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A WORD FROM...

Juan C. Meza, Former Director of the Division of Mathematical Sciences at the NSF

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Introduction

Photo courtesy of UC Merced.

It's been over four years since I was interviewed by AMS after being selected for the position of Director of the Division of Mathematical Sciences (DMS). At that time, I was asked about my expectations for the position, and I responded that it was too early to tell. Here, I'd like to recap some of my experiences as division director, including some of the highlights and challenges we faced. Through these experiences, I also discovered some strategies that I think the math community could use to more effectively partner with DMS for the benefit of both.

DMS is the biggest supporter of mathematical sciences research in the United States and accounts for more than 60% of federally funded basic mathematics research. These areas include algebra, topology and geometric analysis, number theory, applied mathematics, analysis, combinatorics, probability and statistics, computational mathematics, and mathematical biology. The annual budget for DMS was \$233M when I started in 2018 and grew to \$244M in fiscal year 2021. About 73% of these funds go to Individual PIs. Looking at it another way, every year, DMS supports close to 6500 researchers including over 2600 senior researchers, 2300 graduate students, 380 postdocs, and 1200 undergraduate students. In terms of the number of proposals, DMS also handles one of the largest portfolios at NSF, managing over 3000 proposal reviews annually, yielding around 750 awards per year.

Juan C. Meza is a professor of mathematics at the University of California, Merced. His email address is `jcmeza@ucmerced.edu`.

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Research Highlights

An important role for the division director is to champion the value of mathematics research. Throughout my time at DMS, I found that developing new partnerships with other directorates was an excellent means of achieving this. One important initiative was the 10 Big Ideas started by the NSF Director France Cordova. DMS responded by actively participating in three of the Big Ideas: Harnessing the Data Revolution (HDR), the Rules of Life, and Quantum Leap. In the HDR Big Idea, DMS was involved in the Transdisciplinary Research in Principles of Data Science (TRIPODS) and the TRIPODS Institutes. By design, each of the awards had mathematicians playing a major role in the team. We also participated in the Rules of Life Big Idea through the NSF-Simons Research Centers for Mathematics of Complex Biological Systems program. Partnerships weren't restricted to the US either. We initiated two new partnerships, one with the UK and another with Israel to jointly fund proposals that supported collaborations between PIs in those two countries.

One unexpected challenge was the COVID-19 pandemic, and it would be hard to overstate the effect that it had on the operations of DMS. We had to transition to full virtual panels and the staff moved to full remote telework within a few weeks. As the normal means of meeting with PIs was no longer viable, we sought out new ways to engage with the math community. One initiative was a new monthly DMS Virtual Office Hours. Starting in October 2020, we averaged about 150 participants at each session. Topics ranged across all the DMS programs and included news updates from NSF, MPS, and DMS.

In direct response to the pandemic, NSF was provided with additional funds to address the challenges the country faced. In turn, NSF issued a call for proposals in the spring of 2020 to address COVID-19. Watching the math community respond to our request for proposals was one of the most rewarding events I've ever participated in. In the end, we were able to fund 20 proposals that led to novel approaches to understanding the spread of the pandemic as well as new methods for understanding the disease itself. Because of the overwhelming response to this call, the following year DMS cohosted a workshop with the Social, Behavioral, and Economic Sciences directorate that sought to develop new epidemiological models in collaboration with social scientists.

Two other initiatives that I believe will have a long-term influence came out of an earlier partnership with the Simons Foundation and involved the rapidly growing field of Deep Learning. This led to a new program on the Mathematical and Scientific Foundations of Deep Learning (MoDL) in 2020. While other programs existed in this area, none of them addressed the foundational or theoretical aspects of the field.

Training Opportunities and Broadening Participation

One recurring comment I heard from the community was the difficulty in providing support for graduate students. This troubled me, as I've always believed that we must provide support for our early-career mathematicians to ensure a healthy profession. As a result, we decided to increase funds in this area. In 2019–2020, DMS invested \$20M toward graduate student support, which resulted in an additional 675 graduate students funded on research fellowships.

DMS also participated in two new programs aimed at early-career PIs, the ASCEND and LEAPS programs. The ASCEND program supports postdocs who will help broaden the participation of groups that are underrepresented in the mathematical and physical sciences. The LEAPS program, supports the research of early-career faculty and is focused on those working at institutions such as minority-serving institutions, predominantly undergraduate institutions and R2 universities. I believe that both programs are a great step toward increasing the diversity of the math community.

One observation I had when I arrived at DMS was that there was a lack of diversity in program officers. During my initial discussions with the program officers, this subject kept popping up, especially with regard to gender diversity. This led to several discussions at our weekly staff meetings, where everyone agreed that we needed to emphasize gender diversity within our own ranks. When I arrived, there were three women program directors. By 2022, we had a 50-50 split of men/women program directors. In my last year, we hired seven rotators, of whom four were women and two were underrepresented minorities. In total, during my time at NSF, we hired 26 rotators, 16 of whom were women and four of whom were underrepresented minorities. I'm happy to see that this trend has continued, and today the division stands at 27 program directors, with 16 being women—a true team effort by everyone at DMS.

Recommendations

Based on my experiences, I found that DMS has done an exceptional job in serving the mathematics research community. I take heart that the Committee of Visitors, which reviewed the programs in DMS agrees with that assessment. Nonetheless, I believe that both DMS and the mathematical community can do better in terms of making the case for mathematics research in the country. In what follows, I make several recommendations that I hope will start some discussions between all the parties involved.

My first suggestion is that the math community should take additional steps to partner more effectively with NSF, and DMS in particular. From the perspective of NSF, input from the community is highly valued and important, especially during budget discussions. Other directorates have active constituencies—astronomy and computing sciences for example. In astronomy, the community produces a decadal survey prioritizing their large facilities. In computing, the Computing Research Association (CRA) produces quadrennial papers that provide potential research directions, challenges, and recommendations. AMS should consider doing something similar. Many of the internal discussions at NSF revolve around the needs of the scientific communities and having such reports provides important reference points.

The math community already engages DMS through various science policy committees and the Government Relations Office. The science policy committees are a good start, but they generally only meet once a year. More frequent and active partnerships would enhance relations. Professional societies could play an important role in collecting and championing new research areas. Congressional briefings are another excellent idea, and the work of Karen Saxe, head of government relations at AMS, is always appreciated. AMS has a unique ability to take the pulse of the math research community and to elevate important new research areas to the attention of NSF and DMS.

Another avenue is through convening workshops on current research trends. I already mentioned the DMS-funded workshop on Research on Enhancing Socially and Behaviorally Modulated Mathematical Models for Human Epidemiology. Other examples included the workshops on Statistics at a Crossroads: Who is for the Challenge?, the workshops on Rules of Life in the Context of Future Mathematical Sciences, and the Workshop Report for Research Training Group Grants. All these reports were instrumental in helping DMS define its funding portfolio.

Finally, I'll point to the important role that program directors play in the operations of DMS. I encourage those of you who might be interested in serving as a program director to contact DMS about opportunities. I found my time at NSF to be rewarding and I learned much about the state of research (both in mathematical sciences as well as other areas) in the US.

My four years at NSF were a time of great change and many challenges. And they were also some of the best times in my career. Let me end with my heartfelt thanks and deep appreciation to everyone who helped me while I was at DMS. Together we were able to accomplish much, and it was all because of the many people who came together under challenging times to maintain a vibrant and exciting mathematics research environment.

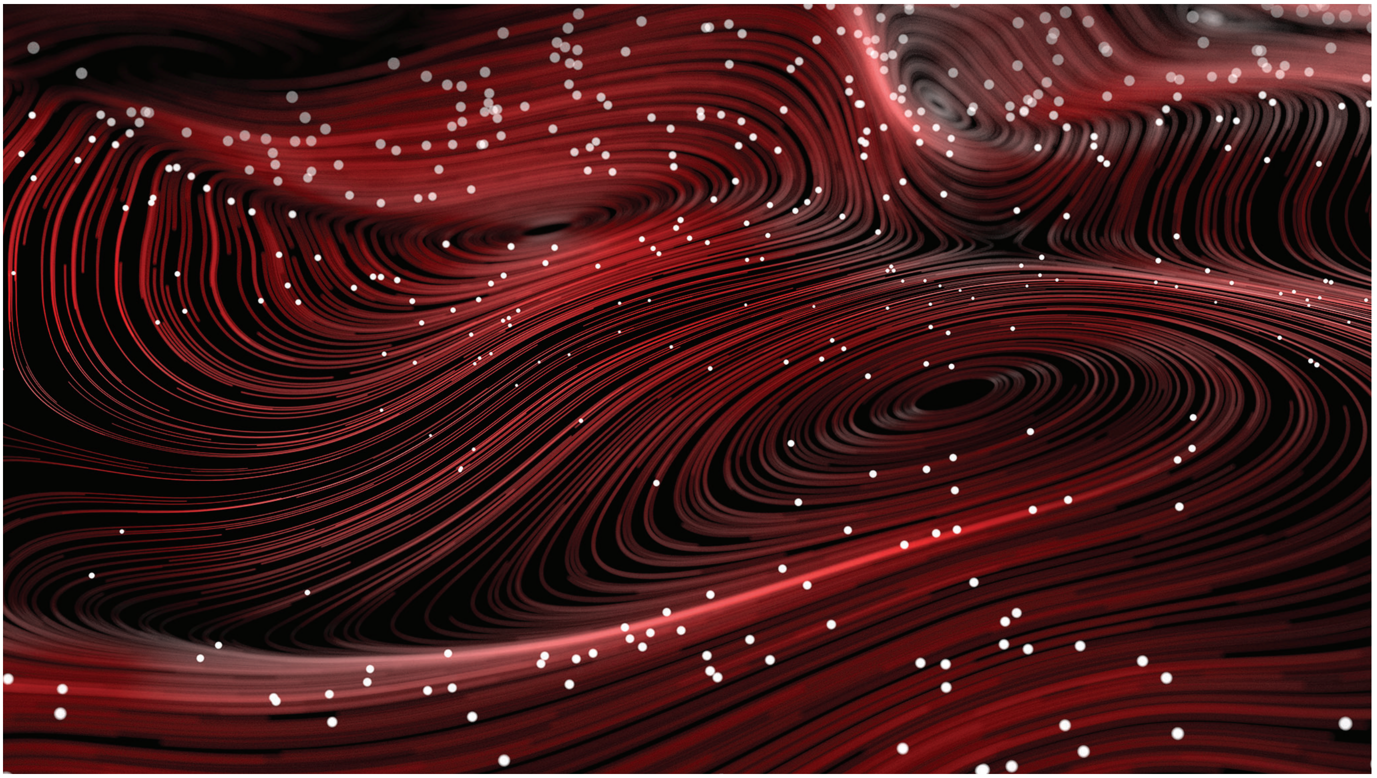
References

[1] NSF's 10 Big Ideas - Special Report, `https://www.nsf.gov/news/special_reports/big_ideas/`

[2] Harnessing the Data Revolution (HDR) at NSF, `https://www.nsf.gov/cise/harnessingdata/`

[3] Activities in NSF's Division of Mathematical Sciences, AMS Special Event, JMM, January 15-18, 2020, Denver, CO, `https://www.juancmeza.com/s/JMM-2020-Final.pdf`

Definability and Arithmetic



Hector Pasten

The purpose of this article is to give an overview of some very active interactions between first order definability and arithmetic. However, even to get started we need to clarify what we mean by "first order definability". The reader who is already familiar with these notions can safely skip directly to the second section.

For the experts, we would like to clarify that this introductory note is not intended to be a comprehensive survey, and many interesting topics are left out by space limitations: uniformity and effectivity in unlikely intersections, arithmetic of complex function fields, Diophantine sets of infinite extensions of $\mathbb{Q}$, etc. Furthermore, at several points we prefer to discuss less-general theorems for the sake of clarity of the exposition. Nevertheless, further reading will be indicated at the relevant points.

Hector Pasten is an associate professor at the Faculty of Mathematics of the Pontificia Universidad Católica de Chile. His email address is `hector.pasten@mat.uc.cl`.

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Languages and structures. For us, a *language* will be a set $\mathcal{L}$ consisting of symbols for constants, for relations, and for functions. An $\mathcal{L}$*-structure* is a set endowed with some distinguished constants, relations, and functions that give meaning to the symbols in $\mathcal{L}$. For instance, the language of arithmetic is $\mathcal{L}_{ar} = \{0, 1, +, \times, =\}$ and any (semi-)ring is an $\mathcal{L}_{ar}$-structure in the obvious way.

Given a language $\mathcal{L}$, a (first order) $\mathcal{L}$*-formula* is a finite string of symbols that only uses the quantifiers $\exists$ and $\forall$, the conjunction $\wedge$, the disjunction $\vee$, implications $\to$ and $\leftrightarrow$, the negation symbol $\neg$, parentheses, and symbols from $\mathcal{L}$ (we may use commas to improve readability). One requires that such a formula is well-written: it can be read. For instance, the string of symbols $\forall\exists x + =$ is not an $\mathcal{L}_{ar}$-formula, while $\forall x \exists y x + y = 0$ is an $\mathcal{L}_{ar}$-formula.

An $\mathcal{L}$-formula may or may not have free variables (variables not under a quantifier) and both cases are of interest to us.

Formulas without free variables are called *sentences*, and given an $\mathcal{L}$-structure they can be true or false in the given structure. For instance, recall our $\mathcal{L}_{ar}$-formula $\forall x \exists y, x + y = 0$; this is true over $\mathbb{Z}$ but it is false over $\mathbb{N} = \{0, 1, 2, \dots\}$. If ϕ is an $\mathcal{L}$-sentence and M is an $\mathcal{L}$-structure, we write $M \vDash \phi$ to express that ϕ is true on M.

Let M be an $\mathcal{L}$-structure. Formulas with n free variables can be used to define subsets of M^n. Indeed, we say that $S \subseteq M^n$ is defined by an $\mathcal{L}$-formula $\phi(x_1, \dots, x_n)$ with free variables x_i if for every tuple $\mathbf{a} \in M^n$, we have that $\mathbf{a} \in S$ if and only if $M \vDash \phi(\mathbf{a})$.

Let us clarify this important idea with an example. Consider the $\mathcal{L}_{ar}$-structure $\mathbb{N}$ and the inequality relation

$$R = \{(a, b) : a \leq b\} \subseteq \mathbb{N}^2.$$

This is $\mathcal{L}_{ar}$-defined by the formula

$$\phi(x, y) : \quad \exists z, y = x + z$$

because the pairs $(a, b) \in \mathbb{N}^2$ such that $\mathbb{N} \vDash \phi(a, b)$ holds are precisely the elements of R.

A final disclaimer: for the sake of exposition we will often identify the symbols in a language with their interpretation in a structure.

Equipped with these notions, let us discuss a first example of how definability interacts with arithmetic.

What can be expressed using addition? As a warm-up, let us discuss a classical problem: to understand the sets over the integers that can be defined using addition.

For instance, one can ask whether we can define multiplication in $\mathbb{Z}$ using addition. To have a more precise question we need to be clear about what we mean by "define". For our purposes, we will restrict our attention to first order formulas over a language. In our case the language will be $\mathcal{L}_{Pres} = \{0, 1, +, =, \leq\}$ which is interpreted in $\mathbb{Z}$ in the usual way. Then the question can be formulated as: *Is multiplication in $\mathbb{Z}$ definable by a first order formula over the language $\mathcal{L}_{Pres}$?*

The key to answer this question is the following classical result of Presburger, which follows from a general result on elimination of quantifiers [16]:

Theorem 1 (Presburger). *A subset $S \subseteq \mathbb{N} \subseteq \mathbb{Z}$ is first order definable over the language $\mathcal{L}_{Pres}$ if and only if S is eventually periodic: there is $M \in \mathbb{N}$ such that $\{x \in S : x \geq M\}$ is periodic in the set $\{M, M+1, M+2, \dots\}$.*

As a consequence we finally get

Corollary 2. *Multiplication in $\mathbb{Z}$ is not first order definable over the language $\mathcal{L}_{Pres}$.*

Proof. Suppose it is. Then so is the set of squares $S = \{n^2 : n \in \mathbb{N}\}$, but this set is not eventually periodic. □

The connection between definability and arithmetic in Presburger's theorem is rather direct. Let us discuss another kind of definable sets that enjoy a less obvious connection with arithmetic.

O-minimal structures. Consider $\mathbb{R}$ as a structure over the language $\mathcal{L}_{rc}^{\mathbb{R}} = \{0, 1, +, \times, =, \leq\} \cup \mathbb{R}$ of real-closed fields expanded by allowing constants from $\mathbb{R}$ in our formulas.

It is a classical theorem of Tarski that $\mathbb{R}$ over the language $\mathcal{L}_{rc}^{\mathbb{R}}$ admits elimination of quantifiers: every $\mathcal{L}_{rc}^{\mathbb{R}}$-formula ϕ is equivalent over $\mathbb{R}$ to an $\mathcal{L}_{rc}^{\mathbb{R}}$-formula without quantifiers. For instance, consider the formula

$$\phi(x) : \quad \exists y, x^2 + y^2 = 1$$

(where $x^2 = x \times x$). Over $\mathbb{R}$ this formula defines the interval $[-1, 1]$ and, in fact, the same set is defined by the quantifier-free formula

$$0 \leq x + 1 \wedge x \leq 1.$$

The $\mathcal{L}_{rc}^{\mathbb{R}}$-definable sets in $\mathbb{R}^n$ are called *semi-algebraic* and, due to elimination of quantifiers, they are exactly the sets defined by finitely many equations and inequalities between polynomials. In particular we get the following consequence of Tarski's theorem:

Corollary 3. *All the $\mathcal{L}_{rc}^{\mathbb{R}}$-definable subsets of $\mathbb{R}$ are finite unions of points and (possibly unbounded) intervals.*

The concept of o-minimality —introduced by van den Dries [2]— comes as a generalization of this structural result for definable subsets of $\mathbb{R}$. This was further developed by Steinhorn, Marker, and many others.

Consider $\mathbb{R}$ as a structure over a language $\mathcal{L}$ expanding $\mathcal{L}_{rc}^{\mathbb{R}}$. We say that this $\mathcal{L}$-structure is *o-minimal* if all $\mathcal{L}$-definable subsets of $\mathbb{R}$ are finite unions of points and intervals.

As an example of a structure which is *not* o-minimal, consider $\mathbb{R}$ with the language $\mathcal{L}_{\sin}$ consisting of $\mathcal{L}_{rc}^{\mathbb{R}}$ and the sine function. Then the (quantifier-free) formula $\sin(x) = 0$ defines a discrete infinite set in $\mathbb{R}$.

One can ask about o-minimal structures beyond $\mathbb{R}$ over $\mathcal{L}_{rc}^{\mathbb{R}}$. A first example, due to Denef and van den Dries after work of Gabrielov, is the following: let $\mathbb{R}_{an}$ be the structure $\mathbb{R}$ over the language $\mathcal{L}_{an}$ consisting of $\mathcal{L}_{rc}^{\mathbb{R}}$ augmented by all functions $f : [0, 1]^n \to \mathbb{R}$ (for all n) that are real analytic on some open neighborhood of $[0, 1]^n$.

Theorem 4. *The structure $\mathbb{R}_{an}$ is o-minimal.*

On the other hand, let $\exp : \mathbb{R} \to \mathbb{R}$ be the exponential function and let $\mathbb{R}_{\exp}$ be the structure $\mathbb{R}$ over the language $\mathcal{L}_{\exp} = \mathcal{L}_{rc}^{\mathbb{R}} \cup \{\exp\}$. Then we have the following celebrated theorem of Wilkie:

Theorem 5. *The structure $\mathbb{R}_{\exp}$ is o-minimal.*

Van den Dries and Miller generalized the previous two results by showing o-minimality over $\mathcal{L}_{an} \cup \{\exp\}$. In fact, o-minimality has been proved over several other languages, but we do not intend to present a survey here.

Definable sets over o-minimal structures are very well-behaved, which allows for a rich theory that generalizes real algebraic geometry (which is the case of $\mathbb{R}$ over $\mathcal{L}_{rc}^{\mathbb{R}}$). Some of the results that can be proved in this generality are existence of cell-decompositions, a good theory of dimension, finiteness of connected components, and parametrization results.

We refer the reader to [3] for an exposition of the theory of o-minimal structures.

Counting rational points. Having discussed the notion of o-minimal structures, let us now turn into connections with arithmetic.

For a rational number $q = a/b$ with a, b coprime integers, the height of q is $H(q) = \max\{|a|, |b|\}$. The height of a tuple $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Q}^n$ is defined as $H(\mathbf{q}) = \max_j H(q_j)$. For a subset $\Gamma \subseteq \mathbb{R}^n$ and $T > 1$ we define

$$N(\Gamma, T) = \#\{\mathbf{q} \in \Gamma \cap \mathbb{Q}^n : H(\mathbf{q}) \le T\}.$$

The story begins with the following counting result of Pila which builds on an earlier result by Bombieri and Pila.

Theorem 6. *Let $f : [0,1] \to \mathbb{R}$ be a transcendental real analytic function on an open set containing $[0,1]$ and let $\Gamma \subseteq [0,1] \times \mathbb{R}$ be its graph. Let $\epsilon > 0$. There is a number $c(f, \epsilon) > 0$ such that for all $T > 1$ we have*

$$N(\Gamma, T) \le c(f, \epsilon) T^{\epsilon}.$$

The hypothesis that f is transcendental cannot be dropped, as can be easily seen by considering, for instance, $f(x) = x^2$.

Naturally, counting rational points is a topic of great arithmetic interest and the previous result provides a very general tool valid for rational points in curves. Pila and Wilkie [12] managed to prove a far-reaching generalization. For this we need to define the transcendental part of a set.

Given a set $X \subseteq \mathbb{R}^n$, we denote by X^{alg} the union of all connected positive dimensional semi-algebraic subsets of X (i.e., those definable over $\mathcal{L}_{rc}^{\mathbb{R}}$ or, equivalently, those definable by finitely many equations and inequalities between polynomials). The *transcendental part* of X is $X^{tr} = X - X^{alg}$. We remark that, in general, X^{alg} is not semialgebraic.

With this notation, the Pila–Wilkie theorem is

Theorem 7. *Let $\mathbb{R}$ over $\mathcal{L}$ be an o-minimal structure and let $X \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable. Let $\epsilon > 0$. There is $c(X, \epsilon) > 0$ such that for all $T > 1$ we have*

$$N(X^{tr}, T) < c(X, \epsilon) T^{\epsilon}.$$

Observe that Theorem 6 is a special case when we consider the o-minimal structure $\mathbb{R}_{an}$.

We remark that in order to get the T^{ϵ} bound it is unavoidable to discard X^{alg} as this subset can accumulate too many rational points.

While all of this is of course very interesting in its own right, it turns out that since the late 2000s, the point-counting theorems in o-minimal structures have found remarkable and unexpected applications in Diophantine Geometry, in the context of *unlikely intersections*.

Unlikely intersections. Let X be an algebraic variety over $\mathbb{C}$ endowed with a countable set S of "special" points and let Y be a properly contained subvariety of X. Counting dimensions, the intersection $Y \cap S$ is unlikely to happen even if S is Zariski dense in X. Thus, one might expect $Y \cap S$ to be small, even finite, unless there is a good reason for this to be otherwise.

Here is a concrete instance of this phenomenon. Consider $X = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ the two-dimensional multiplicative group and S its set of torsion points, namely, pairs (η_1, η_2) where η_j are roots of unity. Then S is countable and Zariski dense in X. Now consider $Y \subseteq X$ an irreducible curve. Is $Y \cap S$ infinite? A good reason for this to happen would be that Y is a multiplicative subgroup of X, or even a multiplicative translate by a point in S. For example, the hyperbola $Y = \{(x, y) \in X : xy = 1\}$ is a multiplicative subgroup and it contains all the points (η, η^{-1}) for η a root of unity. Lang conjectured that, in fact, this is the only way that $Y \cap S$ can be infinite. The result was later proved, independently, by Ihara, Serre, and Tate.

Theorem 8. *Let $Y \subseteq \mathbb{C}^{\times} \times \mathbb{C}^{\times}$ be an irreducible curve containing infinitely many torsion points (i.e., points of the form (η_1, η_2) with η_j roots of unity.) Then Y is the translate by a torsion point of a multiplicative subgroup of $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$.*

The previous result has been generalized in a number of ways and there are several other important problems in the subject of unlikely intersections, but here we will restrict our attention only to two of them: the *Manin–Mumford conjecture* and the *André–Oort conjecture*.

In its simplest form, the Manin–Mumford conjecture asked the following:

Conjecture 9. *Let Y be a smooth projective curve of genus $g \ge 2$ defined over a number field k, contained in its Jacobian J. Then Y contains at most finitely many torsion points of $J(\mathbb{C})$.*

This is a remarkable statement; note that the torsion points of $J(\mathbb{C})$ are dense in J even in the complex topology, but somehow the curve Y manages to avoid almost all of them!

The Manin–Mumford conjecture was proved by Raynaud in 1983. Several other proofs came afterward. In 2008 Pila and Zannier gave a new proof via point counting in o-minimal structures. Let us briefly outline the argument.

Sketch of proof. The abelian variety J, over $\mathbb{C}$, is biholomorphic to $\mathbb{C}^g/\Lambda$ for a suitable lattice Λ. Thus, there is a real-analytic uniformization $\phi : \mathbb{R}^{2g} \to J$ which descends to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, hence, having $[0,1]^{2g}$ as a fundamental domain.

The points in $[0,1]^{2g}$ corresponding to torsion points of $J(\mathbb{C})$ are precisely those with rational coordinates, and the height of such rational points in $[0,1]^{2g}$ is essentially the same as the order of the corresponding torsion point in $J(\mathbb{C})$.

On the other hand, the complex curve $Y \subseteq J$ corresponds, via ϕ, to a certain $\mathcal{L}_{an}$-definable set $Z \subseteq [0,1]^{2g}$. One wishes to show that Z has only finitely many rational points. For the moment, *let us assume that* $Z^{alg} = \emptyset$, so that $Z^{tr} = Z$. By the Pila–Wilkie theorem on the o-minimal structure $\mathbb{R}_{an}$, for every $\epsilon > 0$ we have

$$N(Z,T) < c(Z,\epsilon)T^{\epsilon}.$$

But if $P \in J(\mathbb{C})$ is a torsion point of large order and it belongs to Y, then its Galois orbit over the base number field k has to be considerably large (a precise estimate was proved by Masser; alternatively, one can use results of Serre), giving too many torsion points in Y: the whole Galois orbit of P. A careful analysis of the situation leads to a lower bound of the form

$$N(Z,T) > c' \cdot T^{\delta}$$

for certain fixed $\delta > 0$ and $c' > 0$. This contradicts the upper bound coming from the Pila–Wilkie theorem. Then one has to justify why $Z^{alg} = \emptyset$ to conclude the argument. □

The approach in the previous sketch of proof is known as the *Pila–Zannier strategy*.

This leads to the following general question: Given a transcendental analytic uniformization $\phi : \Omega \to X$ of an algebraic variety X, how to characterize those sets $V \subseteq \Omega$ such that both V and $\phi(V)$ are algebraic? Such sets might be called *bi-algebraic* for ϕ. Of course this is only a vague question, but in many applications of the Pila–Zannier strategy it is a crucial step of the argument in order to compute Z^{alg}. Nowadays there is a general approach to computing bi-algebraic sets by means of generalizations of the Ax–Schanuel theorem from functional transcendence.

Another important problem in the setting of unlikely intersections is the André–Oort conjecture. This concerns the special points (in a technical sense) of a Shimura variety; in the simplest case, an example of Shimura variety is the affine line over $\mathbb{C}$ (seen as the modular curve $Y(1)$), and its set of special points are the j-invariants of elliptic curves with complex multiplication.

Conjecture 10. *Let X be a Shimura variety and let Y be an irreducible subvariety of X. If the set of special points of X contained in Y is Zariski dense in Y, then Y is a special subvariety.*

After partial progress by André (unconditional) and Edixhoven (under GRH), a breakthrough occurred in 2011 when Pila unconditionally proved the André–Oort conjecture for products of modular curves using the Pila–Zannier strategy.

These ideas led to a series of advances on the André–Oort conjecture by several authors. It soon became clear that the two main difficulties were: necessary results on bi-algebraic sets for the relevant analytic uniformizations, and lower bounds on the size of Galois orbits of special points. The first difficulty was solved by Klingler, Ullmo, and Yafaev in 2014. The issue on sizes of Galois orbits remained as the final obstruction. Very recently, a preprint by Pila–Shankar–Tsimerman finally proves the André–Oort conjecture in full generality by bringing p-adic Hodge theory into the study of heights of special points. This builds on work of Binyamini–Schmidt–Yafaev (based on an idea of Schmidt) and an earlier result by Tsimerman for $\mathcal{A}_g$ (which crucially relies on work of Masser–Wüstholz, Andreatta–Goren–Howard–Madapusi Pera, and Yuan–Zhang). But this is not the end of the story; the Zilber–Pink conjecture still awaits on the horizon!

There are several excellent expositions about the connections between o-minimality and Diophantine geometry. See for instance [4] or [20].

Diophantine equations. Let us go back to the basics: Diophantine equations. These are polynomials equations in possibly many variables and having integral coefficients, for which integral or rational solutions are sought.

The topic is very old. It seems that the earliest written record of a (non-linear) Diophantine equation is the Babylonian clay tablet Plimpton 322, written about 1800 BC. This tablet displays several integral solutions of the Diophantine equation $x^2 + y^2 = z^2$.

However, Diophantine equations are named after the greek mathematician Diophantus of Alexandria (3rd century AD), due to his series of books *Arithmetica* where he presents a systematic study of several of these equations. In *Arithmetica*, Diophantus provides examples of integral and rational solutions of Diophantine equations along with a detailed explanation of the method by which the solutions were found.

In modern language, here are some remarkable examples from *Arithmetica*:

- Book II, problem 8: Parametrization of the conic $x^2 + y^2 = z^2$ by rational functions, using the chord method.
- Book IV, problem 24: Duplication of points in the elliptic curve $y(6 - y) = x^3 - x$ by the tangent method.
- Book VI, problem 17: Construction of rational points in the genus 2 curve $y^2 = x^8 + x^4 + x^2$ (birational to $y^2 = x^6 + x^2 + 1$) by specializing an algebraic identity. This equation was fully solved by Wetherell in the nineties using p-adic methods.

Diophantus's *Arithmetica* deeply influenced several scholars who wrote notes on the margin of their copies of the book. Just to mention two:

Around 1637 Fermat wrote in the margin of Book II, problem 8 (quoted above) the statement of his famous "Last Theorem" (solved in the nineties by Wiles) along with an apology for not having enough space for writing his proof.

On the other hand, two centuries earlier than Fermat, Chortasmenos wrote in the margin of the same problem "May your soul, Diophantus, be with Satan because of the difficulty of your other theorems and, particularly, of this one." And in fact, Diophantine equations are truly difficult in a technical sense.

In 1900 Hilbert proposed a famous list of 23 problems. The tenth problem in the list was the following:

Hilbert's Tenth Problem (HTP): *Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers.*

The notion of "process" in the statement of HTP was later formalized as *Turing machine* (or "algorithm" for short). After the work of M. Davis, H. Putnam, and J. Robinson, in 1970 Y. Matiyasevich gave a negative solution to this problem [8].

Theorem 11. *HTP is unsolvable: there is no algorithm to decide solvability in $\mathbb{Z}$ of Diophantine equations.*

Diophantine sets of integers. Let us return to the subject of definability. The key to Theorem 11 is the notion of *Diophantine set*: those subsets of $\mathbb{Z}^m$ that can be defined using Diophantine equations. The precise definition is the following:

A set $S \subseteq \mathbb{Z}^m$ is *Diophantine* if there are polynomials $F_1, \dots, F_r \in \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_n]$ such that for each $\mathbf{a} \in \mathbb{Z}^m$ we have $\mathbf{a} \in S$ if and only if there is $\mathbf{b} \in \mathbb{Z}^n$ with $F_j(\mathbf{a}, \mathbf{b}) = 0$ for all $1 \leq j \leq r$.

(We remark that one can always take $r = 1$ by considering $F = F_1^2 + \cdots + F_r^2$.)

As a first example of Diophantine set we have the odd integers: $a \in \mathbb{Z}$ is odd if and only if there is $b \in \mathbb{Z}$ with $a = 2b + 1$. Here, $r = 1$ and $F_1 = x - 2y - 1$.

A more interesting example is given by the $\leq$ relation $R = \{(a_1, a_2) \in \mathbb{Z}^2 : a_1 \leq a_2\} \subseteq \mathbb{Z}^2$: by the 4-squares theorem we have that $a_1 \leq a_2$ if and only if there are $b_1, \dots, b_4 \in \mathbb{Z}$ with $a_2 = a_1 + b_1^2 + \cdots + b_4^2$.

Here is an alternative way to think about Diophantine sets. Consider an affine algebraic variety $X \subseteq \mathbb{A}^{m+n}$ defined by equations over $\mathbb{Z}$ and take its set of integral points $X(\mathbb{Z})$. Now, project $X(\mathbb{Z})$ onto its first m coordinates. The set constructed in this way is Diophantine, and every Diophantine set arises from this construction.

For instance, in our example of odd integers we can take $X \subseteq \mathbb{A}^2$ as the line defined by $x = 2y + 1$, consider all integer solutions $(a, b) \in \mathbb{Z}^2$, and then project onto the x-coordinate.

There is yet a third equivalent way to define Diophantine sets. Recall the language $\mathcal{L}_{ar} = \{0, 1, +, \times, =\}$ and consider $\mathbb{Z}$ as an $\mathcal{L}_{ar}$-structure. A set $S \subseteq \mathbb{Z}^m$ is Diophantine if and only if it is defined by a positive existential $\mathcal{L}_{ar}$-formula.

This requires some explanation. A formula is *positive existential* if the only quantifier in it is $\exists$, it contains no negations, and the only connectives that it uses are $\vee$ and $\wedge$.

For instance, odd integers are defined by the formula $\exists y, x = 2y + 1$.

One can use Diophantine sets to construct new ones. For instance, we can exhibit the $\neq$ relation $\{(a_1, a_2) \in \mathbb{Z}^2 : a_1 \neq a_2\} \subseteq \mathbb{Z}^2$ as a Diophantine set by the positive existential formula

$$\phi(x_1, x_2) : \quad (x_1 + 1 \leq x_2) \vee (x_2 + 1 \leq x_1)$$

where each $\leq$ is hiding four existential quantifiers.

At first one might find the fact that the relation $\neq$ can be expressed over $\mathbb{Z}$ without using negations quite impressive. It makes us wonder what else can be presented as a Diophantine set over $\mathbb{Z}$.

Certainly, not every subset of $\mathbb{Z}^m$ can be Diophantine: there are only countably many Diophantine sets (count the equations). A more interesting restriction is the following:

Lemma 12. *Every Diophantine set S is listable: there is an algorithm that produces all of its elements and nothing else.*

Proof. Consider a Diophantine set $S \subseteq \mathbb{Z}^m$ defined by the formula

$$\phi(x_1, \dots, x_m) : \quad \exists y_1 \dots \exists y_n, F(\mathbf{x}, \mathbf{y}) = 0,$$

where F is a polynomial (by our earlier remarks on multiplication of polynomials and sums of squares of polynomials, this suffices). Then let us algorithmically enumerate the elements $\mathbf{c} \in \mathbb{Z}^{m+n}$ and at each step evaluate $F(\mathbf{c})$. If one gets a non-zero value, we ignore this $\mathbf{c}$, while if we get 0 we print the first m coordinates of $\mathbf{c}$. This process algorithmically produces all the elements of S and nothing else. $\square$

The previous lemma might seem to provide a restriction on Diophantine sets that is too weak. For instance, the set of factorials is certainly listable, while there is no obvious reason for it to be Diophantine. It turns out that, actually, there is no other restriction: Diophantine sets over $\mathbb{Z}$ are the same as listable sets!

Theorem 13 (DPRM). *Over $\mathbb{Z}$, listable sets are the same as Diophantine sets.*

Thus, for instance, the sets of factorials, powers of 2, primes, etc. are indeed Diophantine!

The DPRM theorem is named after M. Davis, H. Putnam, J. Robinson, and Y. Matiyasevich. The core of the proof is to use Diophantine sets to simulate *any* Turing machine; Davis, Putnam, and Robinson reduced the problem to showing that the graph of the exponential function $(a,b) \mapsto a^b$ $(a, b \geq 1)$ is Diophantine, while Matiyasevich provided the final step by showing that this is indeed the case [8].

While the knowledge of a precise description of all Diophantine sets is a remarkable achievement in its own right, it was obtained as a tool to give a negative solution to HTP. Let us explain this point:

Proof of Theorem 11 using DPRM. Thanks to a theorem of Turing, the Halting problem from computability theory provides a set $H \subseteq \mathbb{Z}$ which is *listable* but it is *not decidable*: there is no algorithm to decide membership to H.

Nevertheless, H is listable, so it is Diophantine by the DPRM theorem. Let $F \in \mathbb{Z}[x, y_1, \dots, y_n]$ be a polynomial such that H is defined by

$$\phi(x): \quad \exists y_1 \dots \exists y_n, F(x, \mathbf{y}) = 0.$$

Now, for each $a \in \mathbb{Z}$ let $F_a \in \mathbb{Z}[y_1, \dots, y_n]$ be the polynomial $F(a, \mathbf{y})$. For the sake of contradiction suppose that HTP has a positive solution and let $\mathcal{A}$ be an algorithm that decides solvability of Diophantine equations over $\mathbb{Z}$.

Given $a \in \mathbb{Z}$ apply $\mathcal{A}$ to F_a; if the output is YES then $F_a = 0$ has a solution in integers and $\mathbb{Z} \vDash \phi(a)$, hence, $a \in H$. Otherwise, $a \notin H$. This decides membership to H; a contradiction. □

The DPRM theorem is not an isolated result and analogues have been obtained in other contexts. For instance we mention the work of Demeyer over $\mathbb{F}_q[z]$.

Extensions of Hilbert's tenth problem. Let A be a (commutative, unitary) ring which is computable. Roughly speaking, this means that A and its operations can be encoded in $\mathbb{N}$ in a computable way, thus, one can use elements from A as inputs for an algorithm (Turing machine).

Hilbert's tenth problem can be seen as a special case of the following more general question:

Problem 14 ($HTP(A)$)**.** *Is there an algorithm to decide the following? Given a system of polynomial equations over A as input, to decide whether the system has solutions over A.*

In this generality one does not always have a negative solution. However, we are interested in rings A with rich arithmetic, so that one can expect a negative solution to $HTP(A)$.

As we will see, the key to approach the generalized HTP is the notion of a Diophantine set over A: the definition is the same as over $\mathbb{Z}$ but now the polynomials have coefficients in A. Alternatively (and more convenient in practice) we say that $S \subseteq A^m$ is Diophantine (over A) if there is a positive existential formula $\phi(x_1, \dots, x_m)$ over $\mathcal{L}_{ar} = \{0, 1, +, \times, =\}$ with parameters from A (i.e., elements of A are allowed in ϕ) such that for each $\mathbf{a} \in A^m$ we have

$$\mathbf{a} \in S \text{ if and only if } A \vDash \phi(\mathbf{a}).$$

We refer the reader to the excellent surveys [5] and [14] for more information on extensions of HTP, as well as the books [19] and [10]. Here we will only focus in the case of rings of integers and global fields.

The case of rings of integers. A natural extension of Hilbert's tenth problem is to ask the analogous question not only for $\mathbb{Z}$ but also for rings of integers of number fields, such as $\mathbb{Z}[\sqrt{2}]$ or the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$.

Let K be a number field, that is, a finite algebraic extension of $\mathbb{Q}$. Let O_K be its ring of integers. Denef and Lipshitz [1] proposed the following conjecture in the seventies:

Conjecture 15. *For every number field K we have that $HTP(O_K)$ has a negative solution.*

But, how to approach this problem? Should we follow the DPRM strategy by simulating Turing machines over O_K? It turns out that we don't have to reinvent the wheel.

Lemma 16. *Let K be a number field. If $\mathbb{Z} \subseteq O_K$ is Diophantine over O_K, then $HTP(O_K)$ has a negative solution.*

Proof. Assume that $\mathbb{Z}$ is Diophantine in O_K. For the sake of contradiction, suppose that there is an algorithm $\mathcal{A}$ that decides solvability over O_K of systems of polynomial equations over O_K. Take any Diophantine equation $F(x_1, \dots, x_m) = 0$ over $\mathbb{Z}$. Now consider the system S over O_K formed by the equation $F = 0$ seen over O_K and, for each variable x_j, the requirement that $x_j \in \mathbb{Z}$ (which by assumption can be done with Diophantine equations over O_K!).

Apply $\mathcal{A}$ to S. If the output is YES then $F = 0$ has a solution over O_K with all the x_j in $\mathbb{Z}$, and if the output is NO then such a solution does not exist. Thus, we have decided solvability of $F = 0$ over $\mathbb{Z}$. We get a contradiction with the negative solution to $HTP(\mathbb{Z})$. □

So, the key is to show that $\mathbb{Z}$ is Diophantine in O_K. We remark that, actually, it is a theorem of J. Robinson dating back to the fifties that $\mathbb{Z}$ is first order definable in O_K for every number field K; it is the Diophantine definability what remains open.

In the seventies and eighties, Conjecture 15 was proved in the following cases:

- K is totally real or quadratic extension of totally real (Denef–Lipshitz)
- K has exactly one pair of complex conjugate embeddings (Pheidas, Shlapentokh, Videla)
- K is abelian over $\mathbb{Q}$ (Shapiro–Shlapentokh).

After a long hiatus, some new families of cases were proved by Mazur and Rubin in 2015 and by Garcia-Fritz and Pasten in 2019.

In addition, Conjecture 15 is known to follow from standard conjectures on elliptic curves: Mazur and Rubin proved this under the finiteness conjecture for Shaferevich–Tate groups, Murty and Pasten proved this under the BSD conjecture, and most recently Pasten proved this under certain well-known conjecture on ranks of fibres of elliptic surfaces. See [9] and the references therein for more details.

All the recent progress on Conjecture 15 has followed from an elliptic curve criterion proved independently by Poonen and Shlapentokh [18]:

Theorem 17. *Let L/K be an extension of number fields. If there is an elliptic curve E over K with positive rank over K that remains the same over L, then O_K is Diophantine in O_L.*

This result has some predecessors in the work of Denef, Poonen, and Cornelissen–Pheidas–Zahidi. Currently, it is the most promising approach to showing that $\mathbb{Z}$ is Diophantine in O_K for every number field K, which in view of Lemma 16 would prove Conjecture 15.

Finally, we mention a surprising result of Shlapentokh [9] that considerably simplifies the problem:

Theorem 18. *Suppose that for every quadratic extension of number fields L/F one has that O_F is Diophantine in O_L. Then for every number field K we have that $\mathbb{Z}$ is Diophantine in O_K.*

Global fields. A *global field* is a field K which is either a number field (finite extension of $\mathbb{Q}$) or the function field of a curve over a finite field. For the sake of exposition, let us restrict our attention to the two most basic cases: $\mathbb{Q}$ and $\mathbb{F}_q(z)$ for a prime power $q = p^s$ (p prime).

First, we have the following celebrated theorem of Pheidas [11]:

Theorem 19. *If q is odd, then $HTP(\mathbb{F}_q(z))$ has a negative answer.*

(The case of q even is due to Videla.) Pheidas proved this result by constructing an interpretation (in a technical sense) of $(\mathbb{Z}; 0, 1, +, \times, =)$ in $\mathbb{F}_q(z)$ using positive existential formulas. Then one can transfer the negative answer of $HTP(\mathbb{Z})$ to a negative answer for $HTP(\mathbb{F}_q(z))$ very much as in the proof of Lemma 16. The proof of Theorem 19 requires definability of *Frobenius orbits* and of *valuations*. As of today, all the results on Hilbert's tenth problem for positive characteristic function fields require in some way these two key steps. This technique has been generalized by several authors, and in particular, Eisentraeger and Shlapentokh extended Phedias's theorem to all global function fields.

As the arithmetic of $\mathbb{F}_q(z)$ is believed to be analogous to that of $\mathbb{Q}$, one expects that $HTP(\mathbb{Q})$ also has a negative solution, but this remains as a main open problem in the field. As in Lemma 16 the strategy is to show that $\mathbb{Z}$ is Diophantine in $\mathbb{Q}$. There are several partial results in the literature, and we can highlight:

(i) (J. Robinson [17]) There is a first order definition of $\mathbb{Z}$ in $\mathbb{Q}$.

(ii) (Poonen [13]) There is a computable set of primes S with density 1 in the primes such that $HTP(\mathbb{Z}[S^{-1}])$ has a negative solution.

(iii) (Koenigsmann, [6]) The set of non-integers $\mathbb{Q} - \mathbb{Z}$ is Diophantine in $\mathbb{Q}$. Furthermore, there are definitions of $\mathbb{Z}$ in $\mathbb{Q}$ with quantifiers of the form $\forall \cdots \forall$ and $\forall\exists \cdots \exists$ (a Diophantine definition would be of the form $\exists \cdots \exists$.)

On the other hand, it might be the case that $\mathbb{Z}$ is *not* Diophantine in $\mathbb{Q}$, which would require an alternative approach to $HTP(\mathbb{Q})$. First, we have the following conjecture of Mazur proposed in 1992:

Conjecture 20. *Let X be an algebraic variety over $\mathbb{Q}$ and let U be the real closure of its set of rational points $X(\mathbb{Q})$ in $X(\mathbb{R})$. Then U is semi-algebraic; in particular, it has finitely many connected components.*

This conjecture implies at once that $\mathbb{Z}$ is not Diophantine in $\mathbb{Q}$; otherwise, one would get an affine algebraic variety $X \subseteq \mathbb{A}^{n+1}$ such that the projection of $X(\mathbb{Q})$ onto the fist coordinate is $\mathbb{Z}$, contradicting Mazur's conjecture.

In another direction, we have the following conjecture of the author:

Conjecture 21. *Let $D \subseteq \mathbb{Q}$ be a Diophantine set. Then the supremum of D in $\mathbb{R}$, if finite, is algebraic.*

The connection with $HTP(\mathbb{Q})$ is given by the following folklore result:

Lemma 22. *If $\mathbb{Z}$ is Diophantine in $\mathbb{Q}$, then listable sets and Diophantine sets of $\mathbb{Q}$ are the same.*

Thus, if $\mathbb{Z}$ is Diophantine in $\mathbb{Q}$ then the set $D = \{3, 3.1, 3.14, 3.141, \ldots\}$ of increasing decimal approximations of π would be Diophantine, contradicting Conjecture 21.

In addition to the previous conjectures there is the following result of Kollár [7]:

Theorem 23. $\mathbb{C}[z]$ *is not Diophantine in* $\mathbb{C}(z)$.

Thus, if one believes that the arithmetic of $\mathbb{Z}$ in $\mathbb{Q}$ is analogous to that of $\mathbb{C}[z]$ in $\mathbb{C}(z)$, then this can be considered as evidence toward the non-Diophantineness of $\mathbb{Z}$ in $\mathbb{Q}$.

Elementary equivalence versus isomorphism. So far we have discussed the connection between arithmetic and definability of sets *inside* a structure. But first order formulas over a language $\mathcal{L}$ can also be used to define a structure across a class of $\mathcal{L}$-structures.

For instance, let $\mathcal{L}_{gp} = \{e, *, =\}$ be the language of groups and let us define $\mathcal{G}$ as the class of $\mathcal{L}_{gp}$-structures given by groups in the obvious way. Then the trivial group, up to isomorphism, can be defined by the $\mathcal{L}_{gp}$-sentence:

$$\phi : \quad \forall x, x = e.$$

That is, a group $G \in \mathcal{G}$ is isomorphic to the trivial group if and only if $G \vDash \phi$.

Two $\mathcal{L}$-structures that cannot be distinguished by the truth of first order $\mathcal{L}$-sentences are called *elementarily equivalent*: to so say, they have the same theorems (expressible by first order formulas). In our example, the trivial group seen as an $\mathcal{L}_{gp}$-structure is only elementary equivalent to groups that are isomorphic to itself.

While it is obvious that isomorphic $\mathcal{L}$-structures are elementarily equivalent, the converse is false in general. For instance, over the language $\mathcal{L}_{ar} = \{0, 1, +, \times, =\}$ the fields $\overline{\mathbb{Q}}$ and $\mathbb{C}$ are elementarily equivalent (by a result of Tarski) but they are not isomorphic: one is countable and the other is not.

The natural fields that one considers when doing arithmetic geometry are *finitely generated fields*. For instance, global fields such as $\mathbb{Q}$ and $\mathbb{F}_q(z)$ are finitely generated. Let us call this class of fields $\mathcal{F}_{fin}$ and consider them as $\mathcal{L}_{ar}$-structures. The following conjecture seems to have appeared in print by the first time in [15], although it was around already in the seventies:

Conjecture 24. *Within the class $\mathcal{F}_{fin}$ of $\mathcal{L}_{ar}$-structures, isomorphism is the same as elementary equivalence.*

There has been considerable progress on this problem and one can refer to the work of Duret, Pierce, Poonen, Pop, Rumely, Sabbagh, Scanlon, and Vidaux, among others.

A recent preprint by Dittmann and Pop finally proves Conjecture 24, provided that one excludes the fields of characteristic 2, or that one assumes resolution of singularities in characteristic 2. This is very exciting news and it opens the door to other problems in the field. For instance, one can ask what are the subclasses of $\mathcal{F}_{fin}$ that can be defined by a first order formula over $\mathcal{L}_{ar}$ (a question first asked by Poonen), or the same for positive existential formulas.

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Hector Pasten

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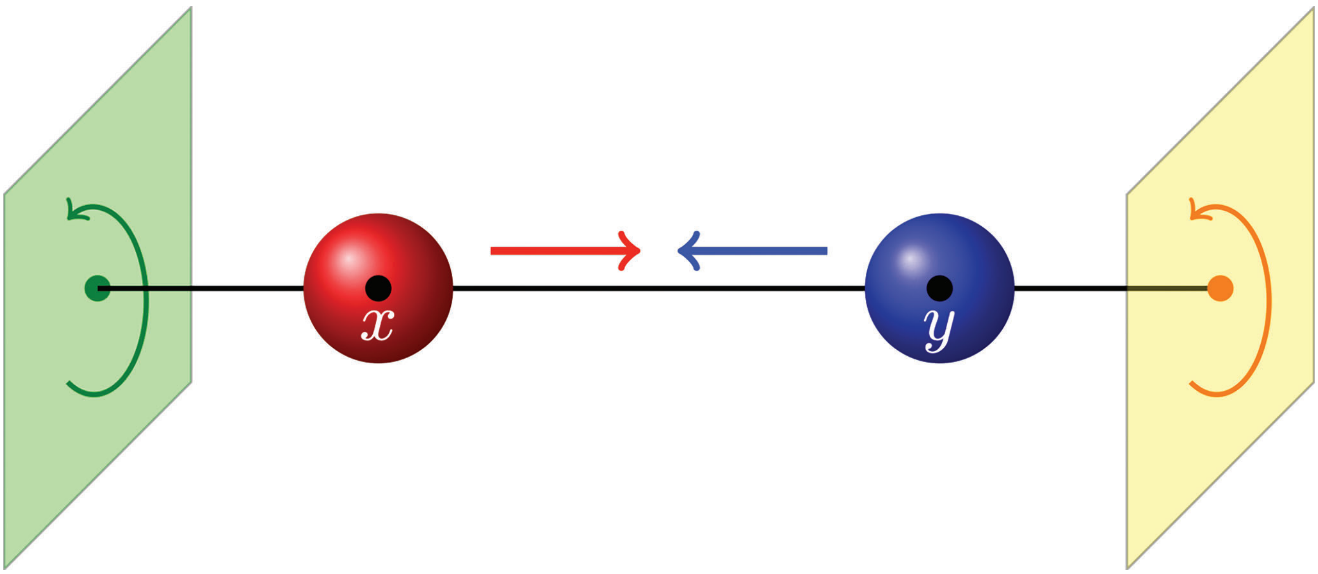
3-Dimensional Mirror Symmetry



Ben Webster and Philsang Yoo

1. Introduction

1.1. The House of Symplectic Singularities. Some have compared research in mathematics to searching through a dark room for a light switch.[1] In other circumstances, it can be like walking through the same house during the day—one can see all the furniture, but can still look through the drawers and cupboards for smaller nuggets of treasure. As enjoyable as such a treasure hunt is (and easier on the shins), discovering new rooms we haven't seen before may lead to even greater rewards. In some fields, this is just a matter of walking down the hall; the hard part is simply knowing which door to open. But even more exciting is finding a secret passage between two rooms we already thought we knew.

Of course, if you are not playing a game of *Clue*, secret passages can be hard to find. You cannot just go tearing out walls and expecting them to be there. However, in the late 20th and early 21st centuries, mathematicians found one remarkable source of such secret passages: quantum field theory (QFT).

What are called "dualities" in QFT often provide connections between mathematical objects that were totally unexpected beforehand. For example, **(2-dimensional) mirror symmetry** has shown that algebraic and symplectic geometers were actually living in the same house, though the passage between them is still quite poorly lit and harder to traverse than we would like. Unfortunately, employing these dualities in mathematics is not just a matter of bringing in a physicist with their x-ray specs; it is more like receiving an incomplete and weather-worn set of blueprints, possibly written in an unknown language, that hint at the right place to look. Still, we get some very interesting hints.

Ben Webster is an associate professor in the Department of Pure Mathematics at the University of Waterloo and an associate faculty at Perimeter Institute. His email address is `ben.webster@uwaterloo.ca`.

Philsang Yoo is an assistant professor in the Department of Mathematical Sciences and a member of the Research Institute of Mathematics at Seoul National University. His email address is `philsang.yoo@snu.ac.kr`.

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[1] *Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated and you can see exactly where you were.* —Andrew Wiles.

For representation theorists, the most splendid and best explored of all mansions is the house of simple Lie algebras; while it is more than a century old, it still has many nooks and crannies with fascinating surprises. It also has a rather innocent-looking little pass-through between rooms, called Langlands duality. After all, it is just transposing the Cartan matrix; most of us cannot keep the Cartan matrix straight from its transpose without looking it up anyway. The Langlands program has revealed the incredible depths of this simple operation.

Many new wings have been found to this manor: Lie superalgebras, representations of algebraic groups in characteristic p, quiver representations, quantum groups, categorification, etc. Despite their diversity, they all rely on the same underlying framework of Dynkin diagrams. But in recent years, researchers have found a new extension more analogous to the discovery of many new series of Dynkin diagrams: the world of **symplectic resolutions** and **symplectic singularities**. According to an oft-repeated *bon mot*, usually attributed to Okounkov: "symplectic singularities are the Lie algebras of the 21st century."

Interesting results about this particular annex started appearing around the turn of the 21st century, based on work of Kaledin, Bezrukavnikov, and others. Some time in 2007, my[2] collaborators Tom Braden, Nick Proudfoot, Tony Licata, and I noticed hints of another secret passage, connecting pairs of rooms (i.e., symplectic resolutions) there. Many coincidences were needed for the different rooms to line up precisely, making space for a secret passage. However, we were not able to step into the passage itself. Nevertheless, we found one very intriguing example: the secret passages we were looking for would generalize Langlands duality to many new examples.

Of course, you can guess from the earlier discussion what happened. After I gave a talk at the Institute for Advanced Study in 2008, Sergei Gukov pointed out to me that physicists already knew that these secret passages should exist based on a known duality: **3-dimensional mirror symmetry**. As explained above, this definitely did not resolve all of our questions; to this day, an explanation of several of the observations we had made remains elusive. More generally, this duality was poorly understood by physicists at the time (and many questions remain), but at least it provided an explanation of why such a passage should exist and a basis to search for it.

In the 15 years since that conversation, enormous progress has been made on the connections between mathematics and 3-dimensional QFT. The purpose of this article is to give a short explanation of this progress and some of the QFT behind it for mathematicians. It is, of necessity, painfully incomplete, but we hope that it will be a useful guide for mathematicians of all ages to learn more.

[2] *All pronouns in this section are from the perspective of BW.*

1.2. Plan of the paper.

Let us now discuss our plan with a bit more precise language. A **symplectic resolution** is a pair consisting of

1. a singular affine variety X_0; and
2. a smooth variety X with an algebraic symplectic form which resolves the singularities of X_0.

The singular affine variety X_0 is a special case of a **symplectic singularity**, which is a singular affine variety where the smooth locus is equipped with a symplectic form that is well-behaved at singularities.

The most famous example of a symplectic resolution is the **Springer resolution**, where X_0 is the variety of nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$, and X is the cotangent bundle of the flag variety of $\mathfrak{g}$. You can reconstruct $\mathfrak{g}$ from the geometry of this resolution. Thus, one perspective on the house of simple Lie algebras is that the Springer resolution is really the fundamental object in each room of a simple Lie algebra, with all other aspects of Lie theory determined by looking at the Springer resolution from various different angles.

Thus, simple Lie algebras lie at one end of a hallway, with many other doors that lead to other symplectic resolutions and singularities. This leads to the natural question of whether any given notion for Lie algebras generalizes to other symplectic resolutions if we treat them like the Springer resolution of a new Lie algebra that we have never encountered. For example, each symplectic singularity has a "universal enveloping algebra" which generalizes the universal enveloping algebra of a Lie algebra.

Two examples accessible to most mathematicians are:

- The cotangent bundle $X_n^{(A)} = T^*\mathbb{CP}^{n-1}$ of complex projective space. This can be written as
$$T^*\mathbb{CP}^{n-1} = \{(\ell,\phi) \in \mathbb{CP}^{n-1} \times M_{n\times n}(\mathbb{C}) \mid \phi(\mathbb{C}^n) \subset \ell, \phi(\ell) = \{0\}\}.$$
Projection to the second component is a resolution of $M^{\mathrm{rk}\le 1}_{n\times n}(\mathbb{C})$, the space of $n\times n$ matrices of rank ≤ 1. This cotangent bundle has a canonical symplectic form, which makes this resolution symplectic.
- The cyclic group $\mathbb{Z}/n\mathbb{Z}$ acts on $\mathbb{C}^2$, preserving its canonical symplectic form, by the matrices
$$k \mapsto \begin{bmatrix} \exp(2\pi i k/n) & 0 \\ 0 & \exp(-2\pi i k/n) \end{bmatrix}.$$
The quotient $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ has a unique symplectic resolution $X_n^{(B)}$ whose exceptional fiber is a union of $n-1$ copies of $\mathbb{CP}^1$'s that form a chain.

We have an isomorphism $X_2^{(A)} \cong X_2^{(B)}$, but for $n > 2$, these varieties have different dimensions. There are some intriguing commonalities when we look at certain combinatorial information coming out of these varieties. Central to this are two geometric objects:

- The action of a maximal torus $T^{(*)}$ on $X_n^{(*)}$ for $* \in \{\mathrm{A}, \mathrm{B}\}$ which preserves the symplectic structure. One obvious invariant is the set of its fixed points of this torus.[3]
- The affine variety X_0 has a unique minimal decomposition into finitely many smooth pieces with induced symplectic structures, generalizing the decomposition of nilpotent matrices into Jordan type.

There are some intriguing coincidences between this pair of varieties:

1. We have isomorphisms
$$\mathfrak{t}^{(\mathrm{A})} \cong H^2(X_n^{(\mathrm{B})}) \qquad \mathfrak{t}^{(\mathrm{B})} \cong H^2(X_n^{(\mathrm{A})}).$$
We can make this stronger by noting that we match geometrically defined hyperplane arrangements on these spaces.[4]
2. Both torus actions have the same number of fixed points, which is n; this also shows that the sum of the Betti numbers of $X_n^{(*)}$ is n.
3. The stratifications on $X_0^{(\mathrm{A})}$ and $X_0^{(\mathrm{B})}$ have the same number of pieces, which is 2.[5]

It would be easy to dismiss these as not terribly significant, but they are numerical manifestations of a richer phenomenon. That is,

4. the "universal enveloping algebra" of $X_n^{(*)}$ has a special category of representations that we call "category $\mathcal{O}$" (see [BLPW16, §3]) and the categories $\mathcal{O}$ of $X_n^{(\mathrm{A})}$ and $X_n^{(\mathrm{B})}$ are **Koszul dual**; the homomorphisms between projective modules in one category describe the extensions between simple modules in the other.

The other reason that we should not dismiss these "coincidences" is that the same statements 1.–4. apply to many pairs of symplectic singularities, which are discussed in [BLPW16, §9]. These include all finite and affine type A quiver varieties and smooth hypertoric varieties. Some examples are self-dual:

- $Y_n^{(\mathrm{A})} = Y_n^{(\mathrm{B})} = T^*\,\mathrm{Fl}_n$, the cotangent bundle of the variety of complete flags in $\mathbb{C}^n$.
- $Z_n^{(\mathrm{A})} = Z_n^{(\mathrm{B})} = \mathrm{Hilb}^n(\mathbb{C}^2)$, the Hilbert scheme of n points in $\mathbb{C}^2$.

After suitable modification[6] of 3., it also includes the Springer resolutions of Langlands dual pairs of Lie algebras.

This mysterious duality on the set of symplectic singularities and their resolutions has obtained the name of "symplectic duality" for its connection of two apparently unrelated symplectic varieties.

Question 1.1. Is there an underlying principle that explains statements 1.-4., that is, which explains the symplectic duality between these pairs of varieties?

As discussed above, work on QFT in dimension 3 suggests that the answer to this question is closer to "yes" than it is to "no." Our aim in this article is to explain the basics of why this is so and what it tells us about mathematics.

We can break this down into two sub-questions:

Q1. What are 3d $\mathcal{N} = 4$ SUSY QFTs and their topological twists?

Q2. What do they have to do with symplectic duality?

In Section 2, we will provide an answer to the questions, which we now briefly summarize.

First, every 3-dimensional topological quantum field theory (TQFT) gives us a Poisson algebra. In many cases, this ring is the coordinate ring of a symplectic singularity X_0, and all the examples discussed above can be constructed in this way. Given a QFT, a choice of a topological twist gives rise to a TQFT. In fact, for a 3d $\mathcal{N} = 4$ theory $\mathcal{T}$, there are two such choices, called the A-twist and the B-twist. Hence each 3d theory gives two symplectic singularities $\mathfrak{M}_A(\mathcal{T})$ and $\mathfrak{M}_B(\mathcal{T})$ called the **Coulomb branch** and **Higgs branch** of the theory.

The pairs of symplectic varieties $X^{(\mathrm{A})}$ and $X^{(\mathrm{B})}$ (similarly, Y, Z, etc.) all turn out to be the Coulomb and Higgs branches of a single theory $\mathcal{T}$. Then statements 1.–4. can be understood in terms of the physical duality referenced in Section 1.1, called "3-dimensional mirror symmetry."

This is a very large topic, and due to constraints on the length and number of references, we will concentrate on the relationship to symplectic resolutions of singularities, giving relatively short shrift to the long and rich literature in physics on the topic; the introduction of [BDGH16] will lead the reader to the relevant references, starting from the original work of Intrilligator–Seiberg and Hanany–Witten, which laid the cornerstone of this theory.

Just as the 2-dimensional mirror symmetry known to mathematicians suggests that complex manifolds and symplectic manifolds (with extra structure) come in pairs whose relationship is hard to initially spot, 3-dimensional mirror symmetry rephrases our answer to Question 1.1: the Coulomb branch of one theory can also be thought of as the Higgs branch of its dual theory: $\mathfrak{M}_A(\mathcal{T}) = \mathfrak{M}_B(\mathcal{T}^\vee)$. Thus, we can also describe our dual pairs of symplectic varieties as the Higgs branches of dual theories $(\mathfrak{M}_B(\mathcal{T}), \mathfrak{M}_B(\mathcal{T}^\vee))$.

[3] *$T^{(A)}$ is the diagonal matrices in $PGL_n(\mathbb{C})$; $T^{(B)}$ the diagonal matrices in $SL_2(\mathbb{C})$ modulo the n torsion.*

[4] *In $\mathfrak{t}$, the vectors where the vanishing set of the corresponding vector field jumps in dimension; in $H^2(X_n^{(B)})$, the Mori walls that cut out the ample cones of the different crepant resolutions of the same affine variety.*

[5] *The smooth locus is one stratum, and in both cases, the other one is a single point.*

[6] *In this case, the strata are the adjoint orbits of nilpotent elements, and the number of these is different for types B_n and C_n. We can recover a bijection by only considering* **special orbits**, *of which there are the same number.*

This answer is not as complete as we would like, since we cannot construct 3-dimensional QFTs as rigorous mathematical objects. We can only work with mathematical rigor on certain aspects of some classes of theories, the most important of which are **linear gauge theories**. In these cases, we have mathematical definitions of the Higgs and Coulomb branches and thus can prove mathematical results about them.

In Section 3, we will review these constructions of the Higgs and Coulomb branches in the case of linear gauge theories. The former of these constructions has been known to mathematicians for many decades [HKLR87], but the construction of Coulomb branches was a surprise even to physicists when it appeared in 2015 [BFN18], and is key to the progress we have made since that time.

These varieties are the keystones of a rapidly developing research area that combines mathematics and physics. In particular, they point the way to understanding a mirror symmetry of 3-dimensional theories that is not only a counterpart to the mirror symmetry known to mathematicians (which is 2-dimensional mirror symmetry) but also provides an enrichment of the geometric Langlands program (which comes from a duality of 4-dimensional theories).

We will conclude the article in Section 4 with a brief discussion of interesting directions of current and future research to give the interested reader guidance on where to turn next.

2. Physical Origin

2.1. QFT. In this section, we will give a very short introduction to (Euclidean) QFT. Typically, a QFT has the following input data:

1. (**spacetime**) a d-dimensional Riemannian manifold (M, g);
2. (**fields**) a fiber bundle B over M and the space $\mathcal{F} = \mathcal{F}(M) = \Gamma(M, B)$ of sections of B over M;
3. (**action functional**) a functional $S : \mathcal{F} \to \mathbb{R}$.

In very rough terms, $\mathcal{F}$ should be viewed as the space of all possible states of a physical system, while the function S controls which states will likely be physically achieved.

In a classical physical system, we want to think about measuring quantities, such as the velocity or position of a particle. We can formalize this in the notion of an **observable**, which is, by definition, a functional $O : \mathcal{F} \to \mathbb{R}$. A particularly important type is **local operators** at x that depend only on the value of a field or its derivatives at x.

Example 2.1 (Free scalar field theory).

1. a (compact) Riemannian manifold (M, g);
2. $B = M \times \mathbb{R}$ so that $\mathcal{F} = C^\infty(M)$;
3. $S : C^\infty(M) \to \mathbb{R}$ given by $S(\phi) = \int_M \phi \Delta_g \phi \mathrm{Vol}_g$, where Δ_g is the Laplacian of the metric g and Vol_g is the volume form associated to g.

In the case of $M = \mathbb{R}$, for any point $x \in \mathbb{R}$, the functionals $O_x, O_x^{(1)} : C^\infty(\mathbb{R}) \to \mathbb{R}$ defined by $O_x(\phi) = \phi(x)$ and $O_x^{(1)}(\phi) = \phi'(x)$ are local operators at x.

Two other types of field theories play an important role for us:

1. Let G_c be a compact Lie group. When $\mathcal{F}$ consists of connections on a principal G_c-bundle over M, such a field theory is called a **gauge theory** and G_c is called the **gauge group** of the theory.
2. Let X be a manifold. When $\mathcal{F}$ consists of maps from M to X, such a theory is called a σ-**model** and X is called the **target** of the σ-model. In this case, $B = M \times X$.

One insight of the quantum revolution in physics is that a physical system cannot be described by a single field, which would have a well-defined value for each observable. Instead, we can only find the expectation values of observables as integrals, where a measure depending on the action accounts for how probable states are. These integrals are often written notionally in the form

$$\langle O \rangle := \int_{\phi \in \mathcal{F}(M)} e^{-S(\phi)/\hbar} O(\phi) D\phi.$$

However, in many cases, these integrals do not make sense because the space $\mathcal{F}(M)$ is often infinite-dimensional, and as a result, the Lebesgue measure $D\phi$ cannot be defined.

More generally, given observables O_i's which only depend on the values of the fields on open sets that do not overlap, we consider the integrals of the following form

$$\langle O_1, \cdots, O_n \rangle := \int_{\phi \in \mathcal{F}(M)} e^{-S(\phi)/\hbar} O_1(\phi) \cdots O_n(\phi) D\phi.$$

These are called the **correlation functions** of the theory and the main objects of study in a QFT. One may also understand the integral as the correlation function of a single observable, as the notion of **operator product** allows one to express products of $O_1, \cdots, O_n$ as a single observable.

2.2. TQFT. In the framework of Atiyah and Segal, a d-dimensional **topological quantum field theory (TQFT)** is a symmetric monoidal functor Z from the category $(\mathrm{Bord}_d, \amalg, \emptyset)$ to the category $(\mathrm{Vect}_{\mathbb{C}}, \otimes, \mathbb{C})$ of complex vector spaces. Objects of Bord_d are closed oriented $(d-1)$-manifolds N, a morphism from N to N' is a diffeomorphism class of a d-dimensional bordism M from N to N', and the monoidal structure is given by disjoint union $\amalg$ with the empty set $\emptyset$ being the unit object.

Regarding a closed d-manifold M as a bordism from $\emptyset$ to $\emptyset$ yields a complex number $Z(M)$. Physically, one should imagine that $Z(M) = \int_{\phi \in \mathcal{F}(M)} e^{-S(\phi)/\hbar} D\phi$. On the other hand, the complex vector space $Z(N)$ attached to a closed

$(d-1)$-manifold N is the Hilbert space of states on N. The most important case is that of $N = S^{d-1}$. In this case, the vector space $Z(S^{d-1})$ will be the vector space of local operators in the TQFT. The principle that these spaces coincide is called "the state-operator correspondence."

Suppose $d = 2$. Since any closed oriented 1-manifold is a disjoint union of copies of circles, it is enough to describe $Z(S^1)$. Moreover, the map associated to a pair of pants yields a linear map $m\colon Z(S^1) \otimes Z(S^1) \to Z(S^1)$ and the one associated to a disk is a linear map $u\colon Z(\emptyset) \to Z(S^1)$:

pair of pants disk

Topological arguments show that these maps and others from the reversed picture induce a commutative Frobenius algebra structure on $Z(S^1)$.

Note that one can apply a similar idea to any d-dimensional TQFT Z to show that $Z(S^{d-1})$ obtains a commutative algebra structure for $d \geq 2$ using the cobordism where we remove two disjoint d-balls from the interior of a d-ball. When we interpret $Z(S^{d-1})$ as the space of local operators of the theory, this product has a physical meaning: it is precisely the operator product introduced above.[7]

Since this is a commutative $\mathbb{C}$-algebra, $Z(S^{d-1})$ can be interpreted as the coordinate ring of an algebraic variety. In fact, the spectrum $\mathfrak{M} = \operatorname{Spec} Z(S^{d-1})$ has a physical interpretation as well: it is the moduli space of vacua of the theory. This reflects the fact that at a vacuum state, which by definition is a linear map $\langle - \rangle\colon Z(S^{d-1}) \to \mathbb{C}$, measurements at distant points cannot interfere so that $\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle$. Thus, $O \mapsto \langle O \rangle$ defines a ring map $Z(S^{d-1}) \to \mathbb{C}$, and a point in the spectrum.

In many examples of applications of the idea of physics to mathematics, the perspective of TQFT provides a useful guiding principle. Before discussing how to use the idea, let us explain how one may obtain a TQFT starting from a QFT.

2.3. From QFT to TQFT. There are two well-known ways to construct a TQFT, that is, a theory which is independent of a metric of the spacetime manifold. One is to begin with a space of fields and action functional which do not depend on a metric. For example, Chern–Simons theory is one such theory. This approach is quite limited and leads to relatively few examples. Many more examples arise from applying a **topological twist** to a supersymmetric field theory (which depends on a metric). Let us briefly review the latter idea.

Consider $M = \mathbb{R}^d$ with the standard metric. In this case, the isometry group $\mathrm{ISO}(d, \mathbb{R}) = \mathrm{SO}(d, \mathbb{R}) \ltimes \mathbb{R}^d$ is called the **Poincaré group** and acts on $\mathbb{R}^d$ by rotation and translation. We will only consider field theories on $\mathbb{R}^d$ where the action functional is equivariant under the induced action on the space of fields.

We will also only consider theories where the space of sections $\mathcal{F}$ is $\mathbb{Z}/2\mathbb{Z}$-graded; this arises physically from the spin angular momentum of particles, and thus the natural classifications of particles into bosons (even) and fermions (odd). We call a field theory **supersymmetric (SUSY)** if it admits nontrivial "odd symmetries," which one calls **supercharges**.

More precisely, this means that the space $\mathcal{F}$ carries an action of a Lie superalgebra $\mathfrak{a}$ called a **super-Poincaré algebra** whose even part is the Poincaré algebra $\mathfrak{a}_0 = \mathfrak{so}(d) \ltimes \mathbb{R}^d$ and whose odd part $\mathfrak{a}_1 = \Sigma$ consists of copies of spin representations of $\mathfrak{so}(d)$. A Lie bracket is given by the action of $\mathfrak{so}(d)$ on Σ, as well as a symmetric[8] pairing $\Gamma\colon \Sigma \otimes \Sigma \to \mathbb{R}^d$ of $\mathfrak{so}(d)$-representations.

For simplicity, we work with a complexification of the supersymmetry algebra from now on; this is mostly harmless for the purpose of discussing twists.

Example 2.2.

1. The $d = 2$, $(\mathcal{N}_+, \mathcal{N}_-)$ supersymmetry algebra $\mathfrak{a}_{d=2}$ has odd part $S_+ \otimes W_+ \oplus S_- \otimes W_-$, where $S_\pm \cong \mathbb{C}$ are the two spin representations of $\mathfrak{so}(2)$ and $\dim W_\pm = \mathcal{N}_\pm$. The pairing $\Gamma\colon \Sigma \otimes \Sigma \to \mathbb{C}^2$ is induced by the isomorphism $S_+^{\otimes 2} \oplus S_-^{\otimes 2} \cong \mathbb{C}^2$ as $\mathfrak{so}(2)$-representations.
2. The $d = 3$, $\mathcal{N} = k$ supersymmetry algebra $\mathfrak{a}_{d=3}$ has odd part $S \otimes W$, where $S \cong \mathbb{C}^2$ is the spin representation of $\mathfrak{so}(3)$ and $\dim W = \mathcal{N}$. The pairing $\Gamma\colon \Sigma \otimes \Sigma \to \mathbb{C}^3$ is induced by the isomorphism $\mathrm{Sym}^2 S \cong \mathbb{C}^3$ as $\mathfrak{so}(3)$-representations.

Finally, in order to extract a TQFT from a SUSY theory, suppose that one has chosen a supercharge Q of a SUSY algebra such that $[Q, Q] = 0$. Since Q is odd, this means $\frac{1}{2}[Q, Q] = Q^2$ acts as zero in any representation of $\mathfrak{a}$. Hence, one can consider $\mathcal{F}$ or even $\mathfrak{a}$ itself as a $\mathbb{Z}/2\mathbb{Z}$ graded complex, and take its Q-cohomology. Necessarily, this procedure results in a simpler theory, which one calls a **twist** or a **twisting**.

If an element $x \in \mathbb{R}^d \subset \mathfrak{a}$ is in the image of $[Q, -]$, then translation by x will be trivial in the twisted theory. The most important case for us is if the image of Q fills in all of $\mathbb{C}^d$. In this case, the dependence on position vanishes and the theory becomes topological; consequently, the twisted theory is called a **topological twist** of the original theory.

[7] *An important warning for the reader: we will considering topological twists of QFTs below, which do not always produce TQFTs in the framework above, since the maps defined by some cobordisms may not converge. For example, $Z(S^{d-1})$ will not be finite-dimensional in the examples we consider.*

[8] *In the world of super Lie algebras, a Lie bracket is symmetric if both inputs are odd!*

Whether a topological twist exists is purely dependent on the super-Poincaré algebra $\mathfrak{a}$, and thus on d and $\mathcal{N}$. Let $\mathfrak{a}_{d=3}$ (resp. $\mathfrak{a}_{d=2}$) be the supersymmetry algebra with $d = 3$ (resp. $d = 2$) and $\mathcal{N} = n$ (resp. $\mathcal{N} = (n_+, n_-)$) supersymmetry. By a standard argument (see, e.g., [ESW22, §§11.2 & 12.1]), we have:

- In the case $d = 3$ (resp. $d = 2$) there is a topological twist if and only if $n \geq 4$ (resp. $n_\pm \geq 2$).
- In the case where $n = 4$ (resp. $n_\pm = 2$), there are exactly 2 topological twists up to appropriate symmetry, which we denote by Q_{A} and Q_{B}.

2.4. **Mirror symmetry.** When X is a Calabi–Yau manifold, there is a physics construction of a 2-dimensional $\mathcal{N} = (2,2)$ SUSY σ-model $\mathcal{T}^{d=2}(X)$ with target X. If we twist with respect to $Q_{\mathrm{A/B}}$, the resulting TQFT is called the A/B-model $\mathcal{T}^{d=2}_{\mathrm{A/B}}(X)$. The A model depends on the symplectic topology of X, and the B-model on the complex geometry of X.

There is a remarkable duality, called **mirror symmetry**, on the set of such SUSY σ-models, which identifies $\mathcal{T}^{d=2}(X)$ and $\mathcal{T}^{d=2}(X^\vee)$ for another **mirror dual** Calabi–Yau manifold $X^\vee$. Moreover, this duality is compatible with topological twists: the identification of $\mathcal{T}^{d=2}(X)$ and $\mathcal{T}^{d=2}(X^\vee)$ is compatible with an involution of the $d = 2$, $\mathcal{N} = (2,2)$ SUSY algebra which exchanges Q_{A} and Q_{B}. Therefore, the $d = 2$ TQFTs $\mathcal{T}^{d=2}_{\mathrm{A}}(X)$ and $\mathcal{T}^{d=2}_{\mathrm{B}}(X^\vee)$ should be equivalent. This idea has resulted in several marvelous predictions. The most famous is that the numbers of rational curves of degree d on a quintic 3-fold, understood as the correlation functions of $\mathcal{T}^{d=2}_{\mathrm{A}}(X)$, should be equal to the correlation functions of $\mathcal{T}^{d=2}_{\mathrm{B}}(X^\vee)$, which can be more easily computed.

The remarkable success of $d = 2$ mirror symmetry motivates the consideration of an analogous duality, called **3d mirror symmetry**, for $d = 3$, $\mathcal{N} = 4$ SUSY field theories, which identifies two superficially different theories, say $\mathcal{T}$ and $\mathcal{T}^\vee$. Just as before, there are still two interesting topological twists Q_{A} and Q_{B} in the super-Poincaré algebra, and an automorphism of $\mathfrak{a}$ which switches these. By the same logic, we have an equivalence of topologically twisted theories between $\mathcal{T}_{\mathrm{A}}$ and $\mathcal{T}^\vee_{\mathrm{B}}$, which we write $Z^{\mathcal{T}}_{\mathrm{A}}$ and $Z^{\mathcal{T}^\vee}_{\mathrm{B}}$, respectively, to emphasize the TQFT perspective.

We will focus on understanding the algebras $\mathcal{A}_{\mathrm{A/B}}(\mathcal{T}) = Z^{\mathcal{T}}_{\mathrm{A/B}}(S^2)$. As discussed in Section 2.2, the algebraic varieties

$$\mathfrak{M}_{\mathrm{A}}(\mathcal{T}) = \operatorname{Spec} Z^{\mathcal{T}}_{\mathrm{A}}(S^2) \qquad \mathfrak{M}_{\mathrm{B}}(\mathcal{T}) = \operatorname{Spec} Z^{\mathcal{T}}_{\mathrm{B}}(S^2)$$

are the moduli spaces of vacua of the respective theories. We will call these the **Coulomb branch** $\mathfrak{M}_{\mathrm{A}}(\mathcal{T})$ and **Higgs branch** $\mathfrak{M}_{\mathrm{B}}(\mathcal{T})$ of the theory $\mathcal{T}$. Of course, the identification of local operators $\mathcal{A}_{\mathrm{A}}(\mathcal{T})$ in one theory with $\mathcal{A}_{\mathrm{B}}(\mathcal{T}^\vee)$ in the mirror theory is one of the most important features of mirror symmetry in this case as well:

$$\mathfrak{M}_{\mathrm{A}}(\mathcal{T}) \cong \mathfrak{M}_{\mathrm{B}}(\mathcal{T}^\vee) \qquad \mathfrak{M}_{\mathrm{B}}(\mathcal{T}) \cong \mathfrak{M}_{\mathrm{A}}(\mathcal{T}^\vee).$$

Thus we call the varieties $\mathfrak{M}_{\mathrm{A}}(\mathcal{T})$ and $\mathfrak{M}_{\mathrm{B}}(\mathcal{T})$ **mirror** to each other, or symplectic duals in the terminology of [BLPW16]. These varieties have the virtue of being familiar types of mathematical objects, while still carrying much of the structure of the theory $\mathcal{T}$.

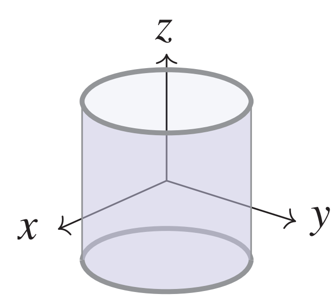
3. Higgs and Coulomb Branches

This section focuses on the Coulomb and Higgs branches in one particularly important case: the $d = 3$, $\mathcal{N} = 4$ SUSY σ-model into $\mathbb{H}^n$, gauged by the action of a subgroup $G_c \subset U(n, \mathbb{H})$. The fields corresponding to the map to $\mathbb{H}^n$ are often called the "matter content" of the theory. It is often more convenient to forget the coordinates on $\mathbb{H}^n$ and think of it as a general $\mathbb{H}$-module X with a choice of norm and an action of G_c. It will also simplify things for us to consider X as a $\mathbb{C}$-vector space with complex structure I and the induced action of the complexification G of G_c; we can encode the action of the quaternions J and K in the holomorphic symplectic form $\Omega(x, y) = \langle Jx, y\rangle + i\langle Kx, y\rangle$. We will denote the corresponding theory by $\mathcal{T}(X, G)$ and denote the Higgs and Coulomb branches by $\mathfrak{M}_{\mathrm{A/B}}(X, G)$.

