## A Perspective on the Regularity Theory of Degenerate Elliptic Equations



## Héctor A. Chang-Lara

## 1. Introduction

Ellipticity is a well-studied characteristic of some partial differential equations which enforces regularity on its solutions. The prototype problem is the Laplace equation $\Delta u:=\partial_{1}^{2} u+\ldots+\partial_{n}^{2} u=0$ whose solutions are known as the harmonic functions. To illustrate its usefulness, consider an arbitrary sequence of uniformly bounded functions over a compact set. In contrast to numerical

[^0]sequences, a bounded sequence of functions does not necessarily have a uniformly convergent subsequence, even if we assume that each function in such sequence is smooth. However, as soon as we assume that the functions are harmonic, then such uniform limits are always guaranteed.

The property we have just described is known as compactness. We will see that it can be derived from regularity estimates for harmonic functions. It is a powerful tool that can be used in many fundamental results, such as the existence theorem for harmonic functions with prescribed boundary values or the convergence of numerical schemes.

To fix some ideas, let us consider a general second-order partial differential equation (PDE) of the form

$$
F\left(D^{2} u, D u, u, x\right)=0 \text { in } \Omega \subseteq \mathbb{R}^{n}
$$

where $F=F(M, p, z, x): \mathbb{R}_{\text {sym }}^{n \times n} \times \mathbb{R}^{n} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ determines the nonlinear operator on the left-hand side,
and the second-order differentiable function $u: \Omega \rightarrow \mathbb{R}$ is the unknown of the problem. Our focus lies on elliptic problems, which are defined by the requirement that $F$ is nondecreasing in the Hessian variable $(M) .{ }^{1}$

Uniform ellipticity arises when we further assume some quantitative control on the monotonicity of $F$ with respect to $M$. In the case that the function $F$ is differentiable, we say that the operator is uniformly elliptic if

$$
\lambda I \leq D_{M} F \leq \Lambda I,
$$

for some fixed constants $0<\lambda \leq \Lambda$. In the previous expression, $I \in \mathbb{R}^{n \times n}$ is the identity matrix and $D_{M} F=\left(\partial_{m_{i j}} F\right) \in$ $\mathbb{R}_{\text {sym }}^{n \times n}$ is the matrix of partial derivatives of $F$ with respect to the Hessian variable $M=\left(m_{i j}\right) \in \mathbb{R}_{\text {sym }}^{n \times n}$.

Operators that are elliptic, but not necessarily uniformly elliptic, are called degenerate elliptic. These find practical applications in diverse fields such as material sciences, fluid dynamics, finance, and image processing. Some famous examples of degenerate equations include the minimal surface equation

$$
\operatorname{div}\left(\frac{D u}{\sqrt{1+|D u|^{2}}}\right)=0,
$$

the $p$-Laplace equation ( $p \geq 1$ )

$$
\operatorname{div}\left(|D u|^{p-2} D u\right)=0,
$$

and, in the time-dependent case, the porous medium equation ( $m>1$ )

$$
\partial_{t} u=\Delta\left(u^{m}\right)
$$

In general, it has been observed that solutions of degenerate elliptic equations are not always guaranteed to have the same regularity estimates as uniformly elliptic equations.

Motivated by the observation for harmonic functions, we may wonder the following:

1. Have solutions of uniformly elliptic PDEs a compactness property, similar to the harmonic functions?
2. Which additional hypotheses could complement the degenerate ellipticity in order to recover a compactness property for the solutions?

The analysis of PDEs relies significantly on bounds for the modulus of continuity of a solution and its derivatives, these estimates constitute the cornerstones of the regularity theory. The development of the regularity theory for elliptic equations has a rich history with numerous authors. This survey concerns the regularity theory of uniformly elliptic equations that originated during the 1980s and 1990s, with main contributions due to Krylov, Safonov, Evans, and Caffarelli, among many others. This regularity theory is now commonly known as the Krylov-Safonov theory.

[^1]

Figure 1. The mean value theorem: If $\Delta u \leq 0$ in $\Omega$, then for any $B_{r}\left(x_{0}\right) \subseteq \Omega$ it holds that $\frac{1}{\left|B_{r}\left(x_{0}\right)\right|} \int_{B_{r}\left(x_{0}\right)} u \leq u\left(x_{0}\right)$.

We aim to provide a perspective on ongoing developments in the regularity theory of degenerate elliptic problems. First, we give an overview of the regularity theory for the Laplacian and uniformly elliptic equations. This discussion will shed light on the fundamental strategies employed in classical scenarios, enabling us to appreciate better the challenges posed by degenerate equations.

Afterward, we focus on the case of degenerate ellipticity, where the absence of uniform ellipticity is complemented with some additional hypotheses. These assumptions could be interpreted as some sort of alternative regularizing mechanism for the solution. It is the interplay of these phenomena what we find to be a quite attractive venue of current research.

We focus on three types of degeneracies which can be loosely described in the following way:

1. Elliptic equations that hold only where the gradient is large: Either the solution obeys a uniformly elliptic equation or its gradient is bounded.
2. Small perturbations: Uniform ellipticity holds in a neighborhood of a given profile.
3. Quasi-Harnack: Uniformly ellipticity holds at macroscopic scales.

Each one of the previous problems will be presented in a different section, titled accordingly. These section titles make reference to the articles [IS16], [Sav07], and [DSS21], where the respective results were originally studied.

## 2. The Laplacian

The Laplacian $\Delta:=\partial_{1}^{2}+\ldots+\partial_{n}^{2}$ is the fundamental differential operator that describes the ellipticity phenomenon in $\mathbb{R}^{n}$. Solutions of the equation $\Delta u=0$, also known as harmonic functions, are abundant in pure and applied mathematics.

Perhaps the most characteristic features of uniformly elliptic problems are the Harnack inequalities. In the case of the Laplacian, these arise as consequences of the divergence theorem through the mean value theorem (Figure $1)$.


Figure 2. The key observation is that for any $x_{1} \in B_{r / 3}\left(x_{0}\right)$ we always get the inclusions $B_{r / 3}\left(x_{0}\right) \subseteq B_{2 r / 3}\left(x_{1}\right) \subseteq B_{r}\left(x_{0}\right)$.

The weak Harnack inequality states that for any $u \geq 0$ satisfying $\Delta u \leq 0$ in $B_{r}\left(x_{0}\right)$, and any value $\mu>0$, it holds that

$$
\begin{equation*}
\frac{\left|\{u \geq \mu\} \cap B_{r / 3}\left(x_{0}\right)\right|}{\left|B_{r / 3}\left(x_{0}\right)\right|} \leq 2^{n} \mu^{-1} \inf _{B_{r / 3}\left(x_{0}\right)} u . \tag{2.1}
\end{equation*}
$$

Figure 2 illustrates a geometric argument from where to establish this estimate from the mean value theorem.

This control on the distribution of the solution is quite powerful. In particular, it can be used to prove an interior Hölder estimate for harmonic functions in the following form

$$
\begin{equation*}
r^{\alpha}[u]_{C^{\alpha}\left(B_{r / 3}\left(x_{0}\right)\right)} \leq C\|u\|_{\left.L^{\infty}\left(B_{r}\left(x_{0}\right)\right)\right)}, \tag{2.2}
\end{equation*}
$$

for some $\alpha \in(0,1)$ and $C>0$ depending only on the dimension.

