MEMORIAL TRIBUTE

Christine Bessenrodt, 1958–2022

Jørn B. Olsson

On Monday, January 25, 2022, Professor Dr. Christine Bessenrodt passed away at the age of 63 after a short serious illness. Mathematics has lost an original, dedicated, and prolific scientist and teacher of international class, who will be sorely missed. Christine will be remembered for her unwavering devotion to all of her professional activities, both scientific and administrative. She was the author of more than 100 scientific papers with more than 40 collaborators. Her papers were on a wide variety of topics centered around representation theory of finite groups and finite dimensional algebras, algebraic and enumerative combinatorics, and additive number theory. She valued and maintained contacts, both professional and social, with numerous colleagues. Her collaborators benefitted from having a perfect coauthor. She was good at finding interesting research problems and at giving new insights into existing ones, often making vague ideas precise. Her many conjectures were usually correct. She was a very good host for all of her visitors, making sure that the social as well as the mathematical part of a visit was rewarding. It was always a pleasure to be in her company. Christine will also be remembered for her involvement in supporting and promoting equality for female mathematicians, which started early in her career. This included her commitment to European Women in Mathematics as well as her many years of dedication to diversity and equal opportunities at the German Mathematical Society. The Emmy Noether Lecture at the society's annual conferences was Christine's initiative.

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Christine Bessenrodt was born in Ahlten, Germany, on March 18, 1958. Both her parents were physicists, her father was a professor of physics in Düsseldorf. After her Abitur in 1975, she studied mathematics and physics at the universities of Düsseldorf (1975–1977) and Essen (1977–1979). In 1980, she received her doctoral degree at the University of Essen, under the supervision of Professor Gerhard Michler. Her dissertation was on a topic in modular representation theory of finite groups: "Indecomposable lattices in blocks with cyclic defect groups."

During her studies she was supported by the Studienstiftung des Deutschen Volkes, Germany's scholarship foundation for exceptionally gifted students. From 1980 to 1990, she held positions as scientific associate or research assistant at the universities of Essen and Duisburg. She visited the University of Illinois, Urbana-Champaign on a postdoc stipend from 1982 to 1983. In 1988, she obtained her Habilitation in Mathematics at the university of Duisburg: "Behaviour of some representation theoretical invariants under change of the group basis of a modular group algebra". Then in 1990, she was awarded a prestigious five-year Heisenberg grant, and she was a member of the Institute of Experimental Mathematics at the University of Essen from 1990 to 1993.

Christine accepted an offer of a professorship in algebra at the Otto-von-Guericke-Universität Magdeburg in the former East Germany in 1993. The universities of the former East Germany were at the time in a period of transition after the German reunification in 1990. The new professors from West Germany were expected to contribute in several ways. The mathematics department in Magdeburg was at the time traditionally strong in applied areas of mathematics. Christine's professorship in algebra was supposed to be the starting point for the development of new strengths in pure mathematics. With her constant commitment, Christine laid the foundation for a still-thriving institute in Magdeburg. Initially she was also responsible for teaching the courses in algebra and related areas at all levels. In the years following her appointment, new professors were hired in discrete mathematics, algebraic coding theory, and geometry. In 1994, she was appointed vice dean for the faculty of mathematics, a challenge she met with dedication and discipline for several years. She regularly invited colleagues for research visits. A particular highlight was the ICM Satellite Conference on "Representations of finite groups and combinatorics," which took place in Magdeburg in 1998.