Both of these varieties have concrete mathematical descriptions, which we will describe here as best we can in limited space. Both can be derived from manipulations in infinite-dimensional geometry, using the principle that the Hilbert space of a physical theory is obtained by geometric quantization of the phase space of the theory. This geometric quantization is easiest if $X = T^*N$ for a G-representation N. In incredibly rough terms, this phase space comes from maps of S^2 into the cotangent bundle of the quotient N/G satisfying certain properties. These are easiest to explain if we deform our S^2 to be the boundary of the cylinder

$$\mathsf{Cyl}_\delta = \{(x, y, z) \mid x^2 + y^2 \leq \delta, |z| \leq \delta\}$$

for some real number $\delta > 0$.



We will frequently refer to the top, bottom, and sides of this cylinder, by which we mean the unit disks in the $z = \delta, -\delta$ planes, and the portion of the boundary in between.

(A) The algebra $\mathcal{A}_{\mathrm{A}}$ is the algebra of *locally constant* functions on the space of maps of the cylinder to N/G which are constant on the sides and *holomorphic* on the top and bottom.
(B) The algebra $\mathcal{A}_{\mathrm{B}}$ is the algebra of *holomorphic* functions on the space of maps of the cylinder to N/G which are constant on the sides and *locally constant* on the top and bottom.

We have phrased this to emphasize the parallelism, that is, how the difference between the A- and B-twists is reflected by the placement of "locally constant" and "holomorphic." In the sections below, we will unpack more carefully how we interpret the concepts in the formulations (A) and (B), since some generalization is necessary.

3.1. **Higgs branches.** First, we consider the B-twist. While second in alphabetical order, the associated Higgs branch is easier to precisely understand, and thus generally attracted more attention in the mathematical literature. According to the description (B), the algebra $\mathcal{A}_{\mathrm{B}}$ should be functions on constant maps $S^2 \to N/G$. Here in addition to a point in N/G, one should also consider a covector to this quotient (see [BF19, §7.14]).

We can define this more concretely using the moment map $\mu : X \to \mathfrak{g}^*$ of G on the symplectic vector space (X, Ω). If $X = T^*N$, then $\mu^{-1}(0)$ consists of all pairs of $n \in N$ and covectors ξ that vanish on the tangent space to the orbit through n (and thus can be considered covectors on the quotient).

Definition 3.1. The Higgs branch $\mathfrak{M}_{\mathrm{B}}(X, G)$ is defined as a **holomorphic symplectic quotient**, that is, one has $\mathcal{A}_{\mathrm{B}}(X,G) = Z_{\mathrm{B}}(S^2) = \mathbb{C}[\mu^{-1}(0)]^G$, the complex polynomial functions on $\mu^{-1}(0)$ which are G-invariant, and $\mathfrak{M}_{\mathrm{B}} = \operatorname{Spec} \mathcal{A}_{\mathrm{B}}$. The points of this space are in bijection with closed G-orbits in $\mu^{-1}(0)$.

The resulting variety is typically singular symplectic. One can reasonably ask if this variety has a symplectic resolution. This does not happen in all cases, but in some cases it does.

There are two particular examples that we will focus on in this article: abelian and quiver gauge theories. In both cases, the target space $X = T^*N$ is the cotangent bundle of a $\mathbb{C}$-representation N of G.

3.1.1. *Abelian/hypertoric gauge theories.* Assume that G is abelian. Since it is connected and reductive, this means $G \cong (\mathbb{C}^\times)^k$ for some k. For any $\mathbb{C}$-representation N of G, we can choose an isomorphism $N \cong \mathbb{C}^n$ such that $G \hookrightarrow D$ is a subgroup of the full group D of $n\times n$ diagonal matrices.

These ingredients are typically used in the construction of a toric variety: the GIT quotient $N /\!\!/_\lambda G$, at any regular value of the moment map will give a quasi-projective toric variety for the action of the quotient $F = D/G$.

The construction of the Higgs branch $\mathfrak{M}_{\mathrm{B}}(T^*N, G)$ of this theory is thus a quaternionic version of the construction of toric varieties. The resulting variety is called a **hypertoric variety** or **toric hyperkähler variety**. This variety has complex dimension $2(\dim N - \dim G)$. Probably the most familiar examples for readers are the following:

Example 3.2. Let $G = \{\varphi I \mid \varphi \in \mathbb{C}^\times\} \subset D$ be the scalar $n\times n$ matrices. We can consider the elements of T^*N as pairs of an $n\times 1$ column vector $\mathbf{a}$ and a $1\times n$ row vector $\mathbf{b}$, with the group $G \cong \mathbb{C}^\times$ acting by

$$\varphi \cdot (\mathbf{a}, \mathbf{b}) = (\varphi \mathbf{a}, \varphi^{-1}\mathbf{b}).$$

The outer product $\mathbf{ab}$ is thus an $n \times n$ matrix of rank ≤ 1, invariant under the action of G. Thus, $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{ab}$ defines a map $\mathfrak{M}_{\mathrm{B}}(T^*\mathbb{C}^n, G) \to M_{n\times n}(\mathbb{C})$. The moment map $\mu(\mathbf{a}, \mathbf{b}) = \mathbf{ba}$ is defined by the dot product, so $\mu(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{ab}$ is nilpotent. Thus, we have a map $\mathfrak{M}_{\mathrm{B}}(T^*\mathbb{C}^n, G) \to M^{\mathrm{rk}\leq 1}_{n\times n}(\mathbb{C})$ to the space of nilpotent matrices of rank ≤ 1.

The G-orbit through $(\mathbf{a}, \mathbf{b})$ is closed if and only if $\mathbf{a}$ and $\mathbf{b}$ are both nonzero or both zero; you can see from this that the map above is a bijection. Thus we find $\mathfrak{M}_{\mathrm{B}}(T^*\mathbb{C}^n, G) \cong M^{\mathrm{rk}\leq 1}_{n\times n}(\mathbb{C})$.

Example 3.3. Let $G \subset D$ be the diagonal matrices of determinant 1. We can again think of T^*N as pairs $(\mathbf{a}, \mathbf{b})$. In this case, the moment map condition guarantees that $a_i b_i = a_j b_j$ for all i, j, and the closed orbit condition that if $a_i = 0$ for some i, then $a_j = 0$ for all other j, and similarly with b_*'s. By multiplying with a diagonal matrix, we can assume that $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$ as well. This defines a surjective map $\mathbb{C}^2 \to \mathfrak{M}_{\mathrm{B}}$, sending (x, y) to $\mathbf{a} = (x, \dots, x)$ and $\mathbf{b} = (y, \dots, y)$. However, this is not injective: the diagonal matrices $e^{2\pi i k/n} I$ have determinant 1 and define an action of the cyclic group $\mathbb{Z}_n$ on the image of this map from $\mathbb{C}^2$. This matches the action of the matrix $\operatorname{diag}(e^{2\pi i k/n}, e^{-2\pi i k/n})$ on $\mathbb{C}^2$. Thus, we find $\mathfrak{M}_{\mathrm{B}}(T^*\mathbb{C}^n, G) \cong \mathbb{C}^2/\mathbb{Z}_n$.

While the hypertoric varieties for other tori are less familiar and more complicated, they still have a very combinatorial flavor, and typically questions about them can be reduced to studying hyperplane arrangements, much as toric varieties can be studied using polytopes. Notably, they all possess symplectic resolutions, constructed with GIT quotients or equivalently hyperhamiltonian reduction at nonzero moment map values. For Example 3.2 above, this resolution is $T^*\mathbb{CP}^{n-1}$ and for Example 3.3, it is the unique crepant resolution obtained by iterated blowups at singular points.

3.1.2. *Quiver gauge theories.* The most famous examples of these reductions are **Nakajima quiver varieties**. That is, we fix a directed graph Γ, and a pair of vectors $\mathbf{v}, \mathbf{w}$ whose components are indexed by the vertex set I. The group

$G = \prod_{i \in I} \mathrm{GL}(v_i; \mathbb{C})$ has representations $\mathbb{C}^{v_i}$ for each $i \in I$. The representation we will consider is

$$N = \Big(\bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})\Big) \oplus \Big(\bigoplus_{i \in I} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})\Big).$$

We want a left group action, so $(A, B) \in \mathrm{GL}(v_i; \mathbb{C}) \times \mathrm{GL}(v_j; \mathbb{C})$ acts on $M \in \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$ by BMA^{-1}. We call gauge theories $\mathcal{T}(T^*N, G)$ for this choice of G and N **quiver gauge theories**, and the Higgs branches $\mathfrak{M}_{\mathrm{B}}(T^*N, G)$ are Nakajima quiver varieties. These are geometric avatars of the μ weight space of a representation of highest weight λ for the Kac–Moody algebra with Dynkin diagram Γ.

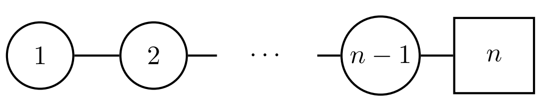
Physicists will typically draw two copies of each node, one in a circle filled with v_i, and one in a square filled with w_i, and draw in the edges of Γ between the first copies, and then edges between the circle and square copies of vertex (not drawing vertices with 0's).

Example 3.4. If we have a quiver with a single vertex so that we have a single v and w, then $N = \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^w)$, and $X = T^*N = \mathrm{Hom}(\mathbb{C}^v, \mathbb{C}^w) \oplus \mathrm{Hom}(\mathbb{C}^w, \mathbb{C}^v)$, that is a pair of matrices (A, B) which are $w \times v$ and $v \times w$ respectively. The moment map in this case is $\mu(A, B) = BA$. If $v = 1$, then this reduces to Example 3.2; more generally, the matrix AB is unchanged by the action of the $\mathrm{GL}(v)$, and defines an isomorphism

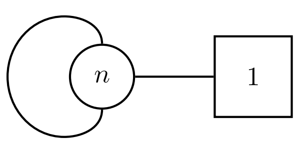
$$\mathfrak{M}_{\mathrm{B}}(T^*N, G) \cong \{C \in M_{w \times w}(\mathbb{C}) \mid C^2 = 0,\ \mathrm{rk}(C) \leq v\}.$$

Other important examples:

E1. This quiver gives the space of $n \times n$ nilpotent matrices:



E2. This quiver gives the symmetric power $\mathrm{Sym}^n \mathbb{C}^2 = (\mathbb{C}^2)^n / S_n$.



There are many variations on these, but these will suffice as our main examples for the rest of this article. All of these examples have symplectic resolutions obtained by replacing the affine quotient with a GIT quotient: the space of rank $\leq v$ matrices is resolved by $T^* \mathrm{Gr}(w, v)$ if $v \leq w/2$, the nilpotent cone is resolved by the cotangent bundle of the flag variety (this is a special case of the Springer resolution), and the symmetric power $\mathrm{Sym}^n \mathbb{C}^2$ is resolved by the Hilbert scheme of n points on $\mathbb{C}^2$.

3.2. Coulomb branches.

Compared to Higgs branches, Coulomb branches are harder to describe. In older papers, one will generally see the statement that the "classical" Coulomb branch is $T^{*L}T/W$ for ${}^LT$ Langlands dual to the maximal torus T of G. However, this is not the true answer, as there are nontrivial "quantum corrections." In certain special cases, the true Coulomb branch could be determined by other methods:

- Work of Hanany–Witten and extensions identified the Coulomb branches of the quiver gauge theories E1. and E2. using string dualities.
- It is implicit in [Tel14] that the Coulomb branch of a pure gauge theory (meaning $N = 0$) with gauge group G is the (Lie algebra) regular centralizer variety for $G^\vee$, called the Bezrukavnikov–Finkelberg–Mirković space $\mathrm{BFM}(G^\vee)$ there.

Work of Braverman, Finkelberg, and Nakajima [BFN18] gives an explicit mathematical definition of the Coulomb branch based on the geometry of affine Grassmannians when $X = T^*N$. Let us try to roughly explain the source of this construction, which can look quite intimidating.

By the description (A), we should consider maps to N/G that are holomorphic on the top and bottom disks of the cylinder and constant along the sides. We will shrink the parameter δ that defines the cylinder to be infinitesimally small and only consider the Taylor expansion at the origin in the top and bottom planes $z = \pm\delta$. If we let $t = x + iy$, then each holomorphic map $D \to N/G$ corresponds to a Taylor series $n(t) \in N[[t]]$, and two give the same map if they are in the same orbit of the G-valued Taylor series $G[[t]]$.

Finally, the map should be constant along the sides of the cylinder. Mapping to a quotient means that two things are equal if they are in the same orbit for a group-valued function on the circle. Since this is only on the sides of the cylinder, the group-valued function comparing the top and bottom might have a pole at the origin. Thus, if $n_\pm(t)$ is the Taylor expansion at $(0, 0, \pm\delta)$, then we must have $g_\pm n_-(t) = n_+(t)$ for some G-valued Laurent series $g_\pm(t) \in G((t))$.

Definition 3.5. The **BFN space** for (G, N) is the quotient of the set $\{(n_+, n_-, g_\pm) \mid g_\pm n_-(t) = n_+(t)\}$ by the action of $G[[t]] \times G[[t]]$ given by

$$(h_+, h_-) \cdot (n_+, n_-, g_{pm}) = (h_+ n_+, h_- n_-, h_+ g_\pm h_-^{-1}).$$

By (A), we should consider the locally constant functions on this space. We have to be careful, and in fact, we need to consider the Borel–Moore homology (very carefully defined) of this quotient. That is:

Definition 3.6. The algebra $\mathcal{A}_{\mathrm{A}}(T^*N, G)$ is the Borel–Moore homology of the BFN space.

While the action of $G[[t]] \times G[[t]]$ is not free, we can still interpret the homology of the quotient using equivariant topology; the interested reader should refer to [BFN18] for a more precise discussion. We can still compute using usual methods from finite-dimensional topology and, in particular, identify the Coulomb branches in many cases.

The Coulomb branch comes equipped with a $\mathbb{C}^\times$ action, induced by the homological grading on Borel–Moore homology. Unfortunately, there can be operators of negative degree. We call such theories **bad**. Most other cases are called **good**.

One bad example which had already attracted the interest of mathematicians was the case of pure gauge theory where $N = 0$. The Higgs branch is quite degenerate in this case, but $\mathfrak{M}_{\mathrm{A}} = \mathrm{BFM}(G)$ is the phase space of the rational Toda lattice, that is, the universal (Lie algebra) centralizer, restricted to the Kostant slice; see [Tel14, §5.1] for a longer discussion of this variety. The other interesting cases we know fall into the two cases discussed before:

3.2.1. *Abelian/hypertoric gauge theories.* If G is an abelian group and acts faithfully on N, then the Coulomb branch will coincide with the Higgs branch of another good abelian theory.

Recall that we can assume that $G \subset D$ is a subgroup of the group D of diagonal matrices. The Langlands dual group ${}^LF$ of $F = D/G$ can be realized as the connected subgroup of D whose Lie algebra is the perpendicular to $\mathfrak{g} \subset \mathfrak{d} = \mathbb{C}^n$; this is also isomorphic to $(\mathbb{C}^\times)^k$, but the isomorphism $F \cong {}^LF$ is not canonical. We thus have two Langlands dual short exact sequences of tori:

$$1 \to G \to D \to F \to 1$$
$$1 \to {}^LG \leftarrow D \leftarrow {}^LF \leftarrow 1.$$

Theorem 3.7. *We have isomorphisms* $\mathfrak{M}_{\mathrm{A/B}}(T^*\mathbb{C}^n, G) \cong \mathfrak{M}_{\mathrm{B/A}}(T^*\mathbb{C}^n, {}^LF)$.

This isomorphism was widely expected earlier, but seems to have first been checked using the mathematical definition by Dimofte and Hilburn; see [BDG17, §3.3] for a physical discussion of this isomorphism, and [BFN18, §4(vii)] for an elegant mathematical proof.

Let us give a sketch of it. First, the algebras $\mathcal{A}_{\mathrm{A/B}}$ carry a grading by the weight lattice of F/coweight lattice of ${}^LF$: (1) On $\mathcal{A}_{\mathrm{B}}(T^*\mathbb{C}^n, G)$, induced by the D-action on $T^*\mathbb{C}^n$; (2) On $\mathcal{A}_{\mathrm{A}}(T^*\mathbb{C}^n, {}^LF)$, induced by the bijection of components of the affine Grassmannian of ${}^LF$ to the coweight lattice. Second, in both algebras, the degree 0 elements form a copy of $\mathrm{Sym}(\mathfrak{f})$: (1) On $\mathfrak{M}_{\mathrm{B}}(T^*\mathbb{C}^n, G)$, the polynomial functions that factor through the moment map $\mathfrak{M}_{\mathrm{B}}(G, T^*\mathbb{C}^n) \to \mathfrak{f}^*$; (2) In $\mathcal{A}_{\mathrm{A}}(T^*\mathbb{C}^n, {}^LF)$, the Borel–Moore homology of a point modulo the action of ${}^LF[[t]]$, isomorphic to $\mathrm{Sym}(\mathfrak{f}) \cong H^*_{{}^LF}(\mathrm{pt})$. The isomorphism of algebras is close to being determined by matching these two aspects of the Higgs and Coulomb branches.

3.2.2. *Quiver gauge theories.* The other class of theories we discussed before are the quiver gauge theories. The resulting Coulomb branches are thus the mirrors of the Nakajima quiver varieties. Those which are good in the sense discussed above correspond to pairs of a highest weight λ for the Kac–Moody Lie algebra with Dynkin diagram given by the quiver, and a dominant weight μ. These are characterized by:

1. The highest weight vector has weight w_i for the root $\mathfrak{sl}_2$ for the node i (that is, $\alpha_i^\vee(\lambda) = w_i$).
2. For simple roots α_i, we have $\lambda - \mu = \sum v_i \alpha_i$.

When Γ is an ADE Dynkin diagram, these Coulomb branches have an interpretation in terms of the affine Grassmannian of the finite-dimensional group G_Γ whose Dynkin diagram is Γ. We can interpret λ and μ as coweights of G_Γ, and thus consider the closure of the orbit $\overline{\mathsf{Gr}}^\lambda = \overline{G_{\mathcal{O}} t^\lambda G_{\mathcal{O}}/G_{\mathcal{O}}} \subset G_{\mathcal{K}}/G_{\mathcal{O}}$.

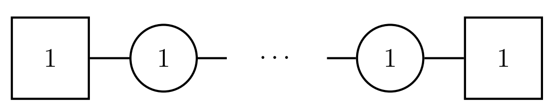
Proposition 3.8 ([BFN19, Th. 3.10])**.** *The Coulomb branch* $\mathfrak{M}_{\mathrm{A}}(T^*N, G)$ *for an ADE quiver gauge theory is isomorphic to the transverse slice to a generic point of* $\overline{\mathsf{Gr}}^\mu$ *inside* $\overline{\mathsf{Gr}}^\lambda$.

Since these affine Grassmannian slices are not familiar to most readers, let us discuss a few examples:

Example 3.9. The quivers E1., E2. both satisfy $\mathfrak{M}_{\mathrm{A}} \cong \mathfrak{M}_{\mathrm{B}}$! This is coincidental and usually doesn't happen.

Example 3.10. In the case of the quiver $\textcircled{1}\!-\!\boxed{n}$, where $\mathfrak{M}_{\mathrm{B}} \cong M^{\mathrm{rk}\leq 1}_{n\times n}(\mathbb{C})$, the Coulomb branch $\mathfrak{M}_{\mathrm{A}}$ is the affine variety $\mathbb{C}^2/\mathbb{Z}_n$, with the cyclic group $\mathbb{Z}_n$ acting by the matrices $\mathrm{diag}(e^{2\pi i k/n}, e^{-2\pi i k/n})$.

Example 3.11. In the quiver below, the Higgs and Coulomb branches are reversed from the previous example: $\mathfrak{M}_{\mathrm{B}} \cong \mathbb{C}^2/\mathbb{Z}_n$, $\mathfrak{M}_{\mathrm{A}} \cong M^{\mathrm{rk}\leq 1}_{n\times n}(\mathbb{C})$.



These are special cases of a much more general result. For good quiver theories where Γ is an affine type A Dynkin diagram (that is, a single cycle), including the Jordan quiver (a single loop), the Coulomb branch is also a Higgs branch for a theory of an affine type A Dynkin diagram, but potentially of a different size, as proven in [NT17]. The combinatorics of this correspondence is a little complicated, but it matches the previously known combinatorics of rank-level duality. This suggests that the corresponding theories are mirror to each other.

4. Advanced Directions

Having given the definition of Higgs and Coulomb branches, the reader will naturally wonder what mathematics these lead to. There are a number of directions which are too deep to discuss in full detail, but which the interested reader might want to explore further:

4.1. **Stable envelopes.** Aganagić and Okounkov [AO20], building on earlier work of Maulik–Okounkov, define classes called **elliptic stable envelopes** on each symplectic resolution with a Hamiltonian $\mathbb{C}^*$-action. There are many examples of these which arise as $\mathfrak{M}_{A/B}$ for different 3d, $\mathcal{N} = 4$ gauge theories.

The equivariant stable envelopes are classes in equivariant elliptic cohomology which correspond to the thimbles flowing to the different $\mathbb{C}^*$-fixed points on the resolution (equivalently, the stable manifolds of the real moment map, thought of as a Morse function). These play an important role in the study of enumerative geometry and are expected to be one of the key mathematical manifestations of 3d mirror symmetry. The elliptic stable envelopes of mirror varieties are expected to be obtained from the specialization of a natural "Mother" class on the product $\mathfrak{M}_A \times \mathfrak{M}_B$; this is confirmed in the case of Example 3.4 by Rimányi–Smirnov–Varchenko–Zhou [RSZV22]. This identification switches two classes of parameters in the physical theory:

1. "masses" which index resolutions of $\mathfrak{M}_A$, and $\mathbb{C}^*$-actions on $\mathfrak{M}_B$, and
2. "Fayet–Iliopoulos (FI) parameters" which play the opposite role of indexing $\mathbb{C}^*$-actions on $\mathfrak{M}_A$ and resolutions of $\mathfrak{M}_B$.

4.2. **Koszul duality of category $\mathcal{O}$'s.** One of the mathematical phenomena which has attracted attention to 3-dimensional mirror symmetry is Koszul duality between categories $\mathcal{O}$. These are based on a deformation quantization of the algebras $\mathcal{A}_{A/B}$ to noncommutative algebras. These deformations can be understood as incorporating the action of the rotation of $\mathbb{R}^3$ by S^1 around the z-axis; in physics terms, this is called an Ω-background. The resulting algebra is noncommutative, since only the z-axis is invariant under the S^1-action, and two invariant points cannot switch places while staying on the z-axis.

1. The algebra $\mathcal{A}_A$ is deformed by considering the S^1-equivariant homology of the BFN space.
2. The algebra $\mathcal{A}_B$ is deformed by replacing $\mathbb{C}[X] = \operatorname{Sym} X^*$ by its Weyl algebra, which is defined by the relations $[x, y] = \hbar\Omega(x, y)$; we can replace the operations of taking the G-invariant functions on $\mu^{-1}(0)$ with a noncommutative analogue of Hamiltonian reduction.

Category $\mathcal{O}$ is a category of special modules over these noncommutative algebras. This can be regarded as a categorification of the stable envelopes, in that instead of considering the homology classes of the thimbles flowing into fixed points, we consider sheaves of modules over a deformation quantization of $\mathfrak{M}_{A/B}$ supported on these thimbles. See [BLPW16, §3] for more details.

It was noticed by Soergel that the principal block of category $\mathcal{O}$ for a semisimple Lie algebra has an interesting self-duality property: It is equivalent to the category of ungraded modules over a graded algebra and inside the derived category $D^b(\tilde{\mathcal{O}})$ of graded modules over that algebra, there is a second "hidden" copy of the original category. For variations, such as singular blocks of category $\mathcal{O}$ or parabolic category $\mathcal{O}$, a similar phenomenon occurs, but it is a copy of another category that appears; for example, the singular and parabolic properties interchange. That is, the graded lifts of these categories are **Koszul** and their Koszul dual is another category (sometimes different, sometimes the same) of a similar flavor.

As discussed in the introduction, we can pretend that another symplectic singularity is the nilpotent cone of a new simple Lie algebra. The definition of category $\mathcal{O}$ for a general symplectic singularity with a $\mathbb{C}^*$-action was given by Braden, Licata, Proudfoot, and the first author in [BLPW16]. Computing numerous examples led these authors to the conjecture:

Conjecture 4.1. *The categories $\mathcal{O}$ of mirror dual symplectic singularities (i.e., the Higgs and Coulomb branch of a 3d $\mathcal{N} = 4$ supersymmetric gauge theory) are Koszul dual.*

A version of this conjecture (obviously, requiring more careful stating) is confirmed in [Web]. The physical interpretation of this Koszul duality is still uncertain, though one is proposed in [BDGH16, §7.5].

4.3. **Line operators.** Just as local operators stand for observations one can make at a single point, there are line operators that describe observations one can make along a single line. Studying these is a natural way to extend our study of 3d mirror symmetry beyond the definition of the Higgs and Coulomb branches.

We can describe this category using the framework of d-dimensional extended TQFT, which assigns not just a Hilbert space to a $(d-1)$-manifold, but more generally a k-category to each manifold of codimension $k+1$. We can then generalize the description of the local operators as the space $Z(S^{d-1})$ by identifying the operators supported on a k-plane with the k-category $Z(S^{d-k-1})$. In particular, the category of line operators should be given by $Z(S^{d-2})$.

For the 3d $\mathcal{N} = 4$ theories of interest to us, these can be understood after passing to the A- or B-twisted theory. Indeed, algebraic descriptions of these categories have been proposed by Hilburn and the second author. Here we only provide a rough description of $Z(S^1)$ with a similar flavor to (A) and (B) (see [BF19] for a more precise statement of this proposal):

- In the A-twist, we obtain *locally constant* sheaves (that is, D-modules) on *holomorphic* loops $N((t))/G((t))$ in the quotient N/G.

- In the B-twist, we obtain *holomorphic* sheaves (that is, quasi-coherent sheaves) on a version of the *locally constant* loops (that is, the small loop space) in N/G.

This proposal is actually a good way to derive Definition 3.6: the trivial line is given by the pushforward D-module from $N[[t]]/G[[t]]$, and naively computing the endomorphisms of this pushforward as the Borel–Moore homology of the fiber product gives precisely Definition 3.6.

It is an intriguing but challenging problem to identify these categories in the already known dual pairs, which one may call the *de Rham 3d homological mirror symmetry* (see below for more context for the name). Recent work of Hilburn–Raskin [HR22] confirms this in the case where $N = \mathbb{C}$, $G = \{1\}$.

4.4. Connections to 4d field theory and the Langlands program. Seminal work of Kapustin and Witten [KW07] interprets a version of the geometric Langlands correspondence in terms of a physical duality between 4-dimensional field theories with $\mathcal{N} = 4$ supersymmetry: the supersymmetric Yang–Mills theory in 4 dimensions for a pair of Langlands dual groups are related by S-duality. Elliott and the second author [EY18] developed a mathematical framework to describe a variant of their proposal which yields the geometric Langlands correspondence upon (categorified) geometric quantization. Moreover, by applying this procedure to the A- and B-twists of a 3d $\mathcal{N} = 4$ theory, one can obtain the aforementioned categories of line operators.

This connects to the 3-dimensional perspective discussed earlier in this paper, since 3d $\mathcal{N} = 4$ theories appear as boundary conditions on 4d $\mathcal{N} = 4$ super Yang–Mills theory. In particular, S-duality of these boundary conditions, as studied by Gaiotto and Witten [GW09], is one of our most powerful tools for finding mirror theories. The theories associated to Nakajima quiver varieties for linear or cyclic (finite or affine type A) quivers arise this way, and this is the quickest route to understanding the duality of these theories discussed in Section 3.2.2.

The mathematical understanding of this perspective is still an emerging topic. Hilburn and the second author proposed a new relationship between the global/local geometric Langlands program and the statement of de Rham 3d homological mirror symmetry. In independent work, Ben-Zvi, Sakellaridis, and Venkatesh realized the physical perspective in the context of the relative Langlands program and have announced a number of interesting conjectures relating periods and special values of L-functions.

4.5. Betti 3d mirror symmetry. The Betti (singular) cohomology and de Rham cohomology of an algebraic variety are, of course, isomorphic, but they have different nonabelian generalizations. They manifest as the moduli spaces of local systems and of flat connections on a given variety, which are analytically isomorphic (via the Riemann–Hilbert correspondence) but algebraically different.

Most importantly for us, the complex structure on the de Rham moduli space depends on the complex structure of the underlying curve, whereas the Betti space does not. Analogously, Ben-Zvi and Nadler propose a "Betti" version of the geometric Langlands correspondence which gives an automorphic description of the quasi-coherent sheaves on the Betti moduli space, to complement the "de Rham" version of the geometric Langlands correspondence.

Our discussions in the earlier sections of this paper also belong to the de Rham world. Namely, in our description of moduli spaces of vacua, objects attached to S^2 depend on a complex structure on this curve; the appearance of structures which are holomorphic in one plane and constant on an orthogonal line is a sort of degenerate complex structure on S^2. On the other hand, for physicists, this perspective looks somewhat artificial, compared to treating all directions in $\mathbb{R}^3$ equally.

Indeed, the proposal of Kapustin and Witten [KW07] is already phrased from a Betti perspective: It does not depend on the complex structure of the curve and needs to be modified to fit with the usual (de Rham) Langlands conjecture (as is done in [EY18]). Another key feature of Higgs and Coulomb branches, which is physically expected, but hard to see from a de Rham perspective, is the existence of a hyperkähler metric. These have been constructed for Higgs branches $\mathfrak{M}_{\mathrm{B}}(T^*N, G)$ using hyperhamiltonian quotients, but it is hard to imagine the construction of such a metric on the Coulomb branch in the framework of [BFN18]. Possibly the most intriguing aspect of the Betti perspective is that, as it does not depend on a complex structure, it is better suited to the approach of extended TQFT. Hence, this is the framework in which one can push the approach of homological mirror symmetry to the fullest.

In the case of 2-dimensional mirror symmetry, Kontsevich made the striking realization that we can capture the equivalence of the A-model of one theory and the B-model of another as an equivalence of two triangulated (dg/A_∞) categories: from the Fukaya category of a symplectic manifold to the derived category of coherent sheaves on a complex variety. These are the categories of boundary conditions of the respective twisted theories, and hence the equivalence of theories can be reconstructed from the equivalence of categories. In terms of extended 2d TQFT, this is an equivalence of $Z(\mathrm{pt})$'s of the dual theories.

This provides an enticing model to follow in the 3d case. Ideally, we would assign a 2-category $Z(\mathrm{pt})$ of boundary conditions to the A- and B-twist of each theory $\mathcal{T}$ and conjecture the equivalence between those for dual theories, which we would call the *Betti 3d homological mirror symmetry*. This program was put forward by Teleman [Tel14], based on a proposal of Kapustin–Rozansky–Saulina [KRS09] for $\mathcal{T}(T^*N, G)$. Significant progress on the A-model 2-category has been made in the abelian case in recent work of Gammage–Hilburn–Mazel-Gee [GHM22] and Doan–Rezchikov [DR22] suggesting an ambitious program for a more general case.

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Ben Webster


Philsang Yoo

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Some Applications of $\mathbb{G}_a$-actions on Affine Varieties

$$V = x^m y = z^{p^e} + t + t^{sp}$$

$$V \times \mathbb{A}_k^1 \cong \mathbb{A}_k^4$$

Neena Gupta

1. Introduction

Over a field k, an affine n-space $\mathbb{A}_k^n$ simply refers to k^n with certain mathematical (topological and sheaf) structures, and affine varieties refer to irreducible closed subspaces of $\mathbb{A}_k^n$ defined by the zero locus of polynomials. Over an algebraically closed field k, affine spaces correspond to polynomial rings and the affine varieties to affine domains, i.e., finitely generated k-algebras which are integral domains.

In affine algebraic geometry (AAG), researchers are mainly interested in the study of certain affine varieties, especially the affine spaces, equivalently, the polynomial rings. There are many fascinating and fundamental problems on polynomial rings which can be formulated in an elementary mathematical language but whose solutions remain elusive. Any significant progress requires development of new and powerful methods and their ingenious applications. The most celebrated problems on polynomial rings include the Jacobian problem (first asked by Ott-Heinrich Keller in 1939), the Zariski cancellation problem (ZCP), the epimorphism or embedding problem of Abhyankar–Sathaye, the affine fibration problem of Dolgačev–Veĭsfeĭler, the linearization problem of Kambayashi, the problem of characterization of polynomial rings by a few chosen properties, and the study of the automorphism groups of polynomial rings. In this article, we will discuss how $\mathbb{G}_a$-actions and their invariants have been employed in recent decades to achieve breakthroughs on some of the above challenging problems on polynomial rings.

The concept of a $\mathbb{G}_a$-action and its equivalent formulations *exponential map* and *locally nilpotent derivation* will be defined in Section 2 along with a brief overview on $\mathbb{G}_a$-actions.

In Section 3, we recall Miyanishi's algebraic characterization of the affine plane obtained in the 1970s using $\mathbb{G}_a$-action and then Fujita–Miyanishi–Sugie's theorem on the

Neena Gupta is a professor in the Statistics and Mathematics Unit at the Indian Statistical Institute. Her email address is `neenag@isical.ac.in`.

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ZCP for the affine plane using this characterization. We shall also discuss a more recent elementary solution of ZCP for the affine plane by Crachiola and Makar-Limanov using tools of $\mathbb{G}_a$-action.

In Section 4, we revisit an invariant of $\mathbb{G}_a$-actions introduced by Makar-Limanov in the 1990s, now known as the Makar-Limanov invariant. We discuss how Makar-Limanov used it to distinguish a well-known threefold from the affine three space. This result led to the solution of the linearization conjecture of Kambayashi for $\mathbb{C}^3$.

Finally in Section 5, we see how the tools of $\mathbb{G}_a$-action were used in the last decade to establish counterexamples to the ZCP in positive characteristic in higher dimensions.

Due to the restriction on the size of the bibliography, many important results and references on $\mathbb{G}_a$-actions and allied topics had to be omitted. Interested readers may refer to the monograph of Freudenburg ([7]) and the survey articles [18], [6] and [13].

Throughout the article, k will denote a field. By a ring, we mean a commutative ring with unity. For a ring R, R^* denotes the group of all units of R. For an algebra A over a ring R, the notation $A = R^{[n]}$ will mean that A is isomorphic to a polynomial ring in n-variables over R, and for a prime ideal P of R, A_P denotes $S^{-1}A$, where $S = R \setminus P$. Capital letters like $X, Y, Z, W, T, X_1, \dots, X_n$, etc., will mean indeterminates over respective rings or fields. For a ring R and R-algebras A and B, the notation $A \cong_R B$ means that A is isomorphic to B as R-algebras. We shall denote the set of all maximal ideals of a ring R by $\operatorname{Max}(R)$.

We recall below the concept of morphisms between affine varieties over k and their relationships with certain k-algebra homomorphisms.

Remark 1.1. Recall that for two affine algebraic varieties $V \subseteq \mathbb{A}_k^n$ and $W \subseteq \mathbb{A}_k^m$, a morphism or a polynomial function $\Phi : V \to W$ is simply a function defined by m polynomials

$$\phi_1(X_1, \dots, X_n),\ \phi_2(X_1, \dots, X_n),\ \dots,\ \phi_m(X_1, \dots, X_n)$$

such that $(\phi_1(a_1, \dots, a_n), \phi_2(a_1, \dots, a_n), \dots, \phi_m(a_1, \dots, a_n)) \in W$ for all $(a_1, \dots, a_n) \in V$. In particular, any polynomial $\phi(X_1, \dots, X_n)$ gives rise to a polynomial function $\phi : V \to k$. The coordinate ring $k[V]$ refers to the ring of polynomial functions on V. Any morphism $\Phi : V \to W$ induces a k-algebra homomorphism $\widetilde{\Phi} : k[W] \to k[V]$ defined by $\widetilde{\Phi}(f) = f \circ \Phi$ for all $f \in k[W]$.

Remark 1.2. For an affine variety V over an algebraically closed field k, by the celebrated Hilbert Nullstellensatz (1893), the maximal ideals of the coordinate ring $k[V]$ are in one to one correspondence with the set of points of V. Hence $\operatorname{Max}(k[V])$ is identified with V itself. Moreover, as a consequence of Hilbert Nullstellensatz, for any k-algebra homomorphism $\Psi : k[W] \to k[V]$ and any maximal ideal m of $k[V]$, $\Psi^{-1}(m)$ is a maximal ideal of $k[W]$. Thus any k-algebra homomorphism $\Psi : k[W] \to k[V]$ induces a map $\operatorname{Max}(k[V]) \to \operatorname{Max}(k[W])$ and hence induces a map $\widehat{\Psi} : V \to W$ which can be shown to be a morphism of varieties. Further, any morphism $\Phi : V \to W$ can be recovered from its induced k-algebra homomorphism $\tilde{\Phi}$, i.e., $\widehat{\widetilde{\Phi}} = \Phi$ and conversely, for any k-algebra homomorphism $\Psi : k[W] \to k[V]$, $\widetilde{\widehat{\Psi}} = \Psi$.

2. $\mathbb{G}_a$-action, Exponential Map, and LND

In this section we shall recall the concept of $\mathbb{G}_a$-actions on affine algebraic varieties and its connection with the concept of exponential maps on affine domains (finitely generated k-algebras which are integral domains) and the concept of locally nilpotent derivations in characteristic zero. We first recall the concept of the group $\mathbb{G}_a$.

Recall that an affine algebraic group G over a field k is an affine variety over k such that G is a group compatible with the underlying variety structure. This means that the binary group operation $\mu : G \times G \to G$ and the inverse group operation $\iota : G \to G$ are morphisms of affine varieties. For example, k is an affine algebraic group with $+$ as the group operation and is denoted by $\mathbb{G}_a^k$. More precisely:

Definition. The group $\mathbb{G}_a^k$ over a field k is the affine algebraic group $(k, +)$ comprising k as an affine variety together with the additive group structure '+', i.e., the binary group operation

$$\mu : k \times k \to k, \text{ defined by } \mu(a, b) = a + b$$

and the inverse operation

$$\iota : k \to k, \text{ defined by } \iota(a) = -a$$

are morphisms of affine varieties. When the underlying field k is understood, the simplified notation $\mathbb{G}_a$ is used in place of $\mathbb{G}_a^k$. We note that $\mathbb{G}_a$ is isomorphic to the affine algebraic group

$$\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(k) \mid a \in k\},$$

and hence is a unipotent group (that is, a matrix group whose eigenvalues are all 1).

Definition. A $\mathbb{G}_a$-*action* on an affine variety V over an algebraically closed field k is a morphism of affine varieties $\theta : \mathbb{G}_a \times V \to V$ satisfying

(i) $\theta(0, x) = x$, for all $x \in V$, and
(ii) $\theta(\alpha, \theta(\beta, x)) = \theta(\alpha + \beta, x)$ for all $x \in V$, $\alpha, \beta \in \mathbb{G}_a$.

Thus, for each $\lambda \in k$, the $\mathbb{G}_a$-action θ induces an isomorphism $\theta_\lambda : V \to V$ given by $\theta_\lambda(x) = \theta(\lambda, x)$ for all $x \in V$. Let B denote the coordinate ring $k[V]$ of V. Then, we get a k-algebra automorphism $\widetilde{\theta_\lambda}$ of the ring B defined by $\widetilde{\theta_\lambda}(f) = f \circ \theta_\lambda$ for $f \in B$.

Let B^θ denote the subset of B comprising elements which are fixed by these induced automorphisms $\widetilde{\theta_\lambda}$, i.e.,

$$B^\theta := \left\{f \in B \,|\, \widetilde{\theta_\lambda}(f) = f \ \text{ for all } \ \lambda \in k\right\}.$$

It is easy to see that the set B^θ is a k-subalgebra of B. The ring B^θ is called the *ring of invariants* of the $\mathbb{G}_a$-action θ.

Definition. Let B be an integral domain containing a field k and $\phi : B \to B^{[1]}$ be a k-algebra homomorphism. For an indeterminate U over B, let ϕ_U denote the map $\phi : B \to B[U]$. Then ϕ is said to be an *exponential map* on B, if the following conditions are satisfied:

(i) $\epsilon_0\phi_U = id_B$, where $\epsilon_0 : B[U] \to B$ is the evaluation map at $U = 0$.

(ii) $\phi_V\phi_U = \phi_{U+V}$, where $\phi_V : B \to B[V]$ is extended to a k-algebra homomorphism $\phi_V : B[U] \to B[U, V]$, by setting $\phi_V(U) = U$.

The *ring of invariants* of the exponential map ϕ is defined to be the subring B^ϕ of B defined as

$$B^\phi := \{f \in B \,|\, \phi(f) = f\}.$$

An exponential map ϕ is said to be nontrivial if $B \neq B^\phi$.

Let $\mathrm{EXP}(B)$ denote the set of all k-linear exponential maps on B. The Makar-Limanov invariant of B, denoted by $\mathrm{ML}(B)$, is a subring of B defined as

$$\mathrm{ML}(B) := \bigcap_{\phi \in \mathrm{EXP}(B)} B^\phi.$$

If B admits no nontrivial exponential map, then $\mathrm{ML}(B) = B$. The invariant $\mathrm{ML}(B)$ was introduced by L. Makar-Limanov and, as we shall see, it has turned out to be a powerful tool in solving certain problems on polynomial rings.

There is a connection between $\mathbb{G}_a$-action and exponential map as stated below.

Theorem 2.1. *When k is algebraically closed, B is an affine domain over k and $V := \mathrm{Max}(B)$, then any $\mathbb{G}_a$-action θ on V gives rise to an exponential map ϕ on B and conversely. Further, $B^\theta = B^\phi$.*

Proof. Any morphism $\theta : \mathbb{G}_a \times V \to V$ of algebraic varieties induces a k-algebra homomorphism $\phi : B \to B \otimes_k k[U] = B[U]$ of their corresponding coordinate rings by Remark 1.1. Further, it is easy to see that the conditions (i) and (ii) of the morphism θ correspond algebraically to the conditions (i) and (ii) of the k-algebra homomorphism ϕ. The converse also follows similarly (cf. Remark 1.2). □

For illustration we consider an example.

Example 2.2. Let V be the affine surface in $\mathbb{A}^3_k$ defined by $\{(a, b, c) \in \mathbb{A}^3_k \mid ac = b^2 - 1\}$ and $B = k[V] = k[X, Y, Z]/(XZ - Y^2 + 1)$ be the coordinate ring of V. Consider the morphism

$$\theta : \mathbb{G}_a \times V \to V \text{ defined by } \theta(\lambda, (a, b, c)) = (a, b + a\lambda, c + 2b\lambda + a\lambda^2).$$

Then,

(i) $\theta(0, (a, b, c)) = (a, b, c)$ for all $(a, b, c) \in V$ and

(ii) $\theta(\alpha, \theta(\beta, (a, b, c))) = \theta(\alpha + \beta, (a, b, c))$ for all $(a, b, c) \in V$ and $\alpha, \beta \in k$.

Hence θ is a $\mathbb{G}_a$-action on V.

Let x, y, and z denote the images of X, Y, and Z in B. Now θ induces the ring homomorphism $\phi : B \to B[U]$ defined by

$$\phi(x) = x,\ \phi(y) = y + xU \ \text{ and } \ \phi(z) = z + 2yU + xU^2.$$

One can see that ϕ is an exponential map on B and $B^\theta = B^\phi = k[x]$.

We now define locally nilpotent derivations.

Definition. Let B be an integral domain containing a field k of characteristic zero. A k-linear derivation D on B is said to be a *locally nilpotent derivation* (or LND) if, for any $a \in B$ there exists an integer n (depending on a) satisfying $D^n(a) = 0$. Thus a k-linear map $D : B \to B$ is said to be an LND if

(i) $D(xy) = xD(y) + yD(x)\ \forall\ x, y \in B$ and

(ii) $D^n(a) = 0$ for some $n \geq 1$ (depending on a) for each $a \in B$.

For an LND D, let $\mathrm{Ker}(D)$ denote the kernel of the derivation D, i.e.,

$$\mathrm{Ker}(D) := \{f \in B \,|\, D(f) = 0\}.$$

Then $\mathrm{Ker}(D)$ is a subring of B.

For example, the partial derivations $\partial/\partial(X)$, $\partial/\partial(Y)$, and $\partial/\partial(Z)$ on the ring $k[X, Y, Z]$ are LNDs with kernels $k[Y, Z]$, $k[X, Z]$, and $k[X, Y]$ respectively. Thus an LND may be thought of as a generalization of partial derivation on a polynomial ring B. The famous Jacobian conjecture can also be formulated as a problem in LND ([7, Chapter 3]).

We now see that, over an algebraically closed field of characteristic zero, the study of exponential maps (equivalently the study of $\mathbb{G}_a$-actions on an affine variety) is equivalent to the study of locally nilpotent derivations.

Theorem 2.3. *Let k be an algebraically closed field of characteristic zero and B be a k-domain. An exponential map $\phi : B \to B[U]$ induces a locally nilpotent derivation D on B and conversely.*

Proof. Let $\phi : B \to B[U]$ be an exponential map on B. For any $b \in B$, let

$$\phi(b) = \phi_0(b) + \phi_1(b)U + \phi_2(b)U^2 + \cdots + .$$

Note that since ϕ is a k-algebra homomorphism, we have

(I) $\{\phi_n\}$ are k-linear maps on B.

(II) $\phi_n(ab) = \sum_{i+j=n} \phi_i(a)\phi_j(b)$ for all $a, b \in B$ and $n \geq 0$

(III) For each $b \in B$, there exists $n \in \mathbb{N}$ such that $\phi_m(b) = 0$ for all $m \geq n$.

Further, as ϕ is an exponential map, we have

(IV) ϕ_0 is the identity map on B, by property (i) of exponential maps.

(V) $\phi_i\phi_j = \binom{i+j}{i}\phi_{i+j}$ for all $i, j \geq 0$, by property (ii) of exponential maps.

Therefore, by (V), in characteristic zero, we have

$$\phi_2 = \frac{1}{2!}\phi_1^2, \ \ldots, \ \phi_n = \frac{1}{n!}\phi_1^n \ \ldots. \tag{1}$$

Let $D := \phi_1$. Then $D : B \to B$ is a k-linear map satisfying $D(ab) = aD(b) + bD(a)$ by property (II) and (IV). Further, by equation (1), $D^n(b) = n!\,\phi_n(b)$, and hence $D^n(b) = 0$ for some $n \geq 1$ by (III). Thus D is a locally nilpotent derivation on B. Further, it follows that $b \in \operatorname{Ker}(D) \Longleftrightarrow \phi(b) = b$, i.e., $b \in B^{\phi}$.

Conversely, let $D : B \to B$ be an LND. Then it is easy to see that the map $\phi : B \to B[U]$ defined by

$$b \longmapsto b + D(b)U + \frac{D^2(b)}{2!}U^2 + \frac{D^3(b)}{3!}U^3 + \ldots$$

induces an exponential map on B. We note that the image of ϕ is actually a polynomial in U since $D^n(b) = 0$ for some $n \geq 1$. Also, $D(b) = 0 \Longleftrightarrow \phi(b) = b$ for any $b \in B$. □

As an example we see below the LND induced by the exponential map defined in Example 2.2.

Example 2.4. Let the notation and hypothesis be as in Example 2.2. The exponential map ϕ induces the locally nilptent derivation D on B defined by

$$D(x) = 0,\ D(y) = x, \ \text{ and } \ D(z) = 2y.$$

Thus by Theorems 2.1 and 2.3, over an algebraically closed field of characteristic 0, we have

$$\mathbb{G}_a\text{-action on } V \Longleftrightarrow \text{ Exponential map on } B \Longleftrightarrow \text{ LND of } B.$$

and

$$\text{Ring of invariants of } \mathbb{G}_a\text{-action} = \text{Ring of invariants of exponential maps} = \operatorname{Ker}(D).$$

Hence, going forward we shall be using $\mathbb{G}_a$-action, exponential map, and LND interchangeably.

M. Nagata (in 1959) had constructed an example of a unipotent group action over a polynomial ring $k^{[n]}$ whose ring of invariants is not finitely generated. Thus the concept of studying unipotent group actions came to be perceived as pathological, not of interest to many geometers.

It is now known that the ring of invariants of even a $\mathbb{G}_a$-action on a polynomial ring need not be finitely generated. In 1990, P. Roberts constructed a nonfinitely k-algebra A over a field of characteristic zero, as the symbolic blow-up of a prime ideal P of $k^{[3]}$. Later, A. A'Campo-Neuen realized the ring A in Roberts's example as the ring of invariants of an LND on $k^{[7]}$. This was the first example of a nonfinitely generated ring of invariants of a $\mathbb{G}_a$-action over a polynomial ring $k^{[7]}$. For subsequent examples, one can see [7, Chapter 7].

In spite of the apparently pathological nature of unipotent group actions, a few mathematicians like P. Gabriel, Y. Nouaźe, R. Rentschler, and J. Dixmier proved some early fundamental results in the 1970s on $\mathbb{G}_a$-actions. M. Miyanishi began to systematically investigate $\mathbb{G}_a$-actions and LND. He highlighted the concepts, proved important theorems on them and applied them to mainstream problems in AAG like classification of surfaces, algebraic characterization of affine plane, etc. His algebraic characterization of the affine plane (1975) eventually led to the solution of the ZCP for the affine plane by Miyanishi–Sugie and Fujita in 1980. In 2008, L. Makar-Limanov and A. Crachiola ([3]) gave an elementary proof of the cancellation theorem for the affine plane using exponential maps. The proof is discussed in Section 3.

Since the 1990s, many algebraists including L. Makar-Limanov, J. Deveney, D. Finston, A. van den Essen, S. Kaliman, D. Daigle, G. Freudenburg, H. Derksen, A. Dubouloz, S.M. Bhatwadekar, A.K. Dutta, and subsequent researchers have been contributing regularly in the study of $\mathbb{G}_a$-actions on affine varieties. With passage of time, there has been an increasing recognition of the importance of the concept of $\mathbb{G}_a$-actions in its own right as well as for application to problems in AAG, some of them longstanding.

A major breakthrough in AAG was obtained during the 1990s when, using LND, L. Makar-Limanov distinguished the famous *Koras–Russell threefold* from the affine three space by showing that the Makar-Limanov invariant (named after him) of this threefold does not coincide with the affine three space. This led to the solution of the linearization conjecture of Kambayashi that "Every faithful algebraic $\mathbb{C}^*$-action on $\mathbb{C}^3$ is linearizable". Thus, even for the study of a "good" action ($\mathbb{C}^*$-action), one had to study a so called "bad" $(\mathbb{C}, +)$-action (see Section 4).

Another breakthrough was obtained by the author in 2010s when she used exponential maps to obtain counterexamples to the ZCP in positive characteristic in higher dimensions (see Section 5).

In the above two cases, the tools and techniques of $\mathbb{G}_a$-actions were the cornerstones in the solutions of the respective problems. Certain invariants of $\mathbb{G}_a$-actions could distinguish between two rings, when all hitherto known methods had failed. These episodes illustrate that "$\mathbb{G}_a$-action" can be a part of the general armoury of algebraists and geometers, and not confined to specialists. More details are given in [7].

3. Algebraic Characterization of the Plane and ZCP

The ZCP for polynomial rings can be posed as follows (cf. [8], [7, Chapter 10], [13, Section 2]):

Question 1. Let A be an affine k-algebra. Suppose that $A[X] \cong_k k[X_1, \dots, X_{n+1}]$. Does it follow that $A \cong_k k[X_1, \dots, X_n]$? In other words, is the polynomial ring $k[X_1, \dots, X_n]$ cancellative?

For an algebraically closed field k, Question 1 is equivalent to the following geometric version:

Question 1′. Let k be an algebraically closed field and let V be an affine k-variety such that $V \times \mathbb{A}^1_k \cong_k \mathbb{A}^{n+1}_k$. Does it follow that $V \cong_k \mathbb{A}^n_k$? In other words, is the affine n-space $\mathbb{A}^n_k$ cancellative?

Question 1 has inspired many fruitful explorations over the past 50 years. Some of the major research accomplishments during the 1970s, like the characterization of the affine plane, originated from the efforts to investigate the question. It is not very difficult to show that the polynomial ring $k[X]$ is cancellative over a field k of any characteristic. A more general result was shown by S. S. Abhyankar, P. Eakin, and W. J. Heinzer (1972). But for the polynomial ring $k[X, Y]$, the problem is much more intricate.

In an attempt to solve the cancellation problem for $\mathbb{C}[X, Y]$, C.P. Ramanujam established in 1971 his celebrated topological characterization of the affine plane over $\mathbb{C}$. In 1975, M. Miyanishi gave a characterization of the polynomial ring $k[X, Y]$ using $\mathbb{G}_a$-action (quoted below as Theorem 3.1). This characterization was used by T. Fujita, M. Miyanishi, and T. Sugie ([8], [17]) to prove the cancellation property of $k[X, Y]$ over fields of characteristic zero and by P. Russell ([19]) over perfect fields of arbitrary characteristic. Later (in 2002), using methods of D. Mumford and C.P. Ramanujam, R.V. Gurjar gave a topological proof of the cancellation property of $\mathbb{C}[X, Y]$. More recently, a simplified proof of the cancellation property of $k[X, Y]$ for an algebraically closed field k was given by A. Crachiola and L. Makar-Limanov in [3] using tools of exponential maps. The arguments in this paper were used by S.M. Bhatwadekar and the author to establish the cancellation property of $k[X, Y]$ over any arbitrary field k. For more details one can see [13, Sections 2 and 3].

In 1987, T. Asanuma constructed a three-dimensional affine ring over a field of positive characteristic (see Example 5.3) as a counterexample to the $\mathbb{A}^2$-fibration problem (defined in Section 5) over a PID not containing $\mathbb{Q}$. Later, in 1994, this ring was envisaged as a possible candidate for a counter-example to either the ZCP or *the linearization problem* (discussed in Section 4) for the affine 3-space in positive characteristic. In 2014, the author showed that Asanuma's ring is indeed a counter-example to the cancellation problem (see Section 5). Thus, when ch. $k > 0$, the affine 3-space $\mathbb{A}^3_k$ is not cancellative. Subsequently, the author showed that when ch. $k > 0$, the affine n-space $\mathbb{A}^n_k$ is not cancellative for any $n \geq 3$. Thus, over a field of positive characteristic, the ZCP has been completely answered in all dimensions ([12]).

Question 1 is still open in characteristic zero for $n \geq 3$ and is of great interest in the area of AAG.

We now return to the two-dimensional case. We first state Miyanishi's algebraic characterization of the affine plane ([18, Theorem 2.2.3]).

Theorem 3.1. *Let k be an algebraically closed field and B be a finitely generated k-algebra of dimension 2 such that*

(i) *B is a UFD.*
(ii) *$B^* = k^*$.*
(iii) *There exists a nontrivial exponential map on B.*

Then $B = k^{[2]}$.

Theorem 3.1 had led to following fundamental result of T. Fujita, M. Miyanishi, and T. Sugie ([17] and [8]) proving the cancellation property of $k[X, Y]$ over any field of characteristic zero and the result of P. Russell ([19]) proving it over any perfect field of arbitrary characteristic.

Theorem 3.2. *Let k be a field and B be a k-domain such that*

$$B^{[1]} = k^{[3]}.$$

Then $B = k^{[2]}$.

The original proof uses many ideas making them not quite self-contained for a large class of mathematicians. A simplified proof of Theorem 3.2 for the case k is algebraically closed was given by L. Makar-Limanov and A. Crachiola ([3]) using exponential maps (equivalently $\mathbb{G}_a$-action). Their proof merely uses the following lemma on exponential maps whose proof is elementary. The simplification indicates the power of the tools introduced by Makar-Limanov comprising the concept of ML-invariant of $\mathbb{G}_a$-action and elementary results on the invariant.

Lemma 3.3. *Let B be a finitely generated k-algebra such that there exists an exponential map ϕ of $B[T]$ with $(B[T])^{\phi} \neq B$. Then there exists an exponential map ψ of B such that $B^{\psi} \neq B$, i.e., if $\mathrm{ML}(B) = B$ then $\mathrm{ML}(B[T]) = B$.*

We note that the ring $C = k[X_1, \dots, X_n]$ admits several exponential maps. More specifically, for each i, $1 \leq i \leq n$, $\psi_i : C \to C[U]$, given by

$$\psi_i(X_i) = X_i + U \text{ and, for } i \neq j,\ \psi_i(X_j) = X_j$$

are exponential maps on C. We see that

$$C^{\psi_i} = k[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n] \text{ and } \bigcap_i C^{\psi_i} = k.$$

Therefore, $\mathrm{ML}(C) = k$.

We now show how Theorem 3.2 follows easily from Theorem 3.1 and Lemma 3.3.

Proof of Theorem 3.2. Since $B^{[1]} = k^{[3]}$, it is easy to see that B is a finitely generated k-algebra, B is a UFD and $B^* = k^*$. Hence by Theorem 3.1, it is enough to show that B admits a nontrivial exponential map. Again as $B^{[1]} = k^{[3]}$, the ring $B[T]$ admits an exponential map ϕ such that $(B[T])^{\phi} \neq B$. Hence the result follows from Lemma 3.3. □

In view of the importance of Theorem 3.1, some characterizations of the affine three space have been obtained by Miyanishi (1984, 1987), Kaliman (2002) and the author with Nikhilesh Dasgupta (2021). So far, no suitable characterization of the affine n-space for $n \geq 4$ is known to the author. For more details on the characterization problem one can see [18, Section 2.2] for a detailed survey and [13, Section 3] for a more updated survey.

4. The Russell–Koras Threefold

A special case of the linearization conjecture of Kambayashi ([15]) asserts the following:

Conjecture. *Every algebraic action of k^* on an affine n-space $\mathbb{A}^n_k$ is linearizable.*

By the equivalence of k^*-action on affine varieties and $\mathbb{Z}$-graded structure on their respective coordinate ring, the conjecture asserts that *given any $\mathbb{Z}$-graded structure of the polynomial ring $B = k^{[n]}$, there exists a homogeneous set of coordinates for B, i.e., $B = k[X_1, \dots, X_n]$, where $X_1, \dots, X_n$ are homogeneous elements with respect to the given $\mathbb{Z}$-grading.*

By a result of Kambayashi ([15]), it follows that any k^*-action on $k^{[2]}$ is linearizable. However the case $n = 3$ turned out to be highly nontrivial. An affirmative answer was finally obtained in characteristic zero as a culmination of efforts of several researchers over a period of nearly three decades.

We now discuss the solution of $\mathbb{C}^*$-action on $\mathbb{C}^{[3]}$ which had been eluding mathematicians for many years. While studying $\mathbb{C}^*$-action on $\mathbb{C}^{[3]}$, M. Koras and P. Russell encountered a family of contractible threefolds which shared several properties of $\mathbb{A}^3_{\mathbb{C}}$ and observed that if any such threefold was isomorphic to $A^3_{\mathbb{C}}$, then it would lead to a nonlinearisable $\mathbb{C}^*$-action on $\mathbb{C}^{[3]}$. Any member of this family was known as a Koras–Russell threefold. One such example (also known as Russell's cubic) is the following ring:

$$A = \mathbb{C}[X, Y, Z, T]/(X^2Y + X + Z^2 + T^3).$$

Let $V = \mathrm{Max}(A)$ and let x denote the image of X in A. The ring A satisfies several properties of the polynomial ring $\mathbb{C}^{[3]}$, namely, it is a regular UFD with $A^* = \mathbb{C}^*$. Moreover, V is diffeomorphic to $\mathbb{R}^6$ with respect to Euclidean topology and there exists a dominant morphism from $\mathbb{A}^3_{\mathbb{C}}$ to V. By C.P. Ramanujam's or by M. Miyanishi's characterization of the affine plane, the corresponding properties suffice for a two-dimensional surface to be the affine plane. There was an excitement, especially in the AAG community, regarding the possibility that V is isomorphic to $\mathbb{A}^3_{\mathbb{C}}$. For, if $V = \mathbb{A}^3_{\mathbb{C}}$, i.e., $A = \mathbb{C}^{[3]}$, then the ring A would have led to a counterexample to the Abhyankar–Sathaye epimorphism conjecture for $\mathbb{C}^{[3]}$ which asserts that if $F \in \mathbb{C}^{[3]} = B$ is such that $B/(F) = \mathbb{C}^{[2]}$, then $B = \mathbb{C}[F]^{[2]}$. Now note that $A/(x - \lambda) = \mathbb{C}^{[2]}\ \forall \lambda \in \mathbb{C}^*$ but $A/(x) \neq \mathbb{C}^{[2]}$. Thus, if $A = \mathbb{C}^{[3]}$, then $A \neq \mathbb{C}[x - \lambda]^{[2]}$ for any $\lambda \in \mathbb{C}$, contradicting the Abhyankar–Sathaye conjecture.

In 1994, in a conference at McGill University, P. Russell discussed the open problem about the triviality of the above ring A and the linearization conjecture. L. Makar-Limanov, who was attending the conference, announced in the meeting that the ring A is not $\mathbb{C}^{[3]}$. His proof was elementary (though not easy) and involved an ingenious idea. He showed that $x \in \mathrm{ML}(A)$ ([16]), i.e., for every $\mathbb{G}_a$-action on V (equivalently any LND D on A), the regular function determined by x on V is fixed (i.e., $x \in \mathrm{Ker}(D)$). On the other hand it is easy to see that $\mathrm{ML}(\mathbb{C}[X, Y, Z]) = \mathbb{C}$. As $x \notin A^* = \mathbb{C}^*$, it follows that $\mathrm{ML}(A) \neq \mathbb{C}(= \mathrm{ML}(\mathbb{C}^{[3]}))$. Thus, $A \ncong \mathbb{C}^{[3]}$. Later, Kaliman and Makar-Limanov proved that none of the Russell–Koras threefolds is isomorphic to the affine 3-space. This led to the solution of the linearization conjecture for $\mathbb{A}^3_{\mathbb{C}}$ by Kaliman–Koras–Makar-Limanov–Russell ([14]).

The ring A is now a potential threat to the Zariski cancellation conjecture for the affine three space in characteristic zero. For if $A^{[1]} = \mathbb{C}^{[4]}$, then A would be a counterexample to the ZCP in characteristic zero for $n = 3$. A. Dubouloz has shown in 2009 ([4]) that $\mathrm{ML}(A^{[1]}) = \mathbb{C}$ and A. Dubouloz and J. Fasel have shown in 2018 that V is "$\mathbb{A}^1$-contractible" ([5]). The variety V is in fact the first example of an $\mathbb{A}^1$-contractible threefold which is not algebraically isomorphic to $\mathbb{C}^3$.

5. The Asanuma Threefold

We first recall the affine fibration problem. For a ring R and any prime ideal P of R, the notation $k(P)$ denotes the residue field of the local ring R_P. It is also the same as the field of fractions of the integral domain R/P.

Definition. Let R be a ring. A finitely generated flat R-algebra B is said to be an $\mathbb{A}^n$-*fibration* over R if $B \otimes_R k(P) = k(P)^{[n]}$ for each prime ideal P of R.

A major problem in the area of affine fibrations is the following question of Dolgačev and Veĭsfeĭler.

Question 2. Let R be a regular local ring of dimension d and A be an $\mathbb{A}^n$-fibration over R. Is A necessarily a polynomial ring over R?

For a survey on the above problem one may see [6, Section 3.1]. T. Kambayashi, M. Miyanishi, and David Wright have shown that Question 5 has an affirmative answer for $\mathbb{A}^1$-fibrations, i.e., any $\mathbb{A}^1$-fibration over a regular local ring is necessarily a polynomial ring. Their results were further refined by A.K. Dutta who showed that it is enough to assume the fiber conditions only on generic and codimension one fibers.

In 1983, A. Sathaye obtained the following major breakthrough on $\mathbb{A}^2$-fibrations ([20]):

Theorem 5.1. *Let R be a PID containing $\mathbb{Q}$ and A be an $\mathbb{A}^2$-fibration over R. Then $A = R^{[2]}$.*

In 1987, T. Asanuma made the next major breakthrough on the affine fibration problem: a deep structure theorem on $\mathbb{A}^n$-fibrations ([1, Theorem 3.4]). From his main structure theorem, he deduced the following stable structure theorem for any affine fibration over a regular local ring ([1, Corollary 3.5]).

Theorem 5.2. *Let R be a regular local ring of dimension $d \geq 1$ and A be an $\mathbb{A}^n$-fibration over R. Then $A^{[m]} = R^{[m+n]}$ for some integer $m \geq 0$.*

In the same paper, Asanuma also constructed the first counter-example to the $\mathbb{A}^2$-fibration problem over a PID not containing $\mathbb{Q}$ ([1, Theorem 5.1]). We present below a version of Asanuma's example.

Example 5.3. Let k be a field of characteristic $p > 0$ and let

$$A = k[X,Y,Z,T]/(X^mY + Z^{p^e} + T + T^{sp}),\ m,e,s \in \mathbb{N} \\ \text{and}\ \ p^e \nmid sp,\ sp \nmid p^e. \tag{2}$$

Let x denote the image of X in A. Then $k[x] \subset A$ and the following properties are satisfied by A (cf. [1, Theorem 5.1]):

(1) $A \otimes_{k[x]} k(P) = k(P)^{[2]}$ for every prime ideal P of $k[x]$.
(2) $A^{[1]} = k[x]^{[3]} = k^{[4]}$.
(3) $A \neq k[x]^{[2]}$.

We note that (3) means that "A is not a polynomial ring over its subring $k[x]$". It does not imply that "A is not a polynomial ring over k", i.e., $A \neq k^{[3]}$.

(1) shows that A is an $\mathbb{A}^2$-fibration over $k[x]$; (2) shows that A is a stably polynomial ring over $k[x]$, in particular, a stably polynomial ring over k; and (3) shows that A is not a polynomial ring over $k[x]$. Thus A is a nontrivial $\mathbb{A}^2$-fibration over $k[x]$, apart from being a nontrivial stably polynomial ring over $k[x]$.

This ring A was soon to acquire a wider significance. In a subsequent paper ([2, Theorem 2.2]), using the ring A, Asanuma constructed nonlinearizable k^*-actions on $\mathbb{A}^n_k$ over any infinite field k of positive characteristic when $n \geq 4$. He then asked whether A is a polynomial ring and explained the significance of his question as follows ([2, Remark 2.3]):

> *"If A is a polynomial ring then it will give an example of a nonlinearizable torus action on k^3 in positive characteristic. On the other hand if A is not a polynomial ring then it will clearly give a counter-example to the cancellation problem."*

Thus either way one would answer a major problem in AAG. This dichotomy has been popularized by P. Russell as "Asanuma's Dilemma".

In [10], using some basic results on exponential maps, the author has shown that the ring A in Example 5.3 is not a polynomial ring when $m \geq 2$. Consequently, it follows that the ZCP does not have an affirmative answer in positive characteristic. In [11], the author made further investigations on the Asanuma ring discussed below.

Recall that a polynomial $g \in k[Z,T]$ is called a *line* if $k[Z,T]/(g) = k^{[1]}$ and a line g is called a *nontrivial line* if $k[Z,T] \neq k[g]^{[1]}$. The famous epimorphism theorem of S.S. Abhyankar and T. Moh (also proved independently by M. Suzuki for $k = \mathbb{C}$) asserts that there does not exist any nontrivial line over any field of characteristic zero. However, as early as in 1957, B. Segre had exhibited an example of a nontrivial line over any field of positive characteristic. Later M. Nagata gave a family of examples of such nontrivial lines. For more details on the epimorphism problem, see [6, Section 2].

Asanuma's three-dimensional ring in Example 5.3 can be considered as a special case of the general class of threefolds in $\mathbb{A}^4_k$ defined by the zero locus of a polynomial of the form $X^mY - f(Z,T)$, where $f(Z,T)$ is a Segre–Nagata nontrivial line. This led us to a problem which was asked to the author independently by P. Russell.

Question 3. Let f be any nontrivial line in $k[Z,T]$ and let A be a ring defined by the relation $x^ry = f(z,t)$. Is the ring A necessarily not a polynomial ring over the field k?

As classification of nontrivial lines is still an open problem, Russell's question has to be approached abstractly. In [11], the author has considered a more general threefold and answered Russell's question affirmatively as a part of a general theory which is independent of the characteristic of the field. This generalization is more transparent and conceptual, and has simplified the earlier proof by the author in [10] of the noncancellative property of $k^{[3]}$ in positive characteristic. The precise statement proved by the author is ([11, Theorem 3.11]):

Theorem 5.4. *Let k be a field of any characteristic and $F \in k[X,Z,T]$. Let $f(Z,T) = F(0,Z,T)$, $G = X^rY - F(X,Z,T) \in k[X,Y,Z,T]$, where $r > 1$ and $A = k[X,Y,Z,T]/(G)$. Then the following statements are equivalent:*

(i) *$f(Z,T)$ is a variable in $k[Z,T]$, i.e., $k[Z,T] = k[f]^{[1]}$.*
(ii) *$A = k[x]^{[2]}$, where x denotes the image of X in A.*
(iii) *$A = k^{[3]}$.*
(iv) *G is a variable in $k[X,Y,Z,T]$, i.e., $k[X,Y,Z,T] = k[G]^{[3]}$.*
(v) *G is a variable in $k[X,Y,Z,T]$ along with X, i.e., $k[X,Y,Z,T] = k[X,G]^{[2]}$.*

Theorem 5.4 answers in one statement several very different looking questions that had been of long interest in the field. The equivalence of (i) and (iii) answers Russell's question affirmatively. It also explains the nontriviality of the Russell–Koras threefold defined in Section 4. On the other hand the following theorem of the author shows that if f is a line, then A is a stably polynomial ring. More precisely ([11, Theorem 4.2]):

Theorem 5.5. *Let k be a field of any characteristic, $F \in k[X,Z,T]$ and let*

$$A = k[X,Y,Z,T]/(X^rY - F(X,Z,T)), \text{ where } r \geq 1.$$

Then A is a stably polynomial ring if $F(0,Z,T)$ is a line in $k[Z,T]$. That is, $A^{[1]} = k[x]^{[3]} = k^{[4]}$ if $k[Z,T]/(F(0,Z,T)) = k^{[1]}$.

By the equivalence of (i) and (iii) in Theorem 5.4, it follows that A is not a polynomial ring if $k[Z,T] \neq k[F(0,Z,T)]^{[1]}$. Thus, if $F(0,Z,T)$ is a nontrivial line in $k[Z,T]$, then A is a stably polynomial ring but not a polynomial ring over k. This gives a recipe for constructing counter-examples to the ZCP. We emphasize again that the proofs of Theorems 5.5 and 5.4 are independent of the characteristic of the field. However, we know by the Abhyankar–Moh–Suzuki theorem that a nontrivial line never exists in characteristic zero. Thus, for obtaining counter-examples to the cancellation problem for affine 3-space, an application of Theorems 5.5 and 5.4 can be made only in positive characteristic.

Theorem 5.4 was achieved through the study of another invariant of a ring arising out of exponential maps that was introduced by H. Derksen, and is now known as the Derksen invariant. For an affine k-algebra R, the Derksen invariant of R is the subring of R, denoted by $\mathrm{DK}(R)$ and defined by

$$\mathrm{DK}(R) := k[f \in R^{\phi} \mid \phi \in \mathrm{Exp}(R) \text{ and } B^{\phi} \neq B].$$

A major result used in the proof of Theorem 5.4 is the following proposition.

Proposition 5.6. *Let A be an integral domain defined by*

$$A = k[X,Y,Z,T]/(X^mY - F(X,Z,T)), \text{ where } m > 1.$$

Set $f(Z,T) := F(0,Z,T)$. Let x, y, z, and t denote, respectively, the images of X, Y, Z, and T in A. Suppose that $\mathrm{DK}(A) \neq k[x,z,t]$. Then the following statements hold.

(i) *There exist $Z_1, T_1 \in k[Z,T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z,T] = k[Z_1,T_1]$ and $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$.*
(ii) *If $k[Z,T]/(f) = k^{[1]}$, then $k[Z,T] = k[f]^{[1]}$.*

From Proposition 5.6, it follows that in case A is as in Example 5.3 with $m \geq 2$, then $\mathrm{DK}(A) = k[x,z,t] \subsetneq A$. Thus $\mathrm{DK}(A) \neq A$ and hence $A \neq k^{[3]}$, as $\mathrm{DK}(k^{[3]}) = k^{[3]}$.

Subsequently, the author generalized the Asanuma threefold to obtain counterexamples to the ZCP in positive characteristic in higher dimensions ([12]). Recently with Parnashree Ghosh, the author has used exponential maps to obtain the following generalization of Theorem 5.4 in higher dimensions ([9]).

Theorem 5.7. *Let $F = F(X_1,\dots,X_m,Z,T) = f(Z,T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1,\dots,X_m,Z,T]$, $G := X_1^{r_1} \cdots X_m^{r_m}Y - F$ and*

$$A = \frac{k[X_1,\dots,X_m,Y,Z,T]}{(X_1^{r_1} \cdots X_m^{r_m}Y - F(X_1,\dots,X_m,Z,T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m, \tag{3}$$

Then the following statements are equivalent:

(i) *$k[X_1,\dots,X_m,Y,Z,T] = k[X_1,\dots,X_m,G]^{[2]}$.*
(ii) *$k[X_1,\dots,X_m,Y,Z,T] = k[G]^{[m+2]}$.*
(iii) *$A = k[x_1,\dots,x_m]^{[2]}$.*
(iv) *$A = k^{[m+2]}$.*
(v) *$k[Z,T] = k[f(Z,T)]^{[1]}$.*

Moreover, each of the above conditions is equivalent to each of the following four conditions involving the Derksen invariant; and each of the subsequent four conditions involving the Makar-Limanov invariant.

(vi) *$A^{[l]} = k^{[l+m+2]}$ for some $l \geqslant 0$ and $k[x_1,\dots,x_m,z,t] \subsetneq \mathrm{DK}(A)$.*
(vii) *A is an $\mathbb{A}^2$-fibration over $k[x_1,\dots,x_m]$ and $k[x_1,\dots,x_m,z,t] \subsetneq \mathrm{DK}(A)$.*
(viii) *$f(Z,T)$ is a line in $k[Z,T]$ and $k[x_1,\dots,x_m,z,t] \subsetneq \mathrm{DK}(A)$.*
(ix) *A is a UFD, $k[x_1,\dots,x_m,z,t] \subsetneq \mathrm{DK}(A)$ and $\left(\frac{A}{x_iA}\right)^* = k^*$, for every $i \in \{1,\dots,m\}$.*
(x) *$f(Z,T)$ is a line in $k[Z,T]$ and $\mathrm{ML}(A) = k$.*
(xi) *$A^{[l]} = k^{[l+m+2]}$ for $l \geqslant 0$ and $\mathrm{ML}(A) = k$.*
(xii) *A is an $\mathbb{A}^2$-fibration over $k[x_1,\dots,x_m]$ and $\mathrm{ML}(A) = k$.*
(xiii) *A is a UFD, $\mathrm{ML}(A) = k$ and $\left(\frac{A}{x_iA}\right)^* = k^*$, for every $i \in \{1,\dots,m\}$.*

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Neena Gupta

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On the Geometry of Metric Spaces

Manuel Ritoré

1. Introduction

The measurement of distances because of practical reasons, as the delimitation of parcels of farmland after floodings or those related to construction problems, lies behind the origins of mathematics in ancient civilizations. It was Euclid, in Proposition 20 of the first book of the *Elements*, who proved that the sum of two sides of a triangle is greater than the remaining one. This fact was probably used by Archimedes to state as assumption, in his work *On the sphere and cylinder*, that the straight line is the shortest between two points. Both achievements are now proven in basic courses on Linear Algebra.

Manuel Ritoré is a professor of mathematics at the University of Granada and editor-in-chief of Analysis and Geometry in Metric Spaces. *His email address is* ritore@ugr.es.

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Presently, the notion of *distance* on a nonempty set X is formalized as a function $d : X \times X \to \mathbb{R}$ satisfying

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

These conditions imply $d \geqslant 0$. The second property is the *symmetry* one. In Gromov's words this assumption "[...] unpleaseantly limits many applications." Removing this symmetry condition gives rise to the concept of *asymmetric distance*, see Busemann's *Local metric geometry* (1958). The third property is the well-known *triangle inequality*.

The pair (X, d), where X is a nonempty set and d a distance on X, is called a *metric space*.

Metric spaces were introduced by Fréchet in 1906 in the context of functional analysis. Shortly after, a comprehensive treatment of the theory was presented by Hausdorff in the second part of his monograph [Hau14] published in 1914, where the notion of topological space was introduced.

Metric spaces appear everywhere in modern mathematics. They are an invaluable tool in functional analysis, geometry of manifolds, graph theory, artificial intelligence, and many other fields. The aim of this note is to give a glance of the role of metric spaces in several fields of mathematics without any intention of being exhaustive. We focus on different notions of curvature in metric spaces, compactness results for Riemannian manifolds, and an application of metric graph theory to group theory.

2. Length Spaces

Given a subset of a Euclidean space we may consider the restriction of the Euclidean distance to the set. For instance, the distance between two antipodal points on a unit sphere is 2. However, this distance is not intrinsic in the sense that cannot be measured *inside* the subset.

Given a continuous curve $\gamma : [a,b] \to X$ on a metric space, its *length* $L(\gamma)$ can be defined as the supremum of the quantities

$$\sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

over all partitions $a = t_0 < t_1 < \cdots < t_k = b$, $k \in \mathbb{N}$, of the interval $[a,b]$. When $L(\gamma)$ is finite the curve γ is said to be *rectifiable*. Hence the length of curves, sometimes infinite, can be measured in metric spaces.

A metric space (X,d) is a *length space* if the distance $d(x,y)$ between any pair of points $x,y \in X$ can be computed as

$$\inf_{\gamma} L(\gamma),$$

where $\gamma : [a,b] \to X$ is any rectifiable curve connecting x and y (i.e., such that $\gamma(a) = x$, $\gamma(b) = y$). A length space (X,d) is *geodesic* if for every pair of points $x,y \in X$, there exists a rectifiable curve γ connecting x,y, such that $d(x,y) = L(\gamma)$. A *geodesic* in X is a rectifiable curve $\gamma : [a,b] \to X$ such that $d(\gamma(t),\gamma(s)) = L(\gamma|_{[s,t]})$ for all $a \leqslant s < t \leqslant b$. The interested reader is referred to Burago, Burago and Ivanov's [BBI01] for a complete presentation of length spaces.

Examples of length spaces are *Riemannian*, *Finsler* and *sub-Riemannian* manifolds. A *Riemannian* manifold (M,g) is a manifold M together with an scalar product g on the fiber bundle (a continuously varying scalar product on each tangent space). In a Finsler manifold the scalar product is replaced by a smooth norm. A *sub-Riemannian* manifold $(M,\mathcal{H},h)$ is a manifold M together with a nonintegrable horizontal distribution $\mathcal{H}$ and a positive definite scalar product h on $\mathcal{H}$. The classical example of a sub-Riemannian manifold is the Heisenberg group $\mathbb{H}^1$: the 3-dimensional Euclidean space $\mathbb{R}^3$ together with a noncommutative product and the left-invariant vector fields

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}$$

induce a left-invariant Riemannian metric g_L making X,Y,Z an orthonormal basis. The vectors fields X,Y generate a nonintegrable distribution $\mathcal{H}$ that, together with the restriction of g_L to $\mathcal{H}$, provide a sub-Riemannian structure on $\mathbb{H}^1$. On a connected sub-Riemannian manifold M, Chow–Rashevskii theorem, see §0.4 and §1.1, §1.2 in Gromov [Gro96], implies that every pair of points can be connected by a smooth *horizontal* curve on M (i.e., everywhere tangent to the horizontal distribution).

To define a distance we consider, in the Riemannian and Finsler cases, the class of piecewise smooth curves and, in the sub-Riemannian case, the class of piecewise smooth horizontal curves. In all cases, the length of a curve $\gamma : I \to M$ can be computed as

$$\int_I \|\gamma'(s)\| \, ds,$$

where $\|\cdot\|$ is the norm associated to g in the Riemannian case, the Finsler norm, or the one associated to h in the sub-Riemannian case. Associated to this length we can define a distance between two points as the infimum of the length of the curves joining both points. This way we obtain the Riemannian distance, the Finsler distance, or the *Carnot-Carathéodory* distance. Geodesics are curves of minimum distance connecting two given points.

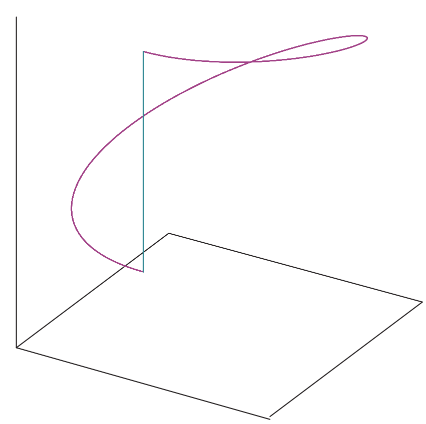

Figure 1. Geodesics in the Riemannian and sub-Riemannian Heisenberg group $\mathbb{H}^1$ connecting two points in the same vertical line. The vertical line is a Riemannian geodesic. The helicoidal curve is a sub-Riemannian geodesic, forced to be tangent to the horizontal distribution.

3. Metric Spaces with Bounded Curvature

Curvature in Riemannian geometry was introduced as a measure of how a Riemannian space deviates from Euclidean space, see Gromov [Gro91] for an enlightening discussion. For surfaces embedded in Euclidean space, the curvature coincides with the product of the principal curvatures of the surface and it is referred to as the Gauß curvature. The classical Gauß–Bonnet theorem provides a first hint on how the curvature K affects the behavior of geodesic triangles on a surface. If T is a triangle delimited by geodesics making inner angles α, β and γ then

$$\int_T K = (\alpha + \beta + \gamma) - \pi.$$

Hence the sum of the inner angles of a triangle on a surface depends on the sign of the Gauß curvature: it is equal to π on surfaces with $K = 0$, larger than π on surfaces with $K \geqslant 0$ and smaller than π when $K \leqslant 0$.

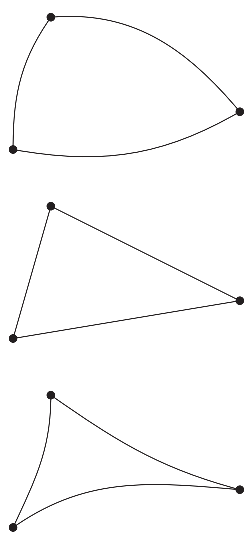

Figure 2. Behavior of geodesics triangles according to Gauß–Bonnet theorem.

