on. You are allowed to and should ask for 2 months of summer salary per year (based on your 9 month university salary). Other common things that you might want to ask for are money for travel for collaboration or to attend conferences, money to reimburse research and seminar visitors, money for publication costs, money to purchase books or an office computer, laptop, or tablet. Be imaginative. You may request academic year and summer salary for graduate RAs. The likelihood of receiving this increases if you name the students in the justification and discuss which problems are appropriate for graduate students in the Project Description. Although your budget requests should be reasonable, do not worry too much about the total dollar amount. When your grant is awarded, NSF program officers may ask you to revise your budget downward, if necessary.

Senior personnel docs (5% effort). For an individual proposal, the only senior personnel is the PI. You must complete three documents.

1. Biosketch (1–2 pages). A biosketch is a very short CV with a specific structure and 2-page limit. To enforce compliance, the NSF now requires you to use a website called SciENcv: Science Experts Network Curriculum Vitae to produce the biosketch. The website will walk you through required elements and build a PDF that you can save and upload to research.gov.

2. Current and Pending Support. Using the NSF supplied template, describe the project goals and list the dollar amounts of any current or requested research funding from the NSF, other government agency (US or foreign), or private foundation. Also list the proposal that you are preparing as pending support. This document helps the NSF avoid double-funding the same research.

3. Collaborators and Other Affiliations. To help NSF identify potential conflicts-of-interest between you and reviewers of your proposal, fill out the NSF supplied template. Follow the instructions and list your PhD advisor(s), PhD students, postdoctoral mentors and mentees, collaborators, as well as their current affiliations (university or employer).

Miscellaneous docs (4% effort). 1. Facilities, Equipment, and Other Resources (1–2 pages). For mathematics, this document is rather anodyne. You should convey that you have adequate resources to carry out proposed activities. Items that you may wish to describe are office space,
Computing resources, journal and library access, staff support, and availability of local lodging (for visitors).

2. Data Management Plan (1/2–1 page). State whether or not you expect your project will produce any data sets. Describe how you will preserve and disseminate any research products (papers) produced by the project (PI website, arXiv, etc.) Your DMP does not need to be long, but take it seriously.

3. Postdoctoral Mentoring Plan (conditional). If you request any salary for a postdoc on the budget, then you must upload a comprehensive single page mentoring plan. Otherwise, leave this section blank.

4. Letters of Collaboration (optional). If parts of the project are joint work, then you may upload a signed letter from each collaborator attesting to their willingness to support the project. The letter must use the specific language described in the PAPPG and does not function as a letter of recommendation. These letters are optional.

5. List of Suggested Reviewers / List of Reviewers Not to Include (optional). Barring exceptional circumstances, it is customary to leave these blank.

Submitting your proposal. When all required and any optional documents that you want to submit have been finished and uploaded to research.gov, click the link to Share Proposal with SPO/AOR. This will alert the office at your university in charge of proposals to review your documents. Depending on local procedure, it is possible that you should also contact the SPO directly. Remember that you must send your proposal for review in advance of the NSF deadline. This is often the most stressful part of the process, because you will now wait on someone else to “click submit” on something that you have spent many hours working on. Stay calm, monitor your email, and be prepared to make any last minute changes to the proposal requested by the SPO. After everything is reviewed, the AOR can submit the proposal to NSF.

Congratulations! You did it!!

General advice. Your proposal will be judged based only on what you put in the proposal. Reviewers are not required to read your papers or preprints or look at references. Unlike when you are applying for a job, there are no letters of recommendation. This means that in your proposal, you need to be your own advocate! It is up to you to explain the importance of your past work and your proposed research and say why you can succeed (but not in a boastful way). By taking time to write a better exposition, you can make it easier for reviewers to write a quality review of your proposal and improve the chances of getting funded. The reality is that not every quality grant can be funded. If your grant proposal is not funded, do not be discouraged from applying the following year.

Advice for MSPRF. The Mathematical Sciences Postdoctoral Research Fellowship (commonly called the NSF postdoc) is an award that is available for US citizens and permanent residents who have held the PhD for less than 2 years. Typically you apply in October of the year that you expect to graduate. If you do not receive the award after your first attempt, it is a good idea to apply again the following year. The Project Description for the MSPRF application is limited to 5 pages and you must discuss the choice of institution and sponsoring scientist. This does not leave much space to talk about mathematics! An effective proposal will convey several interesting mathematical ideas in a concise way. You must request confidential letters of recommendation from your PhD advisor and 2 other researchers. If at all possible, secure a letter from a notable mathematician in your field that is at a different university than where you earned your PhD. In addition, you will upload a supporting letter from the sponsoring scientist that you can read. Does this letter of support indicate that they know what you are about or does it sound generic?

Advice for CAREER. The NSF CAREER award is for untenured assistant professors who are on the tenure-track. Applications are due in July. It can be observed that most recent CAREER recipients previously received the MSPRF or a standard grant, but this is not a requirement. One challenge is that this is a 5-year grant and in addition to research you must propose educational projects—that means you need to describe more activity in the Project Description within the same 15-page limit. If possible, tie in your educational projects with prior broader impacts. An effective proposal will devote at least one-third of the PD to the educational component and also explicitly discuss Integration of Research and Education. It is reasonable to expect reviewers on CAREER panels to be mathematicians who are farther from your speciality than reviewers for standard grants. You may apply for the CAREER award at most three times. Do not feel pressured to apply before you are ready.

Matthew Badger

Credits

Photo of Matthew Badger is courtesy of Matthew Badger.
Writing a Teaching Statement for a Liberal Arts College

Jennifer Biermann

Writing a teaching statement was one of the hardest parts of putting together my first set of job applications. Fresh out of graduate school, I found it difficult to decide what the “right” teaching statement would look like. I am fortunate, after two years in a post-doc and three years as a visiting professor, to have ended up in a tenure-track job that includes close interaction with my students and a department full of wonderful colleagues. I teach in a joint mathematics and computer science department at Hobart and William Smith Colleges, a small liberal arts school in western New York. During my seven years at HWS we have hired three times for visiting professors (two math and one computer science) and once for a tenure-track computer scientist, which has given me experience on both sides of the hiring process. While I now know that there is no magical “right” way to put together a teaching statement, I do have some advice to share for someone writing a teaching statement for a job at a liberal arts college like HWS.

In my view, the teaching statement is the most important part of an application to a liberal arts school. The truth is that any posting for a mathematics position gets hundreds of applicants. On a first pass through reading applications, I focus most of my time on the cover letters and teaching statements. Partially this is due to time constraints, but it also reflects the values of an institution like mine. I care very much, if you are going to be part of my department, that you are an effective and thoughtful teacher. Here are some guidelines to keep in mind as you put together a teaching statement

- **Be yourself.** You know the advice that your mom gave you about making friends when you were in elementary school: “Just be yourself”? This is my number one piece of advice for writing a teaching statement for a liberal arts college as well. You are applying for jobs where you will be working closely with a small number of people. These future colleagues are trying to get to know you through your application materials to see if you would be someone they would like to work with for the next twenty years. There’s not a lot you can get about someone’s personality from a CV or a research statement, but the teaching statement is more personal. Let your personality show through.

- **Be honest.** Your goal with your application is to get an interview, sure, but your ultimate goal is to find a position where you will be happy. Don’t put things in your teaching statement because they are trendy or you think it is what people want to hear. Only put something in there if you truly believe in it and want to implement it in your classroom. If you have no interest in doing undergraduate research, for instance, then you don’t want to sell yourself on your great student research plans and end up in a job that expects you to spend every summer mentoring students.

- **Be honest (part 2).** It is easy for teaching statements to come off as generic—filled with platitudes about teaching that all sound the same. Your best defense against this is to write about the things that you are actually passionate about. Don’t worry about what you think the school will want to hear—you can’t know that anyway. Write about what actually gets you excited about teaching.

- **Be specific.** Wherever possible, back up your statements with anecdotes from your own teaching experiences. Include stories of successes you’ve had in the classroom or challenges that led to changes in the way you teach. This shows that you are reflective—that you have really thought about how you teach, and it is another way to avoid sounding too generic. If you have some activity or classroom technique that you are particularly proud of, make sure to highlight that.

- **Be enthusiastic.** What excites you about working at a liberal arts college? This one is often addressed in a cover letter, but it could also have a place here. What about your teaching is specific to the liberal arts environment? Maybe you have ideas for cross-disciplinary classes. Maybe you are excited about teaching a class on voting theory to non-majors. In any case, there is a reason you are applying to jobs at a school like mine and I want to know what that is. We are concerned that a candidate knows what they are getting into with this job because we want them to be happy and to stay with us. This is especially true if you did your undergraduate and graduate training at large public institutions, and so have no first-hand experience with a liberal arts school. What excites you about this type of job, specifically?

The main point is that the teaching statement is your chance to show the person reading your application what kind of colleague you will be. Use the space to express your values about teaching and about education in general—specifically the education on offer at a liberal arts college. Be your wonderful self. You’re going to do great, I believe in you.

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Jennifer Biermann
Writing a proposal for funding may seem like a daunting task. While competition is often stiff for what seems like not enough funding, we believe that the process of preparing a proposal will be of great value to you and to the community. Below, we lay out the steps involved: whether or not to apply, getting started with due diligence, choosing an appropriate opportunity and reading the solicitation carefully, reading the solicitation (again) while preparing your proposal, and proofreading. While we have each served as an NSF program officer, our goal is to provide a guide to applying for a variety of opportunities.

1. Should You Apply?

If you have never applied for funding through your institution, talk to your department head or others who have been successful obtaining funding. There may be internal incentives for writing proposals to outside funders or reasons to apply for internal funding. Look at funding opportunities to decide which grants are the best match for you. You need to come up with new ideas and write a persuasive proposal that addresses everything the funder requests. Two of the biggest funders of basic research in mathematics and the physical sciences in the US are the National Science Foundation and the Simons Foundation, but there are several other agencies, private as well as government, that fund research. Once you know which to apply for, there may be specific requirements. The proposal may be restricted to a few pages, you may need to create a budget or you may need letters of support. That’s asking a lot, and, to top it all off, you may not get it. But more on that later.

2. Getting Started

Start early. Even if you have an idea and a strong track record, it takes time to write a compelling proposal. You have to establish your expertise, demonstrate your passion and motivate your problem with clarity. You need to be sure that your references are complete and the proposal is error free. Allow yourself enough time to find an appropriate funding opportunity and to write a great proposal.

2.1. The Office of Sponsored Research. Once you have decided to seek support, visit your Office of Sponsored Research (OSR). They will be critically involved in the submission of your proposal. If your school has no OSR, talk to others on your campus and see what they suggest.

The people in this office can do a lot for you. They can tell you about funding opportunities, work with you on your budget, help you fill out the institutional information, and help you submit the proposal. If you have trouble staying on track, they can develop a schedule with you. They can find people to parse the solicitation and read proposal drafts. They may also supervise group writing sessions. If this is offered, take advantage of it—it will keep you on track and other people in the sessions may have interesting ideas for writing. If you get the award, there is a lot more that an OSR can do on the finance end.

Your OSR (or equivalent) will also have knowledge of a wide range of possible grants that you can apply for, from some of the usual suspects to places you may not have considered like the Department of Energy Visiting Faculty Program or one of the MAA Grants. The NSF has many opportunities. The Mathematical Science Institutes have many possibilities, too. The NSF also has curricular or educational initiatives, like the NSF REU or NSF DUE curricular grant opportunities. If these seem like a better fit, the Center for Undergraduate Research in Mathematics (CURM, also funded by the NSF but on hiatus in 2022) has minigrants and workshops. The CURM website offers this useful reason for applying: “Last year you applied for a grant related to promoting undergraduate research and [were] denied because the reviewers felt that you did not have enough experience. A CURM mini-grant would help you gain more experience and make a stronger proposal…” But more on that later.

“Should You Apply?” A Guide to Writing Effective Proposals

Pamela Gorkin and Krishnan Shankar

Introduction

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2 For example, they helped us write this article!

https://beta.nsf.gov/funding/opportunities?sort_bef_combine=nsf_funding_upcoming_due_dates_DESC&f%5B0%5D=division%3A43

https://www.maa.org/programs-and-communities/outreach_initiatives

This article reflects our views, not those of the NSF.

Credits

Photo of Jennifer Biermann is courtesy of Chris Scheper.


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The people in this office can do a lot for you. They can tell you about funding opportunities, work with you on your budget, help you fill out the institutional information, and help you submit the proposal. If you have trouble staying on track, they can develop a schedule with you. They can find people to parse the solicitation and read proposal drafts. They may also supervise group writing sessions. If this is offered, take advantage of it—it will keep you on track and other people in the sessions may have interesting ideas for writing. If you get the award, there is a lot more that an OSR can do on the finance end.

Your OSR (or equivalent) will also have knowledge of a wide range of possible grants that you can apply for, from some of the usual suspects to places you may not have considered like the Department of Energy Visiting Faculty Program or one of the MAA Grants. The NSF has many opportunities. The Mathematical Science Institutes have many possibilities, too. The NSF also has curricular or educational initiatives, like the NSF REU or NSF DUE curricular grant opportunities. If these seem like a better fit, the Center for Undergraduate Research in Mathematics (CURM, also funded by the NSF but on hiatus in 2022) has minigrants and workshops. The CURM website offers this useful reason for applying: “Last year you applied for a grant related to promoting undergraduate research and [were] denied because the reviewers felt that you did not have enough experience. A CURM mini-grant would help you gain more experience and make a stronger proposal…” But more on that later.

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DOI: https://doi.org/10.1090/noti2732

2 For example, they helped us write this article!

https://beta.nsf.gov/funding/opportunities?sort_bef_combine=nsf_funding_upcoming_due_dates_DESC&f%5B0%5D=division%3A43

https://www.maa.org/programs-and-communities/outreach_initiatives

This article reflects our views, not those of the NSF.
3. Choosing Your Match

Think about your strengths and see if the program is a good match for you. Let’s consider two NSF opportunities for early-career faculty: Launching Early-Career Academic Pathways in the Mathematical Sciences (LEAPS) and Faculty Early Career Development Program (CAREER). To figure out if one of these is right for you, read the solicitation. LEAPS “has an emphasis to help launch the careers of pre-tenure faculty in Mathematical and Physical Sciences (MPS) fields at institutions that do not traditionally receive significant amounts of NSF-MPS funding, such as some minority-serving institutions (MSIs), predominantly undergraduate institutions (PUIs), and Carnegie Research 2 (R2) universities.” CAREER supports those who “have the potential to serve as academic role models in research and education and to lead advances in the mission of their department or organization.” But perhaps a straightforward research proposal with broader impacts is better for you than integrating a detailed educational plan. If the opportunity sounds good, see if there is a webinar you can attend. If it still sounds like a possibility, it’s time to get started.

3.1. Read the solicitation. Seriously. Read it. The solicitation tells you whether this opportunity is right for you, what you need to do, and it also offers guidance for the reviewers, telling them how to evaluate the proposals. For example, the CAREER program explicitly notes that “all CAREER proposals should describe an integrated path that will lead to a successful career as an outstanding researcher and educator.” Do you have a strong track record in research and education? Do you have new ideas for the education of students? Are you as passionate about that as you are about research? If not, perhaps there’s a better match. The AMS-Simons Travel Grants, for example, are also for early-career faculty, specifically individuals who have earned a PhD within four years of the grant start date and have a mentor, and they provide support for a total of $6000 over a two-year period for travel.

Once you have your match(es), find out as much as possible about your funder and your audience. Talk to people who have served on panels; they have valuable experience, even if it isn’t in your discipline. Some things are universal, and they can tell you what those are. Ask people who have been successful to share their experience or proposals (your OSR may be helpful, too; they may even have a list of successful proposals). Likely, all the proposals you see will have the following commonalities: a strong track record, persuasive motivation for new problems, a demonstrated passion for the work, well-developed ideas for successful solutions, and potential for strong impact.

4. How to Write Your Proposal

Writing is hard work, especially writing that you want others to appreciate. A proposal, at its core, is an invitation to the reader to follow you on a journey; you want to convince them it is a worthwhile trip for everyone. It’s possible that your work, both past and proposed future, may be so exceptional that reviewers are willing to overlook some lack of clarity. Or, your writing may be beautiful but your proposed projects seem not as exciting for the future of science. Reviewers may allow some margin for either happenstance, but this is not something you will want to count on. So how do you write a good proposal?

4.1. Getting started writing. There is undoubtedly more than one way to do this, but if you do not have a lot of experience you may benefit from writing by hand a short draft of a selection of research problems that you plan to investigate. These could be continuations of your current work, a project you put on the back burner, or a completely new problem or research direction. It is a good rule of thumb to have about three projects—it’s rare that one project will be successful, and describing too many projects will not allow the space to fully develop them. Divide your proposal into sections and write down provisional titles for these sections. For example, a typical NSF proposal could have four sections: background, prior work, proposed projects (intellectual merit), broader impact activities. But, one may equally well divide sections as: introduction, project 1 (background + prior work + future project), project 2, project 3, broader impacts. This is not meant to be prescriptive; it’s just a reminder that you should have a plan and a good story.

4.2. Organization. Describing the actual projects is the heart of the proposal. Reviewers are committed to reading your proposal thoroughly, but everyone’s time is at a premium and your proposal is one of many that they may be evaluating. Your organization and attention to details (Are there uncompiled references? Is the proposal riddled with typos? Are sentences vague? Are there missing references? Is there a page limit?) all go a long way to convince a reviewer that your project is worthwhile and you are the right person to do it. Using sections and subsections can guide the reviewer and indicate that you have done everything that you were asked to do.

4.3. Track record, depth, and strategy. The actual content of your proposed projects is, obviously, the most important thing in your proposal. Think of this as a line being drawn from what you have done in the area of the project (your track record), through the depth and importance of your proposed problem, to your proposed innovation or strategy to tackle this problem. Giving short shrift to depth or strategy risks a lower rating by the reviewer.

Depth. This is often difficult to get right and you should spend time revising to achieve the right balance. If you, for

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5There are unsolicited proposals too, but they have their own set of requirements that you must adhere to.
example, suggest that you plan to make progress on the Riemann Hypothesis with little to no track record, then your proposal will be viewed, as expected, with great skepticism. On the other hand, if you suggest a small improvement on a moderately impactful result in your area, then this may be viewed as not far-reaching (remember, competition is stiff). Look for that "sweet spot."

**Strategy.** Aiming for an ambitious, but possibly attainable, result must be backed up with a plausible strategy. Too often, researchers are cagey about how they expect to solve a problem. We understand why someone might be reluctant to share details. But, your colleagues are less likely to recommend an award without cogent evidence of a viable path forward. Reviewers need something convincing and correct, but not necessarily complete. However, be wary that this is the most likely place for you to introduce an error and you may not have all the details down. Try to find a balance between too little and too much.

4.4. Broader impacts. Everything that we have said also applies to what are known as “broader impacts” in NSF parlance, namely, projects that have an impact on people. If you are proposing an activity that benefits students or the community, then reviewers will want to see a track record. They want to see depth and importance. If you can, say who will be impacted and how. Use your proposal to convince reviewers that you mean what you say: If you say that you will mentor students and take them to conferences, have you discussed funding for their travel? Have you suggested appropriate venues for presentations? Specifics are always more convincing than generalities.

5. Encouraging Good Practices

**Citations.** We begin with advice that is good for all writing, but especially scientific writing: If you cite references selectively, whether by accident or by design, your proposal may be viewed poorly. If you were the first to discover something, say so. But if you gave a new proof of an old result, put that into context. Scrupulously cite all relevant work and be honest about where your work fits in the big picture.

**Proposal, not person.** Focus on the proposal and not on any aspect of personality. For example, you may believe that you are the only person capable of solving a given problem. But simply stating that you can solve a problem because you have solved a problem like this before is not convincing. Instead, demonstrate your expertise by describing your innovative approach or strategy. Keep the focus on your proposal and don’t issue value judgments about other researchers or their work.

**Words matter.** Words are powerful as we all know. But a proposal is different from an equation or theorem. If you claim that your work has broader impacts simply because one of your collaborators is a (senior) mathematician from an under-represented group, this will get noticed and it will be discussed. But probably not in the way you were hoping: reviewers can spot tokenism or efforts to marginalize important aspects of proposal review. Have an experienced reviewer look at your proposal before submission to make sure that this doesn’t happen to you.

6. Reviews and Declinations

Whether your proposal is recommended for funding or not it is good practice to read the reviews carefully. Look for common themes in reviews and summaries: is more than one person saying that your program sounds unrealistic? or are they saying that the work could be more ambitious? Take that as good advice delivered candidly. If you can, discuss your reviews with experienced reviewers. What we encourage you not to do is to read reviews word by word, sentence by sentence to convince yourself that a particular reviewer derailed your proposal. That is unlikely to be true and will distract from improving your proposal.

While there is no guarantee in a very competitive funding landscape, we encourage you to keep trying. Sometimes reviewers identify weak points in the proposal, but sometimes funders have to decline worthy proposals. And very few proposals receive only stellar reviews. Strengthen your track record, present at conferences, talk to established colleagues, review and referee papers, and volunteer to serve on panels (https://www.nsf.gov/bfa/dias/policy/merit_review/Reviewer.jsp). Each of the authors of this article has been funded fewer times than we have applied. For us, the process ensures that the community is aware of our efforts, it helps us maintain focus on our research program, and sometimes we receive valuable feedback.

The benefits of applying include—but are not limited to—the possibility that your proposal will be funded. As the saying goes, “The secret of getting ahead is getting started.”

Pamela Gorkin

Krishnan Shankar

**Credits**

Photo of Pamela Gorkin is courtesy of Pamela Gorkin.

Photo of Krishnan Shankar is courtesy of Krishnan Shankar.
Some Things I’ve Learned About How to Hire

Alice Silverberg

The editor of the Early Career section of the AMS Notices has asked me to give advice for more senior mathematicians on the theme “How to hire a mathematician.” I’ve told stories about hiring (interspersed with some advice) at https://numberlandadventures.blogspot.com, often of what went wrong, but sometimes of what went right. I recently read some of these stories while asking myself “what can we learn from this?” I’ve collected below some of what I’ve learned. I hope it will be useful for those who hire.

1. Write job ads that say what they mean and mean what they say.

I’m frequently contacted by people who ask me for advice about how or whom to hire, or want me to help them spread the word about their job search. They tell me what they’re looking for in applicants.

Before responding, I look up the job ad and compare it to what I was told they’re looking for. It’s surprising how different those two can be.

My first recommendation is that job ads should say what they mean and mean what they say. The ad should give the true criteria on which you’ll base your decision. Make the criteria and hiring procedures public and clear, and stick to them.

It’s not good when an “inner circle” of applicants has access to information, or to the “unwritten rules,” that the rest of the applicants don’t have. The inner circle knows the real rules, and knows which rules and deadlines they can ignore and get away with. People who know people in your department or on the hiring committee, or whose advisors have friends on the faculty, shouldn’t have an unfair advantage.

The “rules of the game” should also be reasonable and make sense. Sometimes, job ads are so specific that it’s clear that the people who wrote it already know whom they want to hire. They’re just going through the motions, following the letter of the rules but not the spirit. If you have already decided whom you want to hire, don’t waste the time of other applicants by posting a job ad.

If a goal is to hire good people, try to write a job ad that gives you maximum flexibility, and doesn’t needlessly tie your hands. That helps you avoid a situation where the applicant you want to hire doesn’t satisfy the criteria in the job ad.

Here’s a story that illustrates some of the above:


2. Advertise widely.

Don’t rely on an “old boy network” of people you know (even if that’s a diverse group).

3. Put together a diverse hiring committee.

There’s a natural tendency (which we should all be fighting) to hire people who remind us of ourselves. We all have blind spots. While we can and should work hard to overcome our own subconscious biases, a diverse hiring committee makes it easier to hire the best people and not overlook them.

Here’s a (hopefully amusing) story where a diverse hiring committee might have been helpful:

https://numberlandadventures.blogspot.com/2018/01/hire-me.html

4. Choose the best applicants for the job.

When you decide to whom to make an offer, choose the best applicants, taking into account what the ad says you’re looking for. This might seem obvious. But the below list includes some stories where that didn’t happen, with various rationalizations that I didn’t think were reasonable. (I do understand that “best” is subjective, and perhaps impossible or unreasonable to pin down. That’s why it’s helpful to think in advance about what your goals are, and to write an ad that helps you achieve them.) In particular, resist the temptation to base hiring decisions on guesses about the candidates’ personal lives, and whether or not a candidate will take a job, or will stay.

https://numberlandadventures.blogspot.com/2021/09/but-she-misled-us-or-hire-people-based.html


https://numberlandadventures.blogspot.com/2021/09/we-assumed-you-were-not-interested-or-how.html

https://numberlandadventures.blogspot.com/2017/08/but-is-she-on-trajectory-to-get-fields.html

https://numberlandadventures.blogspot.com/2017/08/the-meritocracy.html

5. Follow best practices.

Train faculty and staff in best practices for hiring. Best practices include not asking irrelevant personal questions during job interviews. And being prepared to intervene if others who haven’t been trained (such as faculty spouses) ask those questions, or say or do something they shouldn’t. Here are some stories that illustrate this:

https://numberlandadventures.blogspot.com/2018/01/personal-questions.html

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6. Behave professionally, ethically, and legally, and hold people accountable.

Behave professionally, not just to the applicants during their job interviews, but all through the process, to the applicants, to other faculty and staff who are involved in hiring, etc. This isn’t your personal life, it’s your job. Do your job professionally.

Put in place good practices and policies that make it easy for people to do the right thing, and hold people accountable when they break the rules.

At many times in our professional lives, but especially in hiring, it can be helpful to ask ourselves “Is this professional? Is this ethical? Is this legal?”

Here are some relevant stories, which could just as well have been included in the section above on choosing the best applicants:

7. Be honest.

Be honest, not just in job ads, though that too. Most importantly, don’t mislead job candidates. For example, if they are promised something during the hiring process, they have a right (possibly a legal one) to expect such promises to be honored.

Further, don’t bring an applicant to your campus under false pretenses. I know of cases where applicants were told they were being invited for a job interview, and didn’t learn until halfway through the visit that it wasn’t actually a job interview. To get the job, they would have needed to visit again. This isn’t fair to applicants who are traveling a lot for job interviews, and need to decide which trips are worth taking. I know of other cases where candidates were told at the interview that they weren’t really being considered for the job they had been told they were interviewing for.

Honesty needs to extend further than just the job candidates. The people doing the hiring also need to be given accurate information, so they can deal honestly with the candidates.

Two stories:

8. Be kind.

Well, obviously we should be kind to job applicants. Early career applicants are at a vulnerable moment in their lives, and we should make them feel welcome and wanted.

Going the extra distance to do or say something nice, even if it’s small, can make a tremendous difference and be remembered for a long time. Negative interactions will also be remembered for a long time, possibly longer!

Perhaps less obviously, we should also be kind to our colleagues (including staff) who are involved in hiring. Some of the nastiest interactions I’ve seen have involved hiring. We can disagree, and disagreeing is often necessary in order for us to do our jobs well. But it’s easier to do our jobs well, and our communities are better and happier places, when we argue respectfully.

These stories of welcoming and unwelcoming experiences during the hiring process might help to illustrate what I mean:

To summarize:

1. Write job ads that say what they mean and mean what they say.
2. Advertise widely.
3. Put together a diverse hiring committee.
4. Choose the best applicants for the job.
5. Follow best practices.
6. Behave professionally, ethically, and legally, and hold people accountable.
7. Be honest.
8. Be kind.

I hope that keeping these goals in mind will help our community behave professionally, fairly, legally, and kindly.

ACKNOWLEDGMENT. I thank David Pollack for helpful feedback.
Interactions between Elasticity and Fluid Mechanics

Maurizio Garrione, Politecnico di Milano, Italy, and Filippo Gazzola, Politecnico di Milano, Italy, Editors

This book collects different points of view on these phenomena and is addressed both to junior researchers entering the field as well as to experienced professionals aiming to expand their scientific knowledge to closely related disciplines. The book also aims to bring the mathematical and engineering communities closer to create a common language and to encourage future collaborations.

EMS Industrial and Applied Mathematics, Volume 3; 2022; 248 pages; Hardcover; ISBN: 978-3-98547-027-3; List US$75; AMS members US$60; Order code EMSIAM/3

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On Monday, January 25, 2022, Professor Dr. Christine Bessenrodt passed away at the age of 63 after a short serious illness. Mathematics has lost an original, dedicated, and prolific scientist and teacher of international class, who will be sorely missed. Christine will be remembered for her unwavering devotion to all of her professional activities, both scientific and administrative. She was the author of more than 100 scientific papers with more than 40 collaborators. Her papers were on a wide variety of topics centered around representation theory of finite groups and finite dimensional algebras, algebraic and enumerative combinatorics, and additive number theory. She valued and maintained contacts, both professional and social, with numerous colleagues. Her collaborators benefitted from having a perfect coauthor. She was good at finding interesting research problems and at giving new insights into existing ones, often making vague ideas precise. Her many conjectures were usually correct. She was a very good host for all of her visitors, making sure that the social as well as the mathematical part of a visit was rewarding. It was always a pleasure to be in her company. Christine will also be remembered for her involvement in supporting and promoting equality for female mathematicians, which started early in her career. This included her commitment to European Women in Mathematics as well as her many years of dedication to diversity and equal opportunities at the German Mathematical Society. The Emmy Noether Lecture at the society’s annual conferences was Christine’s initiative.

Christine Bessenrodt was born in Ahlten, Germany, on March 18, 1958. Both her parents were physicists, her father was a professor of physics in Düsseldorf. After her Abitur in 1975, she studied mathematics and physics at the universities of Düsseldorf (1975–1977) and Essen (1977–1979). In 1980, she received her doctoral degree at the University of Essen, under the supervision of Professor Gerhard Michler. Her dissertation was on a topic in modular representation theory of finite groups: “Indecomposable lattices in blocks with cyclic defect groups.”