Although we have assumed that the solution is $C^{2}$ regular, the point of the estimate is that its own continuity gets controlled by its size rather than its derivatives. Notably, a uniformly bounded family of harmonic functions is automatically equicontinuous on any compact subset. By the Arzelá-Ascoli theorem, this family always contains a sequence that converges locally uniformly to a limit. Moreover, it can be shown that the limit is also a harmonic function.
2.1. The diminish of oscillation. The Estimate (2.2) can be derived through a strategy known as diminish of oscillation. This elegant argument exemplifies the geometric approach in elliptic PDEs. Let us explain it in detail:

1. Assume that $u \in C^{2}\left(B_{1}\right)$ is a harmonic function taking values between 0 and 1 . For $x_{0} \in B_{1 / 2}$ and $r \in(0,1 / 3]$, we aim to show that

$$
\sup _{B_{r}\left(x_{0}\right)}\left|u-u\left(x_{0}\right)\right| \leq C r^{\alpha} .
$$

This can be proved recursively if there exists some small
$\theta \in(0,1)$ such that for any $r \in(0,1 / 3]^{2}$

$$
\begin{equation*}
\underset{B_{r / 3}\left(x_{0}\right)}{\operatorname{osc}} u \leq(1-\theta) \underset{B_{r}\left(x_{0}\right)}{\operatorname{osc}} u . \tag{2.3}
\end{equation*}
$$

Indeed, (2.3) implies that the oscillation of $u$ has a geometric decay in triadic balls

$$
\underset{B_{3-k}\left(x_{0}\right)}{\operatorname{osc}} u \leq(1-\theta)^{k-1} .
$$

By conveniently fixing $\alpha:=\log _{1 / 3}(1-\theta)$ and $C:=(1-\theta)^{-2}$ we now get the desired estimate in the following form and for every $r \in(0,1 / 3)$ and $k=\left\lfloor\log _{1 / 3} r\right\rfloor$, such that $3^{-(k+1)}<$ $r \leq 3^{-k}$

$$
\begin{aligned}
\sup _{B_{r}\left(x_{0}\right)}\left|u-u\left(x_{0}\right)\right| & \leq \underset{B_{r}\left(x_{0}\right)}{\text { osc }} u \\
& \leq \underset{B_{3-k} \text { osc }\left(x_{0}\right)}{ } u \\
& \leq(1-\theta)^{k-1} \\
& =C 3^{-\alpha(k+1)} \\
& \leq C r^{\alpha} .
\end{aligned}
$$

2. To get the diminish of oscillation (2.3), we apply the weak Harnack inequality to a given translation of $u$. Keep in mind that $u$ oscillates between $m_{r}:=\inf _{B_{r}\left(x_{0}\right)} u$ and $M_{r}:=\sup _{B_{r}\left(x_{0}\right)} u$ over $B_{r}\left(x_{0}\right)$, and consider as well $\mu_{r}:=\left(m_{r}+M_{r}\right) / 2$, the level set that sits just in the middle. Hence, at least one of the following alternatives must be true:

$$
\frac{\left|\left\{u \geq \mu_{r}\right\} \cap B_{r / 3}\left(x_{0}\right)\right|}{\left|B_{r / 3}\left(x_{0}\right)\right|} \geq \frac{1}{2}
$$

or

$$
\frac{\left|\left\{u \leq \mu_{r}\right\} \cap B_{r / 3}\left(x_{0}\right)\right|}{\left|B_{r / 3}\left(x_{0}\right)\right|} \geq \frac{1}{2} .
$$

In the first case we apply the weak Harnack inequality (2.1) to the positive harmonic function $\left(u-m_{r}\right)$ in $B_{r}\left(x_{0}\right)$ to get that

$$
\inf _{B_{r / 3}\left(x_{0}\right)}\left(u-m_{r}\right) \geq 2^{-(n+1)}\left(\mu_{r}-m_{r}\right)
$$

This estimate raises the lower bound on $u$ from $m_{r}$ over the ball $B_{r}\left(x_{0}\right)$, to

$$
m_{r}+2^{-(n+1)}\left(\mu_{r}-m_{r}\right)=m_{r}+2^{-(n+2)} \underset{B_{r}\left(x_{0}\right)}{\text { osc }} u
$$

over $B_{r / 3}\left(x_{0}\right)$ (Figure 3).
For the other alternative, we apply the weak Harnack inequality to $\left(M_{r}-u\right)$ to get a similar improvement on the upper bound instead. In conclusion, either option implies the diminish of oscillation Estimate (2.3) for $\theta:=2^{-(n+2)}$.

[^2]

Figure 3. Diminish of oscillation: The lower bound of the solution improves if the measure of $\left\{u \geq \mu_{r}\right\} \cap B_{r / 3}\left(x_{0}\right)$ is at least half of the measure of $B_{r / 3}\left(x_{0}\right)$.

## 3. Uniformly Elliptic Equations

Let us now consider the second-order equation

$$
\begin{equation*}
F\left(D^{2} u, D u, x\right)=0 . \tag{3.1}
\end{equation*}
$$

Assuming that $F$ is differentiable and $F(0,0, x)=0$, we get by the fundamental theorem of calculus that $u$ satisfies a homogeneous linear equation of the following form ${ }^{3}$

$$
\begin{equation*}
A: D^{2} u+b \cdot D u=0 . \tag{3.2}
\end{equation*}
$$

Indeed, we just need to integrate the derivative $(d / d t) F\left(t D^{2} u, t D u, x\right)$ from $t=0$ to $t=1$ to notice that the coefficients $A=\left(a_{i j}(x)\right) \in \mathbb{R}_{\text {sym }}^{n \times n}$ and $b=\left(b_{i}(x)\right) \in \mathbb{R}^{n}$ are given by

$$
\begin{aligned}
& a_{i j}(x):=\int_{0}^{1} \partial_{m_{i j}} F\left(t D^{2} u(x), t D u(x), x\right) d t, \\
& b_{i}(x):=\int_{0}^{1} \partial_{p_{i}} F\left(t D^{2} u(x), t D u(x), x\right) d t .
\end{aligned}
$$

Even though these coefficients depend on the solution as well, under suitable hypotheses on the derivatives of $F$ we can overlook this dependence and understand the equation in a broad sense.

To start, we can just assume that the coefficients are uniformly bounded. In particular, $|b| \leq \Lambda$, which would follow from a Lipschitz assumption on $F$. Uniform ellipticity requires that for some constants $0<\lambda \leq \Lambda$, it holds that $\lambda I \leq D_{M} F \leq \Lambda I$, which means that $\lambda I \leq A \leq \Lambda I$. We summarize our hypotheses on $F$ as

$$
\left\{\begin{array}{l}
\lambda I \leq D_{M} F \leq \Lambda I, \\
\left|D_{p} F\right| \leq \Lambda, \\
F(0,0, x)=0 .
\end{array}\right.
$$

[^3]By considering the extreme cases in (3.2) given under these assumptions, we obtain that

$$
\left\{\begin{array}{l}
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0,  \tag{3.3}\\
\mathcal{P}_{\lambda, \Lambda}^{+}\left(D^{2} u\right)+\Lambda|D u| \geq 0,
\end{array}\right.
$$

where

$$
\begin{aligned}
& \mathcal{P}_{\lambda, \Lambda}^{-}(M):=\min _{\lambda I \leq A \leq \Lambda I} A: M \\
& \mathcal{P}_{\lambda, \Lambda}^{+}(M):=\max _{I I \leq A \leq \Lambda I} A: M .
\end{aligned}
$$

are known as the Pucci extremal operators.
The main result of the Krylov-Safonov regularity theory established in [KS79], states that solutions of the uniformly elliptic problem (3.3) have an interior Hölder estimate as in (2.2).