According to the historical account in the preface of Alexander, Kapovitch, and Petrunin's monograph, see arXiv:1903.08539, the first synthetic description of curvature is due to A. Wald in a paper published in 1936, followed by a paper by Alexandrov in 1941. It is generally considered that Rauch's comparison results for Jacobi fields in the early 1950s can be thought of as an *infinitesimal* triangle comparison result which was soon followed by the celebrated Toponogov's comparison theorem in the late 1950s.

A version of triangle comparison not involving angles can be stated as follows. Assume we have a point $p \in M$ in a Riemannian manifold and a unit-speed geodesic $\gamma : [0, T] \to M$ in a small neighborhood of p. We consider a second unit-speed geodesic $\bar{\gamma} : [0, T] \to \mathbb{R}^2$ in the plane and a point $\bar{p} \in \mathbb{R}^2$ such that $|\bar{p} - \bar{\gamma}(0)| = d(p, \gamma(0))$, $|\bar{p} - \bar{\gamma}(T)| = d(p, \gamma(T))$. This way we construct a comparison triangle whose vertices $\bar{\gamma}(0)$, $\bar{\gamma}(T)$, $\bar{p}$ lie at the same distance as the ones of the original triangle $\gamma(0), \gamma(T), p$. We define the functions $f(t) = d(p, \gamma(t))$, $\bar{f}(t) = |\bar{p} - \bar{\gamma}(t)|$. Then we have the following.

Theorem. *Under the conditions above*

1. *If M has nonnegative sectional curvatures then $f(t) \geqslant \bar{f}(t)$.*
2. *If M has nonpositive curvature then $f(t) \leqslant \bar{f}(t)$.*

This result is typically proven by means of Jacobi fields, see Cheeger and Ebin (1975), but perhaps the reader could find useful the following alternative argument. Assume that the sectional curvatures of M are no larger than 0. Take $p \in M$, a normal neighborhood U of p where the exponential map is a diffeomorphism, and a geodesic $\gamma : [0, T] \to U$ parameterized by arc length. A basic result on Jacobi fields, similar to the one used in the proof of Bishop's volume comparison, implies that the Hessian ∇^2 of the function $d^2/2$, where d is the distance to p, satisfies

$$\nabla^2\left(\tfrac{1}{2}d^2\right)(v, v) \geqslant 1 \tag{1}$$

for any $v \in T_qM$, $q \in U$, and $|v| = 1$. In Euclidean space we would have equality in (1) replacing d by the Euclidean distance $\bar{d}$ to $\bar{p}$. Then the functions $h(t) = d^2(\gamma(t))/2$ and $\bar{h}(t) = \bar{d}^2(\bar{\gamma}(t))/2$ satisfy

$$\begin{aligned} h''(t) &= \nabla^2\left(\tfrac{1}{2}d^2\right)(\gamma'(t), \gamma'(t)) \\ &\geqslant \nabla^2\left(\tfrac{1}{2}\bar{d}^2\right)(\bar{\gamma}'(t), \bar{\gamma}'(t)) = \bar{h}''(t). \end{aligned}$$

Hence $(h - \bar{h})'' \geqslant 0$ on $[0, T]$ and, since the function $h - \bar{h}$ vanishes at the endpoints of the interval, we get $h \leqslant \bar{h}$, which implies $f \leqslant \bar{f}$.

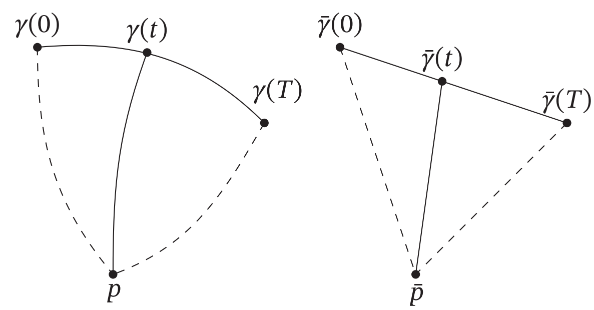

Figure 3. Illustration of triangle comparison. The one to the right lies in a Riemannian manifold M. The one to the left is a comparison triangle in the Euclidean plane. If the sectional curvature of M is nonnegative then $d(p, \gamma(t)) \geqslant |\bar{p} - \bar{\gamma}(t)|$. The opposite inequality holds when M has nonpositive sectional curvature.

This comparison can be stated also in terms of angles. Consider a small geodesic triangle in a Riemannian manifold M. If M has nonnegative curvature then, at every vertex, the angle made by the sides of the triangle is larger than or equal to the comparison angle at the vertex (i.e., the Euclidean angle of a comparison triangle). For nonpositive curvature we have the opposite inequality. Similar comparisons with a simply connected surface of constant curvature $\varkappa$ hold when the sectional curvature of M is no smaller or no larger than $\varkappa$.

The triangle comparison for Riemannian manifolds allows to introduce a synthetic notion of metric space X with curvature bounded, above or below, in terms of properties of geodesic triangles on the metric space.

Note that angles can be defined in a metric space using the law of cosines. Given three different points x, y, z in a metric space X, we can build a comparison Euclidean triangle of vertices $\bar{x}, \bar{y}, \bar{z}$ by requiring that the distances $d(x,y), d(y,z), d(x,z)$ equal $|\bar{x}-\bar{y}|$, $|\bar{y}-\bar{z}|$, $|\bar{x}-\bar{z}|$. The *comparison angle* $\tilde{\measuredangle}xyz$ is defined as the Euclidean angle $\measuredangle\bar{x}\bar{y}\bar{z}$, that is,

$$\tilde{\measuredangle}xyz = \arccos \frac{d(y,x)^2 + d(y,z)^2 - d(x,z)^2}{2\,d(y,x)\,d(y,z)}.$$

If we have two curves $\alpha, \beta : [0, \varepsilon) \to X$ emanating from the same point x, the angle $\measuredangle(\alpha, \beta)$ is

$$\measuredangle(\alpha,\beta) = \lim_{t,s\to 0} \measuredangle\alpha(s)x\beta(t)$$

when the limit exists.

The term CAT(k) spaces is used for metric spaces with k as an upper curvature bound (CAT stands for Cartan–Alexandrov–Topogonov) and the term Alexandrov spaces for spaces with lower curvature bounds. The reader should be aware that sometimes a different terminology is used depending on the direction of the bound.

The theory of CAT(k) spaces and the one of Alexandrov spaces differ in many respects. In the case of curvature bounded above, *Hadamard manifolds*, complete simply connected length spaces of nonpositive curvature, play a central role. Almost all classical results for Riemannian manifolds have been extended to Hadamard manifolds, like uniqueness of geodesics connecting two given points, the validity of triangle comparison for large triangles, the fact that the squared distance function is globally concave. Essential references on CAT(k) spaces are Bridson and Haeffliger [BH99], and Ballman, Gromov, and Schroeder [BGS85]. As for Alexandrov spaces, there are two main points that make the theory quite different from the one of CAT(k) spaces. The first is that triangle comparison holds for arbitrarily large triangles without additional assumptions. This is the content of Toponogov's theorem. The second one concerns the local structure: in an Alexandrov space the Hausdorff dimension coincides with the topological dimension, which is an integer or infinite. An Alexandrov space is a manifold except on a small set of points. Examples of Alexandrov spaces are the boundaries of convex sets in Euclidean spaces. Numerous results valid for Riemannian manifolds also hold in the class of Alexandrov spaces with curvature bounded from below by k. To cite just a few, an upper bound on the diameter when $k > 0$, a splitting theorem, Gromov–Bishop inequalities for the volume of balls. Petrunin also proved a Levy–Gromov type isoperimetric inequality (see the author's monograph [Rit23] for the Riemannian case). The reader is referred to Burago, Burago and Ivanov [BBI01], the already mentioned monograph by Alexander, Kapovitch, and Petrunin, and the references cited therein, for excellent introductions to the theory of metric spaces with bounds on curvature.

4. Convergence of Metric Spaces

The introduction of curvature bounds on metric spaces might seem an abstract construction with no special interest. However, it is related to the behavior of sequences of Riemannian manifolds. One can consider the space of all Riemannian manifolds with some topology and asks whether some given subset is compact or precompact with this topology. In many cases the limit of the sequence is not a Riemannian manifold. This is one of the situations where metric spaces with bounds on curvature play a role.

A standard way of measuring the distance between two sets A, B in a metric space (X, d) is the *Hausdorff distance* $d_H(A, B)$, defined by

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon, B \subset A_\varepsilon\},$$

where for any set $C \subset X$, $C_\varepsilon = \{x \in X : d(x, C) < \varepsilon\}$ is the open tubular neighborhood of C of radius ε.

Gromov, see [Gro81, Gro07] introduced a way of measuring the distance between two metric spaces X, Y, nowadays known as the *Gromov–Hausdorff distance*, denoted by $d_{GH}(X, Y)$. To compute it, we take all metric spaces Z which contain isometric copies X' of X and Y' of Y, compute their Hausdorff distance $d_H(X', Y')$ on Z, and take the infimum over all possible metric spaces Z.

Gromov–Hausdorff distance is really a distance on the space of isometric classes of metric spaces since two isometric metric spaces have d_{GH}-distance equal to 0. To measure the Gromov–Hausdorff distance using its definition is not very practical. However, for compact metric spaces X, Y, should we have two ε-nets $\{x_1, \dots, x_n\}$ on X, and $\{y_1, \dots, y_n\}$ on Y such that

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| < \delta \quad \text{for all } i, j = 1, \dots, n,$$

then we would have $d_{GH}(X, Y) < 2\varepsilon + \delta$.

Important classes of Riemannian manifolds which are precompact in the Gromov–Hausdorff distance are the following:

- For $m \in \mathbb{N}$ and $R, V > 0$, the class of m-dimensional Riemannian manifolds with volume $|M| \leqslant V$ and injectivity radius $\operatorname{inj}(M) \geqslant R$.
- For any $m \in \mathbb{N}$ and $\varkappa \in \mathbb{R}$, $D > 0$, the class of m-dimensional Riemannian manifolds with $\operatorname{diam} M \leqslant D$ and sectional curvature $\geqslant \varkappa$. The same result holds assuming instead that the Ricci curvature is $\geqslant \varkappa$.

While the Gromov–Hausdorff distance is useful when considering *compact* metric spaces, it is usually a very strong convergence for unbounded metric spaces. In this case it should be replaced by the *pointed* Gromov–Hausdorff convergence.

Natural measures which can be considered on metric spaces are the *Hausdorff measures*. Given $s \geqslant 0$, the s-dimensional Hausdorff measure associated to a set $A \subset X$ is given, up to a constant, by

$$\mathcal{H}^s(A) = \inf_{\delta>0} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A)$ is the supremum of $\sum_i \operatorname{diam}(E_i)^s$ taken over all coverings of A by sets E_i satisfying $\operatorname{diam}(E_i) \leqslant \delta$. For a metric space X there exists a constant $N \in [0, +\infty]$ such that $\mathcal{H}^s(X) = 0$ for $s > N$ and $\mathcal{H}^s(X) = \infty$ for all $s < N$. The quantity N is called the *Hausdorff dimension* of X and is denoted by $\dim_H(X)$.

It is also important to note that Gromov–Hausdorff limits (see next section) of Alexandrov spaces of curvature $\geqslant \kappa$ are themselves Alexandrov spaces of curvature $\geqslant \kappa$. There are also compactness results for Alexandrov spaces. Gromov himself proved that the space $\mathcal{M}(n, \kappa, D)$ composed of the Alexandov spaces with curvature $\geqslant \kappa$, diameter $\leqslant D$ and Hausdorff (topological) dimension $\leqslant n$ is compact in the Gromov–Hausdorff topology.

Although the Hausdorff measure $\mathcal{H}^N$ on a metric space (X, d), with $N = \dim_H(X)$, is a natural measure to consider, sometimes it is more interesting to take a different one, see Appendix 2 in [CC97]. This leads to the notion of *metric measure space* (X, d, μ), which is nothing but a metric space (X, d) together with a Borel measure μ on X. If (X, d) is a length (geodesic) space we refer to (X, d, μ) as a *length (geodesic) measure space*.

A sequence (X_i, d_i, μ_i) of metric measure spaces converges in measured Gromov–Hausdorff topology to a metric measure space (X, d, μ) if there is a sequence of measurable maps $f_i : X_i \to X$ and a sequence of real numbers ε_i converging to 0 such that the maps f_i are $(1, \varepsilon_i)$-quasi-isometries:

$$\left| d(f_i(x), f_i(x')) - d_i(x, x') \right| < \varepsilon_i \qquad x, x' \in X_i,$$

the tubular neighborhood of radius ε_i of $f_i(X_i)$ is X, and the push-forward measures $(f_i)_* \mu_i$ converge in weak topology to μ, cf. Fukaya (1987).

5. Metric Spaces with a Lower Bound on Ricci Curvature

That the curvature of a Riemannian manifold implies metric properties of triangles was the key to introduce a synthetic notion of metric spaces with bounded curvature. An analogous approach should be possible for the Ricci curvature on a Riemannian manifold.

It looks like the first ones to discuss the possibility of such generalization were Cheeger and Colding in Appendix 2 in [CC97], who suggested to use lower bounds on the volume growth of sectors. See also §5.44 in Gromov [Gro07]. In a Riemannian manifold M, the classical Bishop–Gromov theorem implies that if $\mathbb{M}_\kappa$ is a model Riemannian manifold (complete simply connected with constant sectional curvature κ) of the same dimension as M and satisfying $\mathrm{Ric}_M \geqslant \mathrm{Ric}_{\mathbb{M}_\kappa}$, we have

$$\frac{|A(p, r_2, r_3)|}{|A(p, r_1, r_2)|} \leqslant \frac{|A_0(r_2, r_3)|}{|A_0(r_1, r_2)|}$$

for arbitrary $r_1 \leqslant r_2 \leqslant r_3$, where $A(p, r, s)$ is the annulus $\{q \in M : r \leqslant d(q, p) \leqslant s\}$, and $A_0(r, s)$ the corresponding annulus in M_0, whose volume is independent of the base point. The symbol $|\cdot|$ denotes the Riemannian volumes. This inequality is also valid if we consider a *geodesic sector $S \subset M$ with focal point p*, which is nothing but a set S such that, for each $q \in S$, there is a geodesic segment connecting p and q. Let $S(r, s) = S \cap A(p, r, s)$. Then we have

$$\frac{|S(r_2, r_3)|}{|S(r_1, r_2)|} \leqslant \frac{|A_0(r_2, r_3)|}{|A_0(r_1, r_2)|}$$

for arbitrary $r_1 < r_2 < r_3$.

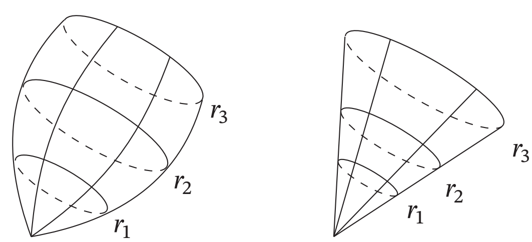

Figure 4. Comparison of volume of geodesic sectors.

Another synthetic approach to Ricci curvature mentioned in Cheeger and Colding [CC97] is the use of the Laplacian of the distance function. The classical Bishop volume comparison for balls in an m-dimensional manifold M with $\mathrm{Ric} \geqslant (m-1)\kappa$ follows from comparison of the mean curvatures of geodesic spheres with the same radii in M and the space form $\mathbb{M}_\kappa$ of constant sectional curvature κ. The mean curvature of a geodesic sphere is the Laplacian of the distance function d to a fixed point. So, for instance, when $\kappa = 0$, this translates into the equation

$$\Delta d \leqslant (m-1)\frac{1}{d} \tag{2}$$

at least for small radius. Note that $(m-1)/d$ is the mean curvature of a ball of radius d in the Euclidean space $\mathbb{R}^m$. Equation (2) was shown to hold in a weak sense in Riemannian manifolds by Calabi (1958). In order for this approach to work we need the extension of the notion of Laplacian to a metric space. Observe that for a Lipschitz function on a metric measure space (X, d, μ) we can define

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)},$$

and the *Cheeger energy*

$$\mathrm{Ch}(f) = \int_X |\nabla f|^2 \, d\mu.$$

By approximation of measurable functions by Lipschitz functions, this energy can be extended to wider classes. In general, Ch is not a quadratic form. We say that a metric measure space is *infinitesimally Hilbertian* if the Cheeger energy is a quadratic form in $L^2(X, \mu)$. On such spaces it is possible to define a weak notion of Laplacian, thus giving sense to inequality (2). This extension of differential calculus to metric spaces has been carried out by Gigli [Gig15]. See also Ambrosio's lecture at ICM2018 [Amb18]. A recent monograph by Heinonen, Koskela, Shanmugalingam, and Tyson (2015) is a very good introduction to analysis on metric spaces.

Yet another generalization of the notion of Ricci curvature comes from the well-known Bochner's formula. Recall that Bochner's formula in a Riemannian manifold reads

$$\frac{1}{2}\Delta|\nabla u|^2 = \nabla u(\Delta u) + |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u). \tag{3}$$

If we assume $\mathrm{Ric} \geqslant K$ and use the estimate $|\nabla^2 u|^2 \geqslant (\Delta u)^2/m$ we arrive at Bochner's inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geqslant \nabla u(\Delta u) + \frac{(\Delta u)^2}{m} + K|\nabla u|^2.$$

Here $m = \dim M$ and, of course, can be replaced by a larger number. All the operators appearing in this formula can be defined weakly on a metric measure space assuming some extra hypotheses, thus providing a notion of Ricci curvature bounded below.

5.1. An approach based on mass transport.

In the first decade of the 21st century, Lott and Villani [LV09] and Sturm [Stu06a, Stu06b] introduced independently equivalent notions of metric measure spaces with Ricci curvature bounded below. These notions were based on mass transportation properties on Riemannian manifolds with Ricci curvature bounded below.

Let us introduce first a few concepts, see Villani [Vil09] for an excellent exposition. Given a metric space (X, d) we consider the space $\mathcal{P}(X)$ of probability measures on X and $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ the subset of those $\mu \in \mathcal{P}(X)$ satisfying

$$\int_X d^2(x, y) \, d\mu(y) < \infty$$

for some (all) $x \in X$. Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, a *transport plan* between μ_0 and μ_1 is a probability measure μ on $X \times X$ with marginals μ_0, μ_1 (i.e., the push-forward measures $(p_1)_*(\mu)$, $(p_2)_*(\mu)$ by the projections $p_i : X \times X \to X$, $i = 1, 2$, are μ_0 and μ_1). We denote the set of transport plans between μ_0 and μ_1 by $\mathrm{Plan}(\mu_0, \mu_1)$. The *Wasserstein distance* W_2 on $\mathcal{P}_2(X)$ is defined by

$$\mathrm{W}_2^2(\mu_0, \mu_1) = \min_{\mu \in \mathrm{Plan}(\mu_0, \mu_1)} \int_{X \times X} d^2(x_0, x_1) \, d\mu(x_0, x_1).$$

A minimizer of this problem is called an *optimal* transport plan between μ_0 and μ_1. The metric space $(\mathcal{P}_2(X), \mathrm{W}_2)$ inherits many properties of (X, d). The equivalence between geodesics in Wasserstein space and certain optimal transport plans was established by Lott and Villani [LV09, §2.3] and Sturm [Stu06a, §2.3].

Given (X, d, μ), we define the *Shannon* and *Rényi entropy functionals* for a measure ν by

$$\mathrm{Ent}(\nu) = \begin{cases} \int_X \rho \log(\rho) \, d\mu, & \text{if the integral exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Ent}_N(\nu) = -\int_X \rho^{1-1/N} \, d\mu,$$

for $N \in [1, \infty)$. Here $\nu = \rho\mu + \nu_s$ is the decomposition of ν as the sum of a measure absolutely continuous with respect to μ and a singular measure $\nu_s \perp \mu$.

We say that a function $f : \mathcal{P}_2(X) \to \mathbb{R}$ is *K-convex* if for any Wasserstein geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(X)$,

$$\begin{aligned} f(\Gamma(t)) \leqslant{} & (1 - t) f(\Gamma(0)) \\ & + t f(\Gamma(1)) - \frac{K}{2}(1 - t)\mathrm{W}_2^2(\Gamma(0), \Gamma(1)). \end{aligned}$$

Observe that 0-convex just means convex. Then we say that (X, d, μ) satisfies the $\mathrm{CD}(0, \infty)$ property (or that has curvature $\geqslant 0$) if, for any pair $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there is a Wasserstein geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(X)$ connecting μ_0, μ_1 such that $\mathrm{Ent} \circ \Gamma$ is convex. We say that (X, d, μ) satisfy the $\mathrm{CD}(0, N)$ property for $N \in [1, \infty)$ if, for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, there exists a Wasserstein geodesic $\Gamma : [0, 1] \to \mathcal{P}_2(X)$ connecting μ_0, μ_1 such that $\mathrm{Ent}_{N'} \circ \Gamma$ is convex for all $N' \geqslant N$.

Property $\mathrm{CD}(K, \infty)$ is defined the same way as $\mathrm{CD}(0, \infty)$ replacing convexity by K-convexity of $\mathrm{Ent} \circ \Gamma$ for any Wasserstein geodesic. Property $\mathrm{CD}(K, N)$, for $N \in [1, \infty)$, is defined by replacing K-convexity by a more involved condition in which the functions $1 - t$, t, are replaced by functions $\tau_{K,N'}^{(1-t)}$, $\tau_{K,N'}^{(t)}$, see Definition 5.4 in Ambrosio [Amb18].

The $\mathrm{CD}(K, N)$ property is satisfied for N-dimensional Riemannian manifolds with $\mathrm{Ric} \geqslant K$ and is stable under measured Gromov–Hausdorff convergence.

Several refinements were made later to the theory of $\mathrm{CD}(K, N)$ spaces. The spaces $\mathrm{CD}^*(K, N)$ satisfy locally the CD condition. Equivalence of CD and CD^* notions for nonbranching metric measure spaces has been proved by Cavalletti and Milman [CM21]. A geodesic space (X, d) is *nonbranching* if the map $f_t : \mathrm{Geo}(X) \to X^2$, taking any geodesic $\gamma : [0, 1] \to X$ to the pair $(\gamma(0), \gamma(t))$, is injective for all $t \in (0,1]$. Finally $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$ spaces (the R stands for Riemannian) mean that the *infinitesimally Hilbertian* hypothesis is added. Recently Erbar, Kuwada, and Sturm [EKS15] and Ambrosio, Mondino and Savaré [AMS19] obtained the validity of Bochner's inequality in a $\mathrm{RCD}(K,\infty)$ metric measure space.

Petrunin (2011) checked the compatibility of these notions with the one of curvature bounded below by proving that m-dimensional Alexandrov spaces with nonnegative curvature satisfy the curvature dimension condition $\mathrm{CD}(0,m)$ for $m < \infty$.

Another approach to a synthetic notion of lower bound on the Ricci curvature was introduced by Ohta (2007) in terms of the *metric contraction property*. This property is also equivalent, on a Riemannian manifold, to having a a lower bound on the Ricci curvature. Also Alexandrov spaces satisfy this property. Roughly speaking, the measure contraction property implies that for every set $A \subset X$ with $0 < \mu(A) < \infty$ and for every $x \in X$, the normalized restriction of the measure μ to A (i.e., $\mu|_A/\mu(A)$) can be transported in a controlled way along geodesics to the Dirac measure δ_x.

It is worth mentioning that the notion of $\mathrm{CD}(K,N)$ and the metric contraction property do not hold in sub-Riemannian manifolds, even in the simplest case of the Heisenberg groups, as shown by Juillet (2009). Recently Milman (2021), inspired by Barilari and Rizzi (2019), has suggested a quasi-convex relaxation $\mathrm{QCD}(Q,K,N)$ of the $\mathrm{CD}(K,N)$ condition for sub-Riemannian manifolds which coincides with the latter when $Q = 1$.

6. Metric Structures on Graphs

A *graph* $\mathcal{G} = (V,E)$ is composed of a set of vertices V and a set of edges E connecting two given vertices. We assume that no more than one edge connects two given vertices and that there are no loops, that is, edges connecting the same vertex. A graph can be *undirected*, meaning that every edge has no orientation, and in this case we denote an edge connecting the vertices e, v by $\{e,v\}$. A graph can also be *directed*, meaning that every edge has an initial and a final point. In this case, it is denoted by (e,v). The vertex e is the origin of the edge and v the endpoint. We say that e, v are *incident* to (e,v) (or $\{e,v\}$). Of course (e,v) and (v,e) are different edges in a directed graph.

Given two vertices e, v in a graph, a *path* connecting e and v is a sequence of vertices $x_0 = e, x_1, \dots, x_k = v$ so that $\{x_{i-1}, x_i\}$ (or (x_{i-1}, x_i)) is an edge for all $i = 1, \dots, k$. A *weight* function $\omega : E \to \mathbb{R}^+$ assigns to each edge ℓ a positive measure $\omega(\ell)$ of the edge. A graph is connected if any pair of vertices can be connected by a path.

We can define a distance on the set of vertices of an undirected connected graph with an arbitrary weight function by defining

$$d(e,v) = \inf \sum_{i=1}^{k} \omega(\{x_{i-1}, x_i\}),$$

where the infimum is taken over all paths $x_0, \dots, x_k$ connecting e and v. *Geodesics* or *shortest length paths* are defined as the ones that realize the distance between its extreme points. The portion of a path connecting two vertices in a geodesic is also a geodesic.

6.1. Asymmetric distances on graphs. If $\mathcal{G} = (V,E)$ is a directed graph with a weight function $\omega : E \to \mathbb{R}$, we can define a "distance" between two vertices e, v simply by setting

$$d(e,v) = \inf \sum_{i=1}^{k} \omega((x_{i-1}, x_i))$$

in case there is a path connecting e and v. The infimum is taken over all paths $x_0, \dots, x_k$ connecting e and v. If there is no such path we define

$$d(e,v) = \infty.$$

We also define $d(e,e) = 0$ for all $e \in V$. This way we have defined a map $d : V \times V \to \overline{R} = \mathbb{R} \cup \{\infty\}$ such that the following properties hold

1. $d(e,v) \geqslant 0$ for all $e, v \in V$,
2. $d(e,v) \leqslant d(e,w) + d(w,v)$ for all $e, v, w \in V$.

In the triangle inequality we have used $\infty + a = a + \infty = \infty + \infty = \infty$ for all $a \geqslant 0$. In case $\omega(\ell) \geqslant \varepsilon > 0$ for all $\ell \in E$ or some another alternative hypothesis is assumed we also have

1′. $d(e,v), d(v,e) > 0$ if $e \neq v$.

A map $d : V \times V \to \overline{R}$ satisfying 1,1′, and 2 is called an *asymmetric distance* on V. The pair (V,d) is called an *asymmetric metric space*.

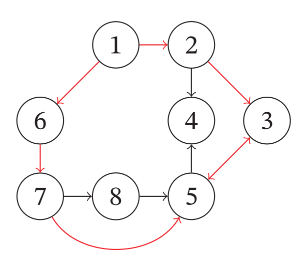

Figure 5. Geodesics or shortest length paths between vertices 1 and 5 in a directed graph depicted in red. The weight function here is $\omega(\ell) = 1$ for any $\ell \in E$.

Metric structures on graphs are commonly used in computer science. For instance, the breadth-first search algorithm, an exploratory algorithm in graphs, consist essentially on the computation of the metric balls on an undirected graph, where the weight function is $\omega(\ell) = 1$ for any edge ℓ. Dijkstra's algorithm computes shortest length paths between two vertices in a directed graph with positive weights using metric projections on metric balls recursively calculated, see [CLRS09].

6.2. **Curvature-dimension conditions on graphs.** A notion of curvature-dimension curvature on graphs has been introduced by Lin and Yau [LY10]. See also Ollivier (2009) for a notion of Ricci curvature on Markov chains on graphs, and Erbar and Maas (2012) for finite Markov chains. For the Lin and Yau notion, consider an undirected graph $\mathcal{G} = (V, E)$ with weight function ω. We assume that $\mathcal{G}$ is *locally finite* in the sense that the number of edges connecting every vertex to others is finite. Define

$$\mu(x) = \sum_{\{x,y\}\in E} \omega(\{x,y\}),$$

and a measure of a set $\Omega \subset V$ by

$$\mu(\Omega) = \sum_{x\in\Omega} \omega(x).$$

For a function $f : V \to \mathbb{R}$ the Laplacian Δf and squared gradient $|\nabla f|^2$ are defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{\{x,y\}\in E} \omega(\{x,y\})(f(y) - f(x)),$$
$$|\nabla f|^2(x) = \frac{1}{\mu(x)} \sum_{\{x,y\}\in E} \omega(\{x,y\})(f(y) - f(x))^2,$$

for every $x \in V$. In addition, for $f, g : V \to \mathbb{R}$, we define the operators

$$\Gamma(f,g) = \frac{1}{2}\{\Delta(fg) - f\Delta g - g\Delta f\},$$
$$\Gamma_2(f,g) = \frac{1}{2}\{\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)\}.$$

According to Lin and Yaus's paper [LY10] the graph $\mathcal{G}$ satisfies the curvature-dimension condition $\mathrm{CD}(m,K)$ if

$$\Gamma_2(f,f) \geqslant \frac{1}{m}(\Delta f)^2 + K\Gamma(f,f),$$

and the $\mathrm{CD}(\infty,K)$ condition if

$$\Gamma_2(f,f) \geqslant K\Gamma(f,f).$$

The $\mathrm{CD}(m,K)$ curvature-dimension condition is the discrete counterpart of Bochner's inequality. Lin and Yan used these notions to prove that a locally finite graph satisfies the $\mathrm{CD}(2, \frac{1}{d} - 1)$ curvature condition, where d is the weighted degree of the graph, given by

$$d = \sup_{x\in V} \sup_{\{x,y\}\in E} \frac{\mu(x)}{\omega(\{x,y\})}.$$

They also obtained a lower bound of the first nonzero eigenvalue of the Laplacian on the graph in terms of the degree d and the diameter of the graph.

6.3. **The Cayley graph.** An interesting metric structure can be introduced on groups, allowing the use of geometric arguments to obtain results on the algebraic structure. Assume we have a group G with a set of generators S satisfying $S^{-1} = S$, that is, for every $s \in S$ we have $s^{-1} \in S$. We define the undirected *Cayley graph* associated to G as (G, E), where

$$E = \big\{\{e, v\} : e^{-1}v \in S\big\}.$$

Hence the vertices are the elements of the group and the edges incident to $e \in G$ are of the form $\big\{\{e, es\} : s \in S\big\}$. We take as weight function $\omega(\ell) = 1$ for any edge ℓ. If $x_0 = e, x_1, \dots, x_k = v$ is a path connecting e and v then there exist $s_1, \dots, s_k$ such that $x_{i-1}s_i = x_i$ for $i = 1, \dots, k$. Hence $v = x_k = e(s_1 \cdots s_k)$ and

$$e^{-1}v = s_1 \cdots s_k.$$

Reciprocally, if we can express $e^{-1}v$ as a product of k elements of S then there is a path $e = x_0, x_1, \dots, x_k = v$ connecting e and v. Hence the associated distance is the minimum number of generators needed to express $e^{-1}v$. This is called the *word metric* on G.

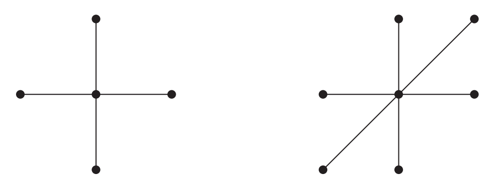

Figure 6. Unit balls for the group $\mathbb{Z}^2$ with set of generators $S = \{\pm(1,0), \pm(0,1)\}$, and $S = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$, respectively.

Since left-translations are isometries of the word metric, all metric balls $B(r)$ of a fixed radius $r > 0$ have the same number of elements. A group has *polynomial growth* if there are constants $C > 0$ and $d > 0$ such that

$$\operatorname{card} B(r) \leqslant Cr^d.$$

The first striking application of metric theory to group theory was the following result by Gromov

Theorem ([Gro81]). *If a finitely generated group has polynomial growth then it contains a nilpotent subgroup of finite index.*

This result is usually considered as the starting point of *Geometric group theory*. See also Gromov's *Hyperbolic groups* (1987) and *Asymptotic invariants of infinite groups* (1993) for influential works on the subject.

Final Comments

The theory of metric spaces is a lively research subject in current Mathematics. The techniques developed to face the problems in this area have had a deep, unifying and simplifying influence in related areas of Mathematics. Despite the fact we have only covered in this note only a few of the geometric aspects, there is an enormous development in progress of the theory of analysis on metric spaces. The reader is referred to Semmes's papers (2003) in these *Notices* for an overview of the subject.

A version of this manuscript with full references can be found on the author's personal webpage: `http://www.ugr.es/~ritore`.

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Manuel Ritoré

Credits

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THE
NEXT GENERATION

Early-career AMS members take a moment

Maria Amarakristi Onyido
Assistant Professor, Northern Illinois University

Favorite memory from an AMS event: The excitement of seeing the big mathematicians whose papers I've read.

Were you inspired by a mathematician?: I was inspired to study math just in order to make it simpler for others. This is because I was the only one that passed math in my graduating high school class and I thought this could be simpler because I believed my classmates were smart too.

Favorite color: Blue

Field of research: Differential Equations

What does the AMS mean to you?: The AMS is a good resource for staying up-to-date on the latest research and networking.

Dony Varghese
Visiting Professor, University of North Alabama

Hobby: Playing Soccer

Favorite color: Green

Favorite food: Biryani

Field of research: Matroid Theory

Describe a situation when you first fell for math: When I won the math competition at my school during the 7th grade.

What is the best piece of advice you received that helped you get through graduate school?: Think about the end product and stay positive.

Were you inspired by a mathematician?: Yes, Srinivasa Ramanujan

Mariam Khachatryan
Assistant Professor, Augustana University

Favorite memory from an AMS event: Meeting my friends and seeing old colleagues.

Field of research: Statistics and Applied Mathematics

Describe a situation when you first fell for math: I was at middle school when I first took a geometry class. I was totally fascinated with the elegance and the beauty of mathematics. Geometry helps discover patterns and understand the world around us.

OF MATHEMATICS

to share a little about themselves:

#AMSMember

Hobby:
Harpist
(I've played
for a former
prime minister
of England)

Field of research:
Probability
and Statistics

PhD Candidate, University of Louisville

Vicki Modisette

What does the AMS mean to you?: Math has always been a way to connect with a community of people who love what I love. The AMS has enabled that connection at both my institution and more broadly at conferences and online.

Hobby:
Soccer

What does the AMS mean to you?: Opportunity, Growth, Success

What do you think is the most important service the AMS offers?: Conferences, Opportunities

Field of research:
Applied
Mathematics

Assistant Professor, University of Nevada, Las Vegas

Rachidi B. Salako

Favorite number:
Mertens
Constant

Field of research:
Analytic
Number Theory

PhD Candidate, Dartmouth College

Grant Molnar

How would you describe math to a non-math person?:
The rhythm of a metered rhyme:

The fractal shape of falling snow
A contour changing over time,
both swift and slow
The logos of the wise and great
The mind unlocked, alert, and free
The dance of chaos, facing fate.

www.ams.org/membership

Almost Sufficiently Large

Danny Calegari

1. The Engineer

Wolfgang Haken, one of the greatest and most original topologists of the 20th century, was born June 21, 1928, in Berlin, Germany. He obtained a PhD in mathematics at the University of Kiel in 1953, and then worked for almost a decade as an electrical engineer designing microwave devices at Siemens. While working there he solved what was perhaps the most fundamental unsolved problem in knot theory: the *unknot recognition problem* (hereafter URP) — i.e., the problem of giving an algorithm to determine whether a knot K in S^3 is the unknot.

Danny Calegari is a professor in the Department of Mathematics at the University of Chicago. His email address is dannyc@uchicago.edu.

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1.1. Knots and the URP.

For our purposes a *knot* K is a smooth embedded S^1 in S^3. Two knots K, K' are *equivalent* if one may be moved to the other by a smooth isotopy. An *unknot* is any knot equivalent to a round circle in S^3 (e.g., a great circle in the round metric).

Some people find it difficult to understand why the URP is a difficult problem. Indeed, it is not easy to draw a diagram of an unknot that can't be obviously simplified by moving one or two strands. But it is easy to be mislead if one thinks of knots that one can easily draw, that have maybe a dozen crossings or fewer. Maybe one gets a better sense of the complexity of the problem by contemplating the act of untangling a string of Christmas lights (or a long garden hose). Figure 1 is one of Haken's examples of "complicated" unknots; the reader who can see easily how to begin to untangle it is a better topologist than I.

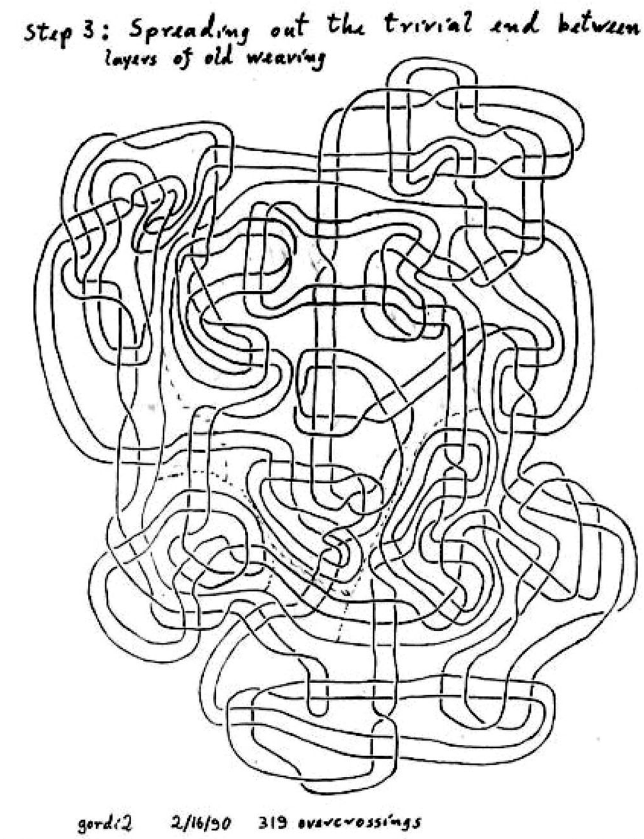

Figure 1. One of Haken's unknots.

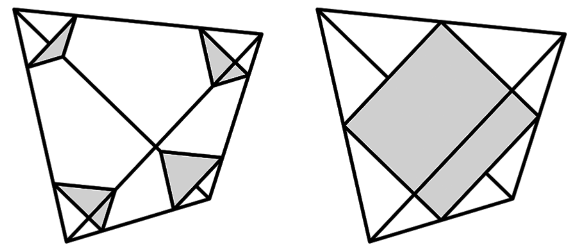

Figure 2. The four triangles and one of three quadrilaterals that might make up the components of $\Sigma \cap \Delta$ for a normal surface Δ and a tetrahedron Δ.

Unknots may be characterized in several ways:

1. K is an unknot if and only if $\pi_1(S^3 - K) \cong \mathbb{Z}$; or
2. K is an unknot if and only if it is the boundary of an embedded D^2 in S^3.

Any knot K in S^3 is the boundary of some compact oriented embedded surface Σ (a so-called *Seifert surface*) for K. The least genus of a Seifert surface is called the *genus* of the knot. Thus: a knot is the unknot if and only if its genus is zero.

1.2. Linear programming. Haken's solution to the URP was revolutionary. It depends on the characterization of an unknot as one that bounds an embedded disk in its complement (a Seifert disk). Haken's algorithm either finds such a disk, or certifies that none exists.

Let's suppose we want to find a Seifert disk D. If it exists, it may be found among the set $\mathcal{S}$ of properly embedded surfaces in the knot complement $S^3 - K$. The problem of course is that $\mathcal{S}$ is an enormously complicated set with little apparent structure. The key is to replace $\mathcal{S}$ by a more manageable collection of surfaces, as follows. Choose a triangulation τ of $S^3 - K$. A properly embedded surface Σ in $S^3 - K$ is said to be *normal* with respect to τ if the intersection of Σ with each tetrahedron Δ of τ is a finite disjoint collection of triangles and quadrilaterals, each of which has the combinatorial form depicted in Figure 2.

There are seven combinatorially distinct triangles and quadrilaterals in each tetrahedron; thus a normal surface Σ determines a nonnegative integral vector $v(\Sigma)$ in $\mathbb{Z}^{7|\tau|}$, where $|\tau|$ is the number of tetrahedra in τ. This enjoys the following rather miraculous properties:

1. any properly embedded surface Σ may either be isotoped to be normal, or may be simplified by an operation (*compression* — we shall discuss this in § 2) that reduces its topological complexity;
2. the set $\mathcal{V}$ of vectors v of the form $v(\Sigma)$ for some normal surface Σ is precisely the subset of $\mathbb{Z}^{7|\tau|}$ satisfying a finite constructible set of integer linear equalities and inequalities; and
3. normal surfaces may be recovered from their vectors; i.e., given $v \in \mathcal{V}$ we may recover Σ with $v(\Sigma) = v$ uniquely up to isotopy.

These properties, though easy to prove, have powerful consequences: they reduce topological questions about embedded surfaces in 3-manifolds — at least, the *incompressible* ones — to linear programming.

A Seifert disk D for K, if one exists, may not be compressed (it is *incompressible*) so it may always be isotoped to be normal, and represented by a vector $v(D) \in \mathcal{V}$. For any triangulation τ the set $\mathcal{V}$ consists of the integer lattice points in a finite union of rational convex polyhedral cones in $\mathbb{R}^{7|\tau|}$; projectivizing this cone gives a finite set of rational projective polyhedra. If a Seifert disk D exists, one may be found so that the projectivization of $v(D)$ is a vertex of one of these polyhedra. Thus the URP comes down to the (routine, though computationally expensive) problem of determining these polyhedra, enumerating their finite set of vertices, and checking whether any of these corresponds to the desired D.

1.3. A "just so" story. The URP is not Haken's best-known theorem in topology. In 1976, with the collaboration of Kenneth Appel, he proved the Four Color Theorem: the conjecture — first made in the 1850s — that one may label the vertices of a planar graph with at most four colors in such a way that no two adjacent vertices receive the same label. Famously (or perhaps notoriously) the proof depended essentially on the use of a computer. The argument proceeds by first constructing and certifying a finite *unavoidable set* of graphs: a collection of graphs with the property that any minimal counterexample must contain one of these as a subgraph. Next one shows that each of the graphs Γ in this collection is *reducible*: it may be simplified to a new graph Γ' in such a way that if Δ contains Γ, and Δ' is the result of replacing Γ in Δ with Γ', then Δ may be four-colored if Δ' can. These two properties of a finite collection — unavoidability and reducibility — together show by induction that no minimal counterexample can exist, and therefore the Four Color Theorem is true.

The essential use of a computer for the verification of such a famous conjecture generated a storm of controversy, centered on the philosophical question of *surveyability*: whether one should accept the validity of a proof whose details, while humanly checkable one by one, were too numerous to be checkable in aggregate. If this strikes the contemporary reader as a rather frivolous point to get hung up on, it might be because the urgency of the controversy always had more to do with the sociology of mathematics than anything else.

Why was Haken able to solve such significant and longstanding problems when others failed? I do not know the answer, and probably it is unknowable.

Here is a guess. Haken appears to have been unusually open-minded about the use of tools (linear programming, computers) that were not part of the traditional mathematical repertoire. Linear Programming was largely invented by economists, and plays a significant role in many highly applied fields such as network flow problems and supply chain management. And although the first computers were invented by mathematicians, arguably as a natural development of the formalization program of Hilbert as pursued, e.g., by Russell–Whitehead, Gödel, and others, their use as a theoretical tool did not conform to Courant's description of mathematics as "an expression of the human mind reflect(ing) the active will, the contemplative reason, and the desire for aesthetic perfection." Is it possible that Haken's early career as an engineer made him more familiar with, and unprejudiced about tools such as these? What other tools, perspectives, metaphors, work-habits might prove of value to contemporary mathematicians if we were able to broaden our conception of our discipline enough to include them?

2. Hierarchies

Let's take a closer look at incompressibility. If Σ is an embedded two-sided surface in a closed 3-manifold M, a *compressing disk* for Σ is a properly embedded disk D in $M - \Sigma$ so that $\partial D \subset \Sigma$ is an essential simple loop in Σ (one that does not bound a disk in Σ). One may cut out an annular neighborhood of ∂D in Σ and glue in two parallel copies of D to produce a simpler surface, obtained from Σ by *compression*. See Figure 3. If Σ admits no compressing disk, it is *incompressible* (one also says Σ is *essential*).

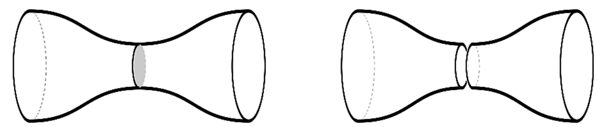

Figure 3. A compressing disk in a surface and the result of compression.

If D is a compressing disk for Σ, the conjugacy class represented by ∂D in $\pi_1(\Sigma)$ is in the kernel of the inclusion map $\pi_1(\Sigma) \to \pi_1(M)$. A remarkable theorem of Max Dehn gives the converse: if $\Sigma \subset M$ is incompressible, then $\pi_1(\Sigma) \to \pi_1(M)$ is injective.

It follows that in the universal cover $\tilde{M}$ the preimages of Σ are properly embedded planes that cut up the universal cover into simply-connected complementary regions which each (universally) cover a component of $M - \Sigma$. A graph Γ with one vertex for each complementary region and one edge for each plane is in fact a tree (because $\tilde{M}$ is simply-connected), and by construction this tree comes with an action of $\pi_1(M)$ for which the edge stabilizers are the conjugates of $\pi_1(\Sigma)$ and the vertex stabilizers are the conjugates of π_1 of the components of $M - \Sigma$.

Friedhelm Waldhausen introduced the term *sufficiently large* for an irreducible 3-manifold M (one in which every smoothly embedded 2-sphere bounds a 3-ball) that satisfies the first (and therefore all) of the following equivalent conditions:

1. M contains an incompressible surface Σ;
2. $\pi_1(M)$ splits nontrivially as an amalgam $A *_C B$ or HNN extension $A*_C$;
3. $\pi_1(M)$ acts nontrivially on a tree Γ.

Cutting M along Σ produces a new (possibly disconnected) 3-manifold M' with boundary. There is a notion of essential surface in a manifold with boundary. If $\partial M'$ contains a surface of positive genus, then $H_2(M', \partial M')$ is nontrivial, and any embedded surface Σ' representing a nontrivial class may be repeatedly compressed or "boundary compressed" until it becomes essential. Thus M' is also sufficiently large, and may be cut along Σ' to produce M'' and so on, unless it happens at some point that every boundary component is a 2-sphere, which implies (because of the irreducibility of M) that the resulting 3-manifold is a union of 3-balls. Technically, after two or more cuts, one should think of the result as a manifold "with corners," and the 3-balls obtained at the end of the procedure have the structure of (combinatorial) polyhedra.

Haken proved that this sequence of cuts — called a *hierarchy* — must indeed necessarily terminate in a union of 3-balls after an effectively computable number of steps, and therefore one may understand and prove theorems about sufficiently large 3-manifolds by induction. This has turned out to be an extremely powerful perspective on 3-manifold topology.

Over the years the terminology changed, so that "sufficiently large" manifolds began to be referred to as "Haken manifolds"; as a benchmark, Hempel's book from 1976 [8] uses the former term, whereas Jaco's book from 1980 [9] uses the latter. Thus a *Haken manifold* is an irreducible 3-manifold that contains an incompressible two-sided embedded surface.

2.1. Topological rigidity. One of the first major applications of the theory of hierarchies, due to Waldhausen [19], was the proof that Haken manifolds are "topologically rigid": a homotopy equivalence between Haken manifolds (satisfying certain further conditions if the manifolds have boundary) is homotopic to a homeomorphism. Examples of (necessarily non-Haken) 3-manifolds for which this fails are Lens spaces (quotients of S^3 by cyclic isometry groups): the Lens spaces $L(p;q_1)$ and $L(p;q_2)$ are homotopy equivalent if and only if one of $\pm q_1 q_2$ is congruent to a square mod p, but are homeomorphic if and only if q_1 is congruent to $\pm q_2^{\pm 1}$ mod p. Waldhausen straightens a homotopy equivalence surface by surface, reducing to the Alexander trick in the case of a ball.

In the same paper Waldhausen showed that the universal cover of a Haken manifold is homeomorphic to a 3-ball. He went on to remark that "of those irreducible manifolds, known to me, which have infinite fundamental group and are not sufficiently large, some (and possibly all) have a finite cover which is sufficiently large" ([19], page 87); the implication being that the universal covers of such manifolds are likewise homeomorphic to a 3-ball. If one interprets "(and possibly all)" as raising a question, this is the first appearance in print of what became known as the *Virtual Haken Conjecture* (hereafter VHC), the question of whether every irreducible 3-manifold with infinite fundamental group is finitely covered by a Haken manifold.

2.2. Geometrization. In the late 1970s 3-manifold topology was turned upside down by William Thurston's *Geometrization Conjecture* — that every prime 3-manifold has a canonical decomposition along a (possibly empty) family of embedded essential tori into *geometric pieces*.

Here a *geometric* 3-manifold is one admitting a complete Riemannian metric of finite volume modeled on one of the eight "3-dimensional geometries," which refer to the possible homogeneous simply-connected complete Riemannian 3-manifolds X which admit a transitive maximal unimodular group G of isometries. Of these eight geometries, by far the most interesting and important is hyperbolic 3-space. A closed hyperbolic 3-manifold is necessarily irreducible, and furthermore its fundamental group is infinite but does not contain a $\mathbb{Z}^2$ subgroup; the geometrization conjecture implies that these necessary conditions are also sufficient. The fundamental group of a closed Haken manifold contains $\mathbb{Z}^2$ if and only if it contains an essential embedded torus; thus the conjecture says that a closed Haken manifold is hyperbolizable if and only if it contains no incompressible tori. Likewise, the conjecture says that a Haken manifold with boundary is hyperbolizable if and only if every incompressible torus is isotopic to a boundary component, and it is not *Seifert fibered* (roughly, a circle bundle over a surface orbifold).

Thurston gave an enormous amount of evidence for this conjecture, proving it for several classes of manifolds, including (most significantly) for Haken manifolds. Since every knot complement in S^3 is Haken, it follows from his proof that any knot K falls into one of the following three classes:

1. *torus knots*: those that lie on an unknotted torus in S^3 (such knot complements are Seifert fibered);
2. *satellites*: those that lie in an essential way in a solid torus neighborhood of a nontrivial knot (the boundary of the solid torus is essential but not boundary parallel); and
3. *hyperbolic knots*: those whose complements admit a complete hyperbolic structure.

The hyperbolization theorem for Haken manifolds proceeds by induction. Even the base step of the argument is rather subtle: one needs to "hyperbolize" the polyhedra into which a Haken manifold is decomposed by a hierarchy, or exhibit a combinatorial obstruction why it cannot be done.

Each such polyhedron P can be thought of as a 3-ball together with a trivalent graph Γ embedded in the boundary, and the goal is to find a hyperbolic metric on the 3-ball for which the edges of Γ are geodesics, and the complementary regions $\partial P^3 - \Gamma$ are totally geodesic, and meet at right angles along Γ. One necessary condition concerns the dual graph $\Gamma' \subset \partial P^3$ to Γ: it says that every loop γ in Γ' of length 3 should be in the link of a vertex of Γ, and every loop of length 4 should be in the link of an edge of Γ.

To see why these conditions are necessary, let D be a disk in P bounding γ, and suppose Q is the closed 3-manifold obtained from P by *reflection* in the sides. This means the following: if P has n faces $F_1, \cdots, F_n$ we take 2^n copies of P indexed by maps from $\{1, \cdots, n\}$ to $\{0, 1\}$ and glue two copies whose associated maps differ only at i by identifying their respective copies of the face F_i.

If ∂D has length 3 then 8 copies of D fit together in Q to form a sphere; similarly, if ∂D has length 4 then 4 copies of D fit together in Q to form a torus. Such spheres and tori will be essential unless the necessary condition holds; thus they are an obstruction to hyperbolization. If P is not a tetrahedron, these necessary conditions turn out to be sufficient, and P may be hyperbolized.

The induction step is rather complicated, depending on the quasiconformal deformation theory of hyperbolic structures on 3-manifolds with incompressible boundary, and we do not discuss it here. For details see, e.g., Misha Kapovich's book [11].

2.3. On proof and progress in mathematics. Thurston's proof of the hyperbolization theorem for Haken manifolds was written up and distributed as a series of preprints, only one of which was ever formally published. Far more influential than the details of the proof was the vision of 3-manifold theory implied by the geometrization conjecture, and a suite of associated conjectures and questions laid out in a famous *Bulletin* article [18].

This article contains few proofs and many examples. The geometrization conjecture and its proof for Haken manifolds is not the end but a starting point. One can take the conclusion of the conjecture as a hypothesis, and hyperbolic 3-manifolds as objects of interest in their own right. Apparently disparate fields — complex and quasiconformal analysis, bounded cohomology, arithmetic lattices, geometric group theory — are revealed as parts of a deeper and more unified whole.

The article ends with a list of 24 questions/projects, of which the geometrization conjecture was only the first, which dominated research in 3-manifolds for the next 30 years. One can see the formulation of this list as both an act of generosity, and of cultivation: it gifted the mathematical community at once with new tools, and new lands to explore with them.

An article Thurston wrote in 1994 [17] explains:

> I concentrated most of my attention on developing and presenting the infrastructure in what I wrote and in what I talked to people about ...
>
> The result has been that now quite a number of mathematicians have what was dramatically lacking in the beginning: a working understanding of the concepts and the infrastructure that are natural for this subject. There has been and there continues to be a great deal of thriving mathematical activity. By concentrating on building the infrastructure and explaining and publishing definitions and ways of thinking but being slow in stating or in publishing proofs of all the "theorems" I knew how to prove, I left room for many other people to pick up credit. ...
>
> What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the geometrization conjecture for Haken manifolds. It is unlikely that the proof of the general geometrization conjecture will consist of pushing the same proof further.

Number 16 on the problem list is the Virtual Haken Conjecture (posed as a question). One may have viewed — I and many colleagues I knew did view — this conjecture as an intermediate step towards a possible approach to geometrization: given M closed and irreducible with infinite π_1, if one knew M had a finite (regular) cover $\hat{M}$ which was Haken, one could geometrize $\hat{M}$ and then analyze the action of the deck group on $\hat{M}$ to show that M was geometric too (the details of how this argument might go emerged as a corollary of work of Dave Gabai in 1997 [7]).

But how to get started? Suppose M is a closed irreducible 3-manifold. If the only thing we know about its fundamental group is that it is infinite, how do we construct any nontrivial finite cover of M at all?

A curious chicken-and-egg situation emerges. Suppose one *already* knew M to be hyperbolic (say). Because the group of isometries of hyperbolic 3-space is a matrix group, we may deduce $\pi_1(M)$ is linear and therefore (by a well-known lemma of Atle Selberg) *residually finite*; this means that for every nontrivial element $\gamma \in \pi_1(M)$ there is a finite index subgroup of $\pi_1(M)$ that does not contain γ. In particular, any hyperbolic 3-manifold admits *many* finite coverings, and one can start to explore whether any Haken manifolds may be found among them. Turning the conventional view on its head, one could think of the geometrization conjecture itself as a stepping stone to the VHC.

As Thurston's quote anticipates, the geometrization conjecture was proved round the turn of the millenium using tools from completely different fields (geometric PDE methods — specifically Ricci flow — and a deep analysis of its singularity formation) by Grisha Perelman [12, 13] (a short-cut to the Poincaré Conjecture spun off in a separate paper in [14]). Thurston's theorem for Haken manifolds (and for that matter, most of 3-manifold topology) plays almost no logical role in the argument.

3. Of LERF and RAAGs

After the work of Perelman the Virtual Haken Conjecture narrowed to the question of whether every closed hyperbolic 3-manifold M has a finite cover that contains an incompressible embedded surface. This question factorizes naturally into two subquestions:

1. does $\pi_1(M)$ contain a subgroup isomorphic to $\pi_1(S)$ for S a closed surface (of genus at least 2)? and
2. does every immersed surface $S \to M$ injective on π_1 lift to an embedding in a finite cover?

Immersed π_1-injective surfaces in hyperbolic 3-manifolds come in two distinct kinds: those that are (virtual) fibers of a fibration of the manifold over the circle; and all the rest. This distinction exactly captures a key geometric property of $\pi_1(S)$ as a subgroup of $\pi_1(M)$. If S is a (virtual) fiber, then $\pi_1(S)$ is exponentially distorted in $\pi_1(M)$, in the sense that one may write certain elements of $\pi_1(S)$ in terms of a (fixed) generating set for $\pi_1(M)$ far more efficiently than they may be written in terms of a (fixed) generating set for $\pi_1(S)$. On the other hand, if S is *not* a virtual fiber, $\pi_1(S)$ is undistorted in $\pi_1(M)$: translating elements of $\pi_1(S)$ back and forth between generating sets for $\pi_1(S)$ and for $\pi_1(M)$ does no more than multiply word length by a bounded multiplicative factor; one says in this situation that $\pi_1(S)$ is a *quasiconvex* subgroup of $\pi_1(M)$.

Topologically, an incompressible immersed surface S lifts to an embedded plane $\tilde{S}$ in the universal cover $\tilde{M}$, which is homeomorphic to $\mathbb{R}^3$. But geometrically, if one thinks of $\tilde{M}$ conformally as the interior of the round unit ball in $\mathbb{R}^3$, when S is a virtual fiber the plane $\tilde{S}$ becomes wilder and wilder near infinity, and limits to a sphere-filling (Peano) curve; whereas when S is not a virtual fiber, the plane $\tilde{S}$ may be compactified to a closed disk in the closed unit ball whose boundary is a topological circle (technically, it enjoys the analytic regularity condition of being a *quasicircle*).

Work of Peter Scott from 1978 [16] shows that the question of lifting an immersed π_1-injective surface to an embedding can be re-expressed in purely algebraic terms. A group G is said to be *LERF* — an acronym for "locally extended residually finite" — if every finitely generated subgroup H is the intersection of some family of finite index subgroups of G; equivalently, if, for every $g \in G - H$ there is a finite index subgroup H' of G containing H but not g. Weaker than LERF is *QCERF*, or "quasiconvex extended residually finite," which asks for the LERF property only for quasiconvex subgroups H of G (the analogous property for hyperbolic 3-manifolds with (torus) boundary is GFERF).

3.1. Nearly geodesic surfaces. How can one find (build?) injective surfaces in a hyperbolic 3-manifold? There are two problems: to find (build) the surface, and to show that it is π_1-injective. An *embedded* 2-sided surface that is not π_1-injective has a simple essential loop in the kernel. For an immersed surface this is unknown (this is the so-called *Simple Loop Conjecture*), and in general it seems very hard to give a purely topological criterion guaranteeing that an immersion $S \to M$ is π_1-injective. However it is possible to give a *geometric* criterion.

In Euclidean space, submanifolds that are nearly flat on a small scale may be badly distorted on a large scale. But in hyperbolic space, quasiconvexity is a *local* condition. Because hyperbolic space is negatively curved, geodesics diverge from each other exponentially quickly. This divergence dominates the behavior of normal geodesics to a submanifold whose extrinsic principal curvatures have absolute value bounded away from 1, and (by comparison with the endpoint map at infinity) certifies that such submanifolds are π_1-injective.

In 2009 Jeremy Kahn and Vladimir Markovic announced their proof [10] that every hyperbolic 3-manifold contains (many!) π_1-injective immersed surfaces. In fact if M is a closed hyperbolic 3-manifold, if $p \in M$ is arbitrary, and if $v \in T_pM$ is an arbitrary vector of length 1, they showed one may find for any ϵ an immersed surface $S \to M$ with principal curvatures bounded in absolute value by ϵ, so that $p \in S$ and the normal to S at p is v. The argument has two parts. Let's fix a 3-manifold M and a real number $\ell > 0$. Say that an immersed pair of pants $P \to M$ is *good* if the cuff lengths are all closed geodesics of length very close to ℓ, and if the extrinsic curvature of P is very close to 0 (i.e., it is very nearly totally geodesic). The first part of the argument uses ergodic theory (technically: exponential mixing of the frame flow) to demonstrate the existence of a proliferation of good pants: if ℓ is large enough (depending on M) then every geodesic γ of length close to ℓ is the boundary of some good pair of pants in many ways. The second part of the argument explains how to glue up these good pants in such a way that the resulting closed surface has extrinsic curvatures very close to 0 and is therefore π_1-injective.

To get good enough estimates to control the distribution of good pants one needs to know a lot about the frame flow on a hyperbolic manifold. Negative curvature once more plays a key role. So this step of the argument depends essentially on knowing that the 3-manifold is hyperbolic (or, at least, negatively curved) and therefore builds on Perelman's proof of geometrization.

3.2. Right-angled pentagons. Suppose we have a hyperbolic 3-manifold M and an immersed, π_1-injective quasigeodesic surface $S \to M$. How may we find a finite cover in which a lift of S embeds (up to homotopy)? One dimension lower the analogous question is: given a hyperbolic surface S and an immersed essential loop $\gamma \to S$ (let it be an immersed simple geodesic for simplicity) how may we find a finite cover in which a lift of γ embeds (up to homotopy)?

Peter Scott solved this problem in 1978, in the paper [16] cited above. His argument, on the face of it, has a rather ad hoc flavor, and depends on the fact that any surface with negative Euler characteristic admits a hyperbolic metric in which it may be tiled by regular right-angled pentagons. How does this help? Since $\gamma \to S$ is injective, there is an (infinite index) cover $\hat{S}$ of S with $\pi_1(\hat{S})$ equal to the image of $\pi_1(\gamma)$ (which is infinite cyclic) in $\pi_1(S)$. This cover is topologically an annulus, and γ lifts to a curve $\hat{\gamma} \subset \hat{S}$ which is homotopic to the core curve of this annulus, so that after a homotopy it may be taken to be *embedded*.

The tiling of S by right-angled pentagons lifts to a tiling of $\hat{S}$, and we may find a compact convex region $P \subset \hat{S}$, containing $\hat{\gamma}$, entirely tiled by right-angled pentagons. To see this, let's first lift all the way to the universal cover $\tilde{S}$, and let $\tilde{\gamma}$ be the preimage of $\hat{\gamma}$ under $\tilde{S} \to \hat{S}$, so that $\tilde{\gamma}$ is a bi-infinite curve in $\tilde{S}$ stabilized by the deck group $\mathbb{Z}$. The convex hull $\tilde{C}$ of $\tilde{\gamma}$ is the intersection of all the (closed) half-spaces of $\tilde{S}$ that contain it. One may obtain a bigger (but still functorial) convex subset $\tilde{P}$ by taking the intersection of all the (closed) half-spaces of $\tilde{S}$ containing $\tilde{\gamma}$ that are tiled perfectly by right-angled pentagons. There are "enough" such half-spaces that $\tilde{P}$ and $\tilde{C}$ are a finite Hausdorff distance apart; since both $\tilde{P}$ and $\tilde{C}$ are evidently invariant under the deck group, they project to convex subsets P and C in $\hat{S}$ respectively, each of which may be seen to be compact.

Now, P is not a closed surface, but (just as we did in § 2.2 in the base step for the inductive proof of hyperbolization for Haken manifolds) we may obtain a closed surface Q by reflecting in the sides of P. The surface Q, like S, is tiled by right-angled regular pentagons, so their fundamental groups are commensurable in the group of isometries of the hyperbolic plane. Thus, one may find a common finite cover of both S and Q in which some lift of γ covers $\hat{\gamma}$ in Q and is therefore embedded.

3.3. RAAGs and cube complexes. Already in Scott's paper he observed that the argument could be generalized to a quasiconvex $\pi_1(S)$ in a 3-manifold group $\pi_1(M)$ if one could find a 3-dimensional right-angled hyperbolic polyhedron X for which $\pi_1(M)$ is commensurable with the group Γ_X generated by reflections in the sides of X.

Unfortunately, 3-manifolds with this property are rare. One can get further by finding a suitable injection $\pi_1(M) \to \Gamma_X$ where X is a higher-dimensional right-angled hyperbolic polyhedron, and in fact Ian Agol, Darren Long, and Alan Reid [3] used this idea to prove GFERF for the fundamental groups of the Bianchi manifolds — certain noncompact finite volume hyperbolic 3-manifolds important for number theory. But this trick has its limits: finite volume right-angled hyperbolic polyhedra do not exist in dimensions above 12 [4] (compact ones do not exist above dimension 4).

Having learned since Thurston the profound importance of hyperbolic geometry for 3-manifold topology, mathematicians now had to unlearn it. What was ultimately most important in Scott's argument about the right-angled hyperbolic pentagons was not their hyperbolicity per se, but their *right-angledness*.

A *right-angled Artin group* — or RAAG for short — is a group Γ_Δ associated to a finite graph Δ by taking one free generator for each vertex, and imposing the relations that two generators commute if and only if the associated vertices share an edge (and no other relations that do not follow from these). Starting around 2000, Dani Wise and his collaborators developed a program to prove that certain classes of groups G are QCERF by showing that they contain a finite index subgroup G' that embeds suitably in a RAAG.

If Γ_Δ is a RAAG then it is the fundamental group of a cell complex K_Δ which is the union (in a natural way) of a torus T^k for each k-tuple of vertices of Δ that span a clique. This cell complex may be built in a natural way from Euclidean cubes, all of side length 1, which are glued isometrically along their faces. With this metric the universal cover $\tilde{K}_\Delta$ enjoys a form of nonpositive curvature expressed by saying that it is CAT(0) (these letters stand for Cartan, Alexandrov, and Topogonov, who all proved significant theorems in comparison geometry). Distance between parameterized geodesics in a CAT(0) space is a convex function, and this convexity, together with a proliferation of totally geodesic separating subspaces (obtained by gluing together codimension one cubes in the dual complex) lets one simulate Scott's argument and separate quasiconvex subgroups.

Suppose G may be exhibited as $\pi_1(K)$ for some Euclidean cube complex whose universal cover is CAT(0). In 2008 Wise and Frédéric Haglund [5] gave a combinatorial criterion (called *special*) for a cube complex K to immerse isometrically in K_Δ for some RAAG Γ_Δ, thereby realizing G as a quasiconvex subgroup of Γ_Δ. If a finite cover of K is special, one says that K (and by abuse of notation G) is "virtually special"; this is evidently good enough to show that G is QCERF.

3.4. Cubulating 3-manifold groups. All well and good. But how on Earth to exhibit $\pi_1(M)$ as $\pi_1(K)$ for some locally CAT(0) cube complex?! Rather astonishingly, a cube complex of exactly the sort we need falls into our laps by feeding the Kahn–Markovic surfaces into a beautiful construction due to Micah Sage'ev [15].

First let's work one dimension lower and consider a proper configuration L of lines in the plane. If the lines are disjoint, L is dual to a tree with one edge for each line in L and one vertex for each complementary region. If the lines cross in general position, L is dual to a 2-dimensional complex whose cells are squares. This complex is not typically CAT(0); if there are three lines ℓ_1, ℓ_2, ℓ_3 that intersect in pairs, these intersections give rise to three squares that share a common vertex with an "atom" of positive curvature, violating the CAT(0) condition. This can be ameliorated by going up a dimension: we may fill in the three squares with a 3-dimensional cube in the obvious way; and likewise, for every configuration of n lines that pairwise intersect we may help ourselves to an n-cube. The resulting complex $\tilde{K}$ will be CAT(0), and if there is an upper bound on the size of the cliques in the incidence graph of L, it will be finite dimensional. If the configuration L is invariant under the action of a group G, then G acts (combinatorially) on $\tilde{K}$ and under the right circumstances, some finite index subgroup of G will act freely with quotient K.

The Kahn–Markovic construction gives rise to a large but finite collection of almost totally geodesic immersed surfaces S in a hyperbolic 3-manifold M. Lifting to the universal cover one obtains a proper collection $\tilde{S}$ of almost totally geodesic planes in hyperbolic 3-space. Roughly speaking, one may obtain a cube complex $\tilde{K}$ with one n-cube for each n-tuple of planes in $\tilde{S}$ that mutually intersect. Elementary coarse properties of hyperbolic geometry imply that $\tilde{K}$ is finite dimensional, and that $\pi_1(M)$ acts on it effectively and properly.

3.5. The long goodbye. Thus things stood in March 2012: the VHC was "reduced" to Wise's conjecture, that every locally CAT(0) cube complex K with $\pi_1(K)$ (word)-hyperbolic was virtually special. In a blog post [20] written March 6 2012, Henry Wilton wrote that

> (Wise's conjecture is) such a remarkable conjecture that it's difficult to believe it's true, but it's also a win–win in the sense that either a positive or a negative answer would be a huge advance in geometric group theory ... implausible or not, I think this conjecture is already a major open problem.

In a remarkable development, only six days later Ian Agol announced a proof of Wise's conjecture (and, consequently, of the VHC) in a talk at the Institut Henri Poincaré in Paris. Two weeks later Agol gave lectures outlining the details of his argument in a workshop at the IHP, and his preprint (containing an appendix written jointly with Daniel Groves and Jason Manning) was posted to the arXiv on April 12.

It is beyond the scope of this article to explain the main ideas of the argument, except to say that one first constructs an *infinite* regular cover $\hat{K}$ of K in which the "hyperplanes" (which correspond in a suitable way to the quasiconvex subgroups to be separated) are compact, two-sided and embedded, and then one builds a finite cover of K "modeled" (in some sense) on $\hat{K}$; and that the construction of $\hat{K}$ rests on a certain technical tool in geometric group theory (hyperbolic Dehn surgery) with origins in hyperbolic 3-manifold topology and the work of Thurston.

Combined with work of Wise and previous work of Agol and others, Agol's work resolved most of the remaining problems (numbers 15–18) on Thurston's list, marking the end of an era in 3-manifold topology.

What determines which questions become central in a field? What constitutes progress, and how can we recognize it when we see it? Academic research, whatever else it is, is a social activity, and the forces that shape social activities are complicated and have an enormous psychological component. Fashion, value-judgments, familiarity, taste all play a role.

Agol has been unusually thoughtful about the social psychology of mathematical practice, sometimes describing conjectures with which he wrestled over many years as "friends" whose company he was pleased to spend so much time with. He writes [2]:

> Mathematicians use part of the social wiring of the brain to engage with mathematical ideas and objects. I certainly feel like this for myself, where the figure 8 knot complement plays the role in my mind that a celebrity (like Bob Dylan) might in others. I think this helps us engage with abstract mathematics, which can be socially isolating because there's very few that we can communicate with about these ideas on a regular basis. I don't think this is special to mathematics, for example astronomers might have a social connection to the Andromeda Galaxy.

The long and meandering route that led from the resolution of one conjecture (the URP) to another (the VHC) winds in and out of topology, geometry, combinatorics, group theory, PDE, and many other fields. Along the way objects and theories emerged, grew, evolved; seedlings transplanted to foreign soils became established and bore brave new fruit. If one could plot progress over time (if one could even agree on an axis along which to measure it) the graph would be the Devil's Staircase. Headway, when it came, came suddenly, like a fire alarm or the finale of Beethoven's Fifth Symphony, and caught most of us naked and napping.

A partial entity-relationship graph of the situation might look like Figure 4. Is the situation as irreducibly complex and muddled as the graph suggests, or might there be some hitherto undreamed of principle that would untangle the skeins of this picture like one of Haken's unknots and let us really see clear to the bottom of the well?

Wolfgang Haken died in his home in Champaign, Illinois, on October 2, 2022, at the age of 94.

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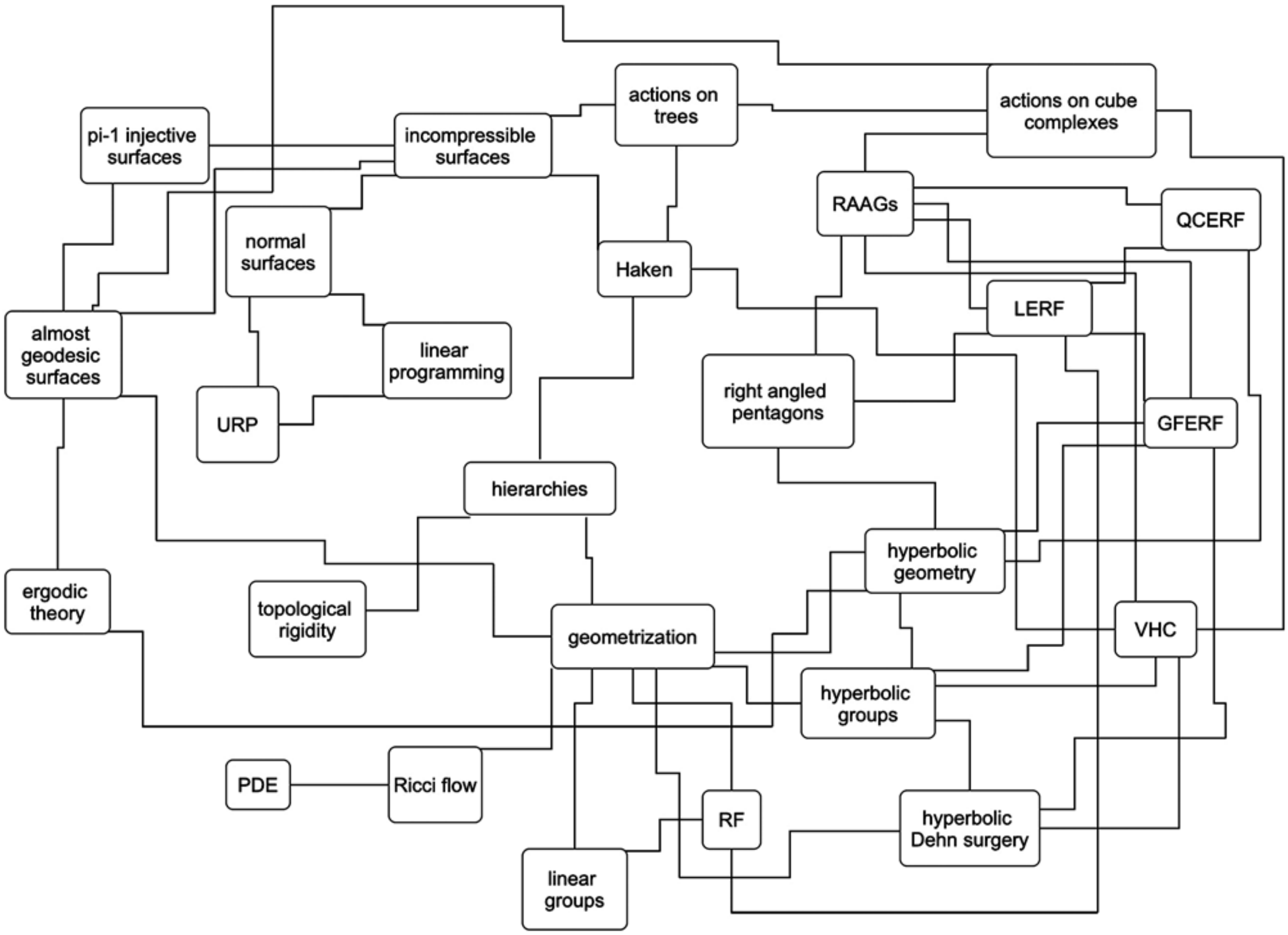

Figure 4. An entity-relationship graph connecting URP to VHC.

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Danny Calegari

Credits

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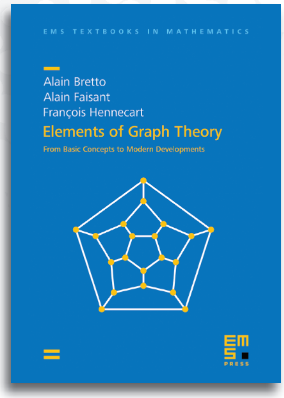
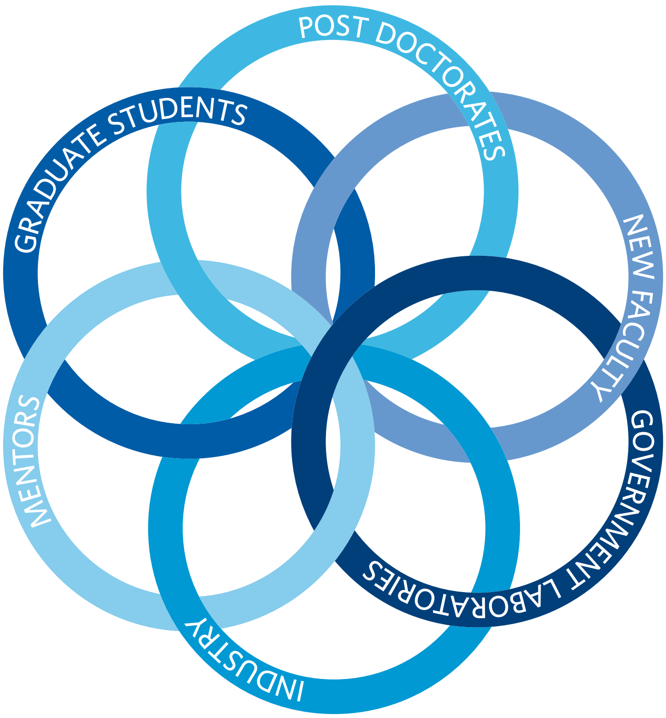


The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Math and the Real World.



Moving Forward

On Challenges and Opportunities of Graduate Advising

Alex Iosevich

1. Introduction

One of the most fulfilling aspects of my career in mathematics has been the opportunity to serve as the thesis advisor for a wonderful group of people over the past 30 years. This includes undergraduate Honors theses, Masters theses, and PhD dissertations, though in this article I am going to focus on doctoral advising as it poses a series of challenges that are coming into an even sharper focus in recent years. One of the reasons for this is increasing efforts by the mathematical community to address the issues of gender and cultural diversity in their ranks. Another is the effort to keep the PhD degree relevant in the age of uncertain job prospects in academia and increasing opportunities in the finance, high tech, and big data sectors of industry. In this essay, I shall describe my approach to graduate advising—the process that goes far beyond the supervision of the PhD dissertations. I shall also endeavor to describe some of the challenges and opportunities that have repeatedly arisen since I started advising graduate students 23 years ago.

I did not enter the world of graduate advising with a solid idea of what the process is supposed to be like. As a graduate student, I was exposed to a wide variety of advising strategies. My own PhD advisor, Chris Sogge, who shaped my views on research mathematics and many other things in profound ways, always emphasized the importance of independence from the early stages of the educational process. This approach played into my strengths as I always viewed mathematics as a creative process that is much closer to my lifelong interest in reading and writing poetry than to a purely technical pursuit. As a result, I developed an independent research program very early and credit this fact to my survival in this turbulent and uncertain profession. To this day, I consider developing an independent perspective an essential part of the development of a graduate student. In addition, once I encountered my own PhD students with a wide variety of backgrounds, interests, and goals, I came to the conclusion that every student requires an individual approach in order to bring out the best in them. This raises the question about the balance of responsibilities between the advisor and the advisee and all the complexities that it entails.

2. My Approach to Advising

2.1. The guiding paradigm. When I was a small child, my grandmother described to me how shoemakers were trained in the small Jewish shtetl she came from. A fairly young person would start out by sweeping the floors and observing the master shoemaker at work. They would then be given simple tasks to do, like putting the shoes into boxes or measuring the shoe size of potential clients. Slowly, over a period of time, the apprentice would be taught how to cut the materials, create basic designs, and,

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Alex Iosevich is a professor of mathematics at the University of Rochester. His email address is iosevich@math.rochester.edu.

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eventually, create their first pair of shoes, first in collaboration with the master, and, ultimately, on their own. Several decades later I recalled these stories when I started advising graduate students and quickly realized that the process need not be all that different. While every student is different and requires a different approach, the apprenticeship scheme is the most compelling model I have ever found to inform my advising process.

In practice, the apprenticeship scheme works as follows. A typical student starts working with me during their second or third year of graduate school, after they have passed most or all of their preliminary examinations. The first thing I do is sit down with a student several times over a period of several days or even weeks to figure out what types of problems they may be passionate about. Once the general class of problems is identified, I typically run a reading course on the background material, at the end of which the student is given a few more papers to read and a concrete problem to solve. This first problem is typically something I have an idea of how to solve, and it forms a foundation of a basic creditable PhD dissertation. If the student is able to complete the first stage in a reasonable amount of time, they are given a second, harder problem, where they are expected to exhibit a greater degree of independence. Strong students who pass the second stage are set loose on open problems with much more uncertain outcomes.

Every graduate student I work with is encouraged to develop practical skills like programming, applied statistics, and basic data analytics techniques. On one hand, this puts them in a position to make a good living if this is where their preferences or the vicissitudes of the job market take them. Moreover, technology has been playing an increasing role in my own research, and that of my graduate students, both due to forays into data science and an increasing computer experimentation to shed more light on my theoretical projects.

2.2. The pitfalls of the "sink or swim" approach. One of many age-old dilemmas in mathematics is whether the "sink or swim" approach works. Also known as the "survival of fittest," or, in the immortal words of Dolph Lundgren, "if he dies, he dies!" This paradigm occasionally pops up in both elite and not-so-elite institutions alike. Sometimes it works, and sometimes it leads to serious problems for all involved, with literally everything in between, but the idea never seems to go away and is probably destined to be debated forever. For obvious reasons, the draconian approach is less likely to fail in top graduate programs, but even there it has been known to lead to unwarranted complications. Among the many problems I have with the "sink or swim" approach to advising is that it does not, in my opinion, lead to the discovery of the most productive or talented mathematicians. The background, psychology, and previous experiences of the individual play a tremendous role in determining how they respond to a particular advising style, independent of their mathematical ability. In many cases, a student with a considerable amount of talent and drive simply crumbles in the face of excessive early expectations and insensitive treatment (see, e.g., [1], [3] and the references contained therein). Students who have faced hidden or open discrimination due to their gender, race, ethnicity, or LGBTQ+ status are statistically particularly vulnerable. Some of my top students, who have developed into strong and productive research mathematicians, may not have survived in the "sink or swim" world that still too often pervades academia.