During her studies she was supported by the Studienstiftung des Deutschen Volkes, Germany’s scholarship foundation for exceptionally gifted students. From 1980 to 1990, she held positions as scientific associate or research assistant at the universities of Essen and Duisburg. She visited the University of Illinois, Urbana-Champaign on a postdoc stipend from 1982 to 1983. In 1988, she obtained her Habilitation in Mathematics at the university of Duisburg: “Behaviour of some representation theoretical invariants under change of the group basis of a modular group algebra”. Then in 1990, she was awarded a prestigious five-year Heisenberg grant, and she was a member
of the Institute of Experimental Mathematics at the University of Essen from 1990 to 1993.

Christine accepted an offer of a professorship in algebra at the Otto-von-Guericke-Universität Magdeburg in the former East Germany in 1993. The universities of the former East Germany were at the time in a period of transition after the German reunification in 1990. The new professors from West Germany were expected to contribute in several ways. The mathematics department in Magdeburg was at the time traditionally strong in applied areas of mathematics. Christine’s professorship in algebra was supposed to be the starting point for the development of new strengths in pure mathematics. With her constant commitment, Christine laid the foundation for a still-thriving institute in Magdeburg. Initially she was also responsible for teaching the courses in algebra and related areas at all levels. In the years following her appointment, new professors were hired in discrete mathematics, algebraic coding theory, and geometry. In 1994, she was appointed vice dean for the faculty of mathematics, a challenge she met with dedication and discipline for several years. She regularly invited colleagues for research visits. A particular highlight was the ICM Satellite Conference on “Representations of finite groups and combinatorics,” which took place in Magdeburg in 1998.

In 2002, she accepted the call for a chair for algebra and number theory at Leibniz Universität in Hannover, and she remained in Hannover until her death. Substantial changes in pure mathematics were lying ahead for the institute. Several professors were about to retire and this was seen as an opportunity to establish new research areas. Christine was the first professor in Hannover working in representation theory, but with her strong combinatorial background she could connect well to some existing groups in discrete mathematics and related areas. She soon became a driving force in restructuring the institute, a move supported by many others. The former “Institut für Mathematik” was split up into four smaller institutes for pure mathematics and Christine acted successfully for more than 15 years as director of the new “Institut für Algebra, Zahlentheorie, und Diskrete Mathematik.” With her tireless energy and impressive management skills she took care of the scientific development of the institute and of all the administrative challenges. Under her directorship two new professors were appointed in number theory and in discrete mathematics, and two further permanent positions were filled in the area of representation theory. All these efforts resulted in a very active institute with numerous postdocs and regular visitors over many years. It is due to Christine’s academic leadership that Hannover became recognized as a center for representation theory and algebraic combinatorics.

We present here an overview of most of Christine’s research, consisting mainly of surveys of some central topics. Christine’s first published papers around 1980 “On blocks of finite lattice type I–II” were based on her dissertation. They were followed in the period until 1991 by a number of other papers on various topics in modular representation theory, e.g., on modular invariants and on the Auslander–Reiten quiver of a modular group algebra. Other work of Christine in this period arose from a collaboration with G. Törner and H.H. Brungs on right chain rings, done primarily during her time in Duisburg. This resulted in a monograph in three parts (1985–1986).

The ordinary and modular representation theory of the finite symmetric and alternating groups \( S_n, A_n \) and their covering groups \( \hat{S}_n, \hat{A}_n \) is involved directly or indirectly in a majority of Christine’s papers after 1990. About half of these were written in collaboration with Jörn B. Olsson.

The covering groups \( \hat{S}_n, \hat{A}_n \) are non-split extensions of \( S_n, A_n \) by a central subgroup of order 2, e.g.,

\[
1 \to \gen{z} \to \hat{S}_n \to S_n \to 1.
\]

Irreducible representations/characters of the symmetric and alternating groups and their covering groups are labelled in a canonical way by classes of partitions of integers and these labels carry important information about the representations. This means that questions about partitions, both combinatorial and number theoretic, play a role in most papers. The combinatorics may, for example, involve bijections between sets of partitions or properties of a single partition (hooks, \( p \)-cores, \( p \)-quotients) and the number theory may involve partition identities.

We present some of these concepts in a short survey.

Partitions, Young diagrams, hooks: A partition \( \lambda \) of \( n \) is a sequence of natural numbers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0 \) and \( \lambda_1 + \lambda_2 + \ldots + \lambda_m = n \). The \( \lambda_i \)'s are the parts of \( \lambda \). For example \( \lambda = (5,4,4,1) = (5,4^2,1) \) is a partition of 14. We may use an exponential notation for repeated parts in a partition. A partition \( \lambda = (a_1, a_2, \ldots, a_m) \) is visualized in its Young diagram \( \text{Y}(\lambda) \) of boxes, arranged in left-justified rows, with the row lengths equal to the parts of \( \lambda \) in non-increasing order.

The Young diagram of \( \lambda = (5,4,4,1) \) and of its transposed partition \( \lambda^T = (4,3,3,3,1) \):

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 \\
\end{array}
\]

Generally, by definition, \( \text{Y}(\lambda^T) \) is obtained by reflecting \( \text{Y}(\lambda) \) along its diagonal.

A bar partition of \( n \) is a partition \( \lambda \) with distinct parts. It is visualized in its shifted Young diagram \( \text{SY}(\lambda) \), as illustrated...
in this example with \( \lambda = (6, 4, 2) \):

To each box in a Young diagram is associated a hook and a hook length. We show a hook of length 4 in \( Y(5, 4, 4, 1) \) and list all hook lengths:

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All hook lengths:

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8 6 5 4 1
6 4 3 2
5 3 2 1
3 1
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We illustrate the removal of a hook in a partition by an example:

Hook of length 4:

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4-hook removed:

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To each box in a shifted Young diagram, we associate a bar and a bar length. We show a bar of length 5 in \( SY(6, 4, 2) \) and list all bar lengths:

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All bar lengths:

```
10 8 6 5 3 1
6 4 3 1
2 1
```

If \( \ell \) is a positive integer, a partition \( \lambda \) is called \( \ell \)-regular if no part is repeated \( \ell \) or more times. Thus \( (6,4^2,1) \) is \( \ell \)-regular for \( \ell \geq 3 \). The bar partitions are the 2-regular partitions.

Labels of irreducible characters: The ordinary irreducible characters \( [\lambda] \) of \( S_n \) are labelled by the partitions \( \lambda \) of \( n \). The sign character \( (sgn) \) of \( S_n \) is equal to \( [\Pi^n] \) and it is known that \((sgn) \cdot [\lambda] = [\lambda^T]\) for all partitions \( \lambda \).

For a prime, the \( p \)-modular irreducible characters \( [\lambda] \) of \( S_n \) are labelled by the \( p \)-regular partitions of \( n \).

The ordinary irreducible characters \( (\lambda) \) of \( \hat{S}_n \) do not contain the central element \( z \) in their kernel are labelled by the bar partitions of \( n \). They are referred to as spin characters.

Branching rules describe restrictions of irreducible characters from \( S_n \) to \( S_{n-1} \) or from \( \hat{S}_n \) to \( \hat{S}_{n-1} \) in term of their labels. For example, the restriction of \([5, 4^2, 1] \) to \( S_{13} \) is \([4^3, 1] + [5, 4, 3, 1] + [5, 4^2] \). The partition labels of the constituents are obtained by removing a 1-hook from \([5, 4^2, 1] \).

Partition identities: Suppose that \( A \) and \( B \) are properties/conditions, which a partition may satisfy. Then the following statement is a partition identity: The number of partitions of \( n \) satisfying \( A \) equals the number of partitions of \( n \) satisfying \( B \). As an example we mention an identity due to Euler: The number of partitions of \( n \) with distinct parts \( (A) \) equals the number of partitions of \( n \) with only odd parts \( (B) \). A partition identity may be verified by constructing a bijection between the sets of partitions satisfying \( A \) and \( B \) or by using generating functions.

Christine’s first paper involving partitions was on the Andrews–Olsson partition identity, a result on partitions with congruence conditions [1]. This identity was discovered while looking at fixed points of the involutary Mullineux map \( M \) on the set of \( p \)-regular partitions and it provided evidence for the Mullineux conjecture [2].

The Mullineux conjecture states that \((sgn) \cdot [\lambda] = [\lambda^M]\) for all \( p \)-regular partitions \( \lambda \). Thus \( M \) should be a \( p \)-analogue of the transposing map \( T \). The conjecture was proved around 1995 after being reduced to a combinatorial statement by A. Kleshchev (in his series of papers on modular branching rules). Some of the earlier papers written by Bessenrodt–Olsson focussed on the Mullineux symbols and their variant residue symbols which led to a simpler proof of the Mullineux conjecture in 1996 [5]. The residue symbols were applied in two later papers by Bessenrodt–Olsson to study Jantzen–Seitz partitions (which label the \( p \)-modular irreducible representations which remain irreducible upon restriction to \( S_{n-1} \)) and modular branching in alternating groups.

Christine obviously enjoyed working with partition theory and this resulted in a number of papers that often drew inspiration from representation theory. This includes papers on hooks in Young diagrams [3], on partition identities, on partition congruences with I. Pak, and on multiplicative properties of partitions with K. Ono [4] and O. Beckwith.

Christine’s first paper on the covering groups of \( S_n \) was on decomposition matrices for spin characters of symmetric groups in characteristic \( 3 \), a collaboration with A. O. Morris and J. B. Olsson, published in 1993 [6]. One of the challenges in the paper was that the modular irreducible spin representations and canonical labels for them were not known until 2002 (Brundan–Kleshchev). The investigations of possible labels in [6] led to a conjecture of a partition identity, which was proved by Christine in a paper in collaboration with G. E. Andrews and J. B. Olsson [7]. A later paper by the same authors included a refinement of the identity.

Character degrees: The degrees of irreducible characters of \( S_n \) are given by the celebrated hook formula: If \( f_{\lambda} \) is the degree of the character \([\lambda]\), labelled by the partition \( \lambda \) of \( n \) and \( H(\lambda) \) is the product of all hook lengths in \( Y(\lambda) \), then \( f_{\lambda} = \frac{n!}{H(\lambda)} \).

For the spin characters there is an analogous bar formula for the degree which also involves a power of 2.

In 1998, Christine posed the problem of classifying the irreducible characters of \( S_n \) of prime power degree. This problem was solved by Christine in collaboration with J. B. Olsson and the number theorists A. Balog and K. Ono [8].
The proof involved an analysis of hook lengths and a new number theoretical result on prime factors in consecutive integers. The method was later also applied to the covering group $\tilde{S}_n$ for odd primes.

\textbf{p-cores, p-blocks and generalizations}: The $p$-core $\lambda(p)$ of a partition $\lambda$ is obtained by removing repeatedly $p$-hooks from $\lambda$ for as long as possible. Thus $(5,4^2,1)_2 = (3,1)$:

\begin{align*}
(5,4,4,1): & \quad \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array} \\
(3,3,2,1): & \quad \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot
\end{array} \\
(3,1): & \quad \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
\end{align*}

Analogously you may define the $p$-bar core of a bar partition of $n$ when $p$ is odd.

The distribution of irreducible characters of $S_n$ into $p$-blocks for all primes is determined by the $p$-cores of the labelling partitions (The "Nakayama conjecture"): Two characters $[\lambda]$ and $[\rho]$ are in the same $p$-block if and only if $\lambda(p) = \rho(p)$. When $p$ is odd, a similar statement holds for the spin characters of $\tilde{S}_n$.

The case of spin characters in characteristic 2 is essentially different since the 2-blocks of $\tilde{S}_n$ may contain both ordinary and spin characters. It was solved by Christine and J. B. Olsson in [9]: A spin character $[\lambda]$ labelled by a bar partition $\tilde{\lambda}$ is in the 2-block containing the ordinary character $[\text{dbl}(\lambda)]$. Here $\text{dbl}(\lambda)$ is obtained from $\lambda$ by breaking each of its parts into two (almost) equal parts.

In 2003, B. Külshammer, J. B. Olsson, and G. R. Robinson [10] presented a generalised $\ell$-block theory for symmetric groups $S_n$ for arbitrary $\ell > 0$ by character theoretic methods. This was closely related to the Iwahori–Hecke algebra $\mathcal{H}_n(q)$, where $q$ is a primitive $\ell$-th root of unity. The $\ell$-blocks satisfied a "Nakayama Conjecture" and there were analogues of basic sets, decomposition matrices, and Cartan matrices and their invariant factors. There was also a conjecture on the invariant factors. This work was the background for some of Christine’s papers. In 2010, Christine and D. Hill formulated a block-wise refinement of the conjecture in [10], and in 2015 this was proved by by A. Evseev. It is well-known that the determinant of the $p$-Cartan matrix of $S_n$ is a power of $p$. In 2001 in collaboration with J. B. Olsson, Christine provided a simple explicit formula for the exact values of these $p$-Cartan determinants. A conjecture of A. Mathas (in light of work of Donkin) stated that the $p$-power property should generalise to determinants of $\ell$-Cartan matrices. In [11], Christine and J. B. Olsson refined this to a conjecture for the exact values of these $\ell$-Cartan determinants. Both of these conjectures were proven by Brundan and Kleshchev in 2002; an alternative proof was later given by Christine, J. B. Olsson, and R. P. Stanley in [12]. This led to a number of papers concerning the determinants and Smith normal forms for submatrices of character tables, culminating in a paper by Christine and R. P. Stanley in which they provide a considerable generalisation of a result of L. Carlitz, D. P. Roselle, and R. A. Scoville from 1971 on ballot type sequences [13].

\textbf{Block equalities}: For a prime $p$ and a $p$-block $B_p$ of a finite group $G$ let $\text{Irr}(B_p)$ be the set of irreducible complex characters of $B_p$. In 1997, G. Navarro and W. Willems conjectured that if for different primes $p, q$ we have a block equality $\text{Irr}(B_p) = \text{Irr}(B_q)$ then $|\text{Irr}(B_p)| = 1$. They verified the conjecture for all blocks in solvable groups. Christine found a counterexample to the conjecture in the extension group $6.A_7$ of the alternating group. However the Navarro–Willems conjecture also holds for all blocks in the symmetric groups (J. B. Olsson, D. Stanton [14]) and their covering groups (C. Bessenrodt, J. B. Olsson [15]). In addition, the conjecture was verified for principal blocks in all finite groups by Christine, G. Navarro, J. B. Olsson, and P. H. Tiep in 2007. The more general question of block inclusions was also treated in [14] and [15]. In a paper by Christine with J. Zhang, it was shown that $G$ is nilpotent if and only if $\text{Irr}(B_p^n) \cap \text{Irr}(B_q^n) = \{1_G\}$ for all principal blocks and all pairs of primes $p, q$ dividing $|G|$ [16].

\textbf{Kronecker products}: Decomposing the Kronecker product (tensor product) of two irreducible complex representations of a symmetric group $S_n$ is considered to be one of the definitive open problems in algebraic combinatorics. It is referred to as the Kronecker problem and a general solution is not in sight. Starting in 1999, Christine and her collaborators wrote a series of papers on Kronecker products in $S_n$ and related groups. In a first paper by Christine and A. Kleshchev from 1999, it was shown that a nontrivial Kronecker product in $S_n$ is always reducible and never homogeneous. They also classified Kronecker products with at most three homogeneous components and made a conjecture for four homogeneous components. This conjecture was later verified by Christine in a paper with S. van Willigenburg in 2014. Other papers with A. Kleshchev dealt with Kronecker products of representations of $A_n$, of modular representations of $S_n$, and of spin representations of the covering groups $\tilde{S}_n, \tilde{A}_n$. A main theme was the classification of homogeneous and irreducible Kronecker products. Christine had conjectured a classification of multiplicity-free Kronecker products in $S_n$ in 1999. This conjecture was verified in 2017 in an impressive paper with C. Bowman [17].

An important part of the Kronecker problem is the positivity problem, i.e., deciding the positivity of Kronecker coefficients. In 2013 it was conjectured by G. Heide, J. Saxl, P. H. Tiep, and A. E. Zalesski, that for $n \neq 2, 4, 9$ there exists an irreducible character of $S_n$ whose Kronecker square contains all irreducible characters as constituents (The HSTZ-conjecture). J. Saxl conjectured that in the case $n = k(k + 1)/2$ the character $[\rho(k)]$, labelled by the staircase partition $\rho(k) = (k, k − 1, …, 2, 1)$ would be a
candidate (the Saxl conjecture). Christine applied a result on the product of spin characters to give a contribution to the Saxl conjecture and she formulated an analogous conjecture for spin characters [18]. In a paper with C. Bowman and L. Sutton, Christine verified Saxl’s conjecture for all irreducible characters of $S_n$ of odd degree [19]. This was a by-product of a result showing that Specht modules labelled by so-called 2-separated partitions are semisimple as $H^1_2(n)$-modules. The paper also contained a strengthened and a generalized Saxl conjecture. The latter involves symmetric $p$-cores. One of Christine’s final papers (with C. Bowman) was announced after her death [20]. It deals with the symmetric and anti-symmetric part of a Kronecker square. A main result is the classification of the partitions $\lambda$ for which the symmetric part (resp. anti-symmetric part) of the square of $[\lambda]$ is multiplicity-free (or zero). In addition, there are several new conjectures in the paper including refinements of the Saxl- and HSTZ-conjectures for symmetric and anti-symmetric parts.

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References


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Jørn B. Olsson
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In 1960, I heard a radio interview with an extraordinary man. He was a mathematician and was also a science fiction writer. He was about to be imprisoned because of his refusal to testify in front of the House Un-American Activities Committee (HUAC). His interview was articulate, eloquent, and reflected a deep commitment to his beliefs. Later, when I came to know him well, I understood what an inspiring and principled man he was.

The man was Horace Chandler Davis (widely known as “Chandler Davis” or “Chan Davis”). On September 24, 2022, he passed away in Toronto at the age of 96 from a probable stroke.

Chandler was a wonderful husband, father, and grandfather, an excellent mathematician, an extremely active political activist, an author of very interesting science fiction stories, a staunch feminist, and a fine poet and composer. He never seemed defeated by or bitter about the obstacles he encountered. He worked tirelessly towards a more egalitarian world, participating in many progressive activities throughout his long life.

Chandler was born on August 12, 1926 in Ithaca, New York, the eldest of five children of Marian R. Davis and Horace Bancroft Davis. His parents were economists whose political views were very left-wing. Like Chan, his father was fired from his position at a university because he refused to answer questions asked by HUAC.

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In 1953, Chandler was subpoenaed to be a witness before the House Un-American Activities Committee, which investigated allegations of communist activity in the United States. Chandler refused to answer the committee’s questions. Unlike most uncooperative witnesses, he invoked the First Amendment of the United States Constitution, which guarantees free speech, rather than using the Fifth Amendment’s protection against self-incrimination. Chandler wanted to establish a precedent that HUAC had no right to ask witnesses questions about their political beliefs. He knew that he risked being cited for contempt of Congress and sent to jail, but he wanted to raise awareness of the dangers of HUAC.

Chandler was then fired by the University of Michigan. On December 3, 1959, the Supreme Court refused to hear his case. Chandler surrendered to serve six months in federal prison.

Chandler continued his research in mathematics before, during, and after his incarceration. He retained his sense of humour throughout: A footnote to a mathematics paper [Dav63a] that he wrote while incarcerated reads:

Research supported in part by the Federal Prison System. Opinions expressed in this paper are the author’s and are not necessarily those of the Bureau of Prisons.

During the several years between his dismissal from Michigan and his imprisonment, Chandler applied for many different positions. It became apparent that he was blacklisted. The blacklist continued even after Chandler got out of prison in 1960.

In 1962, with support from the distinguished Canadian mathematician H.S.M. Coxeter, Chandler accepted a position as professor of mathematics at the University of Toronto.

Chan flourished at U of T. He was an excellent teacher, supervised fifteen PhD theses and continued to make significant research contributions to mathematics, especially to linear algebra and operator theory.

Chan’s teaching inspired many students to become mathematicians. James Arthur, a University Professor and Mossman Chair at the University of Toronto who served a term as president of the American Mathematical Society, writes:

Chandler Davis was my colleague for over forty years. I admired him greatly. His course in real and complex analysis, which I took as a third-year undergraduate at Toronto, was a transformative experience for me, and, I would say, for every other student in the course.

Chandler Davis was a left-wing radical who participated in a huge number of progressive causes, both on campus and off, throughout his long life. Chandler opposed the American–Vietnamese war and was chairman of the Toronto Anti-Draft Program. He was active in Science for Peace and often participated in the Toronto Vigil against the Occupation of the Territories. He regularly attended the Davis, Markert, Nickerson Lecture in Academic and Intellectual Freedom, established in the 1990s by the University of Michigan Faculty Senate in answer to the University’s treatment of faculty, including Chandler, who had been attacked by HUAC.

A few weeks before his death, Chandler co-organized and spoke from his hospital bed at an online event in support of imprisoned dissident Russian mathematician Azat Miftakhov. Chan began his talk as follows:

It is a pleasure to welcome you to this panel in support of our young colleague Azat Miftakhov and other political prisoners; in support, in particular, of Russians courageously speaking out against the war, and, more generally, in support of freedom of conscience and peace. It means a lot to me to be opening this session because I have a special bond to Azat Miftakhov. I was a political prisoner myself, years ago, not in Russia but in the USA. I was not much older than he is now; like him I had a wife standing by me outside; and like him I tried to go ahead doing mathematics while in prison. It was hard, but not as hard as Azat’s imprisonment, and it was only half a year.

Chandler often raised political issues within the community of mathematicians. Many mathematicians resented such activities, arguing that it was wrong to “politicize” mathematics.

Chandler was a very staunch feminist (see the contribution from Mary Gray later in this article). Chan and his wife, the distinguished historian Natalie Zemon Davis, agreed that their marriage would be based on gender equality. They shared care of their three children, even during periods of their lives when they held professorships at universities on opposite sides of North America, Chan at Toronto and Natalie at Berkeley and then Princeton.

When Chan turned 65, he was mandatorily retired and became a professor emeritus. That did not change his life very much. He still maintained his research, taught some courses, and supervised PhD students. He continued to serve as editor-in-chief of the Mathematical Intelligencer.

In 2010, Josh Lukin compiled and edited It Walks in Beauty, a compilation of some of Chan’s essays and stories. This book is in the Aqueduct Press series of Heirloom Books, which aims to bring back into print and preserve work that has helped make feminist science fiction what it is today.
John J. Benedetto

Chandler arrived in Toronto in 1962. I arrived in Toronto in 1962 with an MA from Harvard and having been raised in a working class suburb of Boston. So, at that point we had a little bit of Harvard in common, but not much else. I taught a section of calculus and Chandler was in charge. I became his PhD student the very first day we met—a separate story. It was the wisest, perhaps luckiest, happening of my mathematical life.

Chan seemed old to me. I had just turned 23, and he was 36. I had already decided on Laplace transforms, topological vector spaces, and Schwartz’s theory of distributions for a general thesis area. We met every week in the adviser/advisee dance. He taught me all of the things he could, that I could understand. I loved those meetings, and learned so much from him in them; but, dutifully, as any rebellious child would behave, I did nothing about it at the time. This was a real error on my part, since in later years I understood more and more how deep and ingenious and knowledgeable he was.

When Chandler told me about Naimark’s theorem and its importance and his creative contributions in this area, and all the wonderful mathematics he knew and did, I should have pursued all of it more actively. In fact, so many of his ideas and contributions play a major role in the theory of frames that I have been working on for 25 years. Frames, going back to Paley and Wiener, and then Duffin and Schaeffer, reemerged in the early 1990s as a vehicle for extending and applying wavelet and time-frequency systems both in terms of generality and genuine applicability. My weekly meetings with Chandler were replete with the linear algebra and operator theory necessary for such generality and applicability.

I learned so much from Chandler, without doing much about it when I was a graduate student. However, I knew from the beginning that Chandler was brilliant. What certainly stuck from our meetings were the breadth and overall appreciation and excitement of mathematics—I cannot imagine a better adviser. In any case, he let me run where I wanted to go, and filled in mathematical gaps prodigiously—a protective father, who kept me on schedule.

Early on in Toronto, I found out Chandler was a legend in a cause célèbre. His mathematics paper [Dav63a] was written while he was incarcerated as a result of his courageous testimony to the House Un-American Activities Committee. His red badge of courage (sic) (and humor) in this paper became a badge of pride and honor for me. In [Dav63a], my adviser thanked the Federal Prison System in the acknowledgments, further noting that his opinions there were not necessarily those of the Bureau of Prisons. My fellow North American graduate students only had advisers who could acknowledge national scientific support organizations. Wow, I was so fortunate! By the way, in [Dav63a] Chan solved the so-called second Ungar conjecture.

Knowing, I suppose, that I had pried into his past, he told me that his political activism was a thing of the past. Thank goodness he had second thoughts. He was brave and so principled, and it was his lifelong mantra. Our relation evolved and deepened over the years; and this is a beautiful experience I’ve had with many of my own graduate students. In the process, I learned so many things about the political environment that I still try to understand, and of which I am so bewildered. Most important we’ve had a decades-long correspondence on such matters, all to my benefit.

So when Schrecker’s No Ivory Tower [Sch86] appeared in 1986, along with the many reviews, I pounced on it and some of them. It was authoritative and gripping and compelling. And then Chandler’s own The Purge [Dav88] appeared. I knew some of these folks in that article! Raoul Bott and Ed Moise were my instructors at Harvard. Hans Lewy and I became friends through daily lunches and regular dinners while on our sabbaticals at the Scuola Normale Superiore in Pisa. And Lee Lorch and I became friends through Ray Johnson. Lucky me, reminding me of what Chan wrote in The Purge in a different context: “The experience of marginality is good for the soul and better for the intellect.” I never met Nate Coburn, whom Chan also highlighted in the The Purge, but his son Lew and I were office mates at NYU in 1964–1965 and are good friends.

Naturally, I met Natalie and was dazzled. What a couple—beyond anything I had ever imagined, and a diamond anniversary love affair. Her gift to Chandler on his 60th birthday was a book of his poems (Having Come This Far, 1986) selected by Natalie and their daughters, Hannah and Simone. Their son, Aaron, had set some of the poems to music. I just reread the poems. They are mostly beautiful and loving, perceptive and lyrical; and they are about Natalie and their family, but also about the pool-playing freedom rider and there is the eccentric seafood song and everything in between. Besides an amazing career, Natalie entertained graduate students in Toronto, e.g., me, and she and Chandler stayed with Cathy and me in Maryland. We had been in correspondence, and when Chandler died, she wrote that several days earlier she and Chandler found out that they would be having another
great-grandchild. She added that “life continues, with sorrow and with hope.”

Chandler introduced me to Laurent Schwartz when Schwartz visited Toronto. The three of us had a memorable (for me) lunch together. Close to graduation time, Chan gave some fatherly insights to me. He noted that I was not getting any younger (I was 24 when I received my PhD in 1964); and therefore I should work very hard. Lest this body blow was not sufficient, he also noted that an outsider might construe that anything in my thesis of worth was due to my adviser; and therefore I should work very hard. Chan was a very subtle fellow! His telephone call to NYU got me a tenure-track position.

Chandler has continued to be my hero through all the years, whether it was because of his poetry, his principles, or his mathematics—I never did read his science fiction. At a mathematical fest at the University of Maryland in 1999, he was virtuosic and humble and original and thoughtful as always. In 2019, he was scheduled to speak at another mathematical fest at Maryland. A month before the event, he wrote: “Whom am I kidding, John. I just can’t travel;” at the same time he wrote a long letter that I treasure. And then I was going to visit Toronto in May 2022, and I, too, had to cancel because I couldn’t travel. Alas. Bottom line and my last line: Chandler was extraordinary, and I am truly proud to be his student!

Rajendra Bhatia

Although the behaviour of eigenvalues of a hermitian matrix under perturbation is well understood, there has been almost nothing done on the behaviour of the eigenvectors. It is well known that they vary analytically under analytic perturbations, but for some purposes one would prefer sharp bounds on the distance between the eigenvectors of a matrix and those of a matrix approximating it.

—From the opening paragraph of Chandler Davis’s paper [Dav63b].

With admirable clarity Chandler Davis set himself the goal of finding such sharp bounds. His efforts culminated in the famous “sin ϑ theorem” of Davis and Kahan [DK70]. Let A and B be two hermitian matrices. Let E be the eigenprojection of A corresponding to its eigenvalues lying inside an interval [α, β], and F the eigenprojection of B corresponding to its eigenvalues lying outside (α − δ, β + δ).

Then the Davis-Kahan theorem is the inequality

\[ \|EF\| \leq \frac{1}{\delta} \|A - B\|. \tag{1} \]

This inequality captures several essential features of the problem. When \( A = B \), the projections E and F are mutually orthogonal (because the eigenvectors of A corresponding to distinct eigenvalues are mutually orthogonal). So \( \|EF\| = 0 \). If A is close to B, one might expect \( \|EF\| \) to be close to 0. The inequality (1) is a realisation of this. The dependence on \( \delta \) is dictated by several examples. Elegant in formulation and powerful in applications, the Davis-Kahan theorem is one of the best-known results in numerical linear algebra.

Among other things, Davis and Kahan recognized the connections between this problem and another involving the Sylvester equation \( AX - XB = Y \), of great importance in several areas [BR97]. When the spectra of A and B are disjoint, this equation always has a unique solution X for every Y. The problem is to find good bounds for the solution X. If A and B are hermitian, and the spectrum of A

![Figure 2. Rajendra Bhatia and Chandler Davis at the Grand Canyon, 1989.](image)
lies inside $[\alpha, \beta]$ and that of $B$ outside $(\alpha - \delta, \beta + \delta)$, then

$$\|X\| \leq \frac{1}{\delta} \|AX - XB\|, \quad (2)$$

and the inequality (1) can be derived from this [DK70].