In 2002, she accepted the call for a chair for algebra and number theory at Leibniz Universität in Hannover, and she remained in Hannover until her death. Substantial changes in pure mathematics were lying ahead for the institute. Several professors were about to retire and this was seen as an opportunity to establish new research areas. Christine was the first professor in Hannover working in representation theory, but with her strong combinatorial background she could connect well to some existing groups in discrete mathematics and related areas. She soon became a driving force in restructuring the institute, a move supported by many others. The former "Institut für Mathematik" was split up into four smaller institutes for pure mathematics and Christine acted successfully for more than 15 years as director of the new "Institut für Algebra, Zahlentheorie, und Diskrete Mathematik." With her tireless energy and impressive management skills she took care of the scientific development of the institute and of all the administrative challenges. Under her directorship two new professors were appointed in number theory and in discrete mathematics, and two further permanent positions were filled in the area of representation theory. All these efforts resulted in a very active institute with numerous postdocs and regular visitors over many years. It is due to Christine's academic leadership that Hannover became recognized as a center for representation theory and algebraic combinatorics.

We present here an overview of most of Christine's research, consisting mainly of surveys of some central topics.

Christine's first published papers around 1980 "On blocks of finite lattice type I–II" were based on her dissertation. They were followed in the period until 1991 by a number of other papers on various topics in modular representation theory, e.g., on modular invariants and on the Auslander–Reiten quiver of a modular group algebra. Other work of Christine in this period arose from a collaboration with G. Törner and H.H. Brungs on right chain rings, done primarily during her time in Duisburg. This resulted in a monograph in three parts (1985–1986).

The ordinary and modular representation theory of the finite symmetric and alternating groups S_n, A_n and their covering groups $\hat{S}_n$, $\hat{A}_n$ is involved directly or indirectly in a majority of Christine's papers after 1990. About half of these were written in collaboration with Jørn B. Olsson.

The covering groups $\hat{S}_n$, $\hat{A}_n$ are non-split extensions of S_n, A_n by a central subgroup of order 2, e.g.,

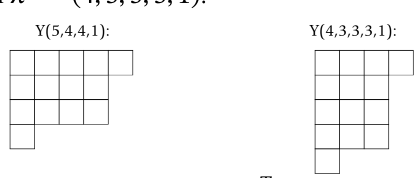
$$1 \to \langle z \rangle \to \hat{S}_n \to S_n \to 1.$$

Irreducible representations/characters of the symmetric and alternating groups and their covering groups are labelled in a canonical way by classes of partitions of integers and these labels carry important information about the representations. This means that questions about partitions, both combinatorial and number theoretic, play a role in most papers. The combinatorics may, for example, involve bijections between sets of partitions or properties of a single partition (hooks, p-cores, p-quotients) and the number theory may involve partition identities.

We present some of these concepts in a short survey.

Partitions, Young diagrams, hooks: A *partition* λ of n is a sequence of natural numbers $\lambda = (a_1, a_2, \dots, a_m)$ such that $a_1 \geq a_2 \geq \dots \geq a_m > 0$ and $a_1 + a_2 + \dots + a_m = n$. The a_i's are the *parts* of λ. For example $\lambda = (5, 4, 4, 1) = (5, 4^2, 1)$ is a partition of 14. We may use an exponential notation for repeated parts in a partition. A partition $\lambda = (a_1, a_2, \dots, a_m)$ is visualized in its *Young diagram* $Y(\lambda)$ of boxes, arranged in left-justified rows, with the row lengths equal to the parts of λ in non-increasing order.

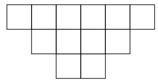
The Young diagram of $\lambda = (5, 4, 4, 1)$ and of its *transposed* partition $\lambda^T = (4, 3, 3, 3, 1)$:



Generally, by definition, $Y(\lambda^T)$ is obtained by reflecting $Y(\lambda)$ along its diagonal.