2.3. Generating ideas. One of the greatest challenges I have faced as a PhD advisor is getting the students to generate their own ideas. Some were naturally predisposed toward creative reasoning, but the point I make to all my students is that one can significantly improve one's ability to come up with new directions through frequent practice. I ask my students to come to my office with at least two or three new thoughts, and I stress that it is absolutely fine if those ideas turn out to be wrong or well-known. I describe the graduate, or even upper-division undergraduate, experience to my students as the process of moving from being a consumer to a creator of mathematical ideas. The degree of resistance I face in these situations varies from student to student, but persistence typically pays off and often results in graduate students developing a well-developed independent perspective by the time they receive their PhDs. This serves them very well whether they end up going into academia or industry.

The key question implicit in the discussion of generating ideas is, how does one begin this process? A typical approach is for a student to examine the research papers they are reading in the area where they are conducting their investigations and look for alternate directions and approaches. This is certainly an important part of the process, but the point I continually emphasize to students of all levels is that creative questions can already be asked about elementary aspects of mathematics going back as far as high school algebra or even before! In an ideal world, in which we do not live, mathematics graduate students would have had years of generating ideas and anticipating deeper concepts behind them, but the current state of mathematics education in the US and the lack of support structures for mathematical activities in elementary school, high school, and sometimes even college means that many rudimentary skills need to be learned later. The good news is, this can be done successfully, and an important part of this process is to continually review elementary mathematics, derive all the basic formulas from scratch, and learn to be creative with simple concepts.

2.4. **Undergraduate research and outreach.** Another way graduate students can sharpen their skills and improve their creative processes is by getting involved in supervising undergraduate research. Universities all over the world are increasingly recognizing the importance of undergraduate research for all students, not just the most talented ones, as a way of preparing them in a more realistic fashion for the world that lies beyond the college walls. While such activities should be organized and run by experienced faculty, graduate students can contribute tremendously by assisting the students with the projects, and by helping to design the various research directions. This process plugs directly into the creativity theme we have been discussing. Mathematics-related community outreach can also help by forcing graduate students to conceptualize their mathematical understanding in a way that can be communicated to people from less-privileged backgrounds who were never exposed to mathematical ideas in a coherent and systematic way. The vertical integration concept implicit in these recommendations should, in my view, increasingly become the guiding paradigm behind university curricula (see, e.g., [4]).

2.5. **Recruiting graduate students.** From the very beginning of my advising experiences, I have been deeply convinced that recruiting makes a tremendous difference in terms of bringing in quality graduate students. If an undergraduate student has the credentials to get into a PhD program in mathematics at Princeton, Harvard, MIT, or another school of that level, no amount of recruiting is likely to persuade them to go to a school of a lower level, but this encompasses only a small sliver of potential graduate students. A large number of very talented students remain after top schools have had their pick, and waiting for these students to apply to your institution on their own initiative is not the strategy I recommend. The experience of the University of Missouri math department, which in the 2000s recruited two PhD students, Roman Vershynin and Svitlana Mayboroda, who have gone on to speak at the International Congress of Mathematics, as well as Atanas Stefanov, Milena Stanislavova, Xiaochun Li, Dimitry Bilyk, Peter Honzig, Roman Shvidkoy, Vlad Yaskin, Doowon Koh, Krystal Taylor, and a number of others who are now well-known tenured faculty at strong PhD-granting institutions, shows that recruitment can fundamentally transform the level and quality of a mathematics graduate program.

The obvious question is, how does one go about recruiting graduate students? Sending out posters, creating an easy-to-navigate departmental web page, and bringing in prospective students for on-site visits are certainly essential, but one should not stop there. An effective recruitment strategy is not limited to the graduate director and encompasses as many people in the department as possible. Virtually every active research mathematician has numerous friends and acquaintances throughout the world of mathematics who have regular contact with strong undergraduate students at their institutions. Contacting these people and asking them to spread the word about your PhD program can go a long way, but one can even go further. Speaking to prospective students directly and getting across your level of enthusiasm for your graduate program and your own research program is a very effective recruiting tool in my experience. This can be accomplished in several different ways. For example, when a research mathematician gives a seminar or a colloquium talk, they can request to meet with upper-class mathematics majors to tell them about graduate studies in general and their home institution in particular. Another way is to work with the graduate director to get in touch with the students who have been accepted to the graduate program but have not yet made the final decision on whether to accept.

I cannot emphasize enough that graduate recruitment is not a zero-sum game. In the process of recruitment, excellent students who may not have received the encouragement and support that is commensurate with their talent and accomplishments may be persuaded to pursue their research dreams. This is especially important with regard to prospective students who are women or members of underrepresented groups. One of the most tired, feeble, and disingenuous excuses often heard in the mathematics world is that it is difficult or impossible to recruit women and members of other underrepresented groups at various levels because there are allegedly so few candidates and everybody else is trying to recruit them as well. In addition to lamenting the relative absence of diversity in mathematics graduate programs, an activity the mathematics world got exceedingly good at over the decades, we should put in a much greater effort to do something about it. This effort must involve every single one of us, and should not just be delegated to the Diversity and Inclusion administrators at our home institutions.

3. Conclusion

I believe that effective PhD advising is a multifaceted process that is not limited to overseeing the content and structure of the dissertation. Initial recruitment before, job placement after, and facilitation of intellectual development and psychological support during the PhD years are all essential components of advising in the modern university. In an ideal academic world, in which we do not live, much better support structures would exist that would make this process much easier. In the meantime, it is up to the advisor to guide the student toward filling the existing gaps in their mathematical background, and facilitating the creation of a work environment that would allow the student to succeed. No matter what support structures and safeguards are put in place, the PhD advisor is destined to remain the central figure in the development of their graduate students.

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Alex Iosevich

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Supporting Faculty in Mentoring Students for Careers Beyond Academia

Lee DeVille, Tegan Emerson, Skip Garibaldi, Mary Lynn Reed, Talitha M. Washington, and Suzanne L. Weekes

Career opportunities for mathematicians in business, government, and industry have never been better. At the 2023 Joint Mathematics Meetings in Boston, MA, the AMS Committee on the Profession sponsored a panel entitled "Supporting Faculty in Mentoring Students for Careers Beyond Academia." The goal of the panel was to provide actionable advice for faculty who seek to increase their ability to mentor students in finding nonacademic employment. To enable this information to reach a wider audience, we offer this interview-style report between our moderator and panelists.

The Panel

Reed: *Welcome! I'm happy to be moderating this discussion on behalf of the AMS Committee on the Profession. To kick things off, I'd like each of the panelists to introduce themselves, briefly mentioning what programs/institutions/activities they've been involved with related to mentoring students for nonacademic careers.*

For myself, I spent the majority of my mathematical career in government and industry, including twenty years at the National Security Agency. My last position at NSA was as the chief of the Mathematics Research Group, which included oversight of the mathematics hiring process at NSA. Panelists?

Lee DeVille is a professor of mathematics and the director of undergraduate studies at the University of Illinois at Urbana-Champaign. His email address is `rdeville@illinois.edu`.

Tegan Emerson is a senior data scientist at the Pacific Northwest National Labs. Her email address is `tegan.emerson@pnnl.gov`.

Skip Garibaldi is the director of the Center for Communications Research in La Jolla, California. His email address is `garibaldi@ccr-lajolla.org`.

Mary Lynn Reed is a professor of mathematics at the Rochester Institute of Technology. Her email address is `mlrsma@rit.edu`.

Talitha M. Washington is a professor of mathematics at Clark Atlanta University and the director of the Atlanta University Center (AUC) Data Science Initiative. Her email address is `twashington@aucenter.edu`.

Suzanne L. Weekes is the executive director of the Society for Industrial and Applied Mathematics (SIAM) and holds a professorship at Worcester Polytechnic Institute. Her email address is `weekes@siam.org`.

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DeVille: I was one of the co-PIs and directors of the PI4 program (Program for Industrial and Interdisciplinary Internships at Illinois) from 2014–2021, and am now the director of the PhD summer internship program at Institute for Mathematical and Statistical Innovation (IMSI)—both programs are aimed at giving PhD students the training they need to be competitive in nonacademic careers after graduation.

Emerson: I am a senior data scientist and mathematics of data science team leader at Pacific Northwest National Laboratory. I finished my PhD in 2017 and made the transition out of academia directly after my PhD. I hold two joint appointments in the Department of Mathematical Sciences at the University of Texas El Paso and the Department of Mathematics at Colorado State University. Additionally, I am one of the organizers and founders of the Topology, Algebra, and Geometry in Data Science (TAGDS) research community (`www.tagds.com`).

Garibaldi: Out of graduate school, I followed the traditional academic career path from postdocs up through being a full professor. A few years ago, I left academia to work at CCR La Jolla, a math research institute that is part of a nonprofit corporation—the AMS calls it a BEGIN[1] employer. Most of my experience mentoring students was during my time as a professor, where I advised undergraduates and PhD students who were interested in nonacademic careers.

Washington: I am the director of the Atlanta University Center (AUC) Data Science Initiative[2] which supports the development of data science innovations across Clark Atlanta University, Morehouse College, Morehouse School of Medicine, and Spelman College. As data science evolves in the industry space, it is important that our work includes inviting our industry partners to work alongside us to develop data science research, curriculum, and student career pathways. As an applied mathematician, I enjoy the interplay between various sectors to solve problems of societal interest.

Weekes: In graduate school, I spent a summer working at IBM; and at WPI, we've made solid connections to industry via our Center for Industrial Mathematics and Statistics. Our students work on research problems that come directly from our industry partners. That sort of experience (for both students and faculty advisors) has been brought to participants all over the US via the MAA & SIAM PIC Math[3] program. Now, as executive director of the Society for Industrial and Applied Mathematics, this connection between academia and industry is core.

Best Advice

Reed: *What's the best advice you have for faculty on how to mentor students for careers beyond academia?*

DeVille: Let me start by pointing out that it is neither easy nor natural for most academic mathematicians to mentor students for careers beyond academia, because of how we ourselves were trained. I don't mean for this to be daunting—it's more to say that if someone starts in this direction, and finds it difficult going at the beginning—this is completely normal. A key point is that academia and industry are two different worlds: they have different values and different incentives. The types of skills, habits, and values that we want to instill in our students are different based on what their lifepaths will be. For a student going on to an academic career, it makes a lot of sense to think slowly and deeply about a problem to come up with the best possible solution; for a student going toward industry, it can be more important to inculcate breadth and flexibility. Another note: the topic of a student's PhD dissertation is often not directly relevant to a student's career in industry. Most employers who are hiring math PhDs want to know that a student can carry out a major research program; if they're convinced the student cracked a hard abstract problem, they often don't care for the details. So, it happens that a student writes a very "pure math" thesis, but still does well in the nonacademic job market because they have coding skills, soft skills, etc. Also, one last thing: it is probably not necessary for every PhD advisor to be good in this space. A lot of the mentoring in this direction can be done centrally by a small number of faculty (e.g., by the director of graduate studies or local equivalent, and/or those faculty with existing experience in this area). As long as you have a few strong faculty mentors in your program, you should be fine!

Emerson: One of the most important skills for someone to have at a national lab is being able to create a narrative around your work. Challenge your students to explain their approach and the impact at different levels. Incorporate literature reviews and analysis for students as part of their learning experience to quickly assess the quality of work being done and learn how to reproduce results. Finally, outside of academia you rarely work alone—being a part of a team is really important. Have students work in groups and learn how to handle the types of conflicts that arise.

Garibaldi: Encourage students to learn diverse subjects. And connect them with resources to prepare for job interviews—SIAM has online videos about interviewing and your campus' Career Center may offer practice interviews.

Washington: One technique I like to utilize is to apply project management techniques within the framework of a course. This includes not only working in teams

[1] *BEGIN stands for "Business, Entrepreneurship, Government, Industry, and Nonprofit."*

[2] `https://datascience.aucenter.edu/`

[3] `https://www.maa.org/programs-and-communities/professional-development/pic-math`

but also talking about the stages of team formation developed in the mid-1960s by Bruce W. Tuckman to optimize team functionalities. I also intentionally expose students to aspects of professional development in the context of a course. This includes creating a resume, writing a personal introduction, and developing SMART (Specific, Measurable, Achievable, Relevant, and Time-Bound) goals. This allows me to get to know the students better and in turn, this helps students better define their mathematical trajectory. In my current role, we have opportunities to engage with data science professionals and we provide guidance on how to network. Incorporating team work along with professional development as a part of the mathematical pursuit is beneficial since it is such an important part of every workplace.

Weekes: I would add that it is important to encourage your students to pursue internships or to get experience with industry. You don't need to have hands-on experience yourself with this. Your university's career services are able to provide help in searching and preparing for internships. Also, direct students to resources from our professional societies about careers; SIAM's career resources[4] are wonderful. Make sure you take a look at our careers brochure.[5] The BIG Math Network[6] is an excellent joint effort with the mission to promote careers in business, industry and government to students and departments of the mathematical sciences.

Common Mistakes

Reed: *Are there things you'd like to advise faculty to avoid doing to best support their students interested in nonacademic careers? What are some common mistakes people make in advising mathematicians seeking nonacademic careers?*

DeVille: I think the biggest mistake I see both faculty mentors and students make is assuming that industry is the "backup plan"; that the "best" students go on to postdocs and industry is for the ones who "don't make the cut"—and a corollary of this is that getting a nonacademic job is somehow easier. This is a prejudice that is widely held in academia, and especially in mathematics, in my experience. One main problem is that many mentors and students have the mindset that a PhD student should focus on writing a strong thesis, and then maybe throw in some coding in year five, and it will all work out. This is not true—students need to train for their desired career throughout the PhD, and this training should be integrated with their education throughout.

Emerson: I echo many of the sentiments of the other panelists. Encourage coding early, on real data, to keep their options open. Don't only value becoming the world expert in a very niche topic. Many careers outside of academia, I believe, are served by a "the more you know" attitude. A broad base builds the best foundation, in my opinion, for a career beyond academia where flexibility and curiosity are necessary for longevity.

Garibaldi: I assume no one reading this article would denigrate the choice to pursue nonacademic careers. Still, the very nature of how math departments operate enforces the idea that the people doing mathematics are professors of mathematics. This message is often transmitted by the learning materials we provide ("here, read this math paper" written by a math professor) and by telling students to attend seminars (typically given by professors from math departments). We would better support our students interested in nonacademic careers by also exhibiting nonacademic mathematical careers as relevant to math.

Washington: When assigning group work, provide guidance on how to work in a group. There are models, such as Tuckman's Model [Tuc65] that can easily be incorporated into the class discussion so students understand how to participate effectively in a team. I have also found that by defining and assigning each student a role on the team (lead, recorder, questioner, timekeeper, etc.) we help students take agency in ways they not have otherwise. Having high expectations while also giving students the tools they need, both mathematical and social, better positions students to learn and excel.

Weekes: I hear of faculty who discourage their students from taking internships as it takes them away from their PhD program. I don't think that this is fair to the students. New PhDs are not obliged to become university faculty as their advisors are, and faculty should respect and support students' career goals.

Programming and Data Science

Reed: *How do you view computing/programming in conjunction with preparation for nonacademic careers? What about statistics and data science?*

DeVille: The ability to code is crucial for a student looking to pursue a research industry career, and it is important for a student to have a record of this (gold standard: a GitHub that solves some interesting problems!). That being said, I don't think it is necessary for mathematics PhD programs to revamp their PhD curriculum or courses to include coding—many campuses have courses or bootcamps in a variety of languages and contexts, not to mention coding contests, etc. So, motivated students can learn in this area, but outside of your department. For data science, as for coding, there are many resources and opportunities outside of math. (Of course, if a PhD program is able and willing to include coding and data science in their PhD training, that is great! But it is not strictly necessary.) I think it's also worth pointing out here that one

[4]https://www.siam.org/careers/resources

[5]https://www.siam.org/students-education/programs-initiatives/thinking-of-a-career-in-applied-mathematics

[6]https://bigmathnetwork.org/

thing a student can do during the PhD that will really make their applications more competitive after graduation is an internship (typically during the summer). A successful internship (or even better, multiple ones) is one sign to a potential employer that a student will be able to fit well into their environment—and typically gives a student one or more valuable letters and references for the job search. Students who are interested in industry should be strongly encouraged to pursue such internships. One worry I have heard from faculty is that a student's dissertation research will go more slowly if they are gone for the summer. I find that often enough this isn't the case—a student who works on something completely different for a while comes back to their original problem refreshed and ready to make progress, whereas students who don't do a second project can often become discouraged and despondent if all they ever do is bump their heads into the same brick wall over and over...

Emerson: Use real world data sets! Have them struggle with "data wrangling" a bit so they get a sense of how real-world problem solving begins. Also, encourage students to prepare presentations with no equations. They should be able to describe what they did and why it matters.

Garibaldi: Include coding and data science organically in the classes you teach. For example, when you teach a class in pure mathematics, there are typically homework exercises that amount to a hand computation. Replacing one of those with a somewhat larger version is often an easy way to create an entry-level use-a-computer-to-do-real-math kind of homework problem.

Washington: Coding nowadays is not limited to one language. Our students are entering a workplace, both inside and outside of the academy, where there may be a high expectation to be able to code in different languages. I enjoy providing opportunities for students to develop programming skills. When I taught Calculus III, I utilized MATLAB to create the 3-dimensional visualizations. I first did a hands-on, follow along where I engaged the class in walking through how to code on MATLAB. Then, each homework assignment included an exercise on creating a visualization in MATLAB. Throughout the course, I did embed coding and found it to be a fruitful experience as students found interesting ways to create the plots. In my current role as Director of the AUC Data Science Initiative, we provide opportunities for all faculty and staff to upskill in data science and coding and have over 200 participate in our summer workshops. I am seeing coding become a meaningful skill across the curriculum from public administration to chemistry to social work and mathematics, just to name a few. Having students experience coding in different disciplines, I think, will position students well for the technology-driven future they will enter.

Weekes: Every student—at the very least every math student—should get experience programming. Basic statistics knowledge via a formal class or in the process of a research experience is also important.

Building Networks with Industry

Reed: *What advice do you have for faculty on how to build networks with industry? Are there specific conferences we should go to? How do we meet people in industry, and what can we do to keep lines of communication open to be able to get information we should share with our students?*

DeVille: Probably, the most valuable connection is through alumni networks. Keep track of what your PhD alumni do and encourage current students to get in touch with them. Alumni who are working at (or, even better, recruiting for) an academic employer are the absolutely best resource for students looking for a job in a closely-related industry. Departments can also have career days where they invite alumni to speak and give feedback to students (many organizations like their employees to do this—if they're happy with someone, they always want more candidates just like that person!). Also, don't forget your department's undergraduate alumni; many employers looking for mathematics BS graduates are also looking for mathematics PhD graduates. Admittedly, alumni networks suffer from a "chicken and egg" problem—what if you don't yet have a robust alumni network in your program? If your institution has an engineering school, this can be a valuable resource—many of the same employers who would hire an EE PhD would hire a Math PhD. More generally, most campuses have career centers and will have a lot of contacts at STEM-hiring outfits.

The communication (both up- and down-stream) can be a challenge. One system which seems to work is if there is a faculty member in charge of all of this (networking, establishing recruiting events, etc.). In many departments this is also handled by the director of graduate studies, or local equivalent. But be forewarned—this is in itself a full-time job, and it is important that departments recognize and reward this work.

Emerson: SIAM conferences are a great place to interface with industrial mathematicians. NeurIPS (Conference on Neural Information Processing) is also a major event that draws a mix of mathematically-oriented academics and industry researchers. TAG-DS (Topology, Algebra, and Geometry in Data Science) is a scientific research community led by researchers at national labs and faculty at universities. There are a series of TAG-DS conferences and workshops at top data science events that facilitate networking between academic, industrial, and government researchers.

Garibaldi: First step is to use the contacts you already have. Your department has alumni working at companies.

Talk to them! Most people are flattered when someone else takes an interest in what they are doing and wants to learn from what they have learned. You can ask industry folks to give talks to your students. And if they do, faculty can attend! The information from the presentation is ore to be mined for mentoring next year's students, who may not benefit from getting to hear the same talk themselves.

Also, when you hear someone in industry talk about the scientific or technical challenges they face, it is natural to listen like a mathematician who wants to solve their problem and focus on the technical details of the problem. But for mentoring, it is better to pull back your focus and think also about the person and how they are describing their challenges, the words they are using, and what that context says about their greater view of the technical landscape. Later on, when talking to students about careers in industry, you will have a better sense about how people in that industry view their world.

Washington: I have built many networks with those in industry via LinkedIn. This is a great way to network with industry folks and as well as alumni. Some LinkedIn tips: include an e-mail address that works, a profile that is updated and matches the resume, and a customized URL. When engaging with industry, it's important to get the industry representatives to engage with faculty, not just students. The industry partners will be eager to recruit students for jobs but they need to remember that the students come and go but faculty stay, and can influence many different student cohorts over their careers. If our faculty understand the expectations from industry about what content knowledge would position students for success in these constantly evolving workspaces, this would provide an opportunity to spice up a course and connect student learning to future opportunities. I really enjoyed attending UIDP's events[7] because they focus on strengthening university-industry partnerships through engagement, research, and other synergies. As we move forward with the AUC Data Science Initiative, we are working to strengthen our relationship with our sponsors to build out data science with our faculty to create more robust pathways for our students.

Weekes: Set up an industry advisory board at your department level. Alumni connections are also very useful. Have university staff contact alumni and ask if they would like to visit campus, give a talk, meet with students.

New Opportunities

Reed: *In your experience, how is non-academic mathematical work changing? Are there new opportunities in this area, or other information, you'd like to make faculty aware of?*

Emerson: Research in data science is changing very fast. The timescales are changing. Outside of academia if an idea is going to fail, you want it to fail fast. This concept is somewhat at odds with the inherently rigorous nature of mathematics and can be a challenge for students fresh out of mathematics graduate programs.

Garibaldi: Do your students include their GitHub on their resumes when they are applying for jobs? Lots of people do these days, and that is a new thing to me.

Washington: Many companies and nonprofits need technology development but some may have limited budgets to carry out this work. This is where those in academia can provide insight and also engage in fun problems. The AUC Data Science Initiative is partnering with the AUC police chiefs and the Atlanta police to find data-driven solutions to enhance public safety in our neighborhoods. I am really excited about this project because of the value added to the community. By looking both at the university or college and nearby enterprises, one may be able to find some interesting mathematical and data science challenges that could provide great learning experiences for students while providing societal benefits.

Weekes: Faculty and students should join a professional society and read the news and newsletters that come out from these organizations. This is a way that they can stay updated on the latest and greatest. I mentioned some resources earlier and would add to that SIAM News.[8]

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[7] `https://uidp.org/`

[8] `https://sinews.siam.org`


Lee DeVille


Tegan Emerson


Skip Garibaldi


Mary Lynn Reed


Talitha M. Washington


Suzanne L. Weekes

Credits

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Leaving Academia

Karen Saxe

Some academic friends tell me that my career path has been "alternative." In fact, it was all but that for most of my career.

I completed my PhD; enjoyed a two-year post-doc; then settled down for 28 years of teaching and leadership at one institution. At all points in that 30-year run, everything seemed "conventional," if extremely fortunate. I appreciate that it is hard to land jobs in academia that are as satisfying and wonderful as mine have been. I am still surprised that I was able to raise three children, stay married, sustain meaningful friendships outside of math, get tenure, be promoted to professor, and become department chair. In retrospect, I've succeeded in achieving my graduate school hopes. But it was hard: I worked nonstop at my job and struggled to be a good mom. I know many of you are familiar with working all day, then heading home to prepare dinner; spending the evening with your family until the kids go to bed, then working again from 10 p.m. until midnight (or as long as you can manage to stay up writing lectures, grading, and answering emails).

The career I have now began in 2013, when I served as the AMS Congressional Fellow. During the academic year 2013–14, I worked for Senator Al Franken in his Washington, DC, office. I worked on education policy concerns: to establish a national STEM master teacher corps, to provide education stability for youth in foster care, to strengthen connections between community colleges and local industries, and to improve the net price calculator that higher education institutions must make publicly available. I also had the opportunity to support Franken's work on the Senate Indian Affairs Committee.

One part of that position that I really enjoyed was the writing. It was mostly of two kinds: (1) background materials for committee hearings, which included providing suggestions for questioning witnesses, or (2) drafting speeches. These speeches were, typically, ones the senator would give when visiting a high school. After that year, I returned to my academic job and, honestly, had no idea I would end up back in Washington! Then in 2015, I learned that Sam Rankin, my predecessor at the American Mathematical Society, was planning to retire. Now here was a job I wanted! I began full time with the AMS in January 2017 as director of government relations. I did not apply for this position because I was trying to leave my academic

Karen Saxe is an associate executive director and director of government relations at the American Mathematical Society. Her email address is kxs@ams.org.

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job—I loved my academic job—but I was wondering what else I could do, in particular how I could give back to the math community. Significant volunteer work for professional societies has always been important to me. As mentioned, I love writing, and—for me—writing for different audiences is a fun challenge. I also enjoy meeting and talking to people, all sorts of people. And, truth be told, I am into politics.

Since this essay is part of the Early Career theme, and I am now toward the end of my career, I will end with some advice:

1. In an academic job, we think we should do everything all at once and excellently—publish good mathematics, be a great teacher, be a department and perhaps campus leader, be a wonderful partner and parent. The list goes on (and obviously, some of us only engage in a subset of those activities). But, whatever is on your list, I encourage you to think of your career as an arc. You cannot excel in all of these roles *at the same time*. You can focus on research for some years, parenting for others. Go to your kids' games and recitals, but don't attend every evening event on campus—you can participate in these later, when your kids are older. During a sabbatical, focus on research and parenting—you can develop a new course another time. You don't have to be on a major department or campus committee every year. Some of this, of course, involves letting go and letting others lead (incidentally, this is generally best for your department). You can have a messy house. Be kind to yourself.
2. In academia, many of us strive to land that *one* job that we plan to keep and love for our entire working life. Keep in mind, though, that hardly anyone outside academia thinks this way. Leaving a job you know well is scary—I know this first hand. In my opinion, a university or college administration should be happy and proud when good faculty members leave for important work outside of the academy. Higher education is under fire in this country, and we academics bear some responsibility for this predicament; one way to alleviate this is to stop isolating ourselves.
3. Think about what you like to do best, and what sort of work environment you prefer. I decided I like writing, I like being with people, I like talking to people outside of math about how wonderful and useful math is and working with them to be *comfortable* with math (not necessarily to understand it).
4. Remember that there are many mathematicians (both with undergraduate and advanced degrees) working outside academia. Indeed, according to the NSF's *Survey of Doctorate Recipients, 2021*, of the 36,600 PhD mathematicians and statisticians who reside and are employed in the US about 59% work in academia. Another 35% work in the private sector (including all of us who work for the professional organizations that support the math community), and roughly 4% work for the government (local, state, and federal).[9] The AMS Congressional Fellowship, and also the AMS Mass Media Fellowship, are just two of many roads into "alternative" careers that—as the data show—engage a large minority of us. Mathematicians who work outside academia are, I have found, very willing to talk about their experiences. . . .seek them out!

Karen Saxe

Credits

Photo of Karen Saxe is courtesy of Macalester College/David Turner.

Disappointment

Danny Calegari

In high school I took part in many math competitions; the hardest (and therefore to my teenage mind the only ones that counted) were the competitions relating to the international mathematical olympiad (IMO). In Melbourne there was a program of competition and training that culminated in a nine hour exam spread over two days to determine the makeup of the Australian IMO team. I remember very well the first time I took the exam. It was 1987, and the IMO was to be held that year in Cuba. As I sat at my carrel in the Morris library, I took the February sunlight for fortune smiling on me, inspiration spilled liberally from my Pelikano steel nib fountain pen, and I went home at the end of the second day in a blur of fatigue and self-congratulation. Six weeks later a pregnant manila envelope arrived in the mail. From its girth alone I knew I had aced the exam and won my rightful spot on the team. Before even opening the envelope I could see myself in the green woolen team blazer with the Australian coat of arms

[9] *See Table 12-1 at* `https://ncses.nsf.gov/pubs/nsf23319`.

Danny Calegari is a professor of mathematics at the University of Chicago. His email address is dannyc@math.uchicago.edu.

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embroidered on the breast pocket, and by the time I found a letter opener I was shaking hands with Fidel Castro.

The envelope contained... sixty-odd sheets of loose-leaf paper, no invitation, no cover letter: my exam papers, bloodstained with question marks, lines through paragraphs, squiggles of uncomprehension, Xs and Os. My stomach fell. I blushed. In a fraction of a second I rewrote or recolored dozens of memories and fantasies from the recent past and future, and became intensely conscious of and embarassed by my vanity and foolishness. What I now find remarkable was the speed and scope of my transformation; the analogy that comes to mind is being struck by a speeding car.

So how does one deal with disappointment? Freud, in *Civilization and Its Discontents* identifies three typical measures:

> powerful deflections, which cause us to make light of our misery; substitutive satisfactions, which diminish it; and intoxicating substances, which make us insensible to it.

If these are the typical responses, are there any others? Before trying to answer this it might be helpful first to articulate what disappointment actually *is*, and then to ask what it's *for*. Evidently, disappointment is a form of mental suffering. It is so unpleasant that we can experience it in a host of physiological dimensions. Profound mental suffering involves a complex array of interactions between any number of processes and subsystems, both conscious and unconscious, involving both the brain and the limbic system. The suffering that arises from disappointment is that that accompanies disruption: disappointment causes a certain kind of shake-up or realignment of our worldview and self-image and consequently of our priorities; this disruption can be so great that we sometimes emerge from it a very different person.

According to certain schools of cognitive science (e.g., Minsky's *Society of Mind* model) the idea of a "self" as a unified, indivisible entity is an oversimplification; rather (they suggest) a self is an uneasy federation of simpler subsystems (sometimes termed "agents") with their own local goals and interests, which are frequently in competition with one another. Under ordinary circumstances stability is achieved by a complicated system of temporary alliances, detentes, three-way standoffs, and so forth. Our subjective sense of the unified self is—in itself!—also a source of stability. Sometimes a dramatic change in (real or perceived) external circumstances—an unforseen event, an unpleasant discovery—can lead to a cascade of disruptions to this order. This is the mechanism of disappointment, and why it is so painful; it is both a crisis and an opportunity—in Homer Simpson's inspired terminology, a crisitunity.

Disappointment measures in pain the gap between reality and what we want the world to be. Disappointment matters. It matters because we don't actually live in the real world. We live in our heads, in a mental world of assumptions, recollections, anticipations, desires and conjectures. And even when we do meet reality, it's a mistake to think that what our senses feed us is objective, unfiltered, unsorted. Rather we operate according to an interrogative protocol—we ask the world questions to confirm what we already "know" (or, more accurately: hope), and only when we get an unpleasant surprise do we take a closer look. As Proust says,

> The real voyage of discovery consists not in seeking new landscapes, but in having new eyes. Happiness is beneficial for the body, but it is grief that develops the powers of the mind.

In his famous paper, "How not to prove the Poincaré Conjecture," John Stallings writes about an adventure in his mathematical life, how he discovered a proof of the Poincaré Conjecture but later found it to be mistaken. He goes so far as to describe this episode as a "sin;" but the sin was not in the mistake per se, rather it was his resistance to recognizing it as such. He writes,

> There are two points about this incorrect proof worthy of note ... (t)he second... is that I was unable to find flaws in my 'proof' for quite a while, even though the error is very obvious. It was a psychological problem, a blindness, an excitement, an inhibition of reasoning by an underlying fear of being wrong. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician.

Few mathematicians are as honest or as generous as Stallings in sharing their own stories of disappointment. This is because disappointment often comes wrapped in shame, because our goals are inextricable from our personal and social attachments and relationships. Convention and social norms dictate that any display of human weakness or failing is "unprofessional." We're not supposed to admit it when we feel stupid, or underappreciated, or jealous, or that we cared so much about something that when it didn't go our way we felt shattered. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician.

Disappointment takes many forms; a partial list from my personal history includes:

(1) being un- or under-acknowledged in a colleague's paper or talk;
(2) being scooped;
(3) missing out on a job/prize/conference invitation;
(4) having a prospective student work with someone else;
(5) having a potential advisor turn me down as a student;
(6) having a promising line of attack on a problem fail to pan out;

(7) discovering an error in an amazing proof;
(8) having a paper go unread or a book go unreviewed;
(9) seeing an admired senior colleague behave badly;
(10) realizing that I haven't lived up to my own standards of behavior;
(11) discovering that success, when it came, was not all I hoped it would be.

The last one, perhaps, deserves elaboration. Some acute disappointments in my career were the result of getting what I thought I wanted: a paper in a fancy journal; a job offer; tenure; an invitation to talk at a fancy conference. I don't mean to diminish the value of such things at all, or the challenges (personal or structural) many people must overcome to achieve them; much about the way such "rewards" are distributed in academic culture is unfair, often in systematic ways, and it should be the goal of all of us to point this out and work to change it wherever we can. I also don't mean to suggest that success has been joyless; the opposite is true. Nevertheless it is the case that sometimes when we get what we think we want, we discover that these things weren't what we thought they were, and (more importantly) that we are not who we though we were. When disappointment accompanies success it is worth paying special attention to. If we get what we want but it doesn't bring us fulfillment, then what's really going on? In my experience, it has only been at the point of my posing this question that I have acquired insight, and the agency to really change things or come to terms with them.

It took a month of pain after the manila envelope arrived before curiosity got the better of me and I opened it again. And a remarkable thing happened. The exam pages: my answers, the blots, the corrections, the red ink, the comments, were exactly as before. But time and some strange alchemy of which disappointment itself was the catalyst had altered their meaning. An actual human being had taken the time to read my work and share valuable feedback with me. My annotated exam was no longer a certificate of failure, it was a how-to manual: it was about how to prove an inequality by leveraging the convexity of a cleverly chosen auxiliary function, or how to recast a geometric figure in terms of complex numbers and understand it with algebra. These math problems weren't "problems" at all: they were windows into mathematics itself. And the manila envelope wasn't a slap in the face, it was a gift; but to see it as a gift I had to see it with new eyes. I never got to Cuba, but I'd taken my first steps on a longer and far more interesting and rewarding journey that continues to this day.

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Danny Calegari

Credits

Photo of Danny Calegari is courtesy of Danny Calegari.

Dear Early Career

If I give a talk on work that I haven't yet finished, should I worry that someone might use my ideas to prove my result before I do?

—*Concerned*

Dear Concerned,

It's good to hear you have some research progress you are excited about! If this work is joint with anyone else, then certainly ask them for their opinion—this is a slightly touchy subject. It is also likely that attitudes toward giving talks before a preprint is available differ from subfield to subfield, and so consulting your mentors is advisable.

With those caveats out of the way, presenting results before the preprint is available is done pretty routinely in mathematics. It can be very positive to create some buzz for a result or let people know it is coming, and as an early career researcher every opportunity to present is valuable.

It seems a little unlikely to us that someone in the audience would merely copy your proof (and if they did, then your public talk on the topic would likely establish priority from the perspective of the community), but it is plausible that someone could find an extension of it, or find a simpler proof, in which case your forthcoming paper might gain an author. If that possibility does not appeal to you at all, then maybe it is best to wait for the preprint to appear. (A more unlikely outcome that cannot be ruled out is someone finding a simpler proof and submitting it immediately by themselves. While we think that one could still publish the original proof in these circumstances, we have to admit that if this had ever happened to us then this answer would be very different.)

We may be a particularly cautious editorial team, but in the absence of a preprint, our main question to ourselves in this circumstance is this: Have we really proved the result, or are there substantial issues that remain? It can be a bit problematic if one presents a forthcoming result and no preprint ever appears, as interested researchers in your field would be left in limbo. Therefore, we would recommend that before presenting you have a detailed account of the proofs that could be circulated (this is a long way from a finished preprint, but could also help establish priority if has been circulated to a few trusted people prior to your talk). Furthermore, if your work solves a problem with some reputation, then having some discussions with mentors and experts would also be a great idea before going full steam into a talk.

In our experience, the opportunity that comes from presenting your ideas outweighs the risks of people acting in an unprofessional manner.

—Early Career editors

Have a question that you think would fit into our Dear Early Career column? Submit it to `Taylor.2952@osu.edu` or `bjaye3@gatech.edu` with the subject Early Career.

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MEMORIAL TRIBUTE

Wolfgang Haken, 1928–2022

Patrick Callahan, Ilya Kapovich, Marc Lackenby, Peter Shalen, and Robin Wilson

Introduction

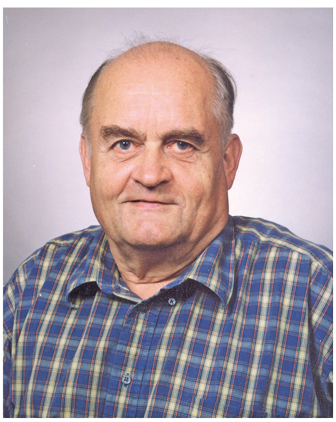

Figure 1.

In the fall of 1947, when Wolfgang Haken was a 19-year-old undergraduate at the University of Kiel, he took an introductory topology course from Karl-Heinrich Weise. In this course, Weise mentioned a number of famous open problems in topology, including the Unknotting Problem, the Four Color Problem, and the Poincaré Conjecture. Haken spent his entire career working on these three problems and their ramifications; he solved the first two, and, in the case of the first, showed that the techniques he had developed for the solution could be applied far beyond the original problem.

The section by Marc Lackenby and Peter Shalen describes Haken's approach to the Unknotting Problem through his theory of normal surfaces, and the far more general results which he obtained using this theory and his closely related theory of hierarchies. This section also gives an indication of the very diverse ways in which other researchers have exploited these theories, both in 3-manifold theory and in geometric group theory. The section by Robin Wilson recounts the fascinating saga of the successful attack on the Four Color Problem by Kenneth Appel and Wolfgang Haken, which was groundbreaking in its extensive involvement of computers.

In contrast to his spectacular successes with the Unknotting Problem and the Four Color Problem, Haken's huge efforts in connection with the Poincaré Conjecture did not yield a proof. However, as Lackenby and Shalen point out, in the years following Grigori Perelman's proof of William Thurston's Geometrization Conjecture—of which the Poincaré Conjecture is a special case—a number of the most striking consequences of Perelman's work involved normal surfaces and hierarchies. This is a further illustration of the enormous influence that Haken's ideas have had on 3-manifold theory.

Haken's distinctive approach to mathematical research was described by one of his colleagues at the University of Illinois Urbana-Champaign:

> Mathematicians usually know when they have gotten too deep into the forest to proceed any further. That is the time Haken takes out his penknife and cuts down the trees one at a time.

Given Haken's magnificent output, it appears that this could be a very effective approach. Of course, in order to succeed with this approach, one had to be Wolfgang Haken.

Those of us who knew Haken remember him as a wonderfully kind and generous person with a delightful low-key sense of humor. These traits come through in Patrick Callahan's section about his days as a PhD student under Haken's direction. Ilya Kapovich's biographical section gives insight into the obstacles that Haken had to overcome on the way to his phenomenal mathematical successes, and into a family legacy that is as impressive as his mathematical legacy.

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Wolfgang Haken's Life

Ilya Kapovich

Wolfgang[1] Haken[2] was born in Berlin on June 21, 1928. His father was a physicist working for the German Patent Office and his mother stayed home to take care of the family and household. His two older brothers died of scarlet fever in 1927, and Wolfgang grew up as an only child. During his childhood in Berlin, Wolfgang developed an early interest in mathematics. At the age of 4, he made what he thought was his first important mathematical discovery: that counting should start with 0 rather than with 1, and that 0 is the first natural number. He tried to convince his father to patent this fact but did not succeed at the time. Wolfgang's mother died in August 1939, several days before the start of World War II, and the family remained in Berlin for most of the war. In 1944, at the age of 15, Wolfgang was drafted to serve in an anti-aircraft battery. He was soon transferred from Berlin to Dessau and from there to Soest, where he remained until the end of the war. After the war, Wolfgang first worked as a farmhand, and passed a high school GED exam in 1946.

In the summer of 1946, he started his undergraduate studies at the University of Kiel. At the age of 17, Haken was the youngest student at Kiel, as the universities in Germany then had a rule not to admit anyone under the age of 23. Initially, Haken wanted to become a physicist but his interests gradually changed to mathematics. At the time, Kiel had only two mathematics faculty members: a professor of mathematics and a professor of geometry, which were regarded as different subjects when the university was founded in the 17th century. The professor of mathematics at Kiel was Karl-Heinrich Weise; most of Haken's mathematics classes were taught by Weise. In the Fall of 1947, while Haken was still an undergraduate, he attended a topology course by Weise. In this course, Weise stated several famous open problems in topology, including the Poincaré Conjecture, the Four Color Problem, and the Unknotting Problem. This experience marked the start of Haken's interest in topology. Remarkably, as was mentioned in the Introduction, of the three major mathematical problems that motivated Haken's interest in topology, Haken eventually solved two—the Unknotting Problem and the Four Color Problem.

Haken received a pre-diploma (roughly equivalent to the Bachelor of Science degree) in physics and mathematics at Kiel in 1948. He then started his doctoral studies in mathematics at Kiel, with Weise as his thesis advisor. Haken obtained his doctorate from Kiel in 1953, with the dissertation entitled "Ein topologischer Satz über die Einbettung $(d-1)$-dimensionaler Mannigfaltigkeiten in d-dimensionale Mannigfaltigkeiten."

Haken met his future wife, Anna-Irmgard Freiin von Bredow, at Kiel in 1950 where she was also studying mathematics as an undergraduate. They were married in 1953. In 1959, Anna-Irmgard also received a doctorate in mathematics with Weise as her advisor.

After getting his doctorate, Haken obtained a job at Siemens in Munich as an electrical engineer, where he worked on designing microwave devices until 1962. The first three of Wolfgang and Anna-Irmgard's six children were born during this period: Armin in 1957, Dorothea in 1959, and Lippold in 1961.

In 1956, Haken sustained a near-fatal accident while mountain climbing in the German Alps. He fell more than 30 feet and remained in a coma for several days. The accident significantly damaged Wolfgang's foot but did not dampen his enthusiasm for the outdoors.

While at Munich, Haken continued doing mathematical research in combinatorial topology. He solved the long-standing Unknotting Problem by producing an algorithm for deciding whether a knot diagram represents the trivial knot. The solution of the Unknotting Problem got Haken's work noticed by several mathematicians in the United States. Ralph Fox, a topologist at Princeton, had his graduate students go over Haken's proof in detail, and, somewhat to Fox's surprise, they found the proof to be correct.

Bill Boone, a group theorist at the University of Illinois at Urbana-Champaign (UIUC), also became intrigued by Haken's paper. At the time, Boone was working on topics related to the unsolvability of the word problem for finitely presented groups, and he understood that there were close connections between algorithmic problems in group theory and algorithmic problems in low-dimensional topology. Since by then it was known that the word problem for finitely presented groups is, in general, undecidable, Boone expected the Unknotting Problem to be undecidable as well. Therefore Haken's proof came as a considerable surprise to him. Just six weeks after the publication of Haken's 1961 paper [Hak61b] on the Unknotting Problem in *Acta Mathematica*, Boone invited Haken to come to UIUC for a year.

Ilya Kapovich is a professor of mathematics at Hunter College of CUNY. His email address is `ik535@hunter.cuny.edu`.

[1]*With gracious permission from the University of Illinois, this section incorporates substantial portions of the article by Ilya Kapovich, "Wolfgang Haken: a biographical sketch,"* Illinois J. Math. ***60*** *(2016), no. 1, iii–ix.*

[2]*Born Wolfgang Rudolf Günther Haken. He dropped his middle names and changed his legal name to just "Wolfgang Haken" in 1976, when he became a US citizen.*

Haken came to Urbana-Champaign with his family in 1962 and spent the 1962–1963 academic year at UIUC as a visiting professor. When preparing for his year at UIUC, Mahlon Day, who was the Mathematics Department Head, suggested that Haken obtain a US immigrant visa (which was relatively simple to do at the time). Haken followed this advice, which made it easier for him to eventually settle permanently in the US. During his year at Illinois, Haken applied for and obtained a temporary membership at the Institute for Advanced Study at Princeton. He spent two years, 1963–1965, at Princeton. In 1965, Haken joined the faculty at the Department of Mathematics at UIUC as a tenured professor. The last three children of Wolfgang and Anna-Irmgard were born in the US: Agnes in 1964, Rudolf in 1965, and Armgard in 1968.

In Haken's paper [Hak61b] and his later papers [Hak61a, Hak62, Hak68] he went far beyond the Unknotting Problem, making huge inroads into the more general *Equivalence Problem* for knots, and the essentially still more general *Homeomorphism Problem* for 3-manifolds. In doing so he introduced the concepts of *normal surface, incompressible surface*, and *hierarchy.* All these terms will be explained in the next section, on knots and 3-manifolds, where an account of the enormous influence of these concepts—extending even far beyond Haken's original applications of them—will be given. It will be seen that these concepts, and Haken's work in this area, are still bearing fruit today.

In the late 1960s, Haken began to work on the Four Color Problem, which had fascinated him ever since the 1947 topology course by Weise and a subsequent 1948 lecture at Kiel by Heinrich Heesch. In 1976, Haken and Kenneth Appel (who was also a professor at UIUC then) proved the Four Color Theorem. Their proof included a substantial computer-aided component and marked the first time that a major mathematical result of this level of importance was solved with the help of a computer.

After an announcement in the *Bulletin of the AMS* in 1976 [AH76a], the proof was published in 1977 in the *Illinois Journal of Mathematics* [AH77, AHK77].

Inevitably, the proof generated much discussion and controversy in the mathematical community. In retrospect, the proof was to a large extent responsible for the birth of computational and experimental mathematics as significant directions in modern mathematical research.

Shortly after Appel and Haken announced their proof in 1976, the UIUC Department of Mathematics put the phrase "Four Colors Suffice" on its official postmark, which remained in use until the mid-1990s: see Figure 2.

Wolfgang Haken delivered an invited address at the International Congress of Mathematicians in Helsinki in 1978. In 1979, Haken and Appel shared the Fulkerson Prize from the American Mathematical Society for their solution of the Four Color Problem.

Figure 2. The "Four Colors Suffice" postmark used by the UIUC Department of Mathematics after the Appel–Haken proof.

Haken remained a professor in the UIUC Department of Mathematics until his retirement in 1998. He was also a member of the University of Illinois Center for Advanced Study from 1993 to 1998. While at UIUC, Haken was a thesis advisor for seven PhD students: Richard Rempel (1973), Thomas Osgood (1973), Mark Dugopolski (1977), Howard Burkom (1978), Robert Fry (1979), Patrick Callahan (1994), and Scott Brown (1995).

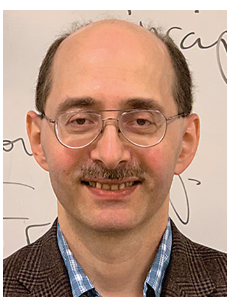
The "Saturday hike" is a delightful UIUC tradition going back to 1909 and having a long association with the mathematics department; the hike was for many years led by the late Joseph Leo Doob. From the 1960s through the rest of his life, Wolfgang Haken was a constant participant in the Saturday hike, as were many other members of the Haken clan. Wolfgang's wife Anna-Irmgard was an informal leader of the hike from 1993 to 2005, and continued to come to the hike in subsequent years, while her health allowed. Anna-Irmgard, the beloved matriarch of the Haken clan, passed away on April 4, 2017.

In retirement, much of Wolfgang's scientific interests concerned thinking about fundamental problems in cosmology. He also remained keenly interested in low-dimensional topology and was extremely pleased to see the tremendous progress in the field, including Grigori Perelman's proof of the Poincaré Conjecture and the proof of the Virtual Haken Conjecture by Ian Agol and Daniel Wise. In 2016, the *Illinois Journal of Mathematics* published a special Haken volume honoring Haken's mathematical contributions and influence. In November 2017, the UIUC Mathematics Department hosted a Four Color Fest to celebrate the 40th anniversary of the proof of the Four Color Theorem by Appel and Haken.

Three of Wolfgang Haken's six children live in the Urbana-Champaign area. Rudolf Haken, a renowned musician and a composer, is a professor of viola in the UIUC School of Music. Lippold Haken designs electronic musical instruments and equipment and owns a company manufacturing a unique "Continuum Fingerboard." He is also retired from the position of teaching professor in the Department of Electrical and Computer Engineering at UIUC, where he conducted research related to sound. Armgard Haken received BS and MS degrees in biology from UIUC, and is currently a research coordinator at the Midwest Big Data Innovation Hub, housed within the National Center for Supercomputing Applications at UIUC.

Haken's eldest son, Armin, obtained a PhD degree in mathematics from UIUC in 1984, specializing in complexity theory and problems related to theoretical computer science. He is now a retired software engineer in San Francisco. Dorothea Blostein, née Haken, received a PhD degree in computer science from UIUC in 1987 and is currently a professor in the School of Computing at Queen's University in Kingston, Ontario. Agnes Debrunner, née Haken, received a BS degree in animal science from UIUC. She is a leader in the US underwater hockey community and lives near Denver, Colorado. In addition to their six children, Wolfgang and Anna-Irmgard had 13 grandchildren, and the Haken clan continues to grow.

Wolfgang Haken passed away on October 2, 2022, at the age of 94, in Champaign, Illinois, surrounded by his family.



Ilya Kapovich

Knots and 3-Manifolds

Marc Lackenby and Peter Shalen

Haken's first major mathematical achievement was the solution to the *Unknotting Problem*, which appeared in his 1961 paper [Hak61b]. The problem had first been raised by Dehn in 1910, and was one of the most fundamental questions in knot theory.

Knots are just simple closed curves smoothly embedded in 3-dimensional space. A knot is usually represented by means of a *diagram*, which is a generic projection to a plane, with over/under information given at each crossing. In the Unknotting Problem, one is given a diagram of a knot, and the challenge is to determine whether it is the trivial knot, in other words whether it can be deformed, without crossing through itself, into a round circle. What is required is an algorithm that can provide a completely reliable answer.

Since the work of Alan Turing in the 1930s, it had been known that there are some problems that admit no algorithmic solution. Indeed, in his final published paper in 1954, Turing wrote "No systematic method is yet known by which one can tell whether two knots are the same." Although Turing did not explicitly say this, it was clear that he was raising the possibility that this problem might not be solvable. After some experimentation, it quickly becomes clear that the Unknotting Problem is certainly not straightforward, as it is possible to produce diagrams of the trivial knot that admit no immediate simplification. An example, due to Haken (with a small correction due to Ian Agol), is given in Figure 3.

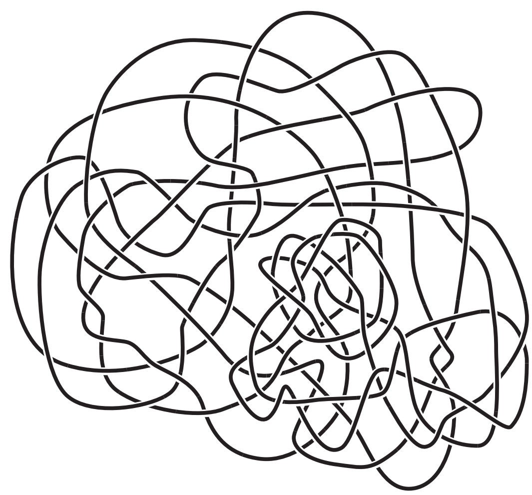


Figure 3. A trivial(!) knot.

A round circle in 3-space is the boundary of a disk. Indeed, this is a characterization of the trivial knot. For if one were to deform the round circle without passing through itself forming a knot K, then one could at the same time deform the disk. Thus any trivial knot forms the boundary of a smoothly embedded disk in 3-space. Conversely, if a knot K bounds a smoothly embedded disk, then one can deform the knot within this disk until it is a nearly-round curve that is visibly unknotted. The challenge, therefore, is to decide whether a given knot in 3-space bounds a smoothly embedded disk. This is called a *spanning disk*.

Normal surfaces. It is technically convenient to consider the given knot K not as lying in $\mathbb{R}^3$, but in its one-point compactification, the 3-sphere S^3. The reason why this is helpful is that if we thicken K to form an open solid torus $N(K)$, then the space $S^3 \setminus N(K)$, which is called the *exterior*

Marc Lackenby is a professor of mathematics at the University of Oxford. His email address is `lackenby@maths.ox.ac.uk`.

Peter Shalen is a professor emeritus in the Department of Mathematics, Statistics, and Computer Science at the University of Illinois at Chicago. His email address is `shalen@uic.edu`.

of the knot K, is compact. The exterior is a 3-manifold with boundary[3], which means that it is locally modeled on the closed upper-half space $\mathbb{R}^3_+$. If K is a trivial knot, a spanning disk for K can be chosen so that it intersects the exterior M of K in an "essential" disk D. To say that D is *essential* means that it is *properly embedded* in the sense that $D \cap \partial M = \partial D$, and is not *boundary-parallel*—i.e., is not obtained from a disk in ∂M by pushing the interior of the disk into the interior of M. The Unknotting Problem is then reduced to determining whether the exterior of a knot contains an essential disk.

Haken's approach to this problem used the notion of a "normal surface." Normal surfaces had already appeared in the work of Kneser in the 1930s to prove results about spheres in 3-manifolds, but Haken developed a systematic theory of such surfaces and recognized their huge power as a tool for addressing algorithmic questions.

The context for normal surface theory is a *triangulation* of a given 3-manifold M (with boundary), which is just a description of M as a collection of tetrahedra with some of their faces glued in pairs. A triangulation of the exterior of a knot K can easily be built from a given diagram of K. If a compact 3-manifold with boundary contains an essential disk, one can consider how such a disk intersects the tetrahedra of the triangulation. By suitably modifying the disk, one can always arrange that it intersects each tetrahedron in a collection of *triangles* and *quadrilaterals*, as shown in Figure 4. A properly embedded surface which meets the tetrahedra in this way is said to be *normal*. (In fact, Haken used an alternative formulation of normal surface theory, using handle structures rather than triangulations, but we will focus on the triangulated version here.)

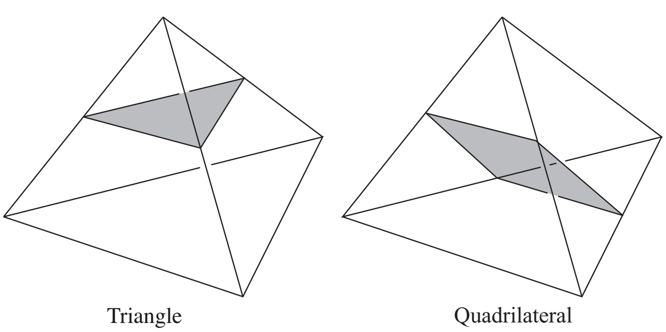


Figure 4. Components of intersection of a normal surface with a tetrahedron.

Thus, the Unknotting Problem reduces to the question of whether the exterior of K contains an essential disk which is a normal surface. This does not immediately solve the problem, as there may well be infinitely many normal surfaces in a given triangulation. However, Haken was able to show that one only needs to check a finite list of possible normal surfaces. His method was to encode a normal surface by combinatorial data in the following way. In each tetrahedron, there are four possible types of normal triangles and three types of normal quadrilaterals. Thus, if there are t tetrahedra, then a normal surface S determines a *vector* (S) whose entries are $7t$ non-negative integers, called *coordinates*, that count the number of triangles and quadrilaterals of each type; the vector (S) determines the surface S up to a harmless equivalence relation. Haken observed that (S) satisfies some simple restrictions. One restriction is that, within each tetrahedron, there can be at most one type of quadrilateral, as otherwise the surface could not be embedded. Thus there are 3^t possibilities for the set of types of quadrilaterals that appear in a given surface. When one has fixed such a set, $2t$ of the coordinates are constrained to equal 0. The other restriction is that when two tetrahedra are glued along a face F, then the requirement that the triangles and quadrilaterals in the adjacent tetrahedra patch together correctly along F imposes three linear constraints on (S). It turns out that these two sets of constraints are sufficient as well as necessary for an element of $\mathbb{N}^{7t}$ to be the vector of a normal surface. Thus the set $\mathfrak{X}$ consisting of all vectors of normal surfaces is a union of 3^t "cones" in $\mathbb{N}^{7t}$, each of which is defined by a system of linear equations with integer coefficients. In particular, if S_1 and S_2 are normal surfaces such that (S_1) and (S_2) lie in the same cone, we may define the *sum* S of S_1 and S_2 by $(S) = (S_1) + (S_2)$.

From the description of $\mathfrak{X}$ as a finite union of cones, Haken deduced, by very general arguments about solutions to systems of integer linear equations, that $\mathfrak{X}$ contains a finite set of "fundamental" vectors such that every vector in $\mathfrak{X}$ is a non-negative integer linear combination of fundamental vectors. Furthermore, the fundamental vectors can be found algorithmically from the equations; the surfaces corresponding to fundamental vectors are also said to be fundamental. Using a topological interpretation of the sum of two surfaces, Haken was able to show that if some surface is an essential disk, then some fundamental surface is an essential disk. Thus, one can decide whether a knot is the trivial knot, by going through each of the fundamental surfaces and checking whether any of them is an essential disk.

This was a brilliant and elegant solution to the Unknotting Problem.

Incompressible surfaces, hierarchies, and the Homeomorphism Problem. After solving the Unknotting Problem in [Hak61b], Haken went on to the more general

[3] *The manifolds we consider will in fact be smooth or piecewise linear, as will their submanifolds. In three dimensions, the transition between smooth and piecewise linear structures on a manifold is well-understood and elementary, and contains no surprises.*

problem, highlighted by Turing, of deciding whether two knots are equivalent, in the sense that one can be deformed into the other without crossing through itself. This *Knot Equivalence Problem* was a much greater challenge.

Just as a trivial knot bounds a spanning disk in S^3, a general knot K always bounds a compact, connected, orientable surface in S^3, called a *Seifert surface*. Compact orientable surfaces are classified up to homeomorphism by two numerical invariants: their number of boundary components (which in this case is 1) and their genus (which is their number of "handles"). Necessarily, when a knot is non-trivial, the genus of any of its Seifert surfaces is greater than zero.

Not all Seifert surfaces are interesting. For example, one can modify a given Seifert surface by adding a large number of handles in a tiny neighborhood of a point. In order to focus on surfaces that really reflect the topological structure of a given 3-manifold (such as a knot exterior), Haken introduced the notion of an "incompressible surface." A *compressing disk* for a properly embedded surface S in a 3-manifold M is defined to be a disk D contained in the interior of M with $D \cap S = \partial D$, such that the boundary of D is not "trivial" in the sense that it already bounds a disk in S. The significance of this notion is that a compressing disk for S can be used to modify S by the operation shown in Figure 5 below, called a *compression*, which will produce a surface that is "simpler" than S in a useful sense.

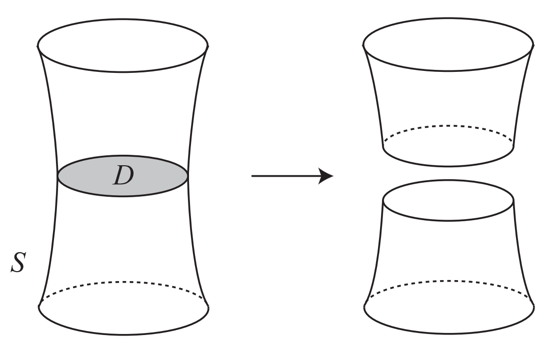


Figure 5. A compression.

For example, if S is connected and has connected boundary, like the surfaces in a knot exterior arising from Seifert surfaces, a surface obtained from S by a compression will have a component having the same boundary as S but having smaller genus.

We may define an *incompressible surface* in a compact, orientable 3-manifold M to be a properly embedded orientable 2-manifold S in M which is not boundary-parallel, is not a 2-sphere bounding a ball, and has no compressing disks. For example, if for a given knot K we choose a Seifert surface F whose genus is minimal among all Seifert surfaces for K, the properly embedded surface in the exterior of K that arises from F will be incompressible. This minimal genus is classically called the *genus* of K.

Any incompressible surface can be isotoped (i.e., deformed through a continuous family of embedded surfaces) to a normal surface. Using this fact, it is possible to use Haken's method of fundamental normal surfaces to find the genus of a knot algorithmically. However, this does not solve the Knot Equivalence Problem, because there are infinitely many inequivalent knots of any given positive genus.

Haken dealt with this issue by introducing a completely new idea in [Hak62]: the notion of a "hierarchy." Suppose that one starts with a 3-manifold with boundary M_1, for example the exterior of a given knot. One provides an incompressible surface S_1 in M_1. Then one removes an open regular neighborhood of S_1 from M_1, forming a new 3-manifold with boundary, called M_2 say. Then in M_2, one finds a new properly embedded incompressible surface S_2, one removes an open regular neighborhood of that, and so on. If the process terminates, in the sense that for some n the 3-manifold M_n is a disjoint union of 3-balls, then the manifolds $M_1, \dots, M_n$ are said to constitute a *hierarchy* for M_1.

Haken proved, making strong use of normal surface theory, that every knot exterior has a hierarchy; one can take the surface S_1 to be the incompressible surface that arises from a minimal-genus Seifert surface. But what he proved is far more general: a compact, orientable 3-manifold M with non-empty boundary always admits a hierarchy, provided that M is *irreducible*, in the sense that it is connected and that every 2-sphere in M bounds a 3-ball. Every knot exterior is irreducible. What is even more important is that, thanks to a result known as the prime decomposition theorem [Hem04][4], most questions about arbitrary compact, orientable 3-manifolds can be reduced to the special case of irreducible manifolds, so that irreducibility is not a serious restriction. (The prime decomposition theorem is often attributed to Kneser and Milnor, but Milnor's part had been done independently by Haken in his paper [Hak61a].)

The version of Haken's theorem on the existence of hierarchies that we have mentioned includes the hypothesis that the manifold M has non-empty boundary. But if an irreducible, orientable 3-manifold M is closed, i.e., is compact and has no boundary, and if M contains some (necessarily closed) incompressible surface, the theorem immediately implies that M has a hierarchy.

The great thing about hierarchies is that they permit the use of *inductive methods*. One can view the manifolds

[4]*To keep the number of references under control, and to benefit non-expert readers who want to learn more, we will often cite texts and survey articles that contain references to the original papers.*

further down the hierarchy as "simpler" than the original one, and one can often use our knowledge of the simpler manifolds to establish information about the original manifold. Haken's theorem on the existence of hierarchies has turned out to be a tremendously powerful tool in 3-manifold theory, as we shall explain in more detail later. For this reason, a compact, orientable, irreducible 3-manifold M that satisfies the hypothesis of the theorem—that either $\partial M \neq \emptyset$, or M is closed and contains an incompressible surface— is now called a *Haken manifold*, a term introduced by William Jaco. A sufficient condition for a closed irreducible manifold M to be a Haken manifold is that its first betti number $\dim H_1(M;\mathbb{Q})$ be strictly positive.

Haken used hierarchies in [Hak62] and [Hak68] to provide a solution to the Homeomorphism Problem for Haken manifolds, and the very closely related Equivalence Problem for knots, with some exceptions. He was able to show that a Haken manifold can be recovered from information about the surfaces in a hierarchy and the way that they glue together. Thus, one can reformulate the Homeomorphism Problem, by asking whether the two given Haken manifolds admit hierarchies that use the same surfaces glued together in the same way. This is useful, since the translation into a question about surfaces permitted Haken to use the theory of normal surfaces that he developed for the Unknotting Problem.

By 1968, the only case of these problems with which Haken could not deal was the case of "fibered 3-manifolds" or "fibered knots." A 3-manifold is said to be *fibered* if it can be obtained from a product $F \times [0,1]$, where F is a 2-manifold, by gluing $F \times \{0\}$ to $F \times \{1\}$ by some homeomorphism. Fibered manifolds can be thought of as Haken manifolds with particularly simple hierarchies; from a slightly fancier point of view, they are 3-manifolds that admit locally trivial fibrations over a circle. A knot is *fibered* if its exterior is fibered. In the late 1970s, Geoffrey Hemion (see [Lac22]) succeeded in solving the Homeomorphism Problem for fibered manifolds, using quite different methods from Haken's. This is a neat example of how two different approaches to a problem can perfectly complement each other.

Haken's proof was both lengthy and delicate, and in many places, his argument was sketchy. Indeed, it was not until 2003 that Sergei Matveev gave a full account of Haken's proof, filling in many of the details and dealing with some of the cases that Haken had omitted. (See [Lac22].) While this was an important contribution, it confirmed the essential correctness of Haken's work in this area, which constitutes a stunning achievement.

In the decades following Haken and Hemion's solution to the Homeomorphism Problem for Haken manifolds, William Jaco, Hyam Rubinstein and others further illustrated the power of Haken's methods by extending them to other kinds of 3-manifolds. A particularly striking example, developed through the combined efforts of Rubinstein and Abigail Thompson, is an algorithm to determine whether a triangulated 3-manifold is homeomorphic to the 3-sphere. This is a long way from the theory of hierarchies that was developed by Haken, since the 3-sphere is known not to contain any closed embedded orientable incompressible surfaces. Thus, there is no obvious surface to place into normal form. Instead, Rubinstein and Thompson used "almost normal" surfaces, which are embedded surfaces that intersect each tetrahedron of the triangulation in a collection of triangles and quadrilaterals, *except* in exactly one tetrahedron, where exactly one piece is not a triangle or quadrilateral, but is an "octagon" in a suitable sense. They were able to show that the 3-sphere could be detected by searching for normal and almost normal spheres, and using Haken's method of encoding normal surfaces by vectors satisfying linear constraints. For an account of these developments, with further references, see [Lac22].

Normal surfaces and the fundamental group. Remarkably, while Haken was developing his ideas about incompressible and normal surfaces in the early and middle 1960s, other researchers, notably David Epstein and John Stallings, were working with incompressible surfaces from a radically different point of view. Rather than thinking about algorithms, which were the focus of Haken's work, these researchers were concerned with the very classical question of the extent to which the algebraic invariants of a 3-manifold determine its topological type. Their results depended strongly on work by Christos Papakyriakopoulos from the 1950s, which implies that a connected orientable incompressible surface F in a connected orientable 3-manifold M is π_1*-injective* in the sense that the inclusion homomorphism from $\pi_1(F)$ to $\pi_1(M)$ is injective. This is a powerful tool for studying the way that the fundamental group of a 3-manifold controls its topological structure. Perhaps the most celebrated result from this period is Stallings's fibration theorem, which implies that a compact, orientable, irreducible 3-manifold is fibered if and only if its fundamental group admits a homomorphism onto $\mathbb{Z}$ with a finitely generated kernel.

It was apparently Friedhelm Waldhausen who recognized the connection between these two very different strains of research involving incompressible surfaces. He exploited the connection to prove a series of extraordinary theorems, one of which implies—among other things—that two closed Haken manifolds are homeomorphic if their fundamental groups are isomorphic. Needless to say, his proofs involved a great many ingenious ideas, but it seems fair to say that the main *tools* that he used are the ingredients in the proof of the Stallings fibration theorem on the one hand, and Haken's result on the existence of hierarchies on the other.

A good reference for these developments is [Hem04].

The interaction between the existence of hierarchies and the π_1-injectivity of incompressible surfaces led to an explosion of work on the topological structure of Haken manifolds. The so-called characteristic submanifold theory was developed by Klaus Johannson [Joh79] and, independently, by William Jaco and Peter Shalen [JS79]; the two approaches were quite different, but the use of hierarchies, and of the π_1-injectivity of incompressible surfaces, was ubiquitous in each. Rather than attempting to explain the characteristic submanifold theory here, we shall emphasize one by-product of the theory: if M is a Haken manifold, which for simplicity we will take to be closed, then there is a canonical (possibly empty) system of incompressible tori in M, and the components of the submanifold obtained by splitting M along these tori have nice properties.

Just how nice these "pieces" of M are was later revealed by William Thurston's geometrization theorem [Mor84]: the interior of each piece admits a geometric structure, locally modeled on a 3-dimensional homogeneous space. There are several different homogeneous spaces that arise in this context; one is Euclidean 3-space, and another is hyperbolic 3-space, the non-Euclidean space discovered by Gauss, Bolyai and Lobachevski. Of the various classes of locally homogeneous manifolds, those modeled on hyperbolic spaces are by far the richest. Indeed, the deep part of Thurston's result was a characterization of those Haken manifolds that admit a hyperbolic structure. And his proof of this revolutionary result, whose influence on 3-manifold theory is still being felt, was an induction on the length of a hierarchy, in which the induction step makes crucial use of π_1-injectivity.

Whereas the use of hierarchies in the work that we have been describing followed the same basic pattern as their use in Waldhausen's work, other developments in the late 20th century involved unexpected twists in the application of Haken's ideas. David Gabai discovered a very surprising refinement of the notion of a hierarchy, called a sutured manifold hierarchy, which turned out to be the key to solving a number of previously intractable problems in 3-manifold theory (see [Sch90]). William Floyd and Ulrich Oertel showed that much of normal surface theory can be reinterpreted in terms of geometric objects in a 3-manifold called branched surfaces; their work elucidated the geometric meaning of normal surfaces. Oertel and Allen Hatcher, working in the context of branched surfaces, initiated the study of 2-dimensional measured laminations in 3-manifolds; these are generalizations of normal surfaces given by solutions of the normal surface equations in which the coordinates are real numbers rather than integers. John Morgan and Peter Shalen used incompressible measured laminations in their work on actions of 3-manifold groups on real trees, which gave a new perspective on one of the main steps in Thurston's geometrization theorem. (The expository article [MS85] gives references for both the Floyd–Oertel paper and the Morgan–Shalen papers.)

Before Waldhausen's work, it is unclear to what extent people whose work used incompressible surfaces from the perspective of algebraic topology were aware of the connection with Haken's work. Haken himself seems to have been quite unaware of the connection. One of the authors of this section, Peter Shalen, was astonished when Haken told him that, before seeing Waldhausen's paper, it had never occurred to him that incompressible surfaces might have anything to do with fundamental groups. Having first encountered Haken's work through Waldhausen's papers, Shalen's own perspective was that incompressible surfaces were important precisely *because* of their connection with fundamental groups. After thinking it over, he realized that this difference in perspective is itself a reflection of the power and versatility of normal surface theory. Its role in the algorithmic side of 3-manifold topology is very different from its role in the side of the subject involving algebraic invariants and geometric structures, and yet it is central to both.

Recent developments. Both aspects of the theory of normal surfaces and hierarchies have continued to flourish, although not in the way that might have been predicted. Three-manifold theory has taken quite unexpected directions because of a huge development in 2002–2003.

We have mentioned that Thurston's proof of his geometrization theorem for Haken manifolds was based on an induction on the length of a hierarchy. Thurston conjectured an extension of his theorem to arbitrary compact 3-manifolds, which includes the Poincaré Conjecture as a special case. (The Poincaré Conjecture, which asserts that every compact, simply connected 3-manifold without boundary is homeomorphic to S^3, was one of the three problems that Haken was introduced to in Weise's lecture. It remained one of the most significant unsolved problems in topology throughout the twentieth century, and in particular, it resisted Haken's considerable efforts to prove it.) Grigori Perelman's proof of this conjecture of Thurston's was perhaps the most important breakthrough in the history of the subject. Perelman used radically different techniques from Thurston's, based on work by Richard Hamilton (see [Mor09]), and his proof did not involve hierarchies or normal surfaces. However, far from rendering the ideas of hierarchies or normal surfaces obsolete, Perelman's work opened vast new opportunities for exploiting these ideas of Haken's.

A spectacular example of how the use of hierarchies interacts with geometrization was provided by the so-called virtual Haken conjecture and virtual fibering conjecture. These were established by Ian Agol as the culmination of a program developed by Daniel Wise, and building on work by Jeremy Kahn and Vladimir Markovic. (Agol's paper [Ago13] provides references to the relevant papers by Wise and Kahn-Markovic.) The virtual Haken conjecture, first hinted at by Waldhausen, asserts that every closed irreducible 3-manifold with infinite fundamental group is finitely covered by a Haken manifold. The virtual fibering conjecture, first speculated about by Thurston (and so strong that it was once widely considered implausible), asserts that every hyperbolic 3-manifold which is closed (or even has finite volume) is finitely covered by a fibered manifold. Wise's program is very largely group-theoretical, and his and Agol's results have major group-theoretical implications beyond 3-manifold theory. The point to be stressed here is that a group-theoretical counterpart of the notion of a hierarchy is central to the program. Thus we see the ideas that Haken developed to study knots and 3-manifolds bearing fruit in a wider context, as well as being central to the most remarkable recent developments in 3-manifold theory.

Another remarkable interaction between Haken's work and geometrization involves the Homeomorphism Problem. Gregory Kuperberg, using Perelman's theorem as a starting point, has written down a completely general solution to the Homeomorphism Problem for 3-manifolds, which he says is essentially due to Thurston and Robert Riley (both of whom assumed the geometrization conjecture). This solution does not involve normal surface theory. However, by combining geometrization with normal surface theory, Kuperberg has established a much stronger result: that the Homeomorphism Problem is solvable by an algorithm with execution time bounded by a tower of exponentials. (See [Lac22] for an account of this work.) Thus we see Haken's ideas being applied in ever stronger ways to the problems for which he first designed them.

Marc Lackenby

Peter Shalen

The Four Color Theorem

Robin Wilson

The Four Color Theorem states:

> *The regions of every plane map can be colored with just four colors so that neighboring regions receive different colors.*

First posed as a problem by Francis Guthrie in 1852, it was not until 1976 that the theorem was proved, by Wolfgang Haken and Kenneth Appel of UIUC. Their proof was one of the earliest to make substantial use of a computer.

In 1879, it was shown that it is sufficient to consider cubic maps, where exactly three regions meet at each intersection. In the same year, Alfred Kempe claimed a proof that was widely accepted until a fatal flaw was exposed in 1890. Kempe's paper, although incorrect, contained several ideas that would resurface in the eventual proof.

Over the next 50 years, it gradually became clear that the problem splits into two parts. In the following, a *configuration of ring-size k* is a collection of regions surrounded by an external ring of k regions: see Figure 6.

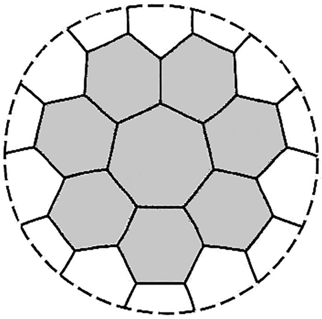


Figure 6. A configuration with ring-size 14.

Unavoidable sets of configurations: Kempe proved that every cubic map must contain at least one region bounded by at most five edges: a digon, triangle, quadrilateral, or pentagon. Any set of configurations (such as these four) is *unavoidable* if every cubic map must contain at least one of them.

Reducible configurations: A configuration is *reducible* if any coloring of the surrounding ring can be extended to the interior regions, either directly or after interchanging pairs of colors (known as *Kempe-interchanges*). Note that no reducible configuration can appear in a minimal counterexample to the Four Color Theorem.

Robin Wilson is a professor emeritus in the Department of Mathematics at The Open University. His email address is robin.wilson@open.ac.uk.

Kempe proved that digons, triangles, and quadrilaterals are reducible, but failed to do so for pentagons. In 1904, Paul Wernicke replaced the pentagon by two adjacent pentagons and a pentagon adjacent to a hexagon, giving a new unavoidable set, and this result was further extended by Philip Franklin in 1922, and by Henri Lebesgue (of Lebesgue integral fame) in 1940.

In 1913, George Birkhoff gave a systematic treatment of reducible configurations, showing that every set of configurations with ring-size up to 5 (other than the pentagon) is reducible, and proving the reducibility of a particular configuration of four pentagons with ring-size 6, known as the *Birkhoff diamond:* see Figure 7.

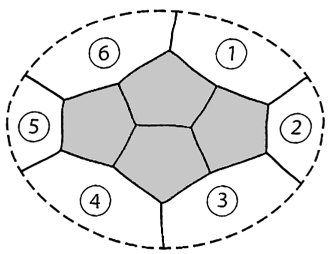


Figure 7. The Birkhoff diamond.

Several mathematicians then built on these ideas, obtaining many reducible configurations. In particular, Franklin introduced new ones in a counting argument which proved that every cubic map with up to 25 regions can be 4-colored. By 1940, this number had increased to 35.

Further details on, and references for, these early developments and those described below can be found in the author's book *Four Colors Suffice* [Wil14]; the quotations by Haken and others are taken from an article by Donald MacKenzie [Mac99] much of which was based on unpublished interviews that were conducted in 1994.

Enter Heesch and Haken. Much of this work was piecemeal, with attempts to find unavoidable sets and reducible configurations being largely independent of each other. It was not until the 1940s that Heinrich Heesch entered the fray. Heesch had contributed to the solution of the 18th of Hilbert's celebrated problems, and around 1935 became interested in map coloring. He realized that to prove the Four Color Theorem, it was enough to find an *unavoidable set of reducible configurations:* every map must include at least one of them, but none can appear in a minimal counterexample.

In the late 1940s, Heesch lectured on his findings at the University of Kiel, and in 1948 one student who attended was Wolfgang Haken, who was studying mathematics, philosophy, and physics, and who had been aware of the Four Color Problem since hearing about it in Weise's topology course. Haken later recalled Heesch's lecture, much of which he did not understand at the time, and remembered Heesch's claim that there might be some 10,000 cases to be investigated.

Over the next 20 years, Haken solved the Unknotting Problem, and moved to the University of Illinois Urbana-Champaign, where he did much of his fundamental work on the Equivalence Problem for knots and the Homeomorphism Problem for 3-manifolds. During this period he also spent much time on the Poincaré Conjecture. In 1966, unable to complete a proof of this conjecture, he began thinking about the Four Color Problem. He contacted Heesch, who was still working on the problem and had discovered thousands of reducible configurations.

Unavoidable sets were still rather scarce, and in order to produce them, Heesch invented his *method of discharging* (as it was later named by Haken). Here, to investigate whether certain configurations form an unavoidable set, he assumed the contrary. He then assigned a "charge" of $6 - k$ to each k-sided region; it then follows from Euler's polyhedron formula that the total charge over the whole map is positive. He next attempted to move these charges around the map in such a way that no charge was created or destroyed; this is called *discharging the map.* In many cases this could be done so that every region received a non-positive charge. This contradiction confirmed that the given configurations did indeed form an unavoidable set.

Heesch also contributed to the theory of reducible configurations, and defined a configuration to be *D-reducible* if every coloring of the surrounding ring can be extended to the interior regions (either directly or after Kempe-interchanges of colors), and to be *C-reducible* if this can be carried out after the configuration had been simplified in some appropriate way. His aim was to develop systematic methods for generating reducible configurations, looking at both D-reducible and C-reducible cases. If a configuration was not D-reducible, he frequently saw how to modify it so as to determine its C-reducibility, and in this way he could restrict his attention to a smaller number of possible colorings. Over the years he developed an uncanny knack of recognizing reducible configurations with at least 80 percent accuracy: as Haken remarked:

> What fascinated me most was that Heesch looked at the configuration, and he said either "No, there is no chance: that cannot be reducible" or "But this one: that is certainly reducible."

Haken invited Heesch to lecture at UIUC, and asked whether computers might help in the examination of large numbers of configurations. Heesch had already obtained the help of a mathematics graduate who developed a method for testing D-reducibility that was sufficiently routine to be implemented on a computer, even though this might take a long time.

The complexity of a configuration can be measured by its ring-size; for example, the Birkhoff diamond has ring-size 6 and there are 31 essentially different colorings of the surrounding regions to consider, but for configurations with ring-size 14 there are nearly 200,000 colorings. Larger ring-sizes were way beyond the capacity of computers of the time.

Haken bid to the University of Illinois for time on a new supercomputer whose construction was nearing completion, but it was not yet ready for use. Eventually, they were referred to the Atomic Energy Commission's Brookhaven laboratory in Long Island, where there was a Cray Control Data 6600 machine, the most powerful machine of its day, which could test configurations of up to ring-size 14.

In 1970, Heesch sent Haken the results of a new discharging experiment which, if applied to a general map, would yield about 8900 "bad" configurations extending up to ring-size 18, in which some regions would still have positive charge. Haken, however, was pessimistic about dealing with so many configurations, especially since several were fairly large. For some time, he had felt that the complexity of the problem would be substantially simplified by better discharging methods. By restricting his attention to maps without hexagons or heptagons, he obtained a much simpler procedure and communicated his findings to Heesch; it is at this stage that Haken began to contribute to the eventual solution of the problem.

Impressed by Haken's findings, Heesch invited Haken to collaborate with him, and in 1971 sent him three "obstacles" whose presence seemed to prevent configurations from being reducible—a 4-*legger region* adjoining four consecutive regions of the surrounding ring, a 3-*legger articulation region* adjoining three surrounding regions that are not all adjacent, and a *hanging* 5-5 *pair* of adjacent pentagons that adjoin a single region inside the surrounding ring: see Figure 8.

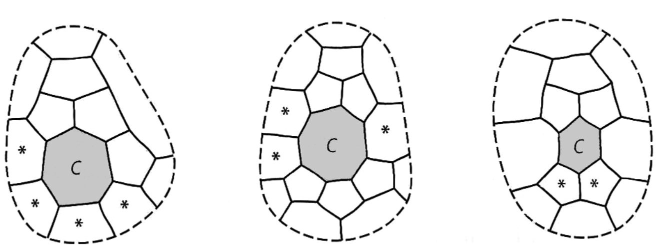


Figure 8. Heesch's obstacles to reducibility.

But by now, Haken was changing his approach. Unlike everyone else whose objective seemed to be to collect reducible configurations by the hundreds before packaging them into an unavoidable set, Haken's primary motivation was to aim directly for an unavoidable set. In order to avoid wasting expensive computer time checking configurations that would eventually be of no interest, this set was to contain only configurations that were *likely to be reducible*—in particular, they should contain none of the obstacles. Any configuration that subsequently proved not to be reducible could then be dealt with individually. As he later commented:

> If you want to improve something, you should not improve that part which is already in good shape. The weakest point is always the one you should improve. This is a very simple answer to why we got it and not the others.

By this time, most workers on the Four Color Problem were using the "dual formulation" of coloring the vertices of the corresponding graph. In particular, Heesch devised a useful notation for representing regions by appropriate symbols so that they can be easily distinguished, such as • for pentagons, · for hexagons, ∘ for heptagons, and ∇ for nonagons.

Enter Appel. With little knowledge of computing, Wolfgang Haken considered giving up the problem until more powerful machines had become available to deal with the massive calculations that would clearly be necessary. Informed that his ideas could not be programmed, he announced:

> The computer experts have told me that it is not possible to go on like that. But right now I'm quitting. I consider this to be the point to which and not beyond one can go without a computer.