The spectra of $A$ and $B$ need to be separated in a rather special way for the inequalities (1) and (2) to hold. If $K_1$ and $K_2$ are two arbitrary subsets of the real line with $\text{dist}(K_1, K_2) = \delta > 0$ and $E, F$ the eigenprojections of $A, B$ corresponding to them, then these inequalities break down. This was noted by Davis and Kahan, and the first open question posed by them was what best could be said in this case. In [BDM83] Chandler returned to this question with new collaborators to provide a decisive answer. They showed that there exists a universal constant $c_1$ (independent of the dimension of $A$ and $B$) such that instead of (2) we have

$$\|X\| \leq \frac{c_1}{\delta} \|AX - XB\|,$$

and a similar inequality holds in place of (1). Further, these authors showed that $c_1 < 2$. This was achieved by obtaining a new form of solution of the Sylvester equation expressed as a Fourier integral and then expressing $c_1$ as the solution of a minimal extrapolation problem for the Fourier transform. Unknown to the authors, this problem for the Fourier transform had been considered earlier in a totally different context (number theory), where it had been shown by Sz.-Nagy [SN53] that $c_1 = \pi/2$.

The authors of [BDM83] also considered an analogue of these problems when $A$ and $B$ are normal matrices. Now $K_1$ and $K_2$ are subsets of the complex plane with $\text{dist}(K_1, K_2) = \delta > 0$. In this case they showed that there exists a constant $c_2$ such that

$$\|X\| \leq \frac{c_2}{\delta} \|AX - XB\|,$$

where $c_2$ is the solution of a minimal extrapolation problem for the Fourier transform in the plane. This turns out to be a harder problem than that of determining $c_1$. It was shown in [BDK89] that $c_2 < 2.91$. (Later Hormander and Bernhardsson [HB93], in a completely different context showed that $2.903887282 < c_2 < 2.903887282$.)

This estimate for eigenvectors led to major progress on a longstanding problem about eigenvalues. In 1912, H. Weyl had shown that the eigenvalues of hermitian matrices $A$ and $B$ can be enumerated as $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ in such a way that

$$\max |\alpha_j - \beta_j| \leq \|A - B\|. \quad (3)$$

For several years many mathematicians tried to prove that the same result would be true for normal matrices $A$ and $B$. Davis and coauthors [BDM83] showed that a slightly weaker version with $c_2\|A - B\|$ instead of $\|A - B\|$ on the right-hand side of (3) is true. Later it was shown by Holbrook [Hol92] that $c_2$ here cannot be replaced by 1 (as had been believed for years). In the four decades since the publication of [BDM83] no further progress has been made on this problem, nor any other method found to handle it with success. To complete this story, in another paper coauthored by Davis [BD84], it was shown that the inequality (3) does hold when both $A$ and $B$ are unitary.

It was my good fortune that I met Chandler soon after my PhD and got introduced to these problems. Our collaboration began in 1980 and lasted until his death. As a collaborator he was generous and gracious. He was both intense and relaxed. After every discussion he typed the salient points and sent a note to his coworkers. But he would not hurry them on to publish. The three-author collaborations in [BDK89] and [BDM83] were coordinated by Chandler—I first met my coauthors McIntosh and Koosis much after the papers had been published.

Chandler devoted a tremendous amount of energy to various progressive causes. Observing him, I was struck by two things. He always treated people with the opposite view with respect and patience, and no activity that could advance a good cause was too small for his attention. As he would say, losing one’s illusions does not mean one should lose one’s hopes.

**Man-Duen Choi**

The work of Chandler Davis influenced a large number of mathematicians. For more than 40 years, Chandler was the mainstay of the Toronto operator theory seminars that meet Monday afternoons. Here I will informally describe Chandler’s research interests in operator theory. Serious readers are referred to a longer article [CR94] with a large bibliography listing 80 papers written by Chandler.

Chandler received his PhD from Harvard University in 1950. His doctoral thesis, written under the supervision of Garrett Birkhoff, was titled "Lattices of Modal Operators." Birkhoff was famous for the book *Lattice Theory* and for developing connections with quantum mechanics.

Three forerunners in operator theory had special impact on Chandler:

- **Mark Krein**: Chandler could read Russian, a big advantage for operator theorists of Chandler’s age. Chandler’s early research showed broad interest in modern analysis as practiced by the Soviet school of Mark Krein, continued by David Milman, Mark Naimark, Israel Gohberg, Vadym Adamyan, Mikhail Livsic, and others.
• Bela Sz.-Nagy: Bela Sz.-Nagy was a leader of Hungarian school of analysis, in charge of the journal Acta Sci. Math (Szeged). Chandler’s favorite book was Analyse harmonique des opérateurs de l’espace de Hilbert [SNF67], coauthored by Sz.-Nagy and Foias.

• Paul Halmos: Halmos was born in Hungary and had greatest impact in North America because of his excellent expository lectures and books. Indeed, Halmos’s book A Hilbert Space Problem Book [Hal67] has always been an inspiration for everybody in the field. Halmos visited Toronto in the 1970s. As highly motivated by Halmos, Chandler used 2-by-2 matrix techniques. Namely, the setting of a single operator $T$ in terms of four operators of the form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is indispensable for Chandler’s major research articles concerning norm structure (e.g., joint work with Kahan and Weinberger [DKW82], compression problems, dilation problems in line with Sz.-Nagy’s work, extension problems, and J-unitary structure as developed by Krein).

Chandler introduced the notion of the shell of a Hilbert-space operator [Dav70]. This 3-dimensional analogue of the numerical range has intriguing relations with various geometric properties. Chandler also dealt with the interesting Toeplitz–Hausdorff theorem on numerical ranges [Dav71].

As generalization of the triangle inequality, the Cauchy–Schwarz-type inequalities related to different convex structures appeared often in Chandler’s research. In particular, a deep theory was developed for the Kantorovič inequality [Dav80], which is a useful tool in numerical analysis and statistics for establishing the rate of convergence of the method of steepest descent.

Chandler claimed that two subspaces are easy, while three subspaces are much harder. This could be translated to fruitful results if a linear subspace was replaced by an orthogonal projection (as alias). In other words, the geometry of subspaces can be transformed to the algebraic features of the projections. Thus, the algebra generated by two projections is completely manageable, while the algebra generated by three projections becomes intractable.

Chandler proceeded further to describe the angle between two subspaces, by means of the subtle cosine and sine functions. On the other hand, Chandler established the following two beautiful results in one of his earliest papers [Dav55]:

• There exist three projections that do not have a nontrivial common invariant subspace.

• The algebra of all bounded operators on a separable Hilbert space is generated (as a weakly closed self-adjoint algebra) by three projections.

These two propositions are logically the same result connecting noncommutability with transitivity.

Chandler’s research also concerned the effect of perturbations on eigenvectors of a Hermitian matrix. Related successful papers with Kahan, Bhatia, McIntosh, and others dealt with normal matrices. This topic is discussed by Rajendra Bhatia earlier in this memorial article.

Aaron Davis

I knew Chandler Davis as a father first and foremost, and as a man of faith. He was not religious but had a deep abiding faith in humanity. He was also driven by a love of life and an insatiable curiosity about it, whether looking through a scientific, mathematical, musical, literary, or poetic lens.

Chandler’s American roots went back to the Mayflower and the early settlers of the Massachusetts Bay Colony, and also the first Quaker settlers of Pennsylvania. The tradition of political commitment ran deep on the abolitionist Quaker side of the family. Chandler’s great-grandfather Norwood Penrose Hallowell took a bullet as a captain in the Union army at the Battle of Antietam. The Hallowells ran a station of the Underground Railroad in Philadelphia. When my sisters and I were children, Chandler and Natalie would take us to civil rights and then anti-war demonstrations. After Chandler’s stand against HUAC, prison sentence, and subsequent blacklist, we moved to Canada, but he didn’t stop with political organizing, and we took in a succession of draft dodgers at our home in the late sixties.

Although keenly analytical, Chandler was also a very expressive man, and his poetry conveyed his deep feelings. In his 1968 poem Toronto Home, Chandler wrote of what I remember as a tranquil scene: our study where we would do our homework around the fireplace while Natalie and Chan prepared their university classes and the cat purred. He saw that “we are not secure...” and saw the cat as “timing our instability with neutral, implacable switching tail.” He saw the fire as casting “pseudopods that fade so fast” as the clock ticks. “We will not survive”, Chandler wrote, as our deaths are inevitable. But knowing that, he wrote: “Die with me, and in the waiting time, our life, caress and kiss, as if you meant to keep it, this temporary love....”

Even in his poems as a young man, Chandler dwelt on the impermanence of life but also of the longer arc of human generations and regeneration and of love and the fight for freedom being passed down somehow, of ideas

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and life being reborn. He will be missed but left a legacy of hope in his words and deeds.

**Natalie Zemon Davis**

I had seventy-four years of marriage with my beloved partner Chandler Davis, and they were filled with conversation. Indeed, the first day we met—in our student days in 1948—we talked the night away in Harvard Yard. We talked of our hopes for the future—for ourselves and for the world. We talked of our intellectual interests and our literary favorites. But there was an asymmetry in our exchange. I could tell him all about the Senior thesis I was starting on at Smith College. It was about a Renaissance Aristotelian philosopher and Chandler could ask knowing questions about him. He could tell me about the poems and science-fiction stories he had been writing and I could follow along and comment with interest. When it came to his doctoral studies, however, I could make little headway. Yes, I'd heard of algebra and took a class in it in high school. But what was linear algebra? I asked for classes in the history of science at Smith, and then established a field in the history of early modern science for my graduate studies, but they didn’t leave me with anything pertinent to discuss about Hilbert spaces.

The conversational asymmetry in regard to our work lasted throughout our marriage. Chandler patiently listened to or read my history manuscripts over the decades and made telling suggestions—for which I thanked him in my acknowledgments. I listened politely as he told me about a new idea or a new theorem, his eyes sparkling. I stood in admiration as he and other mathematicians exchanged information and made mathematical discoveries as they talked or reached for a piece of paper or a piece of chalk.

Especially, I stood in wonder as conversation among mathematicians would suddenly stop and silence would prevail for several minutes. That would never happen with historians and anthropologists—we just went rushing on with our facts, ideas, and associations. Chandler and the other mathematicians would stop simply to think and patiently wait for each other’s conclusions.

The other thing that struck me about Chandler’s scientific world was how international it was. I was used to have contact with historians from Europe who were interested in the same period and the same problems as I. But it was many years before I had links to scholars from Asia or North Africa—and then only because I had turned to problems in their regions. With Chandler, he was in communication with scholars from India and Japan already very early along. When I would see him and others at their international meetings, I was struck by how patient they were with linguistic barriers, how they did what was necessary to communicate.

Chandler was deeply committed to helping mathematicians in different lands, including mathematicians suffering from political persecution. He also applauded the younger generations for their efforts and achievements. Even while pondering the philosophical question of uses of mathematics for society, Chandler never lost his delight in its beauty.

**Simone Weil Davis**

It’s a pleasure to pause and reflect on how it felt, as a daughter who did not go into math, to stand near to Chandler’s life as a mathematician. So much else of our parents’ pursuits were discussed and available to us. How strange it was, to be almost “black box” uncomprehending and yet to notice how this part of his life showed up in our home. From a kid’s point of view, at the height of a tabletop, I remember the abacus, its painted wood marbles impaled on parallel metal rods, all held forever in a wooden frame. With respect, I understood that this toy I played with was somehow also a tool, a way of figuring, a manifestation of questions and answers. The dreamily smooth slide rule pulled me in, too; at times as a grade schooler I would know, briefly, how to use it, due to Dad’s patient, inventive instruction.

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Chandler brought that same creative, gentle practice to the math enrichment classes he helped lead on Saturday mornings; I was proud to see his clear-spoken encouragement of my classmates and friends.

He believed in numeric ideas, and in the intellect of those he addressed (regardless of age). With a pencil and paper, he would let the logic of the numbers take us along. Beauty would follow, if we could stay upright in our mental kayaks long enough to appreciate it. In the early 1960s, he let me watch an animated short he was working on with a colleague in Madison, Max in Mathemagick Land, and quietly explained to me the rules of existence if there were to be a culture that lived in two dimensions. That was a burst of beauty.

In her remarks above, Natalie mentioned Chandler’s many trips—sometimes extended journeys and often across the globe—to connect with his mathematical communities and project partners. As a child, while I’d miss him terribly, I also took note of the very particular kind of trust and interconnection, the bond, between him and his math-making colleagues. Most striking perhaps was his working friendship with Rajendra Bhatia over so many decades. We would pour over his letters from India, and I had a sense of him at ease in this other home, in this other land, climbing paths and walking under flowering trees that I’d never seen or breathed in. I would imagine him and Rajendra talking as they walked, or sitting in silence, jumping up to write on a chalkboard. It seemed such a peculiar, particular thing to me, his mathematical friendships. For one, it made the wide world appear traversable, and interconnection between people everywhere as natural, inevitable, even urgent. To an extent, the connections looked like an emotionally neutral partnership, shaped by rigor and large, shared puzzles. But against the backdrop of that cool profundity, everyone’s particular personalities stood out like wonderful, life-packed anomalies. Your Lee Lorch, your Peter Rosenthal, your Marjorie Senechal! Each person seemed outsized to me and with wonderfully internecine idiosyncrasies, extraordinary because of the bond they shared with my father.

The largest invisible manifestation I sensed, when I looked at Chandler’s engagement with mathematics and his mathematical friends, was love. Love married to inquiry, logic married to imagination, and a bottomless curiosity, as shown in this excerpt from his poem Whether, which Chan was still revising just months before his passing.

It could be someone’s looking at our world
From distant worlds moving at near-light-speed,
But is it meaningful to feel akin to them?
When we can’t invariably distinguish whether
They’re close in time or in our distant future?

And yet I hope they watch and wonder, whether
They think contemporaneously or not.
If nothing in the cosmos tells us whether
It opens out forever in space and time
Or curls upon itself in space and time
So that geodesics long enough are closed,
How could it ever be conceived of whole?
And yet I fondly love it whole, whether
My concept of it has some truth or not.

Hannah Davis Taïeb

My earliest memories of my father as a mathematician come with a feeling of joy.

My father sat on the floor with us when we were little, sharing ideas: matrices, sets, imaginary and real numbers, bases, different ways of looking at the finite and infinite, at the 4th dimension, at time. Somehow, the way he taught, it wasn’t “difficult;” he made it seem charmingly complex, delightfully intricate and precise, and therefore easy, pure, self-evident, each idea emerging from the previous one. His students, later, must have felt some of that same ease, I imagine.

I remember doing the dishes with my father; I was nine or so; he challenged me to prove that 3, 5, 7 were the only examples of consecutive odd primes. I dried a few dishes; he had given me the tools to answer him, and so I could. Sometimes I say that “he taught us mathematics as children,” but it was more like a kind of play, as if his own pleasure at the way things fit together was just too much for him to keep to himself, and so it poured out on us.

One biographer referred to my father as a “polymath.” I love this. His pleasure in abstraction—and in life—meant that he did not isolate his mathematical self, his mathematical reasoning, from all the other ways he had of engaging with the world. His science fiction writing, his poetry, his music—each domain of creation bordered on the others, and enriched the others. Going through his papers in recent weeks, I found clever skits about mathematical subjects, written when he was a student at Harvard. The link between poetic and mathematical expression carried throughout his life, culminating in the workshops on Creative Writing in Mathematics and Science at the math institute at Banff. The way he linked mathematics to every other aspect of his immense creativity made him a perfect person to be the editor-in-chief of the Mathematical Intelligence.

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And, of course, his mathematical self also intersected with his politics. Chandler could have lived out his passion for justice in so many ways; he could have felt a conflict between his political ideals and his profession. Instead, he linked the two, by showing solidarity with mathematicians around the world, standing up for those with whom he shared a profession, a way of life.

Along with my admiration for all that he has done is my pleasant memory of sitting on the floor as a tiny child, playing with numbers, colors, and ideas, and feeling that sharing of joy.

John Friedlander

I am honoured (okay, honored) to have been included amongst the invitees to contribute to this memorial article about H. Chandler Davis. Perhaps I did not know him as well as some of the other authors, but I did know him for a very long time and I do have some memories of him that I hope are worth recounting.

As I recall, Chandler arrived at the University of Toronto and began teaching here sixty years ago, in the autumn of 1962. I took his third-year analysis course the very next year. It was a wonderful experience. Like almost all U. of T. math courses at the time, it was a full-year course. Unlike every one of the many other undergraduate math courses at the time (through all four undergraduate years), it met three hours per week whereas all the others met for only two. This of course was Chandler's doing. In the Fall term we studied metric spaces, Banach spaces, then Hilbert spaces. I don't recall any specific text though there were a number of them floating about. In the winter term the course covered an introduction to complex analysis using volume one of Hille's two volume set. My preference for functions of a “complex” variable started right then.

We had lots of talented students in the course, also before and after my year. Stephen Fienberg, who became a prominent statistician, took it in its first year (and let me hang onto his beautiful lecture notes during the following year). James Arthur took it the year after me. But there were several others who went on to distinguished careers.

As the year progressed, we gradually picked up bits about Chandler’s history in the US that had taken place shortly prior to his coming to Canada. Being Canadian and being university-age students, you can guess where our sympathies lay, how we admired him for his courage.

One day there was, amongst the students, a great buzz of excitement. It seems that one of the “guys” in the class had phoned Chandler at home, something unheard of in those days. He related the following: “A woman answered and said ‘Hello’,” “There was a pause “May I please speak to Professor Davis?” Response without a pause “Which one?” I realize that these days such an exchange would create no excitement at all. But this was a very long time ago. We were all atwitter.

It was quite a number of years before our paths crossed again. I returned to Toronto for good in 1980 and suddenly we were colleagues. Then, within a few years, I was Chandler’s chairman. This might have been a problem. I had always thought of Chandler as one who is tough on authority figures, although not on others, so I was a bit leery about being his chairman. But any genuine cause for this worry never did transpire.

He was tenacious however. I remember being in Kyoto for the 1990 ICM and being engaged in mathematical conversation, catching up with my former postdoc Andrew Granville and being blissfully happy about it, when suddenly Chandler rushed up from out of a crowd, pointing his finger at me with the words “Some Departmental Business!” I was not kind: “Not here, not now!”

There was one incident near the end of my term that I remember very well. It was then the law in the Province of Ontario, and most of the other provinces as well, that professors (and many others) faced mandatory retirement at the age of sixty-five. The Faculty Association and numerous individuals had been pushing hard for the abolition of this requirement for quite a number of years.

So, Chandler came to my Chairman’s office one day and the conversation went almost verbatim like this. He began:

I shall be sixty-five soon and I very much do not want to retire. So, I went to see the dean and he told me the following: ”You do not need to retire. This is up to your chairman to decide. He can continue to pay your salary if he chooses to.”

I should at this point state that I completely believed Chandler’s story. This dean was a person whose definition of the truth was “whatever you can get someone to believe”.

So here is what I told Chandler: “On June 30 of the year your retirement is due, the Provost will take your salary out of the department budget. There is nothing that anybody can do about that. I have an annual departmental budget of some three to four million dollars. Out of that, I am unable to touch all but one account which has about one hundred and fifty thousand dollars. I use that account to help support our faculty in hiring postdoctoral fellows. I could give you that money but we would then not be getting postdoctoral fellows.”

To Chandler’s credit, he both believed me (he too knew that dean) and he didn’t argue the point. This may sound obvious but some faculty have been willing to volunteer to

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What more can be said about Chandler Davis, a hero to me and many others from diverse backgrounds around the world? I can add that he was a feminist in the true sense of the word, in his life and work embodying a deep respect and appreciation for the aspirations and achievements of women and men, all the while campaigning in his multitalented way for us all to take responsibility to make a better world.

And perhaps most important, Chandler provided the respect and support that bolstered our own confidence in ourselves, confidence and motivation to work on a multiplicity of human rights in the US and around the world. That Chandler remained steadfast in his determination, but with respect for the less enlightened, with his eyes on the prize—be the goal justice and equity for all or just moving along to the next step in the right direction—meant having a champion in our own community.

We remember how when I and a handful of other women in mathematics came to recognize more than 50 years ago that while we faced obstacles we needed to organize to take responsibility for improvement, he was there, one of the first male members—and an active one—of the Association for Women in Mathematics, repeatedly challenging the establishment. When we hear talk about double-blind refereeing, I remember Chandler's disdain for the prominent male mathematician who asked "How could we know that a paper was any good if we did not know who wrote it?"

And when the AMS Council debated support for a young woman mathematician identified as among the "disappeared" in the dirty war in Argentina, Chandler did not join the skepticism about whether her mathematical output merited our support, a question not asked about the similarly situated male victims of human rights violations.

When many mathematicians were demonstrating against the Vietnam War, Chandler not only went to Vietnam to meet with mathematicians in the North but succeeded in bringing recognition to a woman researcher who had taught mathematics for years in the midst of fighting for her country.

When Chandler lost his case in the Supreme Court we knew that it was for the free speech rights of all of us—all genders, all colors. To have on our side someone, really an icon, who never gave up the struggle—a mathematician and a person who led a committed life in a way to which we might aspire—was an inspiration for which many will always be grateful.

Mary Gray

What more can be said about Chandler Davis, a hero to me and many others from diverse backgrounds around the world? I can add that he was a feminist in the true sense of the word, in his life and work embodying a deep respect and appreciation for the aspirations and achievements of women and men, all the while campaigning in his multitalented way for us all to take responsibility to make a better world.

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Stephen Halperin

My connection with Mathematics at U of T began as a freshman in 1959 and continued for 40 years (with a break for graduate school at Cornell) until I moved to Maryland. So I never took a course from Chandler and I don’t think we met or knew each other until I returned as a young faculty member. Even then, we taught at different campuses and our mathematical interests were remote. And by the time I moved to the main campus, Chandler had retired.

But I was a very interested member of the large audience in the spring of 1965 when Chandler hosted a teach-in (possibly the first at Toronto) on the war in Vietnam. He had invited speakers who were well known for their positions (both pro and anti the war), and was very clear that he expected an honest presentation, not a political rally.

However, before he could open the event, a small group from the Berkeley Free Speech Movement appeared on the stage and announced that unless the distinguished pro-war professor was removed from the speaker list, they would shut down the teach-in. I will never forget Chandler’s response—he said:

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I spent time in a US prison because I would not provide the House Committee (HUAC) witch hunt with the names of people I knew. When you have done as much to protect free speech, then I will listen, but in the meantime just go and sit down.

And they did.

Chandler’s teach-in opened my eyes to responsibilities outside my discipline and really started me on my path to anti-war activism at Cornell. I may not have learned much math from Chandler, but his example of courageous integrity has stuck with me, and there have been times in my professional life when I have needed to remember it.

**Sheldon Axler**

One day in August 1983, I nervously walked several blocks in downtown Warsaw, carrying in my backpack several hundred leaflets produced by Polish mathematicians associated with the then-banned Polish union Solidarity (Solidarność). Martial law aimed at suppressing Solidarity had ended in Poland only one month earlier. Chandler Davis had asked me to deliver the leaflets I was carrying to a Polish mathematician who would arrange for their distribution to mathematicians attending the International Congress of Mathematicians, and of course I was happy to do so.

Chandler’s involvement with the Polish mathematicians was typical of his lifelong support for human rights. I first heard of Chandler when I was an undergraduate, doing a summer research project on convexity. I came across Chandler’s paper on plane convex curves [Dav63a] that contains his now famous footnote about support from the Federal Prison System, as quoted earlier in this article by Peter Rosenthal. I was intrigued, and even more so after I learned about why the author was in prison.

Then a bit later I read Chandler’s beautiful paper [Dav71] that explains why the numerical range of every operator on a Hilbert space is convex. Paul Halmos’s review of this paper in *Mathematical Reviews* refers to “the elegance of the proof” that Chandler had constructed.

Finally I met Chandler in person when I was a graduate student at Berkeley. Chandler’s wife Natalie, an eminent historian of early modern France, was a history professor at Berkeley at that time, so Chandler often visited Berkeley from Toronto. We had many conversations, and I came to greatly admire Chandler’s insight about multiple subjects, both mathematical and nonmathematical.

In 1987, I became editor-in-chief of the *Mathematical Intelligencer*. The Reviews section of the *Mathematical Intelligencer* was responsible for publishing a few reviews in each issue, mostly of books that would interest mathematicians but also of other relevant items such as mathematical art. Because of Chandler’s deep knowledge of so many aspects of the history and culture of mathematics, I invited him to be the reviews editor of the *Mathematical Intelligencer*. I was delighted when he accepted this invitation.

Of course Chandler did a terrific job as reviews editor of the *Mathematical Intelligencer*. This was the time when Chandler and I started the custom of having dinner together, just the two of us, one night each year at JMM to discuss multiple topics of mutual interest.

When my term as editor-in-chief of the *Mathematical Intelligencer* entered its final year, I was happily surprised when Chandler was willing to become the next editor-in-chief. I think he had a lot of fun with his magnificent handling of the *Mathematical Intelligencer*.

At about the same time, Chandler was elected vice president of the American Mathematical Society, quite a reversal from the McCarthy period when he was blacklisted by American universities. The AMS had behaved decently during part of that time, hiring Chandler as an associate editor at *Mathematical Reviews* at a time when American universities were too frightened to offer him employment.

Chandler leaves a huge legacy as the model for someone who does the right thing in difficult circumstances, while continuing to make important contributions to mathematics.

**Marjorie Senechal**

It was a great privilege, and a great pleasure, to work closely with Chandler Davis for many years. He was an outstanding mathematician, a committed activist, a celebrated fiction and science fiction writer, a fine poet, a composer of gracious art songs, and the visionary editor of the international quarterly journal, *The Mathematical Intelligencer*.

Chandler’s many brilliant facets reflected a single *sui generis* whole. This unity was especially evident in the week-long creative writing workshops in mathematics and science that he and I co-organized at the Banff International Research Station for Mathematical Innovation and Discovery (BIRS) in Canada in 2003, 2004, and 2006 (the last together with the poet and philosopher Jan Zwicky). You can see it in our workshop anthology, *The Shape of Content*.

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We organized the workshops to encourage practitioners who engage this content in their work, to give them opportunities to discuss important issues, to learn what others are doing, to encourage each other, to critique current work, to welcome young writers into the field, to spark collaborations, and to forge networks and build community. In that sense, the creative writing workshops’ goals were the same as any other BIRS workshop’s.

But we based these workshops on the premise that Chandler himself personified: mathematics and science are part of world culture, part of the human spirit and, as such, are fitting subjects and themes for poetry, drama, short stories, novels, nonfiction, comic books, essays, film, and even music.

As we noted in our workshop proposals, creative writers don’t coerce their audiences to eat mathematics and science like medicine hidden in jam, they convey these ideas through art instead of formalism. True, plays like *Proof* and biographies like *A Beautiful Mind* and *The Man Who Loved Only Numbers* might have been less successful had the mathematician character been less idiosyncratic, but the play *Copenhagen* was also a great success. The novel *Einstein’s Dreams* conveys the scientific creative process in a beautiful way and *Arcadia*, a funny and chaotic play whose leitmotif is chaos theory, is a modern classic, and the mathematical formalism is symbolized in its structure.

Chandler’s egalitarian spirit infused the workshops. There were no leaders: everyone learned from everyone else, mathematicians and non-mathematicians alike. And creative writing was sparked by cross-genre insights: a poet helped a fiction writer find a better way to end his story, a mathematician nonfiction writer helped a dramatist extend the ideas of her play, ideas a filmmaker sitting in on their discussions recast in doggerel form. A novelist had insightful comments on poetry.

As Chandler explained ([Dav08]),

I remember Norberto had’t brought his white cane so going out for coffee on unfamiliar streets he gladly held for guidance my gladly offered elbow. And in mathematics, it was the blind leading the blind! Whenever one of us had guidance to give the other, it was a gladness to be giver, it was gladness to be receiver.

"I see, I see," Norberto murmured.

References


Credits

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Figures 2 and 3 are courtesy of Natalie Davis.
Figure 4 is courtesy of Marjorie Senechal.
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Caroline Seely was the first mathematician to be employed full-time by the American Mathematical Society. She worked as a clerical assistant to Frank Nelson Cole and R.G.D. Richardson while they served as secretaries of the Society. Beginning in 1920, while serving as Richardson’s clerk in the New York office, Seely also functioned as an editorial assistant for the Society. She served as an assistant editor of the *Bulletin* from 1925 to 1934, and as a cooperating editor of the *Transactions* from 1924 to 1936.

Initially hired in 1913 as a clerical and editorial assistant for the fledgling American Mathematical Society, Caroline Eustis Seely spent 22 years supporting the Society’s mission. During Seely’s tenure, her duties grew with the Society and her support in the work of its board members became indispensable.

Seely, born on August 3, 1887 in Delhi, New York, was the youngest of four children, eight years younger than her next youngest sibling. In 1901, following the death of her father, a retired naval officer, Caroline and her mother moved from their home in Philadelphia to New York City. A few years later, Caroline matriculated at nearby Barnard College, and is listed with the class of 1908. The only apparent evidence of involvement with college clubs or activities appears in the 1907 Barnard yearbook where Seely is listed as a member of the Barnard Chapter of the Church Students’ Missionary Association. Her eventual connection with the American Mathematical Society may have been initiated when she took courses in theory and practice of teaching, as well as the history of mathematics, from David Eugene Smith who was the Society’s librarian at the time. For reasons not ascertained, Seely took a leave from Barnard before concluding her final year, later returning to complete her studies and graduate with the class of 1911 (Figure 1). She graduated with general honors, was awarded departmental honors in mathematics and astronomy, and was elected into the prestigious Phi Beta Kappa honor society. She subsequently enrolled in a graduate program at Columbia University, where from 1911 to 1913 she also served as Smith’s assistant in mathematics at the university’s Teachers College.

Seely received her master’s degree from Columbia in 1912 with a thesis on the notion of limit prevailing in the modern theory of functions of a real variable. She continued her graduate work under the supervision of Edward A. August 2023 Notices of the American Mathematical Society

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1The Society was founded in 1888 as the New York Mathematical Society having the following objectives: mathematical conversations, solutions of problems, criticism of current mathematical literature, bibliographical and historical notes, and readings of original papers.

2Located about 30 miles south of Cooperstown.