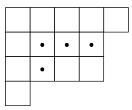
A *bar partition* of n is a partition λ with distinct parts. It is visualized in its *shifted Young diagram* $SY(\lambda)$, as illustrated in this example with $\lambda = (6,4,2)$:

SY(6,4,2):



To each box in a Young diagram is associated a *hook* and a *hook length*. We show a hook of length 4 in $Y(5,4,4,1)$ and list all hook lengths:

Hook of length 4:

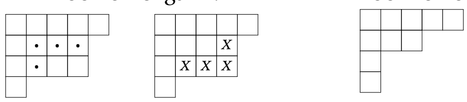


All hook lengths:

8	6	5	4	1
6	4	3	2	
5	3	2	1	
1				

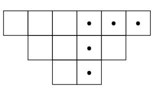
We illustrate the *removal of a hook in a partition* by an example:

Hook of length 4: 4-hook removed:



To each box in a shifted Young diagram, we associate a bar and a bar length. We show a bar of length 5 in $SY(6,4,2)$ and list all bar lengths:

Bar of length 5:



All bar lengths:

10	8	6	5	3	1
	6	4	3	1	
		2	1		

If ℓ is a positive integer, a partition λ is called *ℓ-regular* if no part is repeated ℓ or more times. Thus $(6,4^2,1)$ is ℓ-regular for $\ell \geq 3$. The bar partitions are the 2-regular partitions.

Labels of irreducible characters: The ordinary irreducible characters $[\lambda]$ of S_n are labelled by the partitions λ of n. The sign character (sgn) of S_n is equal to $[1^n]$ and it is known that $(sgn)\cdot[\lambda] = [\lambda^T]$ for all partitions λ.

For p a prime, the p-modular irreducible characters $\{\lambda\}$ of S_n are labelled by the p-regular partitions of n.

The ordinary irreducible characters $\langle\lambda\rangle$ of $\hat{S}_n$ which do not contain the cental element z in their kernel are labelled by the bar partitions of n. They are referred to as *spin characters*.

Branching rules describe restrictions of irreducible characters from S_n to S_{n-1} or from $\hat{S}_n$ to $\hat{S}_{n-1}$ in term of their labels. For example, the restriction of $[5,4^2,1]$ to S_{13} is $[4^3,1]+[5,4,3,1]+[5,4^2]$. The partition labels of the constituents are obtained by removing a 1-hook from $(5,4^2,1)$.

Partition identities: Suppose that A and B are properties/conditions, which a partition may satisfy. Then the following statement is a *partition identity*: The number of partitions of n satisfying A equals the number of partitions of n satisfying B. As an example we mention an identity due to Euler: The number of partitions of n with *distinct* parts (A) equals the number of partitions of n with only *odd* parts (B). A partition identity may be verified by constructing a bijection between the sets of partitions satisfying A and B or by using generating functions.

Christine's first paper involving partitions was on the Andrews–Olsson partition identity, a result on partitions with congruence conditions [1]. This identity was discovered while looking at fixed points of the involutary *Mullineux map M* on the set of p-regular partitions and it provided evidence for the Mullineux conjecture [2].

The *Mullineux conjecture* states that $(sgn)\cdot\{\lambda\} = \{\lambda^M\}$ for all p-regular partitions λ. Thus M should be a p-analogue of the transposing map T. The conjecture was proved around 1995 after being reduced to a combinatorial statement by A. Kleshchev (in his series of papers on modular branching rules). Some of the earlier papers written by Bessenrodt–Olsson focussed on the Mullineux symbols and their variant *residue symbols* which led to a simpler proof of the Mullineux conjecture in 1996 [5]. The residue symbols were applied in two later papers by Bessenrodt–Olsson to study Jantzen–Seitz partitions (which label the p-modular irreducible representations which remain irreducible upon restriction to S_{n-1}) and modular branching in alternating groups.

Christine obviously enjoyed working with partition theory and this resulted in a number of papers that often drew inspiration from representation theory. This includes papers on hooks in Young diagrams [3], on partition identities, on partition congruences with I. Pak, and on multiplicative properties of partitions with K. Ono [4] and O. Beckwith.

Christine's first paper on the covering groups of S_n was on *decomposition matrices for spin characters of symmetric groups in characteristic 3*, a collaboration with A. O. Morris and J. B. Olsson, published in 1993 [6]. One of the challenges in the paper was that the modular irreducible spin representations and canonical labels for them were not known until 2002 (Brundan–Kleshchev). The investigations of possible labels in [6] led to a conjecture of a partition identity, which was proved by Christine in a paper in collaboration with G. E. Andrews and J. B. Olsson [7]. A later paper by the same authors included a refinement of the identity.