Attending this lecture was his colleague Kenneth Appel, a computer programmer with much practical experience. Afterward, Appel told Haken that he considered the experts' view to be nonsense, and offered to work on implementing the discharging procedures. Haken was delighted to accept Appel's offer, and they decided to concentrate their search on unavoidable sets, without taking time to check the configurations for reducibility; in particular, they focused on "geographically good configurations" that contain neither of Heesch's first two obstacles, and would check for reducibility when an entire unavoidable set had been constructed.

Their first exploratory computer runs provided much useful information, but the computer output was enormous, with some configurations appearing many times; they would clearly need to keep such duplications under control if the eventual list were to be manageable. Fortunately, the computer program had run in just a few hours, and so they could experiment as often as necessary. Any changes to the program were easily implemented and the

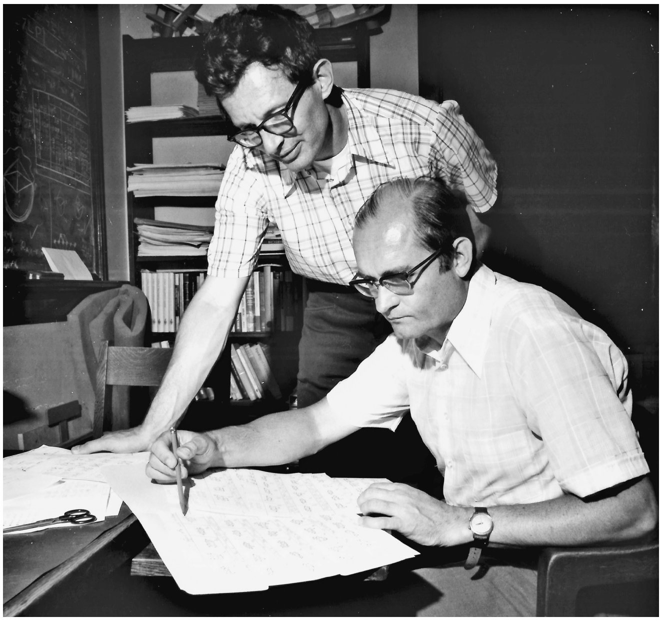


Figure 9. Wolfgang Haken and Kenneth Appel.

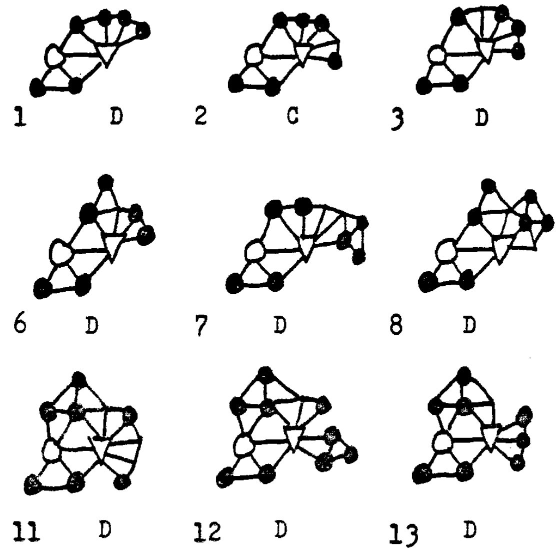


Figure 10. Some of Haken and Appel's configurations.

paper stacks of outputs of later runs were much reduced in thickness; eventually they would come down to a fraction of an inch.

From then on, they continually modified the discharging algorithm or the computer program so that the program grew while the output shrank. This two-way dialog with the computer continued, as problems were sorted out and new ones arose. Within six months of experimenting and improving their procedures, they realized that their method for producing a finite unavoidable set of geographically good configurations in reasonable time was feasible.

In early 1975, they introduced the third of Heesch's obstacles; this inevitably involved changes in procedure, but was carried out successfully with only a doubling in the size of the unavoidable set.

As soon as it seemed that they could probably find an obstacle-free unavoidable set of configurations which were likely to be reducible, they started the massive detailed check for reducibility. Inevitably, a few "rogue" reducible configurations would appear in the list, but their hope was that these would be relatively few in number.

In mid-1974, realizing that they needed help with the reducibility programs, they had enlisted the help of John Koch, a graduate student, to work on the *C*-reducibility of configurations. Appel and Haken were particularly interested in two types of modification that were relatively easy to implement, and Koch discovered that most of his configurations were fortunately of these types.

By early 1976, Haken and Appel could work on the final details of the discharging procedure. To do this, they sought "problem configurations" and immediately tested them for reducibility—this could usually be done fairly quickly. In the event, the final process involved 487 discharging rules, requiring the investigation by hand of about 10,000 neighborhoods of regions with positive charge and the reducibility testing by computer of some 2000 configurations. All configurations were of ring-size 14 or less.

The last few months were extremely heavy on computer time, but Appel, Haken, and Koch were fortunate. Few institutions would have given them 1200 hours on the computer, but the university's computer center was very supportive, and in March 1976 a powerful new machine was bought by the university's administrators. This proved to be so powerful that everything proceeded far more quickly than they had expected, saving them much time on the reducibility testing. Meanwhile, with the help of Haken's daughter Dorothea, they spent months of exhausting and stressful effort working through the 2000 or so configurations that would eventually form the unavoidable set.

Suddenly by late June, almost before they realized what was happening, the entire job was finished: the Hakens had completed the construction of the unavoidable set. Within two days Appel tested the final configurations for reducibility, and celebrated their achievement by announcing on the department's blackboard:

> Modulo careful checking it appears that four colors suffice.

By this time, Haken and Appel knew that they were safe: even if a few configurations proved to be irreducible, there was more than enough self-correction in the system for them to be quickly replaced: no single faulty configuration could destroy the entire edifice. In fact, their system included so much self-correction that they effectively had many thousands of proofs of the Four Color Theorem, instead of just one!

Armed with this confidence, they went public. On July 22, 1976, they formally informed their colleagues and sent complete preprints to everyone in the field. One recipient was Bill Tutte, who waxed eloquent, comparing their achievement with the slaying of a fabled Norwegian sea monster:

> Wolfgang Haken
> Smote the Kraken
> One! Two! Three! Four!
> Quoth he: "The monster is no more."

They quickly wrote short reports for the *Bulletin of the American Mathematical Society* [AH76a] and *Discrete Mathematics* [AH76b], but decided to submit their full solution to the *Illinois Journal of Mathematics,* partly because they wanted it to appear locally, but mainly so that they could suggest suitable referees. By December they were able to refine the proof and prepare it for publication. The result substantially improved on the rough-and-ready preprint they had sent out in July: in particular, their preprint had contained duplications and configurations contained within others, and by eliminating these they reduced their original list of 1936 reducible configurations to 1482.

Their solution appeared in two parts in the December 1977 issue of the *Illinois Journal.* Part I [AH77] on *Discharging* outlined the overall strategy of their proof, while Part II [AHK77] on *Reducibility,* written with John Koch, listed the entire set and described the computer implementation. These were supplemented by microfiche containing 450 pages of further diagrams and detailed explanations.

Wolfgang Haken and Kenneth Appel had achieved their goal: the Four Color Theorem was proved.

Aftermath. The Appel–Haken proof was greeted with enthusiasm—a longstanding problem had at last been solved—but also with skepticism, great disappointment, or outright rejection. Their extensive use of computers was widely criticized, and raised philosophical questions as to whether a proof is valid if it cannot be checked by hand. At a Joint AMS-MAA Summer meeting in Toronto a lecture by Haken to a capacity audience received only polite applause, and when explaining their proof to mathematics departments, its authors were often made to feel unwelcome; in one case, they were even barred from meeting graduate students for having introduced totally inappropriate methods into mathematics. Since then, with the passage of time, the use of computers in mathematical proof has become more widespread.

Inevitably, there were typos in their paper, and also a minor error that needed two weeks for Haken to correct, but the proof emerged largely unscathed. Haken and Appel wrote several further papers on the subject, including one in 1986 in *The Mathematical Intelligencer* discussing such purported errors. In 1989, they followed this with a hefty tome [AH89], published by the AMS, which was entitled *Every Planar Map is Four Colorable.* This gave further details and included a printed version of their earlier microfiche.

In 1994, Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas took a more systematic approach to the problem. Using essentially Appel and Haken's method, they produced an unavoidable set of 633 configurations, reducing the number of discharging rules from 487 to just 32. Interestingly, they chose to use computers in both the unavoidability and reducibility parts of their proof, believing that such an approach was more reliable than hand calculation. Their proof could be externally verified on a home computer in just a few hours. Shortly after this, Robin Thomas wrote an article in the *AMS Notices* linking the Four Color Theorem to the divisibility of integers, the algebra of 3-dimensional vectors, and results on matrices and tensors.

Two further events are worthy of mention. Because *D*-reducible configurations are simpler to deal with than *C*-reducible ones, John P. Steinberger gave a proof in 2008 that was based on only the former; it involved 2832 configurations with ring-size up to 16 and used 42 discharging rules. Meanwhile, in 2004, the French computer scientist Georges Gonthier had provided a fully machine-checked proof of the theorem, which was a formal language implementation and machine verification of the approach of Robertson and his coworkers.

But for graph theorists this was by no means the end of the line, as the Four Color Theorem is just one special instance of some much harder problems; these include finding proofs for Hadwiger's conjecture and the five-flow conjecture, and on these problems good progress has already been made. With these in mind, we leave our final poetic musings to Bill Tutte:

> The Four Color Theorem is the tip of the iceberg, the thin end of the wedge, and the first cuckoo of Spring.

Robin Wilson

Haken as Thesis Advisor

Patrick Callahan

I was one of Haken's last PhD students, completing my degree in 1994, a few years before his retirement.

When I started graduate school in mathematics in 1989, the University of Illinois Urbana-Champaign was (and still is) one of the largest graduate math programs in the United States. There were almost 300 graduate students in the mathematics department and almost one hundred of us were first-year students. With over 70 faculty members, finding a good advisor was an important step for all graduate students and UIUC offered many choices. I decided to meet one on one with every mathematics faculty who was involved in topology, algebra, or number theory. Two rules of thumb that were shared among students seeking advisors were: look for recent, successful graduates completing and finding positions and learn who their advisor was, and look for faculty that were currently active and well-networked in their field. Despite this, I ended up choosing Wolfgang Haken who had not had a graduate student in almost 20 years and had not published or participated in conferences in quite some time. Many people thought working with Haken would be a risky choice.

When I met with Haken he told me that he was very excited about the new developments in 3-manifolds and knot theory but that he was not currently active in the field. However, he made me an interesting offer. He had recently been nominated as a member of the University of Illinois Center for Advanced Study, which included an annual travel stipend. He was not up for much travel, so he said he would be my advisor if he could send me to interesting topology meetings about which I would report back to him. This was an incredible arrangement that I gladly accepted. I was fortunate as a first-year graduate student to be able to travel and meet all the amazing researchers in the field. I was sent to meetings in Israel, France, and across the United States.

It was an exciting time for low-dimensional topology. Three-manifolds and knot theory had spawned a large active research community, which was often surprised when I introduced myself as Haken's student since there had not been any such students in a good many years. But Haken's work was foundational, and Haken manifolds were a critical player in the Geometrization Conjecture. I would return from these conferences and report to Haken what current progress was happening in the field. He would always pause for a long time and then say something like "Ja ja... that is most interesting," then go silent again for another long pause and then tell me to continue recounting what I learned at the conference.

Haken was an atypical advisor. Although he had an office, we would rarely meet there. Rather, there were two somewhat unusual places he liked to work, and when I wanted to meet with him, I would go searching for him in those places. The first unusual location was in an empty classroom in the basement of Altgeld Hall, usually with the lights off. Many times he would not have any paper or anything with him. He would just be sitting in a dark basement in deep thought. The other unique location I would look for him was almost the opposite. He would often sit in the loud and crowded student union and be deep in thought surrounded by hordes of noisy students.

Although often taciturn, Haken would sometimes share anecdotes about his life and work. I recall him telling me that during World War II there was a paper shortage so he would take down propaganda posters and write on the back of them to do his research. He would talk about the controversies around the solution to the Four Color Theorem. Some mathematicians thought that the use of a computer could not be trusted or considered a valid proof. Even though 20 years had elapsed, Haken continued many correspondences regarding the Four Color Theorem. He would regularly get letters sent by professionals and amateurs alike claiming to have found some new clever proof. Haken would take the time to read all of them and send back careful and encouraging feedback. I remember him reading a handwritten letter claiming a new proof, around fifty pages long, for which he found an "unrepairable" error on the 48th page. I asked him why he still read all these letters and he replied, "If this person had made it a little further without an error they might have really discovered something. You never know." Haken said that the field had a responsibility to read plausible claims carefully and not dismiss them just because the authors lacked the professional academic credentials.

Conversations with Haken had a slow pace. He would talk slowly and carefully and there were often long pauses. I once thought that this was because English was not his native language, but a German graduate student informed me that Haken was the same way when conversing in German. Haken was one of the most careful thinkers I have ever met. He paid prodigious attention to details. One of my fond memories regarding details was when I submitted my dissertation. I, perhaps pretentiously, thought it seemed like a good idea to start Chapter 1 with an epigraph in the original Greek from Aeschylus' *Agamemnon*. I do not know classical Greek and had made an error when

Patrick Callahan is the CEO of Math ANEX and Callahan Consulting. His email address is `callahan.web@gmail.com`.

copying it. When Haken returned my draft, he had corrected the quote, in classical Greek.

I recently pulled out a copy of my dissertation. I found the acknowledgement:

> I would like to thank my advisor, Professor Wolfgang Haken, for being the exact type of advisor I needed, and for his support and confidence in me.

I cannot speak to what it was like having Haken as an advisor in the 1970s in the heyday of the Four Color Theorem, but in the 1990s, Haken was indeed the exact type of advisor I needed. He was endlessly curious, and provided the means for me to see the world and be part of a vibrant community of researchers in low-dimensional topology. In some small sense I was his eyes and ears at this stage in his career. Haken was also very hands off in that he never gave me a specific problem to do nor did he tell me what I should work on. He didn't even discourage me when I spent a year working on the Lens Space Conjecture, a special case of the Geometrization Conjecture which was then still open and was notorious for its difficulty. He always had an appetite for big unsolved problems. Haken was very patient and carefully critiqued my many unsuccessful attempts. He was in no hurry. He told me that being a mathematician is mostly about having ideas that fail over and over again, only rarely having one actually work. He encouraged me not to shy away from big problems and to keep looking at and thinking about things differently: to keep having ideas. That was just the advisor I needed.

Patrick Callahan

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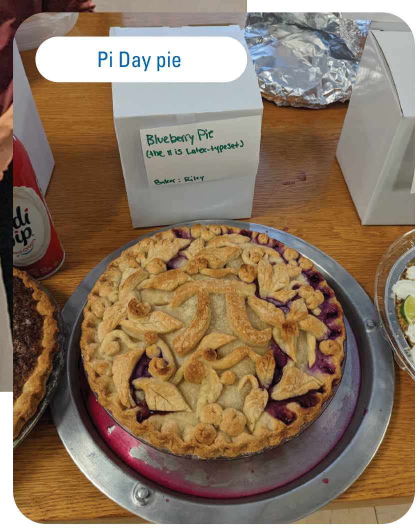
University of Minnesota chapter members



Pi Day pie

University of Minnesota

Chapter members run a variety of events throughout the academic year, including weekly teas, periodic social events, and a seminar series. Past events have included Pi Day pie and trivia competitions and the Unary Day, featuring Guest Speaker Max Engelstein, Assistant Professor.

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Utah State University

The Utah State graduate students invite guest speakers throughout the academic year. One lecture, by Dr. Charles Torre, Professor and Associate Department Head of the Physics Department at Utah State, focused on general relativity and solutions to Einstein's equations.

Utah State University Chapter

Presentation at the Duke University Chapter

Duke University

This Chapter builds community among students by promoting conversation, collaboration, and friendship across research and institutional milieus. They will host the 9th annual Triangle Area Graduate Mathematics Conference (TAGMaC) in 2023, at which graduate students from Duke, University of North Carolina at Chapel Hill, and North Carolina State University give talks about research-level mathematics.

The Chapter Program makes meaningful contributions to the professional development of graduate students in the mathematical sciences and connects graduate students with AMS offerings.

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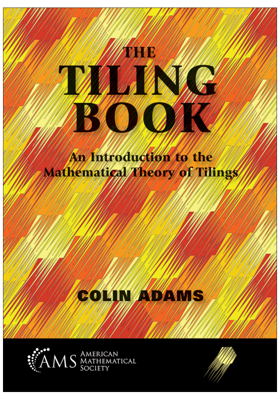
BOOK REVIEW

The Tiling Book

Reviewed by Keiko Kawamuro



The Tiling Book
An Introduction to the Mathematical Theory of Tilings
By Colin Adams. American Mathematical Society, 2022, 298 pp. `https://bookstore.ams.org/mbk-142`

Colin Adams's book, *The Tiling Book*, is the most colorful math book I have ever seen. First of all, the cover attracts our eyes as it is covered with bright red, orange, and yellow lightning bolt shapes. Sooner or later I noticed it is a tiling of the plane!

Once I opened the book, I saw 310 figures and photos listed. Many of them were illustrated by Adams using Adobe Illustrator. Some of them are photos taken by Adams, including his private collection of vintage Kimberly Clark toilet paper, as seen in Figure 1. On the toilet paper a Penrose rhombic tiling is creased. The episode of the toilet paper (p.191) is interesting. Unfortunately, production of the toilet paper was stopped after Sir Roger Penrose sued the company.

Perhaps for a typical mathematician a tiling means the so-called regular tilings (as seen in Figure 2) and the usual subway tile. The book contains all kinds of exotic tilings beyond my imagination. I was fascinated by Casey Mann and his students' counterintuitive tiling.

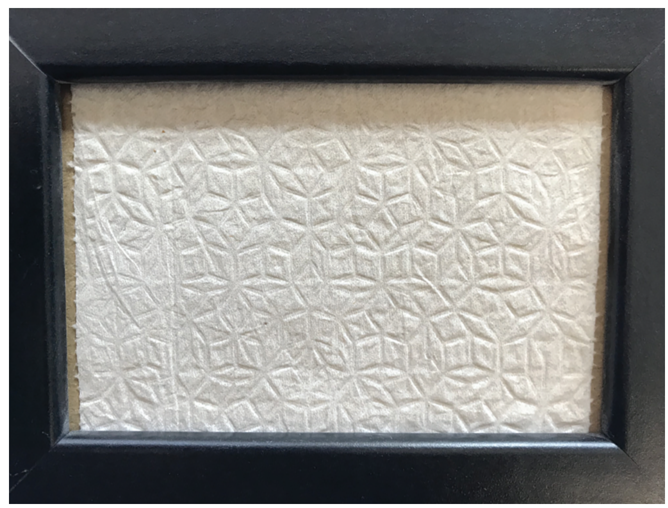


Figure 1. Toilet paper with Penrose tiling.

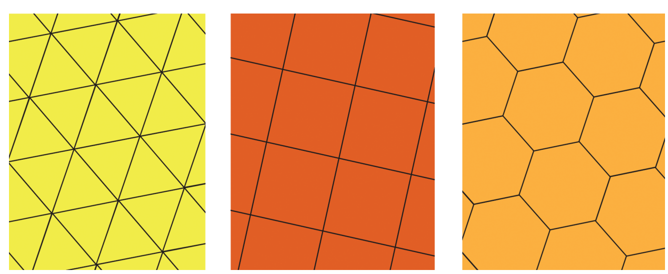


Figure 2. Tilings with regular polygons.

This book is fun to read. Readers do not need to be tile otaku or kitchen-bathroom designers. The book is accessible for general people who love math. As a mathematician studying low-dimensional topology including knots and braids, I had never studied tilings before.

While its target audience may be undergraduate and graduate students, I expect a motivated and interested high school student could read this book. Chapter 0 (12 pages with 8 figures) provides the background that is needed for the entire book and is a friendly introduction to point-set topology and group theory. I read this book seminar-style

Keiko Kawamuro is a professor of mathematics at the University of Iowa. Her email address is `keiko-kawamuro@uiowa.edu`.

Communicated by Notices *Book Review Editor Emily Olson.*

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with five undergraduate students at the University of Iowa and one high school student from Iowa City. Some of them in fact had no previous experience in the material in Chapter 0, but we all enjoyed the book.

The book contains numerous projects suitable for undergraduates to do either independently or in a group. By reading proofs of theorems in the book, students will be excited to see how concepts recently learned in analysis or abstract algebra courses are actually applied.

For instance, in Section 2.6, the Bolzano–Weierstrauss theorem taught in an introductory analysis course, is used to prove the extension theorem. Each section also contains open problems for researchers and graduate students. I can imagine (partially) solving some of those would make an excellent PhD thesis. I want to emphasize that only small advanced training is required to understand the problems, as the background and motivation of the problems are included together with appropriate references.

The Appendix of the book can be useful for children and their teachers. In the Appendix, Adams teaches us how to create our own tilings. Now we have an idea how the bright tiling of the book cover was created. Inspired by his method, I recently taught 3rd and 4th graders a 40-minute origami tile-making course.

Adams uses colors all over the book not only for illustrations but also in narratives. It is a clever idea that he highlights definitions in green, lemmas and theorems in orange, open problems/questions in blue, and projects in red. Glancing at the orange highlights, readers can see that there are only one or two theorems per section, which enables them to focus on what is most important in the section. In addition, almost every definition and theorem comes with concrete examples and figures that I found very helpful.

1. Chapter 1

Chapter 1 starts with a detailed introduction of isometries of a plane and moves into symmetries. If a tiling admits translational symmetries all of which are parallel then its symmetry group is called a *Frieze group*. A *periodic tiling* is a tiling whose symmetry group possesses at least two non-parallel translational isometries. These symmetry groups are called the *wallpaper groups*. Adams presents the complete list of 7 Frieze groups and 17 wallpaper groups along with illustrations of tilings whose symmetry groups are those groups. See Figure 3 for examples.

A convenient feature of Adams's illustrations is that symmetry data are embedded in the tiling pictures. For example, a line of reflection is a solid red line and the center of a 3-fold rotation is a hollow triangle. In addition to the International Union of Crystallography notation, John Conway's orbifold notation is provided for each of the 24 symmetry group types. The orbifold notation records the translation/rotation/reflection/glide reflection

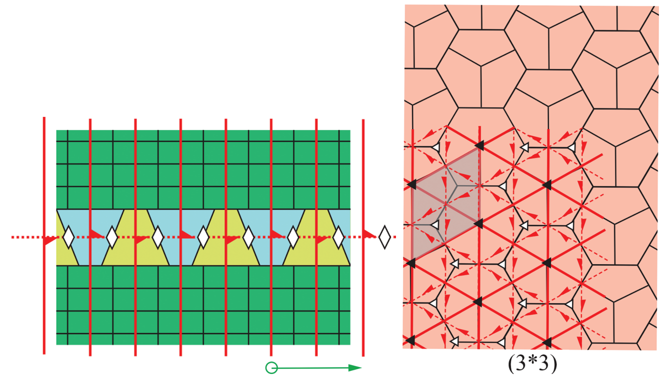


Figure 3. Illustrations of one of the Frieze groups (left) and one of the wallpaper groups (right).

symmetry by a finite word in $\circ, *, \mathrm{x}$, and positive integers. For example, $(3*3)$ in Figure 3 is the orbifold notation for the symmetry group of the tiling. Conway came up a way to assign a certain rational number to each letter in the word and proved an amazing theorem: The orbifold notation symbols add up to exactly 2 if and only if the orbifold notation corresponds to an actual symmetry group of a tiling. In fact for each of the 24 tilings, you can enjoy staring at the image, identifying its orbifold notation and verifying that the sum is exactly 2.

2. Chapter 2

In Chapter 2, Adams shows various kinds of tilings by polygons. An edge-to-edge tiling by regular polygons is called a *uniform tiling* if all vertices have the same type. Using elementary computations, Adams shows that there are 11 such uniform tilings, which included the three regular tilings; see Figure 4. The numerical notation under each illustration describes the vertex type. For example, (4.6.12) means a 4-gon, a 6-gon and a 12-gon are meeting at a vertex with this cyclic order. The fact was originally published by Johannes Kepler in 1619.

The second kind is called Laves tilings, as seen in Figure 5; a monohedral tiling by a polygon (no need to be regular type) is called a Laves tiling if at every vertex angles between adjacent edges are equal ($2\pi/n$ for some n). Adams shows us there are 11 Laves tilings, which also includes the three regular tilings. Adams explains why it is not a coincidence that the number 11 shows up in both uniform tilings and Laves tilings.

After describing random tilings, a general study of periodic tilings is given. The periodicity theorem (2.14) states that if a set of polygonal prototiles (called a protoset) admits an edge-to-edge tiling with a translational symmetry then the same protoset admits a periodic tiling. This statement connects the Frieze groups and the wallpaper groups as follows. A polygonal protoset that generates a tiling with Frieze group symmetry must also generate a tiling with wallpaper group symmetry.

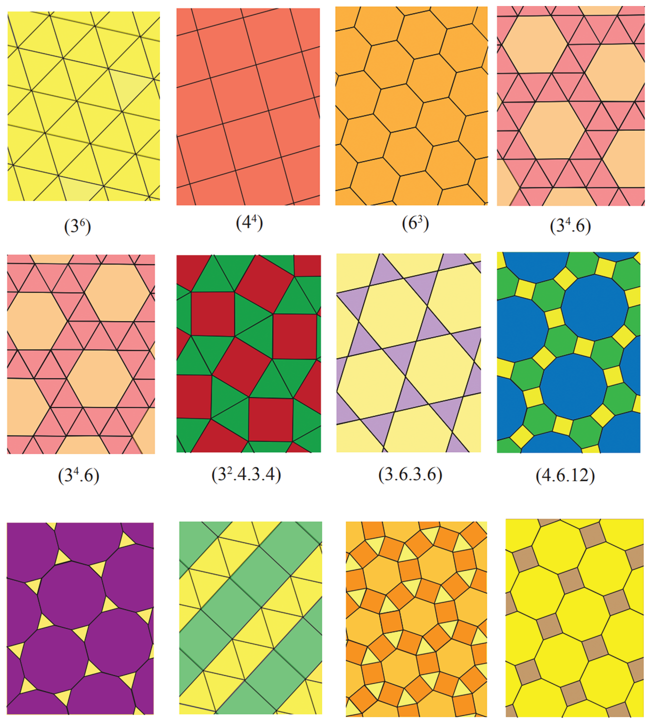


Figure 4. The 11 uniform tilings, where the two instances of $(3^4.6)$ are equivalent via reflection.

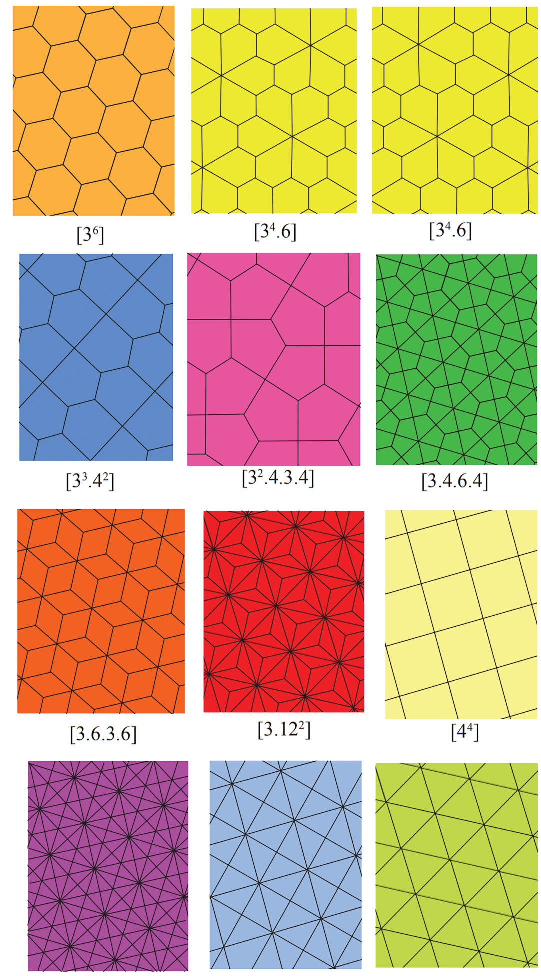


Figure 5. The 11 Laves tilings, where the two instances of $[3^4.6]$ are equivalent via reflection.

The chapter ends with discussion of the problem: Which convex polygons tile? The answer is given: Any convex polygon with seven or more sides cannot tile the plane. In the hexagon case, Karl Reinhardt showed that there are exactly three families of hexagons that can tile the plane, as we see in Figure 6. As for pentagons, there are exactly 15 families that can tile the plane. This result involves at least ten people including Karl Reinhardt, Richard Kerschner, Martin Garner, Richard James, Marjorie Rice, Rolf Stein, Casey Mann, Jennifer McLoud-Mann, David Von Derau, and Michäel Rao. As a corollary, every monohedral tiling by a convex polygon is a periodic tiling.

3. Chapter 3

In Chapter 3, aperiodic protosets are studied. By definition, every tiling generated by an aperiodic protoset is nontranslational.

The substitution method is a method to create nontranslational tilings. Adams provides the chair reptile example (see Figure 7) and the pinwheel tiling example. It's interesting that these examples somehow give me the wrong impression that they have translational symmetry. The Robinson aperiodic protoset, the Penrose aperiodic protosets, and the Taylor-Socolar hexagonal tile generate non-translational tilings.

I was attracted to the two aperiodic protosets by Penrose discussed in Section 3.4. As seen in Figure 8, the first protoset consists of a kite and a dart where $\tau = \frac{1+\sqrt{5}}{2}$ and $\theta = \pi/5$. The second protoset consists of two rhombi. This is the protoset generating the pattern of the toilet paper mentioned earlier. It was a fine surprise to learn that these kite, dart, and two rhombi shapes can be decomposed into two types of triangles. Adams applies the substitution method to these triangles and proves aperiodicity of the two Penrose protosets. The fact $\tau^2 - \tau - 1 = 0$ also plays an essential role in the proof.

4. Chapter 4

In Chapter 4, Adams discusses tilings of the sphere S^2, the hyperbolic plane (Poincaré disk) $\mathbb{H}^2$, the Euclidean 3-space $\mathbb{E}^3$, and more general spaces. This chapter starts with an introduction to non-Euclidean geometries for beginners. Basic comparisons of the three geometries $\mathbb{E}^2$, S^2, and $\mathbb{H}^2$ are presented with detailed studies of triangles and computations of distances. Adams also explains isometries carefully. Classification of isometries of $\mathbb{H}^2$ is nicely contrasted

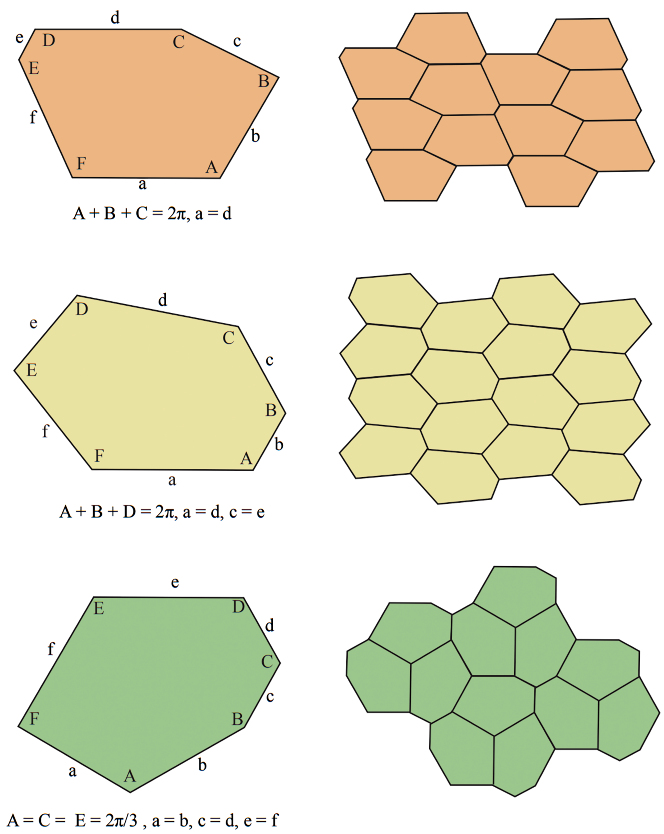


Figure 6. The three families of hexagons that can tile the plane.

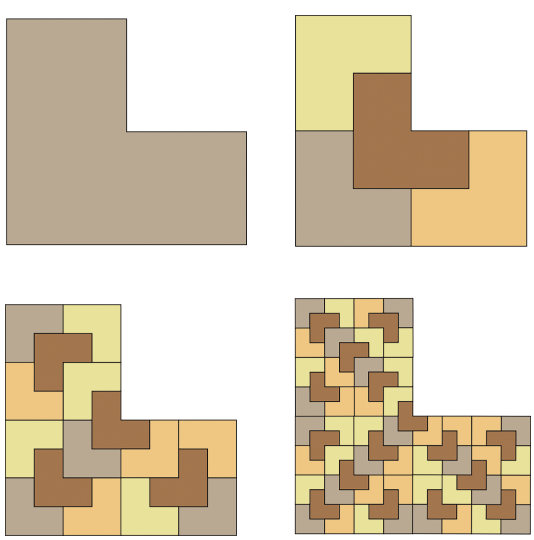


Figure 7. The chair reptile example.

with that of $\mathbb{E}^2$. There are five types: rotation, translation, reflection, glide reflection, and parabolic isometry. The fifth one is unique because it does not have analogy from Euclidean or spherical geometries.

Recall (Lemma 1.12) that every isometry of $\mathbb{E}^2$ is a composition of at most three reflections about lines. Similarly,

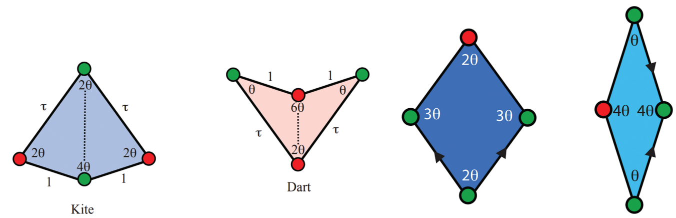


Figure 8. The kite and dart protoset and the two rhombi protoset.

every isometry of $\mathbb{E}^3$ is a composition of at most four reflections about planes. A new type of isometry is called a *screw motion*, which is a composition of four reflections.

4.1. Hyperbolic tilings. Adams shows us five differences between tilings of $\mathbb{E}^2$ and tilings of $\mathbb{H}^2$. For example, for integers $p, q \geq 3$, there is a regular tiling of $\mathbb{H}^2$ by regular p-gons such that q tiles meet at a vertex if and only if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. This was a nice surprise to me. A picture of tiling by 7-gons three of which meet at each vertex ($p = 7$ and $q = 3$, so $\frac{1}{7} + \frac{1}{3} = \frac{10}{21} < \frac{1}{2}$) can be seen in Figure 9, which looks like the surface of a sink full of bubbles. This chapter also contains the more familiar example of the tiling by 8-gons meeting four to a vertex ($p = 8$ and $q = 4$), which is the universal covering space of a genus 2 hyperbolic surface.

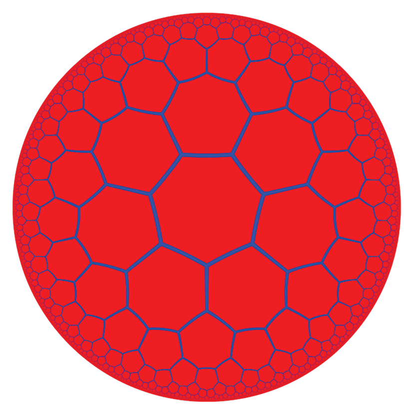


Figure 9. A tiling of $\mathbb{H}^2$ by 7-gons.

Two recent research results that highlight differences between Euclidean tilings and hyperbolic tilings are also introduced here. One is on prototiles for strongly aperiodic tilings and the other is on the realization of arbitrary high Heesch number.

4.2. Tiling of $\mathbb{E}^3$. Tilings of Euclidean 3-space has application to the study of crystals and quasicrystals. A simple way to construct a tiling of $\mathbb{E}^3$ from one of $\mathbb{E}^2$ is by thickening.

The cubical tiling is the only regular tiling of $\mathbb{E}^3$. A simple warm-up exercise is left to readers to show that the regular tetrahedron or the regular octahedron cannot tile $\mathbb{E}^3$. In Figure 10, Adams shows a non-face-to-face tiling of $\mathbb{E}^3$

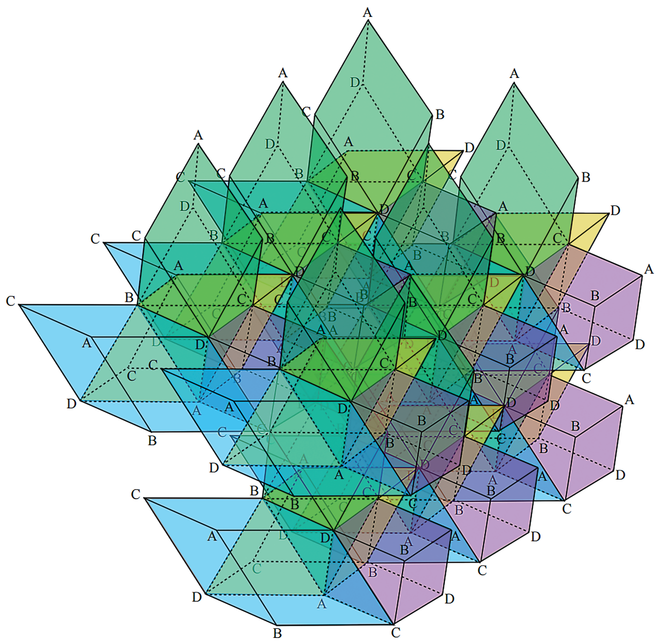


Figure 10. A tiling of $\mathbb{E}^3$ that is non-face-to-face.

by a nonconvex prototile that looks like a precious rock in a science museum. The 3D illustration made by Adams is painted with 4 colors and helps us imagine the complicated structure.

The section ends with the Schmitt-Conway-Danzer aperiodic prototile of $\mathbb{E}^3$ that doesn't admit any $\mathbb{Z}$-action. This shows a clear contrast between $\mathbb{E}^3$ and $\mathbb{E}^2$. There is no such prototile found for $\mathbb{E}^2$.

4.3. Knotted tilings. So far, every prototile is topologically a disk or a ball. We can get rid of this restriction and consider a prototile topologically to be a genus g handlebody. In this case, tiles possibly link together. Adams shows us six examples. He starts with a genus 1 prototile with no linking, as in Figure 11. The most impressive example that blew me away was a trefoil knot in the prototile, as seen in Figure 12.

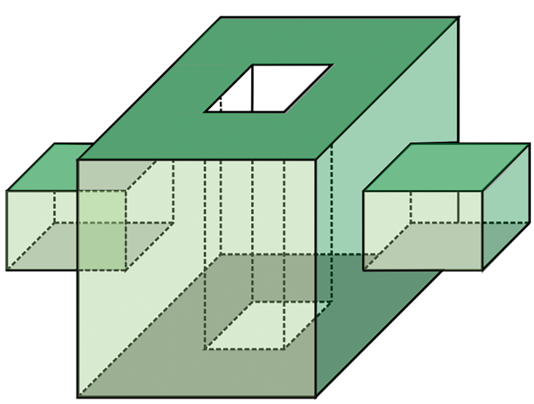


Figure 11. A genus 1 prototile with no linking.

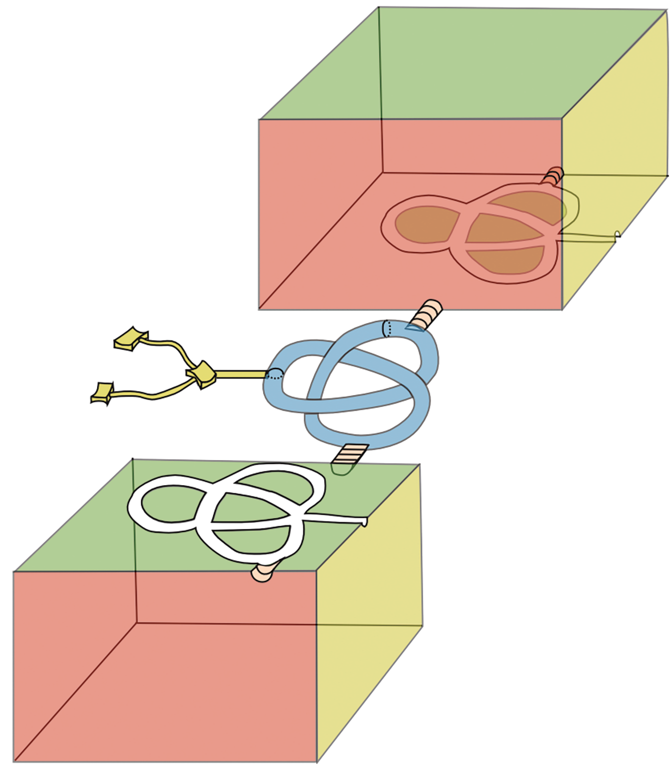


Figure 12. The trefoil knot in the prototile.

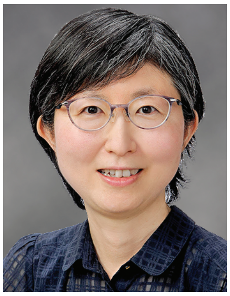
4.4. 3-manifolds. The last section of the book provides a bridge to the study of covering spaces, which is an important topic taught in algebraic topology courses, from the viewpoint of tilings.

Starting with dimension 2, Adams shows us how to obtain infinitely many distinct tori from parallelogram tilings of the plane $\mathbb{E}^2$. We say that $\mathbb{E}^2$ covers a torus. Depending on the corner angles and the edge length ratio of the parallelogram, the resulting torus has different metrics. A rectangular tiling also can yield a Klein bottle.

Increasing the dimension from 2 to 3, Adams discusses compact orientable 3-manifolds that result from tilings of Euclidean 3-space $\mathbb{E}^3$. Six compact 3-manifolds are explicitly defined by identifying faces of cubes and hexagonal prisms. Adams says: "Surprisingly, there are a total of only six such 3-manifolds, including the 3-torus. If the universe if compact and orientable . . . and Euclidean, then our universe must be one of these six possibilities." At the very end we see compact orientable 3-manifolds including the Poincaré manifold and the Seifert-Weber manifold that come from tilings of the 3-sphere S^3 and the hyperbolic 3-space $\mathbb{H}^3$, respectively.

5. Conclusion

In his book, Adams provides many fantastic hiking paths suitable for a wide range of readers. The paths are all walkable, and there is no need to bring ropes or carabiners. The hiking paths contain many scenic points where we can take rest and see interesting tilings. Occasionally along the path, the dimension leaps from 2 to 3. A hiker might notice that the metric system changes from Euclidean to hyperbolic. The paths are open-ended with the possibility that a hiker constructs a brand new path to a new scenic point.



Keiko Kawamuro

Credits

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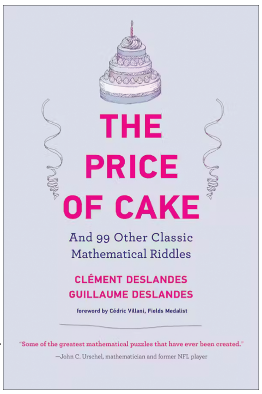

The Price of Cake: And 99 Other Classic Mathematical Riddles
By Clément Deslandes
and Guillaume Deslandes.
The MIT Press, 2023, 215 pp.

I suspect many mathematicians have a long history of solving puzzles. There is a distinctively satisfying moment when a puzzle that has challenged your mind for a minute, an hour, a day, or a week is solved. The recent publication *The Price of Cake* contains 100 inviting and irresistible mathematical riddles and puzzles. The book is ordered not by mathematical knowledge required but by the complexity of solutions, and it contains both familiar and uncommon riddles.

The puzzles make use of the pigeonhole principle, binomial coefficients, equivalence relations, bijections, permutations, induction, congruences, binary expansions, and probability. I expect that many readers could solve the riddles without formal training in mathematics. However, being familiar with these topics provides an advantage, and so there is a brief section at the end of the book that covers the mathematical themes.

I found great fun in the riddles "A Kangaroo on a Staircase," "The Rats and the Bottles," and "Linking the Edges." I think this book would be a great addition to your personal collection of riddles and would be useful when considering puzzles for a math club meeting. As the supplementary mathematics material in the book points to specific riddles, you could even use some of the riddles to enhance your teaching of one of the covered topics.

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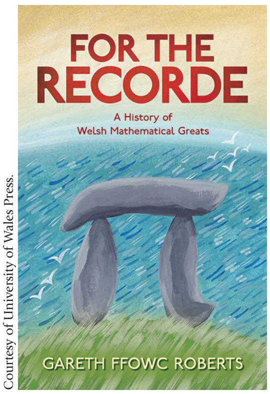

For the Recorde: A Welsh History of Mathematical Greats
By Gareth Ffowc Roberts.
University of Wales Press, 2023, 160 pp.

While this is a history of mathematics book, its focus is unique: Welsh mathematicians who lived during the period from the 16th to 20th century, along with descriptions of prominent locations in Wales at the time. The narrative is lively and offers enough variety to hold a reader's attention, including tales of scandals and mathematicians' travels outside of Wales as well as stories of familiar (non-Welsh) mathematical people and places, such as Pythagoras and Gauss. I was delighted to find that puzzles are sprinkled throughout the text, though some take the form of short mathematical calculations rather than puzzles. In addition, the author includes enough personal anecdotes that the book is part memoir.

The title of the book refers to the Welsh mathematician who used the first known instance of the equals sign, Robert Recorde. The book reveals other "mathematical firsts" credited to Welsh mathematicians; for instance, it details the first mathematician to use π to represent the ratio of the circumference to the diameter of a circle and the development of the first insurance policies and their associated mathematics.

This book will be of interest to mathematicians who appreciate learning historical tidbits for their own enjoyment or to include in their courses. It could supplement a history of mathematics course or provide extra historical context in other math courses. This book is a welcome addition to historical literature not only because it informs the community about mathematical greats from Wales, but also because it adds to the growing inventory of mathematical contributions from a diverse group of people. This was a fun read that I think many mathematicians would enjoy.

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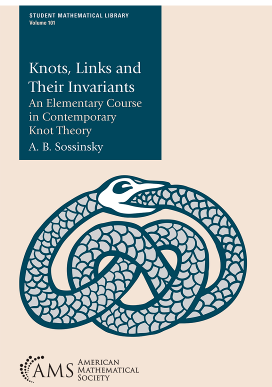

Knots, Links and Their Invariants: An Elementary Course in Contemporary Knot Theory
By A. B. Sossinsky.
STML/101, 2023, 129 pp.

This new addition to the Student Mathematical Library is an excellent concise introduction to knot theory and some of its associated algebra and topology. It is written clearly and simply, and well illustrated with many figures. The book is quite short, though not without demands on the reader. It is based on a course given online to participants in the Math in Moscow program and, because of the format of the course, the book contains rather frequent exercises for the reader, some of which are significant steps in proofs. However, a reader prepared to put in the work will be well rewarded. The exercises are interesting and not too difficult for undergraduates, and they lead the reader to some of the most important and useful knot invariants via surprisingly elementary methods—certainly more elementary than those used when the invariants were originally discovered.

As the book amply illustrates, knot theory is a topic that is easy to approach through elementary, easily visualized, combinatorial arguments. Yet it is also able to grow in many directions due to its connections with algebra, geometry, topology, and even physics. There can be no better example of "seeing the (mathematical) world in a grain of sand."

Lecture 1 begins with some examples of knots, defined as equivalence classes of simple polygons in $\mathbb{R}^3$, and introduces the three Reidemeister moves, which underlie most of the invariance proofs in the lectures that follow. Lecture 2 introduces the Conway polynomial, a variant of the classical Alexander polynomial which is well-suited to illustrate the inductive construction of invariants via *skein relations* (unlike Alexander's construction via covering spaces and homology). Skein relations are also the key to the treatment of the *Jones polynomial* and the *Vassiliev invariants*, which come up next.

The proof of the invariance of the Conway polynomial is only sketched, with some parts left as exercises, but proofs are more complete in Lectures 5 and 6, where we come to the more powerful Jones polynomial. The even more powerful Vassiliev invariants are the subject of Lectures 9, 10, and 11. These too are treated in a mostly elementary manner, and their nonelementary aspect—involving Kontsevich integrals—is nicely explained in Lecture 11. To my knowledge, the Vassiliev material was previously unavailable in a book for undergraduates, so Sossinsky's book is unique for this reason alone.

Most of the "meat" of this book is in the lectures on the knot polynomials and Vassiliev invariants; however, these are interspersed with very attractive informal sketches of related material on knots, braids, and other topics in topology and algebra. Also included is Lecture 13 on the history of knot theory, which mentions all the main contributors to the subject.

Altogether, this book is ideal for self-study by undergraduates with an interest in topology. It is proof that "old school" methods of topology—concrete, visual, and elementary—are still more than capable of providing insight into classical problems.

John Stillwell is a professor emeritus at the University of San Francisco. His email address is `stillwell@usfca.edu`.

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Data Science and Social Justice in the Mathematics Community

Quindel Jones, Andrés R. Vindas Meléndez, Ariana Mendible, Manuchehr Aminian, Heather Zinn Brooks, Nathan Alexander, Carrie Diaz Eaton, and Philip Chodrow

1. Introduction

In the summer of 2021, the Puerto Rico Association of Criminal Defense Lawyers (PRACDL) submitted a brief to the US Supreme Court in the case *Rodríguez-Rivera vs. USA*. At issue was whether a certain class of drug conspiracy crimes should be considered "controlled substance offenses," a status which can carry enhanced sentences. Drug conspiracy charges implicate multiple defendants and carry a low burden of evidence required to obtain a conviction against alleged coconspirators. PRACDL's brief argued that prosecution in drug conspiracy cases reflected discrimination and disparate impact along axes of race, ethnicity, and class. The brief was supported by analysis from the Institute for the Quantitative Study of Inclusion, Diversity, and Equity (QSIDE), including work by present authors Manuchehr Aminian and Philip Chodrow. Working in close dialogue with lawyers at PRACDL, the QSIDE team wrote parsers to extract information from unstructured docket data, combined these results with geographic census data, and tested hypotheses that conspiracy charges disproportionately took place in areas of Puerto Rico impacted by poverty, unemployment, and low rates of education.

Quindel Jones is a PhD candidate in systems modeling and analysis at Virginia Commonwealth University. Her email address is `jonesq2@vcu.edu`.

Andrés R. Vindas Meléndez is an NSF postdoctoral scholar at UC Berkeley. In July 2024, he will start as an assistant professor of mathematics at Harvey Mudd College. His email address is `avindas@msri.org`.

Ariana Mendible is an assistant professor of mathematics at Seattle University. Her email address is `mendible@uw.edu`.

Manuchehr Aminian is an assistant professor of mathematics at California State Polytechnic University, Pomona. His email address is `maminian@cpp.edu`.

Heather Zinn Brooks is an assistant professor of mathematics at Harvey Mudd College. Her email address is `hzinnbrooks@g.hmc.edu`.

Nathan Alexander is an assistant professor of data science and education at Howard University. His email address is `professornaite@gmail.com`.

Carrie Diaz Eaton is an associate professor of digital and computational studies at Bates College. Her email address is `cdeaton@bates.edu`.

Philip Chodrow is an assistant professor of computer science at Middlebury College. His email address is `pchodrow@middlebury.edu`.

The first seven authors are ordered by career stage at the time of article submission.

Due to tight space constraints in the Notices of the AMS, *we are unable to provide a complete set of references to support our discussion in this article. We have compiled a document of notes with additional references, resources, and supporting discussion available at* `https://www.ds4sj.net/`. *In some cases, these references support claims in the text that may appear unsupported in the published article.*

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This work with PRACDL is a small entry in a growing body of scholarship in *data science for social justice* (DS4SJ).[1] This area has received growing attention in several overlapping professional communities, including mathematics, statistics, and computer science. Our focus in this article is how members of the mathematical community can learn, practice, and support this burgeoning area of scholarship and activism.

We highlight ways that DS4SJ work can both appeal to and challenge trained mathematicians. Scholars who pursue DS4SJ may look forward to positive outcomes for served communities, new research questions, new professional connections outside the academy, and opportunities to connect their identity or values to their scholarly work. Realizing these benefits, however, requires confronting a range of challenges and risks. First, impact is never guaranteed. In the PRACDL case, for example, the petition to the court was ultimately denied without comment. Furthermore, the process of justice data science calls on skills in which many of us are untrained. Realizing impact beyond the ivory tower requires engagement beyond the ivory tower, with affected populations, domain experts, and decision-makers. When this work is pursued without sufficient attention to the needs of impacted communities, the results can fail as both scholarship and activism. Attentive engagement with nonscholarly collaborators is a skill which is often not taught in mathematics training programs, perhaps because it is not viewed as "mathematical." Finally, researchers who take on justice data science work must naturally keep an eye on how it aligns with their needs for advancement in their institutions and professional communities.

Our overall aim is to help scholars who wish to begin or deepen their engagement with DS4SJ work, and advocate to the broader mathematics community that this work should be valued and supported. Our core arguments are:

(a). Members of the mathematical community, broadly construed, have contributions to make toward social justice causes through both scholarship and pedagogy.

(b). Making these contributions requires us to engage with collaborators and partners outside the academy, including impacted communities, policy makers, and others who contribute experience, knowledge, and power beyond our own.

(c). Data science work supporting social justice can indeed be a scholarly, scientific endeavor, which should be supported by mathematical institutions at all levels.

In Section 2 we define data science for social justice (DS4SJ), construing this work broadly to include a wide range of computational, mathematical, and qualitative work. We compare and contrast DS4SJ to related bodies of work, including data science for social good and academic data science of human behavior. Based on our definition, we propose a set of four guiding principles for the practice of DS4SJ. These principles describe work that actively engages community members and decision-makers; is aware of its own limitations; intentionally aims to divest from privilege; and grapples with the systemic nature of oppression. We continue in Section 3 with discussion of some features of recent and ongoing work in DS4SJ. We especially focus on considerations for early-career scholars considering greater involvement this work in either research or pedagogy. We also highlight benefits for departments that make a commitment to supporting this work. In Section 4, we illustrate the practice of DS4SJ with a series of vignettes in which early-career scholars describe their work in DS4SJ and its impact on their growth in the broad mathematical community. We conclude in Section 5 with a reflection and discussion on the outlook for future work.

2. Data Science for Social Justice

We take an intentionally broad view of *data science*, referring to the wide set of practices, methods, and activities that support the use of data to gain insight and guide action. This includes standard steps in the data analysis pipeline, such as data collection, exploration, visualization, modeling, and communication. We also deliberately include an expansive range of other activities, including: formulation of mathematical models, development of data analysis and learning algorithms; qualitative research that contextualizes analysis; and deployment of publicly-facing data products such as visualizations and dashboards. In this broad view, data science work takes in many academic disciplines. It is not restricted to departments of statistics, mathematics, or computer science; it is indeed not restricted to STEM fields at all. That said, while our view of data science is broad, our focus in this article is primarily on how the mathematical community can engage with data science work. For this reason, our discussion largely centers on activities and practices in which mathematical training is relevant.

We frame data science for social justice within our broad construal of data science as a whole.

Definition. *Data science for social justice* (DS4SJ) is data scientific work (broadly construed) that actively challenges systems of inequity and concretely supports the liberation of oppressed and marginalized communities.[2]

[1] *We also use the phrase* justice data science *interchangeably to refer to the same scholarly area.*

[2] *Forms of oppression and marginalization include but are not limited to racism, colonialism, classism, sexism, homophobia, transphobia, and ableism.*

DS4SJ acknowledges that some groups of people have been historically oppressed, marginalized, and disenfranchised, and aims to rectify these harms. It does not necessarily aspire to benefit "society as a whole," but specifically aims to correct injustice. DS4SJ is therefore distinct from data science for social *good*, a phrase that often refers to any data scientific work with net positive social impact. Work featured by the organization named Data Science For Social Good, for example, includes efforts toward sustainability, efficient public infrastructure, and reduction of political corruption. These are general social goods which benefit large segments of the world's population. Social *justice* applications also benefit populations, but are distinctive in their explicit focus on the needs of oppressed and marginalized communities harmed by historical wrongs. We emphasize that this distinction, though important, is also porous. Many social good projects also have social justice elements; for example, the improvement of public infrastructure may have an especially large impact on communities that have been historically underserved by that infrastructure. Similarly, justice efforts that rectify systematic wrongs may be appropriately viewed as promoting broader social good. While it is important to distinguish these aims and scopes, we emphasize that neither precludes the other.

DS4SJ is also importantly related to work on bias and harm in large-scale machine learning systems. Recent work has shown that many of these models contain significant biases against members of marginalized identity groups. These effects are often intersectional: in a famous study by Buolamwini and Gebru [4], facial recognition algorithms performed worse on Black women than they did on either Black men or white women, with the faces of white men being recognized most reliably. Contributors to bias in machine learning models include underrepresentation of marginalized identities or overrepresentation of oppressive stereotypes in training data; use of flawed proxy variables for measuring predictive accuracy; or failure to consider the role of an automated system in social and historical context. With the growing prevalence of machine learning models in personal technology, hiring, policing, and healthcare, the stakes for designing nondiscriminatory systems at mass scales are extremely high. We view work toward fair machine learning as DS4SJ: it is data science that challenges systems of inequity and supports the liberation of oppressed and marginalized communities. This is especially true when the work influences the design and assessment of deployed, large-scale systems. That said, the majority of this work is carried out in the computer science community rather than the mathematics community and is therefore outside our scope. We therefore refer the interested reader to Mehran et al. [9] for a survey of recent work in bias and fairness in machine learning, including conceptual foundations, empirical findings, and proposed methods for reducing algorithmic discrimination in a range of machine learning tasks.

Finally, we distinguish DS4SJ as a specific subset of the broad academic area of data science and modeling of social systems. This large and growing area includes work in applied mathematics, computational social science, physics, network science, computer science, and allied fields. Some of this work may indeed actively support the liberation of oppressed and marginalized communities, thereby constituting DS4SJ. Other work—perhaps the majority—in this area has as its primary goals the furthering of quantitative insight into social systems within the scholarly community. An example of the latter kind of work is mathematical analysis of dynamical models of social phenomena, such as segregation or opinion polarization. The long-term impact of such work may be primarily localized in scholarly publications, and may not inform activism, policy, or other mechanisms that affect the lives of those who are not professional researchers. We emphasize that our aim is not to disparage this body of work, and indeed, several of the present authors maintain active research agendas in this broad area.

In summary, our focus here—data science for social justice in the mathematical community—overlaps with but is distinct from data science efforts toward broader social good; research on fairness in machine learning; and the quantitative modeling of social systems.

2.1. The process of DS4SJ. Many of us in the mathematical community are accustomed to measuring the success of our research program in units of theorems, experiments, papers, venues, or citations. The processes by which we conduct our research are often optimized toward these measures. Data science for social justice measures its success in terms of impact on the lives of oppressed and marginalized peoples. This focus requires that we pursue the work in ways likely to realize impact, while also minimizing risk of unintended harm. The process of this work may therefore look very different from the process of much other mathematical scholarship. We propose a series of four guiding principles for the process of DS4SJ, inspired by the writing of Federico Ardila-Mantilla's writing in the *AMS Notices* [2].

Principle 1 (Collaboration). *DS4SJ requires partnership with community members and decision-makers in order to realize intended impact and minimize unintended harm.*

Principle 1 reflects two simple facts. First, as mathematicians, we are often inexperienced in conceiving, pursuing, or applying our work outside of academic or industrial research environments. We often lack relevant training in social science, policy, or activism. We therefore often need collaboration to execute projects and build skills. Second, work that supports a community must be guided by careful engagement *with* that community in order to understand the needs we aim to meet. We identify at least three important elements of effective collaboration for DS4SJ:

- *Perspective* allows us to understand the needs of the people we aim to serve and helps us carry out our work without causing unintended harm.
- *Process* helps direct the work with methods that respect the community, and in directions that are responsive to community perspective with results that will support future action.
- *Power* makes it possible to actually carry out the intervention informed by our data scientific work.

Perspective is an especially important element of collaboration, as it helps us understand the needs of the communities we serve. The best way to build this perspective is through partnership *with the community itself*. This is ultimately the only reliable way to understand the needs of its members and whether they want what we are able to offer; perspective from other scholars or outside experts can inform only insofar as it is grounded in community engagement. The slogan "Nothing about us without us," popularized in English by South African disability activists, offers a concise maxim.

A recent effort at the Institute for Mathematical and Statistical Innovation (IMSI) illustrates these forms of collaboration. A working group led by author Carrie Diaz Eaton worked in partnership with Nuevas Voces, a leadership program by the Woonasquaucket River Watershed Council (WRWC), for residents of a low-income, immigrant and multilingual community in Providence, Rhode Island. This collaboration offered *perspective* by helping the working group understand community needs through the insights of Nuevas Voces, who expressed a need to call attention to the flooding of neighborhoods along the watershed induced by ongoing climate change. The collaborative *process* involved meeting in impacted neighborhoods, conducting interviews in Spanish, and offering support to community members from Nuevas Voces and the WRWC to guide the work as it progressed. Maintaining collaboration through the implementation phase invested the work with *power* to make an impact. The primary deliverable of the intervention with Nuevas Voces was a geospatial dashboard which incorporates not only flooding information but also community resources and stories. Nuevas Voces and WRWC have committed to integrating these tools into their own organizations in order to support their missions.

Principle 2 (Critical Reflection)**.** *DS4SJ requires critical reflection on one's own identity, position, and privilege, and on the role these play in one's scholarly practice.*

We use "critical" in the sense of *critical theory*: a critical perspective is one which views structures of power and privilege with skepticism, and aims toward a more equitable, free, and just society [6]. Principle 2 asks us to reflect on the identities and privileges we inhabit, with a special eye toward how these identities and privileges may tacitly inform our work. Many members of the broad mathematical community inhabit multiple forms of professional privilege. Our profession is culturally respected, often commands above-average salaries, and does not usually place us at risk of physical harm. Mathematical ways of knowing are popularly (and often dangerously) viewed as epistemically privileged: piercing, infallible, and not subject to dispute. Many of us in academia have the luxury of measuring deadlines in units of months or years, rather than days or hours. When projects "don't work out," they can often be safely dropped in favor of new ones. Additionally, a disproportionate number of professional mathematicians benefit from white and male privilege.

It is important to acknowledge and grapple with privilege when aspiring toward justice work of any form, including DS4SJ. For example, how do we reconcile the typical pace of mathematical research with the fact that, as author Nathan Alexander frequently emphasizes, *people are dying* from systemic forms of injustice? The work of interrogating our privileges and understanding how they inform our professional practice is challenging, but there are some aspects of mathematical training that support mathematicians in this work. Privilege can be viewed as a collective social assumption about who deserves freedom, respect, and resources. Assumptions, in the form of theoretical axioms or modeling inputs, are pervasive in mathematical practice. As mathematicians, we are trained in the fundamental skill of articulating assumptions, questioning their applicability, and deriving their consequences. These are exactly the skills that support critical reflection, if they can be turned inward. Author Quindel Jones models reflection of the effects of her own identity and positionality on her scholarly work in her vignette in Section 4.

Principle 3 (Interrogation of Privilege)**.** *DS4SJ asks the researcher to work actively to interrogate and see past their own privilege.*

As discussed under Principle 2, mathematicians often benefit from several forms of systemic privilege, often based on a combination of identity, the social prestige of their profession, and scientific ways of knowing. These forms of privilege can be obstacles to effective DS4SJ work, especially in regard to effective partnership with collaborators from communities that may be very different from our own (Principle 1). Seeing past our privilege therefore involves epistemic humility: the active centering of ways of knowing other than our own. Learning community needs may require that we heed diverse forms of knowledge, such as community narrative, individual experiences, or common wisdom from other professions. Insisting on the primacy of mathematical or scientific ways of knowing over the ways of our served communities is likely to result in failure to synthesize necessary information about community needs or ways to meet them. This can in turn lead to failure to realize intended impact, or even in unintended harm.

In his vignette in Section 4, Andrés R. Vindas Meléndez discusses how his trajectory into DS4SJ as a theoretical mathematician has been aided by awareness of the privilege of academic mathematics and his willingness to engage with other scholarly communities.

Principle 4 (Systemic Perspective)**.** *DS4SJ requires a critical* systemic perspective*: a critical awareness of how systems of power operate to privilege some and oppress others.*

Many mathematicians use sophisticated tools to study the function of complex and interrelated systems in domains ranging from biology to engineering to human society. Systems thinking is a fundamental skill in justice-oriented data work as well, and informs the technical tools we deploy. For example, there is extensive mathematical theory describing mechanisms for fairly distributing resources, but this theory usually lacks a description of historical and present systemic forces of oppression. Rectifying this oppression requires more than fair division of resources; it also requires active work against these systemic forces. In her vignette in Section 4, Heather Zinn Brooks discusses how taking a systemic perspective on the mathematics community can inform data-driven studies of gender representation in mathematical subfields. In some cases, it may also be possible to explicitly model social processes that contribute to disparity. Modeling the social processes of police use-of-force, for example, may ultimately provide insight on how to stop disproportionate police killings of unarmed Black people [15].

3. Challenges and Opportunities in DS4SJ

Justice data science addresses a wide range of problems both within and beyond academia. We roughly divide this work into three overlapping categories.

- **Data Accessibility**: Ida B. Wells famously wrote that "the way to right wrongs is to shine the light of truth on them." Some justice data science work aims simply to make new kinds of information available and interpretable to new audiences. Examples include efforts to compile accessible, online databases of police violence, federal judicial sentencing, and cultural representation.
- **Analysis and Critique**: In some cases, the simple presentation of data in an easily accessible format may be enough to motivate action. In other cases, the sheer quantity of data and the need to account for multiple factors requires explicit, formal data analysis. This analysis can sometimes support controlled or causal claims about the magnitude or impact of bias, and may be published in scholarly journals. In this context, it is important to emphasize that the mere availability of evidence or analysis is often insufficient to change minds or spur change.
- **Implementation**: Impact is ultimately the axis along which justice work, including data science, is measured. Successful DS4SJ work either achieves impact directly through action or supports follow-up efforts by others. In some cases, it may suffice to bring analysis and critique in front of the right parties; for example, work identifying the racial impact of political gerrymandering is analysis and critique, and can contribute toward its intended impact when presented as arguments in court cases. Other examples include the deployment of predictive or decision-making models aimed at explicit equity goals, such as predicting police misbehavior, landlord exploitation, judicial decision-making, or corporate tax abuse. Data scientists and mathematicians, as engaged human beings, also have opportunities to engage in direct action in support of their chosen causes, including community organizing, attendance at protests, and participation in mutual aid societies.

3.1. Getting involved with DS4SJ.

We hope that some of our readers are considering whether to engage in or indirectly support work in data science for social justice. We now highlight some of the reasons why this may be appealing for scholars in mathematics and allied fields. There are a wide range of benefits, including active research prospects, ways to engage undergraduates in the classroom, and opportunities to recruit junior collaborators for research projects.

Justice data science work can motivate the development of new mathematics, data scientific techniques, and methodological perspectives. One well-known example is the development of methods for detecting and quantifying racial bias in electoral maps by defining null distributions on spaces of districting schemes using techniques from metric geometry and using Markov-chain algorithms to sample from them [3]. This body of work has produced analysis and critique in the form of papers quantifying racial bias in existing political maps. There have also been impacts from engagement with power partnerships, with researchers collaborating with state governments and contributing to amicus briefs before the US Supreme Court.

There are also opportunities for methodological advancements in applied statistics, optimization, and network science. Pierson et al. [10], for example, develop novel, highly-scalable statistical tests for detecting racial bias in large-scale datasets of police behavior. Problems inspired by equitable access to schools, grocery stores, and polling places in urban infrastructure networks have led to the development of novel algorithms for editing graph edges to influence diffusion dynamics [11]. A growing body of work uses network-based and statistical methods to quantify gendered inequalities in citation, promotion, and retention across academic disciplines [14]. In her vignette below, author Heather Zinn Brooks describes ongoing related work in mathematics. As author Andrés R. Vindas Meléndez writes in his own vignette, the urgency of social justice applications have helped him both develop an interest in data science techniques and begin to reflect on whether combinatorics, his primary area of research training, might have useful applications toward justice causes.

An orientation toward social justice offers benefits for pedagogy in mathematics, data science, statistics, and computer science. Real-world topics related to issues of injustice and inequity are fruitful sources of problems and examples [7]. Beyond simply furnishing problems, however, justice orientations can support mathematics instructors in welcoming, engaging, and retaining students in undergraduate STEM majors/minors, and higher education more generally. Justice orientations can be especially useful for engaging students who have been historically excluded from or oppressed or marginalized within traditional classrooms. Connecting classroom content to "prosocial communal outcomes" is a recommendation of a recent report focused on improving the persistence of marginalized students in STEM disciplines [5].

It is important to note that justice-oriented pedagogy presents challenges for both students and instructors, and can be harmful to students if cases are presented in a demeaning or tokenizing way. There is potential for harm if instructors incorporate examples that do not promote positive student identity or which fail to acknowledge diversity of experience within cultures and identities [8].

Effective justice-oriented pedagogy seeks to advance students' critical consciousness–their ability to challenge what is framed as normal, whose interests are served by that framing, how system structures are reified, and in what ways they could, potentially, be different. Such pedagogy not only welcomes students who have been historically marginalized in classroom spaces, but also challenges all students to grow as justice-oriented systems thinkers. For example, author Nathan Alexander uses a justice orientation in his introductory statistics classroom when framing the topic of ethical research and data use. The Institutional Review Board (IRB) system was formed in part due to the revelation of the medical abuse of almost 400 Black men in the Tuskegee Syphilis Study. The modern IRB review system aims to prevent future such flagrant breaches of medical ethics. This, however, does not fully address the dimensions of structural racism in the practice of medical research; even today, Black principal investigators are less likely to have grant proposals funded by the National Institutes of Health. These considerations are not only context for the modern practice of statistics and data-informed research, but they are also opportunities to practice core technical course content. Students can study and evaluate, for example, the methods underlying recent findings that the Tuskegee study is *still* shortening the lives of Black men by sowing justified mistrust in medical institutions [1].

These are challenging topics to navigate without tokenizing or otherwise harming students. Doing so requires instructors to show vulnerability, humility, and authentic willingness to learn *with* students in a community built on mutual respect, openness, and collective critical reflection. The reward is an opportunity for students to develop technical skills and critical awareness in tandem, each reinforcing the other.

Similarly, students at all levels are often excited to work on research projects that involve justice work. Undergraduates are increasingly involved in and committed to social justice and are drawn to research experiences with obvious real-world applications [12]. As author Ariana Mendible describes in her vignette below, her early research projects with justice focus have received much higher than expected interest from prospective research students at her institution. This may be in part due to the accessibility of these projects. DS4SJ work often has lower technical barrier to entry when compared to many other problems in academic mathematics, making it possible to involve students across a greater range of experiences and academic backgrounds at earlier stages in their academic careers. Students' personal experiences and interests can also be explicit assets in the pursuit of DS4SJ research projects, especially when they are members of the communities who stand to be impacted by the work.

Finally, we emphasize that DS4SJ work places distinctive demands on researchers and instructors, and is therefore not necessarily for everyone. First, scholars may reasonably have motivations and interests that are not fully aligned with DS4SJ work. DS4SJ work measures itself first and foremost on positive impact for marginalized or oppressed peoples. Scholars pursuing this work therefore need both an orientation toward concrete outcomes and a willingness to devote time toward realizing those outcomes. This in turn requires time to cultivate critical perspective, engage carefully with communities, and develop nontechnical skills. We have nothing but respect for scholars who reflect on the demands of DS4SJ and decide to pursue other paths. Second, we acknowledge that professional incentives in many institutions and societies may not encourage DS4SJ work. For example, some departments may not recognize DS4SJ scholarship as being "sufficiently mathematical" to warrant promotion or recognition, or may view justice-oriented content in the classroom as a distraction. Scholars cannot be expected to commit to this work if they feel it endangers their prospects for advancement in their departments, institutions, or communities.

3.2. DS4SJ and the math community. We therefore call on departments, institutions, and professional communities to support work in DS4SJ. Fortunately, DS4SJ work is aligned with the stated goals of many of these entities, implying both opportunity and incentive to support these efforts. The AMS Statement on Equity, Diversity, and Inclusion states our shared goal of *a community that is diverse, respectful, accessible, and inclusive*. We are still far from this goal in 2023. Academic mathematics still has not reached proportional representation or equity of opportunity along the lines of gender and race. Many departments struggle to recruit and retain minoritized trainees and faculty. Active support of justice data science may be able to contribute to these goals. Based on interviews conducted with underrepresented racial minority students in US STEM graduate programs, Tran [13] recommends that departments seeking to retain underrepresented minority students should legitimize and expand research that has direct applications for social justice. Illustrating this effect, author Quindel Jones discusses in her vignette below how the opportunity to use mathematical models to improve health outcomes for Black women has been a motivating thread in her career trajectory. Jones highlights how the presence of a growing community of mathematical scholars focused on data science and social justice has promoted her scholarship, enabled her to learn from mentors who shared some of her identities, and connect with collaborators who share her interests. Support of justice data science by departments, research institutions, and professional organizations can not only promote scholarship that stands on its own merits, but also further our shared mission toward a more equitable, diverse, and inclusive mathematics profession. We discuss concrete ways that departments, institutes, and societies can support DS4SJ in Section 5.

4. Experiences in DS4SJ

In this section, we highlight the experiences of several pretenure mathematicians in their diverse engagements with data science for social justice. Their stories offer examples of how the principles of DS4SJ described in Section 2.1 can be implemented, and how the benefits and challenges of this work have influenced their trajectories. Ariana Mendible discusses how collaboration has impacted and improved her DS4SJ work (Principle 1). Quindel Jones provides a critical reflection of identity and positionality on scholarship (Principle 2). Andrés R. Vindas Meléndez interrogates the privilege and challenges of doing DS4SJ work as a theoretical mathematician (Principle 3). Heather Zinn Brooks highlights the importance of taking a systemic perspective on research work in DS4SJ (Principle 4). Their experiences highlight the roles of identity, belonging, and justice orientation in the trajectories of early-career scholars working in science for social justice.

Ariana Mendible, Seattle University. I recently returned to my alma mater, an institution with a strong commitment to justice, as an assistant professor of mathematics. I recognize the privilege of this position, offering the freedom to work on unfunded problems that align with my personal values. Pivoting my research to DS4SJ has offered fruitful research collaborations that promise meaningful impact within and outside of academia.

My work as a codirector of QSIDE's Small Town Police Accountability (SToPA) research lab has shown me the eagerness of rising scholars to engage with DS4SJ. We have worked together to tackle tasks like optical character recognition and topic modeling with the goal of empowering small towns to access and understand data from their police departments. The impact and urgency of these problems invites a broad audience to participate in research, including students who may not otherwise have envisioned themselves as researchers.

Sociologists, lawyers, and activists participating in our lab efforts enrich our perspectives beyond our mathematical and technical expertise. Importantly, collaboration with community leaders guarantees direct and effective change from our work. Conversations with one town's racial justice and police reform group provided town-specific history and helped narrow our focus to the most impactful questions. Community members have expressed enthusiasm that our quantitative work will further empower them to reform their local police departments. We hope that their enthusiasm will help our work inform community decision-making and realize its intended impact beyond the ivory tower.

My long-term goal is to continue building a research program that invites students to work on problems that are meaningful to them and their communities. I believe that this program will not only enact my values, but ultimately strengthen my academic career.

Quindel Jones, Virginia Commonwealth University. I am a queer Black woman born and raised in Jackson, Mississippi, a part of the American South.

While I initially took to dance and writing, my mother's affinity for numbers engaged my mathematical curiosity. My own affinity grew as community summer enrichment programs kept me engaged. Throughout high school and college, there were opportunities for me to use mathematics in my personal life and learn from people who looked like or cared about me. I graduated from Jackson State University, one of six historically Black colleges and universities (HBCUs) in the state, under the guidance of Dr. Jana Talley, whom I met in the I.C.STEM program at Jackson State when I was in high school.

I am currently a doctoral candidate in applied mathematics with a concentration in mathematical biology at Virginia Commonwealth University. Since entering graduate school, my professional mission has been to use mathematics to improve the lives of people that look like me—especially Black women. In my dissertation project, I am developing a dynamical model of pain levels during sickle cell disease, a disease that disproportionately affects Black people. We start with self-report data of patient pain, collected by a member of our interdisciplinary team. We use this data to fit an ordinary differential equation (ODE) model of a patient's pain profile over time. This model is already informing the data collection strategies for the next round of our study. Long-term, we hope to embed our model in a wearable app. We hope for patients to be able to log their sleep and pain self-reports, and from these receive warnings about likely upcoming pain crises. Patients could use these warnings to seek prophylactic care or medication.

As I continue on my mathematics journey, I am discovering more and more ways in which mathematics can be applied to address inequities. With my own life having been shaped by inequity due to my marginalized identity in US society, I am thrilled that there's an active community centered on math and data science for social justice. Connecting with this community in venues like the recent program at ICERM has been formative experiences in my mathematical career. These programs and their leaders have helped me see myself and my concerns as important. Part of my excitement for my future as a researcher is knowing that socially impactful mathematical work exists, and that there is a growing community of collaborators with whom to do it.

Andrés R. Vindas Meléndez, UC Berkeley. My mathematical work is mainly in combinatorics. I have always found value in this work from meaningful interactions with collaborators and the joy of mathematical discovery.

I had my "formal" introduction to data science for social justice at a recent ICERM program on this topic. I began working with a number of collaborators to address a deceptively simple question: *who* or *what* is a mathematician? Narrow definitions based on job titles or institutional affiliations can be precise, but can also be used to implicitly or explicitly exclude people from the discipline. In a recent preprint, we examined some of the tensions in several formulations of what it means to be a mathematician, including those based on self-identification, activities, and qualifications. For future work, we aim to design a survey of mathematicians (broadly construed) across a wide range of institutions, personal identities, job titles, and areas of scholarly and pedagogical interest. We will study these data using techniques like text analysis and clustering to extract qualitative insight from survey participant responses. From this work, we hope to deepen our understanding of the identities and needs of mathematicians in order to support interventions at multiple levels to increase the participation of marginalized or underrepresented groups.

My interest in the topic of who is considered a mathematician, and who is welcomed in our community, is informed by my own identity and experience. My identities as a queer, chronically ill, first-generation, Latinx mathematician have shaped my own experience in academic mathematics, including both successes and struggles. I now benefit from privilege in my study and work and well-resourced institutions. My experiences navigating my career so far highlight the need for spaces that allow marginalized people to feel welcomed, comfortable, and supported in the mathematics community.

As someone whose training is primarily in theoretical math, I sometimes feel that I do not have much to contribute to DS4SJ. I remind myself that this work requires diverse experience and expertise and my own identities offer important perspective on critical questions. What is missing from our survey efforts? What identities might be overlooked or mischaracterized by our approach? What voices are centered in extant literature? Whose voices are neglected?

Recently, I have been developing my technical data science skills. At my current institution, I am participating in workshops on Python, R, data visualization, and text analysis as a means to build my skills for my research projects. Furthermore, I have joined the Digital Humanities Working Group, which allows me to have interdisciplinary conversations and reflect on the social prestige that mathematicians carry.

Heather Zinn Brooks, Harvey Mudd College. My identities as a woman and a first-generation college student have impacted my career trajectory and inspired my involvement in research in data science and social justice. Growing up, I didn't know that "mathematician" was a real job title. As a student, I struggled to see myself as a mathematician because of the lack of role models who shared these identities. Now that I have the privilege to carry the title of "professor of mathematics," one of my motivations for becoming involved in work in DS4SJ is to give back to (and improve) a community that has benefited me. I recognize that my positionality as a white, cisgendered, able-bodied individual has contributed toward my positive experiences in the mathematics community. My interest in creating inclusive mathematical spaces has led me to a recent project modeling female gender representation in mathematical subfields and institutions. In this project, my collaborators and I use a data set compiled from the Mathematics Genealogy Project (MGP), combined with algorithmically inferred binary gender from names. This data has important limitations. Binary categories are imperfect representations of gender, an inherently nonbinary facet of identity. Furthermore, gender is not determined by name or appearance. We assume that misgendering takes place in our data set, and we view our results as at most estimates of macroscopic trends. Within these limitations, our goal is to build understanding of how some subfields and institutions have reached relatively high levels of female gender representation, while others remain strongly male-dominated.

We study the roles of homophily and prestige as mechanisms supporting or inhibiting female gender representation in mathematics. A centerpiece of our work is the development of data-informed branching process models for replicating features found in real data. We use machine learning techniques to highlight useful predictors in the data and inform the model mechanisms. In addition to these technical contributions, a long-term aim is to develop actionable strategies to support more inclusive mathematical spaces.

My formal training in applied dynamical systems and mathematical biology has allowed me to connect this motivation into my research in meaningful ways and has challenged me to adopt a systemic perspective on the way our mathematical community functions. Personal identity characteristics, institutional prestige, and perceptions of and the microcultures within mathematical subfields are all salient features that impact the way our community looks, feels, and operates. This systemic perspective allows my collaborators and me to interrogate the MGP data to highlight, explore, and understand the impact of these features.

My primary motivation for engaging in justice-oriented data science work is to fight systemic injustice and oppression. My involvement in this work has also inspired new scholarship, provided a source of joy in my research career, and allowed me to connect my identity to my mathematical pursuits.

5. Outlook

We have argued that many members of the mathematical community can, if they choose, further justice work through the reflective, critical practice of data science. We have highlighted some of the benefits for scholars, especially early-career mathematicians: DS4SJ allows scholars to do work with concrete impact, to relate their identities to their work, to find inspiration for new technical problems, and to connect with their students inside and outside the classroom. There are communities emerging to support this growing subdiscipline, foster collaborations, and build critical expertise. We close with some suggestions for individuals, departments, and institutions to nurture work in this burgeoning area.

While individual motivation may spark engagement in DS4JS scholarship, departments, institutions, and professional societies must offer active support for this scholarship to be successful. This is especially pertinent considering that a large portion of the work in DS4SJ is being spearheaded by early-career mathematicians, including untenured professors, industry professionals, postdocs, and students. Since many of these scholars lack career stability, it is important that institutions align incentives so that these scholars can be confident in their ability to advance their careers while pursuing justice work. At all levels, DS4JS work must be structurally and financially supported to sustain scholarship in this field.