3Carleton (b. 1872), Annie (b. 1873), and Mary (b. 1879).
Kasner\(^4\) and received her PhD degree from Columbia in 1915 with a thesis on nonlinear integral equations. She was one of four American women to earn a mathematics PhD that year, bringing the total to forty-nine. Seely was elected a member of the Society that fall [GL09]. Of the women who received their PhDs before 1940, she was one of only five to serve on a Society Editorial Board, Council, or Board of Trustees.\(^5\)

In 1913, at a salary of $60 a month, Seely was hired as a clerical and editorial assistant to Frank Nelson Cole (Figure 2).\(^6\) professor of mathematics at Barnard and secretary of the American Mathematical Society. At the time of Seely’s hire, the Society’s main office was located in East Hall on Columbia’s Morningside Heights campus. It is the oldest building on campus and the only surviving vestige of the Bloomingdale Insane Asylum. The Society’s library along with records, journals, and official documents resided in Columbia’s University Hall where they had been moved because East Hall was not thought to be fireproof [Spe14]. Ironically, on October 10 of 1914, a fire gutted University Hall. Fortunately no one was injured in the blaze, but ramifications of the destruction were widespread. The offices of the Prison Reform Bureau of the State of New York were destroyed, causing estimated damages of $100,000. Ernest Richard, professor of Germanic history, lost his academic library and all his research notes. The Society also suffered losses, somewhat mitigated by the efforts of Cole and Seely, who the following day combed through piles of debris that had been removed from the building and salvaged the Society’s meeting records from 1901 to 1910, albeit with some charred edges. Most notable among the Society’s losses were Cole’s personal records and many of the Council minutes from 1907 to 1913.\(^7\) It is likely that a good portion of the reorganization of surviving Society records fell on Seely’s shoulders.

The exact extent of Seely’s initial clerical duties for Cole beyond typing the Council minutes and preparing programs of Society meetings is not clear. She later recalled:

> When I first came to work for Cole, one of my jobs every fall was to sort them [Transactions] out into complete volumes and do each volume up in paper and label it and pile it on the shelves. When I ceased to have any odd minutes left for that sort of thing, and paper went up so high in price, we stopped doing this, and the result is, besides the messy appearance of the office, that I don’t have to look in four different places every time I want to fill an order for a single back issue. [See22, 1924]

Fortunately for the Society, Cole recognized Seely’s contributions. At the February 1915 Council meeting,\(^8\) Cole reported that, “His office was likely to lose the services of the present clerk, Miss Seely, unless her salary could be increased [AMS14, 1915].” He argued that her services to the Society were invaluable and she should be given a raise. The Council agreed, voting to increase her salary to $75 a month effective the first of September.\(^9\)

Other projects undertaken by Seely indicate that her work went beyond mere clerical duties and relied heavily on her educational background. For example, her linguistic talents\(^10\) were called upon in 1916 when she translated from the French the testament of Gustav Mittag-Leffler and his wife, Signe, that stipulated the provisions for creating the Mittag-Leffler Mathematical Institute [See16]. In 1915, Smith edited an edition of Augustus De Morgan’s A Budget of Paradoxes and thanked Seely in the preface writing, “To Miss Caroline Eustis Seely for her intelligent and painstaking assistance in securing material for the notes.” Three years later, Smith and Seely prepared a thorough sixty-page catalog of 165 mathematical periodicals [SS18] for the use of research students.\(^11\)

Seely also continued to do research and apply her mathematical talents. For a brief time during the latter part of WWI, Seely served in the ballistics section of the engineering division of the Army Ordnance Department. Under the direction of Major Forest Ray Moulton at the Aberdeen Proving Grounds, she, along with a team of eight other

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\(^4\)Kasner received his PhD from Columbia in 1899 under the supervision of Felix Klein and David Hilbert at Göttingen. Kasner is perhaps best known today, along with J.R. Newman, for coining the term “googol.”

\(^5\)The other women were Olive C. Hazlett, Florence P. Lewis, Clara E. Smith, and Anna Pell Wheeler [GL09, p. 110].

\(^6\)Cole, born in Ashland, Massachusetts, in 1861 and educated at Harvard, had taught at the University of Michigan from 1888 to 1895 before accepting positions at Columbia University and Barnard College.

\(^7\)East Hall, originally Macy Villa (now Buell Hall), is located at 115 West 116th Street.

\(^8\)The North River Insurance Company of New York City covered $847.06 of the Society’s losses [AMS14, 1914]. In addition, the AMS Council voted to keep a copy of its minutes in the Columbia University Library archives.

\(^9\)The Council of the Society normally met five times a year: February, April, September, October, and December, often in East Hall.

\(^10\)A dollar in 1915 had the purchasing power of about $30 in 2022.

\(^11\)During Seely’s time as a student, both French and German were required as part of Barnard’s modern language requirement.

\(^12\)The catalog sold for 10 cents.
women, was tasked with calculating the projectile motion of firearms.\textsuperscript{13}

In 1919, Seely wrote an article for the Annals of Mathematics entitled "Non-symmetric Kernels of Positive Type" in which she considers three well-known theorems concerning the symmetric case and proves analogous theorems for the non-symmetric case. It was only the second article by a woman to appear in the prestigious Annals of Mathematics.\textsuperscript{14} During the next ten years she extended her results culminating in a research presentation at a meeting of the Society held in New York City [See\textsuperscript{27}, See\textsuperscript{30}].

Meanwhile, in her first seven years working for the Society, membership showed very slow but steady growth.\textsuperscript{15} Meeting minutes recorded by Seely indicate that a number of significant decisions were undertaken by the Council during the period. Among the issues, one that may be of contemporary interest concerns a 1915 report to the Council that urged the Society to consider cooperating with mathematical teaching associations that were concerned with the status of preparatory mathematics in secondary schools. The Council ruled that such a move would be undesirable at the present time. Their reasoning is indicated in their answer to a similar resolution from the Chicago Section, where the Council issued the following statement:

\begin{quote}
It is deemed unwise for the American Mathematical Society to enter into the activities of the special field now covered by the American Mathematical Monthly. [AMS\textsuperscript{14}, 1915]
\end{quote}

This action led directly to the establishment of the Mathematical Association of America.

Financially, the years leading up to 1920 were precarious times for the Society. In 1916, even though there was a small surplus from the previous year, the annual operating budget was about $5000 with dues and initiation fees bringing in $3360.\textsuperscript{16} The Bulletin was operating at an annual loss of over $1100, costing twice as much to produce as the revenue it generated.\textsuperscript{17} The Transactions lost about $1000 a year.\textsuperscript{18} Library costs ran to over $100 a year,\textsuperscript{19} and the annual cost of running the Society’s New York office was about $1300 [AMS\textsuperscript{14}, 1916]. Adding to these problems was that their printer, New Era Printing Company of Lancaster, Pennsylvania,\textsuperscript{20} had decided to raise rates. Furthermore, on account of reasons stemming from the war, an alarming number of members were not paying their dues.\textsuperscript{21} By 1920 the annual deficit was around $830.

Even in the midst of the turmoil Seely’s humor came through. In a letter to Cole regarding dealings with a member of the Society she wrote:

\begin{quote}
I interpret Biggerstaff’s letter differently from you and Hedrick. I am reasonably sure that the Transactions he is referring to are the Proceedings of the London Mathematical Society, because he has been subscribing to our Transactions for several years, even before he became a member of the Society, and also he has just finished buying a complete set of all back volumes . . . He seems to be an enthusiast for buying mathematical periodicals, even when he doesn’t know their names. I wish there were more of the same. [See\textsuperscript{22}, 1923]
\end{quote}

In 1919, the Council discussed incorporating the Society\textsuperscript{22} and publishing an annual critical survey of mathematical literature, an English counterpart to the Jahrbuch über Fortschritte der Mathematik [AMS\textsuperscript{14}, 1919]. Ultimately they decided that it was an inopportune time financially to pursue either venture. The Council did agree, as it would turn out to their great benefit, to take part in a national movement towards the organization of research in affiliation with the National Academy of Science. It would not be until 1940 that the Society would publish the Mathematical Reviews.

In 1920, the Society’s Committee on Reorganization, chaired by former Society president Thomas Fiske, proposed an increase in the subscription prices for the Transactions and the Bulletin, as well as increases in annual dues, from $5 to $6, and the fee for life membership, first established in 1898, from $50 to $75. The committee also recommended that the number of printed copies of each edition of the Transactions be decreased, and that the List of Officers and Members be published biennially rather than annually. In addition, in an attempt to secure a more substantial legal status, protect current holdings, and attract gifts and bequests, the committee strongly urged Council

\begin{footnotesize}
\begin{enumerate}
\item Moultou, an astronomer, was Edwin Hubble’s mentor at the University of Chicago and a proponent of the Chamberlin–Moulton planetesimal hypothesis of the origin and evolution of the solar system.
\item An article on integral equations by Laura Guggenbühl of Hunter College had appeared earlier in the same volume.
\item Membership went from 692 in 1913 to 770 in 1920 [Arc\textsuperscript{39}, p. 44].
\item The initiation fee was $5.
\item The Bulletin was first published in 1891 with a goal “to contain, primarily, historical and critical articles, accounts of advances in different branches of mathematical science, reviews of important new publications, and general mathematical news and intelligence” [Arc\textsuperscript{39}, p. 48]. Its page size and original color imitated J.W.L. Glaisher’s Messenger of Mathematics. Its character was influenced chiefly by Darboux’s Bulletin des Sciences Mathématiques and the Zeitschrift für Mathematik und Physik [Fis\textsuperscript{87}].
\item In 1900, after failing to acquire the American Journal of Mathematics from Johns Hopkins University in 1898, the Society began publishing its Transactions in an effort to promote more original mathematical research [Arc\textsuperscript{39}, pp. 56–59].
\item In 1951, the Society’s Library was purchased by the University of Georgia for $66,000 [AMS\textsuperscript{34}, 1951].
\item New Era became the Lancaster Press in 1928.
\item In 1918, the Council decided that members in the military service whose dues are in arrears shall be continued on the rolls but shall not receive the Bulletin.
\item The subject of incorporation was first discussed by the Council in 1892 [AMS\textsuperscript{92}].
\end{enumerate}
\end{footnotesize}
to reverse their previous stance and consider incorporating the Society [AMS14, 1920]. There was also a recommendation to raise Seely’s salary to $1500 a year and broaden her clerical and editorial responsibilities as follows [AMS20]:

Duties for the Treasurer:
- Keep the books, members’ cards, and agents’ accounts.
- Send bills and receipts out.
- Answer routine letters about price, orders, and disputed bills, generally after consultation with the Treasurer.
- Make up the Treasurer’s Report.
- Prepare books for the auditors.
- Keep records of the Life Membership Fund.

Duties for the Librarian:
- Receive periodicals, list them on cards, and take them to the Reading Room.
- Tie up volumes of periodicals when complete, list them and then when there is sufficient accumulation, have them sent over to Columbia University Library, who then sends them to the binders.
- Receive, list, and send over to the library any non-periodical volumes.
- Look over the books after they are bound and catalogued, and check up shelf list.
- Send out requests for missing numbers of exchange periodicals.
- Prepare the Librarian’s Report.

Duties for the Shipping Clerk:
- Fill orders for the Colloquium Lectures,23 and those numbers of the Bulletin and Transactions that are stored in the office.

Duties for the Bulletin and Publication Committee:
- Prepare manuscripts for publication in the Bulletin.
- Read Bulletin proof.
- Write up the Notes, from contributions sent in by Professor Snyder,24 from other periodicals, American and foreign, and other sources.
- Typewrite list of New Publications for Professor Archibald’s cards, also collect as many items as possible for this list and send them to him.
- Receive books for review in the Bulletin; send out periodic list to them; send off books to reviewers, as assigned by the Editor; keep records of this, and send periodic reminders to reviewers.
- Collect material for the Annual List of Published Papers in the July Bulletin, and write it.
- Correct Bulletin and Transactions mailing lists.

Duties for the Secretary:
- Send out copies of the Council minutes, notices of special meetings, copies of committee reports, etc.
- Write letters, under the directions of the Secretary.
- Collect material and make up the List of Members.

The end of 1920 saw a change in Society leadership with the resignations of Cole as secretary and D.E. Smith as librarian.25 The Council selected Roland George Dwight Richardson as Cole’s successor and Raymond Clare Archibald as Smith’s successor. Richardson, born in Dartmouth, Nova Scotia, in 1878, was a direct descendant of the astronomer Simon Newcomb, a former president of the Society. After graduating from Acadia University in Wolfville, Richardson taught in the small fishing village of Margaretsville and served as a high school principal in Westport, Nova Scotia. He received his PhD from Yale University in 1906 under the supervision of James Pierpont, who had just finished a term as vice-president of the Society [Arc50]. Richardson spent a year at the Mathematical Institute in Göttingen with David Hilbert and Felix Klein working primarily in the field of the calculus of variations. He served at Brown as chair of the mathematics department and as dean of the graduate school, roles in which he was particularly supportive in helping women secure teaching positions [GL09, p. 64]. Under his leadership the Society’s membership tripled,26 an endowment fund was established, the Gibbs Lectures initiated, reciprocity memberships offered, and Mathematical Reviews established.

At a meeting in 1921, the Committee on Reorganization felt that with Richardson and Archibald in Rhode Island and Earle Raymond Hedrick, editor of the Bulletin, in Missouri,27 it might be best to decentralize the office work of the Society. The committee asked Richardson and Hedrick (Figure 3) to supply a list of Dr. Seely’s responsibilities. Richardson’s list contained the following items [Ric21b]:

1. Keep up to date:
   a. List of members with dates of withdrawal or dismissal and inform Treasurer and Bulletin editor of changes.
   b. The lists of those proposed for membership.

2. Send out:
   a. Blanks for information for Register.
   b. Application blanks.
   c. Abstract blanks.
   d. Notices of special meetings, Reports of Council meetings, and of numerous committees.

3. Prepare for Secretary’s signature:
   a. Formal letters.
   b. Acknowledgments of applications.
   c. Notices of election to membership.

4. Aid the Secretary:
   a. In keeping calendar up to date.
   b. In preparing business for Council meetings.
   c. In preparing reports of Society meetings for Bulletin and Science.

5. Prepare copy for Register.
6. Answer letters of inquiry on miscellaneous topics from many institutions and individuals.
7. Read proofs of:
   b. Register.
8. Aid committees in investigations.

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23The Colloquium Lectures were inaugurated in 1896 as a supplement to the summer meeting in an effort to stimulate mathematical thought [Arc39, p. 67].
24Virgil Snyder, a member of the Committee of Publications, edited the “Notes” section of the Bulletin. He served as president of the Society, 1927–1928.
25Cole had been secretary since 1896 and Smith the librarian since 1902.
26From 770 to 2314 members.
27In 1924 Hedrick became head of the mathematics department at the Southern Branch of the University of California, now UCLA, later serving as vice-president of the University of California.
9. Stenographic duties:
(a) Write letters under Secretary’s direction.
(b) Mimeograph reports of Council meetings, of committee reports, etc.

Richardson remarked:
It will be noted in only a small measure can these duties be classed as stenographic. They involve work each day for twelve months and must be undertaken on this basis. Whether a Secretary could obtain part-time help on this basis is problematical but if so, he must, it seems to me, add at least $300.00 to the expense of about $50.00 which is at present incurred for the stenographer … There would probably have to be an allowance of $250.00 for office furniture such as a typewriter, desk filing cabinet, if the Secretary’s work were separated from the other. But the main point which should be kept in mind is that a readjustment of the list would throw a good deal more work on the shoulders of the Secretary. He would have to carry more detail himself and spend time in training every year one or two new assistants. A conservative estimate of the added time necessary is one hundred fifty to two hundred hours yearly. [Ric21b]

Seely’s first reaction to the new plan was to resign and look for another job. She applied to the College Employment Bureau at Barnard asking Richardson and Hedrick for references. Hedrick replied, apologizing for her having to give up her vacation waiting for proofs of the June-July issue of the Bulletin from New Era, adding that he would be happy to support her effort to seek employment at Barnard [Hed21]. He wrote to Luther Eisenhart, chair of the Budget Committee and later president of the Society, to underscore Seely’s value to the organization.

As to Miss Seely herself, I can heartily concur in the general opinion of her exceptional ability. She has assisted very much with the work of the Bulletin, in the manner described in a sheet showing all of her duties, distributed by Richardson. If we were paying her an adequate salary or if she would have the slightest difficulty in securing a more desirable position, I would hesitate to speak frankly of the difficulties that are inherent in the present arrangement, and which render it uneconomical. I am sure, that she herself would be the first to emphasize the difficulties under which she has been laboring. To try to do work for a half-dozen men at a great distance, each one ignorant of the demands of the other, must be trying to say the least … If Miss Seely’s services were not available, it would undoubtedly increase the burden of the editorials perceptibly. Some small clerical assistance would doubtless be needed. [Hed21]

Hedrick continued by estimating that thirty-five percent of her efforts went toward Bulletin affairs and included the following with the list of her responsibilities:

1. Read Bulletin proof, and send copies to authors and editors.
2. Collect material for Notes, and send first draft to Hurwitz.
3. Collect material for List of New Publications and send first draft to Young.28
4. Receive books for review, and send card to Young, books to reviewers.
5. Correct Bulletin mailing list.
6. Collect material for annual list of published papers.
7. Prepare index for July number.

In a letter to Seely, Richardson noted that he had received a great number of letters concerning the proposed decentralization of the office work of the Society. Quoting from a letter from W.F. Osgood,29 he added:

The really serious moment in the present situation is Dr. Seely’s resignation. Her services to the Society are of great importance. She has had long experience in the affairs of the Society, and has proved herself efficient. If she feels that the value of her services is called into question, set her right on this point and reassure her. To lose her would mean training up another person to do, as nearly as may be, what Miss Seely is doing so well. [Ric21a]

At the Council meeting in September 1921, it was decided that as long as financial conditions warrant, a permanent office be maintained in New York City with the organization as at present [AMS14, 1921].

From 1920 to 1935 the Society went from an operating budget of $10,000 to one of $40,000. Nevertheless, mainly on account of its publications, the Society operated each year at a loss. Actions taken to reduce costs included reducing the font size of the Transactions, initiating a reciprocal agreement with the London Mathematical Society,30 and, for a while, printing the Transactions in Germany.31 Revenue-building measures included the initiation of an endowment fund with a goal of $100,00032 and the establishment of a Sustaining Member category for corporations, institutions, associations, and even individuals (for

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28W.A. Hurwitz and J.W. Young were members of the Society’s Committee of Publications.
29William Fogg Osgood, professor of mathematics at Harvard, was president of the Society, 1905–1906.
30Similar agreements were made in 1931 with the Unione Matematica Italiana and the Deutsche Mathematiker-Vereinigung.
31The Bulletin was printed by Lütke and Wulff in Hamburg.
32By 1926, $75,000 had been raised.

Figure 3. R.G.D. Richardson (l); E.R. Hedrick (r).
a $100 donation). In addition, member dues were raised along with prices for the Bulletin, Transactions, and Colloquium Lectures.

Seely continued to demonstrate dedication to her work with the Society, even when she was called upon to adjust summer vacation or travel plans to accommodate Society requests. For example, in the summer of 1922, perhaps wishing to visit her sister in Kansas whose husband was working with the Society, even when she was called upon to adjust summer vacation or travel plans to accommodate Society requests. For example, in the summer of 1922, perhaps wishing to visit her sister in Kansas whose husband was looking for a job New York State, she writes to Richardson:

I won’t go off anywhere for more than a week-end until Fite gets back from summer school, which is just after July 4th. He says he will attend to mail for me during the summer session. Eisenhart would prefer that I didn’t get more than a day’s mail away from New York until the page proof for the January Transactions is out of the ways; Fite could forward page proof to me if I was in New York state… Any time I don’t answer promptly or send you what you ask for, will you please remind me, and if I don’t, it means that the letter was snowed under or filed in the wrong place or blew out the window. [See22, 1922]

In late 1922, the Society began to experience and uptrend with an anonymous donation of $4000 to publish an extra volume of the Transactions thereby reducing the backlog of articles to be published [Ric22]. In 1923, then president Oswald Veblen persuaded the Rockefeller Foundation to extend its research fellowships to include mathematics [Par15]. The Carnegie Corporation donated generously to the endowment fund. Also of note was a gift of $10,000 to aid in scientific publication of research from the General Education Board that was allotted to the Society by way of the National Academy of Science.

Yet, keeping printing costs down continued to remain a major concern along with many other problems with the printers. In 1922, Seely worried about the way the Transactions were handled. After seeing the proofs for one of the volumes, she wrote to Richardson that, “It looks very much the reverse of encouraging” adding that the charge from New Era for the latest issue was 25% higher than the previous year’s issue. That winter, a typesetters’ strike at New Era delayed distribution of the Bulletin. Perhaps as a result, in January much of Seely’s time was spent finishing and correcting the galleys of the Index and List of Published Papers for the Bulletin which she referred to as a “horrid job” [See22, 1923].

In May of 1923, Seely proposed a genuine justification for the Society’s incorporation:

One of the young men here, whose Hamburg father is a customs broker, says we will not be able to get the package of Transactions away from the dock before the duty is determined unless we are able to show evidence that we are incorporated; they wouldn’t be willing to let us take it out “in bond” before the duty is paid, and mail it off, unless they were sure they could recover the duty from us, and they wouldn’t be able to recover from an unincorporated society. So Fite thinks he should have some evidence of incorporation here, so that we could show it if necessary. [See22, 1923]

The next month, she wrote Richardson: “I received a statement from Washington to the effect that we could get a copy of the incorporation certificate for $1.35, doubtless through your efforts. So we are sending them a check” [See22, 1923].

In 1924 Seely writes of yet another printing issue:

An awful tragedy has just occurred. I forgot to tell Hedrick that he would have to put “printed in Germany” on the cover of the Bulletin in some form, and now we have a notification that we must go down and sit on the dock and mark each copy of the surplus of the January edition with a rubber stamp, or they won’t admit it. The printer has put “Press of Lütcke & Wulff, Hamburg” on the inside of the title page, and we are going to see if they may accept this as a substitute, but anyway the March-April number will be here shortly and the same trouble will come up. There are more things to think of these days! [See22, 1924]

United States Customs takes failure to mark country of origin quite seriously and Seely may not have been entirely facetious when she wrote:

Fite and I will surely go to jail this time. You remember all that talk about having the surplus stock of January-February Bulletin marked “made in Germany” before it could get through the customs, and that we had the brokers do this; well they did, and got the stock through customs as shipped to Lancaster, and Lancaster sent me 25 copies, as usual, and when the latter arrived I found they had marked them England instead of Germany. I don’t dare mail any of them off as they stand, as that is clearly a misstatement of fact. I suppose they will all have to be stamped over, and the “England” canceled. [See22, 1924]

Concerns were alleviated when the broker agreed to have an office boy mark the Bulletin properly, but the Society would be responsible for an additional 10% duty [See22, 1924].

On a more positive note, in July 1924 Seely wrote Richardson while he was on vacation in Nova Scotia:

A mysterious lady called up this afternoon, wanted your address, so I tried to spell Antigonish for her over the telephone; I hope you get this letter; for I think she said she was from the Rockefeller Institute, which is to give us the $50,000. [See22, 1924]

33In 1934, institutional members paid $2.75 times the average number of pages published in the Bulletin, Transactions, American Journal of Mathematics, and Annals during the period from 1927 to 1933 adjusted to the nearest $25, but at least $25.
34They eventually relocated to Vermont.
35William Benjamin Fite, earned a PhD at Cornell University under the supervision of G.A. Miller. He was professor of mathematics at Columbia University (1911–1932) and served as the Society’s treasurer (1921–1929).
36The Society was incorporated in Washington, DC, on May 3, 1923.
In the 1920s many of the Society’s annual and summer meetings were coordinated with the American Association for the Advancement of Science. Seely’s responsibilities included rechecking abstracts for papers to be presented, getting the program in shape, and addressing any possible issues such as those described in the following note to Richardson:

The programme looks all right to me. It seems to me the women are well taken care of, and get a great deal of tea. I warn you there is always trouble about how we get seated at general dinners. The ladies don’t like to be segregated off in a corner (Fiske made that awful break once in New York), and yet if left to themselves are likely to segregate themselves and then be discontented because they have. Once Slaught37 had a good solution (wasn’t it in Chicago) of announcing that no two ladies were to be allowed to sit together, as there is always a superfluity of men, the women haters can get together in spite of this. I suppose I oughtn’t to give away trade secrets like this, but they all come and confide in me if they don’t like things. [See22, 1922]

In February 1925, Seely seriously considered a teaching job offer from Tomlinson Fort, chair of the mathematics department at Hunter College. A replacement was needed for a teacher who had experienced a serious accident. In a letter to Richardson she wrote,

With regard to that job at Hunter College that Professor Fort called you about last night. He offered $75 a month from now to June for 6 hours a week teaching at Hunter, and called up this morning to confirm the offer. But I had been thinking it over, and decided that his suggestion of my hiring someone to take my place while I was off duty was out of the question. I think that I am using clerical assistance to the extent that it is practicable already, and all my seven-hour day now goes into work that I have to do myself; so I should have to make up time lost by overtime. Professor Fite agreed with this, and said that he would not oppose it if I thought I could certainly carry it. But of course I can’t be sure that I could, and it seems unfair to the Society to risk inconveniencing it by giving out in the middle of the term, So I told Professor Fort that morning that I could not accept his offer. Of course the only reason I ever considered accepting it was that I wanted some teaching experience, so that in case I ever got fired from here I wouldn’t be in the position I found I was in several years ago of being unprepared for anything else. [See22, 1925]

For the next few years Seely’s work focused primarily on the preparation of mathematical manuscripts for the printer and the reading of proofs for the Bulletin and Transactions.38 Very little of her time was spent on office management as by then she had a budget for clerical staff, which consisted mainly of Barnard graduates [See22, 1934]. She cooperated with the Committee on Reorganization, chaired by associate secretary J.R. Kline, suggesting that the time had come for a major office transformation including a move of the Society’s office to Low Library.

By 1934, Seely was ready to retire. Intent on starting a chicken farm, she purchased seventeen acres of land south of Ithaca in the village of Willsboro in Tioga County. She tendered her resignation with an effective date of April 1, 1935 [See22, 1934]. The reorganization committee used all efforts within the dignity of the Society to secure the withdrawal of her resignation, but without success. The resignation of Miss Seely was presented to the Council and was discussed. The following resolution was passed:

It is with great regret that the Council of the Society has learned of Miss Seely’s decision to retire from the position she has held for so many years. No one who has been associated with the work of the Society can fail to have appreciated the unfailing devotion and noble skill with which Miss Seely has carried on her diverse, complicated, and frequently taxing tasks. In regretfully accepting the resignation, the Council expresses to Miss Seely its highest appreciation of her signal contribution to the cause of American mathematics. [AMS14, 1934]

A similar resolution was passed by the Board of Trustees:

That the resignation of Miss Caroline Eustis Seely as managing clerk of the office of the American Mathematical Society be accepted with profound regret and that the secretary be requested to express to her the deep appreciation felt by the Trustees of her constant interest in the cause of mathematics and in the welfare of the Society, and of the valuable and efficient work she has done. [AMS34, 1935]

To fill the void that would be created by Seely’s departure, it was decided that the Society would hire a chief clerk and an associate clerk on a yearly basis with the promise of reappointment for satisfactory performance. Qualifications stipulated that the chief clerk hold a PhD in mathematics and the associate clerk to have training approximately to that of a PhD in mathematics. Both would answer to the secretary [AMS34, 1934]. On January 2, 1935, Miss Evelyn M. Hull39 and Miss Alta Odoms40 were hired in the NYC office. By December 1936 an assistant clerk and another editorial assistant had been added to the staff.

Seely agreed to continue to assist in editing the Transactions until December 1935. In the event it was necessary for her to come to the main offices, the Society agreed to pay her traveling expenses. In addition, the Society agreed to continue through 1935 its payment to the annuity they had set up for her in 1931.41 She began receiving a pension of $25 a month in January 1936. There was no promise made of it lasting beyond one year.

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37 Herbert Ellsworth Slaught (1861–1937) was secretary of the Chicago section of the Society (1906–1916) and also served as local arrangements chair.
38 Which she estimated took 90% of her time [See22, 1934].
39 Hull continued as office manager until October 31, 1951.
40 Odoms (Gray), who received her PhD from the University of Cincinnati in 1936, mainly functioned as an editorial assistant.
41 $270 to TAA annually.
Sixteen years later, the treasurer of the Society, Albert E. Meder, Jr. wrote to Seely on the occurrence of the last payment of her monthly pension.

I am personally particularly pleased that it falls my lot to write this letter, because I am one of the comparatively few among the 4000 members of the Society who can remember when, instead of the extensive headquarters of today and the dozen or so regular employees, there was a small room on the top of East Hall at Columbia and Miss Seely. Being one of the few who have known personally all the secretaries of the Society—Fiske, Cole, Richardson, Kline, and Begel—and almost all of the treasurers, I hope that you will feel that this word of appreciation is not a perfunctory letter written by an officer of the Society as part of his duty, but in addition is an expression from one who can remember conditions as they were actively working for the Society, and therefore writes with both knowledge and understanding.

Though it is true that the phenomenal growth of the Society and the inexorable passage of time has brought about a situation wherein most of the members would perhaps not even recognize your name, it is also incontestably true that these members owe a great debt to you. For years you were the guardian of the Society’s affairs, the adviser of the secretaries, and the person who with her own hand did the work. In your day there was no extensive staff to supervise; you were the staff. You did not sit in an executive’s desk; you sat at a battered typewriter. You did not sign orders on the treasury; you pinched every penny and made it do double duty. …The American Mathematical Society depends in large measure on the careful nurture which the Society received at your hands during your period of active service as clerk. …all of us want to know again, …that there are those of us who understand and appreciate what you did for the Society, indeed in its very spirit, and that we wish for you many years of enjoyable retirement and true happiness.