Theorem 3.1 (Interior Hölder estimate). Given the parameters of uniform ellipticity $0<\lambda \leq \Lambda$ and the dimension $n$, there exist $\alpha \in(0,1)$ and $C>0$ such that the following holds:

Let $u \in C^{2}\left(B_{r}\left(x_{0}\right)\right)$ satisfy

$$
\left\{\begin{array}{l}
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0 \text { in } B_{r}\left(x_{0}\right), \\
\mathcal{P}_{\lambda, \Lambda}^{+}\left(D^{2} u\right)+\Lambda|D u| \geq 0 \text { in } B_{r}\left(x_{0}\right) .
\end{array}\right.
$$

Then

$$
r^{\alpha}[u]_{C^{\alpha}\left(B_{r / 3}\left(x_{0}\right)\right)} \leq C_{B_{r}\left(x_{0}\right)}^{\operatorname{osc}} u .
$$

In the same way as for the Laplacian, this estimate can be deduced by a diminish of oscillation argument from the weak Harnack inequality, also known in this case as the $L^{\varepsilon}$ estimate. The only difference is that, in the general setting, the bound on the distribution becomes of order $\mu^{-\varepsilon}$, for some exponent $\varepsilon>0$, perhaps small and depending on the parameters of uniform ellipticity and the dimension.

From now on, and to simplify the statements of the following lemmas and theorems, we will assume that any constants mentioned in these statements depend by default on the parameters of uniform ellipticity and the dimension.

Lemma 3.2 (Weak Harnack inequality). There exist $\varepsilon, C>$ 0 such that the following holds:

Let $u \in C^{2}\left(B_{r}\left(x_{0}\right)\right)$ be nonnegative and satisfy

$$
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0 \text { in } B_{r}\left(x_{0}\right) .
$$

Then, for any $\mu>0$

$$
\frac{\left|\{u \geq \mu\} \cap B_{r / 2}\left(x_{0}\right)\right|}{\left|B_{r / 2}\left(x_{0}\right)\right|} \leq C\left(\mu^{-1} \inf _{B_{r / 2}\left(x_{0}\right)} u\right)^{\varepsilon} .
$$

For a long time, the challenge to demonstrate this type of result was to find some connection between pointwise and measure quantities on the solution. For the Laplacian, this connection is naturally suggested by the divergence theorem (keep in mind that $\Delta=\operatorname{div} D$ ). In the general case, this link was eventually established by the


Figure 4. The contact set for a function $u$ is the set of points in the domain that admit a supporting graph of the form $\varphi_{y_{0}}+c$ from below.

Alexandrov-Bakelman-Pucci maximum principle, often abbreviated as the ABP lemma.
3.1. The ABP lemma. Before stating the main result of this section we will need some preliminary notions. This presentation showcases constructions due to Cabré [Cab97] and Savin [Sav07].

Consider the family of functions given by translations of a fixed profile

$$
\varphi_{y_{0}}(x):=\varphi\left(x-y_{0}\right) .
$$

For a given function $u$, we say that a vertical translation of $\varphi_{y_{0}}$ touches $u$ from below at $x_{0}$ if and only if $x_{0} \in$ $\operatorname{argmin}\left(u-\varphi_{y_{0}}\right)$. We define the lower contact set as $y_{0}$ varies in some set $B$ in the following way (Figure 4)

$$
A_{B}:=\bigcup_{y_{0} \in B} \operatorname{argmin}\left(u-\varphi_{y_{0}}\right) .
$$

The set $A_{B}$ may be designed to capture important information about $u$. For instance, if $\varphi(x)=-|x|^{2}$ then $A_{B_{r}} \subseteq\left\{u \leq u(0)+r^{2}\right\}$. Indeed, for any $x_{0} \in \operatorname{argmin}\left(u-\varphi_{y_{0}}\right)$

$$
u\left(x_{0}\right) \leq \varphi_{y_{0}}\left(x_{0}\right)+u(0)-\varphi_{y_{0}}(0) \leq u(0)+r^{2}
$$

Consider now the mapping $T: A_{B} \rightarrow B$, such that $T\left(x_{0}\right)=y_{0}$ if $x_{0} \in \operatorname{argmin}\left(u-\varphi_{y_{0}}\right)$. This transformation can be computed by solving for $y_{0}$ in the expression $D u\left(x_{0}\right)=D \varphi\left(x_{0}-y_{0}\right)$.

If $T$ is surjective, we get by the change of variable formula that

$$
|B| \leq \int_{A_{B}}|\operatorname{det}(D T)|
$$

The Jacobian is given by

$$
D T(x)=I-\left[D^{2} \varphi(x-T(x))\right]^{-1} D^{2} u(x)
$$

Hence, the measure of the contact set $A_{B}$ can be compared with the measure of $B$, provided some bound on the Hessian of $u$ over $A_{B}$.

At every contact point $x_{0} \in A_{B}$, we can use the secondderivative test to bound the eigenvalues of $D^{2} u\left(x_{0}\right)$ from
below by ${ }^{4}-\left|D^{2} \varphi_{y_{0}}\left(x_{0}\right)\right|_{o p}$. On the other hand, if we now assume that $u$ satisfies

$$
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0,
$$

then we can also bound the eigenvalues of $D^{2} u\left(x_{0}\right)$ from above.

To see this, we notice first that ${ }^{5}$

$$
\mathcal{P}_{\lambda, \Lambda}^{-}(M)=\sum_{e \in \operatorname{eig}\left(D^{2} u\left(x_{0}\right)\right)}\left(\lambda e_{+}-\Lambda e_{-}\right)
$$

Thanks to the lower bound on the eigenvalues of $D^{2} u\left(x_{0}\right)$ and the first-derivative test, we get that for any $e \in$ $\operatorname{eig}\left(D^{2} u\left(x_{0}\right)\right)$

$$
\lambda e \leq \Lambda(n-1)\left|D^{2} \varphi_{y_{0}}\left(x_{0}\right)\right|_{o p}+\Lambda\left|D \varphi_{y_{0}}\left(x_{0}\right)\right|
$$

The next lemma gives a concrete implementation of this construction. In this result, we fix the family of concave paraboloids

$$
\left\{\begin{array}{l}
\varphi_{y_{0}}(x):=p\left(\left|x-y_{0}\right|\right)-p(5 r / 6) \\
p(\rho):=-\frac{1}{2 r^{2}} \rho^{2}
\end{array}\right.
$$

Lemma 3.3 (ABP). There exists $\eta \in(0,1)$, such that the following holds:

Let $B=B_{r / 6}\left(x_{0}\right)$ and $u \in C^{2}\left(B_{r}\left(x_{0}\right)\right)$ satisfy

$$
\left\{\begin{array}{l}
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0 \text { in } B_{r}\left(x_{0}\right) \\
\inf _{B_{r}\left(x_{0}\right)}\left(u-\varphi_{y_{0}}\right) \leq 0 \text { for every } y_{0} \in B .
\end{array}\right.
$$

Then, $\left|A_{B}\right| \geq \eta r^{n}$.
This previous lemma can be used to bound the distribution of $u$ over $B_{r}\left(x_{0}\right)$, in terms of its infimum over $B_{r / 2}\left(x_{0}\right)$.

Notice first that the particular choice of $\varphi$ and $B$ implies that for every $y_{0} \in B$

$$
\left\{\begin{array}{l}
\varphi_{y_{0}} \geq p(5 r / 6)-p(2 r / 3)=1 / 8 \text { in } B_{r / 2}\left(x_{0}\right) \\
\varphi_{y_{0}}<0 \text { in } \mathbb{R}^{n} \backslash B_{r}\left(x_{0}\right)
\end{array}\right.
$$

Then, a hypothesis of the form $\inf _{B_{r / 2}\left(x_{0}\right)} u \leq 1 / 8$ gives us that $\inf _{B_{r}\left(x_{0}\right)}\left(u-\varphi_{y_{0}}\right) \leq 0$ for every $y_{0} \in B$ (Figure 5).

On the other hand, $A_{B} \subseteq\{u \leq 1\}$, hence the conclusion from Lemma 3.3 let us recover a nontrivial upper bound on the density of the set $\{u \geq 1\}$ with respect to $B_{r}\left(x_{0}\right)$.