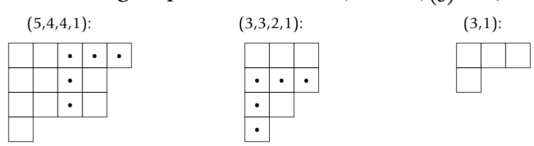
Character degrees: The degrees of irreducible characters of S_n are given by the celebrated *hook formula*: If f_λ is the degree of the character $[\lambda]$, labelled by the partition λ of n and $H(\lambda)$ is the product of all hook lengths in $Y(\lambda)$, then $f_\lambda = \frac{n!}{H(\lambda)}$.

For the spin characters there is an analogous bar formula for the degree which also involves a power of 2.

In 1998, Christine posed the problem of classifying the irreducible characters of S_n of prime power degree. This problem was solved by Christine in collaboration with J. B. Olsson and the number theorists A. Balog and K. Ono [8].

The proof involved an analysis of hook lengths and a new number theoretical result on prime factors in consecutive integers. The method was later also applied to the covering group $\hat{S}_n$ for odd primes.

p-cores, p-blocks and generalizations: The *p-core* $\lambda_{(p)}$ of a partition λ is obtained by removing repeatedly p-hooks from λ for as long as possible. Thus $(5,4^2,1)_{(5)} = (3,1)$:



Analogously you may define the *p-bar core* of a bar partition of n when p is odd.

The distribution of irreducible characters of S_n into p-blocks for all primes is determined by the p-cores of the labelling partitions (The "Nakayama conjecture"): Two characters $[\lambda]$ and $[\rho]$ are in the same p-block if and only if $\lambda_{(p)} = \rho_{(p)}$. When p is odd, a similar statement holds for the spin characters of $\hat{S}_n$.

The case of spin characters in characteristic 2 is essentially different since the 2-blocks of $\hat{S}_n$ may contain both ordinary and spin characters. It was solved by Christine and J. B. Olsson in [9]: A spin character $\langle\lambda\rangle$ labelled by a bar partition λ is in the 2-block containing the ordinary character $[\mathrm{dbl}(\lambda)]$. Here $\mathrm{dbl}(\lambda)$ is obtained from λ by breaking each of its parts into two (almost) equal parts.

In 2003, B. Külshammer, J. B. Olsson, and G. R. Robinson [10] presented a generalised ℓ-block theory for symmetric groups S_n for arbitrary $\ell > 0$ by character theoretic methods. This was closely related to the Iwahori–Hecke algebra $\mathcal{H}_n(q)$, where q is a primitive ℓ-th root of unity. The ℓ-blocks satisfied a "Nakayama Conjecture" and there were analogues of basic sets, decomposition matrices, and Cartan matrices and their invariant factors. There was also a conjecture on the invariant factors. This work was the background for some of Christine's papers. In 2010, Christine and D. Hill formulated a block-wise refinement of the conjecture in [10], and in 2015 this was proved by by A. Evseev. It is well-known that the determinant of the p-Cartan matrix of S_n is a power of p. In 2001 in collaboration with J. B. Olsson, Christine provided a simple explicit formula for the exact values of these p-Cartan determinants. A conjecture of A. Mathas (in light of work of Donkin) stated that the ℓ-power property should generalise to determinants of ℓ-Cartan matrices. In [11], Christine and J. B. Olsson refined this to a conjecture for the exact values of these ℓ-Cartan determinants. Both of these conjectures were proven by Brundan and Kleshchev in 2002; an alternative proof was later given by Christine, J. B. Olsson, and R. P. Stanley in [12]. This led to a number of papers concerning the determinants and Smith normal forms for submatrices of character tables, culminating in a paper by Christine and R. P. Stanley in which they provide a considerable generalisation of a result of L. Carlitz, D. P. Roselle, and R. A. Scoville from 1971 on ballot type sequences [13].