Academic institutions can support justice-oriented scholars at all career stages. Departments can support faculty in their teaching endeavors by encouraging the development of new courses at the intersection of mathematics, data science, and social justice. Programs can provide training to undergraduate and graduate students and facilitate opportunities for community collaborations. Hiring committees can revise rubrics to explicitly place value on justice-oriented work, which may be particularly important in searches related to applied data science. In retention, promotion, and tenure decisions, work must be understood and valued appropriately. For many departments, this necessitates broadening the perspective on what constitutes evidence of scholarship in this area, including both publications and evidence of impact of the scholar's work outside academic venues. In order to meaningfully support justice-focused scientific work, departments must formally recognize its value in training and evaluative processes at all levels.

The structural support of institutes and professional societies is also critical to furthering this growing subfield of mathematics. As with any subfield, workshops and conferences are critical for disseminating work, generating new ideas, and creating and maintaining connections among scholars. Such events can be especially impactful for early-career scholars aiming to make connections and jump-start their scholarship. The Summer 2022 program on "Data Science and Social Justice: Networks, Policy, and Education" at ICERM, at which the present authors began this article, is just one example of a growing number of research community spaces.

Simply making venues available for workshops and collaborations is, however, not sufficient. Adequate, equitable funding is necessary for diverse researchers and community members to fully participate in partnership. Providing this funding may challenge traditional institutions. Some examples of funding to support equitable participation include travel for community participants (not just researchers); upfront financial support to graduate student researchers for travel and meals; and financial support for childcare or partner travel. Measures like these enable the full participation of individuals for whom the typical financial and time costs of workshop participation may be especially heavy. We therefore encourage reflection and creativity on the part of institutes and other funded projects to design funding which supports equitable participation in DS4SJ work.

Professional societies can further research and innovation by forming and supporting professional communities for mathematicians working in DS4SJ. Messaging from these societies is a powerful and important tool for building respect and professional legitimacy for justice data scientists and the scholarship they create. Such messaging also helps scholarly reviewers understand the nature of DS4SJ research, which this article aims to inform.

The intersection of data science and social justice work represents an exciting opportunity for mathematical scientists to work toward a more just and equitable future, both within our own community and beyond. We hope that this article can promote the growth and success of data science for social justice.

Author Positionality

The authors of this manuscript represent a diverse range of identities, experiences, and voices within the mathematical community. Our views on data science and social justice are necessarily shaped by our experiences of both marginalization and privilege. Many of us embody marginalized identities along axes of race, ethnicity, gender, and sexuality. For some of us, our interest in justice work is informed by a passion to combat forms of oppression that we ourselves have experienced. We also acknowledge that many of us occupy positions of privilege within the mathematical community, including job security, access to research funding, and recognition within our professional communities. We also acknowledge the considerable privilege represented by the invited opportunity to share our perspectives in the *Notices*. We look forward to continuing both our learning and our work toward justice.

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Quindel Jones

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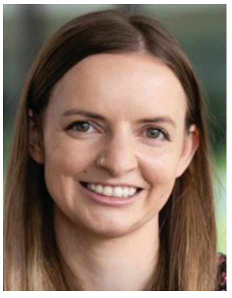

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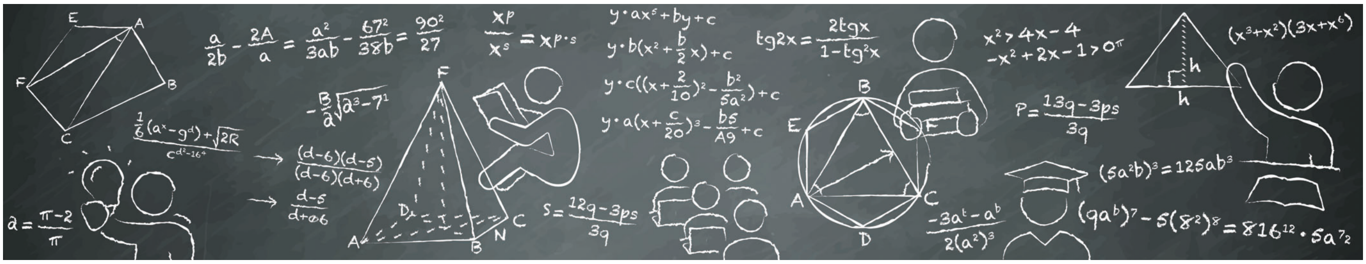

Machine-Human Collaborations Accelerate Math Research

Susan D'Agostino

One might think that earning a Fields Medal would quiet any mathematician's nagging doubts about their proof-writing abilities moving forward. But soon after Peter Scholze earned the 2018 award,[1] he questioned one part of the proof of the liquid vector space theorem he co-authored with Dustin Clausen. With humility, Scholze reached out to the math community.

"I spent much of 2019 obsessed with the proof of this theorem, almost going crazy over it," Scholze wrote on the Xena Project blog[2] in December 2020 about a black box of functional analysis contained within the proof. "I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts. ... this may be my most important theorem to date ... Better be sure it's correct."

Indeed, Adam Topaz, assistant professor of mathematics at the University of Alberta, confessed with nervous laughter that he and his study group had skipped the proof of the theorem when they read the work in 2020.

"He's a very convincing person," Topaz said during a talk at the February 2023 Machine Assisted Proofs conference[3] held at UCLA's Institute for Pure and Applied Mathematics. "If he says something is true, people tend to trust him."

That's why Scholze and a team led by Johan Commelin of the University of Freiburg in Germany launched a large, collaborative project to verify the proof with the free, open-source, Lean Proof Assistant. The community-driven software was developed[4] principally by Leonardo de Moura when he was at Microsoft Research and is supported by a mathematical library built by the Lean community. Topaz and other mathematicians and computer scientists joined Commelin's effort. By May 2021, they hit their first target, and by July 2022, they completed the project, which required 98,000 lines of code, according to Topaz.

"I find it absolutely insane that interactive proof assistants are now at the level that, within a very reasonable time span, they can formally verify difficult original research," Scholze wrote on his blog.

Susan D'Agostino is a mathematician and Spencer Journalism Fellow at Columbia University. She may be reached through her website: `https://www.susandagostino.com`.

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[1] `https://www.britannica.com/science/Fields-Medal`

[2] `https://xenaproject.wordpress.com/2020/12/05/liquid-tensor-experiment/`

[3] `https://www.ipam.ucla.edu/programs/workshops/machine-assisted-proofs/?tab=schedule`

[4] `https://leanprover-community.github.io/`

Carefully checking the proof would have been a "non-starter" without the machine assistance, Topaz said. As a bonus to finding and fixing some inaccuracies and verifying the proof of the main theorem, Topaz reported that the group also proved some auxiliary statements, answered some questions from Scholze's notes, and filled some gaps in Lean's mathematics library, specifically in homological algebra, topology, and category theory.

"It was really a human project, not a robot project," Topaz said.

Since late 2022, academe has been abuzz with talk of ChatGPT, GPT-4, and other AI writing tools that may change research and college writing. But the mathematics community is also grappling with disruption, as evidenced by human-machine collaborations in writing proofs of difficult theorems at the cutting edge of research mathematics. Machines not only verify mathematical results that humans discover but help mathematicians reason, explore new ideas, and even learn new mathematics.

Automated and Interactive Methods

Formal methods, a subfield of computer science, uses techniques based in computational logic to assist with mathematical reasoning, according to Jeremy Avigad, professor of philosophy and mathematical science at Carnegie Mellon University. Within the subfield, there are automated methods and interactive methods. Broadly speaking, in the former, a human asks a question, pushes a button, and gets an answer—or an "I don't know," which can leave the mathematician stuck. In the latter, the user works with the computer to explore or check reasoning. That leaves room for possibility.

"It's early and we're still figuring out what the technology can do," Avigad said. "But there's a lot of promise and a lot of excitement."

When a mathematician works with an interactive proof assistant, they might "teach" it about a new area of mathematics by entering the basic definitions and theorems. They might later "ask" the machine what it knows about these objects. When formalizing a proof, they will try to get the computer to determine whether something follows from previous statements, providing more information if necessary. In this way, machines and humans collaborate to fill in details and nudge the effort toward rigor.

But distinguishing between automated and interactive methods is something of a false dichotomy, according to Avigad.

"Even if you're doing math interactively, you want to automate the tedious parts as much as possible," Avigad said. "And even if you're using automation, whenever you try something that doesn't work, you make a change and try again. So, the two sides of the field kind of grow together."

Michael Kinyon, professor of mathematics at the University of Denver, also blurs this boundary when using the tools. That is, he uses automatic theorem provers, but he uses them as proof assistants. The work has shifted his approach to research questions.

"We may be a little quicker, for example, to try out something using the software first before sitting and staring off into space and trying to figure it out on their own," Kinyon said. "There's more of a willingness to be experimental in the early stages of these things."

That computers "think" differently than humans facilitates the discovery process for humans, according to some.

"That breaks us out of our patterns and deepens our understanding," Jordan Ellenberg, professor of mathematics at the University of Wisconsin, said. "You can think of formalization as teaching a mathematical idea to a machine, and the machine's 'mind' is very different from ours." In 2016, Ellenberg and Dion Gijswijt, professor of mathematics at the Delft University of Technology in the Netherlands, solved the cap set conjecture, a result that was later published[5] in the *Annals of Mathematics*.

Later, in 2019, mathematician Sander Dahmen and computer scientists Johannes Hölzl and Robert Lewis—all from Vrije Universiteit Amsterdam—used the Lean theorem prover to formalize[6] the proof of Ellenberg and Gijswijt's theorem. The authors deemed the combinatorial background necessary to formalize the proof as "less intimidating" than proofs from other math subfields. This formalization work provided some reassurance that, under favorable circumstances, one could computer-check a recent paper from the *Annals of Mathematics*. But a vanishingly small proportion of papers in this esteemed journal include short proofs with "less intimidating" prerequisites. Enthusiasts can hope that the software will improve over time.

On Elegance

Some may wonder whether machines reduce proofs to computations. But such reductionism is not at the heart of this work, according to Heather Macbeth, assistant professor of mathematics at Fordham University. That's because humans write proofs for understanding, and a computation may not offer that.

"You may think that when you move from paper to the computer, we throw [beauty] out the window, that we have decided that we're going to embrace function over form, and we're really going to move to a world in which what matters is getting the proof done," Macbeth said. "That's not the case." (Macbeth spoke at the Institute for Pure and Applied Mathematics conference.)

[5] `https://annals.math.princeton.edu/2017/185-1/p08`
[6] `https://drops.dagstuhl.de/opus/volltexte/2019/11070/`

Macbeth has spent three years writing thousands of lines of Lean code in an effort to formalize proofs. For two years, she has also collaborated with others to maintain the Lean mathematical library. In this role, she has reviewed thousands of lines of code written by others to assess whether the code can be improved and to ensure that it fits with the rest of the library.

"Some of it, I felt, was really beautiful, and some of it I haven't," Macbeth said. "Almost all principles for writing good mathematics on paper extend to the corresponding principles in formal mathematics."

Arguments contained within automated proofs do not always mimic patterns of human thought, according to Macbeth.

"There is such a thing as a good proof that, nonetheless, is not exactly the proof a human would have thought of," Macbeth said. Humans, she observed, in an attempt to be efficient, may prune a search space at various steps in the search for a proof. A machine, however, may check more cases, including those that initially may have seemed less than promising.

"This is really a question about where there are differences and why those differences are interesting," Macbeth said.

Also, some proofs gain elegance during formalization. Thomas Browning and Patrick Lutz, both mathematics graduate students at the University of California, Berkeley, undertook an effort to formalize Galois theory—a subfield many already deem elegant. But at one point in their work, they needed to work with arbitrary finite sequences, which are not among Lean's strengths, according to Browning and Lutz. So, they found a workaround.

"This maneuver is mostly just a way to avoid having to deal with certain types of arguments that don't work very smoothly in Lean," Browning and Lutz wrote in their paper. "But it does have the added benefit that some standard proofs become simpler when rewritten using this new induction principle."

An Accelerating Trend

Decades ago, machine-assisted proofs were viewed as fringe projects by mainstream mathematicians, according to Josef Urban, a distinguished researcher at the Czech Institute of Informatics, Robotics, and Cybernetics in Prague.

"High-profile mathematicians said that people like Tom Hales were wasting their talent by doing the proof of the Kepler conjecture formally," Urban said of the mathematician who, after responding to frustrations with the usual refereeing process, offered a machine-verified proof. "That really changed in the last 15 years, thanks especially to Hales."

To be sure, machine assisted proofs still have naysayers, including Michael Harris, professor of mathematics at Columbia University. Mathematicians who use theorem provers must learn to code and express problems in terms computers understand, which reduces time spent doing math.

"By the time I've reframed my question into a form that could fit into this technology, I would have solved the problem myself," Harris told[7] *Quanta Magazine*.

Doron Zeilberger, professor of mathematics at Rutgers University, is also not a fan, though his distaste begins with human-written proofs and extends to machine-assisted proofs.

"We don't need more 'formal versions' of human-generated mathematics," Zeilberger wrote on his blog[8] last year, while noting his appreciation for the pioneering work of Avigad and others. "Formalizing known proofs is not unlike Pig-Latin. Once you have done it a couple of times, it is no fun anymore. While it is both intellectually and technically challenging, or else such brilliant people would not engage in it, these people are wasting their talents."

"Indeed, our silicon servants, soon to become our masters, can be used much more fruitfully," Zeilberger continues. "Coq [another interactive theorem prover] and Lean continue the pernicious *Greek tradition*, that introduced the axiomatic method and made mathematics a deductive, logic-centric science."

But those views appear to go against the artificial-intelligence headwinds that appear to be driving society in the current moment, including the math community.

Kevin Buzzard, professor of pure mathematics at Imperial College London, gave[9] a talk, "The Rise of Formalism in Mathematics," at the 2022 International Congress of Mathematicians. There, he dubbed the mood not "the beginning of the end" but the "beginning of the beginning." In his view, mathematical researchers' jobs are not at risk. Machines, however, will increasingly help humans prove mathematical theorems not only by working out examples but also with reasoning. Computers will also help humans find proofs or counterexamples in databases, construct simple proofs, and generally make it easier for humans to learn mathematics, according to Buzzard.

The conference circuit appears to support this claim. In addition to the machine-assisted proofs conference at the Institute for Pure and Applied Mathematics held in February, other respected institutions are devoting time and space this year for mathematicians to gather and discuss this accelerating trend.

In May, the Banff International Research Station for Mathematical Innovation and Discovery hosted a

[7] https://www.quantamagazine.org/how-close-are-computers-to-automating-mathematical-reasoning-20200827/

[8] https://sites.math.rutgers.edu/~zeilberg/Opinion184.html

[9] https://www.youtube.com/watch?v=SEID4XYFN7o

Formalization of Cohomology Theories workshop[10] that brought together formalization experts and subject matter experts to make strides towards cutting-edge research. In June, the Simons Laufer Mathematical Sciences Institute introduced graduate students to the technology and ideas behind it in a Formalization of Mathematics summer graduate school program.[11] In July, MIT's Lorentz Center held a Machine-Checked Mathematics workshop[12] targeting mathematicians who have heard about, but not yet tried, the technology.

Outside of the United States, in June, the University of Copenhagen offered a masterclass[13] called Formalisation of Mathematics to build on the Scholtz's liquid tensor experiment. In July, the Lorentz Center in the Netherlands offered[14] Machine-checked Mathematics, a week-long introductory workshop for interested mathematicians intended to spur collaboration. Also in the summer, the Hausdorff Research Institute for Mathematics offered a program,[15] Prospects of Formal Mathematics, to provide a forum for experts and junior researchers to gather, collaborate, and "interface them better with the mathematical mainstream." The same institution offered[16] a week-long lecture series,[17] Formal Mathematics and Computer-Assisted Proving, in September. Many more may be found on the Lean Community website.[18] Also, Lean is not the only interactive theorem prover. Other systems include Coq, Isabelle/HOL, HOL Light, Agda, cubical Agda, Metamath, and Mizar, according to Buzzard.

"They digitized music—CD, MP3—but at the time nobody foresaw the consequences—Napster, Spotify," Buzzard said during his talk. "We're digitizing mathematics, and I believe this will inevitably change mathematics. You are welcome to join us."

[10]https://www.birs.ca/events/2023/5-day-workshops/23w5124
[11]https://www.msri.org/summer_schools/1021
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Susan D'Agostino

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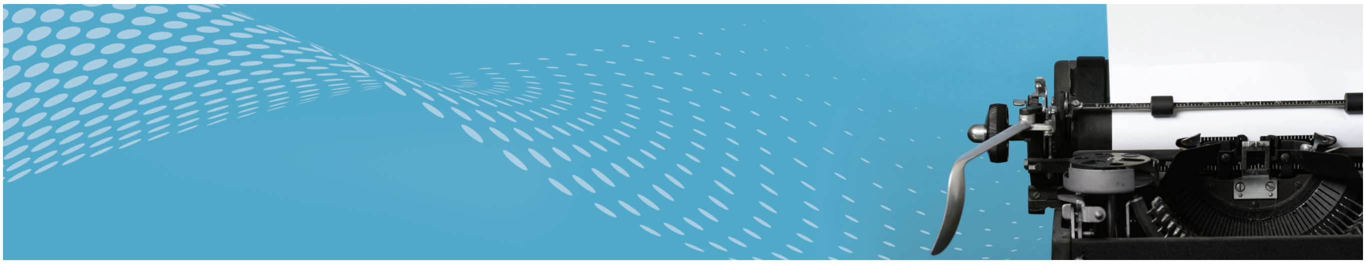

The Beauty of Roots

John C. Baez, J. Daniel Christensen, and Sam Derbyshire

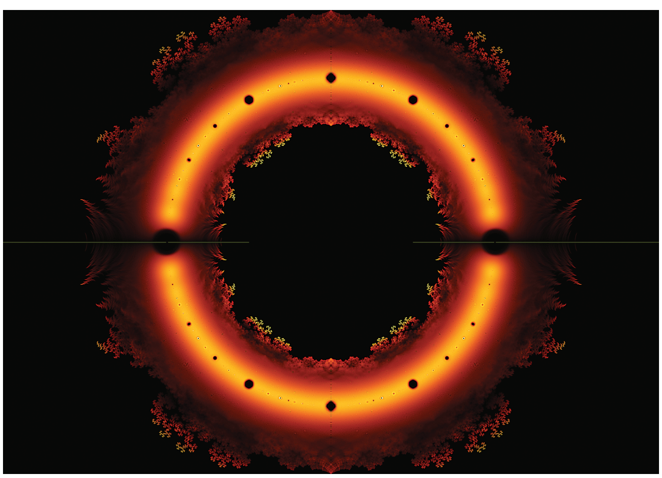

Figure 1. Roots of all polynomials of degree 23 whose coefficients are ± 1. The brightness shows the number of roots per pixel.

One of the charms of mathematics is that simple rules can generate complex and fascinating patterns, which raise questions whose answers require profound thought. For example, if we plot the roots of all polynomials of degree 23 whose coefficients are all 1 or -1, we get an astounding picture, shown in Figure 1.

More generally, define a **Littlewood polynomial** to be a polynomial $p(z) = \sum_{i=0}^{d} a_i z^i$ with each coefficient a_i equal to 1 or -1. Let $\mathbf{X}_n$ be the set of complex numbers that are roots of some Littlewood polynomial with n nonzero terms (and thus degree $n-1$). The 4-fold symmetry of Figure 1 comes from the fact that if $z \in \mathbf{X}_n$ so are $-z$ and $\bar{z}$. The set $\mathbf{X}_n$ is also invariant under the map $z \mapsto 1/z$, since if z is the root of some Littlewood polynomial then $1/z$ is a root of the polynomial with coefficients listed in the reverse order.

It turns out to be easier to study the set

$$\mathbf{X} = \bigcup_{n=1}^{\infty} \mathbf{X}_n = \{z \in \mathbb{C} | z \text{ is the root of some Littlewood polynomial}\}.$$

If n divides m then $\mathbf{X}_n \subseteq \mathbf{X}_m$, so $\mathbf{X}_n$ for a highly divisible number n can serve as an approximation to $\mathbf{X}$, and this is why we drew $\mathbf{X}_{24}$.

Some general properties of $\mathbf{X}$ are understood. It is easy to show that $\mathbf{X}$ is contained in the annulus $1/2 < |z| < 2$. On the other hand, Thierry Bousch showed [2] that the closure of $\mathbf{X}$ contains the annulus $2^{-1/4} \leq |z| \leq 2^{1/4}$. This means that the holes near roots of unity visible in the sets $\mathbf{X}_n$ must eventually fill in as we take the union over all

John C. Baez is a professor of mathematics at UC Riverside. His email address is john.baez@ucr.edu.

J. Daniel Christensen is a professor of mathematics at the University of Western Ontario. His email address is jdc@uwo.ca.

Sam Derbyshire is a Haskell consultant at Well-Typed. His email address is sam@well-typed.com.

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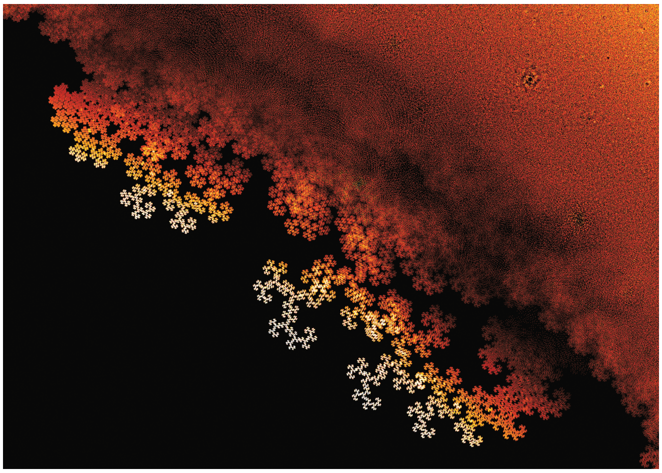

Figure 2. The region of $\mathbf{X}_{24}$ near the point $z = \frac{1}{2}e^{i/5}$.

n. More surprisingly, Bousch showed in 1993 that the closure $\overline{\mathbf{X}}$ is connected and locally path-connected [3]. It is worth comparing the work of Odlyzko and Poonen [7], who previously showed similar result for roots of polynomials whose coefficients are all 0 or 1.

The big challenge is to understand the diverse, complicated and beautiful patterns that appear in different regions of the set $\mathbf{X}$. There are websites that let you explore and zoom into this set online [4, 5, 8]. Different regions raise different questions.

For example, what is creating the fractal patterns in Figure 2 and elsewhere? An anonymous contributor suggested a fascinating line of attack which was further developed by Greg Egan [5]. Define two functions from the complex plane to itself, depending on a complex parameter q:

$$f_{+q}(z) = 1 + qz, \qquad f_{-q}(z) = 1 - qz.$$

When $|q| < 1$ these are both contraction mappings, so by a theorem of Hutchinson [6] there is a unique nonempty compact set $D_q \subseteq \mathbb{C}$ with

$$D_q = f_{+q}(D_q) \cup f_{-q}(D_q).$$

We call this set a **dragon**, or the **q-dragon** to be specific. And it seems that *for* $|q| < 1$, *the portion of the set* $\mathbf{X}$ *in a small neighborhood of the point* q *tends to look like a rotated version of* D_q.

Figure 3 shows some examples. To precisely describe what is going on, much less prove it, would take real work. We invite the reader to try. A heuristic explanation is known, which can serve as a starting point [1, 5]. Bousch [3] has also proved this related result:

Theorem. *For* $q \in \mathbb{C}$ *with* $|q| < 1$, *we have* $q \in \overline{\mathbf{X}}$ *if and only if* $0 \in D_q$. *When this holds, the set* D_q *is connected.*

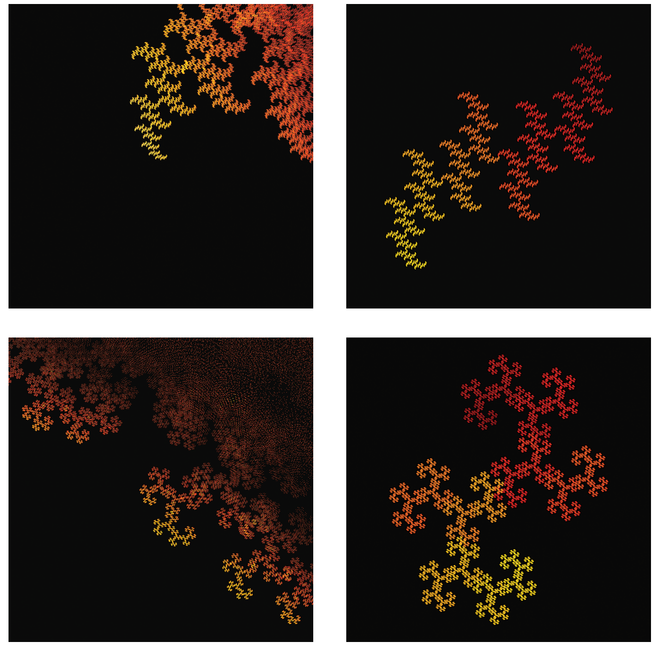

Figure 3. Top: the set $\mathbf{X}$ near $q = 0.594 + 0.254i$ at left, and the set D_q at right. Bottom: the set $\mathbf{X}$ near $q = 0.375453 + 0.544825i$ at left, and the set D_q at right.

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John C. Baez

J. Daniel Christensen

Credits

Figures 1–3 are courtesy of Sam Derbyshire.
Photo of John C. Baez is courtesy of Lisa Raphals.
Photo of J. Daniel Christensen is courtesy of Mitchell Zimmer.

AMS Prizes and Awards

I. Martin Isaacs Prize for Excellence in Mathematical Writing

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

About this Prize

The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in a primary journal and is not a criterion for this prize.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups—such as solvable groups—that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.

Next Prize: January 2025

Nomination Period: The deadline is March 31, 2024.

Nomination Procedure: `www.ams.org/isaacs-prize`

Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination, and a complete bibliographic citation for the article being nominated.

Joint Prizes and Awards

2024 MOS–AMS Fulkerson Prize

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society (AMS). Up to three awards of US$1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 25th International Symposium on Mathematical Programming to be held in Montreal, Canada, in the summer of 2024.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2018 through December 2023). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term "discrete mathematics" is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based only on their mathematical quality and significance.

Previous winners of the Fulkerson Prize are listed here: `www.mathopt.org/?nav=fulkerson#winners`.

Further information about the Fulkerson Prize can be found at `www.mathopt.org/?nav=fulkerson` and `https://www.ams.org/fulkerson-prize`.

The Fulkerson Prize Committee consists of

- Julia Böttcher (London School of Economics), MOS Representative
- Rosa Orellana (Dartmouth College), AMS Representative
- Dan Spielman (Yale University), Chair and MOS Representative

Please send your nominations (including reference to the nominated article and an evaluation of the work) by February 15, 2024 to the chair of the committee:

Professor Daniel Spielman

Email: `daniel.spielman@yale.edu`

POSITION AVAILABLE

Executive Director

AMERICAN MATHEMATICAL SOCIETY

POSITION

The Trustees of the American Mathematical Society invite applications for the position of Executive Director of the Society. The Executive Director has the opportunity to strongly influence all activities of the Society, as well as the responsibility of overseeing a large and diverse spectrum of people, programs, and publications. **The desired starting date is February 1, 2024.**

DUTIES AND TERMS OF APPOINTMENT

The American Mathematical Society, founded in 1888 to further the interests of mathematical research and scholarship, serves the national and international community through its publications, meetings, advocacy, and other programs. The AMS promotes mathematical research and its communication and uses; encourages and promotes the transmission of mathematical understanding and skills; supports mathematical education at all levels; advances the status of the profession of mathematics, encouraging and facilitating the full participation of all individuals; and fosters an awareness and appreciation of mathematics and its connections to other disciplines and everyday life.

These aims are pursued mainly through an active portfolio of programs, publications, meetings, conferences, and advocacy. The Society is a major publisher of mathematical books and journals, including MathSciNet®, an organizer of numerous meetings and conferences each year, and a sponsor of grants and training programs. The Society's headquarters are located in Providence, Rhode Island, and the Executive Director is based there. The society also maintains a print shop in Pawtucket, Rhode Island; an office in Washington, DC, that houses the Office of Government Relations and the Office of Equity, Diversity, and Inclusion; and an office in Ann Arbor, Michigan, that publishes MathSciNet.

The Executive Director is the principal executive officer of the Society and is responsible for the execution and administration of the policies of the Society as approved by the Board of Trustees and by the Council. The Executive Director is a full-time employee of the Society and is responsible for the operation of the Society's offices in Providence, and Pawtucket, RI; Ann Arbor, MI; and Washington, DC. The Executive Director attends meetings of the Board of Trustees, the Council, and the Executive Committee, is an ex-officio (nonvoting) member of the policy committees of the Society, and is often called upon to represent the Society in its dealings with other scientific and scholarly bodies.

The Society employs a staff of over 200 in the four offices. The directors of the various divisions report directly to the Executive Director. Information about the operations and finances of the Society can be found in its Annual Reports, available at www.ams.org/annual-reports.

The Executive Director is appointed by and serves at the pleasure of the Trustees. The terms of appointment, salary, and benefits will be consistent with the nature and responsibilities of the position and will be determined by mutual agreement between the Trustees and the prospective appointee.

DESIRED QUALIFICATIONS

The successful candidate must be a leader, and we seek candidates who additionally have as many as possible of the following:

- A doctoral degree (or equivalent) in mathematics or a closely related field.
- Substantial experience and demonstrated visibility as a professional mathematician in academic, industrial, or governmental employment, with success in obtaining and administering grants.
- Extensive knowledge of the Society, the mathematics profession, and related disciplines and organizations, with a thorough understanding of the mission that guides the Society.
- Excellent communication skills, both written and oral, and an enthusiasm for public outreach.
- Demonstrated sustained commitment to diverse, inclusive, and equitable organizational environments and substantial experience in advancing equity, diversity, and inclusion priorities in the mathematical community.
- Demonstrated leadership ability supported by strong organizational and managerial skills.
- Familiarity with the mathematical community and its needs, and an ability to work effectively with mathematicians and nonmathematicians.
- Strong interest in engaging in fundraising and enjoyment of social interactions.

APPLICATIONS PROCESS

A search committee co-chaired by **Joseph Silverman** (joseph_silverman@brown.edu) and **Bryna Kra** (kra@math.northwestern.edu) has been formed to seek and review applications. All communication with the committee will be held in confidence. Suggestions of suitable candidates are most welcome.

Applicants should submit a CV and a letter of interest on MathJobs. The letter should be at most four pages, explaining your interest in being the Executive Director of the AMS and why you consider yourself to be a compelling candidate. The majority of the letter should discuss your major accomplishments and experiences that illustrate your leadership philosophy and address the desired qualifications for the position. **Applications received by September 15, 2023 will receive full consideration.**

AMS American Mathematical Society

Advancing research. Creating connections.

The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, veteran status, or immigration status.

YOUR VOTE COUNTS! AMS

2024 Election

Call for Suggestions

YOUR SUGGESTIONS ARE WANTED BY:

the Nominating Committee, for the following contested seats in the 2024 AMS elections:

vice president, trustee, and five members at large of the Council.

Deadline for suggestions: November 1, 2023

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the president, for the following contested seats in the 2024 AMS elections:

three members of the Nominating Committee and two members of the Editorial Boards Committee.

Deadline for suggestions: January 31, 2024

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the Editorial Boards Committee, for appointments to various editorial boards of AMS publications.

Deadline for suggestions: Can be submitted any time

Send your suggestions for any of the above to:
Boris Hasselblatt, Secretary
American Mathematical Society
201 Charles Street
Providence, RI 02904-2213, USA
`secretary@ams.org`
or submit them online at `www.ams.org/committee-nominate`

Applications Open for

AMS CONGRESSIONAL FELLOWSHIP 2024–2025

Apply your mathematics knowledge toward solutions to societal problems.

The American Mathematical Society will sponsor a Congressional Fellow from September 2024 through August 2025.

The Fellow will spend the year working on the staff of either a member of Congress or a congressional committee, working in legislative and policy areas requiring scientific and technical input.

The Fellow brings his/her/their technical background and external perspective to the decision-making process in Congress. Prospective Fellows must be cognizant of and demonstrate sensitivity toward political and social issues and have a strong interest in applying personal knowledge toward solutions to societal problems.

Now in its 19th year, the AMS Congressional Fellowship provides a unique public policy learning experience, and demonstrates the value of science–government interaction. The program includes an orientation on congressional and executive branch operations, and a year-long seminar series on issues involving science, technology, and public policy.

"Every day on the Hill is new and exciting! Whether I am writing press releases, synthesizing complicated ideas into one-pagers, or meeting influential individuals, I learn something new every day. "

—Duncan Wright,
AMS Congressional
Fellow 2022–2023

Applicants must have a PhD or an equivalent doctoral-level degree in the mathematical sciences by the application deadline (February 1, 2024). Applicants must be US citizens. Federal employees are not eligible.

The Fellowship stipend is US$100,479 for the Fellowship period, with additional allowances for relocation and professional travel, as well as a contribution toward health insurance.

Applicants must submit a statement expressing interest and qualifications for the AMS Congressional Fellowship as well as a current curriculum vitae. Candidates should have three letters of recommendation sent to the AMS by the February 1, 2024 deadline.

For more information and to apply, please go to **www.ams.org/ams-congressional-fellowship**.

Deadline for receipt of applications: **February 1, 2024**

AMS AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

jmm 2023

Learn more at the JMM 2024 session on DC-based Policy and Communications Opportunities to be held Friday, January 5, 2024 at 4:30 p.m.

How to Throw a Math Party for 500 People

Elaine Beebe

In April 2023, 564 people flocked to the University of Cincinnati (UC) to immerse themselves in math at the American Mathematical Society's spring meeting for the central section.

They represented 42 US states and the District of Columbia. Other guests traveled from Canada (seven), Japan (three), and Haiti, Mexico, and Sweden (one each).

"I heard from so many participants, many of whom hadn't traveled much in recent years, how important this opportunity to reconnect with colleagues was," said AMS Associate Secretary for the Central Section Betsy Stovall, professor of mathematics, University of Wisconsin-Madison. As one of four AMS associate secretaries, Stovall plans two sectional meetings per year.

During two weekend days, 468 speakers presented 481 abstracts. Thirty special sessions were composed of 102 sub-sessions. Two contributed paper sessions were held, and four invited addresses took place.

"As a conference host, I'm just in shock at how much went on in the span of a single weekend," said Michael Goldberg, UC math department chair and professor. "Then again, I've never thrown a party for 500 people before."

"Fantastic weekend indeed," said Eyvindur Ari Palsson, associate professor of mathematics at Virginia Tech, who organized a special session and presented research in Cincinnati.

We asked Palsson, Stovall, Goldberg, and other behind-the-scenes players for their advice to prospective hosts of sectional meetings. Here's what they had to say.

Elaine Beebe is the communication and outreach content specialist at the American Mathematical Society. Her email address is exb@ams.org.

DOI: https://doi.org/10.1090/noti2769

Plan Ahead

Due to the COVID-19 pandemic, UC waited four years to host the spring sectional meeting it had begun to organize in 2019.

"Typically, the AMS chooses a location for its meetings about 18 months to two years in advance. In our case we arranged in 2019 to hold the 2021 Spring Central Sectional meeting at the University of Cincinnati," Goldberg said.

About one year in advance, the AMS visits a meeting location to confirm the suitability of the configuration of classrooms, auditoriums, public space, and facilities. Stovall and her checklist got plans moving.

"I have a questionnaire that I go through with the local organizers," the associate secretary said. "This helps them think about what reservations and other advance planning they might need to do ahead of time—for instance, to ensure accessible facilities. And it also helps the AMS communicate to all of the meeting participants what facilities are available, what to expect in terms of A/V equipment, etc."

"I'd really encourage departments that are considering hosting a sectional to go for it, because they bring a lot to the mathematical community and give the department a chance to shape that," Stovall said. A sectional meeting "brings a great deal of scientific activity to the host department and creates opportunities for local mathematicians to organize sessions, give talks, and attend talks without having to travel," she said.

"I think these meetings can seem really daunting for departments because of the scale, but the AMS meetings staff is able to provide a lot of logistical assistance, and the model of having lots of special sessions means the work is more distributed than at a regular conference," Stovall added.

Palsson and Krystal Taylor of Ohio State University, friends since graduate school, assembled the AMS Special

Session "Interface of Geometric Measure Theory and Harmonic Analysis" in seven months. Creating the special session was a natural progression after Taylor, associate professor of mathematics, was selected to deliver an invited address.

First, Taylor and Palsson brainstormed the precise theme and title of the session. "Once this was decided, we identified and reached out to potential speakers related to this theme with an emphasis on highlighting a diverse group of speakers from a range of institutions and career stages," Palsson said.

"Speakers who accepted submitted their talk title and abstract in February. Once we knew the precise topics, we organized the schedule accordingly, to group thematic areas," he said.

"Eyvi put together a list of old friends combined with many new faces," Taylor said. "He mixed in some unexpected names from tangential areas, but people catered their talks to the name of the session and the crowd so it really worked."

Stovall agreed. "This session was at the interface of a few different areas—harmonic analysis, combinatorics, and geometric measure theory—and, as such, brings together people who might not normally go to the same conferences," she said. "It also had a great mix of early-career researchers and those who are further along in their careers."

Considering your own special session? "I'd encourage potential organizers to go for it and just submit a session proposal," Stovall said. "I especially encourage organizers to plan ahead for a diverse lineup of speakers, along many axes, and to think about including both familiar faces and also those who might not be invited so frequently to speak."

Form a Team

"The AMS has a detailed checklist of what needs to be in place for a sectional meeting," Goldberg said. "My role has been to connect the dots between what's on that checklist and what resources—locations, people, services—are available on our campus."

Math department business manager Nancy Diemler spearheaded logistics. A sign on Diemler's office door reads "UC Mental Health Champion," which means she has trained in methods to support student, faculty, and staff mental health. Diemler proved to be invaluable during the sectional, from managing student workers to coordinating camera crews, dining with luminaries to clearing classrooms of forgotten items after the meeting (because someone had to). That weekend, she made sure to have available 50 gallons of regular coffee, eight gallons of decaf coffee, and five gallons of hot water for tea.

UC Math Professor Robert Buckingham, whose research focuses on the asymptotic analysis of problems from probability and differential equations, took the lead on fundraising and managing the budget for the meeting. The team pitched in to publicize the meeting in the math department and around the university, advertising the AMS Einstein Public Lecture to the broader community.

During the meeting, Goldberg said he hoped "to be just another one of the 500 participants, talking and listening to people at the forefront of mathematical research." But if anything came up, Diemler had him on speed-dial.

Have a Plan B

Nathaniel Whitaker, interim dean of the College of Natural Sciences at the University of Massachusetts Amherst, arrived in Cincinnati the day before he was to deliver the Einstein Public Lecture.

That was the only part of his day's travel plan to be executed seamlessly.

Flights were delayed. Changing planes in Detroit might have stranded Whitaker, a numerical analyst who develops algorithms to solve physical and biological problems described by differential equations.

Upon learning that his midday connecting flight was canceled, Whitaker decided to rent a car and drive the four hours from Detroit to Cincinnati. He arrived an hour before his dinner reservation, tired, but cheerful and unflapped.

Flexibility is key, Palsson said. "We had a speaker whose flight was canceled, so they didn't make it to Cincinnati until after their talk, but due to cancellations from other speakers, we were able to fit in the latecomer later in the day."

That was Brian McDonald of the University of Rochester, who presented his 8:30 a.m. paper, "Paths and cycles in distance graphs over finite fields," at 4:30 p.m. No one in the classroom appeared troubled by this rearrangement.

Clone Yourself?

"Needing to be several places at once is a big part of the AMS meeting experience," Goldberg observed.

"Besides the special session I was involved with organizing, there were somewhere between three and six others I would have enjoyed attending at the same time. In fact, one of my co-organizers had to duck out for a while so he could go present a talk in another session," the UC math chair said. "I presented a talk in another session too, but by some miracle there wasn't a time conflict."

Palsson and Taylor suggested sharing the workload of session organizing. "It would have been easier if we had just one or two more organizers," Taylor said. "I was running back and forth to pump for my baby and to attend some other sessions, and a lot of responsibility fell on Eyvi's shoulders."

"As a participant, the meeting was wildly successful, but too short to do everything," Goldberg said. "There were a few hiccups as always—one speaker had to cancel due to a missed flight connection, the room A/V liked some input types better than others—but the show went on as it's supposed to."

During two days in April, Goldberg accomplished quite a bit. He met with several collaborators he hadn't seen in person for a long time, to plan out their next research projects. He caught up with a friend he hadn't seen in 20 years.

He met a potential future collaborator, who had developed an entirely new tool kit for solving some research problems where Goldberg has always gotten stuck.

He also attended "about a dozen and a half talks" covering different research topics in analysis and differential equations. Goldberg attended plenary talks and learned about connections between numerical analysis and algebraic topology which he "never would have suspected."

And he presented a talk on his favorite research problem and was asked a lot of good questions: "most of which," Goldberg said, "I can't completely answer. Yet."

AMS Updates

Maddock Named AMS Interim Executive Director

Lucy Maddock was appointed interim executive director of the American Mathematical Society (AMS) as of July 1, 2023. Maddock is chief financial officer (CFO) and associate executive director for finance and administration at the AMS.

"We're delighted that Lucy Maddock has agreed to serve as interim executive director of the American Mathematical Society, continuing her outstanding business decision-making leadership for the past three years at the AMS," said Bryna Kra, AMS president.

Maddock began her term as interim executive director serving alongside Executive Director Catherine Roberts, who transitioned into an advisory role before leaving the AMS at the end of August 2023. Upon the installation of a new AMS executive director, Maddock will resume her role of CFO.

"Lucy has served as a senior leader in organizations for more than two decades," said Joseph Silverman, chair of the AMS board of trustees. "We will look to this extensive experience to inform and guide the AMS through the transitions of the coming months."

A key member of the AMS senior leadership team, Maddock has been responsible for maintaining the organization's overall fiscal health, helping to establish financial targets and assessing the results. As the AMS's primary financial advisor, she has managed the processes and procedures to ensure an audit-ready position and has overseen contract creation, management, and compliance. Maddock's duties include supervising the fiscal department and the directors of human resources and of facilities and purchasing at the AMS.

"I am honored to be appointed interim executive director," Maddock said. "I look forward to continuing the work of the AMS to serve the entire mathematical community. Catherine Roberts positioned the AMS for a thriving future in many ways."

—AMS Communications

DOI: https://doi.org/10.1090/noti2788

Call Issued for 2025 MRC Organizer Proposals

The Mathematics Research Communities (MRC) program of the American Mathematical Society (AMS) is accepting organizer proposals for the 2025 summer conferences. We seek proposals focusing in any area of pure, applied, or interdisciplinary mathematics, as well as proposals focused on problems of relevance in the business, entrepreneurship, government, industry, and nonprofit (BEGIN) arenas.

The MRC program is aimed at jumpstarting collaborative groups of early-career researchers in new, rapidly evolving areas of mathematics and its applications. Central to the program are four summer conferences organized by teams of investigators who can engage early-career individuals in hands-on research and guide them in developing as professionals. We are looking for creative proposals that involve groups of 20 or 40 early-career mathematicians for an intensive week of collaborative problem-solving, research, and professional development.

Full proposals are due November 30, 2023. For more information, please refer to the document called "How to Develop a Proposal" at `http://www.ams.org/programs/research-communities/mrc-proposals-25`.

—AMS Programs

Math Matters in Supreme Court Case

Mathematics, gerrymandering, and voting rights were at the heart of a June 2023 decision by the US Supreme Court that the Alabama legislature drew congressional maps that minimized Black voting power in the state.

"In considering redistricting in Alabama, the Supreme Court received several amici briefs and examined alternate maps submitted by mathematical scientists," said Karen Saxe, AMS associate executive director and director of government relations. In the trial, a crucial brief submitted by three computational redistricting experts explained that the number of possible Alabama districting maps is at minimum in the "trillion trillions."

The 112-page decision regularly cited Moon Duchin, a Tufts University geometer, and Kosuke Imai, a Harvard University professor with joint appointments in the departments of government and statistics. Duchin's name was mentioned 46 times, in such contexts as this footnote: "After all, as Duchin explained, any map produced in a deliberately race-predominant manner would necessarily emerge at some point in a random, race-neutral process."

Duchin and Imai helped write the 2018 AMS Policy Statement on Drawing Voting Districts and Partisan Gerrymandering, which the AMS coauthored with the American Statistical Association. Read the full policy statement at `https://www.ams.org/about-us/governance/policy-statements/gerrymandering`.

"That statement notes that modern mathematical, statistical, and computing methods can be used to identify district plans that give one of the parties an unfair advantage in elections," Saxe said. "It is very good to see these methods being presented in court cases, and—specifically and currently--being referred to in the SCOTUS June ruling that Alabama maps likely diluted Black votes."

To read more about how mathematicians are working to end gerrymandering, download a 2022 article from Notices of the AMS at `https://www.ams.org/journals/notices/202204/rnoti-p616.pdf`.

—*AMS Communications*

Deaths of AMS Members

Robert D. Bechtel, of Denver, Colorado, died on May 13, 2018. Born on April 2, 1931, he was a member of the Society for 57 years.

Richard Body, of Canada, died on April 5, 2022. Born on August 2, 1946, he was a member of the Society for 1 year.

Mary V. Connolly, of South Bend, Indiana, died on June 15, 2023. Born on November 1, 1939, she was a member of the Society for 54 years.

Carl C. Ganser, of Vineyard Haven, Massachusetts, died on April 15, 2021. Born on August 7, 1934, he was a member of the Society for 59 years.

Jay R. Goldman, of Cambridge, Massachusetts, died on August 13, 2022. Born on August 2, 1940, he was a member of the Society for 60 years.

Patricia Clark Kenschaft, of Montclair, New Jersey, died on November 20, 2022. Born on March 25, 1940, she was a member of the Society for 58 years.

Ronald M. Mathsen, of Wausau, Wisconsin, died on August 10, 2022. Born on October 6, 1938, he was a member of the Society for 58 years.

Bengt C. H. Nagel, of Sweden, died on September 21, 2016. Born on January 17, 1927, he was a member of the Society for 34 years.

L. Raphael Patton, of Moraga, California, died on December 6, 2021. Born on January 14, 1942, he was a member of the Society for 48 years.

Alan H. Schoen, of Carbondale, Illinois, died on July 26, 2023. Born on December 11, 1924, he was a member of the Society for 42 years.

Herbert A. Steinberg, of Armonk, New York, died on July 28, 2023. Born on September 19, 1929, he was a member of the Society for 71 years.

Michael Voichick, of Madison, Wisconsin, died on July 1, 2021. Born on May 28, 1934, he was a member of the Society for 60 years.

D. Wales, of Pasadena, California, died on July 17, 2023. Born on July 31, 1939, he was a member of the Society for 61 years.

Kenneth G. Whyburn, of Atherton, California, died on June 19, 2022. Born on September 3, 1944, he was a member of the Society for 57 years.

Dorothy W. Wolfe, of Haverford, Pennsylvania, died on April 5, 2023. Born on August 20, 1920, she was a member of the Society for 60 years.

NEWS

Mathematics People

Lieb Wins 2023 Kyoto Prize

Elliott H. Lieb, professor emeritus at Princeton University, won the 2023 Kyoto Prize in Basic Sciences.

According to a press release, Lieb "is one of the intellectual giants in the field of mathematical sciences. ...Primarily through his achievements in many-body physics, [Lieb] established a foundation for mathematical research in fields such as physics, chemistry, and quantum information science. His contributions to the development of mathematical analysis are significant as well."

"I am deeply honored to have been selected for the Kyoto Prize," Lieb said. "Its founder, Dr. Kazuo Inamori, and I share not only a birth year but also a philosophy that has guided my activities in research and education."

Established in 1984 by the Inamori Foundation, the Kyoto Prize is an international award to "individuals who have contributed significantly to the scientific, cultural, and spiritual betterment of humankind." The prize annually honors an individual in the fields of advanced technology, arts and philosophy, and basic sciences (which includes pure mathematics in a five-year cycle).

Lieb will receive the prize, which includes a diploma, a medal made of twenty-carat gold, and a cash award of 100 million yen (approximately US $700,000), at a ceremony in Kyoto, Japan, on November 10, 2023.

"It is especially gratifying to see my work in mathematics and physics recognized in a city that played an important role in my cultural and scientific life," Lieb added. "It was in Kyoto in 1956 that I wrote my first postdoctoral paper and where my immersion in Japanese culture made a profound and continuing impact on my life."

A life member of the American Mathematical Society, Lieb joined the AMS in 1969 and was named to its inaugural class of fellows in 2013.

—AMS Communications

DOI: https://doi.org/10.1090/noti2790

Drinfeld, Yau Receive 2023 Shaw Prize in Mathematical Sciences

The Shaw Prize in Mathematical Sciences was awarded today in equal shares to Vladimir Drinfeld of the University of Chicago and Shing-Tung Yau of Tsinghua University. Drinfeld and Yau were recognized for their contributions related to mathematical physics, to arithmetic geometry, to differential geometry, and to Kähler geometry, according to the prize citation.

Drinfeld, the Harry Pratt Judson Distinguished Service Professor of Mathematics at UChicago, received a PhD from Moscow State University in 1978. He was appointed assistant professor at Bashkir State University in 1978 and lecturer at Kharkov State University in 1980. Drinfeld then served as research fellow at B. Verkin Institute for Low Temperature Physics and Engineering from 1981 to 1998. He has been a professor of mathematics at Chicago since 1998. Drinfeld is a member of the American Academy of Arts and Sciences and a fellow of the Academy of Sciences, Ukraine.

Yau is currently director of the Yau Mathematical Sciences Center at Tsinghua University. He received a PhD in 1971 from the University of California, Berkeley. Yau was a member (1971–1972) and professor (1980–1984) of the Institute for Advanced Study (IAS) at Princeton and assistant professor (1972–1974) at the State University of New York at Stony Brook. Yau joined Stanford University, where he was successively associate professor and full professor (1974–1979). In 1984, Yau moved to the University of California at San Diego as professor (1984–1987). He then joined Harvard University, where he has been a distinguished professor (from 1987), director of the Institute of Mathematical Sciences (from 1994) and also professor in the Department of Physics (from 2013), becoming emeritus in 2022. Yau has been a distinguished professor-at-large at the Chinese University of Hong Kong since 2003. He is a member of the Chinese Academy of Sciences, the US National Academy of Sciences, and the American Academy of Arts and Sciences, and an inaugural fellow and member of the AMS.

In its twentieth year, the Shaw Prize consists of the annual naming of laureates in astronomy, life science and medicine, and mathematical sciences. Each prize bears a monetary award of US $1.2 million. Laureates will receive their awards at a presentation ceremony scheduled for November 12, 2023, in Hong Kong.

—*Shaw Prize press release*

MAA Welcomes New Director of Project NExT

Christine Kelley, University of Nebraska-Lincoln (UNL) mathematics professor, joined the Mathematical Association of America (MAA) as the new director of the professional development program MAA Project NExT (**N**ew **Ex**periences in **T**eaching).

Launched in 1994, MAA Project NExT addresses all aspects of an academic career: improving the teaching and learning of mathematics, engaging in research and scholarship, identifying interesting service opportunities, participating in professional activities, and creating a network of peers and mentors. More than 1,700 fellows from a wide variety of institutions have participated in this hallmark program of the MAA: the world's largest community of mathematicians, students, and enthusiasts.

Kelley participated in Project NExT in 2008–2009 as an early-career faculty member. "Project NExT taught me how to think critically about teaching, exposed me to diverse teaching methods, and gave me practical tools and training to succeed as a new faculty member in mathematics," she said. "Moreover, it gave me an instant community and a sense of belonging in the field."

Kelley joined the AMS in 2010. She also is a member of AWM, MAA, SIAM, and IEEE (Information Theory Society).

—*Mathematical Association of America*

Classified Advertising
Employment Opportunities

ILLINOIS

University of Chicago—Assistant Professor

Position Description

The University of Chicago Department of Mathematics invites applications for the position of Assistant Professor. Successful candidates are typically two to four years past the PhD. These positions are intended for mathematicians whose work has been of outstandingly high caliber. Appointees are expected to have the potential to become leading figures in their fields. The appointment is generally for four years, with the possibility for renewal. The teaching obligation is up to three one-quarter courses per year.

Applicants are strongly encouraged to include additional information related to their teaching experience, such as evaluations from courses previously taught, as well as an AMS cover sheet. If you have applied for an NSF Mathematical Sciences Postdoctoral Fellowship, please include that information in your application, and let us know how you plan to use it if awarded. Questions may be directed to `matwimberly@uchicago.edu`. We will begin screening applications on October 1, 2023. Screening will continue until all available positions are filled.

Qualifications

Completion of a PhD in mathematics or a closely related field is required at the time of appointment.

Application Instructions

Required materials are: (a) a cover letter, (b) a curriculum vitae, (c) three or more letters of reference, at least one of which addresses teaching ability, (d) a description of previous research and plans for future mathematical research and (e) a teaching statement.

Applications must be submitted online through `https://www.mathjobs.org/jobs/list/22662`.

Equal Employment Opportunity Statement

All University departments and institutes are charged with building a faculty from a diversity of backgrounds and with diverse viewpoints; with cultivating an inclusive community that values freedom of expression; and with welcoming and supporting all their members.

We seek a diverse pool of applicants who wish to join an academic community that places the highest value on rigorous inquiry and encourages diverse perspectives, experiences, groups of individuals, and ideas to inform and stimulate intellectual challenge, engagement, and exchange. The University's Statements on Diversity are at `https://provost.uchicago.edu/statements-diversity`.

The University of Chicago is an Affirmative Action/Equal Opportunity/Disabled/Veterans Employer and does not discriminate on the basis of race, color, religion, sex, sexual orientation, gender identity, national or ethnic origin, age, status as an individual with a disability, protected veteran status, genetic information, or other protected classes under the law. For additional information please see the University's Notice of Nondiscrimination.

Job seekers in need of a reasonable accommodation to complete the application process should call 773-834-3988 or email equalopportunity@uchicago.edu with their request.

10

University of Chicago—L.E. Dickson Instructor

Position Description

The University of Chicago Department of Mathematics invites applications for the position of L.E. Dickson Instructor. This is open to mathematicians who have recently completed or will soon complete a doctorate in mathematics or a closely related field, and whose work shows remarkable promise in mathematical research. The initial appointment is for a term of up to three years. The teaching obligation is generally four one-quarter courses per year. If you are two or more years post PhD, we encourage you to apply for our Assistant Professor position as well at https://www.mathjobs.org/jobs/list/22662.

Applicants are strongly encouraged to include additional information related to their teaching experience, such as evaluations from courses previously taught, as well as an AMS cover sheet. If you have applied for an NSF Mathematical Sciences Postdoctoral Fellowship, please include that information in your application, and let us know how you plan to use it if awarded. Questions may be directed to matwimberly@uchicago.edu. We will begin screening applications on October 1, 2023. Screening will continue until all available positions are filled.

Qualifications

Completion of all requirements for a PhD in mathematics or a closely related field is required at the time of appointment.

Application Instructions

Required materials are: (a) a cover letter, (b) a curriculum vitae, (c) three or more letters of reference, at least one of which addresses teaching ability, (d) a description of previous research and plans for future mathematical research and (e) a teaching statement.

Applications must be submitted online through https://www.mathjobs.org/jobs/list/22661.

Equal Employment Opportunity Statement

All University departments and institutes are charged with building a faculty from a diversity of backgrounds and with diverse viewpoints; with cultivating an inclusive community that values freedom of expression; and with welcoming and supporting all their members.

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Job seekers in need of a reasonable accommodation to complete the application process should call 773-834-3988 or email equalopportunity@uchicago.edu with their request.

14

NEW JERSEY

The Mathematics Department of Rutgers University–New Brunswick invites applications for the following positions which may be available starting September 2024.

Tenure Stream Faculty:

Contingent upon the availability of funding, the department plans to hire a named chair at the Full Professor level, and to also hire at the level of tenure-track Assistant Professor with a particular emphasis in the areas of algebra, algebraic geometry, number theory, discrete mathematics, and data science. However, all exceptional candidates at any level will be considered.

Candidates must have a PhD and have a strong record of research accomplishments in mathematics as well as effective teaching. More details on the search, including priority areas, can be found in the forthcoming Rutgers University–New Brunswick ad listing on mathjobs.org. The normal teaching load for research-active faculty is 2–1.

Review of applications begins November 1, 2023; applications after that date may be considered if the positions have not been filled.

Hill and Other Assistant Professorships:

The Hill Assistant Professorship is a three-year non-tenure-track, nonrenewable, postdoctoral appointment. These positions carry a teaching load of 2–1 for research. Contingent upon funding, we will be offering one or more of these positions. Other postdoctoral positions of varying duration and teaching loads may also be available. Candidates should have received a PhD and show outstanding promise of research ability in pure or applied mathematics, as well as a capacity for effective teaching. Review of applications begins December 1, 2023; applications after that date may be considered if the positions have not been filled.

Teaching Faculty:
Teaching Instructor or Assistant/Associate Teaching Professor in Mathematics. Contingent upon the availability of funding we may have one or more renewable non-tenure-track teaching positions at the level of Teaching Instructor, with the possibility of appointment at a higher level (Assistant or Associate Teaching Professor) for exceptionally well-qualified candidates.

This position has a teaching load of 4–4 (four undergraduate courses in each of the fall and spring semesters), with possible teaching reductions for administrative, advising, or other departmental duties. Applicants should normally have received a PhD and show outstanding evidence of teaching at the undergraduate level. Please see our posting on mathjobs.org.

Application Procedure:
The positions will be posted at mathjobs.org by this fall under Rutgers–New Brunswick.

An applicant should complete the application at mathjobs.org. In addition to the mathjobs application, for each position for which you apply, you must also complete a basic employment application on the Rutgers employment website. The mathjobs listing for each position includes a link to this basic application.

Rutgers, the State University of New Jersey, is an Equal Opportunity / Affirmative Action Employer. Qualified applicants will be considered for employment without regard to race, creed, color, religion, sex, sexual orientation, gender identity or expression, national origin, disability status, genetic information, protected veteran status, military service, or any other category protected by law. As an institution, we value diversity of background and opinion, and prohibit discrimination or harassment on the basis of any legally protected class in the areas of hiring, recruitment, promotion, transfer, demotion, training, compensation, pay, fringe benefits, layoff, termination or any other terms and conditions of employment.

11

NEW YORK

Director, Engineering Student Success Program

The Albert Nerken School of Engineering of The Cooper Union invites applications for a non-tenure track staff position to be the inaugural director of a new engineering student success program. Candidates with strong backgrounds in mathematics are particularly encouraged to apply. For a complete description, please visit our website: `https://cooper.edu/work/employment-opportunities/director-engineering-student-success-program`.

8

RHODE ISLAND

Brown University—Mathematics Department

J. D. Tamarkin Assistant Professorship: One or more three-year non-tenured nonrenewable appointments, beginning July 1, 2024. The teaching load is one course one semester, and two courses the other semester and consists of courses of more than routine interest. Candidates are required to have received a PhD degree or equivalent by the start of their appointment, and they may have up to three years of prior academic and/or postdoctoral research experience. Applicants should have a strong research potential, demonstrated excellence in teaching, and a commitment to building a diverse and inclusive community in Mathematics. Field of research should be consonant with the current research interests of the department.

For full consideration, applicants must submit a curriculum vitae, an AMS Standard Cover Sheet, at least three letters of recommendation primarily focused on research, and one letter addressing teaching by November 17, 2023. Applicants are required to identify a Brown faculty member with similar research interests. The cover letter should address the applicant's commitment to diversity in terms of teaching, research, and activities in the math community, OR applicants may attach a diversity statement if desired. (Later applications will be reviewed to the extent possible.)

Please submit all application materials online at `http://www.mathjobs.org`.

As an EEO/AA employer, Brown University provides equal opportunity and prohibits discrimination, harassment and retaliation based upon a person's race, color, religion, sex, age, national or ethnic origin, disability, veteran status, sexual orientation, gender identity, gender expression, or any other characteristic protected under applicable law, and caste, which is protected by our University policies.

13

TEXAS

Baylor University
Postdoctoral Fellow, Mathematics

Baylor University seeks a postdoctoral fellow in Mathematics to start in August 2024. Details for this position can be found at `https://www.mathjobs.org/jobs/list/22670`. Applications must be completed and received by November 26, 2023.

This position is on a renewable twelve-month contract potentially leading to a maximum appointment of three years. Special consideration will be given to applicants with interests aligned with areas of research in the department that include algebra, analysis, applied/computational mathematics, differential equations, mathematical physics, numerical analysis, representation theory, and topology, with potential interdisciplinary applications.

Located in Waco, Texas, Baylor University is the oldest college in Texas. With a population of around 21,000 students, Baylor is one of the top universities in the nation, having just been named an R1 institution by the Carnegie Classification in 2022. Baylor is also on the honor roll of the "Great Colleges to Work For" from The Chronicle of Higher Education, Baylor offers competitive salaries and benefits while giving faculty and staff the chance to live in one of the fastest-growing parts of the state. Our strategic plan, Illuminate Forward, guides the University as we continue to live up to Baylor's mission of educating men and women for worldwide leadership and service by integrating academic excellence and Christian commitment within a caring community.

Baylor University is a private not-for-profit university affiliated with the Baptist General Convention of Texas. As an Affirmative Action/Equal Opportunity employer, Baylor is committed to compliance with all applicable anti-discrimination laws, including those regarding age, race, color, sex, national origin, pregnancy status, military service, genetic information, and disability. As a religious educational institution, Baylor is lawfully permitted to consider an applicant's religion as a selection criterion. Baylor encourages women, minorities, veterans, and individuals with disabilities to apply.

9

WISCONSIN

University of Wisconsin–Madison Department of Mathematics

The Department of Mathematics at UW–Madison is accepting applications for faculty positions beginning August 19, 2024, subject to budgetary approval. Rank will be as assistant professor (tenure-track), associate professor (tenured), or in exceptional cases, professor (tenured). All areas of mathematics will be considered. PhD in mathematics or related field is required prior to start of appointment. Faculty members are expected to contribute to the research, teaching, and service missions of the department. Appointment with tenure requires evidence of excellence in scholarly research, teaching, and service. Candidates for a tenure-track position should exhibit evidence of outstanding research potential, normally including significant contributions beyond the doctoral dissertation. The teaching responsibility is two courses per academic year, including both undergraduate- and graduate-level courses, and a strong commitment to excellence in instruction is also expected. An application packet should include a completed AMS Standard Cover Sheet, a curriculum vitae that includes a publication list, and brief descriptions of research and teaching, and a DEI statement addressing past and potential contributions to an inclusive educational environment through research, teaching, and service. Application packets should be submitted electronically to `https://www.mathjobs.org/jobs/list/22690`. We require three to four letters of recommendation, at least one of which must discuss the applicant's teaching experiences, capabilities, and potential, sent to the above URL. To ensure full consideration, application packets must be received by November 1, 2023. Applications will be accepted until the position is filled. The University of Wisconsin–Madison is an Affirmative Action, Equal Opportunity Employer and encourages applications from women and minorities. Unless confidentiality is requested in writing, information regarding the applicants must be released upon request. Finalists cannot be guaranteed confidentiality. A background check will be required prior to employment.

12

AUSTRIA

Assistant Professor (tenure-track) and Professor (tenured) positions in Mathematics

The Institute of Science and Technology Austria invites applications for several open positions in **all areas of mathematics**.

We offer:

- Thriving **international and interdisciplinary research environment** with English as the working language
- **Highly competitive salary** and **generous start-up package**, ensuring you have the resources necessary to establish and lead a successful research group
- Guaranteed **annual base funding** including dedicated funding for PhD students and postdocs, enabling you to build up a dynamic research team
- **International graduate school** with highly selective admissions criteria and a comprehensive educational program, training the next generation of scientific leaders
- **PhD program with a unique blend of interdisciplinary coursework and research group rotations**, attracting scholars from diverse international backgrounds
- **Light teaching load**, entirely at the level of graduate students
- **Support for acquiring third-party funds**
- **Tailored leadership program** designed to enhance your professional development
- **Collaborative atmosphere** that promotes interaction, knowledge sharing, and interdisciplinary collaborations among research groups
- **Employee Assistance Program**, providing support to help you maintain a healthy work-life balance
- **Dual-career support**, advising your spouse or partner on finding local career opportunities

- **Childcare facilities** on campus (for children aged 3 months till primary school age)
- **Diverse and inclusive working environment**, committed to equal employment opportunities
- **Close proximity to Vienna**, consistently ranked among the most liveable cities worldwide, offering a wealth of cultural and recreational opportunities

Assistant professors receive independent group leader positions with an initial contract of six years, at the end of which they are reviewed by international peers. A positive evaluation leads to promotion to the tenured professor position.

Tenured positions are open to distinguished scientists with several years of experience leading research groups.

ISTA (`www.ista.ac.at`) is an international institute dedicated to basic research and graduate education in the natural, mathematical, and computational sciences.

At ISTA, we promote a diverse and inclusive working environment and are committed to the principle of equal employment opportunity for all applicants, free of discrimination. We strongly encourage individuals from underrepresented groups to apply.

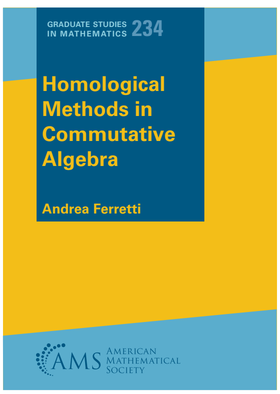
Take the next step in your academic career and apply at: `www.ista.ac.at/jobs/faculty/`.

The closing date for applications is October 25, 2023.

7

New Books Offered by the AMS

Algebra and Algebraic Geometry



Homological Methods in Commutative Algebra

Andrea Ferretti

This book develops the machinery of homological algebra and its applications to commutative rings and modules. It assumes familiarity with basic commutative algebra, for example, as covered in the author's book, *Commutative Algebra*.

The first part of the book is an elementary but thorough exposition of the concepts of homological algebra, starting from categorical language up to the construction of derived functors and spectral sequences. A full proof of the celebrated Freyd-Mitchell theorem on the embeddings of small Abelian categories is included.

The second part of the book is devoted to the application of these techniques in commutative algebra through the study of projective, injective, and flat modules, the construction of explicit resolutions via the Koszul complex, and the properties of regular sequences. The theory is then used to understand the properties of regular rings, Cohen-Macaulay rings and modules, Gorenstein rings and complete intersections.

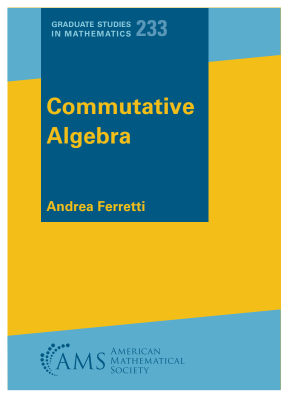
Overall, this book is a valuable resource for anyone interested in learning about homological algebra and its applications in commutative algebra. The clear and thorough presentation of the material, along with the many examples and exercises of varying difficulty, make it an excellent choice for self-study or as a reference for researchers.

Graduate Studies in Mathematics, Volume 234
November 2023, approximately 413 pages, Hardcover, ISBN: 978-1-4704-7128-6, LC 2023012887, 2020 *Mathematics Subject Classification*: 13–01, 13D02, 13D07, 13D45, 13H10, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/234

November 2023, Softcover, ISBN: 978-1-4704-7436-2, LC 2023012887, 2020 *Mathematics Subject Classification*: 13–01, 13D02, 13D07, 13D45, 13H10, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/234.S

`bookstore.ams.org/gsm-234`

`bookstore.ams.org/gsm-234-s`



Commutative Algebra

Andrea Ferretti

This book provides an introduction to classical methods in commutative algebra and their applications to number theory, algebraic geometry, and computational algebra. The use of number theory as a motivating theme throughout the book provides a rich and interesting context for the material covered. In addition, many results are reinterpreted from a geometric perspective, providing further insight and motivation for the study of commutative algebra.

The content covers the classical theory of Noetherian rings, including primary decomposition and dimension theory, topological methods such as completions, computational techniques, local methods and multiplicity theory, as well as some topics of a more arithmetic nature, including the theory of Dedekind rings, lattice embeddings, and Witt vectors. Homological methods appear in the author's sequel, *Homological Methods in Commutative Algebra*.

Overall, this book is an excellent resource for advanced undergraduates and beginning graduate students in algebra or number theory. It is also suitable for students in neighboring fields such as algebraic geometry who wish to develop a strong foundation in commutative algebra. Some parts of the book may be useful to supplement undergraduate courses in number theory, computational algebra or algebraic geometry. The clear and detailed presentation, the inclusion of computational techniques and arithmetic topics, and the numerous exercises make it a valuable addition to any library.

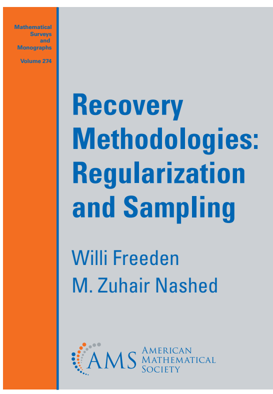
Graduate Studies in Mathematics, Volume 233
November 2023, 373 pages, Hardcover, ISBN: 978-1-4704-7127-9, LC 2023012824, 2020 *Mathematics Subject Classification*: 13–01, 11R04, 13P99, 11–01, 14A10, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/233

`bookstore.ams.org/gsm-233`

November 2023, 373 pages, Softcover, ISBN: 978-1-4704-7434-8, LC 2023012824, 2020 *Mathematics Subject Classification*: 13–01, 11R04, 13P99, 11–01, 14A10, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/233.S

`bookstore.ams.org/gsm-233-s`

Applications



Recovery Methodologies: Regularization and Sampling

Willi Freeden, *University of Kaiserslautern, Germany*, and **M. Zuhair Nashed**, *University of Central Florida, Orlando, FL*

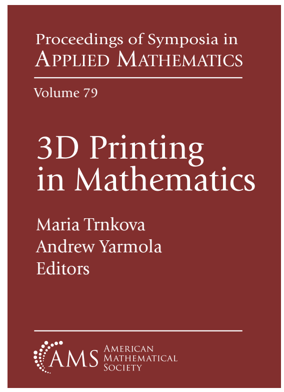
The goal of this book is to introduce the reader to methodologies in recovery problems for objects, such as functions and signals, from partial or indirect information. The recovery of objects from a set of data demands key solvers of inverse and sampling problems. Until recently, connections between the mathematical areas of inverse problems and sampling were rather tenuous. However, advances in several areas of mathematical research have revealed deep common threads between them, which proves that there is a serious need for a unifying description of the underlying mathematical ideas and concepts. Freeden and Nashed present an integrated approach to resolution methodologies from the perspective of both these areas.

Researchers in sampling theory will benefit from learning about inverse problems and regularization methods, while specialists in inverse problems will gain a better understanding of the point of view of sampling concepts. This book requires some basic knowledge of functional analysis, Fourier theory, geometric number theory, constructive approximation, and special function theory. By avoiding extreme technicalities and elaborate proof techniques, it is an accessible resource for students and researchers not only from applied mathematics, but also from all branches of engineering and science.

This item will also be of interest to those working in mathematical physics and probability and statistics.

Mathematical Surveys and Monographs, Volume 274
October 2023, 491 pages, Softcover, ISBN: 978-1-4704-7345-7, LC 2023014083, 2020 *Mathematics Subject Classification*: 65J20, 62D05; 65R32, 78A50, 94A20, 86A22, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/274

`bookstore.ams.org/surv-274`



3D Printing in Mathematics

Maria Trnkova, *University of California, Davis, CA*, and **Andrew Yarmola**, *Princeton University, NJ*, Editors

This volume is based on lectures delivered at the 2022 AMS Short Course "3D Printing: Challenges and Applications" held virtually from January 3–4, 2022.

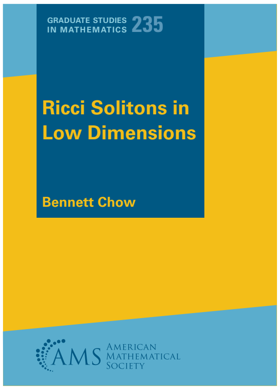
Access to 3D printing facilities is quickly becoming ubiquitous across college campuses. However, while equipment training is readily available, the process of taking a mathematical idea and making it into a printable model presents a big hurdle for most mathematicians. Additionally, there are still many open questions around what objects are possible to print, how to design algorithms for doing so, and what kinds of geometries have desired kinematic properties. This volume is focused on the process and applications of 3D printing for mathematical education, research, and visualization, alongside a discussion of the challenges and open mathematical problems that arise in the design and algorithmic aspects of 3D printing.

The articles in this volume are focused on two main topics. The first is to make a bridge between mathematical ideas and 3D visualization. The second is to describe methods and techniques for including 3D printing in mathematical education at different levels—from pedagogy to research and from demonstrations to individual projects. We hope to establish the groundwork for engaged academic discourse on the intersections between mathematics, 3D printing and education.

Proceedings of Symposia in Applied Mathematics, Volume 79
November 2023, approximately 226 pages, Softcover, ISBN: 978-1-4704-6916-0, LC 2023022844, 2020 *Mathematics Subject Classification*: 00A66, 97D40, 97N80, 97U60; 54C40, 14E20, 97I60, 14–04, 46E25, 20C20, 97D60, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code PSAPM/79

`bookstore.ams.org/psapm-79`

Geometry and Topology



Ricci Solitons in Low Dimensions

Bennett Chow, *University of California, San Diego, La Jolla, CA*

Ricci flow is an exciting subject of mathematics with diverse applications in geometry, topology, and other fields. It employs a heat-type equation to smooth an initial Riemannian metric on a manifold. The formation of singularities in the manifold's topology and geometry is a desirable outcome. Upon closer examination, these singularities often reveal intriguing structures known as Ricci solitons.

This introductory book focuses on Ricci solitons, shedding light on their role in understanding singularity formation in Ricci flow and formulating surgery-based Ricci flow, which holds potential applications in topology. Notably successful in dimension 3, the book narrows its scope to low dimensions: 2 and 3, where the theory of Ricci solitons is well established. A comprehensive discussion of this theory is provided, while also establishing the groundwork for exploring Ricci solitons in higher dimensions.

A particularly exciting area of study involves the potential applications of Ricci flow in comprehending the topology of 4-dimensional smooth manifolds. Geared towards graduate students who have completed a one-semester course on Riemannian geometry, this book serves as an ideal resource for related courses or seminars centered on Ricci solitons.

Graduate Studies in Mathematics, Volume 235
October 2023, 339 pages, Hardcover, ISBN: 978-1-4704-7428-7, 2020 *Mathematics Subject Classification*: 53E20, 53E10, 53E30, 58J05, 58J35, 58J60, 57K30, 57M50, 30F20, 30F45, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code GSM/235

bookstore.ams.org/gsm-235

October 2023, 339 pages, Softcover, ISBN: 978-1-4704-7523-9, 2020 *Mathematics Subject Classification*: 53E20, 53E10, 53E30, 58J05, 58J35, 58J60, 57K30, 57M50, 30F20, 30F45, List US$89, **AMS members US$71.20**, **MAA members US$80.10**, Order code GSM/235.S

bookstore.ams.org/gsm-235-s

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Eigenfunctions of Transfer Operators and Automorphic Forms for Hecke Triangle Groups of Infinite Covolume

Roelof Bruggeman, *Universiteit Utrecht, The Netherlands*, and **Anke Dorothea Pohl**, *University of Bremen, Germany*

Memoirs of the American Mathematical Society, Volume 287, Number 1423
July 2023, 172 pages, Softcover, ISBN: 978-1-4704-6545-2, 2020 *Mathematics Subject Classification*: 11F12, 11F67, 37C30; 30F35, 37D40, 11F72, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1423

bookstore.ams.org/memo-287-1423

Automorphism Orbits and Element Orders in Finite Groups: Almost-Solubility and the Monster

Alexander Bors, *Carleton University, Ottawa, Canada, The University of Western Australia, Crawley, Australia, and Radon Institute for Computational and Applied Mathematics, Linz, Austria*, **Michael Giudici**, *The University of Western Australia, Crawley, Australia*, and **Cheryl E. Praeger**, *The University of Western Australia, Crawley, Australia*

Memoirs of the American Mathematical Society, Volume 287, Number 1427
July 2023, 95 pages, Softcover, ISBN: 978-1-4704-6544-5, 2020 *Mathematics Subject Classification*: 20D60; 20D05, 20D45, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1427

bookstore.ams.org/memo-287-1427

Analysis

Convexity of Singular Affine Structures and Toric-Focus Integrable Hamiltonian Systems

Tudor S. Ratiu, *Shanghai Jiao Tong University, China, Université Genève, Switzerland, and Ecole Polytechnique Fédérale de Lausanne, Switzerland*, **Christophe Wacheux**, *Overflood, Lille, France*, and **Nguyen Tien Zung**, *Université Paul Sabatier, Toulouse, France*

This item will also be of interest to those working in geometry and topology.

Memoirs of the American Mathematical Society, Volume 287, Number 1424
July 2023, 89 pages, Softcover, ISBN: 978-1-4704-6439-4, 2020 *Mathematics Subject Classification*: 37J35, 53A15, 52A01, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1424

bookstore.ams.org/memo-287-1424

Nilspace Factors for General Uniformity Seminorms, Cubic Exchangeability and Limits

Pablo Candela, *Universidad Autónoma de Madrid, Spain, and Ciudad Universitaria de Cantoblanco, Madrid, Spain*, and **Balázs Szegedy**, *MTA Alfréd Rényi Institute of Mathematics, Budapest, Hungary*

This item will also be of interest to those working in probability and statistics.

Memoirs of the American Mathematical Society, Volume 287, Number 1425
July 2023, 101 pages, Softcover, ISBN: 978-1-4704-6548-3, 2020 *Mathematics Subject Classification*: 37Axx, 37A15, 60–XX, 60Bxx; 11B30, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1425

bookstore.ams.org/memo-287-1425

Embeddings of Decomposition Spaces

Felix Voigtlaender, *Catholic University of Eichstätt-Ingolstadt, Germany*

Memoirs of the American Mathematical Society, Volume 287, Number 1426
July 2023, 253 pages, Softcover, ISBN: 978-1-4704-5990-1, 2020 *Mathematics Subject Classification*: 42B35, 46E15, 46E35, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1426

bookstore.ams.org/memo-287-1426

Number Theory

Overlapping Iterated Function Systems from the Perspective of Metric Number Theory

Simon Baker, *University of Birmingham, United Kingdom*

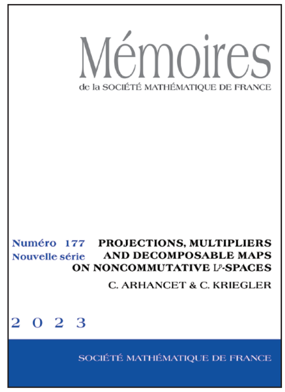
This item will also be of interest to those working in analysis.

Memoirs of the American Mathematical Society, Volume 287, Number 1428
July 2023, 95 pages, Softcover, ISBN: 978-1-4704-6440-0, 2020 *Mathematics Subject Classification*: 11K60, 28A80, 37C45, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/287/1428

bookstore.ams.org/memo-287-1428

New AMS-Distributed Publications

Analysis



Projections, Multipliers and Decomposable Maps on Noncommutative L^p-Spaces

Cédric Arhancet, *Alibi, France*, and **Christoph Kriegler**, *Université Clermont Auvergne, Clermont-Ferrand, France*

The authors introduce a noncommutative analogue of the absolute value of a regular operator acting on a noncommutative L^p-space. They equally prove that two classical operator norms, the regular norm and the decomposable norm, are identical.

The authors also describe precisely the regular norm of several classes of regular multipliers. This includes Schur multipliers and Fourier multipliers on some unimodular locally compact groups that can be approximated by discrete groups in various senses. A main ingredient is to show the existence of a bounded projection from the space of completely bounded L^p operators onto the subspace of Schur or Fourier multipliers, preserving complete positivity.

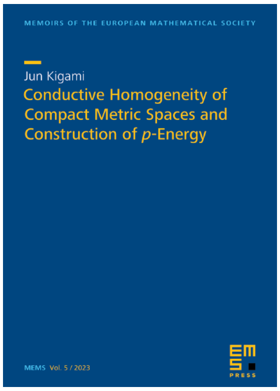
On the other hand, the authors show the existence of bounded Fourier multipliers that cannot be approximated by regular operators, on large classes of locally compact groups, including all infinite abelian locally compact groups. The authors finish by introducing a general procedure for proving positive results on self-adjoint contractively decomposable Fourier multipliers, beyond the amenable case.

This item will also be of interest to those working in number theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 177
July 2023, 186 pages, Softcover, ISBN: 978-2-85629-971-5, 2020 *Mathematics Subject Classification*: 46L51, 46L07, 43A07, 43A22, 43A30, 43A35, 43A40, 46L52, 47L25, 43A15, List US$65, **AMS members US$52**, Order code SMFMEM/177

bookstore.ams.org/smfmem-177



Conductive Homogeneity of Compact Metric Spaces and Construction of p-Energy

Jun Kigami, *Kyoto University, Japan*

In the ordinary theory of Sobolev spaces on domains of $\mathbb{R}^n$, the p-energy is defined as the integral of $|\nabla f|^p$. In this book, the author tries to construct a p-energy on compact metric spaces as a scaling limit of discrete p-energies on a series of graphs approximating the original space. In conclusion, the author proposes a notion called conductive homogeneity under which one can construct a reasonable p-energy if p is greater than the Ahlfors regular conformal dimension of the space. In particular, if $p=2$, then he constructs a local regular Dirichlet form and shows that the heat kernel associated with the Dirichlet form satisfies upper and lower sub-Gaussian type heat kernel estimates. As examples of conductively homogeneous spaces, the author presents new classes of square-based, self-similar sets and rationally ramified Sierpiński crosses, where no diffusions were constructed before.