In conclusion, under the hypotheses of the Lemma 3.2, we recover the following preliminary measure estimate: If

$$
\inf _{B_{r / 2}\left(x_{0}\right)} u \leq 1 / 8
$$

then

$$
\frac{\left|\{u \geq 1\} \cap B_{r}\left(x_{0}\right)\right|}{\left|B_{r}\left(x_{0}\right)\right|} \leq\left(1-\eta\left|B_{1}\right|^{-1}\right)
$$

${ }^{4}$ For $M \in \mathbb{R}^{n \times n}$, we denote $|M|_{o p}:=\sup _{|x|=1}|M x|$. If $M$ is also symmetric then we also have that

$$
|M|_{o p}=\max \{|e| \mid e \in \operatorname{eig}(M)\} .
$$

[^4]

Figure 5. Each paraboloid $\varphi_{y_{0}}$ with $y_{0} \in B_{r / 6}\left(x_{0}\right)$ must be crossed by $u$ if $u$ is less or equal than $1 / 8$ at some point in $B_{r / 2}\left(x_{0}\right)$.
3.2. Summary. Here is a quick summary of the KrylovSafonov theory revisited in this survey, before moving to the degenerate problems in the next sections:

1. Uniform ellipticity $\Rightarrow$ Mean value theorem/ABP lemma: For the Laplacian it follows from the divergence theorem. In the general setting, we used instead the change of variable formula over a contact set for the solution.
2. Mean value theorem/ABP lemma $\Rightarrow$ Weak Harnack: For the Laplacian it is a geometric observation (Figure 2). In general, it follows by an iterative diminish of the distribution. We did not offer any details in this presentation; however, they can be found in [CC95].
3. Weak Harnack $\Rightarrow$ Hölder estimate: In either case it follows by the iterative diminish of the oscillation as discussed for harmonic functions.

## 4. Elliptic Equations That Hold Only Where the Gradient is Large

There are numerous scenarios in which the ellipticity parameter of a given operator depends on the gradient of the solution. This is the case for quasi-linear operators of divergence form, which emerged from problems in the calculus of variations. Among the most widely known problems, we find the minimal surface equation and the $p$-Laplacian.

Another example is the very degenerate equation recently explored in [CLP21]

$$
\max (1-|D u|, \Delta u+1)=0
$$

This problem arises as the Hamilton-Jacobi equation of a zero-sum game. Notice that in this case, $\Delta u=-1$ holds in the region where $|D u|>1$.

These examples raise a natural question. Can we obtain some regularity for the solutions of an elliptic equation for which uniform ellipticity only holds over the set $\{|D u|>\gamma\}$, for some $\gamma \geq 0$ ?

In compact subsets of $\{|D u|>\gamma\}$ we may just invoke the classical estimates, meanwhile in the complementary region $\{|D u| \leq \gamma\}$, the Lipschitz semi-norm is automatically
bounded by $\gamma$. The problem is to understand the behavior of the solution at the interface between these two regimes.

The following interior Hölder estimate due to Imbert and Silvestre gives a positive answer to the previous question [IS16].
Theorem 4.1. Given $\gamma>0$ there exists $\alpha \in(0,1)$ and $C>0$ such that the following estimate holds:

Let $u \in C^{2}\left(B_{r}\left(x_{0}\right)\right)$ and $\Omega_{\gamma}:=\{|D u|>\gamma\}$ satisfy

$$
\left\{\begin{array}{l}
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-\Lambda|D u| \leq 0 \text { in } \Omega_{\gamma} \cap B_{r}\left(x_{0}\right), \\
\mathcal{P}_{\lambda, \Lambda}^{+}\left(D^{2} u\right)+\Lambda|D u| \geq 0 \text { in } \Omega_{\gamma} \cap B_{r}\left(x_{0}\right) .
\end{array}\right.
$$

Then,

$$
r^{\alpha}[u]_{C^{\alpha}\left(B_{r / 3}\left(x_{0}\right)\right)} \leq C_{B_{r}\left(x_{0}\right)}^{\operatorname{osc}} u .
$$

As in the uniformly elliptic setting, the proof relies on the weak Harnack inequality and an ABP-type lemma. The idea consists of using a different family of functions for the ABP lemma, namely $\varphi(x)=-C|x|^{1 / 2}$. The advantage is that the family of functions from this profile can be arranged such that they only have contact with the solution in the region where the gradient is large and the uniform ellipticity is present.

A year later in [Moo15], Mooney offered a second proof of this result which extended the analysis to equations of the form

$$
\left\{\begin{array}{l}
\mathcal{P}_{\lambda, \Lambda}^{-}\left(D^{2} u\right)-b|D u| \leq 0 \text { in } \Omega_{\gamma} \cap B_{r}\left(x_{0}\right), \\
\mathcal{P}_{\lambda, \Lambda}^{+}\left(D^{2} u\right)+b|D u| \geq 0 \text { in } \Omega_{\gamma} \cap B_{r}\left(x_{0}\right) .
\end{array}\right.
$$

with $b \in L^{n}\left(B_{r}\left(x_{0}\right)\right)$, possibly unbounded. This answered one proposed open problem in [IS16].

Notice that without imposing an equation in the region $\{|D u| \leq \gamma\}$, we have that arbitrary functions $u$ with $|D u| \leq \gamma$, are trivial solutions of these equations. By doing so, we prevent the possibility of deriving any continuity estimate on the gradient. In this sense, the previous theorem is quite optimal in terms of the expected regularity.
4.1. Some further developments. The methods in [IS16, Moo15] have proven to be quite flexible to treat other equations as well. The envelope from [IS16] was employed by Silvestre and Schwab to extend regularity estimates for parabolic integro-differential equations in [SS16]. Pimentel, Santos, and Teixeira also used this idea recently to obtain higher-order fractional estimates in [PST22]. In collaboration with Santos, we revisited the regularity theory for the porous medium equation in [CLS23] by adapting Mooney's argument to a particular parabolic setting.
4.2. An open problem in the parabolic setting. Analogous estimates as in Theorem 4.1 for parabolic equations remain unknown. Notice that functions that depend only on time, $u=u(t)$, are automatically solutions of

$$
\partial_{t} u=A: D^{2} u+b \cdot D u \text { in }\{|D u|>\gamma\} .
$$

This indicates that the corresponding estimate should only address the continuity of the solution in the spatial variable $(x)$. This problem was originally proposed in [IS16].

## 5. Small Perturbations

Caffarelli developed in [Caf89] a perturbative approach to higher regularity estimates for solutions of uniformly elliptic equations, sometimes referred to as regularity by compactness or the improvement of flatness. By flatness we mean that a solution is uniformly close to a prescribed profile. The general strategy was inspired by De Giorgi's regularity theorem for minimal surfaces. The idea can be roughly described by saying that if the solution is uniformly close to a smooth solution, then it inherits the estimates of the corresponding linearization.

This approach quickly provided alternative proofs to the regularity estimates for some of the canonical degenerate equations. Caffarelli and Cordoba treated in [CC93] the minimal surface equation, while Wang studied in [Wan94] the estimates for the $p$-Laplace equation.

In [Sav07], Savin demonstrated that these estimates could be extended to allow operators $F=F(M, p)$ which are only required to be uniformly elliptic in a neighborhood of a given profile $\left(M_{0}, p_{0}\right) \in\{F=0\}$. In particular, his result allows us to treat equations that become degenerate as $(M, p)$ is large, complementing the ideas in the previous section. In this case, the alternative mechanism that supplements the equation is a flatness hypothesis on the solution.

For simplicity, we state the following result for $\left(M_{0}, p_{0}\right)=(0,0)$ and over the unit ball.