Block equalities: For a prime p and a p-block B_p of a finite group G let $\mathrm{Irr}(B_p)$ be the set of irreducible complex characters of B_p. In 1997, G. Navarro and W. Willems conjectured that if for different primes p, q we have a *block equality* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ then $|\mathrm{Irr}(B_p)| = 1$. They verified the conjecture for all blocks in solvable groups. Christine found a counterexample to the conjecture in the extension group $6.A_7$ of the alternating group. However the Navarro–Willems conjecture also holds for all blocks in the symmetric groups (J. B. Olsson, D. Stanton [14]) and their covering groups (C. Bessenrodt, J. B. Olsson [15]). In addition, the conjecture was verified for principal blocks in all finite groups by Christine, G. Navarro, J. B. Olsson, and P. H. Tiep in 2007. The more general question of *block inclusions* was also treated in [14] and [15]. In a paper by Christine with J. Zhang, it was shown that G is nilpotent if and only if $\mathrm{Irr}(B_p^0) \cap \mathrm{Irr}(B_q^0) = \{1_G\}$ for all principal blocks and all pairs of primes p, q dividing $|G|$ [16].

Kronecker products: Decomposing the Kronecker product (tensor product) of two irreducible complex representations of a symmetric group S_n is considered to be one of the definitive open problems in algebraic combinatorics. It is referred to as the *Kronecker problem* and a general solution is not in sight. Starting in 1999, Christine and her collaborators wrote a series of papers on Kronecker products in S_n and related groups. In a first paper by Christine and A. Kleshchev from 1999, it was shown that a nontrivial Kronecker product in S_n is always reducible and never homogeneous. They also classified Kronecker products with at most three homogeneous components and made a conjecture for four homogeneous components. This conjecture was later verified by Christine in a paper with S. van Willigenburg in 2014. Other papers with A. Kleshchev dealt with Kronecker products of representations of A_n, of modular representations of S_n, and of spin representations of the covering groups $\hat{S}_n, \hat{A}_n$. A main theme was the classification of homogeneous and irreducible Kronecker products. Christine had conjectured a classification of multiplicity-free Kronecker products in S_n in 1999. This conjecture was verified in 2017 in an impressive paper with C. Bowman [17].

An important part of the Kronecker problem is the positivity problem, i.e., deciding the positivity of Kronecker coefficients. In 2013 it was conjectured by G. Heide, J. Saxl, P. H. Tiep, and A. E. Zalesski, that for $n \neq 2, 4, 9$ there exists an irreducible character of S_n whose Kronecker square contains all irreducible characters as constituents (The HSTZ-conjecture). J. Saxl conjectured that in the case $n = k(k+1)/2$ the character $[\rho(k)]$, labelled by the staircase partition $\rho(k) = (k, k-1, \ldots, 2, 1)$ would be a candidate (the Saxl conjecture). Christine applied a result on the product of spin characters to give a contribution to the Saxl conjecture and she formulated an analogous conjecture for spin characters [18]. In a paper with C. Bowman and L. Sutton, Christine verified Saxl's conjecture for all irreducible characters of S_n of odd degree [19]. This was a by-product of a result showing that Specht modules labelled by so-called 2-separated partitions are semisimple as $H^{\mathbb{C}}_{-1}(n)$-modules. The paper also contained a strengthened and a generalized Saxl conjecture. The latter involves symmetric p-cores. One of Christine's final papers (with C. Bowman) was announced after her death [20]. It deals with the symmetric and anti-symmetric part of a Kronecker square. A main result is the classification of the partitions λ for which the symmetric part (resp. anti-symmetric part) of the square of $[\lambda]$ is multiplicity-free (or zero). In addition, there are several new conjectures in the paper including refinements of the Saxl- and HSTZ-conjectures for symmetric and anti-symmetric parts.

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Jørn B. Olsson

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