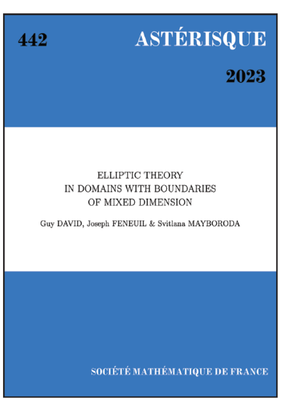
This item will also be of interest to those working in probability and statistics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Memoirs of the European Mathematical Society, Volume 5

July 2023, 138 pages, Softcover, ISBN: 978-3-98547-056-3, 2020 *Mathematics Subject Classification*: 46E36; 31E05, 28A80, 31C45, 31C25, 30L10, List US\$75, **AMS members US\$60**, Order code EMSMEM/5

bookstore.ams.org/emsmem-5



Elliptic Theory in Domains with Boundaries of Mixed Dimension

Guy David, *Université Paris-Saclay, Orsay, France*, **Joseph Feneuil**, *Mathematical Sciences Institute, Australian National University, Acton, Australia*, and **Svitlana Mayboroda**, *School of Mathematics, University of Minnesota, Minneapolis, Minnesota*

Take an open domain $\Omega \subset \mathbb{R}^n$ whose boundary may be composed of pieces of different dimensions. For instance, Ω can be a ball on $\mathbb{R}^3$, minus one of its diameters D, or a so-called saw-tooth domain, with a boundary consisting of pieces of 1-dimensional curves intercepted by 2-dimensional spheres. It could also be a domain with a fractal (or partially fractal) boundary. Under appropriate geometric assumptions, essentially the existence of doubling measures on Ω and $\partial\Omega$ with appropriate size conditions. The authors construct a class of second order degenerate elliptic operators L adapted to the geometry, and establish key estimates of elliptic theory associated to those operators. This includes boundary Poincaré and Harnack inequalities, maximum principle, and Hölder continuity of solutions at the boundary.

This item will also be of interest to those working in differential equations and probability and statistics.

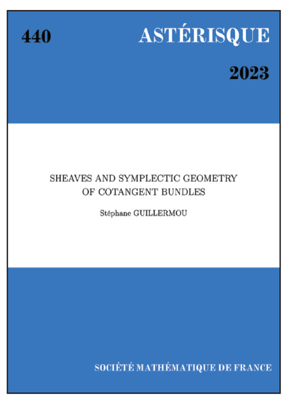
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 442

July 2023, 139 pages, Softcover, ISBN: 978-2-85629-974-6, 2020 *Mathematics Subject Classification*: 28A15, 28A25, 31B05, 31B25, 35J25, 42B37, 35J70, List US\$65, **AMS members US\$52**, Order code AST/442

bookstore.ams.org/ast-442

Differential Equations



Sheaves and Symplectic Geometry of Cotangent Bundles

Stéphane Guillermou, *Laboratoire de Mathématiques Jean Leray, Nantes, France*

The aim of this paper is to apply the microlocal theory of sheaves of Kashiwara-Schapira to the symplectic geometry of cotangent bundles, following ideas of Nadler-Zaslow and Tamarkin.

The author recalls the main notions and results of the microlocal theory of sheaves, in particular the microsupport of sheaves. The microsupport of a sheaf F on a manifold M is a closed conic subset of the cotangent bundle T^*M which indicates in which directions we can modify a given open subset of M without modifying the cohomology of F on this subset. An important theorem of Kashiwara-Schapira says that the microsupport is coisotropic, and recent works of Nadler-Zaslow and Tamarkin study in the other direction the sheaves which have for microsupport a given Lagrangian submanifold A, obtaining information on A in this way.

Nadler and Zaslow made the link with the Fukaya category, but Tamarkin only made use of the microlocal sheaf theory. The author moves in this direction and recovers several results of symplectic geometry with the help of sheaves.

In particular, the author explains how we can recover the Gromov nonsqueezing theorem, the Gromov-Eliashberg rigidity theorem, and the existence of graph selectors. The author also proves a three cusps conjecture of Arnol'd about curves on the sphere. In the last sections, the author recovers more recent results on the topology of exact Lagrangian submanifolds of cotangent bundles.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 440
July 2023, 274 pages, Softcover, ISBN: 978-2-85629-972-2, 2020 *Mathematics Subject Classification*: 18F20, 35A27, 53D12, List US$81, **AMS members US$64.80**, Order code AST/440

`bookstore.ams.org/ast-440`

General Interest

European Congress of Mathematics

Portorož, June 20–26, 2021

Ademir Hujdurović, *University of Primorska, Koper, Slovenia*, **Klavdija Kutnar**, *University of Primorska, Koper, Slovenia*, **Dragan Marušič**, *University of Primorska, Koper, Slovenia*, **Štefko Miklavič**, *University of Primorska, Koper, Slovenia*, **Tomaž Pisanski**, *University of Primorska, Koper, Slovenia*, and **Primož Šparl**, *University of Primorska, Koper, Slovenia*, Editors

The European Congress of Mathematics, held every four years, is a well-established major international mathematical event. Following those in Paris (1992), Budapest (1996), Barcelona (2000), Stockholm (2004), Amsterdam (2008), Kraków (2012), and Berlin (2016), the Eighth European Congress of Mathematics (8ECM) took place in Portorož, Slovenia, June 20–26, 2021, with about 1700 participants from all over the world, mostly online due to the COVID pandemic.

Ten plenary and thirty invited lectures along with the special Abel and Hirzebruch lectures formed the core of the program. As in all the previous EMS congresses, ten outstanding young mathematicians received the EMS prizes in recognition of their research achievements. In addition, two more prizes were awarded: The Felix Klein Prize for a remarkable solution of an industrial problem and the Otto Neugebauer Prize for a highly original and influential piece of work in the history of mathematics. The program was complemented by five public lectures, several exhibitions, and 62 minisymposia with about 1000 contributions, spread over all areas of mathematics. A number of panel discussions and meetings were organized, covering a variety of issues ranging from the future of mathematical publishing and the role of the ERC to increase public awareness of mathematics.

These proceedings provide a permanent record of current mathematics of highest quality by presenting extended versions of seven plenary, six prize, and 14 invited lectures as well as 11 lectures from minisymposia keynote speakers, all of which were delivered during the Congress.

July 2023, 982 pages, Hardcover, ISBN: 978-3-98547-051-8, 2020 *Mathematics Subject Classification*: 00B25, List US$159, **AMS members US$127.20**, Order code EMSEMC/2021

`bookstore.ams.org/emsemc-2021`

Number Theory

Topics in Statistical Mechanics

Cédric Boutillier, *Sorbonne Université, Paris, France*, **Béatrice de Tilière**, *Université Paris Dauphine, Paris, France*, and **Kilian Raschel**, *Laboratoire Angevin de Recherche en Mathématiques, Université d'Angers, Angers, France*, Editors

This volume provides an overview of the "États de la recherche on Statistical Mechanics," organized by the French Mathematical Society, which took place at the Institut Henri Poincaré (Paris) in 2018. It was a successful event bringing together 125 mathematicians, ranging from master students to young and confirmed researchers. There were four mini courses by Francesco Caravenna; Hugo Duminil-Copin; Thierry Bodineau, Isabelle Gallagher, and Laure Saint-Raymond; and Vincent Vargas. These were complemented by 13 research talks, altogether giving an overview of a wide number of models of statistical mechanics, including the Ising model, Potts model, percolation, perfect gases, Coulomb gases, particle systems, kinetically constrained spin models, and the dimer model.

This volume contains an introduction with a summary of the four mini courses and of all the talks. The heart of this publication consists of five original contributions by Djalil Chafaï, Ewain Gwynne, Nina Holden, and Xin Sun; Arnaud Le Ny; Sébastien Ott, and Yvan Velenik; Rémi Rhodes and Vincent Vargas.

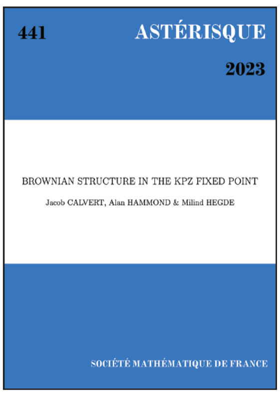
NEW BOOKS

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Panoramas et Synthèses, Number 59
July 2023, 230 pages, Softcover, ISBN: 978-2-85629-970-8, 2020 *Mathematics Subject Classification*: 15B52, 28A80, 60K35, 81T40, 82B20, 82B27, 82B44, 82C05, 82B40, List US$81, **AMS members US$64.80**, Order code PASY/59

`bookstore.ams.org/pasy-59`

Probability and Statistics



Brownian Structure in the KPZ Fixed Point

Jacob Calvert, *University of California, Berkeley, CA*, **Alan Hammond**, *University of California, Berkeley, CA*, and Milind Hegde, *Columbia University, New York, NY*

Many models of one-dimensional local random growth are expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. For such a model, the interface profile in the long time limit is expected—and proved for a few integrable models—to be, when viewed in appropriately scaled coordinates, up to a parabolic shift, the **Airy**$_2$ process $\mathcal{A}: \mathbb{R} \to \mathbb{R}$. This process may be embedded via the Robinson-Schensted-Knuth correspondence as the uppermost curve in an $\mathbb{N}$-indexed system of random continuous curves, the Airy line ensemble.

Among the authors' principal results is the assertion that the **Airy**$_2$ process enjoys a very strong similarity to Brownian motion (of rate two) on unit-order intervals.

The authors' technique of proof harnesses a probabilistic resampling or Brownian Gibbs property satisfied by the Airy line ensemble after parabolic shift, and this book develops Brownian Gibbs analysis of this ensemble begun in the work of Corwin and Hammond (2014) and pursued by Hammond (2019).

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 441
July 2023, 119 pages, Softcover, ISBN: 978-2-85629-973-9, 2020 *Mathematics Subject Classification*: 82C22, 82B23, 60H15, List US$57, **AMS members US$45.60**, Order code AST/441

`bookstore.ams.org/ast-441`

Meetings & Conferences of the AMS

October Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: stovall@math.wisc.edu; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: mamanes@hawaii.edu; telephone: (808) 956-4679.

Meetings in this Issue

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.

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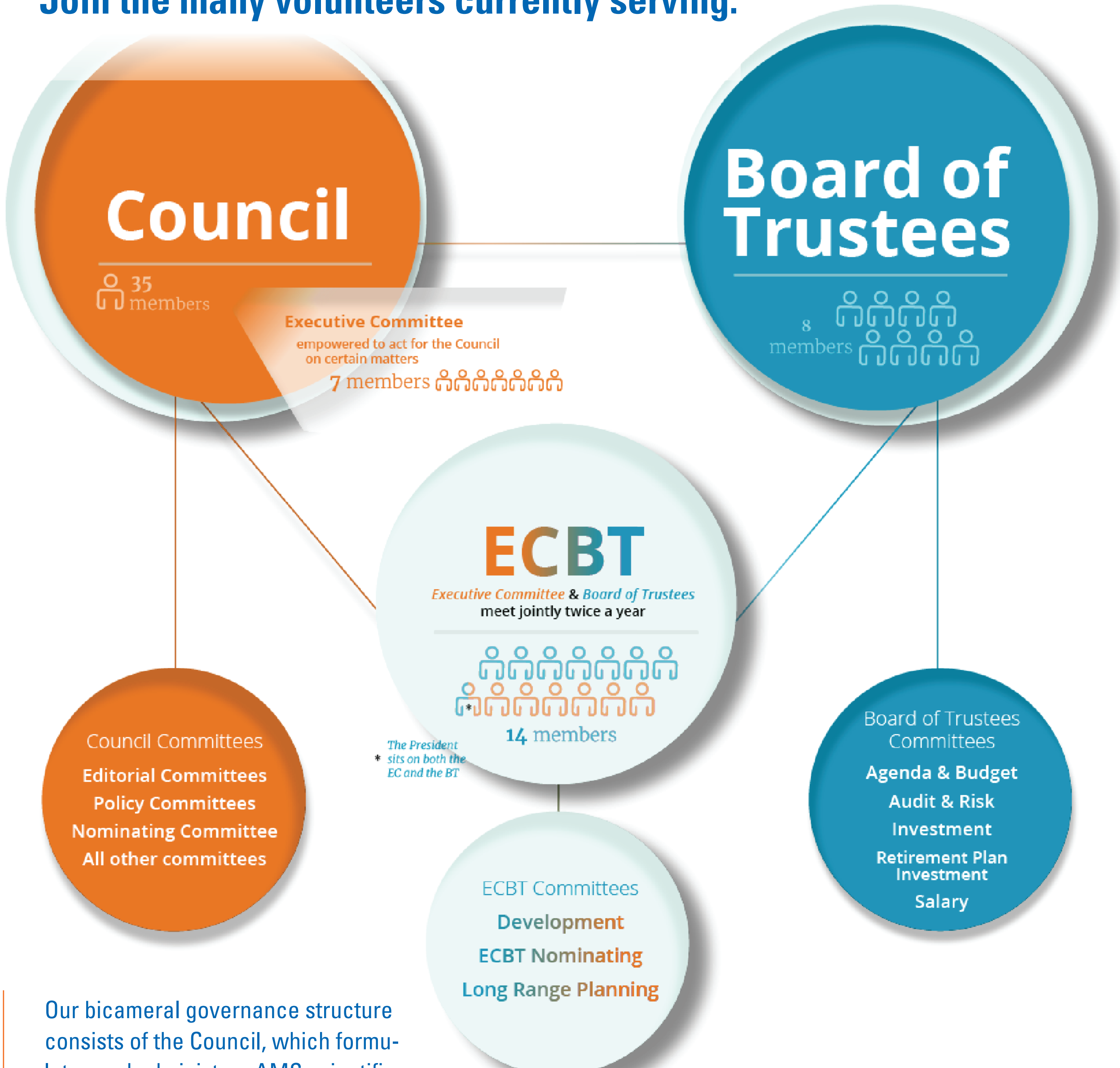

Our bicameral governance structure consists of the Council, which formulates and administers AMS scientific policies, and the Board of Trustees, which receives and administers the funds of the Society.

Governance Leadership consists of Officers, the Council, Executive Committee of the Council, and Board of Trustees.

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www.ams.org/governance

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Buffalo, New York

University at Buffalo (SUNY)

September 9–10, 2023

Saturday – Sunday

Meeting #1188

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 3

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Jennifer Balakrishnan, Boston University, *Mordell's Conjecture: One Century Later*.

Sigal Gottlieb, University of Massachusetts, Dartmouth, *Developing High Order, Efficient, and Stable Time-Evolution Methods Using a Time-Filtering Approach*.

Sam Payne, UT Austin, *Motivic Structures in the Cohomology of Moduli Spaces of Curves*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Automorphic Forms and L Functions, **Xiaoqing Li** and **Joseph A Hundley**, University at Buffalo, SUNY.

Building Bridges Between $\mathbb{F}_1$-Geometry, Combinatorics and Representation Theory, **Jaiung Jun**, SUNY New Paltz, **Chris Eppolito**, The University of the South, and **Alexander Sistko**, Manhattan College.

Combinatorial and Categorical Techniques in Representation Theory, **Nicholas Davidson**, College of Charleston, **Robert Muth**, Duquesne University, **Tianyuan Xu**, Haverford College, and **Jieru Zhu**, Université Catholique de Louvain.

Difference and Differential Equations: Modeling, Analysis, and Applications to Mathematical Biology, **Nhu N. Nguyen** and **Mustafa R Kulenovic**, University of Rhode Island.

Ergodic Theory of Group Actions, **Hanfeng Li**, SUNY at Buffalo, and **Jintao Deng**, University of Waterloo.

Financial Mathematics, **Maxim Bichuch**, University at Buffalo, and **Zachary Feinstein**, Stevens Institute of Technology.

From Classical to Quantum Low-Dimensional Topology, **Adam S Sikora**, University At Buffalo, SUNY, **Roman Aranda**, Binghamton University, and **William W. Menasco**, University at Buffalo.

Gauge Theory and Low-Dimensional Topology, **Cagatay Kutluhan**, University at Buffalo, **Paul M. N. Feehan**, Rutgers University, New Brunswick, **Thomas Gibbs Leness**, Florida International University, and **Francesco Lin**, Columbia University.

Geometry of Groups and Spaces, **Johanna Mangahas**, University at Buffalo, **Joel Louwsma**, Niagara University, and **Jenya Sapir**, Binghamton University.

Geometry, Physics and Representation Theory, **Jie Ren**, SUNY Buffalo.

Homological Aspects of p-adic Groups and Automorphic Representations, **Karol Koziol**, Baruch College, CUNY, and **Anantharam Raghuram**, Fordham University.

Inverse Problems in Science and Engineering, **Sedar Ngoma**, SUNY Geneseo.

Nonlinear Partial Differential Equations in Fluids and Waves, **Qingtian Zhang**, West Virginia University, and **Ming Chen**, University of Pittsburgh.

Nonlinear Wave Equations and Integrable Systems, **Gino Biondini** and **Alexander Chernyavsky**, SUNY Buffalo.

Probability, Combinatorics, and Statistical Mechanics, **Douglas Rizzolo**, University of Delaware, and **Noah Forman**, McMaster University.

Recent Advances in Numerical Methods for Fluid Dynamics and Their Applications, **Daozhi Han**, The State University of New York at Buffalo, **Guosheng Fu**, University of Notre Dame, and **Jia Zhao**, Binghamton University.

Recent Advances in Water Waves: Theory and Numerics, **Sergey Dyachenko**, University at Buffalo, and **Alexander Chernyavsky**, SUNY Buffalo.

Recent Developments in Operator Algebras and Quantum Information Theory, **Priyanga Ganesan**, University of California San Diego, **Samuel Harris**, Northern Arizona University, and **Ivan G. Todorov**, University of Delaware.

Recent Trends in Spectral Graph Theory, **Michael Tait**, Villanova University, and **Shahla Nasserasr** and **Brendan Rooney**, Rochester Institute of Technology.

Representation Theory and Flag Varieties, **Yiqiang Li**, SUNY At Buffalo, and **Changlong Zhong**, SUNY Albany.

Topics in Combinatorics and Graph Theory, **Rong Luo** and **Kevin G. Milans**, West Virginia University, and **Guangming Jing**, University of West Virginia.

Contributed Paper Sessions

AMS Contributed Paper Session, **Steven H Weintraub**, Lehigh University.

Omaha, Nebraska

Creighton University

October 7–8, 2023

Saturday – Sunday

Meeting #1189

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 4

Deadlines

For organizers: To be announced
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Lydia Bieri, University of Michigan, *The Mathematics of Gravitational Waves*.

Aaron J Pollack, UC San Diego, *Exceptional groups and their modular forms*.

Christopher Schafhauser, University of Nebraska–Lincoln, *Classification of amenable operator algebras*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Graph Theory and Combinatorics I, **Bernard Lidický** and **Steve Butler**, Iowa State University.

Advances in Operator Algebras I, **Christopher Schafhauser**, University of Nebraska–Lincoln, and **Ionut Chifan**, University of Iowa.

Analytic Number Theory and Related fields I, **Vorrapan Chandee** and **Xiannan Li**, Kansas State University, and **Micah B. Milinovich**, University of Mississippi.

Applied Knot Theory I, **Isabel K. Darcy**, University of Iowa, and **Eric Rawdon**, University of St. Thomas.

Automorphic Forms, their Arithmetic, and their Applications I, **Aaron J Pollack**, UC San Diego, and **Spencer Leslie**, Boston College.

Commutative Algebra, Differential Operators, and Singularities I, **Uli Walther**, Purdue University, **Claudia Miller**, Syracuse University, and **Vaibhav Pandey**, Purdue University.

Commutative Algebra i, **Thomas Marley**, University of Nebraska-Lincoln, and **Eloísa Grifo**, University of Nebraska-Lincoln.

Discrete, Algebraic, and Topological Methods in Mathematical Biology I, **Alexander B Kunin**, Creighton University.

Enumerative Combinatorics I, **Seok Hyun Byun**, Clemson University, **Tri Lai**, University of Nebraska, and **Svetlana Poznanovic**, Clemson University.

Foliations, Flows, and Groups I, **Ying Hu**, University of Nebraska Omaha, and **Michael Landry**, Washington University in Saint Louis.

Harmonic Analysis in the Midwest I, **Betsy Stovall**, University of Wisconsin-Madison, and **Terence L. J. Harris**, Cornell University.

Homotopy Theory I, **Prasit Bhattacharya**, New Mexico State University, **Agnes Beaudry**, University of Colorado Boulder, and **Zhouli Xu**, University of California, San Diego.

Interactions of Floer Homologies, Contact Structures, and Symplectic Structures I, **Robert DeYeso III** and **Joseph Breen**, University of Iowa.

Mathematical Modeling and Analysis in Ecology and Epidemiology I, **Yu Jin**, University of Nebraska-Lincoln, **Shuwen Xue**, Northern Illinois University, and **Chayu Yang**, University of Nebraska-Lioncoln.

Nonlinear PDE and Free Boundary Problems I, **William Myers Feldman**, University of Utah, and **Fernando Charro**, Wayne State University.

Progress in Nonlinear Waves I, **David M. Ambrose**, Drexel University.

Recent Development in Advanced Numerical Methods for Partial Differential Equations I, **Mahboub Baccouch**, University of Nebraska At Omaha.

Recent Developments in Theory and Computation of Nonlocal Models I, **Anh Vo** and **Scott Hootman-Ng**, University of Nebraska-Lincoln, and **Animesh Biswas**, University of Nebraska Lincoln.

Topology of 3- and 4-Manifolds !, **Alexander Zupan**, University of Nebraska-Lincoln, **Roman Aranda**, Binghamton University, and **David Auckly**, Kansas State University.

Varieties with Unexpected Hypersurfaces, Geproci Sets and their Interactions I, **Brian Harbourne**, University of Nebraska, and **Juan C. Migliore**, University of Notre Dame.

Mobile, Alabama

University of South Alabama

October 13–15, 2023

Friday – Sunday

Meeting #1190

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 4

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtg/sectional.html`.

Invited Addresses

Theresa Anderson, Carnegie Mellon, *Number theory and friends: a mathematical journey*.

Laura Ann Miller, University of Arizona, *Flows around some soft corals*.

Cornelius Pillen, University of South Alabama, *Lifting to tilting: modular representations of algebraic groups and their Frobenius kernels*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Extremal Combinatorics, **Joseph Guy Briggs**, Auburn University, and **Chris Cox**, Iowa State University.

Analysis, Computation, and Applications of Stochastic Models, **Yukun Li**, University of Central Florida, **Feng Bao**, Florida State University, **Xiaobing Henry Feng**, The University of Tennessee, **Junshan Lin**, Auburn University, and **Liet Anh Vo**, University of Illinois, Chicago.

Categorical Representations, Quantum Algebra, and Related Topics, **Arik Wilbert**, University of South Alabama, **Mee Seong Im**, United States Naval Academy, Annapolis, and **Bach Nguyen**, Xavier University of Louisiana.

Combinatorics and Geometry Related to Representation Theory, **Markus Hunziker** and **William Erickson**, Baylor University.

Cyberinfrastructure for Mathematics Research & Instruction, **Steven Craig Clontz**, University of South Alabama, and **Tien Chih**, Oxford College of Emory University.

Discrete Geometry and Geometric Optimization, **Andras Bezdek**, Auburn University, Auburn AL, **Ferenc Fodor**, University of Szeged, and **Woden Kusner**, University of Georgia, Athens GA.

Dynamics and Equilibria of Energies, **Ryan Matzke**, Technische Universität Graz, and **Liudmyla Kryvonos**, Vanderbilt University.

Dynamics of Fluids, **I. Kukavica**, University of Southern California, **Dallas Albritton**, Princeton University, and **Wojciech S. Ozanski**, Florida State University.

Ergodic Theory and Dynamical Systems, **Mrinal Kanti Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and **Joanna Furno**, University of South Alabama.

Experimental Mathematics in Number Theory and Combinatorics, **Armin Straub**, University of South Alabama, **Brandt Kronholm**, University of Texas Rio Grande Valley, and **Luis A. Medina**, University of Puerto Rico.

Extremal and Probabilistic Combinatorics, **Sean English**, University of North Carolina Wilmington, and **Emily Heath**, Iowa State University.

Mathematical Modeling of Problems in Biological Fluid Dynamics, **Laura Ann Miller**, University of Arizona, and **Nick Battista**, The College of New Jersey.

New Directions in Noncommutative Algebras and Representation Theory, **Jonas T. Hartwig**, Iowa State University, and **Erich C. Jauch**, University of Wisconsin - Eau Claire.

Number Theory and Friends, **Robert James Lemke Oliver**, Tufts University, **Theresa Anderson**, Carnegie Mellon, **Ayla Gafni**, University of Mississippi, and **Edna Luo Jones**, Duke University.

Recent Advances in Low-dimensional and Quantum Topology, **Christine Lee**, Texas State University, and **Scott Carter**, University of South Alabama.

Recent Developments in Graph Theory, **Andrei Bogdan Pavelescu**, University of South Alabama, and **Kenneth Roblee**, Troy University.

Recent Progress in Numerical Methods for PDEs, **Muhammad Mohebujjaman**, Texas A&M International University, **Leo Rebholz**, Clemson University, and **Mengying Xiao**, University of West Florida.

Representation Theory of Finite and Algebraic Groups, **Daniel K. Nakano**, University of Georgia, **Pramod N. Achar**, Louisiana State University, and **Jonathan R. Kujawa**, University of Oklahoma.

Rings, Monoids, and Factorization, **Jim Coykendall**, Clemson University, and **Scott Chapman**, Sam Houston State University.

Theory and Application of Parabolic PDEs, **Wenxian Shen** and **Yuming Paul Zhang**, Auburn University.

Topics in Harmonic Analysis and Partial Differential Equations, **Jiuyi Zhu** and **Phuc Cong Nguyen**, Louisiana State University.

Contributed Paper Sessions

AMS Contributed Paper Session, **Brian D. Boe**, University of Georgia.

Tallahassee, Florida

Florida State University in Tallahassee

March 23–24, 2024

Saturday – Sunday

Meeting #1193

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: January 23, 2024

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Meeting #1194

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: September 5, 2023
For abstracts: February 13, 2024

Milwaukee, Wisconsin

University of Wisconsin- Milwaukee

April 20–21, 2024

Saturday – Sunday

Meeting #1195

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: September 19, 2023
For abstracts: February 20, 2024

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Meeting #1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: October 4, 2023
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Recent Advances in Differential Geometry, **Zhiqin Lu**, University of California, **Shoo Seto** and **Bogdan Suceavă**, California State University, Fullerton, and **Lihan Wang**, California State University, Long Beach.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

San Antonio, Texas

University of Texas, San Antonio

September 14–15, 2024

Saturday – Sunday

Meeting #1198

Central Section

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: February 13, 2024

For abstracts: July 23, 2024

Savannah, Georgia

Georgia Southern University, Savannah

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section

Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 5, 2024

For abstracts: August 13, 2024

Albany, New York

State University of New York at Albany

October 19–20, 2024

Saturday – Sunday

Meeting #1200

Eastern Section

Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 19, 2024

For abstracts: August 27, 2024

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Meeting #1201

Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 26, 2024
For abstracts: September 3, 2024

Auckland, New Zealand

December 9–13, 2024

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Hartford, Connecticut

Connecticut Convention Center and Hartford Marriott Downtown

April 5–6, 2025

Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

jmm 2024 Joint Mathematics Meetings

San Francisco • January 3–6

Moscone North/South, Moscone Center,
San Francisco, CA

Section Contents

Please note: The times listed herein were current as of press time. For the most up to date JMM 2024 scheduling information, please see: `https://www.jointmathematicsmeetings.org/meetings/national/jmm2024/2300_timetable.html`.

Welcome to JMM 2024! Reimagined by 16 (and counting!) partners and again happening in person, this largest annual mathematics gathering in the world offers you a broad range of addresses, sessions, posters, presentations, panels, exhibits, minicourses, professional enhancement programs, and social gatherings.

Expect to learn, to celebrate mathematics and mathematicians, and to build and renew relationships.

The reimagined Joint Mathematics Meetings feature new organizations, disciplines, participants, and programming to advance research, pedagogy, inclusion, career opportunities, the arts, recreation, and more.

Scientific exploration and discovery remain at the heart of JMM 2024. Leaf through the robust offerings on the following pages to find how JMM partners (listed below) have poured their collective energy into this gathering. Prepare to come together to connect and collaborate. At JMM 2024, we will learn from each other. We will see old friends, and we will make new ones. Stay tuned for JMM updates via social media and `www.jointmathematicsmeetings.org`. We can't wait to see you January 3–6 in San Francisco!

Best regards,

Boris Hasselblatt, Secretary of the AMS

As of press time, these are the 16 organizations that have joined forces to organize the JMM:

- American Institute of Mathematics
- American Mathematical Society
- American Statistical Association
- Association for Symbolic Logic
- Association for Women in Mathematics
- Centre de Recherches Mathématiques—Pacific Institute for the Mathematical Sciences—Atlantic Association for Research in Mathematical Sciences
- Consortium for Mathematics and its Applications
- International Linear Algebra Society
- Julia Robinson Mathematics Festival
- National Association of Mathematicians
- Pi Mu Epsilon
- Pro Mathematica Arte
- Simons Laufer Mathematical Sciences Institute/MSRI
- Society for Industrial and Applied Mathematics
- Spectra, the Association for LGBTQ+ Mathematicians
- Transforming Post-Secondary Education in Mathematics

San Francisco, CA

Moscone North/South, Moscone Center

January 3–6, 2024

Wednesday – Saturday

Meeting #1192

This meeting includes the annual meetings of the AMS, American Institute of Mathematics (AIM), American Statistical Association (ASA), Association for Women in Mathematics (AWM), and National Association of Mathematicians (NAM), winter meeting of Association for Symbolic Logic (ASL), and sessions/events by them and Centre de Recherches Mathématiques—Pacific Institute for the Mathematical Sciences—Atlantic Association for Research in Mathematical Sciences (CRM-PIMS-AARMS), Consortium for Mathematics and its Applications (COMAP), International Linear Algebra Society (ILAS), Julia Robinson Mathematics Festival (JRMF), Pi Mu Epsilon (PME), Pro Mathematica Arte (PMA), The Simons Laufer Mathematical Sciences Institute (SLMath), formerly Mathematical Sciences Research Institute (MSRI), Society for Industrial and Applied Mathematics (SIAM), Spectra, and Transforming Post-Secondary Education in Mathematics (TPSE).

Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 1

Deadlines

For organizers: Expired
For abstracts: September 12, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/national.html`.

Joint Invited Addresses

Maria Chudnovsky, Princeton University, *What Makes a Problem Hard?* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Anne Schilling, University of California, Davis, *The Ubiquity of Crystal Bases* (AWM-AMS Noether Lecture).

Peter Winkler, Dartmouth College, *Permutons* (AAAS-AMS Invited Address).

Kamuela E. Yong, University of Hawaii West Oahu, *When Mathematicians Don't Count* (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).

AMS Invited Addresses

Ruth Charney, Brandeis University, *From Braid Groups to Artin Groups* (AMS Retiring Presidential Address).

Daniel Erman, University of Wisconsin-Madison, *Title to be announced.*

Suzanne Marie Lenhart, University of Tennessee/Knoxville, *Natural System Management: A Mathematician's Perspective* (AMS Josiah Willard Gibbs Lecture).

Ankur Moitra, Massachusetts Institute of Technology, *Learning from Dynamics* (von Neumann Lecture).

Kimberly Sellers, Georgetown University, *Dispersed Methods for Handling Dispersed Count Data.*

Terence Tao, UCLA, *Machine Assisted Proof* (AMS Colloquium Lecture I).

Terence Tao, UCLA, *Machine Assisted Proof* (AMS Colloquium Lecture II).

Terence Tao, UCLA, *Machine Assisted Proof* (AMS Colloquium Lecture III).

John Urschel, MIT, *Title to be announced* (AMS Erdős Lecture for Students).

Suzanne L. Weekes, SIAM, *Title to be announced* (AMS Lecture on Education).

Melanie Matchett Wood, Harvard University, *An Application of Probability Theory for Group to 3-Manifolds* (AMS Maryam Mirzakhani Lecture).

Invited Addresses of Other JMM Partners

Katherine Ensor, Rice University, *Celebrating Statistical Foundations Driving 21st-Century Innovation* (ASA Invited Address).

Stephan Ramon Garcia, Pomona College, *Title to be announced* (ILAS Invited Address).

Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland; past president of American Physical Society, National Medal of Science, *What Challenges Does Data Science Present to Mathematics Education?* (TPSE Invited Address).

Trachette Jackson, University of Michigan, *Mobilizing Mathematics for the Fight Against Cancer* (PME J. Sutherland Frame Lecture).

Joni Teräväinen, University of Turku, *Title to be announced* (AIM Alexanderson Award Lecture).

Matthew Harrison-Trainor, UIC, *The Complexity of Classifying Topological Spaces* (ASL Invited Address).

Åsa Hirvonen, University of Helsinki, *Games for Measuring Distances Between Metric Structures* (ASL Invited Address).

François Loeser, Institut Universitaire de France, Sorbonne, *Model Theory and Non-Archimedean Geometry* (ASL Invited Address).

Toby Meadows, University of California Irvine, *A Moderate Foundational Argument for the Generic Multiverse* (ASL Invited Address).

Dima Sinapova, Rutgers University, *Combinatorial Principles at Successors of Singular Cardinals* (ASL Invited Address).

Sławomir Solecki, Cornell University, *Descriptive Set Theory and Generic Measure Preserving Transformations* (ASL Invited Address).

Mariana Vicaria, University of California, Los Angeles, *Model Theory of Valued Fields* (ASL Invited Address).

Henri Darmon, McGill University, *Title to be announced* (CRM-PIMS-AARMS Invited Address).

Mariel Vazquez, UC Davis, *Title to be announced* (SIAM Invited Address).

Shelly M. Jones, Central Connecticut State University, *Title to be announced* (NAM Claytor-Woodard Address).

Ranthony A.C. Edmonds, Duke University, *Hidden Figures Revealed* (NAM Cox-Talbot Address).

Joint Prize Session

Join the JMM 2024 Partners in celebrating the achievements of a selection of their prize and award winners at 4:30 p.m. PST on Wednesday. All participants are invited and encouraged to attend.

Special Sessions of the JMM

If you are volunteering to speak in a Special Session, you should submit your abstract as early as possible via the abstract submission form found at www.jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

American Institute for Mathematics Special Sessions

AIM Special Session Associated with the Alexanderson Award and Lecture (Code: AIMSS2A), **Joni Teräväinen**, University of Turku, **Terence Tao**, UCLA, **Kasia Matomäki**, University of Turku, **Maksym Radziwill**, California Institute of Technology, and **Tamar Ziegler**, Hebrew University.

Equivariant Techniques in Stable Homotopy Theory (Code: AIMSS4A), **Michael A. Hill**, UCLA, and **Anna Marie Bohmann**, Vanderbilt University.

Graphs and Matrices (Code: AIMSS5A), **Mary Flagg**, University of St. Thomas, and **Bryan A Curtis**, Iowa State University.

Little School Dynamics: Cool Research by Researchers at PUIs (Code: AIMSS1A), **Kimberly Ayers**, California State University, San Marcos, **Ami Radunskaya**, Pomona College, **Andy Parrish**, Eastern Illinois University, **David M. McClendon**, Ferris State University, and **Han Li**, Wesleyan University.

Math Circle Activities as a Gateway Into Research (Code: AIMSS3A), **Jeffrey Musyt**, Slippery Rock University, **Lauren L Rose**, Bard College, **Tom G. Stojsavljevic**, Beloit College, **Nick Rauh**, Julia Robinson Math Festivals, **Edward Charles Keppelmann**, University of Nevada Reno, **Allison Henrich**, Seattle University, **Violeta Vasilevska**, Utah Valley University, and **Gabriella A. Pinter**, University of Wisconsin, Milwaukee.

American Mathematical Society Special Sessions

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

Advances in Analysis, PDE's and Related Applications (Code: SS 50A), **Tepper L. Gill**, Howard University, **E. Kwessi**, Trinity University, and **Henok Mawi**, Howard University (Washington, DC, US).

Advances in Coding Theory (Code: SS 13A), **Emily McMillon**, Rice University, **Christine Ann Kelley**, University of Nebraska-Lincoln, **Tefjol Pllaha**, University of Nebraska-Lincoln, and **Mary Wootters**, Stanford University.

Algebraic Approaches to Mathematical Biology (Code: SS 71A), **Nicolette Meshkat**, Santa Clara University, **Cash Bortner**, California State University, Stanislaus, and **Anne Shiu**, Texas A&M University.

Algebraic Structures in Knot Theory (Code: SS 69A), **Sam Nelson**, Claremont McKenna College, and **Neslihan Gügümcu**, Izmir Institute of Technology in Turkey.

AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology (Code: SS 46A), **Sarah Blackwell**, Max Planck Institute for Mathematics, **Luya Wang**, University of California, Berkeley, and **Nicole Magill**, Cornell University (AMS-AWM).

Analysis and Differential Equations at Undergraduate Institutions (Code: SS 24A), **Evan Randles**, Colby College, and **Lisa Naples**, Macalester College.

Applications of Extremal Graph Theory to Network Design (Code: SS 91A), **Kelly Isham**, Colgate University, and **Laura Monroe**, Los Alamos National Laboratory.

Applications of Hypercomplex Analysis (Code: SS 3A), **Mihaela B. Vajiac**, Chapman University, **Daniel Alpay**, Chapman University, and **Paula Cerejeiras**, University of Aveiro, Portugal.

Applied Topology Beyond Persistence Diagrams (Code: SS 61A), **Nikolas Schonsheck**, University of Delaware, **Lori Ziegelmeier**, Macalester College, **Gregory Henselman-Petrusek**, University of Oxford, and **Chad Giusti**, Oregon State University.

Applied Topology: Theory, Algorithms, and Applications (Code: SS 35A), **Woojin Kim**, Duke University, **Johnathan Bush**, University of Florida, **Alex McCleary**, Ohio State University, **Sarah Percival**, Michigan State University, and **Iris Yoon**, University of Oxford.

Arithmetic Geometry with a View toward Computation (Code: SS 81A), **David Lowry-Duda**, ICERM/Brown University, **Barinder Banwait**, Boston University, **Shiva Chidambaram**, Massachusetts Institute of Technology, **Juanita Duque-Rosero**, Boston University, **Brendan Hassett**, ICERM/Brown University, and **Ciaran Schembri**, Dartmouth College.

Bridging Applied and Quantitative Topology (Code: SS 62A), **Henry Adams**, University of Florida, and **Ling Zhou**, The Ohio State University.

Coding Theory for Modern Applications (Code: SS 74A), **Rafael D'Oliveira**, Clemson University, **Hiram H. Lopez**, Cleveland State University, and **Allison Beemer**, University of Wisconsin-Eau Claire.

Combinatorial Insights into Algebraic Geometry (Code: SS 73A), **Javier Gonzalez Anaya**, UC Riverside.

Combinatorial Perspectives on Algebraic Curves and their Moduli (Code: SS 17A), **Sam Payne**, UT Austin, **Melody Chan**, Brown University, **Hannah K. Larson**, Harvard University and UC Berkeley, and **Siddarth Kannan**, Brown University.

Combinatorics for Science (Code: SS 52A), **Stephen J. Young**, **Bill Kay**, and **Sinan Aksoy**, Pacific Northwest National Laboratory.

Commutative Algebra and Algebraic Geometry (associated with Invited Address by Daniel Erman) (Code: SS 98A), **Daniel Erman**, University of Wisconsin-Madison, and **Aleksandra C. Sobieska**, University of Wisconsin-Madison.

Complex Analysis, Operator Theory, and Real Algebraic Geometry (Code: SS 42A), **J. E. Pascoe**, Drexel University, **Kelly Bickel**, Bucknell University, and **Ryan K. Tully-Doyle**, Cal Poly SLO.

Complex Social Systems (a Mathematics Research Communities session) I (Code: SS 104A), **Ekaterina Landgren**, University of Colorado, Boulder, **Cara Sulyok**, Lewis University, **Casey Lynn Johnson**, UCLA, **Molly Lynch**, Hollins University, and **Rebecca Hardenbrook**, Dartmouth College.

Computable Mathematics: A Special Session Dedicated to Martin D. Davis (Code: SS 1A), **Valentina S. Harizanov**, George Washington University, **Alexandra Shlapentokh**, East Carolina University, and **Wesley Calvert**, Southern Illinois University.

Computational Biomedicine: Methods - Models - Applications (Code: SS 7A), **Nektarios A. Valous**, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Germany, **Anna Konstorum**, Center for Computing Sciences, Institute for Defense Analyses, **Heiko Enderling**, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, and **Dirk Jäger**, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Germany.

Computational Techniques to Study the Geometry of the Shape Space (Code: SS 97A), **Shira Faigenbaum-Golovin**, Duke University, **Shan Shan**, University of Southern Denmark, and **Ingrid Daubechies**, Duke University.

Covering Systems of the Integers and Their Applications (Code: SS 41A), **Joshua Harrington**, Cedar Crest College, **Tony Wing Hong Wong**, Kutztown University of Pennsylvania, and **Matthew Litman**, University of California, Davis.

Cryptography and Related Fields (Code: SS 86A), **Ryann Cartor**, Clemson University, **Angela Robinson**, NIST, and **Daniel Everett Martin**, Clemson University.

Derived Categories, Arithmetic, and Geometry (a Mathematics Research Communities session) I (Code: SS 105A), **Anirban Bhaduri**, University of South Carolina, **Gabriel Dorfsman-Hopkins**, St. Lawrence University, **Patrick Lank**, University of South Carolina, and **Peter McDonald**, University of Utah.

Developing Students' Technical Communication Skills through Mathematics Courses (Code: SS 36A), **Michelle L. Ghrist**, Gonzaga University, **Timothy P. Chartier**, Davidson College, **Maila B. Hallare**, US Air Force Academy, and **Denise Taunton Reid**, Valdosta State University.

Diffusive Systems in the Natural Sciences (Code: SS 88A), **Francesca Bernardi**, Worcester Polytechnic Institute, and **Owen Lewis**, University of New Mexico.

Discrete Homotopy Theory (Code: SS 25A), **Krzysztof R. Kapulkin**, University of Western Ontario, **Anton Dochtermann**, Texas State University, and **Antonio Rieser**, CONACYT-CIMAT.

Dynamical Systems Modeling for Biological and Social Systems (Code: SS 75A), **Daniel Brendan Cooney**, University of Pennsylvania, **Chadi M. Saad-Roy**, University of California, Berkeley, and **Chris M. Heggerud**, University of California, Davis.

Dynamics and Management in Disease or Ecological Models (associated with Gibbs Lecture by Suzanne Lenhart) (Code: SS 100A), **Suzanne Lenhart**, University of Tennessee, Knoxville, **Christina Edholm**, Scripps College, and **Wandi Ding**, Middle Tennessee State University.

Dynamics and Regularity of PDEs (Code: SS 32A), **Zongyuan Li**, Rutgers University, **Weinan Wang**, University of Arizona, **Xueying Yu**, University of Washington, and **Zhiyuan Zhang**, Northeastern University.

Epistemologies of the South and the Mathematics of Indigenous Peoples (Code: SS 77A), **María Del Carmen Bonilla Tumialán**, National University of Education Enrique Guzman y Valle, **Wilfredo Vidal Alangui**, University of the Philippines Baguio, and **Domingo Yojcom Rocché**, Center for Scientific and Cultural Research.

Ergodic Theory, Symbolic Dynamics, and Related Topics (Code: SS 10A), **Andrew T. Dykstra**, Hamilton College, and **Shrey Sanadhya**, Ben Gurion University of the Negev, Israel.

Ethics in the Mathematics Classroom (Code: SS 6A), **Victor Piercey**, Ferris State University, and **Catherine Buell**, Fitchburg State University.

Explicit Computation with Stacks (a Mathematics Research Communities session) I (Code: SS 103A), **Santiago Arango**, Emory University, **Jonathan Richard Love**, CRM Montreal, and **Sameera Vemulapalli**, Princeton University.

Exploring Spatial Ecology via Reaction Diffusion Models: New Insights and Solutions (Code: SS 20A), **Jerome Goddard II**, Auburn University Montgomery, and **Ratnasingham Shivaji**, University of North Carolina Greensboro.

Extremal and Probabilistic Combinatorics (Code: SS 15A), **Sam Spiro**, Rutgers University, and **Corrine Yap**, Georgia Institute of Technology

Geometric Analysis in Several Complex Variables (Code: SS 9A), **Ming Xiao**, University of California, San Diego, **Bernhard Lamel**, Texas A&M University at Qatar, and **Nordine Mir**, Texas A&M University at Qatar.

Geometric Group Theory (Associated with the AMS Retiring Presidential Address) (Code: SS 38A), **Kasia Jankiewicz**, University of California Santa Cruz, **Edgar A. Bering**, San José State University, **Marion Campisi**, San José State University, and **Tim Hsu** and **Giang Le**, San José State University.

Geometry and Symmetry in Differential Equations, Control, and Applications (Code: SS 93A), **Taylor Joseph Klotz** and **George Wilkens**, University of Hawaiʻi.

Geometry and Topology of High-Dimensional Biomedical Data (Code: SS 67A), **Smita Krishnaswamy**, Yale University, **Dhananjay Bhaskar**, Yale University, **Bastian Rieck**, Technical University of Munich, and **Guy Wolf**, Université de Montréal.

Group Actions in Commutative Algebra (Code: SS 51A), **Alessandra Costantini**, Oklahoma State University, **Alexandra Seceleanu**, University of Nebraska-Lincoln, and **Andras Lorincz**, University of Oklahoma.

Hamiltonian Systems and Celestial Mechanics (Code: SS 23A), **Zhifu Xie**, The University of Southern Mississippi, and **Ernesto Pérez-Chavela**, ITAM.

Harmonic Analysis, Geometry Measure Theory, and Fractals (Code: SS 54A), **Kyle Hambrook**, San Jose State University, **Chun-Kit Lai**, San Francisco State University, and **Caleb Marshall**, University of British Columbia.

History of Mathematics (Code: SS 89A), **Adrian Rice**, Randolph-Macon College, **Sloan Evans Despeaux**, Western Carolina University, **Deborah Kent**, University of St. Andrews, and **Jemma Lorenat**, Pitzer College.

Homological Techniques in Noncommutative Algebra (Code: SS 48A), **Robert Won**, George Washington University, **Ellen E. Kirkman**, Wake Forest University, and **James J. Zhang**, University of Washington.

Homotopy Theory (Code: SS 47A), **Krzysztof R. Kapulkin**, University of Western Ontario, **Daniel K. Dugger**, University of Oregon, **Jonathan Beardsley**, University of Nevada, Reno, and **Thomas Brazelton**, University of Pennsylvania.

Ideal and Factorization Theory in Rings and Semigroups (Code: SS 4A), **Scott Chapman**, Sam Houston State University, and **Alfred Geroldinger**, University of Graz.

Informal Learning, Identity, and Attitudes in Mathematics (Code: SS 44A), **Sergey Grigorian**, **Mayra Ortiz**, **Xiaohui Wang**, and **Aaron Wilson**, University of Texas Rio Grande Valley.

Integer Partitions, Arc Spaces and Vertex Operators (Code: SS 59A), **Hussein Mourtada**, Université Paris Cité, and **Andrew R. Linshaw**, University of Denver.

Interplay Between Matrix Theory and Markov Systems: Applications to Queueing Systems and of Duality Theory (Code: SS 58A), **Alan Krinik** and **Randall J. Swift**, California State Polytechnic University, Pomona.

Issues, Challenges and Innovations in Instruction of Linear Algebra (Code: SS 96A), **Feroz Siddique**, University of Wisconsin-Eau Claire, and **Ashish K. Srivastava**, Saint Louis University.

Joint Meetings Registration (Code: REGSAT), **Radmila Sazdanovic**, NC State University (Joint).

Knots, Skein Modules, and Categorification (Code: SS 66A), **Rhea Palak Bakshi**, ETH Institute for Theoretical Studies, Zurich, **Sujoy Mukherjee**, University of Denver, and **Jozef Henryk Przytycki**, George Washington University.

Large Random Permutations (affiliated with AAAS-AMS Invited Address by Peter Winkler) (Code: SS 101A), **Peter Winkler**, Dartmouth College, and **Jacopo Borga**, Stanford University.

Loeb Measure after 50 Years (Code: SS 12A), **Yeneng Sun**, National University of Singapore, **Robert M. Anderson**, UC Berkeley, and **Matt Insall**, Missouri University of Science and Technology.

Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later (Code: SS 85A), **Younhee Lee**, Southern Connecticut State University, **James Alvarez**, University of Texas Arlington, **Ekaterina Fuchs**, City College of San Francisco, **Tyler Kloefkorn**, American Mathematical Society, **Yvonne Lai**, University of Nebraska-Lincoln, and **Carl Olimb**, Augustana University.

Mathematical Modeling and Simulation of Biomolecular Systems (Code: SS 79A), **Zhen Chao**, Western Washington University, and **Jiahui Chen**, University of Arkansas.

Mathematical Modeling of Nucleic Acid Structures (Code: SS 70A), **Pengyu Liu**, University of California, Davis, **Van Pham**, University of South Florida, and **Svetlana Poznanovic**, Clemson University.

Mathematical Physics and Future Directions (Code: SS 34A), **Shanna Dobson**, University of California, Riverside, **Tepper L. Gill**, Howard University, **Michael Anthony Maroun**, University of California, Riverside, and **Lance W. Nielsen**, Creighton University.

Mathematics and Philosophy (Code: SS 53A), **Tom Morley**, Georgia Tech, and **Bonnie Gold**, Monmouth University.

Mathematics and Quantum (Code: SS 82A), **Kaifeng Bu** and **Arthur M. Jaffe**, Harvard, **Sui Tang**, UCSB, and **Jonathan Weitsman**, Northeastern University.

Mathematics and the Arts (Code: SS 76A), **Karl Kattchee**, University of Wisconsin-La Crosse, **Doug Norton**, Villanova University, and **Anil Venkatesh**, Adelphi University.

Mathematics of Computer Vision (Code: SS 94A), **Timothy Duff** and **Max Lieblich**, University of Washington.

Mathematics of DNA and RNA (Code: SS 87A), **Marek Kimmel**, Rice University, **Chris McCarthy**, BMCC, City University of New York, and **Johannes Familton**, Borough of Manhattan Community College, CUNY.

Metric Dimension of Graphs and Related Topics (Code: SS 18A), **Briana Foster-Greenwood**, Cal Poly Pomona, and **Christine Uhl**, St. Bonaventure University.

Metric Geometry and Topology (Code: SS 33A), **Christine M. Escher**, Oregon State University, and **Catherine Searle**, Wichita State University.

Mock Modular forms, Physics, and Applications (Code: SS 40A), **Amanda Folsom**, Amherst College, **Terry Gannon**, University of Alberta, and **Larry Rolen**, Vanderbilt University.

Modeling Complex Adaptive Systems in Life and Social Sciences (Code: SS 57A), **Yun Kang** and **Theophilus Kwofie**, Arizona State University, and **Sabrina H. Streipert**, University of Pittsburgh.

Modeling to Motivate the Teaching of the Mathematics of Differential Equations (Code: SS 21A), **Brian Winkel**, SIMIODE, **Kyle T. Allaire**, Worcester State University, **Maila B. Hallare**, US Air Force Academy, **Yanping Ma**, Loyola Marymount University, and **Lisa Naples**, Macalester College.

Modelling with Copulas: Discrete vs Continuous Dependent Data (Code: SS 19A), **Martial Longla**, University of Mississippi, and **Isidore Seraphin Ngongo**, University of Yaounde I.

Modern Developments in the Theory of Configuration Spaces (Code: SS 31A), **Christin Bibby**, Louisiana State University, and **Nir Gadish**, University of Michigan.

Modular Tensor Categories and TQFTs beyond the Finite and Semisimple (Code: SS 27A), **Colleen Delaney**, UC Berkeley, and **Nathan Geer**, Utah State University.

Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects (Code: SS 64A), **Vinodh Kumar Chellamuthu**, Utah Tech University, and **Xiaoxia Xie**, Idaho State University.

New Faces in Operator Theory and Function Theory (Code: SS 37A), **Michael R Pilla**, Ball State University, and **William Thomas Ross**, University of Richmond.

Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives (Code: SS 95A), **Armando Roldan**, University of Central Florida, and **Thomas Dombrowski**, Moffitt Cancer Center.

Number Theory in Memory of Kevin James (Code: SS 39A), **Jim L. Brown**, Occidental College, and **Felice Manganiello**, Clemson University.

Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms (Code: SS 92A), **Anastasiia Minenkova**, University of Hartford, and **Gamal Mograby**, University of Cincinnati.

On Topological and Algebraic Approaches for Optimization (Code: SS 80A), **Ali Mohammad Nezhad**, Carnegie Mellon University.

Partition Theory and q-Series (Code: SS 30A), **William Jonathan Keith**, Michigan Technological University, **Brandt Kronholm**, University of Texas Rio Grande Valley, and **Dennis Eichhorn**, University of California, Irvine.

Polymath Jr REU Student Research (Code: SS 106A), **Steven Joel Miller**, Williams College, and **Alexandra Seceleanu**, University of Nebraska-Lincoln.

Principles, Spatial Reasoning, and Science in First-Year Calculus (Code: SS 16A), **Yat Sun Poon** and **Catherine Lussier**, University of California, Riverside, and **Bryan Carrillo**, Saddleback College.

Quantitative Justice (Code: NAMSS1A), **Ron Buckmire**, Occidental College, **Omayra Ortega**, Sonoma State University, and **Robin Wilson**, California State Polytechnic University, Pomona (NAM-SIAM-AMS).

Quaternions (Code: SS 49A), **Chris McCarthy**, BMCC, City University of New York, **Johannes Familton**, Borough of Manhattan Community College, CUNY, and **Terrence Richard Blackman**, Medgar Evers Community College, CUNY.

Recent Advances in Mathematical Models of Diseases: Analysis and Computation (Code: SS 22A), **Najat Ziyadi** and **Jemal S. Mohammed-Awel**, Morgan State University.

Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems (Code: SS 72A), **Tuan A. Phan**, IMCI, University of Idaho, **Nhu N. Nguyen**, University of Rhode Island, and **Jianjun P. Tian**, New Mexico State University.

Recent Developments in Commutative Algebra (Code: SS 45A), **Austyn Simpson** and **Alapan Mukhopadhyay**, University of Michigan, and **Thomas Marion Polstra**, University of Virginia.

Recent Developments in Numerical Methods for PDEs and Applications (Code: SS 2A), **Chunmei Wang**, University of Florida, **Long Chen**, UC Irvine, **Shuhao Cao**, University of Missouri-Kansas City, and **Haizhao Yang**, University of Maryland College Park.

Recent Developments on Markoff Triples (Code: SS 68A), **Elena Fuchs**, UC Davis, and **Daniel Everett Martin**, Clemson University

Recent Progress in Inference and Sampling (Associated with AMS Invited Address by Ankur Moitra) (Code: SS 99A), **Ankur Moitra**, Massachusetts Institute of Technology, and **Sitan Chen**, Harvard University.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs (Code: SS 29A), **Darren A. Narayan**, Rochester Institute of Technology, **John C. Wierman**, Johns Hopkins University, **Mark Daniel Ward**, Purdue University, **Khang Duc Tran**, California State University, Fresno, and **Christopher O'Neill**, San Diego State University.

Research Presentations by Math Alliance Scholar Doctorates (Code: SS 84A), **Teresa Martines**, University of Texas, Austin, and **David Goldberg**, Math Alliance/Purdue University.

Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session) I (Code: SS 102A), **Aleyah Dawkins**, George Mason University, **Xavier Ramos Olive**, Smith College, **Zhaiming Shen**, University of Georgia, **David Harry Richman**, University of Washington, and **Michael G Rawson**, PNNL.

Serious Recreational Mathematics (Code: SS 8A), **Erik Demaine**, Massachusetts Institute of Technology, **Robert A. Hearn**, H3 Labs, and **Tomas Rokicki**, California.

Solvable Lattice Models and their Applications Associated with the Noether Lecture (Code: SS 43A), **Amol Aggarwal**, Columbia, **Benjamin Brubaker**, University of Minnesota-Twin Cities, **Daniel Bump**, Stanford University, **Andrew Hardt**, Stanford University, **Slava Naprienko**, Stanford University and University of North Carolina, **Leonid Petrov**, University of Virginia, and **Anne Schilling**, University of California, Davis.

Spectral Methods in Quantum Systems (Code: SS 26A), **Matthew Powell**, Georgia Institute of Technology, and **Wencai Liu**, Texas A&M University.

Structure-preserving Algorithms, Analysis and Simulations for Differential Equations (Code: SS 5A), **Brian E. Moore**, University of Central Florida, and **Qin Sheng**, Baylor University.

The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors (Code: SS 11A), **Quiyana Murphy**, Virginia Tech, **Sofia Rose Martinez Alberga**, Purdue University, **Kelly Buch**, Austin Peay State University, and **Alexis Hardesty**, Texas Tech University.

The Mathematics of Decisions, Elections, and Games (Code: SS 60A), **David McCune**, William Jewell College, **Michael A. Jones**, Mathematical Reviews | AMS, and **Jennifer M. Wilson**, Eugene Lang College, The New School.

Theoretical and Numerical Aspects of Nonlocal Models (Code: SS 63A), **Nicole Buczkowski**, Worcester Polytechnic Institute, **Christian Alexander Glusa**, Sandia National Laboratories, and **Animesh Biswas**, University of Nebraska Lincoln.

Theta Correspondence (Code: SS 56A), **Edmund Karasiewicz** and **Petar Bakic**, University of Utah.

The Teaching and Learning of Undergraduate Ordinary Differential Equations (Code: SS 14A), **Viktoria Savatorova**, Central Connecticut State University, **Chris Goodrich**, The University of New South Wales, **Itai Seggev**, Wolfram Research, **Beverly H. West**, Cornell University, and **Maila B. Hallare**, US Air Force Academy.

Thresholds in Random Structures (Code: SS 78A), **Will Perkins**, Georgia Tech.

Topics in Combinatorics and Graph Theory (Code: SS 90A), **Cory Palmer**, University of Montana, **Neal Bushaw**, Virginia Commonwealth University, and **Anastasia Halfpap**, University of Montana.

Topics in Equivariant Algebra (Code: SS 65A), **Ben Spitz**, University of California Los Angeles, and **Christy Hazel** and **Michael A. Hill**, UCLA.

Undergraduate Research Activities in Mathematical and Computational Biology (Code: SS 55A), **Timothy D. Comar**, Benedictine University, and **Anne E. Yust**, University of Pittsburgh.

Using 3D-Printed and Other Digitally-Fabricated Objects in the Mathematics Classroom (Code: SS 83A), **Shelby Stanhope**, US Air Force Academy, **Paul E. Seeburger**, Monroe Community College, and **Stepan Paul**, North Carolina State University.

Water Waves (Code: SS 28A), **Anastassiya Semenova** and **Bernard Deconinck**, University of Washington, **John Carter**, Seattle University, and **Eleanor Devin Byrnes**, University of Washington.

Association for Symbolic Logic Special Sessions

Descriptive Methods in Dynamics, Combinatorics, and Large Scale Geometry (Code: ASLSS1A), **Jenna Zomback**, Williams College, and **Forte Shinko**, UCLA.

Association for Women in Mathematics Special Sessions

EvenQuads Live and In-person: The Honorees and the Games (Code: AWMSS2A), **sarah-marie belcastro**, Mathematical Staircase, Inc., **Sherli Koshy-Chenthittayil**, Touro University Nevada, **Oscar Vega**, California State University, Fresno, **Monica D. Morales-Hernandez**, Adelphi University, **Linda McGuire**, Muhlenberg College, and **Denise A. Rangel Tracy**, Fairleigh Dickinson University.

Mathematics in the Literary Arts and Pedagogy in Creative Settings (Code: AWMSS4A), **Shanna Dobson**, University of California, Riverside, and **Claudia Maria Schmidt**, California State University.

Recent Developments in Harmonic Analysis (Code: AWMSS1A), **Betsy Stovall**, University of Wisconsin-Madison, and **Sarah E. Tammen**, Massachusetts Institute of Technology.

Women in Mathematical Biology (Code: AWMSS3A), **Christina Edholm**, Scripps College, **Lihong Zhao**, University of California, Merced, and **Lale Asik**, University of the Incarnate Word.

Consortium for Mathematics and its Applications Special Sessions

Math Modeling Contests: What They Are, How They Benefit, What They Did – Discussions with the Students and Advisors (Code: COMAPSS1A), **Jack A. Picciuto**, COMAP, and **Kayla Blyman**, Saint Martin's University.

International Linear Algebra Society Special Sessions

Generalized Numerical Ranges and Related Topics (Code: ILASSS2A), **Tin-Yau Tam** and **Pan-Shun Lau**, University of Nevada, Reno.

Graphs and Matrices (Code: ILASSS1A), **Jane Breen**, Ontario Tech University, and **Stephen Kirkland**, University of Manitoba.

Innovative and Effective Ways to Teach Linear Algebra (Code: ILASSS6A), **David M. Strong**, Pepperdine University, and **Sepideh Stewart**, University of Oklahoma.

Linear algebra, matrix theory, and its applications (Code: ILASSS3A), **Stephan Ramon Garcia**, Pomona College.

Sign-pattern Matrices and Their Applications (Code: ILASSS4A), **Bryan L. Shader**, University of Wyoming, and **Minerva Catral**, Xavier University.

Spectral and combinatorial problems for nonnegative matrices and their generalizations (Code: ILASSS5A), **Pietro Paparella**, University of Washington Bothell, and **Michael J. Tsatsomeros**, Washington State University.

National Association of Mathematicians Special Sessions

NAM-SIAM-AMS Special Session on Quantitative Justice (Code: NAMSS1A), **Ron Buckmire**, Occidental College, **Omayra Ortega**, Sonoma State University, and **Robin Wilson**, Loyola Marymount University (NAM-SIAM-AMS).

The Simons Laufer Mathematical Sciences Institute (SLMath), formerly MSRI Special Sessions

African Diaspora Joint Mathematics Working Groups (ADJOINT) (Code: SLMSS1A), **Caleb Ashley**, Boston College, and **Anisah Nabilah Nu'Man**, Spelman College.

Summer Research in Mathematics (SRiM): Recent Trends in Nonlinear Boundary Value Problems (Code: SLMSS3A), **Maya Chhetri**, UNC Greensboro, **Elliott Zachary Hollifield**, University of North Carolina at Pembroke, and **Nsoki Mavinga**, Swarthmore College.

The MSRI Undergraduate Program (MSRI-UP) (Code: SLMSS2A), **Maria Mercedes Franco**, Queensborough Community College-CUNY.

National Science Foundation Special Sessions

NSF Special Session Exploring Funding Opportunities in the Division of Mathematical Sciences (Code: NSFSS1A), **Elizabeth Wilmer**, NSF, and **Junping Wang**, National Science Foundation.

NSF Special Session on Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences (Code: NSFED), **Michael Ferrara**, National Science Foundation.

Pro Mathematica Arte Special Sessions

BSM Special Session: Mathematical Research in Budapest for Students and Faculty (Code: PMASS1A), **Kristina Cole Garrett**, St. Olaf College.

Society for Industrial and Applied Mathematics Minisymposia

SIAM Minisymposium on Computational Mathematics and the Power Grid (Code: SIAM5A), **Todd Munson**, Argonne National Laboratory.

SIAM Minisymposium on Current Advances in Modeling and Simulation to Uncover the Complexity of Disease Dynamics (Code: SIAM3A), **Naveen K. Vaidya**, San Diego State University, and **Elissa Schwartz**, Washington State University.

SIAM Minisymposium on Mathematical Methods in Computer Vision and Image Analysis (Code: SIAM6A), **Andreas Mang**, University of Houston.

SIAM Minisymposium on Mathematical Modeling of Complex Materials Systems (Code: SIAM2A), **Maria G Emelianenko**, George Mason University.

SIAM Minisymposium on Mathematics of Bacterial Viruses: From Virus Discovery to Mathematical Principles (Code: SIAM1A), **Javier Arsuaga**, University of California, Davis, **Carme Calderer**, University of Minnesota, and **Ami Bhatt**, Stanford University.

SIAM Minisymposium on Recent Developments in the Analysis and Control of Partial Differential Equations Arising in Fluid and Fluid-Structure Interactive Dynamics (Code: SIAM7A), **George Avalos**, University of Nebraska-Lincoln, and **Pelin Guven Geredeli**, Iowa State University.

SIAM Minisymposium on Scientific Machine Learning to Advance Modeling and Decision Support (Code: SIAM4A), **Erin Acquesta**, Sandia National Laboratories, **Timo Bremer**, Lawrence Livermore National Laboratories, and **Joseph Hart**, Sandia National Laboratories.

SIAM ED Session on Artificial Intelligence and its Uses in Mathematical Education, Research, and Automation in the Industry (Code: SIAM8A), **Kathleen Kavanagh**, Clarkson University, **Alvaro Ortiz Lugo**, Georgia Gwinnett College, and **Sergio Molina**, University of Cincinnati.

NAM-SIAM-AMS Special Session on Quantitative Justice (Code: NAMSS1A), **Ron Buckmire**, Occidental College, **Omayra Ortega**, Sonoma State University, and **Robin Wilson**, Loyola Marymount University (NAM-SIAM-AMS).

SPECTRA Special Sessions

Research by LGBTQ+ Mathematicians (Code: SPECTSS1A), **Devavrat Dabke**, Princeton University, **Joseph Nakao**, Swarthmore College, and **Michael A. Hill**, UCLA.

Contributed Paper Sessions of the JMM

AMS Contributed Paper Sessions

There will be sessions of ten-minute contributed talks. Although an individual may present only one contributed paper at a meeting, any combination of joint authorship may be accepted, provided no individual speaks more than once on the contributed paper program. Contributed papers will be grouped together by related subject classifications into sessions.

ASL Contributed Paper Sessions

ASL Contributed Paper Session (Code: ASLCP1A), **David Reed Solomon**, University of Connecticut.

COMAP Contributed Paper Sessions

COMAP Contributed Paper Session: Integrating Modeling into Established Courses (Code: COMAPCP1A), **Kayla Blyman**, Saint Martin's University.

NAM Contributed Paper Sessions

NAM Haynes-Granville-Browne Session of Presentations by Recent Doctoral Recipients (Code: NAMCPA), **Aris Winger**, Georgia Gwinnett College, **Torina D. Lewis**, American Mathematical Society, and **Omayra Ortega**, Sonoma State University.

PME Contributed Paper Sessions

PME Contributed Session on Research by Undergraduates (Code: PMECP1), **Thomas Wakefield**, Youngstown State University, and **Jennifer Beineke**, Western New England University.

TPSE Contributed Paper Sessions

TPSE Contributed Paper Session on Using Institutional and National Data Sources to Recruit, Retain and Support a Diverse Population of Mathematics Students (Code: TPSECP1A), **Rick Cleary**, Babson College, and **Mitchel T. Keller**, University of Wisconsin-Madison.

Submission of Abstracts for JMM Sessions

Authors must submit abstracts of talks through the JMM abstract submission site.[1] Simply follow the step-by-step instructions through to completion, until you receive confirmation of your successful submission. No submission is complete until you receive this confirmation. The deadline for all submissions is **September 12, 2023**. Late papers cannot be accommodated. Please email `meet@ams.org` if you have questions.

[1] `https://meetings.ams.org/math/jmm2024/cfp.cgi`

Programs of JMM Partners

Please see complete descriptions of these sessions on the JMM website.

American Mathematical Society

AMS Poster Session

AMS-PME Student Poster Session, organized by **Chad Awtrey**, Samford University, and **Frank Patane**, Samford University; Friday, 12–1:30 p.m. and 3:30–5:00 p.m. These sessions provide a venue for undergraduate students to deliver poster presentations based on original research; presentations that are purely expository in nature are not appropriate for these sessions. First-year graduate students are eligible to present if their research was completed while they were still undergraduates. High school students are eligible to present if their research was conducted under the supervision of a faculty member at a post-secondary institution. Presenters need not be members of any particular mathematics or honorary society.

Participants should submit an abstract through the JMM abstract submission portal by **September 26**. Questions regarding this session should be directed to **Chad Awtrey**, `cawtrey@samford.edu` or **Frank Patane**, `fpatane@samford.edu`.

AMS Panels

Please see complete descriptions of these sessions on the JMM website.

AMS Committee on Education Panel Discussion, organized by **Terrence Blackman**, Medger Evans College, **Michael Dorff**, Brigham Young University, **William Yslas Velez**, University of Arizona, and **Erica Walker**, Ontario Institute for Studies in Education; Thursday, 1:00–2:30 p.m. The moderator and panelists are to be announced. This panel is sponsored by the AMS Committee on Education.

AMS Committee on the Profession Panel Discussion: Building a Successful Research Career in Mathematics, organized by **Edray Herber Goins**, Pomona College, and **Pamela E. Harris**, University of Wisconsin at Milwaukee; Wednesday, 1:00–2:30 p.m. The moderator for this panel is **Edray Herber Goins**, Pomona College. Panelists are **Priyam Patel**, University of Utah, **Abbey Bourdon**, Wake Forest University, and **Henok Mawi**, Howard University. This panel is sponsored by the AMS Committee on the Profession.

AMS Committee on Science Policy Panel Discussion, organized by **Gunnar Carlsson**, Stanford University, **Duane Cooper**, Morehouse College, **Carla Cotwright-Williams**, US Department of Defense, **Fern Hunt**, National Institute of Standards and Technology, and **Jerry McNerney**, US Congressman, retired; Friday, 2:30–4:00 p.m. The moderator and panelists are to be announced. This panel is sponsored by the Committee on Science Policy.

AMS Workshops

Please see complete descriptions of these sessions on the JMM website.

*2024 AMS Workshop for Department Chairs and Leaders***.** This annual one-day workshop for department chairs, leaders, and prospective leaders will be held on Tuesday, January 2, 2024, 9:00 a.m.–3:00 p.m., the day before the JMM begins.

The workshop will provide opportunities to share experiences with issues and trends that have an impact on math department chairs, math departments, and colleges and universities. Workshop topics could include, but are not limited to, resources, handling stress (students, staff, and faculty), curriculum, and instructional delivery. The organizers expect the workshop to help build a community of leaders who can continue to exchange ideas and offer each other support and advice.

Registration for this workshop will include breakfast and lunch. More details about registration and associated fees will be available on the workshop web page.[2] Please send questions to `chairsworkshop@ams.org`.

Other AMS Events

Please see complete descriptions of these sessions on the JMM website.

Council, Time and location to be announced.

Business Meeting, Time and location to be announced.

MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Session, organized jointly by the Mathematical Association of America, Society for Industrial and Applied Mathematics, and the American Mathematical Society; Friday, 9:00–10:30 a.m. This year the session will consist of a lecture from 9:00–9:50 a.m. given by **Kamuela Yong**, University of Hawaii–West Oahu, *Title to be announced*, and a short panel discussion, *Title to be announced*, from 9:50–10:30 a.m. Panelists to be announced.

Career Fair, Thursday, 8:30–10:30 a.m. The AMS Career Fair is an opportunity for mathematically trained job seekers at various phases of education and experience—undergraduates, graduate students, postdocs, and others—to interact in

[2] `https://www.ams.org/chairsworkshop`

person with employers in Business, Entrepreneurship, Government, Industry, and Nonprofit (BEGIN). This event is job seekers' chance to discover how their mathematical training makes them strong candidates for BEGIN jobs.

Recruiters can represent their companies or organizations and connect with potential employees. For **US$180/$0 AMS Corporate Member**, recruiters will be provided with a table for print materials, where they will also be welcome to engage personally with interested BEGIN job seekers.

Information is available here: `https://www.ams.org/career-fair`.

Graduate School Fair, Friday, 8:30–10:30 a.m. This event is undergraduate and master's students' chance for one-stop shopping in the graduate school market. January is a great time for college juniors to learn more about applying to graduate school, and seniors may still be able to refine their search. Meet representatives from mathematical sciences graduate programs from universities all over the United States. At JMM 2023, over 300 students engaged with representatives from more than 60 graduate programs.

Colleges and universities that offer graduate programs in the mathematical sciences are invited to exhibit at this event. For **US$200/$140 AMS Institutional Member**, program representatives will be provided with a table on which to display posters and printed materials, and where they will be able to speak directly with interested students.

Information is available here: `https://www.ams.org/gradfair`.

Current Events Bulletin, organized by **David Eisenbud**, Mathematical Sciences Research Institute; Friday, 2:00–6:00 p.m.

AMS Travel Grants

PUI Faculty Travel Grants. The AMS is excited to offer an opportunity for faculty at primarily undergraduate institutions (PUI) to apply for funding to support attendance at the JMM. Grant funds can be used to offset expenses for travel, registration, lodging, and meals. One advantage of this funding is that it can be used to support participation in the Chairs' Workshop. Additional information can be found here: `https://www.ams.org/puifac-tg`.

Graduate Student Travel Grants. With funding from the AMS Next Generation Fund, the AMS will be accepting applications for partial travel support for graduate students attending the JMM in San Francisco, CA, January 3–6, 2024. While the AMS encourages students' institutions to match the award, matching is not required.

Applications will be accepted ONLY from doctoral students in mathematics who are in their last year of study; applicants must not have received their doctoral degrees before the travel takes place but must expect to receive their degrees within twelve months of the JMM. No student shall receive a grant more than once. Information can be found here: `https://www.ams.org/emp-student-JMM`.

Undergraduate Student Travel Grants. With support from the National Science Foundation and an anonymous donor, the AMS is offering travel support to a limited number of undergraduate students who are presenting in the following JMM sessions: Pi Mu Epsilon Undergraduate Poster Session, AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-baccalaureate Programs, and Other Special or Contributed Sessions at the JMM in San Francisco, CA, January 3–6, 2024.

Awards will help undergraduate students defray travel expenses associated with JMM participation. Applications are especially encouraged from students from groups that have been underrepresented in the mathematical sciences and from those with financial need. Additional information can be found here: `https://www.ams.org/undergrad-tg`.

Please also see the section on Child Care Grants.

American Association for the Advancement of Science

The AAAS-AMS Invited Address will be given by **Peter Winkler**, Dartmouth College, *Permutons*; Friday, 4:45 p.m.

American Institute for Mathematics

Please see complete descriptions of these sessions on the JMM website.

AIM has several AIM Special Sessions. A full list of these sessions can be found under the heading AIM Special Sessions above; a joint AMS-AIM Special Session is included.

AIM will host a reception; please see the listing in the Social Events section of the announcement.

American Statistical Association

The ASA Invited Address will be given by **Kathy Ensor**, Rice University, *Celebrating Statistical Foundations Driving 21st-Century Innovation.*

ASA will host a reception; please see the listing in the Social Events section of the announcement.

Association for Symbolic Logic

Please see complete descriptions of these sessions on the JMM website.

Association for Symbolic Logic Tutorial, Parts I & II, organized by **Solomon Reed**, ASL; Wednesday, 9:00–10:00 a.m and 1:00–2:00 p.m. The speaker for these tutorial sessions is to be announced.

The ASL Invited Address program will take place on Friday and Saturday. The program will include invited addresses by **Åsa Hirvonen**, University of Helsinki, **Dima Sinapova**, Rutgers University, **François Loeser**, Institute Universitaire de France, Sorbonne, **Mariana Vicaria**, University of California, Los Angeles, **Matthew Harrison-Trainor**, UIC, **Sławomir Solecki**, Cornell University, and **Toby Meadows**, University of California Irvine.

ASL will also host an ASL Contributed Paper Session on Friday afternoon and two ASL Special Sessions on Thursday; more information on these sessions can be found under the heading Association for Symbolic Logic Special Sessions above.

Association for Women in Mathematics

Please see complete descriptions of these sessions on the JMM website.

AWM Panel: Celebrating Academic Pivots in Mathematics; Friday, 1:00–2:30 p.m. Panel moderator and panelists are to be announced.

Business Meeting, organized by **Darla Kremer**, Association for Women in Mathematics; Time and location to be announced.

The AWM-AMS Noether Lecture will be delivered on Thursday at 9:45 a.m. by **Anne Schilling**, University of California, Davis, *The Ubiquity of Crystal Bases.*

AWM Workshop Poster Presentations and Reception, Friday, 4:00–5:30 p.m. AWM will conduct its workshop poster presentations by women graduate students. This session is open to all JMM attendees. AWM seeks volunteers to serve as mentors for workshop participants. If you are interested, please contact the AWM office at `awm@awm-math.org`.

Association for Women in Mathematics Reception and Award Presentation, Friday, 5:00-6:30 p.m. Please see the listing in the Social Events section of the announcement.

AWM Workshop: Women in Operator Theory, organized by **Catherine A. Beneteau**, University of South Florida, and **Asuman Aksoy**, Claremont McKenna College; Saturday, 8:00 a.m.–12:00 p.m. and 1:00–5:00 p.m. A Poster Session for graduate students and recent PhDs will be held in conjunction with the workshop on Friday. Updated information about the workshop is available at `www.awm-math.org`.

AWM also has a number of AWM Special Sessions. A full list of these sessions can be found under the heading AWM Special Sessions above.

Consortium for Mathematics and its Applications

Please see complete descriptions of these sessions on the JMM website.

COMAP Workshop on Modeling for Educators: Integrate Modeling into Your Classroom, organized by **Michelle Isenhour**, Consortium for Mathematics and Its Applications, **Victor Piercey**, Ferris State University, and **Daniel Teague**, North Carolina School of Science and Mathematics; Saturday, 1:00 p.m.–5:00 p.m.

COMAP Contributed Paper Session: Integrating Modeling into Established Courses I and II, organized by **Kayla Blyman**, Saint Martin's University, **Keith Erickson**, Georgia Gwinnett College, **Marie Meyer**, Lewis University, and **Katherine Pinzon**, Georgia Gwinnett College; Thursday, 8:00 a.m.–12:00 p.m. and 1:00–5:00 p.m.

COMAP Panel on Math Modeling Contests: Trends, Topics, and Tips, organized by **Kayla Blyman**, Saint Martin's University; Friday, 3:00–4:30 p.m.

COMAP also has a COMAP Special Session. More information on this session can be found under the heading Consortium for Mathematics and its Applications Special Sessions.

International Linear Algebra Society

The *ILAS Invited Address* will be given by **Stephan Ramon Garcia**, Pomona College, *Title to be announced.*

ILAS also has a number of ILAS Special Sessions. A full list of these sessions can be found under the heading ILAS Special Sessions above.

Julia Robinson Mathematics Festival

Please see complete descriptions of these sessions on the JMM website.

Julia Robinson Math Festival, organized by **Daniel Kline**, Julia Robinson Mathematics Festival; Saturday, 9:00 a.m–12:00 p.m.

The Simons Laufer Mathematical Sciences Institute (SLMath), formerly MSRI

Please see complete descriptions of these sessions on the JMM website.

NAM/MSRI/SLMath Film Presentation: World Premiere of George Csicsery's film "Journeys of Black Mathematicians: Part 1" and Panel Discussion, organized by **Omayra Ortega**, Sonoma State University, **Johnny Houston**, Elizabeth City State University, **Jenn Murawski**, MSRI/SLMath, and **George Csicsery**, Zala Films; Saturday, 11:30 a.m.–1:00 p.m. Moderator and panelists to be announced.

MSRI/SLMath has a number of MSRI/SLMath Special Sessions. A full list of these sessions can be found under the heading The Simons Laufer Mathematical Sciences Institute (SLMath), formerly MSRI Special Sessions.

MSRI/SLMath will host a reception; please see the listing in the Social Events section of the announcement.

National Association of Mathematicians

Please see complete descriptions of these sessions on the JMM website.

The Haynes-Granville-Browne Session of Presentations by Recent Doctoral Recipients in the Mathematical Sciences organized by **Aris Winger**, Georgia Gwinnett College, **Torina Lewis**, American Mathematical Society, and **Omayra Ortega**, Sonoma State University; Thursday, 8:00 a.m.–12:00 p.m.

The Cox-Talbot Address, **Ranthony A.C. Edmonds**, Duke University, *Hidden Figures Revealed*, organized by **Aris Winger**, Georgia Gwinnett College, **Torina Lewis**, American Mathematical Society, and **Omayra Ortega**, Sonoma State University; Friday, 7:45–8:45 p.m., after the banquet. See details about the banquet on Friday in the Social Events section.

NAM/MSRI/SLMath Film Presentation: World Premiere of George Csicsery's film "Journeys of Black Mathematicians: Part 1" and Panel Discussion, organized by **Omayra Ortega**, Sonoma State University, **Johnny Houston**, Elizabeth City State University, **Jenn Murawski**, MSRI/SLMath, and **George Csicsery**, Zala Films; Saturday, 11:30 a.m.–1:00 p.m. Moderator and panelists to be announced.

The NAM Business Meeting will take place on Saturday, 10:00–11:00 a.m.

NAM Claytor-Woodard Lecture, **Shelly M. Jones**, Central Connecticut State University, *Title to be announced*, organized by **Aris Winger**, Georgia Gwinnett College, **Torina Lewis**, American Mathematical Society, and **Omayra Ortega**, Sonoma State University; Thursday, 2:15–3:20 p.m.

Pi Mu Epsilon

Please see complete descriptions of these sessions on the JMM website.

Pi Mu Epsilon Contributed Sessions on Research by Undergraduates, organized by **Thomas Wakefield**, Youngstown State University, and **Jennifer Beineke**, Western New England University; Thursday, 1:00–5:00 p.m. and Friday, 8:00 a.m.–12:00 p.m.

The PME J. Sutherland Frame Lecture will be delivered on Friday at 2:15 p.m. by **Trachette Jackson**, University of Michigan, *Mobilizing Mathematics for the Fight Against Cancer*.

AMS-PME Student Poster Session, organized by **Chad Awtrey**, Samford University, and **Frank Patane**, Samford University; Friday, 12:00–1:30 p.m. and 3:30–5:00 p.m. These sessions feature research done by undergraduate students. First-year graduate students are eligible to present if their research was completed while they were still undergraduates. Research by high school students can be accepted if the research was conducted under the supervision of a faculty member at a post-secondary institution.

Appropriate content for a poster includes, but is not limited to, a new result, a new proof of a known result, a new mathematical model, an innovative solution to a Putnam problem, or a method of solution to an applied problem. Purely expository material is not appropriate for this session.

Participants should submit an abstract through the JMM abstract submission portal by **September 26**. Questions regarding this session should be directed to **Chad Awtrey**, `cawtrey@samford.edu` or **Frank Patane**, `fpatane@samford.edu`.

PME Panel: What Every Student Should Know about the JMM, organized by **Jennifer Beineke**, Western New England University, **Stephanie Edwards**, Hope College, and **Thomas Wakefield**, Youngstown State University; Wednesday, 1:00–2:30 p.m. and Thursday, 10:30 a.m.–12:00 p.m. This panel is sponsored by Pi Mu Epsilon.