Theorem 5.1. Given $\gamma>0$ and $\alpha \in(0,1)$, there exist $\delta_{0}, C>$ 0 such that the following estimate holds:

Let $F=F(M, p) \in C^{2}\left(\mathbb{R}_{s y m}^{n \times n} \times \mathbb{R}^{n}\right)$ satisfy

$$
\left\{\begin{array}{l}
\lambda I \leq D_{M} F \leq \Lambda I  \tag{5.1}\\
\left|D^{2} F\right| \leq \Lambda \\
F(0,0)=0
\end{array}\right.
$$

Let $u \in C^{2}\left(B_{1}\right)$ and $\Omega_{\gamma}=\left\{\left|D^{2} u\right|+|D u| \leq \gamma\right\}$ satisfy

$$
\left\{\begin{array}{l}
F\left(D^{2} u, D u\right)=0 \text { in } \Omega_{\gamma} \cap B_{1} \\
\|u\|_{L^{\infty}\left(B_{1}\right)} \leq \delta_{0}
\end{array}\right.
$$

Then $u \in C^{2, \alpha}\left(B_{1 / 2}\right)$ with

$$
\|u\|_{C^{2, \alpha}\left(B_{1 / 2}\right)} \leq C\|u\|_{L^{\infty}\left(B_{1}\right)}
$$

The proof of the previous theorem provides another important use of the compactness property derived from the Krylov-Safonov estimate. For this reason, we would like to offer a sketch of the argument, at least in the uniformly elliptic case, that is $\gamma=\infty$ and $\Omega_{\gamma}=B_{1}$. Later on, we will give a few comments on the degenerate case $\gamma<\infty$. The interested reader may refer to [Sav07] for a complete proof.


Figure 6. Improvement of flatness: $u \sim Q_{k} \sim Q_{*}$, both with errors of order $\mu^{(2+\alpha) k}$ over $B_{\mu^{k}}\left(x_{0}\right)$; then also $u \sim Q_{*}$ with the same order of approximation.

1. Given $x_{0} \in B_{1 / 2}$, the goal is to build a multiple scale approximation of $u$ around $x_{0}$ of the following form and for some $\mu \in(0,1 / 2)$ to be chosen sufficiently small

$$
u(x)=\underbrace{P_{0}(x)+\mu^{2+\alpha} P_{1}\left(\mu^{-1} x\right)+\ldots+\mu^{(2+\alpha) k} P_{k}\left(\mu^{-k} x\right)}_{:=Q_{k}(x)}+\ldots
$$

We require that this approximation satisfies:
(1) $P_{0}=Q_{0}=0$,
(2) $P_{i}$ are quadratic polynomials with $\left\|P_{i}\right\|_{L^{\infty}\left(B_{1}\right)} \leq 1$,
(3) $\left.\left\|u-Q_{k}\right\|_{L^{\infty}\left(B_{\mu} k\right.}\left(x_{0}\right)\right) \leq \mu^{(2+\alpha) k}$,
(4) Each $Q_{k}$ satisfies $F\left(D^{2} Q, D Q\left(x_{0}\right)\right)=0$.

From the second item we obtain that

$$
\left\|Q_{k+1}-Q_{k}\right\|_{L^{\infty}\left(B_{1}\right)} \leq C \mu^{\alpha k}
$$

Hence, $Q_{*}:=\lim Q_{k}$ is well defined. Combining now the third item we get

$$
\left\|u-Q_{*}\right\|_{L^{\infty}\left(B_{\mu^{k}}\left(x_{0}\right)\right)} \leq C \mu^{(2+\alpha) k}
$$

This is an equivalent way to state the desired $C^{2, \alpha}$ estimate (Figure 6).
2. The goal now is to find a suitable correction $\mu^{(2+\alpha)(k+1)} P_{k+1}\left(\mu^{-(k+1)} x\right)$ to the quadratic polynomial $Q_{k}$ approximating $u$ over $B_{\mu^{k}}\left(x_{0}\right)$. This correction must satisfy all the items above. However, in this sketch, we will mainly focus on the error estimate given by the third item.

Assume that all the hypotheses are satisfied up to some scale $r:=\mu^{k}$, for some $k \geq 1$, now fixed. Then we consider the small perturbation $v \in C^{2}\left(B_{1}\right)$ such that for $Q=Q_{k}$,

$$
u(x)=Q(x)+r^{2+\alpha} v\left(r^{-1}\left(x-x_{0}\right)\right)
$$

Notice that by the inductive hypothesis $\|v\|_{L^{\infty}\left(B_{1}\right)} \leq 1$.
Under the uniform ellipticity hypothesis we get that $v$ satisfies $G\left(D^{2} v, D v\right)=0$ in $B_{1}$, where

$$
G(M, p):=r^{-\alpha} F\left(r^{\alpha} M+D^{2} Q, r^{1+\alpha} p+D Q\left(x_{0}\right)\right)
$$

satisfies the same hypotheses as $F$ in (5.1). Therefore, $v$ has an interior Hölder estimate that depends exclusively on the parameters of uniform ellipticity and the dimension.
3. If we assume by contradiction that the correcting polynomial can not be found for any $\mu \rightarrow 0^{+}$, we then extract a pair of converging sequences $v_{r} \rightarrow v_{0}$ and $Q_{r} \rightarrow Q_{0}$, locally uniformly in $B_{1}$.

Using that $F\left(D^{2} Q_{r}, D Q_{r}\right)=0$ and $\left|D^{2} F\right| \leq \Lambda$, we also get that the sequence $Q_{r}$ determines a converging sequence of operators

$$
\begin{aligned}
& G_{r}(M, p):= \\
& r^{-\alpha} F\left(r^{\alpha} M+D^{2} Q_{r}, r^{1+\alpha} p+D Q_{r}\left(x_{0}\right)\right)= \\
& \frac{F\left(r^{\alpha} M+D^{2} Q_{r}, r^{1+\alpha} p+D Q_{r}\left(x_{0}\right)\right)-F\left(D^{2} Q_{0}, D Q_{0}\left(x_{0}\right)\right)}{r^{\alpha}}
\end{aligned}
$$

such that $G_{r} \rightarrow G_{0}=G_{0}(M)$, the linear operator with constant coefficients given by

$$
G_{0}(M):=D_{M} F\left(D^{2} Q_{0}, D Q_{0}\left(x_{0}\right)\right): M .
$$

It turns out that the limit function $v_{0}$ also satisfies the linear equation $G_{0}\left(D^{2} v_{0}\right)=0$.
4. As a final step, we notice that by the $C^{3}$ estimates for linear equations with constant coefficients, $v_{0}$ can be approximated by a quadratic polynomial around the origin. This leads to a contradiction of the assumed fact that the corrections did not exist for any small value of $\mu$.

For the degenerate case when $\gamma<\infty$, Savin's remarkable observation is that under the flatness hypothesis, it is possible to reconstruct most of the Krylov-Safonov regularity theory. However, there is a caveat related to the diminish of oscillation argument. As one rescales the equation, the degeneracy becomes more and more pervasive and the argument leading to the improvement on the oscillation eventually breaks down. This means that from an initial $\delta$-flatness hypothesis on the solution, with $\delta \in\left(0, \delta_{0}\right)$, one can only get a truncated modulus of continuity for the solution. Nevertheless, the radius on the truncation also vanishes as the flatness parameter $\delta$ goes to zero. This means that the compactness of solutions in the previous argument still holds by a Cantor diagonal argument.
5.1. Some further developments. Within the scope of this presentation, it is not possible to cite the numerous articles that rely on these techniques. The original idea was developed to answer a celebrated conjecture by De Giorgi about level sets in Ginzburg-Landau phase transition models in [Sav09]. The approach has been extended by De Silva to establish regularity estimates for the Bernoulli free boundary problem starting in [DS11]. Regularity estimates for nonlocal minimal surfaces were established by Caffarelli, Roquejoffre, and Savin in [CRS10]. Armstrong, Silvestre, and Smart also utilized this approach to develop partial regularity results for fully nonlinear equations in [ASS12]. Colombo and Figalli developed regularity estimates for degenerate equations from traffic congestion models in [CF14]. Finally, in collaboration with Pimentel, we demonstrated in [CLP21] the continuity
of $|D u|$, where $u$ solves the gradient-constrained problem $\max (1-|D u|, \Delta u+1)=0$.