Seely responded thanking him for the kind letter of appreciation adding that she, “never expected anything so kind and friendly as your letter. I shall keep it and read it any time when I need cheering up” [SM52]. She noted that she was receiving copies the Proceedings and the Transactions as a nominee of the department of mathematics at Columbia. In addition, as she had access to the Cornell Library, Columbia should feel free to discontinue her subscription any time it needed the membership for some young mathematician [SM52].

In 1942, Hedrick retired from UCLA and accepted a visiting professorship at Brown. He died of a lung infection in early 1943. Richardson retired from Brown in 1948. He died the next year of pneumonia on a fishing expedition in Nova Scotia. Caroline Seely died in 1961 at a nursing home in Ithaca. The resources presented in this paper indicate that Seely realized her worth, as did many of those who worked with her. One hopes that governing bodies in the AMS continue to value the importance of recognizing the seminal contributions that its behind-the-scenes staff have made throughout the years in making the Society what it is today.

ACKNOWLEDGMENTS. The authors would like to thank Martha Tenney, director of the Barnard College Archives, for her assistance. They also thank the referees for their thoughtful suggestions.

References

[AMS92] AMS, Minutes of the annual meeting, 1892. American Mathematical Society Archives, Ms 75.2, Box 12, Folder 5.
[AMS34] AMS, AMS Trustees Minutes, 1934–1951. American Mathematical Society Archives, Ms 75.1, Box 1.
[Fis87] Thomas S. Fiske, The beginnings of the AMS – Reminiscences of Thomas S. Fiske, 1887. AMS Archives, Ms 75.2, Box 21, Folder 128.
[Hed21] E.R. Hedrick, Correspondence: Hedrick to Seely and Hedrick to Eisenhart, 1921. AMS Archives, Ms 75.4, Box 21, Folder 60.
[Ric21a] R.G.D. Richardson, Correspondence: Richardson to Seely, 1921. AMS Archives, Ms 75.4, Box 21, Folder 53.
[Ric21b] R.G.D. Richardson, Memorandum from R.G.D. Richardson concerning the assistance of Dr. Seely in the duties of the Secretary of the American Mathematical Society, 1921. AMS Archives, Ms 75.4, Box 21, Folder 53.
[See30] Caroline Seely, Note on kernels of positive type, Ann. of Math. (2) 31 (1930), no. 1, 32–34. MR1502915

42 Begun in 1950.

[See22] Caroline Seely, *Correspondence: Seely to Richardson*, 1922–1934. AMS Archives, Ms 75.4, Boxes 21, 22, and 24.

[SM52] Caroline Seely and Albert E. Meder, *Correspondence between Seely and Meder*, 1952. AMS Archives, Ms 75.1, Box 1, Folder 114.


**Credits**

Figures 1 and 2 are courtesy of the Barnard Archives and Special Collections.

Figure 3 (left) is courtesy of the AMS.

Figure 3 (right) is courtesy of the Mathematical Association of America Archives.

Photo of James J. Tattersall is courtesy of Terry Tattersall.

Photo of Shawn L. McMurrinan is courtesy of Bret McMurrinan.
The fundamental object of study in this book is a quadratic form, i.e., any expression \( Q(x, y) = ax^2 + bxy + cy^2 \) where \( a, b, c \) are rational integers.\(^1\) This is an area studied by Euler, Lagrange, and Gauss in the eighteenth century and Hatcher juxtaposes this rich theory with number theory in order to get, in his words, “a deeper sense of the beauty and subtlety of number theory.” With a minimal collection of prerequisites the book develops a list of topics including Pythagorean triples, the Euclidean algorithm, Pell’s equation, continued fractions, and Farey sequences, leading to the high-water mark, the Class Group, a notion first introduced by Gauss.

Quadratic forms arise in many contexts in number theory, for example triples of integers which make the form \( x^2 + y^2 - z^2 \) equal to zero are usually called Pythagorean triples, with everyone’s favourite such triple being \((3, 4, 5)\). It’s not difficult to give a formula for all such triples, and this was known to the Ancient Greeks. Of course, one can, and should, make certain simplifying nondegeneracy assumptions. For example, if \((x, y, z)\) is a Pythagorean triple, and if two of the three numbers have a common factor \( p \), then in fact \( p \) divides the remaining number. This means one can restrict attention to primitive triples, which is to say triples with no common factor. In this way one is led to considering the equation \((x/z)^2 + (y/z)^2 = 1\) since for primitive triples, the fractions \(x/z\) and \(y/z\) are now in their lowest terms and this recasts the question of Pythagorean triples into the question of finding rational points on the circle \(X^2 + Y^2 = 1\). Taking this as a starting point makes the analysis of the equation quite easy: There is a trivial solution, \(X = 0\) and \(Y = 1\) and if one considers a line of slope \(m\) through this point, it has equation \(Y = mx + 1\).

Now, draw the picture and you’ll see that with the exception of the cases that the slope \(m\) is one of \(0\) or \(\infty\), this line meets \(X\)-axis at one nonzero point. Denoting this point by \((r, 0)\), some gentle algebra reveals that the line meets the circle at the point \((2r^2 - 1, r^2 + 1)\). Clearly, any rational choice of \(r\) yields a rational point, hence an integral solution to Pythagorus’s equation and one sees easily that the converse is also true. However, the method also shows more, for example, any arc on the circle contains infinitely many rational points hence infinitely many such solutions.

This is the first such instance of a quadratic form which yields interesting and nontrivial mathematics, but number theory is replete with such examples. Another such example is the equation \(x^2 - Dy^2 = 1\) for some fixed nonsquare rational integer \(D\); this is known as Pell’s equation. It arises in multiple contexts in number theory, and its study also goes back hundreds of years. Its subtlety and interest is highlighted by the fact that quite small values of \(D\) only have very large integer solutions. For example when \(D = 61\), the smallest positive integer solution is \((1766319049, 226153980)\). Nonetheless, one can find all the solutions using variations on the ideas presented in the previous paragraph and it is a theme of Hatcher’s book.

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\(^{1}\)These are the integers you learnt about at your mother’s knee \([-\ldots, -1, 0, 1, 2, \ldots]\) and so on. Number theorists use the term integer for any root of a monic equation with rational integer coefficients.
that one can use this kind of geometric thinking to study
the solutions to certain equations.

To this end, he first introduces the Farey diagram, an idea
which goes back to Hurwitz. The Farey diagram is a deceptively simple yet rich way
of organizing the rational numbers together with infinity
(which is regarded as both $1/0$ and $-1/0$). This diagram
may be constructed as follows: If there is an edge connect-
ing $p/q$ and $r/s$, then there is a triangle in the tessellation
whose vertices are $(p/q, (p+r)/(q+s), r/s)$. This provides
an inductive scheme for generating the whole picture, by
beginning with the edge joining $1/0$ and $0/1$ for positive ra-
tionals and taking the same edge regarded as joining $-1/0
and $0/1$ for the negative rationals. It’s easily checked that if
$p/q < r/s$, then $p/q < (p+r)/(q+s) < r/s$ but this kind of in-
ductive description makes it hard to determine whether a
given pair of fractions is connected by an edge in the Farey
diagram, however there is a clean criterion: $p/q$ and $r/s
are connected by an edge if and only if the determinant of the
matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ is equal to $\pm1$. This is the first hint that
something geometric in nature is connected to something
algebraic. Moreover, while the proof is elementary it has
nontrivial implications, for example, it shows the inducti-
ve rule always produces fractions in their lowest terms.
It can also be used to show (it’s a nice exercise using the Eu-
clidean algorithm) that every such fraction occurs in the
Farey diagram.

Another important feature of the Farey diagram is its
symmetry group, by which we mean any invertible trans-
formation which carries vertices to vertices, edges to edges,
and triangles to triangles. It is not hard to show that given

a matrix $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ with integer entries and of determi-
nant $\pm1$, that the transformation given by
$T(x/y) = \frac{px+ry}{qx+sy}$ is a transformation in this sense. All such transformations
have this form (notice that $M$ and $-M$ have the same ac-
tion) and the group of such is usually called linear fractional
transformations. It plays an important role in other areas of
number theory as well as in hyperbolic geometry.

The next ingredient is the topograph. One can superim-
pose a dual tree on the Farey diagram, i.e., take a vertex in
the center of each triangle of the Farey diagram and con-
nect any two such vertices if their Farey triangles are adja-
cent. This dual graph divides the interior of the circle into
regions, each of which is adjacent to one rational number
in the original diagram. Given a quadratic form $Q(x,y)$ one
forms the topograph of $Q$ by labelling the region adja-
cent to $p/q$ with the value $Q(p,q)$. Figure 3 shows the
topograph for $Q(x,y) = x^2 + y^2$, so that, for example, the
region labelled $13 = 2^2 + 3^2$ is adjacent to $2/3$. The power
of this idea derives in part from the fact that there is a simple
inductive rule which generates the labelled picture from a
small amount of initial data: If the values of $Q(x,y)$ are
as shown in the regions adjacent to an edge, then the in-
tegers $p, q + r, s$ form an arithmetic progression (see Fig-
ure 4). It’s not difficult to convince oneself that this rule,
coupled with the three values in the regions adjacent to
a single vertex, determines the value in every region. The
topograph offers a geometric avenue into problems associ-
ated with quadratic forms, and the book explores some of
these. Let us illustrate with the example of Pell’s equation;
here \( Q(x, y) = x^2 - Dy^2 \) and we are interested in regions in the topograph which carry the label 1.

One finds that there is a separator line in the graph separating the positive regions from the negative and the line organises the picture into a pattern which is infinitely repeating with respect to translation along the line. The separator line can be constructed easily from the three values 

\[
\begin{align*}
Q(1, 0) &= 1, \\
Q(0, 1) &= -D \\
Q(1, 1) &= 1 - D.
\end{align*}
\]

We remark that in fact it is closely connected to the continued fraction for the nonsquare integer \( \sqrt{D} \). One can show that every periodic line in the dual tree of the Farey diagram occurs as the separator line for some form.

Not unexpectedly, one can show that in a topograph which has a separator line, as one moves away from this line the values increase monotonically towards \(+\infty\) on the positive side and \(-\infty\) on the negative side, so if the value 1 is to occur it must be on the separator line itself. Moreover, we see from the initial data used in the construction, that the value 1 occurs on the line so that there is a region in the topograph carrying the value 1, and by periodicity there must be other regions which do. These correspond to interesting solutions to Pell’s equation. One way of finding such solutions is to construct the fractional linear translation which stabilizes the separator line. It is not difficult to see that this transformation must have infinite order, so that there are infinitely many solutions to Pell’s equation.

As always, there comes a point where it needs to be decided what it means for two objects under consideration to be equivalent. In this setting, there is a very natural definition, namely, two quadratic forms \( Q_1 \) and \( Q_2 \) are equivalent if there is a fractional linear transformation throwing the topograph of one to the topograph of the other. Given the fact that topographs are locally determined by their values around any vertex, an alternative equivalent definition is that there are vertices \( v_1 \) and \( v_2 \) so that the values of the \( Q_1 \)-topograph around \( v_1 \) are the same as those in the \( Q_2 \)-topograph around \( v_2 \). With this equivalence decided, one asks about obstructions to forms being equivalent and again there is a simple obstruction: a quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \), one defines its discriminant to be \( b^2 - 4ac \). Notice that any discriminant is congruent to a square modulo 4, so it must be either 0 or 1 mod 4. Discriminants play a central role in the theory of quadratic forms and it’s not difficult to see that equivalent forms have equal discriminant. Moreover, it can be shown that there are only a finite number of equivalence classes of forms with a given nonzero discriminant.

However, it’s natural to refine this situation a little further. If the fractional linear transformation can be chosen to be orientation preserving, then one says the forms are properly equivalent. Also, if you multiply a form by some constant \( d > 0 \) its essential features are not significantly changed, and so it’s reasonable to restrict attention to primitive forms, which is to say forms for which \( a, b, c \) have no common divisor. Equivalence preserves primitivity. The number of proper equivalence classes of primitive forms of a given discriminant \( \Delta \) is called the class number, \( h_\Delta \), for that discriminant. This is a number which can be computed algorithmically for any given discriminant \( \Delta \). Despite this, the dependence of \( h_\Delta \) upon \( \Delta \) is mysterious and not well understood.

It is of especial interest to study the discriminants for which all forms are primitive, these are the so-called fundamental discriminants. Every nonsquare discriminant \( \Delta \) can be written \( \Delta = d^2\Delta' \) where \( \Delta' \) is fundamental, and one can relate \( h_\Delta \) to \( h_{\Delta'} \), so that the study of class numbers is largely about the case of fundamental discriminants.

The question of which discriminants have class number 1 has been a magnet for research for a long time; this amounts to finding the discriminants for which all primitive forms are equivalent. For example, the nine fundamental discriminants \( \Delta = -3, -4, -7, -8, -11, -19, -43, -67, -163 \) have class number 1. A conjecture of Gauss from around 1800 was that there were no other negative discriminants of class number 1, this was finally resolved in the affirmative in the sixties. This is to be contrasted with the situation for positive discriminants with class number 1 which is not at all understood. It seems likely that there are in fact infinitely many, however, this is still
unsolved and remains one of the most basic unsolved problems about quadratic forms.

Class numbers and the representation of integers by quadratic forms are the beginning of a rich and interesting theory, which is developed further in the chapters which follow. While the point of view might be a little idiosyncratic, both in terms of the material that is addressed, as well as the manner in which the book chooses to address it, many will find this geometric viewpoint hugely appealing. Certainly this reviewer did. It is a beautiful book.

Credits

Figures 1–4 are courtesy of Allen E. Hatcher. Photo of D. D. Long is courtesy of Chris Leininger.
**Distrust**  
Big Data, Data-Torturing, and the Assault on Science  

Faced with increasingly complex technology, we must consider its responsible and ethical use. The science behind this technology often boasts about how revolutionary it is. Or is it? In *Distrust*, Gary Smith reveals missteps that science and mathematics made when rolling out innovations such as the internet and machine learning models. Through the lenses of disinformation, data torturing, and data mining, this book leads the reader through a history of instances where the public doubts the facts.

The general lack of faith in science is a substantial and unsettling threat. *Distrust* is filled to the brim with examples of those who reject scientific evidence or have reasonable doubt about black-box algorithms. You will gain an appreciation from a historical perspective, as some examples date back hundreds of years. However, we cannot only blame an impressionable population for distrust in science; some scientists compound the problem when they analyze only the subset of data that corroborates the desired result or evaluate too much data and find statistical significance when events are merely coincidental.

Smith also warns against the overuse of artificial intelligence in decision-making. His examples range from medical diagnoses to image recognition where AI has overpromised and underdelivered. As it turns out, there are many ways for the public to distrust science. But what can be done? We could require researchers to publish their data and code along with results to encourage replicability. To reduce the pressure on scientists that leads to extreme data mining, we could de-emphasize statistical significance in favor of more transparent methodology. We could hold social media companies accountable for the content their users post and share. This book could be a supplement to an introductory statistics course or quantitative reasoning course.

**An Introduction to the Math of Voting Methods**  

Most votes for elected positions in the United States allow each voter to assign a single vote to a single candidate. The votes are counted, and the winner is the candidate with the most votes. This method is straightforward, easy to implement, and voters are rarely confused. Voting theory explores other ways to determine a winner; ranked voting, for instance, allows a voter to rank all candidates in order of preference. But then one can ask: how is the winner determined? What is fair or unfair about a given system?

Sullivan’s new book could be used when teaching a unit on voting theory in a quantitative literacy or mathematics for liberal arts course. Voting often occurs outside of politics: choosing where a group will dine, who wins the Heisman trophy in college football, or which agenda items take precedence at a school board meeting. Voting theory can be understood without the prerequisites of algebra or calculus. While a more advanced student may prefer a more advanced text, the high readability of Sullivan’s book makes it compelling for all students.

The text clearly explains a variety of voting methods, including Borda count, pairwise comparisons, and approval voting. The examples are easy to follow and often illustrate surprising winners based on which method is chosen. Each section contains practice problems with solutions, along with homework exercises. The exercises require creativity; in one, students are asked to find a newspaper article about ranked choice voting and analyze the journalist’s point of view. If you find yourself wanting to convince students that understanding mathematics is important, consider this book.
The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world’s leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit bookstore.ams.org to explore the entire collection of AMS titles.

**Geometric Structures on Manifolds**

Surfaces naturally enjoy 3 types of constant curvature metrics and accordingly fall into 3 geometric types: spherical, Euclidean, or hyperbolic. Thurston’s geometrization program from the 1970s extended this idea to the richer setting of 3-dimensional manifolds, providing a pathway for their classification and a solution to the 3-dimensional Poincare conjecture. As later proved by Perelman, 3-manifolds canonically decompose into primitive pieces each having 1 of 8 possible metric geometries.

Influenced by his teachers, Thurston, Sullivan, and Hirsch, Goldman began at an early stage in his career to develop techniques for studying geometric structures, especially affine and projective models, using tools from representation theory and Lie groups. This work evolved into the larger theory of (not necessarily metric) geometric structures on manifolds, sometimes also known as “higher Teichmüller theory.” However, the literature on this subject has been somewhat scattered, and as the field has grown increasingly active, there is a parallel need for a coherent and accessible treatment.

Goldman’s book presents a timely exposition of this fast-developing subject aimed toward students with some familiarity with manifolds, Lie groups, fiber bundles, and differential geometry. The first part of the book introduces affine, projective, and hyperbolic geometry, the second puts these geometries on manifolds, and the third discusses some of the main conjectures driving the subject. Containing many examples and exercises, it is suitable for a topics course, and as a resource for researchers in adjacent fields.

**Groups and Topological Dynamics**
By Volodymyr Nekrashevych. GSM/223, 2022, 693 pp.

There is a rich interplay between groups and the spaces on which they act preserving their geometric structure. In the influential work of Gromov and Thurston, fundamental groups of hyperbolic manifolds are endowed with a geometry that reflects the geometry of their associated spaces and gives rise to the theory of geometric groups. Groups and Topological Dynamics focuses on an analogous study where groups associated with dynamical systems are endowed with a structure that is informed by and gives insight into the fractal geometry that lies within the dynamics.

Nekrashevych’s book gives an engaging introduction to this still evolving field. A motivating example is iterated monodromy groups. These groups carry essential dynamical information associated with a locally expanding covering map from a space to itself. An example is a postcritically finite rational map defined on the Riemann sphere. The preimages of a point in the complement of the postcritical set naturally give rise to an infinite tree on which the iterated monodromy group acts and properties of the group reflect “self-similarity” properties akin to what one sees in the associate Julia set.

The book begins with basic examples of dynamical systems including subshifts, Cantor systems, and hyperbolic and holomorphic dynamical systems, and then sets up technical tools associated with group actions, automata, and groupoids. While iterated monodromy groups are a central example, the book further develops the theory as a tool to understand a broader class of groups.

Containing plentiful illustrations and exercises, this book is accessible to students who have finished standard first year graduate courses and is adaptable to topics courses and independent study.

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Modeling America’s Racial Wealth Disparities
Mathematical Models Help Chart Pathways for Closing Racial Wealth Gaps

Pedro Nascimento de Lima, Jonathan Lamb, Osonde Osoba, and Jonathan Welburn

Background
Will America always be a country marked by racial disparity? Will Black families always be systematically poorer than white families? Can the American society reach a stage where past disparities are mended, economic opportunity is expanded, and where everyone gets their fair share? Are such goals mathematically feasible, and if so, over what timeframe? What policies could achieve these goals?

For many, questions like these evoke a very strong response. Some may immediately object, arguing that America is and has always been a land of opportunity for all races and groups—a leading democracy that delivered prosperity through free markets. Others will say America has never been a land of opportunity for all—particularly for Black Americans. No matter where you lie along this spectrum, we invite you to dive into mathematical facts about racial wealth disparity in America and models that help shed light on America’s wealth disparity struggle.

Building on our recently released report [1] describing the headwinds of a long and incessant racial wealth gap, we introduce a measure of racial wealth disparity and characterize the wealth distribution among Black and white US households. Next, we frame America’s racial wealth disparity challenge within a computational optimization setting where the social objective is to close racial wealth gaps at a minimal cost. Finally, we discuss how models that describe long-term dynamics of wealth accumulation—specifically, overlapping generation (OLG) models—can be used to investigate whether the United States can ever become a country free of racial wealth disparities.

Measuring Racial Wealth Disparities
Consider a nation composed of $N$ households. Because our analysis is focused on the Black-white wealth gap, we consider the set of races $R = \{b, w\}$, Black and white. Over the years, households are created, get older, and are dissolved; but at any point in time $t$, household $i$ has wealth $w_{it}$. While wealth data is not collected for all American households $i$ of race $r$, we can obtain information about the density function of wealth through the Survey of Consumer Finances (SCF), a triannual survey examining American households’ wealth maintained by the Federal Reserve [2].
Let \( f_w(x) \) denote the density function of the wealth distribution of white households and \( f_b(x) \) the density function of the wealth of Black households. In a society without racial wealth disparity, knowing a household’s race would give no information about their wealth; \( f_w(x) \) and \( f_b(x) \) should be similar. To operationalize this idea, let \( F_w(x) \) and \( F_b(x) \) be the cumulative density functions of the white and Black household wealth, respectively. The inverse\(^1\) of those functions \( F_w^{-1}(p) \) and \( F_b^{-1}(p) \) are quantile functions of both distributions and should be similar in a society free of racial wealth inequity. The existing literature commonly uses differences in the median wealth \((F_w^{-1}(0.5) - F_b^{-1}(0.5))\) to characterize wealth gaps, but a more complete representation of the wealth gap can be obtained if one considers multiple percentiles. We therefore propose the following disparity measure:

\[
D = \frac{1}{n_p} \sum_{p=p_0}^{p_n} \frac{|F_w^{-1}(p) - F_b^{-1}(p)|}{F_w^{-1}(p)},
\]

where \( p \) are percentiles and \( D \) is the racial wealth disparity; index is computed using \( n_p \) percentiles of the wealth distribution of white and Black households. That is, \( D \) is constructed from pairwise differences between the wealth of white and Black households at each percentile,\(^2\) \( p \), which defines racial wealth disparity as a normalized average of differences between white and Black household wealth. Specifically, we normalize the difference at each percentile, \( p \), by the level of white wealth \( F_w^{-1}(p) \) to avoid weighting differences at the upper end of the distribution much more than those at the lower end.

This disparity index has some useful properties. First, it is a purely descriptive index and can be used to track racial wealth inequity gaps using available SCF data. It also has a clear interpretation—if there is no disparity and Black households are as wealthy as white households, then \( D = 0 \). In a society where Black households hold no wealth, \( D = 1 \). If Black households hold about 60 percent of the wealth of white households at each percentile in the wealth distribution, \( D = 0.4 \). Moreover, policies or factors that increase wealth disparity increase \( D \) even if they increase wealth overall, and policies that decrease racial wealth disparities decrease \( D \). Finally, this disparity metric simplifies to a difference between median wealth if one only considers the 50th percentile.

Nevertheless, this disparity measure should not be seen as the only approach for measuring racial wealth disparities, nor as a metric that fully accounts for the accumulated injustices committed against Black Americans. To clarify that difference, it is useful to distinguish between disparity—a mathematical difference between groups—and inequity—an unjust difference between groups. Researchers in this field (notably, Darity, Jr. and Mullen \[^3\] have used several methods to calculate the amount that would have to be repaid in reparations to atone for injustices committed against Black Americans throughout US history. In doing so, this literature seeks to provide an account of the monetary value that would compensate for past racial inequity. Rather than being a metric meant to fully capture racial wealth inequities, index \( D \) is simply a disparity metric that sheds light on how racially segregated wealth is in any society at a given time.

Note that a society free of racial wealth disparities may still be highly unequal. Just as we differentiate between inequity and disparity, we may also differentiate between inequity and inequality—which refers to a difference in wealth within a group rather than between groups. Eq. (1) imposes no constraints on what the overall wealth distribution is in this society. Wealth could be highly concentrated with high inequality yet with little disparity. Therefore, \( D \) in eq. (1) is not a wealth inequality metric.

\section*{America’s Racial Wealth Gap is a Persistent Phenomenon}

Figure 1 shows the racial wealth inequity \( D \) metric from 1987–2019—the full range of available time-series data from the SCF. After decreasing from 1989 to 2002, disparities increased from 2002 to 2013 and since then have decreased slightly, from 0.91 to 0.87—meaning that Black households hold only about 13% of the wealth of white households. The data confirms a disappointing trend: wealth disparities have not substantially improved over the past 30 years.

The easiest way to understand why wealth disparity will not go to zero on its own is to realize that white households are positioned to gain wealth at a higher rate than Black households. Let the wealth of household \( i \) at time \( t \) be represented by random variable \( X_{it} \). Wealth grows over time following \( X_{it} = X_{it-1} + x_{it} \) where \( x_{it} \sim F(\mu, \sigma) \) are stochastic gains. Now let the wealth Black households be represented by \( Y_{jt} = Y_{jt-1} + y_{jt} \) where \( y_{jt} \sim F(M, \Sigma) \). Take \( \mu < M \) as given, then as time advances, \( \mathbb{E}[X_{it}] \) only diverges from \( \mathbb{E}[Y_{jt}] \), and the probability that Black households will catch up with Black households in their wealth is \( P(\mathbb{E}[X_{it}] < \mathbb{E}[Y_{jt}]) = 0 \). Thus, as long as \( \mu < M \), Black households will never catch up with white households in wealth. Therefore, a difference in starting conditions in wealth and a difference in the savings is sufficient to cause an indefinite disparity in wealth.

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\(^1\)Note that \( F(x) \) is nondecreasing, so \( F^{-1}(p) \) can be defined as a generalized inverse distribution function as \( F^{-1}(p) = \inf\{x \in \mathbb{R} : F(x) \geq p\} \).

\(^2\)In our report, we use \( W_{w,p} \) as shorthand for \( F^{-1}(p) \) for the wealth distribution of white households and \( W_{b,p} \) for \( F^{-1}(p) \) of the wealth distribution of Black households.

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\[^3\]August 2023 Notices of the American Mathematical Society 1137
One might hope that this simplified model does not capture important features of the real world and that the wealth and income of white and Black households are on track to converge. Perhaps the incomes of white and Black households would converge as education opportunities are leveled, or perhaps white households would spend their savings, naturally closing the wealth gaps. Unfortunately, this optimistic view is neither supported by decades of data nor by models that estimate the long-run dynamics of wealth (discussed later).

The long-run difference in $E[X_i(t)] < E[Y_i(t)]$ shows that disparities can never close unless Black households are able to increase their wealth at a faster rate than white households. Yet, the legacy of past disparity is transferred to younger generations through several mechanisms [4], including lower levels of intergenerational transfer (i.e., inheritance) for Black households relative to white households and Black households earn less income even for the same educational level of their white counterparts. The combined effect results in the strong and enduring headwinds shown in Figure 1.

**Closing the Racial Wealth Gap as an Optimization Problem**

Society has many competing priorities, so what is the minimum amount of wealth that would have to be created or allocated to close the racial wealth gap? Our recent research explores these questions through a sequence of optimization exercises. We start by framing the wealth inequity problem as an optimization problem, where wealth is transferred to eligible households within a wealth transfer policy. We abstract away from the policy mechanisms that create such wealth, and, for the purposes of this exercise, the policy costs approximately the same amount of wealth it creates. For example, debt forgiveness is arguably one of such policies that result in an immediate wealth effect. Other policies, such as tax incentives to home ownership, combined with easy monetary policy (i.e., low interest rates) also affect the distribution of household wealth through home values appreciation. For instance, a policy structure that results in house appreciation well above inflation results in a wealth transfer from young to old households.

In this analysis, we consider two eligibility criteria. The first is the race of the eligible households. Reparations policies exclusively target Black households ($r =$ Black) while so-called race-blind policies, such as baby bonds, allocate wealth to all households ($r =$ all races/ethnicities). The second is the wealth of the eligible household; we consider a threshold maximum value of wealth $\tilde{W}$ below which households would be eligible for the wealth-allocation policy. Finally, the total monetary value allocated to all eligible households $A$ and the number of eligible households $n_e$ determine all households’ immediate post-allocation wealth $W_{i,r}$ of household $i$ of race $r$ as follows:

$$W'_{i,r} = \begin{cases} \frac{A}{n_e}, & \text{if } W_{i,r} \leq \tilde{W} \text{ and } r \in \{\text{Black}, \text{all races}\} \\ W_{i,r}, & \text{otherwise}. \end{cases}$$ (2)

Equation 2 assumes that each eligible household would receive the equal resulting allocation $A/n_e$. It also assumes that eligibility would always be determined by current wealth, such that the least wealthy households (in terms of $n_e$) would receive an equal wealth allocation.

Given this setup, we use a many-objective evolutionary algorithm (NSGA-II) to find the sets of policies (pairs of $\tilde{W}$ and $A$) that minimize racial wealth disparities as measured by $D$ at a minimum cost. We run the optimization exercise both making all households eligible (targeted allocations to all households) and only making Black households eligible (targeted allocations to Black households). We also

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3 Baby bonds are a proposed policy for endowing every US-born child with a government-financed trust account, created at birth and accessible at age 18. Although baby bonds could be given to all children at birth, they are often discussed as a way to endow children from lower-income families with disparity-reducing allocations. Hamilton and Darity [5] propose this progressive implementation as a solution for eliminating the racial wealth gap over time.

4 A policy wherein households receive an equal allocation is only one amongst many potential policy structures. For simplicity, this is the only policy structure we consider in this paper. Future work may relax this assumption and explore other types of policies.
evaluate policies that resemble a reparations policy (equal allocations to all Black households) and policies that allocate wealth to all households irrespective of race (equal allocation to all households). Figure 2 presents the first-order, immediate effect of these policies on the wealth disparity measure $D$.