Pro Mathematica Arte

Please see complete descriptions of these sessions on the JMM website.

The PMA program includes a Budapest Semesters in Math Special Session. Information on this session can be found under the heading Pro Mathematica Arte Special Sessions above.

PMA will also host a reception for BSM Alumni. Please see the listing in the Social Events section of the announcement.

Society for Industrial and Applied Mathematics

Please see complete descriptions of these sessions on the JMM website.

SIAM Minisymposia for JMM 2024 will take place Wednesday–Saturday. There are 8 Minisymposia. A full list of these sessions can be found under the heading Society for Industrial and Applied Mathematics Minisymposia above.

The SIAM Invited Address will be delivered by **Mariel Vazquez**, University of California, Davis, *Title to be announced*; Thursday, 11:10 a.m.

SIAM Panel on Business-Industry-Government Careers for Mathematicians, organized by **Nessy Tania**, Pfizer; Thursday, 8:30–10:00 a.m. Panelists to be announced.

MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Session, organized jointly by the Mathematical Association of America, Society for Industrial and Applied Mathematics, and the American Mathematical Society; Friday, 9:00–10:30 am. This year the session will consist of a lecture from 9:00–9:50 a.m. given by **Kamuela E. Yong**, University of Hawaii West Oahu, *Title to be announced*, and a short panel discussion, *Title to be announced*, from 9:50–10:30 a.m. Panelists to be announced.

MAA-AMS-SIAM Gerald and Judith Porter Public Lecture will be given by **Maria Chudnovsky**, Princeton University, *Title to be announced*; Saturday, 3:30 p.m.

SIAM will also host a reception; please see the listing in the Social Events section of the announcement.

Association for LGBTQ+ Mathematicians (Spectra)

Please see complete descriptions of these sessions on the JMM website.

Spectra Lavender Lecture, Speaker and title to be announced; Thursday, 8:30 a.m. The Spectra Lavender Lecture honors LGBTQ+ mathematicians who have made significant contributions to the mathematical sciences, mathematical education, or the mathematical community at large.

Spectra Business Meeting, organized by **Devavrat Dabke**, Princeton University, Time and location to be announced

Spectra Workshop: Creating an Inclusive Undergraduate Mathematics Curriculum, organized by **Devavrat Dabke**, Princeton University; Thursday, 1:00–5:00 p.m.

Spectra will also host a reception; please see the listing in the Social Events section of the announcement.

Spectra also has a Spectra Special Session on the program; a listing of this session can be found under the heading Spectra Special Sessions.

Transforming Post-Secondary Education in Mathematics

Please see complete descriptions of these sessions on the JMM website.

TPSE Invited Address will be delivered on Friday at 11:00 a.m. by **Sylvester James Gates, Jr**, Clark Leadership Chair in Science, University of Maryland; past president of American Physical Society, National Medal of Science, *What Challenges Does Data Science Present to Mathematics Education?*

TPSE Panel on Grading for Active Learning & Department Change, organized by **Katherine Stevenson**, CSU Northridge, **Rachel Weir**, Allegheny College, **Scott Wolpert**, University of Maryland and TPSE Math, and **Stan Yoshinobu**, University of Toronto; Thursday, 1:00–2:30 p.m.

TPSE Panel on Developing Innovative Upper Division Pathways in Mathematics: Strategies for Enrollment and Inclusion, organized by **Oscar Vega**, California State University, Fresno, and **Padmanabhan Seshaiyer**, George Mason University; Thursday, 3:00–4:30 p.m.

JMM Sessions and Events

Professional Enhancement Programs (PEP)

Professional Enhancement Programs (PEP) are open only to persons who register for the Joint Meetings and pay the Joint Meetings registration fee in addition to the appropriate PEP fee. The AMS reserves the right to cancel any PEP that is undersubscribed. Participants should read the descriptions of each PEP thoroughly as some require participants to bring their own laptops and special software; laptops will not be provided in any PEP. The enrollment in each PEP is limited to 45; the cost is US$125 per program for the member rate (AIM, AMS, AWM, ASA, NAM, or SIAM) and US$175 for the nonmember rate.

Please see complete descriptions of these JMM Professional Enhancement Programs (PEP) on the JMM website.

Professional Enhancement Program (PEP) 1: *Visualizing Projective Geometry Through Photographs and Perspective Drawings*, presented by **Annalisa Crannell**, Franklin & Marshall College, and **Fumiko Futamura**, Southwestern University; Part A, Wednesday, 1:00–3:00 p.m., and Part B, Thursday, 1:00–3:00 p.m.

Professional Enhancement Program (PEP) 2: *GitHub for Mathematicians*, presented by **Steven Clontz**, University of South Alabama; Part A, Wednesday, 1:00–3:00 p.m., and Part B, Thursday, 1:00–3:00 p.m.

Professional Enhancement Program (PEP) 3: *Changing Math Department Culture: Embracing Servingness*, presented by **Ben Ford**, Sonoma State University, **Rochelle Gutiérrez**, University of Illinois, **Brigitte Lahme**, Sonoma State University, **Luis Leyva**, Vanderbilt-Peabody College, **Omayra Ortega**, Sonoma State University, and **Aris Winger**, Georgia Gwinnett College; Part A, Friday, 9:00–11:00 a.m., and Part B, Saturday, 1:00–3:00 p.m.

Professional Enhancement Program (PEP) 4: *Becoming a Math JEDI: Working for Justice, Equity, Diversity, and Inclusion*, presented by **Michael Dorff**, TPSE Math, Brigham Young University, **Abbe Herzig**, TPSE Math, and **Aris Winger**, Georgia Gwinnett College; Part A, Friday, 1:00–3:00 p.m., and Part B, Saturday, 9:00–11:00 a.m.

Professional Enhancement Program (PEP) 5: *Development of Mathematics Programs for Workforce Preparation*, presented by **Rick Cleary**, Babson College, and **Chris Malone**, Winona State University; Part A, Thursday, 1:00–3:00 p.m., and Part B, Friday, 1:00–3:00 p.m.

Professional Enhancement Program (PEP) 6: *Skills and Tools for Communicating your Research to the Public, Policymakers, and Future Funders*, presented by **Sadie Witkowski**, Institute for Mathematical and Statistical Innovation, University of Chicago, and **Sam Hansen**, Acmescience/University of Michigan, Ann Arbor; Part A, Wednesday, 9:00–11:00 a.m., and Part B, Thursday, 9:00–11:00 a.m.

Professional Enhancement Program (PEP) 7: *Effective Technical Advocacy: How to Talk About Mathematics so Policymakers will Hear you*, presented by **Audrey Malagon**, Virginia Wesleyan University, and **Stephanie Singer**, Hatfield School of Government, Portland State University and Campaign Scientific; Part A, Friday, 9:00–11:00 a.m., and Part B, Saturday, 9:00–11:00 a.m.

Professional Enhancement Program (PEP) 8: *Shaping Thoughtful Conversations on the Past and Future of Data*, presented by **Jemma Lorenat**, Pitzer College, and **Deborah Kent**, University of St. Andrews; Part A, Wednesday, 9:00–11:00 a.m., and Part B, Thursday, 9:00–11:00 a.m.

Professional Enhancement Program (PEP) 9: *Developing Learning Activities for Multivariable Calculus using CalcPlot3D and 3D-Printed Surfaces*, presented by **Paul Seeburger**, Monroe Community College, **Shelby Stanhope**, Air Force Academy, and **Stepan Paul**, North Carolina State University; Part A, Friday, 9:00–11:00 a.m., and Part B, Saturday, 9:00–11:00 a.m.

JMM Panels

Please see complete descriptions of these sessions on the JMM website.

JMM Panel: Cal-Bridge: Building Bridges and Diversifying Mathematics, organized by **Suzanne Sindi**, University of California, Merced; Saturday, 2:00–3:30 p.m.

JMM Panel: Regional Math Alliances: Activities and Formation of Regional Groups to Support the Goals of the National Math Alliance, organized by **Teresa Martines**, University of Texas, Austin; Friday, 8:30–10:00 a.m.

JMM Panel: The Future of Graduate Mathematics Textbooks, organized by **Ravi Vakil**, Stanford University; Thursday, 10:30 a.m.–12:00 p.m.

JMM Panel: Decolonizing Mathematics, organized by **Tarik Aougab**, Haverford College; Wednesday, 3:00–4:30 p.m.

JMM Workshops

Please see complete descriptions of these sessions on the JMM website.

JMM Workshop on Leveraging Research-Based Instruction in Introductory Proofs Courses, organized by **Rachel Arnold**, Virginia Tech; Friday, 1:00–2:30 p.m.

JMM Workshop on Teaching Student-Centered Mathematics: Active Learning & the Learning Assistant Model, organized by **Katherine Johnson**, Florida Gulf Coast University; Wednesday, 3:00–4:30 p.m.

JMM Workshop: Building Conceptual Understanding of Multivariable Calculus using 3D Visualization in CalcPlot3D and 3D-Printed Surfaces, organized by **Shelby Stanhope**, US Air Force Academy, **Paul Seeburger**, Monroe Community College, and **Stepan Paul**, North Carolina State University; Wednesday, 10:30 a.m.–12:00 p.m.

Programs of Other Organizations

This section includes scientific sessions. Several organizations or special groups are having receptions or other social events. Please see the Social Events section of this announcement for those details.

Please see complete descriptions of these sessions on the JMM website.

National Science Foundation (NSF)

The NSF will be represented in several sessions and events taking place at the 2024 JMM.

*NSF Special Session on Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Science*s, organized by **Michael Ferrara**, Division of Undergraduate Education, National Science Foundation.

NSF Special Session: Exploring Funding Opportunities in the Division of Mathematical Sciences, organized by **Elizabeth Wilmer** and **Junping Wang**, Division of Mathematical Sciences, National Science Foundation. This interactive session will provide information on a range of DMS programs and offer advice on submitting effective proposals. DMS program officers will be available to answer questions.

MAA Project NExT

MAA Project NExT Workshop, Details to be announced.

MAA Project NExT Lecture on Teaching, Details to be announced.

MAA Project NExT Reception, Details to be announced.

Special Interest Groups of the MAA (SIGMAA)

Please see complete descriptions of these sessions on the JMM website.

SIGMAA on the Philosophy of Mathematics

SIGMAA on the Philosophy of Mathematics Guest Lecture, Title to be announced will be delivered on Friday at 5:30 p.m. by **Arezoo Islami**, San Francisco State University, organized by **Bonnie Gold**, Monmouth University, and **Kevin Iga**, Pepperdine University.

Sessions for Students

Please see complete descriptions of these sessions on the JMM website.

PME Panel: What Every Student Should Know about the JMM, organized by **Jennifer Beineke**, Western New England University, **Stephanie Edwards**, Hope College, and **Tom Wakefield**, Youngstown State University; Wednesday, 1:00–2:30 p.m. and Thursday, 10:30 a.m.–12:00 p.m. This panel is sponsored by Pi Mu Epsilon.

Grad School Fair, Thursday, 3:30–5:00 p.m. Sponsored by the AMS.

AMS-PME Student Poster Session, organized by **Chad Awtrey**, Samford University, and **Frank Patane**, Samford University; Friday, 12:00–1:30 p.m. and 3:30–5:00 p.m. These sessions feature research done by undergraduate students. First-year graduate students are eligible to present if their research was completed while they were still undergraduates. Research by high school students can be accepted if the research was conducted under the supervision of a faculty member at a post-secondary institution.

Appropriate content for a poster includes, but is not limited to, a new result, a new proof of a known result, a new mathematical model, an innovative solution to a Putnam problem, or a method of solution to an applied problem. Purely expository material is not appropriate for this session.

Participants should submit an abstract through the JMM abstract submission portal by **September 26**. Questions regarding this session should be directed to **Chad Awtrey**, `cawtrey@samford.edu` or **Frank Patane**, `fpatane@samford.edu`.

Career Fair, Thursday, 8:30–10:30 a.m. The AMS Career Fair is an opportunity for mathematically trained job seekers at various phases of education and experience—undergraduates, graduate students, postdocs, and others—to interact in person with employers in Business, Entrepreneurship, Government, Industry, and Nonprofit (BEGIN). This event is job seekers' chance to discover how their mathematical training makes them strong candidates for BEGIN jobs.

Other Events

The Mathematical Art Exhibit is organized by **Robert Fathauer**, Tessellations Company, and **Nathan Selikoff**, Digital Awakening Studios, and supported by the Bridges Organization. A popular feature at the Joint Mathematics Meetings, this exhibition provides a break in your day.

On display will be works in various media by artists who are inspired by mathematics and by mathematicians who use visual art to express their love of mathematics. Topology, fractals, polyhedra, and tiling are some of the ideas at play here. Do not miss this unique opportunity for a different perspective on mathematics. The exhibition will be located inside the Joint Mathematics Exhibits and open during exhibit hours.

Submissions will accepted online from September 15 through October 15 at `http://gallery.bridgesmathart.org/`.

For questions about the Mathematical Art Exhibition, please contact Robert Fathauer at `tessellations@cox.net`.

Exhibits

The Joint Mathematics Meetings Exhibits include the country's leading scientific publishers, professional organizations, companies that offer mathematics-enrichment products and services, computer hardware and software companies, and the Mathematical Art Exhibit. It will be open to all registered participants on Wednesday (starting with the Grand Opening Reception) 6:00–8:30 p.m., on Thursday and Friday 9:00 a.m.–5:00 p.m., and Saturday 9:00 a.m.–2:00 p.m. See more details on the JMM website.

Welcoming Environment Policy

The AMS strives to ensure that participants in the JMM, including exhibitors, enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, veteran status, or immigration status.

Harassment is a form of misconduct that undermines the integrity of the AMS, their activities and missions.

The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all participants of JMM activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for the JMM and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to 855-282-5703 or at `www.mathsociety.ethicspoint.com`. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation.

Assistance may also be sought from any staff or volunteer member wearing a MathSafe badge. Learn more about the MathSafe program.[3] Violations may also be brought to the attention of the AMS Director of Meetings & Conferences at the registration desk during the meeting.

MathSafe

MathSafe is a program by and for the mathematical community to support safe and welcoming meetings. MathSafe volunteers will be available at the JMM to listen to and guide participants who experience harassing behavior. Volunteers will be identifiable by their MathSafe buttons. See more details on the JMM website.

How to Reserve Hotel Rooms

See details about hotels and how to reserve a room on the JMM website.

Importance of Staying in an Official JMM Hotel

The importance of reserving a room at one of the official JMM hotels cannot be stressed enough. The AMS makes every effort to keep participants' expenses at the meeting as low as possible and a lot of work and effort goes into negotiating the most affordable hotel rates. When a participant registers for the meeting and reserves a room at an official JMM hotel, they are helping to support not only JMM 2024, but future JMMs as well.

Reserving a Room

Participants are encouraged to register for the JMM in order to reserve hotel rooms at the JMM rates. If a participant needs to reserve a hotel room before they are registered for the JMM, they should contact the Mathematics Meetings Services Bureau (MMSB) at `mmsb@ams.org` or 1-800-321-4267 (ext. 4094 or ext. 4144) for further instructions.

Special rates have been negotiated exclusively for this meeting at the following hotels: San Francisco Marriott Marquis, Hilton San Francisco Union Square, Marriott Union Square, Hotel Spero San Francisco, Hotel Abri, Galleria Park Hotel, and The Barnes Hotel. See details on these hotels and more details on the JMM website.

[3] `https://www.jointmathematicsmeetings.org/jmm-mathsafe`

Reservations must be made through the MMSB. The hotels will not be able to accept reservations directly until after **December 11, 2023**. At that time, rooms and rates will be based on availability. Any rooms reserved directly with the hotels after **December 11, 2023**, will be subject to higher rates.

A link to the JMM 2024 hotel reservation portal will be included in the confirmations of registrations sent by email. If a participant needs the link sent directly to them, they should send a request to `mmsb@ams.org`. If any participant has difficulty reserving a hotel room, they should send email to `mmsb@ams.org` for assistance.

Any participant who needs to reserve a hotel room and does not have a credit card to guarantee it should send email to `mmsb@ams.org` for further instructions. If a check is being used to guarantee a room, the reservation and check must be received by the MMSB no later than **November 30, 2023**.

Miscellaneous

Please see details about audio-visual equipment; email services; information distribution; local information; the JMM Broadcasting, Photographing, and Videotaping Policy; and telephone messages on the JMM website.

Child Care Grants

Please see details about how to apply for child care grants on the JMM website.

Registration Information

Everyone is welcome at the JMM. The American Mathematical Society (AMS) encourages all participants to register for the JMM. The importance of registering for the meeting cannot be overemphasized. Paying a registration fee helps to support a wide range of activities associated with planning, organizing, and executing the meetings.

All participants who wish to attend sessions are expected to register for the JMM and should be prepared to show their badges, if requested. Badges are required to enter the Exhibits and the AMS Employment Center. The Mathematics Meetings Service Bureau (MMSB) is the official registration and housing bureau for the meeting and will be available to assist you with your registration and housing arrangements.

Cancellation Policy

100% of fees paid will be issued for cancellations of any registrations, including the PEP programs and banquet tickets, up to November 6, 2023. 50% refunds will be issued for any cancellations after that date up to December 27, 2023. No refunds can be issued for any cancellations after December 27, 2023. To cancel any registration, send an email to `mmsb@ams.org`.

Deadlines

Register by December 20, 2023, midnight EST to be eligible for discounted registration fees. After this date, registration will continue through the end of the meeting, but increased fees will apply. Updates and corrections received too late to be included in the program books will be included in the online program on the JMM website and in the JMM Mobile App.

Registration for PEPs: Online registration will turn off for PEPs after January 2. After that, registration for a PEP can only be done in person at a cashier station, through January 3. Registration will close after January 3 for the PEPs.

Registration for NAM Banquet: Online registration will turn off for the NAM banquet after December 27. After that, registration for the banquet can only be done in person at a cashier station, through January 3. Registration for the banquet will close after January 3.

Please see detailed information about registration fees and categories on the JMM 2024 website under Registration Fees.[4]

Register for the Meeting

Registration can only be done online until Tuesday, January 2 when the registration desk opens at the meeting. At that date, you can either register online or in person. Paper registration forms are no longer available for online registration. To register for the meeting online, go to the online registration form and choose "Register." You will be asked to enter

[4] `https://www.jointmathematicsmeetings.org/2300_regfees`

your email address and to sign in with your personal AMS web account. If you do not have an AMS web account, you will need to create one. After you have signed in, proceed with completing the registration form.

VISA, MasterCard, Discover, and American Express are the only methods of payment accepted for registrations and charges to credit cards will be made in US funds. Registration acknowledgments will be sent to the email addresses provided.

See details on how to register at `https://www.jointmathematicsmeetings.org/meetings/national/jmm2024/2300_reg`.

Special Registration Codes

To allow for easy tracking of registrations for participants that belong to certain groups and are attending the meeting solely to participate in those groups, a registration code will be sent to them to register. See details at `https://www.jointmathematicsmeetings.org/meetings/national/jmm2024/2300_reg`.

Joint Mathematics Meetings Registration Fees (all fees in US$)

Registration Category (see definitions below)	By Dec. 20 (midnight EST)	After Dec. 20
Member of AIM, AMS, AWM, ASA, NAM, or SIAM	US$411	US$541
Nonmember	652	832
Graduate Student	92	108
Undergraduate Student	92	108
High School Student	92	108
Unemployed	92	108
Retired	92	108
Developing Country Participant	92	108
High School Teacher	92	108
Librarian	92	108
One-day Only—Member (AIM, AMS, AWM, ASA, NAM, or SIAM)	N/A	294
One-day Only—Nonmember	N/A	458
Non-mathematician Guest	31	31
Commercial Exhibitor	0	0

JMM Professional Enhancement Program (PEP)–Per Program	By Dec. 20 (midnight EST)	After Dec. 20
Member of AIM, AMS, AWM, ASA, NAM, or SIAM	US$125	US$125
Nonmember	175	175

Grad School Fair / Career Fair	By Dec. 20 (midnight EST)	After Dec. 20
Grad School Fair Non-Institutional Members	US$200	US$200
Grad School Fair AMS Institutional Members	140	140
Graduate Program Table		fees apply per table for Grad School Fair
Career Fair		180 per table Free for AMS Corporate Members
Career Fair Table		fees apply per table for Career Fair

Department Chairs Workshop	By Dec. 20 (midnight EST)	After Dec. 20
Members of AMS (in person)	US$220	US$220
Nonmember (in person)	330	330

Registration Category Definitions

Full-Time Students

Any person over 16 years of age who is currently working toward a degree or diploma is eligible for this category. Students are asked to determine whether their status can be described as a graduate (working toward a degree beyond the bachelor's), an undergraduate (working toward a bachelor's degree), or high school (working toward a high school diploma) and to mark the registration form accordingly. Any child 16 years and younger can attend the meeting free of charge but must be accompanied by an adult at all times.

Retired

Any person who has been a member of the AMS for twenty years or more and who retired because of age or long-term disability from his or her latest position is eligible for this category.

Librarian

Any librarian who is not a professional mathematician is eligible for this category.

Unemployed

Any person who is currently unemployed, actively seeking employment, and is not a student is eligible for this category. This category is not intended to include any person who has voluntarily resigned or retired from his or her latest position.

Developing Country Participant

Any person who is employed in a developing country, where salary levels are radically not commensurate with those in the US, is eligible for this category. See the most recent list of developing countries at `https://worldpopulationreview.com/country-rankings/developing-countries`.

Non-mathematician Guest

Any family member, friend, or associate, who is not a mathematician, and who is accompanied by a participant in the meeting is eligible for this category. Guests will receive a badge and may attend any session, talk, or other event at the meeting.

Commercial Exhibitor

Any person who is exhibiting in the Joint Mathematics Meetings Exhibits is eligible for this category. This does not include anyone participating in a poster session or the art exhibit. Any exhibitor who is a mathematician and is participating in the scientific program and/or wants to attend sessions, talks, etc. is expected to register separately for the meeting.

Social Events

All events listed are open to all registered participants. It is strongly recommended that for any event requiring a ticket, tickets should be purchased through advance registration. Only a very limited number of tickets, if any, will be available for sale on site. If you must cancel your participation in a ticketed event, you may request a 50% refund by returning your tickets to the Mathematics Meetings Service Bureau (MMSB) by **December 30, 2023**. After that date, no refunds can be made. Special meals are available at banquets upon advance request, but this must be indicated on the Registration/Housing Form. *Please see complete descriptions of these events on the JMM website.*

American Institute for Mathematics Math Circles Dessert and Games Night Reception, Thursday, 8:00–9:30 p.m.

American Statistical Association Reception, Thursday, 6:00–7:00 p.m. An open reception—the opportunity to celebrate the contributions of statistics to science and society and to build community.

AMS Journal Reviewer Appreciation Reception, Thursday, 7:00–8:00 p.m.

Association of Christians in the Mathematical Sciences (ACMS) Reception and Lecture, Day and time to be announced.

Association for Women in Mathematics Reception and Awards Presentation, Friday, 5:00–6:30 p.m. The AWM Reception is open to all JMM participants and will begin at 5:00 p.m., during the poster presentations.

Budapest Semesters in Mathematics Annual Alumni Reunion, Thursday, 6:00–7:00 p.m. Budapest Semesters in Mathematics (BSM) Alumni Reunion Event. BSM alums are invited for light appetizers. The BSM North American Directors and staff will be hosting this event. BSM is the prestigious and essential study abroad program for undergraduates studying mathematics, established in 1985.

Canada/USA Mathcamp Alumni and Friends Gathering, Thursday, 6:00–7:30 p.m.

Estimathon!, organized by **Andrew Niedermaier**, Jane Street Capital; Time to be announced.

Grand Opening Reception, Wednesday, 6:00–8:30 p.m. The JMM officially opens with a brief ribbon-cutting ceremony (at 4:30 p.m.), followed by an Awards Ceremony. Participants will then enjoy festivities to further celebrate our vibrant mathematical community. At the reception, the mathematical art display, vendor, and exhibitor booths will all be available to you, along with hors d'oeuvres, food stations, beverages, and entertainment. ALL are welcome! FREE!

Meet up with friends or explore on your own, but be sure to take in all the fun, refreshments, and special offerings. Travel each aisle—many exhibitors are planning special offerings just for this evening!

Inspiring Stories: How an Academic Rejection Led to Something Amazing, organized by **Allison Henrich**, Seattle University, and **Aaron Wooten**, University of Portland; Friday, 3:00–4:30 p.m.

Knitting Circle, Thursday, 8:15–9:45 p.m. Bring a project (knitting/crochet/tatting/beading/etc.) and chat with other mathematical crafters!

Mathematical Reviews Reception, Friday, 6:00–7:00 p.m. All friends of Mathematical Reviews (MathSciNet®) are invited to join reviewers as well as editors and staff of Mathematical Reviews (past and present) for a reception in honor of all of the efforts that go into the creation and publication of the Mathematical Reviews database. This year we are also celebrating the release of the brand new user interface for MathSciNet. We look forward to seeing old and new friends this year. Refreshments will be served.

Mathematical Institutes Open House, Day and time to be announced.

Mathematically Bent Theater, organized by **Colin Adams**, Williams College; Friday, 6:00–7:00 p.m. When you are trying to prove a theorem, does it help to bang your head against the wall? Why does the Skiponacci Quarterly only produce three issues per year? Did you mistakenly take my tote-bag at the Wisconsin reception at JMM 2023? These are just a few of the questions we will not answer in this presentation of four short humorous math pieces.

Mathematical Variety Show, organized by **Dan Margalit**, Georgia Institute of Technology; Friday, 8:00 p.m. at the Alcazar Theater. Ticket purchases can be made on the JMM registration page: $25 student ticket/$30 general admission.

MEET and SHARE: A Mathematicians' Storytelling Event, presented by The Coalition for the Amplification of Historically Excluded Mathematicians (The Coalition), Day and time to be announced.

National Association of Mathematicians Banquet, Friday, 6:00–9:00 p.m. A cash bar reception will be held at 6:00 p.m., and dinner will be served at 6:30 p.m. The Cox-Talbot Invited Address will be given after the dinner, 7:45–8:45 p.m. Tickets will be available for sale once registration is open for the JMM.

Nevertheless She Persisted: The Daughters of Hypatia, organized by **Karl Schaffer,** De Anza College. Day and time to be announced. Dedicated to the foremothers of mathematics as well as to their leading contemporaries, this exciting six-woman dance concert celebrates great mathematical women throughout the ages, telling their stories with thoughtful dances, dynamic storytelling, colorful projections, and more. The dancers recount intriguing stories from the women's lives and perform powerful dances inspired by their mathematical work.

NSA WiMS Networking Event, Day and time to be announced.

Project NExT Reception, Friday, 8:00–10:00 p.m.

SLMath (MSRI) Reception for Current and Future Donors, Friday, 6:00–7:30 p.m.

Society for Industrial and Applied Mathematics (SIAM) Reception on Industrial Math Modeling, Thursday, 7:00–9:00 p.m.

Spectra Reception for LGBT Mathematicians, Day and time to be announced.

Undergraduate Student Reception, Friday, 6:00–8:00 p.m. A community-building event open to all undergraduate students and their supporters. Join us for activities, games, food, and fun. Organized by AMS and Pi Mu Epsilon, with funding from an AMS anonymous donor.

University of Michigan Alumni and Friends Reception, Day and time to be announced. Please join us for the University of Michigan, Mathematics, Alumni and Friends Reception!

Wrong Answers Only, a science comedy game show; Wednesday, 8:45–9:45 p.m. Created by LabX.

Yearly Gather: Collaborative Puzzle Time!, Wednesday, 8:45–10:00 p.m., organized by **sarah-marie belcastro**, MathILy, **Corinne Yap**, Rutgers University, **Brian Freidin**, Auburn University, and **Jonah Ostroff**, University of Washington.

Travel/Transportation

Please see details about travel and transportation options on the JMM website.

Program Timetable

This document provides a thumbnail sketch of all scientific and social events so you can easily see which events may overlap and better plan your time.

Tuesday, January 02

JMM (JOINT MATHEMATICS MEETINGS)

2:00–7:00 p.m. *Joint Meetings Registration*

Wednesday, January 03

JMM (JOINT MATHEMATICS MEETINGS)

7:00 a.m.–3:00 p.m. *Joint Meetings Registration*

8:00 a.m.–3:00 p.m. *Employment Center*

AIM (AMERICAN INSTITUTE OF MATHEMATICS)

8:00 a.m.–12:00 p.m. *AIM Special Session Associated with the Alexanderson Award and Lecture, I*

AMS (AMERICAN MATHEMATICAL SOCIETY)

8:00 a.m.–12:00 p.m. *AMS Special Session on Advances in Coding Theory, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Algebraic Approaches to Mathematical Biology, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Applications of Hypercomplex Analysis, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Applied Topology: Theory, Algorithms, and Applications, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Combinatorial Insights into Algebraic Geometry, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Commutative Algebra and Algebraic Geometry (associated with the Invited Address by Daniel Erman), I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Computable Mathematics: A Special Session Dedicated to Martin D. Davis, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Developing Students' Technical Communication Skills through Mathematics Courses, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Dynamical Systems Modeling for Biological and Social Systems, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Ethics in the Mathematics Classroom, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Extremal and Probabilistic Combinatorics, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Geometric Group Theory (associated with the AMS Retiring Presidential Address), I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Harmonic Analysis, Geometry Measure Theory, and Fractals, I*

8:00 a.m.–12:00 p.m. *AMS Special Session on Homological Techniques in Noncommutative Algebra, I*

8:00 a.m.–12:00 p.m.	*AMS Special Session on Integer Partitions, Arc Spaces and Vertex operators, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematical Modeling and Simulation of Biomolecular Systems, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics and the Arts, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics of DNA and RNA, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Metric Dimension of Graphs and Related Topics, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modeling Complex Adaptive Systems in Life and Social Sciences, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modeling to Motivate the Teaching of the Mathematics of Differential Equations, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modelling with Copulas: Discrete vs Continuous Dependent Data, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Number Theory in Memory of Kevin James, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Advances in Mathematical Models of Diseases: Analysis and Computation, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Research Presentations by Math Alliance Scholar Doctorates, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Serious Recreational Mathematics, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Theoretical and Numerical Aspects of Nonlocal Models, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Theta Correspondence, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Topics in Equivariant Algebra, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Undergraduate Research Activities in Mathematical and Computational Biology, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Water Waves, I*
8:00 a.m.–12:00 p.m.	*AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, I*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
8:00 a.m.–12:00 p.m.	*AWM Special Session on EvenQuads Live and in person: The honorees and the games*
8:00 a.m.–12:00 p.m.	*AWM Special Session on Women in Mathematical Biology*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Generalized Numerical Ranges and Related Topics*
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Innovative and Effective Ways to Teach Linear Algebra, I*
	SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)
8:00 a.m.–12:00 p.m.	*SLMath (MSRI) Special Session on Summer Research in Mathematics (SRiM): Recent Trends in Nonlinear Boundary Value Problems*
	TPSE (TRANSFORMING POST-SECONDARY EDUCATION)
8:00 a.m.–12:00 p.m.	*TPSE Contributed Paper Session on Using Institutional and National Data Sources to Recruit, Retain and Support a Diverse Population of Mathematics Students*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
9:00–10:00 a.m.	*Association for Symbolic Logic Tutorial*
	JMM (JOINT MATHEMATICS MEETINGS)
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 1A: Visualizing Projective Geometry Through Photographs and Perspective Drawings*

9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 6A: Skills and Tools for Communicating your Research to the Public, Policymakers, and Future Funders*
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 8A: Shaping Thoughtful Conversations on the Past and Future of Data*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
9:30–11:30 a.m.	*SIAM Minisymposium on Scientific Machine Learning to Advance Modeling and Decision Support, I*
	INVITED ADDRESS
9:45–10:50 a.m.	*AMS John von Neumann Lecture - Ankur Moitra, Massachusetts Institute of Technology*
	JMM (JOINT MATHEMATICS MEETINGS)
10:30 a.m.–12:00 p.m.	*JMM Workshop: Building Conceptual Understanding of Multivariable Calculus using 3D Visualization in CalcPlot3D and 3D-Printed Surfaces*
10:30 a.m.–12:00 p.m.	*JMM Workshop on Teaching Student-Centered Mathematics: Active Learning & the Learning Assistant Model*
	INVITED ADDRESSES
11:00 a.m.–12:00 p.m.	*AMS Erdős Lecture for Students - John Urschel*
1:00–2:00 p.m.	*AMS Colloquium Lecture I - Terence Tao, University of California, Los Angeles*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
1:00–2:00 p.m.	*Association for Symbolic Logic Tutorial*
	AMS (AMERICAN MATHEMATICAL SOCIETY)
1:00–2:30 p.m.	*AMS Committee on the Profession Panel Discussion: Building a Successful Research Career in Mathematics*
	PME (PI MU EPSILON)
1:00–2:30 p.m.	*PME Panel: What Every Student Should Know about the JMM*
	JMM (JOINT MATHEMATICS MEETINGS)
1:00–3:00 p.m.	*Professional Enhancement Program (PEP) 2A: GitHub for Mathematicians*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
1:00–4:30 p.m.	*SIAM Minisymposium on Scientific Machine Learning to Advance Modeling and Decision Support, II*
	AIM (AMERICAN INSTITUTE OF MATHEMATICS)
1:00–5:00 p.m.	*AIM Special Session on Little School Dynamics: Cool Research by Researchers at PUIs, I*
	AMS (AMERICAN MATHEMATICAL SOCIETY)
1:00–5:00 p.m.	*AMS Special Session on Advances in Coding Theory, II*
1:00–5:00 p.m.	*AMS Special Session on Algebraic Approaches to Mathematical Biology, II*
1:00–5:00 p.m.	*AMS Special Session on Applications of Hypercomplex Analysis, II*
1:00–5:00 p.m.	*AMS Special Session on Commutative Algebra and Algebraic Geometry (associated with the Invited Address by Daniel Erman), II*
1:00–5:00 p.m.	*AMS Special Session on Computable Mathematics: A Special Session Dedicated to Martin D. Davis, II*
1:00–5:00 p.m.	*AMS Special Session on Dynamical Systems Modeling for Biological and Social Systems, II*
1:00–5:00 p.m.	*AMS Special Session on Ethics in the Mathematics Classroom, II*
1:00–5:00 p.m.	*AMS Special Session on Extremal and Probabilistic Combinatorics, II*

1:00–5:00 p.m.	*AMS Special Session on Harmonic Analysis, Geometry Measure Theory, and Fractals, II*
1:00–5:00 p.m.	*AMS Special Session on Homological Techniques in Noncommutative Algebra, II*
1:00–5:00 p.m.	*AMS Special Session on Integer Partitions, Arc Spaces and Vertex Operators, II*
1:00–5:00 p.m.	*AMS Special Session on Mathematical Modeling and Simulation of Biomolecular Systems, II*
1:00–5:00 p.m.	*AMS Special Session on Mathematics and the Arts, II*
1:00–5:00 p.m.	*AMS Special Session on Mathematics of DNA and RNA, II*
1:00–5:00 p.m.	*AMS Special Session on Metric Dimension of Graphs and Related Topics, II*
1:00–5:00 p.m.	*AMS Special Session on Modeling Complex Adaptive Systems in Life and Social Sciences, II*
1:00–5:00 p.m.	*AMS Special Session on Modeling to Motivate the Teaching of the Mathematics of Differential Equations, II*
1:00–5:00 p.m.	*AMS Special Session on Modelling with Copulas: Discrete vs Continuous Dependent Data, II*
1:00–5:00 p.m.	*AMS Special Session on Number Theory in Memory of Kevin James, II*
1:00–5:00 p.m.	*AMS Special Session on Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms, II*
1:00–5:00 p.m.	*AMS Special Session on Topological and Algebraic Approaches for Optimization, I*
1:00–5:00 p.m.	*AMS Special Session on Recent Advances in Mathematical Models of Diseases: Analysis and Computation, II*
1:00–5:00 p.m.	*AMS Special Session on Recent Progress in Inference and Sampling (associated with AMS Invited Address by Ankur Moitra), I*
1:00–5:00 p.m.	*AMS Special Session on Research Presentations by Math Alliance Scholar Doctorates, II*
1:00–5:00 p.m.	*AMS Special Session on Serious Recreational Mathematics, II*
1:00–5:00 p.m.	*AMS Special Session on The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors, I*
1:00–5:00 p.m.	*AMS Special Session on Theoretical and Numerical Aspects of Nonlocal Models, II*
1:00–5:00 p.m.	*AMS Special Session on Theta Correspondence, II*
1:00–5:00 p.m.	*AMS Special Session on Topics in Equivariant Algebra, II*
1:00–5:00 p.m.	*AMS Special Session on Undergraduate Research Activities in Mathematical and Computational Biology, II*
1:00–5:00 p.m.	*AMS Special Session on Using 3D-Printed and Other Digitally-Fabricated Objects in the Mathematics Classroom, I*
1:00–5:00 p.m.	*AMS Special Session on Water Waves, II*
1:00–5:00 p.m.	*AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, II*
1:00–5:00 p.m.	*AMS-AWM Special Session on Solvable Lattice Models and their Applications Associated with the Noether Lecture, I*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
1:00–5:00 p.m.	*AWM Special Session on EvenQuads Live and in person: The honorees and the games*
1:00–5:00 p.m.	*AWM Special Session on Women in Mathematical Biology*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
1:00–5:00 p.m.	*ILAS Special Session on Generalized Numerical Ranges and Related Topics*

1:00–5:00 p.m.	*ILAS Special Session on Innovative and Effective Ways to Teach Linear Algebra, I*
	NAM (NATIONAL ASSOCIATION OF MATHEMATICIANS)
1:00–5:00 p.m.	*NAM-SIAM-AMS Special Session on Quantitative Justice, I*
	SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)
1:00–5:00 p.m.	*SLMath (MSRI) Special Session on Summer Research in Mathematics (SRiM): Recent Trends in Nonlinear Boundary Value Problems*
	TPSE (TRANSFORMING POST-SECONDARY EDUCATION)
1:00–5:00 p.m.	*TPSE Contributed Paper Session on Using Institutional and National Data Sources to Recruit, Retain and Support a Diverse Population of Mathematics Students*
	INVITED ADDRESS
2:15–3:20 p.m.	*AMS Retiring Presidential Address - Ruth Charney, Brandeis University*
	JMM (JOINT MATHEMATICS MEETINGS)
3:00–4:30 p.m.	*JMM Panel: Decolonizing Mathematics*
4:30–5:45 p.m.	*Awards Celebration*
6:00–8:30 p.m.	*Exhibits and Book Sales*
	SOCIAL EVENTS
6:00–8:30 p.m.	*Grand Opening Reception*
8:45–9:45 p.m.	*Wrong Answers Only*
8:45–10:00 p.m.	*Yearly Gather: Collaborative Puzzle Time!*

Thursday, January 04

	JMM (JOINT MATHEMATICS MEETINGS)
7:30 a.m.–4:00 p.m.	*Joint Meetings Registration*
8:00 a.m.–5:30 p.m.	*Employment Center*
	AIM (AMERICAN INSTITUTE OF MATHEMATICS)
8:00 a.m.–12:00 p.m.	*AIM Special Session on Little School Dynamics: Cool Research by Researchers at PUIs, I*
8:00 a.m.–12:00 p.m.	*AIM Special Session on Math Circle Activities as a Gateway Into Research, I*
	AMS (AMERICAN MATHEMATICAL SOCIETY)
8:00 a.m.–12:00 p.m.	*AMS Special Session on Advances in Coding Theory, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Applications of Extremal Graph Theory to Network Design, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Applied Topology: Theory, Algorithms, and Applications, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Bridging Applied and Quantitative Topology, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Complex Social Systems (a Mathematics Research Communities session) I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Computable Mathematics: A Special Session Dedicated to Martin D. Davis, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Derived Categories, Arithmetic, and Geometry (a Mathematics Research Communities session) I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Developing Students' Technical Communication Skills through Mathematics Courses, II*

8:00 a.m.–12:00 p.m.	*AMS Special Session on Dynamics and Management in Disease or Ecological Models (associated with the Gibbs Lecture by Suzanne Lenhart), I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Ergodic Theory, Symbolic Dynamics, and Related Topics, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Explicit Computation with Stacks (a Mathematics Research Communities session) I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Geometric Group Theory (associated with the AMS Retiring Presidential Address), II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Hamiltonian Systems and Celestial Mechanics, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Harmonic Analysis, Geometry Measure Theory, and Fractals, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Homological Techniques in Noncommutative Algebra, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Homotopy Theory, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Ideal and Factorization Theory in Rings and Semigroups, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Integer Partitions, Arc Spaces and Vertex Operators, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematical Physics and Future Directions, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics of Computer Vision, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics of DNA and RNA, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mock Modular forms, Physics, and Applications, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modeling Complex Adaptive Systems in Life and Social Sciences, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Topological and Algebraic Approaches for Optimization, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Developments on Markoff Triples, I*
8:00 a.m.–12:00 p.m.	*AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Research Presentations by Math Alliance Scholar Doctorates, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session) I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Structure-preserving Algorithms, Analysis and Simulations for Differential Equations, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on The Teaching and Learning of Undergraduate Ordinary Differential Equations, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Theoretical and Numerical Aspects of Nonlocal Models, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Thresholds in Random Structures, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Water Waves, III*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
8:00 a.m.–12:00 p.m.	*ASL Special Session on Descriptive Methods in Dynamics, Combinatorics, and Large Scale Geometry, I*

PMA (PRO MATHEMATICA ARTE)

8:00 a.m.–12:00 p.m. *BSM Special Session: Mathematical Research in Budapest for Students and Faculty*

COMAP (CONSORTIUM FOR MATHEMATICS AND ITS APPLICATIONS)

8:00 a.m.–12:00 p.m. *COMAP Contributed Paper Session: Integrating Modeling into Established Courses, I*

NAM (NATIONAL ASSOCIATION OF MATHEMATICIANS)

8:00 a.m.–12:00 p.m. *NAM Haynes-Granville-Browne Session of Presentations by Recent Doctoral Recipients*

SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)

8:00 a.m.–12:00 p.m. *SLMath (MSRI) Special Session on Summer Research in Mathematics (SRiM): Recent Trends in Nonlinear Boundary Value Problems*

8:00 a.m.–12:00 p.m. *SLMath (MSRI) Special Session on African Diaspora Joint Mathematics Working Groups (ADJOINT)*

INVITED ADDRESS

8:30–9:35 a.m. *The Spectra Lavender Lecture*

SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)

8:30–10:00 a.m. *SIAM Panel on Business-Industry-Government Careers for Mathematicians*

JMM (JOINT MATHEMATICS MEETINGS)

8:30–10:30 a.m. *Career Fair*

9:00–11:00 a.m. *Professional Enhancement Program (PEP) 1A: Visualizing Projective Geometry Through Photographs and Perspective Drawings*

9:00–11:00 a.m. *Professional Enhancement Program (PEP) 6A: Skills and Tools for Communicating your Research to the Public, Policymakers, and Future Funders*

9:00–11:00 a.m. *Professional Enhancement Program (PEP) 8B: Shaping Thoughtful Conversations on the Past and Future of Data*

SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)

9:00–11:00 a.m. *SIAM Minisymposium on Mathematical Modeling of Complex Materials Systems*

JMM (JOINT MATHEMATICS MEETINGS)

9:00 a.m.–5:00 p.m. *Exhibits and Book Sales*

INVITED ADDRESSES

9:45–10:50 a.m. *AWM-AMS Noether Lecture - Anne Schilling, University of California, Davis*

9:45–10:50 a.m. *ILAS Invited Address - Stephan Ramon Garcia, Pomona College*

PME (PI MU EPSILON)

10:30 a.m.–12:00 p.m. *PME Panel: What Every Student Should Know about the JMM*

JMM (JOINT MATHEMATICS MEETINGS)

10:30 a.m.–12:00 p.m. *JMM Panel: The Future of Graduate Mathematics Textbooks*

INVITED ADDRESSES

11:00 a.m.–12:00 p.m. *AIM Alexanderson Award Lecture - Joni Teräväinen*

1:00–2:00 p.m. *AMS Colloquium Lecture II - Terence Tao, University of California, Los Angeles*

AMS (AMERICAN MATHEMATICAL SOCIETY)

1:00–2:30 p.m. *AMS Committee on Education Panel Discussion, I*

TPSE (TRANSFORMING POST-SECONDARY EDUCATION)

1:00–2:30 p.m. *TPSE Panel on Grading for Active Learning & Department Change*

JMM (JOINT MATHEMATICS MEETINGS)

1:00–3:00 p.m. *Professional Enhancement Program (PEP) 2A: GitHub for Mathematicians*

1:00–3:00 p.m. *Professional Enhancement Program (PEP) 5A: Development of Mathematics Programs for Workforce Preparation*

SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)

1:00–4:00 p.m. *SLMath (MSRI) Special Session on The MSRI Undergraduate Program (MSRI-UP)*

AIM (AMERICAN INSTITUTE OF MATHEMATICS)

1:00–5:00 p.m. *AIM Special Session on Graphs and Matrices, I*

1:00–5:00 p.m. *AIM Special Session on Math Circle Activities as a Gateway Into Research, I*

AMS (AMERICAN MATHEMATICAL SOCIETY)

1:00–5:00 p.m. *AMS Special Session on Applications of Extremal Graph Theory to Network Design, II*

1:00–5:00 p.m. *AMS Special Session on Coding Theory for Modern Applications, I*

1:00–5:00 p.m. *AMS Special Session on Combinatorial Insights into Algebraic Geometry, II*

1:00–5:00 p.m. *AMS Special Session on Complex Analysis, Operator Theory, and Real Algebraic Geometry, I*

1:00–5:00 p.m. *AMS Special Session on Complex Social Systems (a Mathematics Research Communities session) II*

1:00–5:00 p.m. *AMS Special Session on Computational techniques to Study the Geometry of the Shape Space, I*

1:00–5:00 p.m. *AMS Special Session on Derived Categories, Arithmetic, and Geometry (a Mathematics Research Communities session) II*

1:00–5:00 p.m. *AMS Special Session on Ergodic theory, Symbolic Dynamics, and Related Topics, II*

1:00–5:00 p.m. *AMS Special Session on Explicit Computation with Stacks (a Mathematics Research Communities session) II*

1:00–5:00 p.m. *AMS Special Session on Geometric Group Theory (associated with the AMS Retiring Presidential Address), III*

1:00–5:00 p.m. *AMS Special Session on Hamiltonian Systems and Celestial Mechanics, II*

1:00–5:00 p.m. *AMS Special Session on Informal Learning, Identity, and Attitudes in Mathematics, I*

1:00–5:00 p.m. *AMS Special Session on Knots, Skein Modules, and Categorification, I*

1:00–5:00 p.m. *AMS Special Session on Loeb Measure after 50 Years, I*

1:00–5:00 p.m. *AMS Special Session on Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later, II*

1:00–5:00 p.m. *AMS Special Session on Mathematical Modeling and Simulation of Biomolecular Systems, III*

1:00–5:00 p.m. *AMS Special Session on Mathematics and Philosophy, I*

1:00–5:00 p.m. *AMS Special Session on Mathematics and Quantum, I*

1:00–5:00 p.m. *AMS Special Session on Mathematics of Computer Vision, II*

1:00–5:00 p.m. *AMS Special Session on Metric Geometry and Topology, I*

1:00–5:00 p.m. *AMS Special Session on Mock Modular Forms, Physics, and Applications, II*

1:00–5:00 p.m. *AMS Special Session on Partition Theory and q-Series, I*

1:00–5:00 p.m. *AMS Special Session on Quaternions, I*

1:00–5:00 p.m. *AMS Special Session on Recent Developments on Markoff Triples, II*

1:00–5:00 p.m. *AMS Special Session on Recent Progress in Inference and Sampling (associated with AMS Invited Address by Ankur Moitra), II*

1:00–5:00 p.m.	*AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, II*
1:00–5:00 p.m.	*AMS Special Session on Research Presentations by Math Alliance Scholar Doctorates, IV*
1:00–5:00 p.m.	*AMS Special Session on Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session) II*
1:00–5:00 p.m.	*AMS Special Session on Structure-preserving Algorithms, Analysis and Simulations for Differential Equations, II*
1:00–5:00 p.m.	*AMS Special Session on The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors, III*
1:00–5:00 p.m.	*AMS Special Session on The Teaching and Learning of Undergraduate Ordinary Differential Equations, II*
1:00–5:00 p.m.	*AMS Special Session on Using 3D-Printed and Other Digitally-Fabricated Objects in the Mathematics Classroom, II*
1:00–5:00 p.m.	*AMS-AWM Special Session on Solvable Lattice Models and their Applications Associated with the Noether Lecture, II*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
1:00–5:00 p.m.	*ASL Special Session on Descriptive Methods in Dynamics, Combinatorics, and Large Scale Geometry, I*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
1:00–5:00 p.m.	*AWM Special Session on Recent Developments in Harmonic Analysis, I*
	COMAP (CONSORTIUM FOR MATHEMATICS AND ITS APPLICATIONS)
1:00–5:00 p.m.	*COMAP Contributed Paper Session: Integrating Modeling into Established Courses, II*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
1:00–5:00 p.m.	*ILAS Special Session on Linear algebra, matrix theory, and its applications*
	PME (PI MU EPSILON)
1:00–5:00 p.m.	*PME Contributed Session on Research by Undergraduates*
	SPECTRA
1:00–5:00 p.m.	*Spectra Workshop: Creating an Inclusive Undergraduate Mathematics Curriculum*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS
1:00–6:00 p.m.	*SIAM Minisymposium on Mathematics of Bacterial Viruses: From Virus Discovery to Mathematical Principles*
	INVITED ADDRESSES
2:15–3:20 p.m.	*AMS Invited Address II - Daniel Erman, University of Wisconsin-Madison*
2:15–3:20 p.m.	*NAM Claytor-Woodard Lecture - Shelly M. Jones, Central Connecticut State University*
	NSF (NATIONAL SCIENCE FOUNDATION)
2:30–5:00 p.m.	*NSF Special Session on Exploring Funding Opportunities in the Division of Mathematical Sciences*
	TPSE (TRANSFORMING POST-SECONDARY EDUCATION)
3:00–4:30 p.m.	*TPSE Panel on Developing Innovative Upper Division Pathways in Mathematics: Strategies for Enrollment and Inclusion*
	INVITED ADDRESSES
3:30–4:30 p.m.	*ASA Invited Address - Kathy Ensor, Rice University*
3:30–4:35 p.m.	*AMS Invited Address I - Kimberly Sellers, Georgetown University*

5:00–6:00 p.m.	*AMS Josiah Willard Gibbs Lecture - Suzanne Lenhart, University of Tennessee*
	SOCIAL EVENTS
6:00–7:00 p.m.	*American Statistical Association Reception*
6:00–7:00 p.m.	*Budapest Semesters in Mathematics Annual Alumni Reception*
6:00–7:00 p.m.	*Nevertheless She Persisted: The Daughters of Hypatia*
6:00–7:30 p.m.	*Canada/USA Mathcamp Alumni and Friends Gathering*
7:00–8:00 p.m.	*AMS Journal Reviewer Appreciation Reception*
7:00–9:00 p.m.	*Society for Industrial and Applied Mathematics (SIAM) Reception on Industrial Math Modeling*
8:00–9:30 p.m.	*American Institute for Mathematics Math Circle Dessert and Games Night Reception*
8:15–9:45 p.m.	*Knitting Circle*

Friday, January 05

	JMM (JOINT MATHEMATICS MEETINGS)
7:30 a.m.–4:00 p.m.	*Joint Meetings Registration*
8:00 a.m.–5:30 p.m.	*Employment Center*
	AIM (AMERICAN INSTITUTE OF MATHEMATICS)
8:00 a.m.–12:00 p.m.	*AIM Special Session Associated with the Alexanderson Award and Lecture, I*
	AMS (AMERICAN MATHEMATICAL SOCIETY)
8:00 a.m.–12:00 p.m.	*AMS-AAAS Special Session on Large Random Permutations (affiliated with the AAAS-AMS Invited Address by Peter Winkler), I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Advances in Analysis, PDE's and Related Applications, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Analysis and Differential Equations at Undergraduate Institutions, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Applied Topology: Theory, Algorithms, and Applications, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Arithmetic Geometry with a View toward Computation, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Bridging Applied and Quantitative Topology, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Coding Theory for Modern Applications, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Complex Analysis, Operator Theory, and Real Algebraic Geometry, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Covering Systems of the Integers and Their Applications, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Cryptography and Related Fields, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Dynamical Systems Modeling for Biological and Social Systems, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Dynamics and Management in Disease or Ecological Models (associated with the Gibbs Lecture by Suzanne Lenhart), II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Ergodic Theory, Symbolic Dynamics, and Related Topics, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Geometry and Topology of High-Dimensional Biomedical Data, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Group Actions in Commutative Algebra, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on History of Mathematics, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Homotopy Theory, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Ideal and Factorization Theory in Rings and Semigroups, II*

8:00 a.m.–12:00 p.m.	*AMS Special Session on Interplay Between Matrix Theory and Markov Systems: Applications to Queueing Systems and of Duality Theory, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Issues, Challenges and Innovations in Instruction of Linear Algebra, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Knots, Skein Modules, and Categorification, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematical Modeling of Nucleic Acid Structures, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics and Philosophy, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics and Quantum, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modular Tensor Categories and TQFTs beyond the Finite and Semisimple, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Thresholds in Random Structures, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Topics in Combinatorics and Graph Theory, I*
8:00 a.m.–12:00 p.m.	*AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, III*
8:00 a.m.–12:00 p.m.	*AMS-AWM Special Session on Solvable Lattice Models and their Applications Associated with the Noether Lecture, III*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
8:00 a.m.–12:00 p.m.	*AWM Special Session on Women in Mathematical Biology*
	COMAP (CONSORTIUM FOR MATHEMATICS AND ITS APPLICATIONS)
8:00 a.m.–12:00 p.m.	*COMAP Special Session on Math Modeling Contests: What They Are, How They Benefit, What They Did – Discussions with the Students and Advisors*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Linear algebra, matrix theory, and its applications*
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Sign-pattern Matrices and Their Applications*
	PME (PI MU EPSILON)
8:00 a.m.–12:00 p.m.	*PME Contributed Session on Research by Undergraduates*
	SPECTRA
8:00 a.m.–12:00 p.m.	*Spectra Special Session on Research by LGBTQ+ Mathematicians*
	JMM (JOINT MATHEMATICS MEETINGS)
8:30–10:00 a.m.	*Graduate School Fair*
8:30–10:00 a.m.	*JMM Panel: Regional Math Alliances: Activities and Formation of Regional Groups to Support the Goals of the National Math Alliance*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
8:30–11:30 a.m.	*SIAM Minisymposium on Computational Mathematics and the Power Grid*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
9:00–10:00 a.m.	*ASL Invited Address*
	INVITED ADDRESS
9:00–10:30 a.m.	*MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture by Kamuela Yong followed by the MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Panel*

JMM (JOINT MATHEMATICS MEETINGS)

9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 3A: Changing Math Department Culture: Embracing Servingness*
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 7A: Effective Technical Advocacy: How to Talk About Mathematics so Policymakers will Hear you*
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 9A: Developing Learning Activities for Multivariable Calculus using CalcPlot3D and 3D-Printed Surfaces*

SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)

9:00 a.m.–12:00 p.m.	*SLMath (MSRI) Special Session on The MSRI Undergraduate Program (MSRI-UP)*

JMM (JOINT MATHEMATICS MEETINGS)

9:00 a.m.–5:00 p.m.	*Exhibits and Book Sales*

INVITED ADDRESS

9:45–10:50 a.m.	*CRM-PIMS-AARMS Invited Address - Henri Darmon, McGill University*

ASL (ASSOCIATION OF SYMBOLIC LOGIC)

10:00–11:00 a.m.	*ASL Invited Address*

AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)

10:30 a.m.–12:00 p.m.	*AWM Workshop: Mathematicians + Wikipedia – A Training Edit-a-thon*

INVITED ADDRESSES

11:00 a.m.–12:00 p.m.	*SIAM Invited Address - Mariel Vazquez, UC Davis*
11:00 a.m.–12:00 p.m.	*TPSE Invited Address - Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland*

PME (PI MU EPSILON)

12:00–1:30 p.m.	*AMS-PME Undergraduate Student Poster Session, I*

ASL (ASSOCIATION OF SYMBOLIC LOGIC)

1:00–2:00 p.m.	*ASL Invited Address*

INVITED ADDRESS

1:00–2:00 p.m.	*AMS Colloquium Lecture III - Terence Tao, University of California, Los Angeles*

AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)

1:00–2:30 p.m.	*AWM Panel: Celebrating Academic Pivots in Mathematics*

JMM (JOINT MATHEMATICS MEETINGS)

1:00–2:30 p.m.	*JMM Workshop on Leveraging Research-Based Instruction in Introductory Proofs Courses*
1:00–3:00 p.m.	*Professional Enhancement Program (PEP) 4A: Becoming a Math JEDI: Working for Justice, Equity, Diversity, and Inclusion*
1:00–3:00 p.m.	*Professional Enhancement Program (PEP) 5A: Development of Mathematics Programs for Workforce Preparation*

AIM (AMERICAN INSTITUTE OF MATHEMATICS)

1:00–5:00 p.m.	*AIM Special Session on Graphs and Matrices, I*
1:00–5:00 p.m.	*AIM Special Session on Math Circle Activities as a Gateway Into Research, I*

AMS (AMERICAN MATHEMATICAL SOCIETY)

1:00–5:00 p.m.	*AIM-AMS Special Session on Applied Topology Beyond Persistence Diagrams, I*
1:00–5:00 p.m.	*AMS Special Session on Algebraic Structures in Knot Theory, I*

1:00–5:00 p.m.	*AMS Special Session on Analysis and Differential Equations at Undergraduate Institutions, II*
1:00–5:00 p.m.	*AMS Special Session on Combinatorial Perspectives on Algebraic Curves and their Moduli, I*
1:00–5:00 p.m.	*AMS Special Session on Combinatorics for Science, I*
1:00–5:00 p.m.	*AMS Special Session on Computational Biomedicine: Methods - Models - Applications, I*
1:00–5:00 p.m.	*AMS Special Session on Diffusive Systems in the Natural Sciences, I*
1:00–5:00 p.m.	*AMS Special Session on Discrete Homotopy Theory, I*
1:00–5:00 p.m.	*AMS Special Session on Dynamics and Management in Disease or Ecological Models (associated with the Gibbs Lecture by Suzanne Lenhart), III*
1:00–5:00 p.m.	*AMS Special Session on Dynamics and Regularity of PDEs, I*
1:00–5:00 p.m.	*AMS Special Session on Epistemologies of the South and the Mathematics of Indigenous Peoples, I*
1:00–5:00 p.m.	*AMS Special Session on Exploring Spatial Ecology via Reaction Diffusion Models: New Insights and Solutions, I*
1:00–5:00 p.m.	*AMS Special Session on Geometric Analysis in Several Complex Variables, I*
1:00–5:00 p.m.	*AMS Special Session on Geometric Group Theory (associated with the AMS Retiring Presidential Address), IV*
1:00–5:00 p.m.	*AMS Special Session on Geometry and Symmetry in Differential Equations, Control, and Applications, I*
1:00–5:00 p.m.	*AMS Special Session on Group Actions in Commutative Algebra, II*
1:00–5:00 p.m.	*AMS Special Session on History of Mathematics, II*
1:00–5:00 p.m.	*AMS Special Session on Ideal and Factorization Theory in Rings and Semigroups, III*
1:00–5:00 p.m.	*AMS Special Session on Informal Learning, Identity, and Attitudes in Mathematics, II*
1:00–5:00 p.m.	*AMS Special Session on Interplay Between Matrix Theory and Markov Systems: Applications to Queueing Systems and of Duality Theory, II*
1:00–5:00 p.m.	*AMS Special Session on Issues, Challenges and Innovations in Instruction of Linear Algebra, II*
1:00–5:00 p.m.	*AMS Special Session on Loeb Measure after 50 Years, II*
1:00–5:00 p.m.	*AMS Special Session on Modern Developments in the Theory of Configuration Spaces, I*
1:00–5:00 p.m.	*AMS Special Session on Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects, I*
1:00–5:00 p.m.	*AMS Special Session on New Faces in Operator Theory and Function Theory, I*
1:00–5:00 p.m.	*AMS Special Session on Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives, I*
1:00–5:00 p.m.	*AMS Special Session on Partition Theory and q-Series, II*
1:00–5:00 p.m.	*AMS Special Session on Principles, Spatial Reasoning, and Science in First-Year Calculus, I*
1:00–5:00 p.m.	*AMS Special Session on Quaternions, II*
1:00–5:00 p.m.	*AMS Special Session on Spectral Methods in Quantum Systems, I*
1:00–5:00 p.m.	*AMS Special Session on The Mathematics of Decisions, Elections, and Games, I*
1:00–5:00 p.m.	*AMS Special Session on Topics in Combinatorics and Graph Theory, II*
1:00–5:00 p.m.	*AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, IV*

	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
1:00–5:00 p.m.	*AWM Special Session on Recent Developments in Harmonic Analysis, II*
	COMAP (CONSORTIUM FOR MATHEMATICS AND ITS APPLICATIONS)
1:00–5:00 p.m.	*COMAP Workshop on Modeling for Educators: Integrate Modeling into Your Classroom*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
1:00–5:00 p.m.	*ILAS Special Session on Sign-pattern Matrices and Their Applications*
	SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)
1:00–5:00 p.m.	*SLMath (MSRI) Special Session on African Diaspora Joint Mathematics Working Groups (ADJOINT)*
	SPECTRA
1:00–5:00 p.m.	*Spectra Special Session on Research by LGBTQ+ Mathematicians*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
1:00–6:00 p.m.	*SIAM Minisymposium on Mathematical Methods in Computer Vision and Image Analysis*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
2:00–3:00 p.m.	*ASL Invited Address*
	INVITED ADDRESS
2:15–3:20 p.m.	*PME J. Sutherland Frame Lecture - Trachette Jackson, University of Michigan*
	AMS (AMERICAN MATHEMATICAL SOCIETY)
2:30–4:00 p.m.	*AMS Committee on Science Policy Panel Discussion: Artificial Intelligence in Mathematics, Science, and Society*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
3:00–6:00 p.m.	*ASL Contributed Paper Session*
	COMAP (CONSORTIUM FOR MATHEMATICS AND ITS APPLICATIONS)
3:00–4:30 p.m.	*COMAP Panel on Math Modeling Contests: Trends, Topics, and Tips*
	INVITED ADDRESS
3:30–4:35 p.m.	*AMS Maryam Mirzakhani Lecture - Melanie Wood, Harvard University*
	PME (PI MU EPSILON)
3:30–5:00 p.m.	*AMS-PME Undergraduate Student Poster Session, II*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
4:00–5:30 p.m.	*AWM Workshop Poster Presentations*
	INVITED ADDRESS
4:45–5:50 p.m.	*AAAS-AMS Invited Address - Peter Winkler, Dartmouth College*
	OTHER ORGANIZATION
5:30–6:30 p.m.	*POMSIGMAA Guest Lecture and Discussion*
	INVITED ADDRESS
7:45–8:45 p.m.	*NAM Cox-Talbot Address - Ranthony A.C. Edmonds, Duke University*
	SOCIAL EVENTS
5:00–6:30 p.m.	*Association for Women in Mathematics Reception and Awards Presentation*
6:00–7:00 p.m.	*Mathematically Bent Theater*
6:00–7:00 p.m.	*Mathematical Reviews Reception*
6:00–7:30 p.m.	*SLMath (MSRI) Reception for Current and Future Donors*

6:00–8:00 p.m.	*Undergraduate Student Reception*
6:00–9:00 p.m.	*National Association of Mathematicians Banquet*
8:00–10:00 p.m.	*Mathematical Variety Show*

Saturday, January 06

JMM (JOINT MATHEMATICS MEETINGS)

7:30 a.m.–2:00 p.m.	*Joint Meetings Registration*

AIM (AMERICAN INSTITUTE OF MATHEMATICS)

8:00 a.m.–12:00 p.m.	*AIM Special Session on Equivariant Techniques in Stable Homotopy Theory, I*

AMS (AMERICAN MATHEMATICAL SOCIETY)

8:00 a.m.–12:00 p.m.	*AIM-AMS Special Session on Applied Topology Beyond Persistence Diagrams, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Advances in Analysis, PDE's and Related Applications, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Arithmetic Geometry with a View toward Computation, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Coding Theory for Modern Applications, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Combinatorial Perspectives on Algebraic Curves and their Moduli, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Combinatorics for Science, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Computational techniques to study the geometry of the shape space, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Cryptography and Related Fields, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Dynamics and Regularity of PDEs, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Epistemologies of the South and the Mathematics of Indigenous Peoples, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Exploring Spatial Ecology via Reaction Diffusion Models: New Insights and Solutions, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Geometric Analysis in Several Complex Variables, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Geometry and Symmetry in Differential Equations, Control, and Applications, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on History of Mathematics, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Knots, Skein Modules, and Categorification, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Large Random Permutations (affiliated with AAAS-AMS Invited Address by Peter Winkler), II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematical Modeling of Nucleic Acid Structures, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mathematics and the Arts, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Metric Geometry and Topology, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Mock Modular Forms, Physics, and Applications, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modern Developments in the Theory of Configuration Spaces, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Modular Tensor Categories and TQFTs beyond the Finite and Semisimple, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on New Faces in Operator Theory and Function Theory, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives, II*

8:00 a.m.–12:00 p.m.	*AMS Special Session on Polymath Jr REU Student Research, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Developments in Commutative Algebra, I*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Recent Developments in Numerical Methods for PDEs and Applications, I*
8:00 a.m.–12:00 p.m.	*AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Serious Recreational Mathematics, III*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Spectral Methods in Quantum Systems, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on The Mathematics of Decisions, Elections, and Games, II*
8:00 a.m.–12:00 p.m.	*AMS Special Session on Undergraduate Research Activities in Mathematical and Computational Biology, III*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
8:00 a.m.–12:00 p.m.	*AWM Special Session on Mathematics in the Literary Arts and Pedagogy in Creative Settings*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Graphs and Matrices*
8:00 a.m.–12:00 p.m.	*ILAS Special Session on Spectral and combinatorial problems for nonnegative matrices and their generalizations*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
8:00 a.m.–12:00 p.m.	*SIAM ED Session on Artificial Intelligence and its Uses in Mathematical Education, Research, and Automation in the Industry*
8:00 a.m.–12:00 p.m.	*SIAM Minisymposium on Recent Developments in the Analysis and Control of Partial Differential Equations Arising in Fluid and Fluid-Structure Interactive Dynamics*
	ASL (ASSOCIATION OF SYMBOLIC LOGIC)
9:00–10:00 a.m.	*ASL Invited Address*
	JMM (JOINT MATHEMATICS MEETINGS)
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 7A: Effective Technical Advocacy: How to Talk About Mathematics so Policymakers will Hear you*
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 9B: Developing Learning Activities for Multivariable Calculus using CalcPlot3D and 3D-Printed Surfaces*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
9:00–11:30 a.m.	*AWM Workshop: Women in Operator Theory*
	JRMF (JULIA ROBINSON MATHEMATICS FESTIVAL)
9:00 a.m.–12:00 p.m.	*Julia Robinson Math Festival*
	JMM (JOINT MATHEMATICS MEETINGS)
9:00 a.m.–2:00 p.m.	*Exhibits and Book Sales*
9:00–11:00 a.m.	*Professional Enhancement Program (PEP) 4A: Becoming a Math JEDI: Working for Justice, Equity, Diversity, and Inclusion*
	INVITED ADDRESS
9:45–10:50 a.m.	*AMS Lecture on Education - Suzanne Weekes, Society for Industrial and Applied Mathematics*

ASL (ASSOCIATION OF SYMBOLIC LOGIC)

10:00–11:00 a.m. *ASL Invited Address*

NAM (NATIONAL ASSOCIATION OF MATHEMATICIANS)

10:00–11:00 a.m. *NAM Business Meeting*

SLMATH (MSRI/SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE)

11:30 a.m.–1:00 p.m. *SLMath (MSRI) - NAM Film Presentation: World Premiere of George Csicsery's film "Journeys of Black Mathematicians: Part 1" and Panel Discussion*

ASL (ASSOCIATION OF SYMBOLIC LOGIC)

1:00–2:00 p.m. *ASL Invited Address*

JMM (JOINT MATHEMATICS MEETINGS)

1:00–3:00 p.m. *Professional Enhancement Program (PEP) 3A: Changing Math Department Culture: Embracing Servingness*

AIM (AMERICAN INSTITUTE OF MATHEMATICS)

1:00–5:00 p.m. *AIM Special Session on Equivariant Techniques in Stable Homotopy Theory, I*

AMS (AMERICAN MATHEMATICAL SOCIETY)

1:00–5:00 p.m. *AIM-AMS Special Session on Applied Topology Beyond Persistence Diagrams, III*

1:00–5:00 p.m. *AMS Special Session on Advances in Analysis, PDE's and Related Applications, III*

1:00–5:00 p.m. *AMS Special Session on Arithmetic Geometry with a View toward Computation, III*

1:00–5:00 p.m. *AMS Special Session on Combinatorial Perspectives on Algebraic Curves and their Moduli, III*

1:00–5:00 p.m. *AMS Special Session on Combinatorics for Science, III*

1:00–5:00 p.m. *AMS Special Session on Computational techniques to study the geometry of the shape space, III*

1:00–5:00 p.m. *AMS Special Session on Cryptography and Related Fields, III*

1:00–5:00 p.m. *AMS Special Session on Diffusive Systems in the Natural Sciences, II*

1:00–5:00 p.m. *AMS Special Session on Discrete Homotopy Theory, II*

1:00–5:00 p.m. *AMS Special Session on Dynamics and Regularity of PDEs, III*

1:00–5:00 p.m. *AMS Special Session on Exploring Spatial Ecology via Reaction Diffusion Models: New Insights and Solutions, III*

1:00–5:00 p.m. *AMS Special Session on Geometric Analysis in Several Complex Variables, III*

1:00–5:00 p.m. *AMS Special Session on Geometry and Symmetry in Differential Equations, Control, and Applications, III*

1:00–5:00 p.m. *AMS Special Session on History of Mathematics, IV*

1:00–5:00 p.m. *AMS Special Session on Informal Learning, Identity, and Attitudes in Mathematics, III*

1:00–5:00 p.m. *AMS Special Session on Mathematics and the Arts, IV*

1:00–5:00 p.m. *AMS Special Session on Metric Geometry and Topology, III*

1:00–5:00 p.m. *AMS Special Session on Mock Modular Forms, Physics, and Applications, IV*

1:00–5:00 p.m. *AMS Special Session on Modular Tensor Categories and TQFTs beyond the Finite and Semisimple, III*

1:00–5:00 p.m. *AMS Special Session on Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects, II*

1:00–5:00 p.m. *AMS Special Session on New Faces in Operator Theory and Function Theory, III*

1:00–5:00 p.m.	*AMS Special Session on Partition Theory and q-Series, III*
1:00–5:00 p.m.	*AMS Special Session on Polymath Jr REU Student Research, II*
1:00–5:00 p.m.	*AMS Special Session on Principles, Spatial Reasoning, and Science in First-Year Calculus, II*
1:00–5:00 p.m.	*AMS Special Session on Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems, III*
1:00–5:00 p.m.	*AMS Special Session on Recent Developments in Commutative Algebra, II*
1:00–5:00 p.m.	*AMS Special Session on Recent Developments in Numerical Methods for PDEs and Applications, II*
1:00–5:00 p.m.	*AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, IV*
1:00–5:00 p.m.	*AMS Special Session on Serious Recreational Mathematics, IV*
1:00–5:00 p.m.	*AMS Special Session on Spectral Methods in Quantum Systems, III*
1:00–5:00 p.m.	*AMS Special Session on The Mathematics of Decisions, Elections, and Games, III*
1:00–5:00 p.m.	*AMS Special Session on Undergraduate Research Activities in Mathematical and Computational Biology, IV*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
1:00–5:00 p.m.	*AWM Special Session on Mathematics in the Literary Arts and Pedagogy in Creative Settings*
	ILAS (INTERNATIONAL LINEAR ALGEBRA SOCIETY)
1:00–5:00 p.m.	*ILAS Special Session on Graphs and Matrices*
1:00–5:00 p.m.	*ILAS Special Session on Spectral and combinatorial problems for nonnegative matrices and their generalizations*
	SIAM (SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS)
1:00–6:00 p.m.	*SIAM Minisymposium on Current Advances in Modeling and Simulation to Uncover the Complexity of Disease Dynamics*
	AWM (ASSOCIATION FOR WOMEN IN MATHEMATICS)
1:30–4:30 p.m.	*AWM Workshop: Women in Operator Theory*
	JMM (JOINT MATHEMATICS MEETINGS)
2:00–3:30 p.m.	*JMM Panel: Cal-Bridge: Building Bridges and Diversifying Mathematics*
	INVITED ADDRESS
3:30–4:35 p.m.	*AMS-MAA-SIAM Gerald and Judith Porter Public Lecture*

AMS Employment Center

Hall B, Moscone North/South, Moscone Center, San Francisco, CA (Meeting #1192) January 3–6, 2024

The AMS Employment Center offers a convenient, safe, and practical meeting place for employers and job seekers attending the Joint Meetings. The focus of the Employment Center is on PhD-level mathematical scientists and representatives from academia, business, and government who seek to hire them. The AMS strongly encourages use of the Employment Center venues for all interviews of prospective employees at society meetings. The use of hotel guest rooms for interviews violates the AMS Welcoming Environment Policy. Interviews must be conducted in public meeting spaces.

AMS Employment Center Web Services

Employment Center registration information should be accessed through the MathJobs.org system. The website and all information will be available in early September 2023 and will remain accessible through January 5, 2024 (the last day of the Employment Center). While some institutions may not set appointments until late December 2023, virtually all scheduling will be done before any Joint Mathematics Meetings (JMM) travel takes place, so applicants should expect few or no additional appointments to be available after arrival.

No Admittance Without a JMM Badge

All applicants and employers who plan to enter the Employment Center—even just for one interview—must present a 2024 Joint Meetings Registration badge. Meeting badges are obtained by registering for the JMM and paying a meeting registration fee. The advanced registration deadline is December 20, 2023 (11:59 p.m. EST). See the JMM website at `www.jointmathematicsmeetings.org/jmm` for registration instructions and rates.

Employers: Choose a Table Type

Three table types are available for employers, based on the number of interviewers who will be present at any given time:

- One or two interviewers per table in the "**Quiet**" area: US$380/$285 AMS Institutional Member, each additional table US$215/$161 AMS Institutional Member.
- Three to six interviewers per table in the "**Committee**" area: US$460/$345 AMS Institutional Member, each additional table US$235/$176 AMS Institutional Member.
- "**One-Day Tables**" allow for on-site interviewing. These tables, which can accommodate up to two interviewers,

2024 AMS Employment Center Schedule

December 20, 2023 is the deadline for table registration. After this date, only "One-Day Tables" will be available for purchase. This is also the deadline to register for the JMM badge, required for admittance, at advance registration prices.

Hours of Operation (Please note there is no access to the EC prior to the opening times listed.):

Wednesday, January 3, 2024—8:00 a.m.–3:00 p.m.

Thursday, January 4, 2024—8:00 a.m.–5:30 p.m.

Friday, January 5, 2024—8:00 a.m.–5:30 p.m.

Location: Hall B, Moscone North/South, Moscone Center, San Francisco, CA (meeting #1192).

Do not schedule an interview to begin until 15 minutes after the EC opens.

may be purchased through January 5, 2024. The fee is US$210/$157 AMS Institutional Member. Please register online at MathJobs.org and choose the "EC One-Day Table purchase" option.

- Electricity is supplied to every table with purchase of a table.

All Employment Center data and registrations must be entered on the MathJobs.org site. An existing account can be used for accessing Employment Center services and for paying applicable fees. If no account exists, participants can start an account solely for Employment Center use.

Employers are expected to create their own interview schedules as far in advance as possible by using the assisted-email system in MathJobs.org, or by using other means of communication. Please do not schedule an interview to begin until 15 minutes after the Employment Center opens.

Employers must mark appointments as "confirmed" in their MathJobs.org account to ensure that appointments display in applicants' schedules. At the time of interview, the employer will meet the applicant in the on-site waiting area and escort the applicant to their table.

Employers: Registering with the AMS Employment Center

- Registration runs from early September 2023 through December 20, 2023 on `www.MathJobs.org`. After December 20, only One-Day Tables will be available. They may also be reserved and paid for through MathJobs.org.
- New users of MathJobs.org should click the "NEW EMPLOYER" link on the Registered Employers page of `www.MathJobs.org`. Choose the appropriate table type and fill out the New Employer Form.
- Existing users of MathJobs.org should go to `www.MathJobs.org` and log into their existing account. They may purchase a table by clicking the "EmpCent" logo in the menus along the top tool bar and using the "buy tables" link.
- Every person who needs to enter the Employment Center area must have a meeting badge (obtained by registering for the JMM and paying a meeting registration fee).

To display an ad on site and use no Employment Center services at all, an employer may submit a one-page paper ad, including departmental contact information, to the Employment Center staff on site in San Francisco. There is no fee for this service.

For complete information, visit `www.ams.org/emp-reg/`.

Applicants: Deciding to Attend

The majority of Employment Center employers are academic departments of mathematical sciences seeking to meet a short list of applicants who applied for their open positions during the fall. Each year, a few government or industry employers are present. Often, they are seeking US citizens only due to existing contracts.

- The Employment Center is restricted to registered interviewers and applicants who have a scheduled appointment.
- The Employment Center offers no guarantees of interviews or jobs. Hiring decisions are not made during or immediately following interviews.
- Research-oriented postdoctoral positions are not ordinarily listed or discussed at the Employment Center.

Please visit the Employment Center website for further advice, information, and program updates at `www.ams.org/emp-reg/`.

Applicants: Signing Up with the AMS Employment Center

Applicants do not pay Employment Center fees, but admission to the Employment Center room requires a 2024 JMM badge, obtainable by registering (and paying a fee) for the JMM. To register for the meeting, go to the website: `www.jointmathematicsmeetings.org/jmm`.

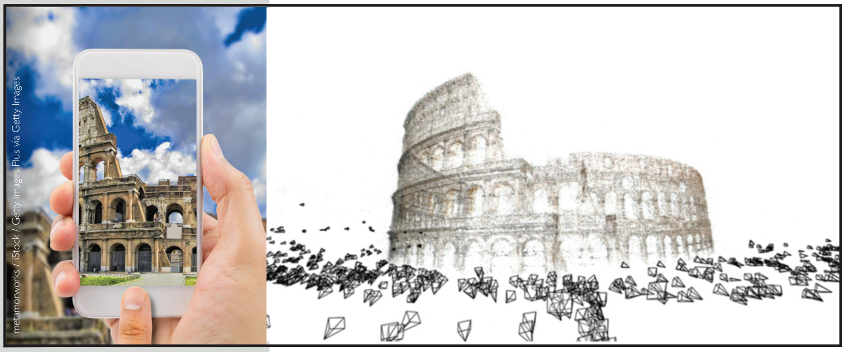
- To let employers know that an applicant will be attending the JMM, applicants should log into their MathJobs.org account or create a new account, look for the "EmpCent" icon across the top tool bar, and mark that they will be attending the JMM by clicking the "Click here if you are attending the Employment Center" link. Applicants can then upload documents and peruse the list of employers attending and the positions available. Applicants do not have the option to request an interview with an employer, but if an applicant is interested in any position, the applicant can apply to the job. The employer will be aware that the applicant is also attending the JMM and will contact the applicant directly if interested in setting up an interview.

Applicants should keep track of their interview schedules. If an applicant is invited for an interview at a time when they are not available, they should ask the employer to offer a new time or suggest one.

For complete information, visit `www.ams.org/emp-reg/`.

Questions about the Employment Center registration and participation can be directed to Rosalynde Vas Dias, AMS Programs Department, at `emp-info@ams.org`.

Bringing Photographs to Life



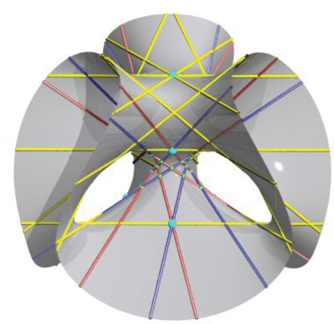
A 3D reconstruction of the Colosseum based on photos from Flickr.

Credit to Sameer Agarwal, Yasutaka Furukawa, Noah Snavely, Ian Simon, Steve Seitz, and Richard Szeliski. *Building Rome in a Day*, https://grail.cs.washington.edu/rome/.

Picture this. Your best friend just returned from a whirlwind vacation to Rome. She's showing you photo after photo of the Colosseum, the famed ancient Roman amphitheater. As you look through her pictures, you wonder: Are these 2D photos really giving you a good sense of the 3D Colosseum?

It turns out you're getting more information than you might think. Tools that build 3D models out of 2D photos are all around us. They blur our Zoom background when we're embarrassed by the mess behind us, and they make the special effects in our favorite movies possible. In fact, in 2009, a team from the University of Washington, Cornell University, and Microsoft Research used images uploaded to Flickr by tourists like your friend to create a 3D model of the Colosseum,as well as other attractions in Venice, Italy; and Dubrovnik, Croatia.

The mathematics of image reconstruction is both simpler and more abstract than it seems. To reconstruct a 3D model based on photographic data, researchers and algorithms must solve a set of polynomial equations. Some solutions to these equations work mathematically, but correspond to an unrealistic scenario — for instance, a camera that took a photo backwards. Additional constraints help ensure this doesn't happen. Researchers are now investigating the mathematical structures underlying image reconstruction, and stumbling over unexpected links with geometry and algebra.



A team of mathematicians found that this shape, which has been studied for centuries, has close ties to an image reconstruction algorithm.

Credit to Alain Esculier, mathcurve.com.

Watch an interview with an expert!

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References:

Sameer Agarwal, Yasutaka Furukawa, Noah Snavely, Ian Simon, Steve Seitz, and Richard Szeliski. *Building Rome in a Day*, https://grail.cs.washington.edu/rome/.

Andrew Pryhuber, Rainer Sinn, Rekha R. Thomas. *Existence of Two View Chiral Reconstructions*, https://arxiv.org/abs/2011.07197.

Joe Kileel, Kathlén Kohn. *Snapshot of Algebraic Vision*, https://arxiv.org/pdf/2210.11443.pdf.