## 6. Quasi-Harnack

Degeneracy can also manifest itself across scales. For example, when modeling a PDE using finite difference schemes, the continuous formulation of uniform ellipticity breaks down at the level of the discretization. However, if the numerical scheme approximates a uniformly elliptic equation, we expect that the discrete solution will approximate the continuous solution over large scales, inheriting with it the classical manifestations of uniform ellipticity.

A recent work by De Silva and Savin in [DSS21] proposes a weak notion of solution for equations where the uniform ellipticity manifests from a given scale onward.

The next definition of solutions relies on the following geometric configuration: Given $u \in C(\Omega)$, and $\varphi \in$ $C\left(B_{r}\left(x_{0}\right)\right)$ with $B_{r}\left(x_{0}\right) \subseteq \Omega$, we say that $\varphi$ touches $u$ from below (above) at $x_{0}$ and over $B_{r}\left(x_{0}\right)$ if

$$
\left\{\begin{array}{l}
\varphi \leq(\geq) u \text { in } B_{r}\left(x_{0}\right), \\
\varphi\left(x_{0}\right)=u\left(x_{0}\right) .
\end{array}\right.
$$

Definition 6.1. Let $F=F(M) \in C\left(\mathbb{R}_{\text {sym }}^{n \times n}\right)$ be nondecreasing. We say that $u \in C(\Omega)$ satisfies

$$
F\left(D^{2} u\right) \leq_{r}\left(\geq_{r}\right) 0 \text { in } \Omega
$$

if for every quadratic polynomial $\varphi$ that touches $u$ from below at $x_{0}$ and over $B_{r}\left(x_{0}\right) \subseteq \Omega$, it holds that $F\left(D^{2} \varphi\right) \leq$ $(\geq) 0$.

The equality $=_{r}$ holds when both inequalities are simultaneously satisfied.

This definition is consistent with the classical notion of the inequality $F\left(D^{2} u\right) \leq 0$ for $u \in C^{2}(\Omega)$. This follows by the second-derivative test and the monotonicity hypothesis on $F$. Notice also that if $r_{1}<r_{2}$ and $F\left(D^{2} u\right) \leq_{r_{1}} 0$, then also $F\left(D^{2} u\right) \leq_{r_{2}} 0$, meaning that this notion of solution is more relaxed as $r$ becomes larger.

To give a concrete example, let us consider a twodimensional numerical approximation of the Laplacian over the two-dimensional lattice $\varepsilon \mathbb{Z}^{2}$. In this scenario, we will work with a continuous function $u$, but the relevant values will be given on the lattice as

$$
u_{i j}:=u(\varepsilon i, \varepsilon j) \text { for }(i, j) \in \mathbb{Z}^{2} .
$$

We can then extend $u$ to each square $Q_{i j}:=[\varepsilon i, \varepsilon(i+1)) \times$ $[\varepsilon j, \varepsilon(j+1))$ in a continuous manner, ensuring that the maximum and minimum of $u$ over $\overline{Q_{i j}}$ are reached at the corners of $Q_{i j}$ (Figure 7).

A classical discretization of the Laplace equation over the lattice is formulated by

$$
\begin{equation*}
u_{i j}=\frac{u_{i-1, j}+u_{i+1, j}+u_{i, j-1}+u_{i, j+1}}{4} . \tag{6.1}
\end{equation*}
$$



Figure 7. An interpolation of a discrete function such that over each square $\overline{Q_{i j}}$, the extremal values are attained over the corners.

In particular, it follows from a straightforward computation, that a quadratic polynomial is harmonic if and only if satisfies (6.1) at one point.

Assume $u$ satisfies (6.1) for $(i, j) \in \varepsilon^{-1} \Omega \cap \mathbb{Z}^{2}$, and consider a quadratic polynomial $\varphi$ touching $u$ from below at $x_{0}$ and over $B_{r}\left(x_{0}\right)$. If $r<\varepsilon$, it is not difficult to come up with interpolations for $u$ for which $D^{2} \varphi$ could be arbitrary.

On the other hand, if $r \geq \varepsilon$ and $x_{0}=\varepsilon\left(i_{0}, j_{0}\right) \in \varepsilon \mathbb{Z}^{2}$ is a lattice point, then

$$
\begin{aligned}
\varphi_{i_{0}, j_{0}} & =u_{i_{0}, j_{0}} \\
& =\frac{u_{i_{0}-1, j_{0}}+u_{i_{0}+1, j_{0}}+u_{i_{0}, j_{0}-1}+u_{i_{0}, j_{0}+1}}{4} \\
& \geq \frac{\varphi_{i_{0}-1, j_{0}}+\varphi_{i_{0}+1, j_{0}}+\varphi_{i_{0}, j_{0}-1}+\varphi_{i_{0}, j_{0}+1}}{4}
\end{aligned}
$$

This implies that $\Delta \varphi \leq 0$, as would be required by the definition of $\Delta u \leq_{r} 0$.

Would it be possible to get a similar result in the general case, when $x_{0}$ is not necessarily a lattice point?

The answer to this question is affirmative. However, it is necessary to modify the operator at hand. Indeed, let us see that if $u_{i j}$ satisfies (6.1) for $(i, j) \in \varepsilon^{-1} \Omega \cap \mathbb{Z}^{2}$, then $u$ satisfies

$$
\left\{\begin{array}{l}
\mathcal{P}_{1, \Lambda}^{-}\left(D^{2} u\right) \leq_{3 \varepsilon} 0 \text { in } \Omega  \tag{6.2}\\
\mathcal{P}_{1, \Lambda}^{+}\left(D^{2} u\right) \geq_{3 \varepsilon} 0 \text { in } \Omega
\end{array}\right.
$$

The parameter $\Lambda$ will be conveniently fixed as a large constant by the end of the argument.

We will show the inequality for $\mathcal{P}_{1, \Lambda^{\prime}}^{-}$as the one corresponding to $\mathcal{P}_{1, \Lambda}^{+}$has a similar analysis.

1. Let $\varphi$ be a quadratic polynomial touching $u$ from below at $x_{0}$ over $B_{3 \varepsilon}\left(x_{0}\right) \subseteq \Omega$. Assume by contradiction that $\mathcal{P}_{1, \Lambda}^{-}\left(D^{2} \varphi\right)>0$. This test function can be written as

$$
\left\{\begin{array}{l}
\varphi=P+L \\
P(x)=\lambda_{1}\left(\xi_{1} \cdot\left(x-x_{0}\right)\right)^{2}+\lambda_{2}\left(\xi_{2} \cdot\left(x-x_{0}\right)\right)^{2} \\
L \text { is affine }
\end{array}\right.
$$

with $\left\{\xi_{1}, \xi_{2}\right\}$ orthonormal.

Assume without loss of generality that $\lambda_{1} \leq \lambda_{2}$ and that the angle subtended by the vectors $\xi_{2}$ and $e_{2}=(0,1)$ is between $-\pi / 4$ and $\pi / 4$.