Figure 2 sheds light on some of the dilemmas policymakers would face when choosing a type of policy to close wealth gaps. First, policies that create an equal amount of wealth to all households (orange line) will reduce wealth disparities, but only at a high cost. Second, a reparations policy (blue line) can reduce disparities, but also may imply a short-term increase in disparities (as measured by disparity measure $D$) across the wealth distribution. This arises because with uniform transfers, selecting a value of $A$ sufficient to eliminate disparities in one quantile will invert any disparity in quantiles with a smaller absolute gap, resulting in a U-shaped function for $D$. Models describing long-term wealth dynamics (discussed later) project these effects to wash away over time and disparities to reverse again. Another way to reduce wealth disparities without any short-term increase in disparity measure $D$ would be to use targeted policies—i.e., allocations targeted to the least-affluent households, with allocation amounts calibrated to decrease wealth disparities at the minimum cost (red and black lines). The black curve demonstrates how much racial wealth disparities could decrease immediately if such policies were race-neutral (targeted allocations to all households) and if they were targeted to Black households (targeted allocations to Black households).

Such a model of the impact of wealth-changing policies on racial wealth disparities can help inform policy decisions that would otherwise be made in the absence of this information. Policymakers interested in understanding the immediate effect of policies that shape wealth outcomes can use this optimization approach to better calibrate the magnitude of policies such that interventions reduce, rather than add to racial wealth inequalities. The analysis presented earlier only shows the immediate effect of policies, but policymakers should look over a longer time-horizon when making policy choices that affect wealth disparities. Looking beyond one time-period however requires both more data and more modeling effort. Hence, we turn to models used in macroeconomics to shed light on long-term dynamics of disparity and the potential impacts of interventions.

The Long-term Dynamics of Racial Wealth Disparities

Overlapping generation (OLG) models are used in macroeconomics to study the long-run dynamics of fiscal policies and the accumulation and distribution of wealth. OLG models have been used to clarify the need for policies that are foundational to society, such as social security. Recently, OLG models have been used to study long-term consequences of policies to reduce racial wealth disparities [6], [7]. Where other common approaches derived from Solow or Ramsey make a simplifying assumption that economic agents are identical and infinitely lived [8], OLG models incorporate multiple subpopulations at different
stages of a finite lifecycle to capture more realistic aggregate economic measures and the evolution of variables like wage levels and savings rates. The simplest OLG formulation follows two generations: “young” wage-earning households, and “old” retired households. Each period, the “young” earn income and save to fund consumption in retirement, while the “old” generation spends remaining savings plus any earned interest. In the next period, the young household transitions to become the new “old,” and a new “young” is born. Wages are determined by a production sector using labor of the “young” and capital supplied by the “old” savings. This simplistic scenario can be extended with additional generations, the introduction of taxation and social safety nets, bequests between generations, investments in productivity, and so on.

Consider the following model, described in more detail in [10]. At time $t$, the young generation, $g$, consumes $c_{g,t}$, chooses whether to attend post-secondary education $e_{g,t}$, chooses whether to buy a house $I_{g-1,t}$, and pays housing rent $h_{g,t}$ to the old generation, $g-1$. Wage levels and savings rates. The simplest OLG formulation [9] follows two generations: “young” wage-earning households, and “old” retired households. Each period, the “young” earn income and save to fund consumption in retirement, while the “old” generation spends remaining savings plus any earned interest. In the next period, the young household transitions to become the new “old,” and a new “young” is born. Wages are determined by a production sector using labor of the “young” and capital supplied by the “old” savings. This simplistic scenario can be extended with additional generations, the introduction of taxation and social safety nets, bequests between generations, investments in productivity, and so on.

Finally, at time $t$, the old generation, $g-2$, consumes $c_{g-2,t}$, chooses whether to buy a house $I_{g-2,t} = \{0 \text{ if no, } 1 \text{ if yes}\}$, pays housing rent $h_{g,t}$ if they do not own, saves $s_{g-2,t}$, and pays a net intergenerational transfer. Old earns income from interest earning savings from the prior period, $s_{g-2,t-1}$, and pre-tax labor income $y_{g-2,t}$ (i.e., retirement income) as follows:

$$c_{g-2,t} + (1 - H_{g-2,t}) h_{g,t} + \sum_{j \neq g-2} (R_{g-2,j,t} - R_{jg-2,t}) \leq y_{g-2,t}(e)(1 - \tau(y_{g-2,t})) + (1 + r) s_{g-2,t-1} + H_{g-2,t}V_t.$$  

(6)

In each period for each generation, households receive a stochastic income stream as a function of the median income for their generation at period $t$, a set of demographic characteristics including race, and an income shock conditional on education and demographics:

$$y_{g,t} = F_y(\mu_{g,t}, \sigma_{g,t}; e_{g,t}, D_g).$$

(7)

Net wealth in period $t$ is given by $W_{g,t} = s_{g,t} + H_{g,t}V_t$ and generations gain utility with constant relative risk aversion

$$u_{g,t}(c_{g,t}) = \beta c_{g,t}^{1-\rho}.$$ 

(8)

We express the value function iteration, known as the Bellman equation, as the Q-function:

$$Q^*(s_t, a_t) = u(s_t, a_t) + \gamma \cdot Q^*(s_{t+1}, a).$$  

(9)

After specifying a model that describes how wealth evolves over time, and how households make decisions, one can set initial conditions for this model—including race heterogeneity using the SCF data described previously to understand how wealth might evolve over the next several decades or even centuries.

While models like these are scarce, they recently clarified potential pathways to reduce racial wealth disparities in the long run [6], [7]. First, no model-based analysis predicts that racial wealth disparities can close absent any policy intervention. One paper [6] finds that absent any other changes, even a reparations policy that would immediately close average wealth gaps would not necessarily lead to convergence of wealth in the long run. In their model, different beliefs about risk returns would prevent Black households from realizing the benefits of reparations transfers and would cause the $D$ disparity measure to increase back to around 0.5 after being reduced to 0 under the reparations policy. Their analysis finds that an alternative policy, investment subsidies (i.e., low-interest loans) targeted to Black households can close wealth gaps in the long run. In their model, such policies would eventually increase the expected returns from investments among Black households and help catalyze a virtuous cycle of Black wealth creation. Another paper based on an OLG
model emphasizes the importance of initial conditions, in particular firm ownership [7]. In this model, an initial gap in firm ownership is sufficient to prevent Black households to catch up in wealth. This result holds even if there was no racism post Jim-Crow era. Those analyses highlight that the US government must carefully consider the potential effects of policy interventions in the long run if it aims to close wealth gap over a meaningful time-horizon.

Conclusion
The extent to which model assumptions would hold in the real world is a subject of debate. For instance, existing OLG models may not fully account for present-day racism that could hinder wealth accumulation amongst Black households. Yet mathematical models agree that racial wealth disparities will be perpetuated absent policy interventions and help chart potential pathways to reduce racial wealth disparities in the United States over the next few centuries.

While mathematical models do not replace fundamental value judgements about the merits of different policies, they can inform the debate about how to close racial wealth disparities by allowing researchers to clearly state model assumptions, anticipate predictable consequences of policy interventions, and ultimately help steer the United States towards a more equitable society.

Data and Code Availability
All the data used by this paper is publicly available. All figures and tables in this paper and our report can be reproduced with free software, and we made our code publicly available at https://github.com/RANDCorporation/racial-wealth-gap

References

Credits
Figures are courtesy of Pedro Nascimento de Lima.
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Full proposals received by November 30, 2023, will receive full consideration.

The 2025 MRC program is funded by the National Science Foundation, the AMS, and private donors.
Paths have a logic and a mystery all their own. Milwaukee Avenue, which diagonally skewers the otherwise largely Cartesian Chicago layout, closely follows a Native American trail, used for centuries by the indigenous inhabitants of Illinois, which may perhaps have been first created by animals seeking the waters of Lake Michigan.

Topologists care about paths and path-connectedness. And it sometimes happens that a topological space of interest can be shown to be path-connected in a constructive way, by exhibiting an explicit path between two given points.

Iterated function systems (hereafter IFSs) are a well-studied class of dynamical systems. One fixes a complete metric space $X$ and a finite collection of uniform contracting maps $f_1, \ldots, f_n$ of $X$, and considers the semigroup $S$ generated by the $f_i$. Associated to an IFS is its attractor $\Lambda$, the unique closed nonempty subset of $X$ with the property that $\Lambda$ is equal to the union of its images $f_i \Lambda$. Because of the uniform contraction of the $f_i$, one may obtain a concrete description of $\Lambda$ as the closure of the set of points $w(p)$ where $p \in \Lambda$ is arbitrary (e.g., we could take $p$ to be the unique fixed point of $f_1$) and where $w$ ranges over the elements of $S$.

Perhaps the simplest nontrivial examples of IFSs are obtained by taking $X = \mathbb{C}$, and considering the IFS generated by a pair $f, g$ of linear contractions with distinct fixed points and common contraction factor $c$ (with $0 < |c| < 1$). Up to conjugacy, we may normalize the generators as $f : z \to cz$ and $g : z \to 1 + cz$ and in this way obtain a natural family of attractors $\Lambda(c) \subset \mathbb{C}$ parameterized by the punctured open unit disk $D^*$. Figure 2 shows some examples of $\Lambda(c)$ for various $c$. In each example, $f\Lambda(c)$ is in blue and $g\Lambda(c)$ is in orange, and one can see quite viscerally how $\Lambda$ is obtained from the union.

For this family of IFSs there is a clean dichotomy for $\Lambda(c)$: either $\Lambda(c)$ is connected (in fact, path-connected) or it is a Cantor set. In 1981 Barnsley–Harrington proposed...
the study of the “Mandelbrot-like” set $M \subset \mathbb{D}^*$ consisting of parameters $c$ for which $\Lambda(c)$ is connected. The set $\mathcal{M}$ is depicted in Figure 3.

In 1988 Thierry Bousch, in an unpublished manuscript, proved that $\mathcal{M}$ is connected path-connected. His proof is rather elegant, and gives rise to explicit paths with their own particular charm, and we reproduce it in what follows.

The first (rather elementary but useful) observation is that $\mathcal{M}$ contains the entire annulus $\{1 > |z| > 1/\sqrt{2}\}$. Here is a sort of heuristic argument, using Lebesgue measure. Let’s suppose the (2-dimensional) Lebesgue measure $\mu$ of $\Lambda(c)$ is positive (it is evidently finite). A linear contraction by the factor $|c|$ multiplies Lebesgue measure by $|c|^2$, so if $f\Lambda(c)$ and $g\Lambda(c)$ are disjoint, then from $\Lambda = f\Lambda(c) \cup g\Lambda(c)$ it follows that $\mu = 2|c|^2\mu$, or $|c|^2 = 1/2$. This suggests (and it turns out to be true) that for $|c| < 1/\sqrt{2}$ the Lebesgue measure of $\Lambda(c)$ is zero, and for $|c| > 1/\sqrt{2}$ the images $f\Lambda(c)$ and $g\Lambda(c)$ intersect nontrivially (in a set of positive Lebesgue measure).

To go beyond this we look at $\mathcal{M}$ in another way. One very pleasant reason to study $\mathcal{M}$ is that it has, perhaps unexpectedly, entirely different characterization from the dynamical one given above in terms of roots. We explain how. First, notice that 0, being the unique fixed point of $f$, is necessarily contained in $\Lambda(c)$, and in fact $\Lambda(c)$ is the closure of the images $w(0)$ as $w$ ranges over the set of finite words in the alphabet $\{f, g\}$. If $w$ is such a word, the image $w(0)$ is the value of a polynomial $p_w$ evaluated at $c$, where the degree of $p_w$ is (most) one less than the length of $w$, and the coefficients of $p_w$ (from lowest to highest) are 0 or 1 according to whether the letters of $w$ are $f$ or $g$.

This is best illustrated by example:

$$
gffg(0) = gffg(1)
= gff(c) = gff(1 + c^2) = g(c + c^3)
= g(c^2 + c^4) = 1 + c^3 + c^5
$$

Words beginning with $f$ correspond to polynomials with coefficients in $\{0, 1\}$ and with constant term 0, and words beginning with $g$ correspond to polynomials with coefficients in $\{0, 1\}$ and with constant term 1. Taking closures, one sees that $\Lambda(c)$ is the set of values of power series with all coefficients in $\{0, 1\}$ evaluated at $c$.

It is not hard to show that $\Lambda(c)$ is connected if and only if $f\Lambda \cap g\Lambda$ is nonempty; equivalently if there are two power series $\phi, \psi$ with coefficients in $\{0, 1\}$, where $\phi$ has constant term 1 and $\psi$ has constant term 0, such that $\phi(c) = \psi(c)$. Equivalently, $c$ is a root of the power series $(\phi - \psi)/2$ which has coefficients in $\{-1, 0, 1\}$. We finally arrive at the characterization of $\mathcal{M}$ as the set of $0 < |c| < 1$ which are roots of power series in one variable $z$ and coefficients in $\{-1, 0, 1\}$ (starting with constant term 1 for concreteness).

OK, now let’s show $\mathcal{M}$ is path-connected. Let $\mathfrak{P}$ denote the set of power series in $z$ (with constant term 1) and coefficients in $\{-1, 0, 1\}$ and let $\mathfrak{Q}$ denote the space of closed subsets of the open unit disk (in the Hausdorff topology). There is a map $\rho : \mathfrak{P} \to \mathfrak{Q}$ which takes a power series to its set of roots. We may topologize $\mathfrak{P}$ in an obvious way as a Cantor set and then ask about the continuity of $\rho$.

How do the roots of a power series depend on its coefficients? Since the coefficients are all bounded, one may show (e.g., by Rouché’s theorem) that for every $\varepsilon$ and every $\lambda < 1$ there is an $n$ so that if $\phi, \psi \in \mathfrak{P}$ agree on the first $n$ coefficients, then $\rho(\phi)$ and $\rho(\psi)$ are $\varepsilon$-close in the Hausdorff distance when restricted to the disk of radius $\lambda$ (this is even true in the sense of counting roots with multiplicity). Since $\mathcal{A} \subset \mathcal{M}$ we only need to consider roots with absolute value at most $1/\sqrt{2}$, and therefore (in this range) $\varepsilon$ may be controlled uniformly by $n$, and $\rho$ is (uniformly) continuous.

Using this observation we now indicate why $\mathcal{M}$ is path-connected. Let $\phi \in \mathfrak{P}$. For every $n$ we will construct a finite

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**Figure 2.** Attractors $\Lambda(c)$ for various $c$.

**Figure 3.** The set $M$. 
sequence of power series

\[ \phi = B_0, A_1, B_1, \ldots, A_m = 1 \]

where each \( A_j \) (\( j > 0 \)) is a polynomial of degree \( \leq n \) obtained by truncating \( B_{j-1} \), throwing away all terms of degree \( > n \); and where

\[ B_j = \frac{A_j}{1 + \alpha(j)z^{n(j)}} \quad (j > 0) \]

where \( n(j) \) is the degree of \( A_j \), and \( -\alpha(j) \) is the coefficient of \( z^{n(j)} \) in \( A_j \).

Since \( 1 + \alpha(j)z^{n(j)} \) is nonzero throughout the open unit disk, \( B_j \) has the same roots as \( A_j \). Furthermore, since \( A_j \) and \( B_{j-1} \) agree on the first \( n \) coefficients, \( \rho(A_j) \) is \( \epsilon \)-close to \( \rho(B_{j-1}) \) outside the annulus \( A \). Since \( A_m \) has no roots outside this annulus (in fact no roots at all) this gives a sequence of successive points in \( M - A \), each at most distance \( \epsilon \) from the previous one, connecting any root of \( \phi \) to \( A \).

Finally, observe that increasing \( n \) gives a new sequence that refines the old, where intermediate terms in the new sequence between successive terms in the old sequence all agree in degree \( < n \).

This raises an interesting question. Recall that \( \Psi \) is our notation for the (Cantor) set of power series, and let \( \mathcal{E} \) denote the total space of the bundle over \( \Psi \) whose fiber over each \( p \in \Psi \) is its set of roots in the open unit disk. We can define an equivalence relation on \( \mathcal{E} \) which identifies the fibers over different \( p, p' \) with the same roots in the open unit disk. Let \( X \) denote the quotient space. Projecting to the unit disk defines a surjection \( X \to M \); since \( X \) is path-connected, so is \( M \). What is the topology of \( X \)?

Call for Nominations for the Ostrowski Prize 2023

The aim of the Ostrowski Foundation is to promote the mathematical sciences. Every second year it provides a prize for recent outstanding achievements in pure mathematics and in the foundations of numerical mathematics. The value of the prize for 2023 is 100,000 Swiss francs.

The prize has been awarded every two years since 1989. The most recent winners are Yitang Zhang in 2013, Peter Scholze in 2015, Akshay Venkatesh in 2017, and Assaf Naor in 2019, and Tim Austin in 2021. See https://www.ostrowski.ch/index_e.php for the complete list and further details.

The jury invites nominations for candidates for the 2023 Ostrowski Prize. Nominations should include a CV of the candidate, a letter of nomination and 2-3 letters of reference.

The Chair of the jury for 2023 is Cameron Stewart of the University of Waterloo, Canada. Nominations should be sent to cstewart@uwaterloo.ca by September, 30th, 2023.
AMS Fellowships

NEW! Stefan Bergman Fellowship

The AMS established the Stefan Bergman Fellowship in 2023 to support the advancement of the research portfolios of research mathematicians who are engaged in the areas of real analysis, complex analysis, or partial differential equations. This fellowship program is made possible by the proceeds of the Stefan Bergman Trust, which had previously supported the Stefan Bergman Prize. One fellowship in the amount of $25,000 will be awarded for the 2024–2025 academic year.

About this Fellowship

The eligibility rules are as follows: The fellowship is open only to research mathematicians who (1) are engaged in the areas of real analysis, complex analysis, or partial differential equations; (2) are AMS members; (3) have not received tenure; and (4) have not held significant fellowship support. This fellowship honors the memory of Stefan Bergman, best known for his research in several complex variables, as well as the Bergman projection and the Bergman kernel function that bear his name. A native of Poland, he taught at Stanford University for many years and died in 1977 at the age of 82. He was an AMS member for 35 years. When his wife Adele Bergman died, the terms of her will stipulated that funds should go toward a special prize in her husband’s honor.

Application Period

Applications will be collected via MathPrograms.org July 15, 2023–October 11, 2023 (11:59 pm EST). Find more application information at https://www.ams.org/bergman-fellow. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; prof-serv@ams.org; 401-455-4129.
Vietnamese Colleagues Translate AMS Mathematical Moments

Thanks to colleagues at the Institute of Mathematics (IM) in Hanoi, Vietnam, readers can enjoy—in Vietnamese—Mathematical Moments, the AMS series of posters and interviews about applications of mathematics in the world.

Dr. Hop Nguyen and contemporaries at IM generated 21 translations while AMS staff artists provided the graphic design. Recently, IM displayed posters of Mathematical Moments at their March event “Mathematics for Everyone.”

The Vietnamese translations and other Mathematical Moments in 15 languages—from Portuguese to Polish, Chinese to Czech—are available for free at https://www.ams.org/mathmoments.

Deaths of AMS Members

Harry Cohn, of Australia, died on May 7, 2021. Born on July 27, 1940, he was a member of the Society for 46 years.

Maurice J. Frank, Jr., of Evanston, Illinois, died on February 13, 2023. Born on December 20, 1942, he was a member of the Society for 52 years.

Bert D. Garrett, of Ft. Worth, Texas, died on December 31, 2011. Born on February 15, 1939, he was a member of the Society for 43 years.

Doris Hinestroza, of Colombia, died in February 2019. Born on May 24, 1954, she was a member of the Society for 30 years.

Robert J. Koch, of Baton Rouge, Louisiana, died on August 11, 2020. Born on April 17, 1926, he was a member of the Society for 70 years.

Sally Irene Lipsey, of New York, New York, died on November 26, 2022. Born on December 31, 1926, she was a member of the Society for 46 years.

Bernard Morin, of France, died on March 12, 2018. Born on March 3, 1931, he was a member of the Society for 56 years.

Duane P. Sather, of Boulder, Colorado, died on November 5, 2016. Born on September 19, 1933, he was a member of the Society for 47 years.

David Singmaster, of the United Kingdom, died on February 13, 2023. Born on December 14, 1938, he was a member of the Society for 59 years.

Paul Jacob Weiner, of Torrance, California, died in December 2012. Born on May 14, 1949, he was a member of the Society for 27 years.

Abdul-Aziz Yakubu, of Silver Spring, Maryland, died in August 2022. He was a member of the Society for 36 years.

Robert J. Zimmer, of Chicago, Illinois, died on May 23, 2023. Born on November 5, 1947, he was a member of the Society for 37 years.
Mathematics People

Nine Mathematicians Elected to National Academy of Sciences

Nine mathematical scientists were among 143 individuals elected to the National Academy of Sciences in 2023 “in recognition of their distinguished and continuing achievements in original research,” according to an NAS press release.

Newly elected mathematical scientists and their affiliations at the time of election:

- Michael Crandall, professor emeritus, Department of Mathematics, University of California, Santa Barbara
- Lisa J. Fauci, professor, Department of Mathematics, Tulane University
- Cameron M. Gordon, professor, Department of Mathematics, University of Texas at Austin
- Gunther Uhlmann, Robert R. and Elaine F. Phelps Endowed Professor, Department of Mathematics, University of Washington
- Akshay Venkatesh, Robert and Luisa Fernholz Professor, School of Mathematics, Institute for Advanced Study
- W. Hugh Woodin, professor of philosophy and mathematics, Department of Philosophy, Harvard University
- Crandall, Fauci, and Uhlmann are AMS Fellows and AMS members. Venkatesh is an AMS member.

Newly elected international mathematical scientists, their affiliations at the time of election, and their countries of citizenship:

- Alison M. Etheridge, professor of probability, department of statistics, University of Oxford (United Kingdom)
- Masaki Kashiwara, emeritus and project professor, Research Institute for Mathematical Sciences, Kyoto University (Japan)
- Dominique Picard, emeritus professor, mathematics, Paris Diderot University (France)

A private, nonprofit institution, the National Academy of Sciences was established under a congressional charter signed by President Abraham Lincoln in 1863. NAS recognizes achievement in science by election to membership. With the National Academy of Engineering and the National Academy of Medicine, NAS provides the federal government and other organizations with policy advice on science, engineering, and health. Read the full list of newly elected members at [http://www.nasonline.org/news-and-multimedia/news/2023-nas-election.html](http://www.nasonline.org/news-and-multimedia/news/2023-nas-election.html).

—National Academy of Sciences

SIAM Announces Class of 2023 Fellows

The Society for Industrial and Applied Mathematics (SIAM) announced the 2023 class of SIAM Fellows. “Through their various contributions, SIAM Fellows are a core group of individuals helping to advance the fields of applied mathematics and computational science,” the society wrote in a press release.

In alphabetical order, the 26 new SIAM Fellows are:

- Rodrigo Bañuelos, Purdue University, for pioneering and fundamental contributions in probability theory and analysis, and for fostering diversity in mathematics and education.
- George Biros, University of Texas at Austin, for development of high-performance scientific computing algorithms and their use in tackling challenging problems in science, engineering, and medicine.
- Ron Buckmire, Occidental College, for broadening participation in mathematics, creating innovative educational materials in applied mathematics, and contributing to the field of finite differences.
- Fioralba Cakoni, Rutgers, The State University of New Jersey, for seminal contributions to inverse scattering theory, the existence of transmission eigenvalues, and non-scattering phenomena.
- Daniela Calvetti, Case Western Reserve University, for outstanding contributions to numerical linear algebra, Bayesian scientific computing, and inverse problems and applications, and for extraordinary mentoring activities.
- Coralia Cartis, University of Oxford, for theoretical and practical developments in continuous optimization.

DOI: https://doi.org/10.1090/noti2745
Alina Chertock, North Carolina State University, for significant contributions to numerical methods for hyperbolic systems of conservation laws and important service to the applied mathematics community.

Lenore Jennifer Cowen, Tufts University, for seminal contributions to computational biology through the design of graph-based algorithms and insights into network distance measures.

Petros Drineas, Purdue University, for pioneering contributions to all aspects of randomized numerical linear algebra: research, applications, advocacy, and outreach.

Aric Hagberg, Los Alamos National Laboratory, for contributions to network science, dynamical systems, and dedicated service to the scientific community.

Chandrika Kamath, Lawrence Livermore National Laboratory, for community leadership and contributions to data mining and its application to real-world problems in science and engineering.

Angela Kunoth, University of Cologne, for fundamental contributions to multi-level and wavelet methods for the numerical solution of partial differential equations and optimal control with partial differential equation constraints.

James Donald Meiss, University of Colorado Boulder, for contributions to the understanding of the onset of chaos and transport in Hamiltonian and volume-preserving dynamical systems.

Andrew M. Odlyzko, University of Minnesota, for fundamental and visionary contributions to analytic and computational number theory, cryptography, and communication and electronic publishing.

Ali Pinar, Sandia National Laboratories, for theoretical, algorithmic, and application impacts, and community leadership in combinatorial scientific computing and network science.

Edward B. Saff, Vanderbilt University, for contributions to approximation theory, potential theory, numerical analysis, particle systems analysis, and inverse problems.

David James Silvester, University of Manchester, for contributions to finite elements and computational fluid dynamics.

Barry Simon, California Institute of Technology, for outstanding originality in contributions to spectral theory, mathematical physics, and orthogonal polynomials, as well as strong research leadership through supervision.

Catherine Sulem, University of Toronto, for numerical and analytical contributions to nonlinear dispersive waves in optics and fluids.

Sivan Toledo, Tel Aviv University, for advances in parallel and randomized numerical linear algebra and location estimation technology in movement ecology.

Konstantina Trivisa, University of Maryland, College Park, for outstanding contributions to analysis of nonlinear partial differential equations, exemplary service, and excellence in mentoring of students and postdocs.

Caroline Uhler, Massachusetts Institute of Technology and Broad Institute, for fundamental contributions at the interface of statistics, machine learning, and biology.

John S. Wettlaufer, Yale University and Nordic Institute for Theoretical Physics, for fundamental contributions to the modeling of interfacial problems, the study of ice, geophysics, and climate dynamics.

Dongbin Xiu, Ohio State University, for pioneering fundamental contributions to the mathematics and applications of uncertainty quantification, and for exceptional service in organizing many workshops.

Laurent Younes, Johns Hopkins University, for fundamental contributions to the theory and computation of shape space in image analysis.

Yongjie Jessica Zhang, Carnegie Mellon University, for pioneering contributions to computational geometry, volumetric parameterization, isogeometric analysis, mesh generation, image processing, and simulation-based engineering applications.

—Society for Industrial and Applied Mathematics

Downey Awarded S. Barry Cooper Prize

The Association for Computability in Europe awarded the 2023 S. Barry Cooper Prize to Rodney G. Downey, Victoria University of Wellington [New Zealand].

The award is given to a researcher who has contributed to a broad understanding and foundational study of computability by (a) outstanding results; (b) seminal and lasting theory building; and (c) exceptional service to the research communities.

“Rod Downey has an impressive record of accomplishment on all three fronts, with a breadth of contribution that is truly outstanding,” the prize committee reported. “He has been involved, inter alia, in establishing important results on the lattice of computably enumerable degrees, on algorithmic randomness, and in parameterized complexity.

“More than the individual results, he is widely recognized as a founding figure in the subject of parameterized complexity and as having inspired a great burgeoning of interest in algorithmic randomness. … At the same time, he has played a pivotal role in building research communities, including not only the international community
in his research areas but the mathematical community in New Zealand.”

The Cooper Prize will be announced formally at the Conference CiE 2023 in Batumi, Georgia. Downey will deliver the award lecture during the Conference CiE 2024 in Amsterdam, the Netherlands.

—Association for Computability in Europe

Sloan Research Fellowships Awarded to 20 Early-Career Mathematicians

Twenty mathematicians were among the 125 early-career scholars awarded 2023 Sloan Research Fellowships by the Alfred P. Sloan Foundation:

Yuansi Chen, Duke University
Duncan Dauvergne, University of Toronto
Edgar Dobriban, University of Pennsylvania
Jeremy Hahn, Massachusetts Institute of Technology
Zheng Tracy Ke, Harvard University
Roy R. Lederman, Yale University
Nicole R. Looper, University of Illinois at Chicago
Bhargav Narayanan, Rutgers, The State University of New Jersey
Sam Raskin, University of Texas at Austin
Jose Israel Rodriguez, University of Wisconsin-Madison
Pedro J. Sáenz, University of North Carolina, Chapel Hill
Will Sawin, Columbia University
Sophie Spirkl, University of Waterloo
Hiro Lee Tanaka, Texas State University
Yunqing Tang, University of California, Berkeley
Caroline Terry, Ohio State University
Xiaochuan Tian, University of California, San Diego
Alexander S. Wein, University of California, Davis
Jonathan J. Zhu, University of Washington
Ziquan Zhuang, Johns Hopkins University

“Their achievements and potential place them among the next generation of scientific leaders in the U.S. and Canada,” according to the prize announcement. Winners receive $75,000, which may be spent during a two-year term on any expense that is supportive of their research.

—Alfred P. Sloan Foundation

Cori, Kucharski Receive 2023 Adams Prize

The University of Cambridge awarded the 2023 Adams Prize to Anne Cori of Imperial College London and Adam Kucharski of the London School of Hygiene and Tropical Medicine for their work modelling the spread of infectious disease outbreaks, from SARS-CoV-2 and influenza to Ebola and HIV.

Since 1850, the Adams Prize has honored UK-based researchers under age 40 who produce first-class international research in the mathematical sciences. In 2023, for the first time since the prize was awarded, the prize honored achievements in the field of mathematical and statistical epidemiology.

Named for the mathematician John Couch Adams, the prize was endowed by members of St John’s College and commemorates Adams’s role in the discovery of the planet Neptune through calculation of the discrepancies in the orbit of Uranus. Former winners include theoretical physicist Professor Stephen Hawking (1966). Cori is the sixth woman to win the prize in its almost 175-year history.

—University of Cambridge

Murty Receives 2023 CMS Jeffery–Williams Prize

For his contributions to mathematical research, the Canadian Mathematical Society (CMS) awarded the 2023 CMS Jeffery–Williams Prize to V. Kumar Murty, director of the Fields Institute for Research in Mathematical Science.