By hypothesis,

$$
\mathcal{P}_{1, \Lambda}^{-}\left(D^{2} \varphi\right)=\sum_{i=1}^{2}\left(\lambda_{i}\right)_{+}-\Lambda\left(\lambda_{i}\right)_{-}>0
$$

so that $\lambda_{2}>0$ and $\lambda_{1} \in\left(-\lambda_{2} / \Lambda, \lambda_{2}\right]$.
2. We get in this way that for $\lambda:=\lambda_{2} / \Lambda$,

$$
\begin{aligned}
\bar{P}(x) & :=-\left(\xi_{1} \cdot\left(x-x_{0}\right)\right)^{2}+\Lambda\left(\xi_{2} \cdot\left(x-x_{0}\right)\right)^{2} \\
& \leq P(x) / \lambda
\end{aligned}
$$

touches $(u-L) / \lambda$ from below at $x_{0}$ and over $B_{3 \varepsilon}\left(x_{0}\right)$. By computing the infimum of $\bar{P}$ over $B_{3 \varepsilon}\left(x_{0}\right)$, we get that

$$
\begin{equation*}
(u-L) \geq-9 \lambda \varepsilon^{2} \text { in } B_{3 \varepsilon}\left(x_{0}\right) \tag{6.3}
\end{equation*}
$$

3. Let $\left(i_{0}, j_{0}\right) \in \mathbb{Z}^{2}$ such that $x_{0} \in Q_{i_{0} j_{0}}$. We will see now that in at least one corner $y=\left(\varepsilon i_{1}, \varepsilon j_{1}\right) \in \partial Q_{i_{0} j_{0}} \cap \varepsilon \not \mathbb{Z}^{2}$ we must have

$$
\begin{equation*}
(u-L)(y) \geq c \lambda_{2} \varepsilon^{2}-9 \lambda \varepsilon^{2} \tag{6.4}
\end{equation*}
$$

for some constant $c>0$ to be fixed.
By considering

$$
\begin{aligned}
& y_{-}:=\left(\varepsilon i_{0}, \varepsilon j_{0}\right) \\
& y_{+}:=\left(\varepsilon i_{0}, \varepsilon\left(j_{0}+1\right)\right),
\end{aligned}
$$

and the angle assumption on $\xi_{2}$, we get that

$$
\begin{aligned}
\left.\mid \xi_{2} \cdot\left(y_{-}-x_{0}\right)\right)\left|+\left|\xi_{2} \cdot\left(y_{+}-x_{0}\right)\right|\right. & \geq \varepsilon\left(\xi_{2} \cdot e_{2}\right) \\
& \geq \varepsilon / \sqrt{2}
\end{aligned}
$$

Hence, in at least one of these two corners we must have that $\left.\mid \xi_{2} \cdot\left(y-x_{0}\right)\right) \mid \geq \varepsilon /(2 \sqrt{2})$, and the desired bound follows for $c=1 / 8$ by using that $(u-L) \geq \lambda \bar{P}$.
4. By the contact given by $(u-L)$ and $\lambda \bar{P}$ at $x_{0}$ we get that $(u-L)\left(x_{0}\right)=\lambda \bar{P}\left(x_{0}\right)=0$. The way in which we considered the continuous extension of $u$ also implies that the minimum of $(u-L)$ over the four corners of $\overline{Q_{i_{0} j_{0}}}$ must be nonpositive. In this final step, we will see how to get a contradiction from this fact, together with (6.3) and (6.4).

The choice on the scale $r=3 \varepsilon$ was made such that the nine closed squares of the form $\overline{Q_{i j}}$ with $\left|i-i_{0}\right| \leq 1$ and $\left|j-j_{0}\right| \leq 1$ are also contained in $B_{3 \varepsilon}\left(x_{0}\right)$ (Figure 8).

Let $Q:=\left(i_{0}-1, i_{0}+2\right) \times\left(j_{0}-1, j_{0}+2\right)$ and $w_{i j}$ be defined for $(i, j) \in \bar{Q} \cap \mathbb{Z}^{2}$ such that the following holds: For the three interior nodes $(i, j) \in\left(Q \cap \mathbb{Z}^{2}\right) \backslash\left\{\left(i_{1}, j_{1}\right)\right\}$

$$
w_{i j}=\frac{w_{i-1, j}+w_{i+1, j}+w_{i, j-1}+w_{i, j+1}}{4}
$$

Meanwhile, it also satisfies the boundary conditions

$$
\left\{\begin{array}{l}
w_{i_{1} j_{1}}=1 \\
w_{i j}=0 \text { in } \partial Q \cap \mathbb{Z}^{2}
\end{array}\right.
$$



Figure 8. The ball $B_{3 \varepsilon}\left(x_{0}\right)$ contains the 9 squares surrounding the square $Q_{i_{0} j_{0}}$ in which falls the center of the ball.

A simple computation determines that $w_{i j}=2 / 7$ in the interior nodes of $Q \cap \mathbb{Z}^{2}$ adjacent to ( $i_{1}, j_{1}$ ), and $w_{i j}=1 / 7$ in the opposite one.

By the discrete comparison principle we get that for any of the four interior nodes $(i, j) \in Q \cap \mathbb{Z}^{2}$

$$
\begin{aligned}
\left(u-L+9 \lambda \varepsilon^{2}\right)_{i j} & \geq \lambda_{2} \varepsilon^{2} w_{i j} / 8, \\
& \geq \lambda_{2} \varepsilon^{2} / 48 .
\end{aligned}
$$

Nevertheless, this contradicts that the minimum of $(u-L)$ over these interior nodes is nonpositive once we choose $\Lambda>432$.

In the recent article of De Silva and Savin [DSS21], the authors get a weak Harnack inequality for the general degenerate problem (6.2). The alternative mechanism used in this case is a measure estimate over any ball of radius $\rho=C r$.

We say that $u \in C(\Omega)$ satisfies the measure estimate with respect to the parameters $\rho, M>0$ and $\delta \in(0,1)$, if for every $B_{\rho}\left(x_{0}\right) \subseteq \Omega$, and $\mu>0$, we get that

$$
u-\inf _{B_{\rho}\left(x_{0}\right)} u \leq \mu \text { in } B_{\rho / 2}\left(x_{0}\right),
$$

implies

$$
\frac{\left|\left\{u-\inf _{B_{\rho}\left(x_{0}\right)} u \geq M \mu\right\} \cap B_{\rho / 2}\left(x_{0}\right)\right|}{\left|B_{\rho / 2}\left(x_{0}\right)\right|} \leq \delta .
$$

Theorem 6.1. There exists a small fraction $\delta \in(0,1)$ depending on the dimension for which the following statement is true:

Given $\Lambda, M \geq 1$, there exist $r_{0}, C, c>0$ and $\alpha \in(0,1)$ such that the following holds:

Let $u \in C\left(B_{1}\right)$ satisfy for some $r \in\left(0, r_{0}\right)$

$$
\left\{\begin{array}{l}
\mathcal{P}_{1, \Lambda}^{-}\left(D^{2} u\right) \leq_{r} 0 \text { in } B_{1}, \\
\mathcal{P}_{1, \Lambda}^{+}\left(D^{2} u\right) \geq_{r} 0 \text { in } B_{1},
\end{array}\right.
$$

and $\pm u$ satisfy the measure estimate with respect to $\mathrm{Cr}, \mathrm{M}$, and ס. Then $u$ has a truncated Hölder estimate of the form

$$
\sup _{\substack{x_{0} \in B_{1 / 2} \\ \rho \in(c r, 1 / 2)}} \rho^{-\alpha} \operatorname{sosc}_{B_{\rho}\left(x_{0}\right)}^{\text {osc }} u \leq C\|u\|_{L^{\infty}\left(B_{1}\right)} .
$$

Besides the already discussed applications to numerical schemes, the previous theorem can also be applied in the homogenization of elliptic problems with degeneracies, as was studied in [AS14]. Finally, it was shown in [DSS21] that integro-differential uniformly elliptic equations of or$\operatorname{der} \sigma$ close to two, also fit in the framework of the previous theorem. In this way, it provides a new proof to the Harnack inequality of Caffarelli and Silvestre [CS09].

## 7. Concluding Remarks

In this note, we have revisited the regularity theory of uniformly elliptic equations under the perspective of degenerate ellipticity. It is our hope to have conveyed some of the beautiful geometric insights of the theory.

While we did not go deeper into the models that have brought up these particular degeneracies, it is important to emphasize that a careful understanding of such natural phenomena has been instrumental in the analysis of the solutions. It can be easily the topic of just one survey to uncover each one of these models in detail, such as minimal surfaces, the $p$-Laplacian, the porous medium equation, etc. For the same reason, many important references have been unfortunately left out.

This article is dedicated to Luis Caffarelli with gratitude and admiration.

ACKNOWLEDGMENT. The author would like to thank his collaborators Néstor Guillén, Edgard Pimentel, and Alberto Saldaña; his colleagues at CIMAT, Octavio Arizmendi and Luis Núñez; and the anonymous referees for their helpful feedback on this manuscript. The author was supported by CONACyT-MEXICO grant A1-S48577.

## References

[AS14] Scott N. Armstrong and Charles K. Smart, Regularity and stochastic homogenization of fully nonlinear equations without uniform ellipticity, Ann. Probab. 42 (2014), no. 6, 25582594, DOI $10.1214 / 13$-AOP833. MR3265174
[ASS12] Scott N. Armstrong, Luis E. Silvestre, and Charles K. Smart, Partial regularity of solutions of fully nonlinear, uniformly elliptic equations, Comm. Pure Appl. Math. 65 (2012), no. 8, 1169-1184, DOI 10.1002/cpa.21394. MR2928094
[Cab97] Xavier Cabré, Nondivergent elliptic equations on manifolds with nonnegative curvature, Comm. Pure Appl. Math. 50 (1997), no. 7, 623-665, DOI 10.1002/(SICI) 1097-0312(199707)50:7<623::AID-CPA2>3.3.CO;2-B MR1447056
[Caf89] Luis A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130 (1989), no. 1, 189-213, DOI $10.2307 / 1971480$ MR1005611
[CC93] Luis A. Caffarelli and Antonio Córdoba, An elementary regularity theory of minimal surfaces, Differential Integral Equations 6 (1993), no. 1, 1-13. MR1190161
[CC95] Luis A. Caffarelli and Xavier Cabré, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995, DOI 10.1090/coll/043. MR1351007
[CF14] Maria Colombo and Alessio Figalli, Regularity results for very degenerate elliptic equations (English, with English and French summaries), J. Math. Pures Appl. (9) 101 (2014), no. 1, 94-117, DOI 10.1016/j.matpur.2013.05.005. MR3133426
[CLP21] Héctor A. Chang-Lara and Edgard A. Pimentel, Nonconvex Hamilton-Jacobi equations with gradient constraints, Nonlinear Anal. 210 (2021), Paper No. 112362, 17, DOI 10.1016/j.na.2021.112362, MR4249793
[CLS23] Héctor A. Chang-Lara and Makson S. Santos, Hölder regularity for non-variational porous media type equations, J. Differential Equations 360 (2023), 347-372, DOI 10.1016/j.jde.2023.02.055, MR4557323
[CRS10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, Nonlocal minimal surfaces, Comm. Pure Appl. Math. 63 (2010), no. 9, 1111-1144, DOI 10.1002/cpa.20331. MR2675483
[CS09] Luis Caffarelli and Luis Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), no. 5, 597-638, DOI 10.1002/cpa.20274. MR2494809
[DS11] D. De Silva, Free boundary regularity for a problem with right hand side, Interfaces Free Bound. 13 (2011), no. 2, 223-238, DOI $10.4171 /$ IFB/255. MR2813524
[DSS21] D. De Silva and O. Savin, Quasi-Harnack inequality, Amer. J. Math. 143 (2021), no. 1, 307-331, DOI 10.1353/ajm.2021.0001 MR4201786
[IS16] Cyril Imbert and Luis Silvestre, Estimates on elliptic equations that hold only where the gradient is large, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 6, 1321-1338, DOI 10.4171/JEMS/614 MR3500837
[KS79] N. V. Krylov and M. V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure (Russian), Dokl. Akad. Nauk SSSR 245 (1979), no. 1, 18-20. MR525227
[Moo15] Connor Mooney, Harnack inequality for degenerate and singular elliptic equations with unbounded drift, J. Differential Equations 258 (2015), no. 5, 1577-1591, DOI 10.1016/j.jde.2014.11.006. MR3295593
[PST22] Edgard A. Pimentel, Makson S. Santos, and Eduardo V. Teixeira, Fractional Sobolev regularity for fully nonlinear elliptic equations, Comm. Partial Differential Equations 47 (2022), no. 8, 1539-1558, DOI 10.1080/03605302.2022.2059676. MR4462186
[Sav07] Ovidiu Savin, Small perturbation solutions for elliptic equations, Comm. Partial Differential Equations 32 (2007), no. 4-6, 557-578, DOI 10.1080/03605300500394405. MR2334822
[Sav09] Ovidiu Savin, Regularity of flat level sets in phase transitions, Ann. of Math. (2) 169 (2009), no. 1, 41-78, DOI 10.4007/annals.2009.169.41. MR2480601
[SS16] Russell W. Schwab and Luis Silvestre, Regularity for parabolic integro-differential equations with very irregular kernels, Anal. PDE 9 (2016), no. 3, 727-772, DOI 10.2140/apde.2016.9.727. MR3518535
[Wan94] Lihe Wang, Compactness methods for certain degenerate elliptic equations, J. Differential Equations 107 (1994), no. 2, 341-350, DOI 10.1006/jdeq.1994.1016. MR1264526


Héctor A .
Chang-Lara
Credits
All figures and the opener are courtesy of Héctor A. ChangLara.
Photo of Héctor A. Chang-Lara is courtesy of CIMAT.

## Stony Brook Lectures in Algebraic Geometry

We are proud to announce the inauguration of an endowed series of annual lectures by an outstanding early-career mathematician working in or around algebraic geometry.

## Transcendence in Algebraic Geometry

## Jacob Tsimerman

(University of Toronto)
Tuesday, Wednesday and Thursday
November 7 -9, 2023
4:00 pm
The talks will be live-streamed.
For more information consult the webpage at math.stonybrook.edu/stony-brook-lectures-algebraic-geometry

These lectures are made possible by a generous gift from the Popescu family.


[^0]:    Héctor A. Chang-Lara is investigador titular in the Department of Mathematics at el Centro de Ivestigación en Matemáticas (CIMAT), Guanajuato, Mexico. His email address is hector.chang@cimat.mx.

    Communicated by Notices Associate Editor Daniela De Silva.
    For permission to reprint this article, please contact:
    reprint-permission@ams.org.
    DOI: https://doi.org/10.1090/noti2800

[^1]:    ${ }^{1}$ We assume the following partial order on $\mathbb{R}_{s y m}^{n \times n}: M \leq N$ if and only if $x \cdot M x \leq$ $x \cdot N x$ for all $x \in \mathbb{R}^{n}$.

[^2]:    ${ }^{2}$ The oscillation of a function measures the variation of the values that it takes in a given set:

[^3]:    ${ }^{3}$ We assume the following inner product in $\mathbb{R}^{n \times n}$

    $$
    A: B:=\operatorname{tr}\left(A^{T} B\right)=\sum_{i, j=1}^{n} a_{i j} b_{i j} .
    $$

[^4]:    ${ }^{5}$ For $e \in \mathbb{R}$, we denote $e_{ \pm}:=\max ( \pm e, 0)$ the positive and negative parts of $e=$ $e_{+}-e_{-}$.