“I am touched and honored to receive a prize named after two dedicated professionals, Jeffery and Williams, who did so much to promote the growth and development of mathematical research in Canada,” Murty said. His mathematical interests include analytic number theory, algebraic number theory, information security, and arithmetic algebraic geometry. His recent work has expanded to mathematical modelling in social, economic, and health contexts, including integrative modelling related to the COVID-19 pandemic.

In 1982 Murty received his doctorate from Harvard University as a student of John Tate. In 1987, he was appointed associate professor at the University of Toronto, and was promoted to full professor in 1991. He was chair of the Department of Mathematics at Toronto during 2008–2013 and again from 2014–2017.
Murty has served the Canadian Mathematical Society on its board of directors and as its vice-president. He was elected a Fellow of the Royal Society of Canada in 1995; Fields Institute Fellow in 2003; Fellow of the National Academy of Sciences (India) in 2011; Senior Fellow of Massey College in 2020; and a Fellow of the American Mathematical Society in 2021. Murty received the Coxeter–James Prize in 1991, the Balaguer Prize (together with M. Ram Murty) in 1996, and the University of Toronto’s Inventor of the Year Award in 2011.

—Canadian Mathematical Society

Nešlehová Awarded 2023 Krieger–Nelson Prize

Johanna G. Nešlehová of McGill University was named the recipient of the 2023 Krieger–Nelson Prize, the Canadian Mathematical Society announced.

“Nešlehová is recognized for her exceptional contributions to statistics, including multivariate analysis, stochastic dependence modeling, and extreme-value theory,” according to a CMS press release. “She is a world leader on copula models and their many ramifications in multivariate statistics, notably in relation to risk analysis and extreme-value theory, an area to which she has made numerous outstanding contributions.”

Nešlehová earned her PhD in mathematics from Carlvon-Ossietzky-Universität Oldenburg in 2004. She is currently editor-in-chief of the Canadian Journal of Statistics and has served as associate editor for journals such as Test, the Journal of Multivariate Analysis, and Statistics & Risk Modeling. In 2011, Nešlehová was named an Elected Member of the International Statistics Institute and in 2020 was named a Fellow of the Institute of Mathematical Statistics. Additionally, in 2019 she received the CRM-SSC Prize in Statistics, and McGill recognized the excellence of her graduate training with the Carrie M. Derick Award for Graduate Supervision and Teaching.

Jointly named for Cecilia Krieger and Evelyn Nelson, the Krieger–Nelson Prize was launched in 1995 to recognize outstanding contributions in the area of mathematical research by a female mathematician.

—Canadian Mathematical Society

JMSJ Outstanding Paper Prize for 2023 Honors Ros

Antonio Ros, Department of Geometry and Topology, University of Granada, has received the 2023 Outstanding Paper Prize from the Journal of the Mathematical Society of Japan (JMSJ) for his paper “On the first eigenvalue of the Laplacian on compact surfaces of genus three” (74 (2022), no. 3, 813–828).

Since 2010, this prize has been awarded annually to the authors of the most outstanding article published in JMSJ in the preceding year. The prize committee comprises JMSJ’s editorial board members. Founded in 1948, JMSJ is published quarterly by the Mathematical Society of Japan (MSJ).

—Mathematical Society of Japan

MIT Wins 2022 MAA Putnam Competition

The Massachusetts Institute of Technology team of Mingyang Deng, Luke Robitaille, and Daniel G. Zhu ranked first in the 83rd William Lowell Putnam Mathematical Competition, held in December 2022 by the Mathematical Association of America.

The 2022 Individual Putnam Fellows—the five highest-ranking individuals, listed in alphabetical order—are Deng, Papon Lapate (MIT), Brian Liu (MIT), Robitaille, and Zhu. The 2022 Elizabeth Lowell Putnam Prize winner is Binwei Yan (MIT).

The international undergraduate competition, which awards cash prizes, drew 3,415 students from 456 institutions. The highest score on the six-hour exam was 101 out of a possible 120 points.

—Mathematical Association of America
New Books Offered by the AMS

Algebra and Algebraic Geometry

Iwasawa Theory and Its Perspective
Tadashi Ochiai, Tokyo Institute of Technology, Japan

Iwasawa theory began in the late 1950s with a series of papers by Kenkichi Iwasawa on ideal class groups in the cyclotomic tower of number fields and their relation to $p$-adic $L$-functions. The theory was later generalized by putting it in the context of elliptic curves and modular forms. The main motivation for writing this book was the need for a total perspective of Iwasawa theory that includes the new trends of generalized Iwasawa theory. Another motivation of this book is an update of the classical theory for class groups taking into account the changed point of view on Iwasawa theory.

The goal of this first part of the two-part publication is to explain the theory of ideal class groups, including its algebraic aspect (the Iwasawa class number formula), its analytic aspect (Leopoldt–Kubota $L$-functions), and the Iwasawa main conjecture, which is a bridge between the algebraic and the analytic aspects.

The second part of the book will be published as a separate volume in the same series, Mathematical Surveys and Monographs of the American Mathematical Society.

This item will also be of interest to those working in number theory.

Mathematical Surveys and Monographs, Volume 272

bookstore.ams.org/surv-272

Discrete Mathematics and Combinatorics

Numbers and Figures
Six Math Conversations Starting from Scratch
Giancarlo Travaglini, Università di Milano-Bicocca, Italy

One of the great charms of mathematics is uncovering unexpected connections. In Numbers and Figures, Giancarlo Travaglini provides six conversations that do exactly that by talking about several topics in elementary number theory and some of their connections to geometry, calculus, and real-life problems such as COVID-19 vaccines or fiscal frauds. Each conversation is in two parts—an introductory essay which provides a gentle introduction to the topic and a second section that delves deeper and requires study by the reader. The topics themselves are extremely appealing and include, for example, Pick’s theorem, Simpson’s paradox, Farey sequences, the Frobenius problem, and Benford’s Law.

Numbers and Figures will be a useful resource for college faculty teaching Elementary Number Theory or Calculus. The chapters are largely independent and could make for nice course-ending projects or even lead-ins to high school or undergraduate research projects. The whole book would make for an enjoyable semester-long independent reading course. Faculty will find it entertaining bedtime reading and, last but not least, readers more generally will be interested in this book if they miss the accuracy and imagination found in their high school and college math courses.

This item will also be of interest to those working in general interest, number theory, and probability and statistics.
**Analytic Number Theory for Beginners**

Second Edition

Prapanpong Pongsriiam, Silpakorn University, Nakhon Pathom, Thailand, and Nagoya University, Japan

This new edition of *Analytic Number Theory for Beginners* presents a friendly introduction to analytic number theory for both advanced undergraduate and beginning graduate students, and offers a comfortable transition between the two levels. The text starts with a review of elementary number theory and continues on to present less commonly covered topics such as multiplicative functions, the floor function, the use of big $O$, little $o$, and Vinogradov notation, as well as summation formulas. Standard advanced topics follow, such as the Dirichlet $L$-function, Dirichlet's Theorem for primes in arithmetic progressions, the Riemann Zeta function, the Prime Number Theorem, and, new in this second edition, sieve methods and additive number theory.

The book is self-contained and easy to follow. Each chapter provides examples and exercises of varying difficulty and ends with a section of notes which include a chapter summary, open questions, historical background, and resources for further study. Since many topics in this book are not typically covered at such an accessible level, *Analytic Number Theory for Beginners* is likely to fill an important niche in today's selection of titles in this field.

This item will also be of interest to those working in analysis.

**Student Mathematical Library, Volume 103**


bookstore.ams.org/stml-103

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**Your Daily Epsilon of Math Wall Calendar 2024**

Rebecca Rapoport, Harvard University, Cambridge, MA, and Michigan State University, East Lansing, MI, and Dean Chung, Harvard University, Cambridge, MA, and University of Michigan, Ann Arbor, MI

Keep your mind sharp all year long with *Your Daily Epsilon of Math Wall Calendar 2024* featuring a new math problem every day and 13 beautiful math images! Let mathematicians Rebecca Rapoport and Dean Chung tickle the left side of your brain by providing you with a math challenge for every day of the year. The solution is always the date, but the fun lies in figuring out how to arrive at the answer, and possibly discovering more than one method of arriving there.

Problems run the gamut from arithmetic through graduate level math. Some of the most tricky problems require only middle school math applied cleverly. With word problems, math puns, and interesting math definitions added into the mix, this calendar will intrigue you for the whole year.

End the year with more brains than you had when it began with *Your Daily Epsilon of Math Wall Calendar 2024*.

August 2023, ISBN: 978-1-4704-7423-2, 2020 Mathematics Subject Classification: 00A07, 00A09, 00A06, 00A08, 00A05, List US$20, **AMS members US$16**, MAA members US$18, Order code MBK/147

bookstore.ams.org/mbk-147

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**Number Theory**
New in Contemporary Mathematics

Algebra and Algebraic Geometry

Number Theory Through the Eyes of Sophie Germain
An Inquiry Course

David Pengelley, Oregon State University, Corvallis, OR

Number Theory Through the Eyes of Sophie Germain: An Inquiry Course is an innovative textbook for an introductory number theory course.

Sophie Germain (1776–1831) was largely self-taught in mathematics and, two centuries ago, in solitude, devised and implemented a plan to prove Fermat’s Last Theorem. We have only recently completely understood this work from her unpublished letters and manuscripts. David Pengelley has been a driving force in unraveling this mystery and here he masterfully guides his readers along a path of discovery. Germain, because of her circumstances as the first woman to do important original mathematical research, was forced to learn most of what we now include in an undergraduate number theory course for herself. Pengelley has taken excerpts of her writings (and those of others) and, by asking his readers to decipher them, skillfully leads us through an inquiry-based course in elementary number theory. It is a detective story on multiple levels. What is Sophie Germain thinking? What do her mathematical writings mean? How do we understand what she knew and what she was trying to do, where she succeeded and where she didn’t?

Classroom Resource Materials, Volume 70

bookstore.ams.org/clrm-70

Algebra and Coding Theory

A. Leroy, Université d’Artois, Lens, France, and S. K. Jain, Ohio University, Athens, OH, Editors

This volume contains the proceedings of the Virtual Conference on Noncommutative Rings and their Applications VII, in honor of Tariq Rizvi, held from July 5–7, 2021, and the Virtual Conference on Quadratic Forms, Rings and Codes, held on July 8, 2021, both of which were hosted by the Université d’Artois, Lens, France.

The articles cover topics in commutative and noncommutative algebra and applications to coding theory. In some papers, applications of Frobenius rings, the skew group rings, and iterated Ore extensions to coding theory are discussed. Other papers discuss classical topics, such as Utumi rings, Baer rings, nil and nilpotent algebras, and Brauer groups. Still other articles are devoted to various aspects of the elementwise study for rings and modules. Lastly, this volume includes papers dealing with questions in homological algebra and lattice theory. The articles in this volume show the vivacity of the research of noncommutative rings and its influence on other subjects.

This item will also be of interest to those working in applications.

Contemporary Mathematics, Volume 785

bookstore.ams.org/conm-785
New in Memoirs of the AMS

Algebra and Algebraic Geometry

Congruence Lattices of Ideals in Categories and (Partial) Semigroups

James East, Western Sydney University, Penrith, Australia, and Nik Ruškuc, University of St Andrews, Fife, United Kingdom

Memoirs of the American Mathematical Society, Volume 284, Number 1408

bookstore.ams.org/memo-284-1408

Analysis

A Proof that Artificial Neural Networks Overcome the Curse of Dimensionality in the Numerical Approximation of Black–Scholes Partial Differential Equations

Philipp Grohs, University of Vienna, Austria, and Austrian Academy of Sciences, Linz, Austria, Fabian Hornung, Karlsruhe Institute of Technology, Germany, and ETH Zürich, Switzerland, Arnulf Jentzen, The Chinese University of Hong Kong, Shenzhen, China, University of Münster, Germany, and ETH Zürich, Switzerland, and Philippe von Wurstemberger, ETH Zürich, Switzerland, and The Chinese University of Hong Kong, Shenzhen, China

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 284, Number 1409

bookstore.ams.org/memo-284-1409

Discrete Mathematics and Combinatorics

The Existence of Designs via Iterative Absorption: Hypergraph F-Designs for Arbitrary F

Canberra, Australia, and David Penneys, The Ohio State University, Columbus, OH

Memoirs of the American Mathematical Society, Volume 284, Number 1405

bookstore.ams.org/memo-284-1405

The Classification of Subfactors with Index at Most 51–4

Narjess Afzaly, Australian National University, Canberra, Australia, Scott Morrison, Australian National University,

Differential Equations

On Pseudoconformal Blow-Up Solutions to the Self-Dual Chern-Simons-Schrödinger Equation: Existence, Uniqueness, and Instability

Kihyun Kim, Korea Advanced Institute of Science and Technology, Daejeon, Republic of Korea, and Soonsik Kwon, Korea Advanced Institute of Science and Technology, Daejeon, Republic of Korea

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 284, Number 1407

bookstore.ams.org/memo-284-1407
University of Birmingham, United Kingdom, and Deryk Osthus, University of Birmingham, United Kingdom

Memoirs of the American Mathematical Society, Volume 284, Number 1406
April 2023, 131 pages, Softcover, ISBN: 978-1-4704-6024-2, LC 2023015045, 2020 Mathematics Subject Classification: 05B05, 05C51; 05B30, 05B40, 05C70, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/284/1406

bookstore.ams.org/memo-284-1406

New AMS-Distributed Publications

Number Theory

A mod $p$ Jacquet-Langlands Relation and Serre Filtration via the Geometry of Hilbert Modular Varieties
Splicing and Dicing
Fred Diamond, King’s College, London, UK, Payman Kassaei, King’s College, London, UK, and Shu Sasaki, Queen Mary University of London, UK

The authors consider Hilbert modular varieties in characteristic $p$ with Iwahori level at $p$ and construct a geometric Jacquet-Langlands relation showing that the irreducible components are isomorphic to products of projective bundles over quaternionic Shimura varieties of level prime to $p$. The authors use this to establish a relation between mod $p$ Hilbert and quaternionic modular forms that reflects the representation theory of $GL_2$ in characteristic $p$ and generalizes a result of Serre for classical modular forms. Finally the authors study the fibers of the degeneracy map to level prime to $p$ and prove a cohomological vanishing result that is used to associate Galois representations to mod $p$ Hilbert modular forms.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 439

bookstore.ams.org/ast-439

Séminaire Bourbaki
Volume 2021/2022 Exposés 1181–1196

This 73rd volume of the Bourbaki Seminar gathers the texts of the 16 lectures delivered during 2021/2022:
• Ricci flow and diffeomorphisms of 3-manifolds
• Structure of limit spaces of non-collapsed manifolds
• Classification of joinings
• Non-density of integral points and variations of Hodge structures
• Subconvexity problem for $L$-functions
• Non-linear Schrödinger equation
• Kannan-Lovász-Simonovits’ conjecture
• Binary additive problems over finite fields
• Crystalline measures
• K(π,1) conjecture for affine Artin groups
• Sets with no three terms in arithmetic progression.

This item will also be of interest to those working in algebra and algebraic geometry and differential equations.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 438

bookstore.ams.org/ast-438

New AMS-Distributed Publications

Number Theory

A mod $p$ Jacquet-Langlands Relation and Serre Filtration via the Geometry of Hilbert Modular Varieties
Splicing and Dicing
Fred Diamond, King’s College, London, UK, Payman Kassaei, King’s College, London, UK, and Shu Sasaki, Queen Mary University of London, UK

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Astérisque, Number 439

bookstore.ams.org/ast-439
Meetings & Conferences of the AMS
August Table of Contents

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: stovall@math.wisc.edu; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: manes@hawaii.edu; telephone: (808) 956-4679.

Meetings in this Issue

2023

September 9–10 Buffalo, New York p. 1159
October 7–8 Omaha, Nebraska p. 1160
October 13–15 Mobile, Alabama p. 1162

2024

January 3–6 San Francisco, California (JMM 2024) p. 1163
March 23–24 Tallahassee, Florida p. 1169
April 6–7 Washington, DC p. 1170
April 20–21 Milwaukee, Wisconsin p. 1170
May 4–5 San Francisco, California p. 1170
July 23–26 Palermo, Italy p. 1170
September 14–15 San Antonio, Texas p. 1171
October 5–6 Savannah, Georgia p. 1171
October 19–20 Albany, New York p. 1171
October 26–27 Riverside, California p. 1171
December 9–13 Auckland, New Zealand p. 1172

2025

January 8–11 Seattle, Washington (JMM 2025) p. 1172

2026

January 4–7 Washington, DC (JMM 2026) p. 1172

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.
AMS Undergraduate Student

TRAVEL GRANTS

Now providing support for undergraduate student travel to the January Joint Mathematics Meetings

If you are presenting in the
• Pi Mu Epsilon Undergraduate Poster Session
• AMS-SIAM Special Sessions on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs
• Other Special or Contributed Sessions at JMM

Apply for partial travel and lodging support to attend the Joint Mathematics Meetings and
• learn new mathematics
• network with other students and mathematics professionals
• find out about job opportunities
• consider your next steps at the Grad School Fair

Submission deadlines and eligibility info at: www.ams.org/undergrad-tg
Meetings & Conferences
of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Buffalo, New York
University at Buffalo (SUNY)

September 9–10, 2023
Saturday – Sunday

Meeting #1188
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 44, Issue 3

Deadlines
For organizers: Expired
For abstracts: July 18, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Jennifer Balakrishnan, Boston University, Title to be announced.
Sigal Gottlieb, University of Massachusetts, Dartmouth, Title to be announced.
Sam Payne, UT Austin, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Automorphic Forms and $L$ Functions (Code: SS 1A), Xiaqing Li and Joseph A. Hundley, University at Buffalo, SUNY. Building Bridges Between $\mathbb{F}_1$-Geometry, Combinatorics and Representation Theory (Code: SS 2A), Jaiung Jun, SUNY New Paltz, Chris Eppolito, The University of the South, and Alexander Sistko, Manhattan College.
Combinatorial and Categorical Techniques in Representation Theory (Code: SS 3A), Nicholas Davidson, College of Charleston, Robert Muth, Duquesne University, Tianyuan Xu, Haverford College, and Jieru Zhu, Université Catholique de Louvain.

Difference and Differential Equations: Modeling, Analysis, and Applications to Mathematical Biology (Code: SS 4A), Nhu N. Nguyen and Mustafa R. Kulenovic, University of Rhode Island.

Ergodic Theory of Group Actions (Code: SS 5A), Hanfeng Li, SUNY at Buffalo, and Jintao Deng, University of Waterloo.

Financial Mathematics (Code: SS 6A), Maxim Bichuch, University at Buffalo, and Zachary Feinstein, Stevens Institute of Technology.

From Classical to Quantum Low-Dimensional Topology (Code: SS 7A), Adam S. Sikora, University At Buffalo, SUNY, Roman Aranda, Binghamton University, and William Menasco, University at Buffalo.

Gauge Theory and Low-Dimensional Topology (Code: SS 8A), Cagatay Kutluhan, University at Buffalo, Paul M. N Feehan, Rutgers University, New Brunswick, Thomas Gibbs Leness, Florida International University, and Francesco Lin, Columbia University.

Geometry of Groups and Spaces (Code: SS 9A), Johanna Mangahas, University at Buffalo, Joel Louwsma, Niagara University, and Jenya Sapir, Binghamton University.

Geometry, Physics and Representation Theory (Code: SS 10A), Jie Ren, SUNY Buffalo.

Homological Aspects of p-adic Groups and Automorphic Representations (Code: SS 11A), Karol Koziol, Baruch College, CUNY, and Anantharam Raghuram, Fordham University.

Inverse Problems in Science and Engineering (Code: SS 12A), Sedar Ngoma, SUNY Geneseo.

Nonlinear Partial Differential Equations in Fluids and Waves (Code: SS 13A), Qingtian Zhang, West Virginia University, and Ming Chen, University of Pittsburgh.


Probability, Combinatorics, and Statistical Mechanics (Code: SS 15A), Douglas Rizzolo, University of Delaware, and Noah Forman, McMaster University.

Recent Advances in Numerical Methods for Fluid Dynamics and Their Applications (Code: SS 16A), Daozhi Han, The State University of New York at Buffalo, Guosheng Fu, University of Notre Dame, and Jia Zhao, Utah State University.

Recent Advances in Water Waves: Theory and Numerics (Code: SS 17A), Sergey Dyachenko, University At Buffalo, and Alexander Chernyavsky, SUNY at Buffalo.

Recent Developments in Operator Algebras and Quantum Information Theory (Code: SS 18A), Priyanga Ganesan, University of California San Diego, Samuel Harris, Northern Arizona University, and Ivan G. Todorov, University of Delaware.

Recent Trends in Spectral Graph Theory (Code: SS 19A), Michael Tait, Villanova University, and Shahla Nasserzadeh and Brendan Rooney, Rochester Institute of Technology.

Representation Theory and Flag Varieties (Code: SS 20A), Yiqiang Li, University at Buffalo, and Changlong Zhong, SUNY Albany.

Topics in Combinatorics and Graph Theory (Code: SS 21A), Rong Luo and Kevin G. Milans, West Virginia University, and Guangming Jing, University of West Virginia.

Omaha, Nebraska
Creighton University

October 7–8, 2023
Saturday – Sunday

Meeting #1189
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 44, Issue 4

Deadlines
For organizers: To be announced
For abstracts: August 8, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.
**Invited Addresses**

Lydia Bieri, University of Michigan, *Title To Be Announced.*

Aaron J. Pollack, University of California, San Diego, *Title To Be Announced.*

Christopher Schafhauser, University of Nebraska–Lincoln, *Title To Be Announced.*

**Special Sessions**

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.*

*Advances in Graph Theory and Combinatorics I* (Code: SS 7A), Bernard Lidicky and Steve Butler, Iowa State University.

*Advances in Operator Algebras I* (Code: SS 10A), Christopher Schafhauser, University of Nebraska–Lincoln, and Ionut Chifan, University of Iowa.

*Analytic number theory and related fields I* (Code: SS 14A), Vorrapan Chandee and Xiannan Li, Kansas State University, and Micah B. Milinovich, University of Mississippi.

*Applied knot theory I* (Code: SS 15A), Isabel K. Darcy, University of Iowa, and Eric Rawdon, University of St. Thomas.

*Automorphic forms, their arithmetic, and their applications I* (Code: SS 8A), Aaron J. Pollack, UC San Diego, and Spencer Leslie, Boston College.

*Commutative algebra, differential operators, and singularities I* (Code: SS 11A), Uli Walther, Purdue University, Claudia Miller, Syracuse University, and Vaibhav Pandey, Purdue University.

*Commutative Algebra I* (Code: SS 9A), Thomas Marley, University of Nebraska–Lincoln, and Eloísa Grifo, University of Nebraska–Lincoln.

*Discrete, Algebraic, and Topological Methods in Mathematical Biology I* (Code: SS 6A), Alexander B. Kunin, Creighton University.

*Enumerative Combinatorics I* (Code: SS 4A), Seok Hyun Byun, Clemson University, Tri Lai, University of Nebraska, and Svetlana Poznanovic, Clemson University.

*Foliations, flows, and groups I* (Code: SS 16A), Ying Hu, University of Nebraska Omaha, and Michael Landry, Washington University in Saint Louis.

*Fractal Geometry and Dynamical Systems!* (Code: SS 5A), Mrinal Kanti Roychowdhury, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Manuj Verma, Indian Institute of Technology Delhi, New Delhi, 110016, India, and Megha Pandey, University of Texas Rio Grande Valley.

*Harmonic Analysis in the Midwest I* (Code: SS 20A), Betsy Stovall, University of Wisconsin–Madison, and Terence L. J. Harris, Cornell University.

*Homotopy theory I* (Code: SS 21A), Prasit Bhattacharya, New Mexico State University, Agnès Beaudry, University of Colorado Boulder, and Zhourli Xu, University of California, San Diego.

*Interactions of Floer Homologies, Contact Structures, and Symplectic Structures I* (Code: SS 19A), Robert DeYeso III and Joseph Breen, University of Iowa.

*Mathematical modeling and analysis in ecology and epidemiology I* (Code: SS 17A), Yu Jin, University of Nebraska–Lincoln, Shuwen Xue, Northern Illinois University, and Chayu Yang, University of Nebraska–Lioncoln.

*Nonlinear PDE and Free Boundary Problems I* (Code: SS 3A), William Myers Feldman, University of Utah, and Fernando Charro, Wayne State University.

*Progress in Nonlinear Waves I* (Code: SS 2A), David M. Ambrose, Drexel University.

*Recent Development in Advanced Numerical Methods for Partial Differential Equations I* (Code: SS 13A), Mahboub Baccouch, University of Nebraska At Omaha.

*Recent Developments in Theories and Computation of Nonlocal Models I* (Code: SS 18A), Anh Vo and Scott Hootman-Ng, University of Nebraska–Lincoln, and Animesh Biswas, University of Nebraska–Lincoln.

*Topology of 3- and 4-Manifolds!* (Code: SS 12A), Alexander Zupan, University of Nebraska–Lincoln, Roman Aranda, Binghamton University, and David Auckly, Kansas State University.

*Varieties with unexpected hypersurfaces, geproci sets and their interactions I* (Code: SS 1A), Brian Harbourne, University of Nebraska, and Juan C. Migliore, University of Notre Dame.
Mobile, Alabama
University of South Alabama

October 13–15, 2023
Friday – Sunday

Meeting #1190
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Deadlines
For organizers: Expired
For abstracts: August 15, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Laura Ann Miller, University of Arizona, *Flows around some soft corals.*
Cornelius Pillen, University of South Alabama, *Lifting to tilting: modular representations of algebraic groups and their Frobenius kernels.*

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

*Advances in Extremal Combinatorics* (Code: SS 17A), Joseph Guy Briggs, Auburn University, and Chris Cox, Iowa State University.

*Analysis, Computation, and Applications of Stochastic Models* (Code: SS 11A), Yukun Li, University of Central Florida, Feng Bao, Florida State University, Xiaobing Henry Feng, The University of Tennessee, and Liet Anh Vo, University of Illinois, Chicago.

*Categorical Representations, Quantum Algebra, and Related Topics* (Code: SS 7A), Arik Wilbert, University of Georgia, Mee Seong Im, United States Naval Academy, Annapolis, and Bach Nguyen, Xavier University of Louisiana.

*Combinatorics and Geometry Related to Representation Theory* (Code: SS 8A), Markus Hunziker and William Erickson, Baylor University.

*Cyberinfrastructure for Mathematics Research & Instruction* (Code: SS 9A), Steven Craig Clontz, University of South Alabama, and Tien Chih, Oxford College of Emory University.

*Discrete Geometry and Geometric Optimization* (Code: SS 13A), Andras Bezdek, Auburn University, Auburn AL, Ferenc Fodor, University of Szeged, and Woden Kusner, University of Georgia, Athens GA.

*Dynamics and Equilibria of Energies* (Code: SS 4A), Ryan Matzke, Technische Universität Graz, and Liudmyla Kryvonos, Vanderbilt University.

*Dynamics of Fluids* (Code: SS 20A), I. Kukavica, University of Southern California, Dallas Albritton, Princeton University, and Wojciech S. Ozanski, Florida State University.

*Ergodic Theory and Dynamical Systems* (Code: SS 5A), Mrinal Kanti Roychowdhury, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and Joanna Furno, University of South Alabama.


*Extremal and Probabilistic Combinatorics* (Code: SS 14A), Sean English, University of North Carolina Wilmington, and Emily Heath, Iowa State University.

*Mathematical Modeling of Problems in Biological Fluid Dynamics* (Code: SS 1A), Laura Ann Miller, University of Arizona, and Nick Battista, The College of New Jersey.

*New Directions in Noncommutative Algebras and Representation Theory* (Code: SS 18A), Jonas T. Hartwig, Iowa State University, and Erich C. Jauch, University of Wisconsin - Eau Claire.

Recent Advances in Low-dimensional and Quantum Topology (Code: SS 16A), Christine Lee, Texas State, and Scott Carter, University of South Alabama.

Recent Developments in Graph Theory (Code: SS 15A), Andrei Bogdan Pavelescu, University of South Alabama, and Kenneth Roblee, Troy University.

Recent progress in numerical methods for PDEs (Code: SS 12A), Muhammad Mohhebujaman, Texas A&M International University, Leo Rebholz, Clemson University, and Mengying Xiao, University of West Florida.

Representation Theory of Finite and Algebraic Groups (Code: SS 3A), Daniel K. Nakano, University of Georgia, Pramod N. Achar, Louisiana State University, and Jonathan R. Kujawa, University of Oklahoma.

Rings, Monoids, and Factorization (Code: SS 10A), Jim Coykendall, Clemson University, and Scott Chapman, Sam Houston State University.

Theory and Application of Parabolic PDEs (Code: SS 19A), Wenxian Shen and Yuming Paul Zhang, Auburn University.

Topics in harmonic analysis and partial differential equations (Code: SS 6A), Jiuyi Zhu and Phuc Cong Nguyen, Louisiana State University.

San Francisco, California
Moscone North/South, Moscone Center

January 3–6, 2024
Wednesday – Saturday

Meeting #1192
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: June 5, 2023

Issue of Abstracts: Volume 45, Issue 1

Deadlines
For organizers: To be announced
For abstracts: September 12, 2023

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/national.html.

Joint Invited Addresses
Maria Chudnovsky, Princeton University, To be announced (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).
Anne Schilling, University of California, Davis, To be announced (AWM-AMS Noether Lecture).
Peter Winkler, Dartmouth College, To be announced (AAAS-AMS Invited Address).
Kamuela Yong, University of Hawai‘i - West O‘ahu, To be announced.

AMS Invited Addresses
Ruth Charney, Brandeis University, To be announced (AMS Retiring Presidential Address).
Daniel Erman, University of Wisconsin–Madison, To be announced.
Suzanne Marie Lenhart, University of Tennessee/ Knoxville, To be announced (AMS Josiah Willard Gibbs Lecture).
Ankur Moitra, Massachusetts Institute of Technology, To be announced (AMS John von Neumann Lecture).
Kimberly Sellers, Georgetown University, To be announced.

Terence Tao, UCLA, To be announced (AMS Colloquium Lecture I - Terence Tao, University of California, Los Angeles).

Terence Tao, UCLA, To be announced (AMS Colloquium Lecture II - Terence Tao, University of California, Los Angeles).

Terence Tao, UCLA, To be announced (AMS Colloquium Lecture III - Terence Tao, University of California, Los Angeles).

John Urschel, MIT, To be announced (AMS Erdös Lecture for Students).

Suzanne L Weekes, SIAM, To be announced (AMS Lecture on Education).

Melanie Matchett Wood, Harvard University, To be announced (AMS Maryam Mirzakhani Lecture).

Invited Addresses of Other JMM Partners
Katherine Ensor, Rice University, Celebrating Statistical Foundations Driving 21st Century Innovation (ASA Invited Lecture - Kathy Ensor, Rice University).

Stephan Ramon Garcia, Pomona College, To be announced (ILAS Invited Address).

Trachette Jackson, University of Michigan, Mobilizing Mathematics for the Fight Against Cancer (PME Invited Address).

Joni Teräväinen, University of Turku, To be announced (AIM Alexanderson Award Lecture).
AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

Advances in Analysis, PDE’s and Related Applications (Code: SS 50A), Tepper L. Gill, Howard University, E. Kwessi, Trinity University, and Henok Mawi, Howard University (Washington, DC, US).

Advances in Coding Theory (Code: SS 13A), Emily McMillon, Rice University, Christine Ann Kelley, University of Nebraska–Lincoln, Tefjol Plaha, University of Nebraska–Lincoln, and Mary Wootters, Stanford University.

Algebraic Approaches to Mathematical Biology (Code: SS 71A), Nicolette Meshkat, Santa Clara University, Cash Bortner, California State University, Stanislaus, and Anne Shiu, Texas A&M University.

Algebraic Structures in Knot Theory (Code: SS 69A), Sam Nelson, Claremont McKenna College, and Neslihan Gügümcu, Izmir Institute of Technology in Turkey.

AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology (Code: SS 46A), Sarah Blackwell, Max Planck Institute for Mathematics, Luya Wang, University of California, Berkeley, and Nicole Magill, Cornell University.

Analysis and Differential Equations at Undergraduate Institutions (Code: SS 24A), Evan Randles, Colby College, and Lisa Naples, Macalester College, Saint Paul MN USA.

Applications of Extremal Graph Theory to Network Design (Code: SS 91A), Kelly Isham, Colgate University, and Laura Monroe, Los Alamos National Laboratory.

Applications of Hypercomplex Analysis (Code: SS 3A), Mihaela B Vajiac, Chapman University, Orange, CA, Daniel Alpay, Chapman University, and Paula Cerejeiras, University of Aveiro, Portugal.


Arithmetic Geometry with a View toward Computation (Code: SS 81A), David Lowry-Duda, ICERM & Brown University, Barinder Banwait, Boston University, Shiva Chidambaram, Massachusetts Institute of Technology, Juanita Duque-Rosero, Boston University, Brendan Hassett, ICERM/Brown University, and Ciaran Schembri, Dartmouth College.


Coding Theory for Modern Applications (Code: SS 74A), Rafael D'Oliveira, Clemson University, Hiram H. Lopez, Cleveland State University, and Allison Beemer, University of Wisconsin-Eau Claire.

Combinatorial Insights into Algebraic Geometry (Code: SS 73A), Javier Gonzalez Anaya, UC Riverside.


Combinatorics for Science (Code: SS 52A), Stephen J Young, Bill Kay, and Sinan Aksoy, Pacific Northwest National Laboratory.

Commutative Algebra and Algebraic Geometry (associated with Invited Address by Daniel Erman) (Code: SS 100A), Daniel Erman, University of Wisconsin–Madison, and Aleksandra C. Sobieska, University of Wisconsin–Madison.

Complex Analysis, Operator Theory, and Real Algebraic Geometry (Code: SS 42A), J. E. Pascoe, Drexel University, Kelly Bickel, Bucknell University, and Ryan K. Tully-Doyle, Cal Poly SLO.

Computable Mathematics: A Special Session Dedicated to Martin D. Davis (Code: SS 1A), Valentina S Harizanov, George Washington University, Alexandra Shlapentokh, East Carolina University, and Wesley Calvert, Southern Illinois University.

Computational Biomedicine: Methods - Models - Applications (Code: SS 7A), Nektarios A Valous, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Im Neuenheimer Feld 267, 69120 Heidelberg, Germany, Anna Konstorum, Center for Computing Sciences, Institute for Defense Analyses, 17100 Science Drive, Bowie, MD, 20715, USA, Heiko Enderling, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, FL, 33647, USA, and Dirk Jäger, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Im Neuenheimer Feld 267, 69120 Heidelberg, Germany.

Computational techniques to Study the Geometry of the Shape Space (Code: SS 99A), Shira Faigenbaum-Golovin, Phillip Griffiths Assistant Research Professor, Department of Mathematics, Duke University, Shan Shan, University of Southern Denmark, and Ingrid Daubechies, Duke University.
Covering Systems of the Integers and Their Applications (Code: SS 41A), Joshua Harrington, Cedar Crest College, Tony Wing Hong Wong, Kutztown University of Pennsylvania, and Matthew Litman, University of California, Davis.

Cryptography and Related Fields (Code: SS 86A), Ryann Cartor, Clemson University, Angela Robinson, NIST, and Daniel Everett Martin, Clemson University.

Current Advances in Modeling Trends of Infectious Diseases across Multiple Scales (Code: SS 98A), Naveen K. Vaidya, San Diego State University, and Elissa Schwartz, Washington State University.

Developing Students’ Technical Communication Skills through Mathematics Courses (Code: SS 36A), Michelle L. Ghrist, Gonzaga University, Timothy P Chartier, Davidson College, Maila B Hallue, US Air Force Academy, USAFA CO USA, and Denise Tauntun Reid, Valdosta State University.

Diffusive Systems in the Natural Sciences (Code: SS 88A), Francesca Bernardi, Worcester Polytechnic Institute, and Owen Lewis, University of New Mexico.

Discrete Homotopy Theory (Code: SS 25A), Krzysztof R. Kapulkin, University of Western Ontario, Anton Dochtermann, Texas State University, and Antonio Rieser, CONACYT-CIMAT.

Dynamical Systems Modeling for Biological and Social Systems (Code: SS 75A), Daniel Brendan Cooney, University of Pennsylvania, Chadi M Saad-Roy, University of California, Berkeley, and Chris M. Heggerud, University of California, Davis.

Dynamics and Management in Disease or Ecological Models (associated with Gibbs Lecture by Suzanne Lenhart) (Code: SS 102A), Suzanne Lenhart, University of Tennessee, Knoxville, Christina Edholm, Scripps College, and Wandi Ding, Middle Tennessee State University.

Dynamics and Regularity of PDEs (Code: SS 32A), Zongyuan Li, Rutgers University, Weinan Wang, University of Arizona, Xueying Yu, University of Washington, and Zhiyuan Zhang, New York University.

Epistemologies of the South and the Mathematics of Indigenous Peoples (Code: SS 77A), María Del Carmen Bonilla Tumialán, National University of Education Enrique Guzman y Valle, Wilfredo Vidal Alangui, University of the Philippines Baguio, and Domingo Yojcom Roché, Center for Scientific and Cultural Research.

Ergodic Theory, Symbolic Dynamics, and Related Topics (Code: SS 10A), Andrew T. Dykstra, Hamilton College, and Shrey Sanadhya, Ben Gurion University of the Negev, Israel.

Ethics in the Mathematics Classroom (Code: SS 6A), Victor Piercey, Ferris State University, and Catherine Buell, Fitchburg State University.


Extremal and Probabilistic Combinatorics (Code: SS 15A), Sam Spiro, Rutgers University, and Corrine Yap, Georgia Institute of Technology.

Geometric Analysis in Several Complex Variables (Code: SS 9A), Ming Xiao, University of California, San Diego, Bernhard Lamel, Texas A&M University At Qatar, and Nordine Mir, Texas A&M University at Qatar.

Geometric Group Theory (Associated with the AMS Retiring Presidential Address) (Code: SS 38A), Kasia Jankiewicz, University of California Santa Cruz, Edgar A. Bering, San José State University, Marion Campisi, San Jose State University, and Tim Hsu and Giang Le, San José State University.

Geometry and Symmetry in Differential Equations, Control, and Applications (Code: SS 93A), Taylor Joseph Klotz and George Wilkens, University of Hawai‘i.

Geometry and Topology of High-Dimensional Biomedical Data (Code: SS 67A), Smita Krishnaswamy, Yale, Dhananjay Bhaskar, Yale University, Bastian Rieck, Technical University of Munich, and Guy Wolf, Université de Montréal.

Group Actions in Commutative Algebra (Code: SS 51A), Alessandra Costantini, Oklahoma State University, Alexandra Seceleanu, University of Nebraska-Lincoln, and Andras Lorincz, University of Oklahoma.

Hamiltonian Systems and Celestial Mechanics (Code: SS 23A), Zhifu Xie, The University of Southern Mississippi, and Ernesto Pérez-Chavela, ITAM.

Harmonic Analysis, Geometry Measure Theory, and Fractals (Code: SS 54A), Kyle Hambrock, San Jose State University, Chun-Kit Lai, San Francisco State University, and Caleb Marshall, University of British Columbia.

History of Mathematics (Code: SS 89A), Adrian Rice, Randolph-Macon College, Sloan Evans Despeaux, Western Carolina University, Deborah Kent, University of St. Andrews, and Jemma Lorenat, Pitzer College.

Homological Techniques in Noncommutative Algebra (Code: SS 48A), Robert Won, George Washington University, Ellen E Kirkman, Wake Forest University, and James J Zhang, University of Washington.

Homotopy Theory (Code: SS 47A), Krzysztof R. Kapulkin, University of Western Ontario, Daniel K. Dugger, University of Oregon, Jonathan Beardsley, University of Nevada, Reno, and Thomas Brazelton, University of Pennsylvania.

Ideal and Factorization Theory in Rings and Semigroups (Code: SS 4A), Scott Chapman, Sam Houston State University, and Alfred Geroldinger, University of Graz.
Informal Learning, Identity, and Attitudes in Mathematics (Code: SS 44A), Sergey Grigorian, Mayra Ortiz, Xiaohui Wang, and Aaron Wilson, University of Texas Rio Grande Valley.

Integer Partitions, Arc Spaces and Vertex operators (Code: SS 59A), Hussein Mourtada, Université Paris Cité, and Andrew R. Linshaw, University of Denver.


Issues, Challenges and Innovations in Instruction of Linear Algebra (Code: SS 97A), Feroz Siddique, University of Wisconsin-Eau Claire, and Ashish K. Srivastava, Saint Louis University.

Knots, Skein Modules, and Categorification (Code: SS 66A), Rhea Palak Bakshi, ETH Institute for Theoretical Studies, Zurich, Sujoy Mukherjee, University of Denver, and Jozef Henryk Przytycki, George Washington University.

Large Random Permutations (affiliated with AAAS-AMS Invited Address by Peter Winkler) (Code: SS 103A), Peter Winkler, Dartmouth College, and Jacopo Borga, Stanford University.

Loeb Measure after 50 Years (Code: SS 12A), Yeneng Sun, National University of Singapore, Robert M Anderson, UC Berkeley, and Matt Insall, University of Science and Technology.

Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later (Code: SS 85A), Yoonhee Lee, Southern Connecticut State University, James Alvarez, University of Texas Arlington, Ekaterina Fuchs, City College of San Francisco, Tyler Kloefkorn, American Mathematical Society, Yvonne Lai, University of Nebraska–Lincoln, and Carl Olimb, Augustana University.

Mathematical Modeling and Simulation of Biomolecular Systems (Code: SS 79A), Zhen Chao, Western Washington University, and Jiahui Chen, University of Arkansas.

Mathematical Modeling of Nucleic Acid Structures (Code: SS 70A), Pengyu Liu, University of California, Davis, Van Pham, University of South Florida, and Svetlana Poznanovic, Clemson University.

Mathematical Physics and Future Directions (Code: SS 34A), Shanna Dobson, University of California, Riverside, Tepper L. Gill, Howard University, Michael Anthony Maroun, University of California, Riverside, CA, and Lance W Nielsen, Creighton University.

Mathematics and Philosophy (Code: SS 53A), Tom Morley, Georgia Tech, and Bonnie Gold, Monmouth University.

Mathematics and Quantum (Code: SS 82A), Kaifeng Bu and Arthur M Jaffe, Harvard, Sui Tang, UCSB, and Jonathan Weitsman, Northeastern University.

Mathematics and the Arts (Code: SS 76A), Karl Kattchee, University of Wisconsin-La Crosse, Doug Norton, Villanova University, and Anil Venkatesh, Adelphi University.

Mathematics of Computer Vision (Code: SS 94A), Timothy Duff and Max Lieblich, University of Washington.

Mathematics of DNA and RNA (Code: SS 87A), Marek Kimmel, Rice University, Chris McCarthy, BMCC, City University of New York, and Johannes Hamilton, Borough of Manhattan Community College, CUNY.

Metric Dimension of Graphs and Related Topics (Code: SS 18A), Briana Foster-Greenwood, Cal Poly Pomona, and Christine Uhl, St. Bonaventure University.

Mock Geometry and Topology (Code: SS 33A), Christine M. Escher, Oregon State University, and Catherine Searle, Wichita State University.

Mock Modular forms, Physics, and Applications (Code: SS 40A), Amanda Folsom, Amherst College, Terry Gannon, University of Alberta, and Larry Rolen, Vanderbilt University.


Modeling to Motivate the Teaching of the Mathematics of Differential Equations (Code: SS 21A), Brian Winkler, SIMIODE, Chardon NY USA, Kyle T Allaire, Worcester State University, Worcester MA USA, Maila B Hallare, US Air Force Academy, LISAFA CO USA, Yanping Ma, Loyola Marymount University, Los Angeles CA USA, and Lisa Naples, Macalester College, Saint Paul MN USA.

Modelling with Copulas: Discrete vs Continuous Dependent Data (Code: SS 19A), Martial Longla, University of Mississippi, and Isidore Séraphin Ngongo, University of Yaounde I.

Modern Developments in the Theory of Configuration Spaces (Code: SS 31A), Christin Bibby, Louisiana State University, and Nir Gadish, University of Michigan.

Modular Tensor Categories and TQFTs beyond the Finite and Semisimple (Code: SS 27A), Colleen Delaney, UC Berkeley, and Nathan Geer, Utah State University.

Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects (Code: SS 64A), Vinodh Kumar Chellamuthu, Utah Tech University, and Xiaoxia Xie, Idaho State University.
New Faces in Operator Theory and Function Theory (Code: SS 37A), Michael R Pilla, Ball State University, and William Thomas Ross, University of Richmond.

Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives (Code: SS 95A), Armando Roldan, University of Central Florida, and Thomas Dombrowski, Moffitt Cancer Center.

Number Theory in Memory of Kevin James (Code: SS 39A), Jim L. Brown, Occidental College, and Felice Manganiello, Clemson University.

Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms (Code: SS 92A), Anastasiia Minenkova, University of Hartford, and Gamal Mograby, University of Cincinnati.


Partition Theory and q-Series (Code: SS 30A), William Jonathan Keith, Michigan Technological University, Brandt Kronholm, University of Texas Rio Grande Valley, and Dennis Eichhorn, University of California, Irvine.

Principles, Spatial Reasoning, and Science in First-Year Calculus (Code: SS 16A), Yat Sun Poon and Catherine Lussier, University of California, Riverside, and Bryan Carrillo, Saddleback College.

Quantitative Justice (Code: SS 96A), Ron Buckmire, Occidental College, and Carrie Diaz Eaton, Bates College.

Quaternions (Code: SS 49A), Chris McCarthy, BMCC, City University of New York, Johannes Hamilton, Borough of Manhattan Community College, CUNY, and Terrence Richard Blackman, Medgar Evers Community College, CUNY.

Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems (Code: SS 72A), Tuan A. Phan, IMCI, University of Idaho, Nhu N. Nguyen, University of Rhode Island, and Jianjun P. Tian, New Mexico State University.

Recent Advances in Mathematical Models of Diseases: Analysis and Computation (Code: SS 22A), Najat Ziyadi and Jemal S Mohammed-Awel, Department of Mathematics, Morgan State University.

Recent Developments in Commutative Algebra (Code: SS 45A), Austyn Simpson and Alapan Mukhopadhyay, University of Michigan, and Thomas Marion Polstra, University of Virginia.

Recent Developments in Numerical Methods for PDEs and Applications (Code: SS 2A), Chunmei Wang, University of Florida, Long Chen, UC Irvine, Shuhao Cao, University of Missouri-Kansas City, and Haizhao Yang, University of Maryland College Park.

Recent Developments on Markoff Triples (Code: SS 68A), Elena Fuchs, UC Davis, and Daniel Everett Martin, Clemson University.


Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs (Code: SS 29A), Darren A. Narayan, Rochester Institute of Technology, John C. Wierman, Johns Hopkins University, Mark Daniel Ward, Purdue University, Khang Duc Tran, California State University, Fresno, and Christopher O’Neill, San Diego State University.

Research Presentations by Math Alliance Scholar Doctorates (Code: SS 84A), Teresa Martines, University of Texas, Austin, and David Goldberg, Math Alliance/Purdue University.

Serious Recreational Mathematics (Code: SS 8A), Erik Demaine, Massachusetts Institute of Technology, Robert A. Hearn, H3 Labs, and Tomas Rokicki, California.

Solvable Lattice Models and their Applications Associated with the Noether Lecture (Code: SS 43A), Anne Schilling, University of California, Davis, Amol Aggarwal, Columbia, Benjamin Brubaker, University of Minnesota - Twin Cities, Daniel Bump, Stanford, Andrew Hardt, Stanford University, Slava Naprienko, Stanford and University of North Carolina, Leonid Petrov, University of Virginia, and Anne Schilling, University of California, Davis.


The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors (Code: SS 11A), Quiyana Murphy, Virginia Tech, Sofia Rose Martinez Alberga, Purdue University, Kelly Buch, Austin Peay State University, and Alexis Hardesty, Texas Tech University.


Theoretical and Numerical Aspects of Nonlocal Models (Code: SS 63A), Nicole Buczkowski, Worcester Polytechnic Institute, Christian Alexander Glusa, Sandia National Laboratories, and Animesh Biswas, University of Nebraska–Lincoln.
MEETINGS & CONFERENCES

Theta Correspondence (Code: SS 55A), Timothy D Comar and Timothy D Comar, Benedictine University, and Anne E. Yust and Anne E. Yust, University of Pittsburgh.

The Teaching and Learning of Undergraduate Ordinary Differential Equations (Code: SS 14A), Viktoria Savatorova, Central Connecticut State University, Chris Goodrich, The University of New South Wales, Ibai Seggev, Wolfram Research, Beverly H. West, Cornell University, and Maila B Hallare, US Air Force Academy, USAFA CO USA.

Thresholds in Random Structures (Code: SS 78A), Will Perkins, Georgia Tech.

Topics in Combinatorics and Graph Theory (Code: SS 90A), Cory Palmer, University of Montana, Neal Bushaw, Virginia Commonwealth University, and Anastasia Halfpap, University of Montana.

Topics in Equivariant Algebra (Code: SS 65A), Ben Spitz, University of California Los Angeles, and Christy Hazel and Michael A. Hill, UCLA.

Undergraduate Research Activities in Mathematical and Computational Biology (Code: SS 55A), Timothy D Comar and Timothy D Comar, Benedictine University, and Anne E. Yust and Anne E. Yust, University of Pittsburgh.


Water Waves (Code: SS 28A), Anastassiya Semenova and Bernard Deconinck, University of Washington, John Carter, Seattle University, and Eleanor Devin Byrnes, University of Wisconsin.

AIM Special Sessions

AIM Special Session Associated with the Alexanderson Award and Lecture (Code: AIMSS2), Joni Teräväinen, University of Turku, Terence Tao, UCLA, Kasia Matomäki, University of Turku, Maksym Radziwill, California Institute of Technology, and Tamar Ziegler, Hebrew University.

Equivariant techniques in stable homotopy theory (Code: AIMSS4), Michael A. Hill, UCLA, and Anna Marie Bohmann, Vanderbilt University.

Graphs and Matrices (Code: AIMSS5), Mary Flagg, University of St. Thomas, and Bryan A Curtis, Iowa State University.

Little School Dynamics: Cool Research by Researchers at PUIs (Code: AIMSS1), Kimberly Ayers, California State University San Marcos, Ami Radunskaya, Pomona College, Andy Parrish, Eastern Illinois University, David M. McClendon, Ferris State University, and Han Li, Wesleyan University.

Math Circle Activities as a Gateway into Research (Code: AIMSS3), Jeffrey Musyt, Slippery Rock University, Lauren L. Rose, Bard College, Tom G. Stojavljevic, Beloit College, Nick Rauh, Julia Robinson Math Festivals, Edward Charles Keppelmann, University of Nevada Reno, Allison Henrich, Seattle University, Violeta Vasilevska, Utah Valley University, and Gabriella A. Pinter, University of Wisconsin, Milwaukee.

ASL Special Sessions

Descriptive methods in dynamics, combinatorics, and large scale geometry (Code: aslss1A), Jenna Zomback, Williams College, and Forte Shinko, UCLA.

AWM Special Sessions

AWM Workshop: Women in Operator Theory (Code: AWMWS1), Catherine A Beneteau, University of South Florida, and Asuman Aksoy, Claremont McKenna College.

EvenQuads Live and in person: The honorees and the games (Code: AWMSS2), sarah-marie belcastro, Mathematical Staircase, Inc., Sherli Koshy-Chenthittayil, Touro University Nevada, Oscar Vega, California State University, Fresno, Monica D Morales-Hernandez, Adelphi University, Linda McGuire, Muhlenberg College, and Denise A. Rangel Tracy, Fairleigh Dickinson University.

Mathematics in the Literary Arts and Pedagogy in Creative Settings (Code: AWMSS4), Shanna Dobson, University of California, Riverside, and Claudia Maria Schmidt, California State University.

Recent developments in harmonic analysis (Code: AWMSS1), Betsy Stovall, University of Wisconsin–Madison, and Sarah E Tammen, Massachusetts Institute of Technology.

Women in Mathematical Biology (Code: AWMSS3), Christina Edholm, Scripps College, Lihong Zhao, University of California, Merced, and Lale Asik, University of the Incarnate Word.

COMAP Special Sessions

Math Modeling Contests: What They Are, How They Benefit, What They Did – Discussions with the Students and Advisors (Code: COMAPSS1), Jack A Picciuto, COMAP, and Kayla Blyman, Saint Martin’s University.
ILAS Special Sessions

Generalized Numerical Ranges and Related Topics (Code: ILASSS2), Tin-Yau Tam and Pan-Shun Lau, University of Nevada, Reno.

Graphs and Matrices (Code: ILASSS1), Jane Breen, Ontario Tech University, and Stephen Kirkland, University of Manitoba.

Linear algebra, matrix theory, and its applications (Code: ILASSS3), Stephan Ramon Garcia, Pomona College.

Sign-pattern Matrices and Their Applications (Code: ILASSS4), Bryan L Shader, University of Wyoming, and Minerva Catral, Xavier University.


MSRI/Simons Laufer Mathematical Sciences Institute Special Sessions

African Diaspora Joint Mathematics Working Groups (ADJOINT) (Code: SLMSS1A), Caleb Ashley, Boston College, and Anisah Nabilah Nu’Man, Spelman College.


The MSRI Undergraduate Program (MSRI-UP) (Code: SLMSS2A), Maria Mercedes Franco, Queensborough Community College-CUNY.

NSF Special Sessions

NSF Special Session Exploring Funding Opportunities in the Division of Mathematical Sciences (Code: NSFSS1), Elizabeth Wilmer, NSF, and Junping Wang, National Science Foundation.

PMA Special Sessions

BSM Special Session: Mathematical Research in Budapest for Students and Faculty (Code: pmass1A), Kristina Cole Garrett, St. Olaf College.

SIAM Minisymposium

Current Advances in Modeling Trends of Infectious Diseases across Multiple Scales (Code: SIAMSS1), Naveen K. Vaidya, San Diego State University, and Elissa Schwartz, Washington State University.

SPECTRA Special Sessions

Research by LGBTQ+ Mathematicians (Code: spectss1A), Devavrat Dabke, Princeton University, Joseph Nakao, Swarthmore College, and Michael A. Hill, UCLA.

Tallahassee, Florida
Florida State University in Tallahassee

March 23–24, 2024
Saturday – Sunday

Meeting #1193
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 2

Deadlines
For organizers: August 22, 2023
For abstracts: January 23, 2024
Washington, District of Columbia
Howard University

**April 6–7, 2024**
Saturday – Sunday

**Meeting #1194**
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 2

**Deadlines**
For organizers: September 5, 2023
For abstracts: February 13, 2024

Milwaukee, Wisconsin
University of Wisconsin- Milwaukee

**April 20–21, 2024**
Saturday – Sunday

**Meeting #1195**
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 2

**Deadlines**
For organizers: September 19, 2023
For abstracts: February 20, 2024

San Francisco, California
San Francisco State University

**May 4–5, 2024**
Saturday – Sunday

**Meeting #1196**
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of Abstracts: Volume 45, Issue 3

**Deadlines**
For organizers: October 4, 2023
For abstracts: March 12, 2024

_The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html._

**Special Sessions**
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

_Recent Advances in Differential Geometry_, Zhiqin Lu, University of California, Shoo Seto and Bogdan Suceavă, California State University, Fullerton, and Lihan Wang, California State University, Long Beach.

Palermo, Italy

**July 23–26, 2024**
Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced
San Antonio, Texas
University of Texas, San Antonio

September 14–15, 2024
Saturday – Sunday

Meeting #1198
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 3

Deadlines
For organizers: February 13, 2024
For abstracts: July 23, 2024

Savannah, Georgia
Georgia Southern University, Savannah

October 5–6, 2024
Saturday – Sunday

Meeting #1199
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

Deadlines
For organizers: March 5, 2024
For abstracts: August 13, 2024

Albany, New York
State University of New York at Albany

October 19–20, 2024
Saturday – Sunday

Meeting #1200
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

Deadlines
For organizers: March 19, 2024
For abstracts: August 27, 2024

Riverside, California
University of California, Riverside

October 26–27, 2024
Saturday – Sunday

Meeting #1201
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of Abstracts: Volume 45, Issue 4

Deadlines
For organizers: March 26, 2024
For abstracts: September 3, 2024
Auckland, New Zealand

**December 9–13, 2024**

Monday – Friday  
Associate Secretary for the AMS: Steven H. Weintraub  
Program first available on AMS website: To be announced  

Issue of Abstracts: To be announced  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced

Seattle, Washington

*Washington State Convention Center and the Sheraton Seattle Hotel*

**January 8–11, 2025**

Wednesday – Saturday  
Associate Secretary for the AMS: Steven H. Weintraub  
Program first available on AMS website: To be announced  

Issue of Abstracts: To be announced  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced

Washington, District of Columbia

*Walter E. Washington Convention Center and Marriott Marquis Washington DC*

**January 4–7, 2026**

Sunday – Wednesday  
Associate Secretary for the AMS: Betsy Stovall  
Program first available on AMS website: To be announced  

Issue of Abstracts: To be announced  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced
MATH POETRY

Poetry is about as ancient as mathematics in human culture. Its language can be whimsical, somber, joyful, beautiful, concise, thought-provoking, and inspirational. Poems can be limericks, raps, sonnets, haiku, acrostics, square stanzas, based on the Fibonacci sequence (with the number of syllables per line based on the sequence), or other types. Imagine and express mathematics in this unique form!

infinity

is a color
only artists can
see / yet I see it
in the one one two
three of Fibonacci / but they burned math at the stake and called her
a heretic / so we ignore the math / because / it
can never be as right as the sound a poet makes when he comes / home after
seeing The North Cape by Moonlight for the first time / a computer could never
paint that / I insist this without evidence because to believe otherwise would
be to lose what makes us human / like my mother /
and God / love does not exist / there is no equation for it / math can never
prove / how much I love you / because / zero is the lover’s / infinity / what
cannot start can never end / the only people who ever die are those who were
born in the first place / infinity / the only people we ever love /
this is what makes us /

—River Oxenreider, The Ohio State University

Sum of Us

See the sum of sequences—of us,
Where each of us is heading.
If only time stopped, our past unlocked,
Without that departure I’m dreading.

See the diverging series—farther,
Farther with every increase.
Our past unlocked if only time stopped,
But come one day, our friendship will cease.

See the converging series—closer,
Closer but never quite there.
If only time stopped, our past unlocked,
With our carefree selves, no tear or tear.

See the sum of sequences—of us,
All we ever amount to.
Our past unlocked if only time stopped,
But one day, I’ll truly forget you.

—Angela Zhou, International Academy East

Forest of Numbers

As I walk in the forest it’s all so dark,
But as I switched on the light it caused a spark.
Numbers and variables floating away.
Oh I wonder, what shall I learn today?
When I passed 0 the vines all grew,
All the way up to 100.
The sky was graphed in ways no one knew.
Clouds were set on points like (0, 5) and (3, 2)
Wings of the birds were copied and pasted.
Symmetry all around, no chance wasted.
Animals with trunks were all so grand.
Rise, Run, and Fall was the name of the band.
The band that played with all the numbers.
Oh, such music caused a slumber.
As all the creatures fell asleep.
Numbers in my head began to shriek!
More things, problems, equations to solve.
What will I do? I can’t do it all
I call on the forest to help me out.
Oh math, I love math, there is no doubt!

—Miranda Jedlinski, Arthur and Polly Mays Conservatory of the Arts

These poems by middle school, high school, and undergraduate students received AMS Math Poetry Contest awards at the 2023 Joint Mathematics Meetings

2024 Math Poetry Contest Opens September 1, 2023.
Learn more: www.ams.org/math-poetry
Contribute to AMS Open Math Notes

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