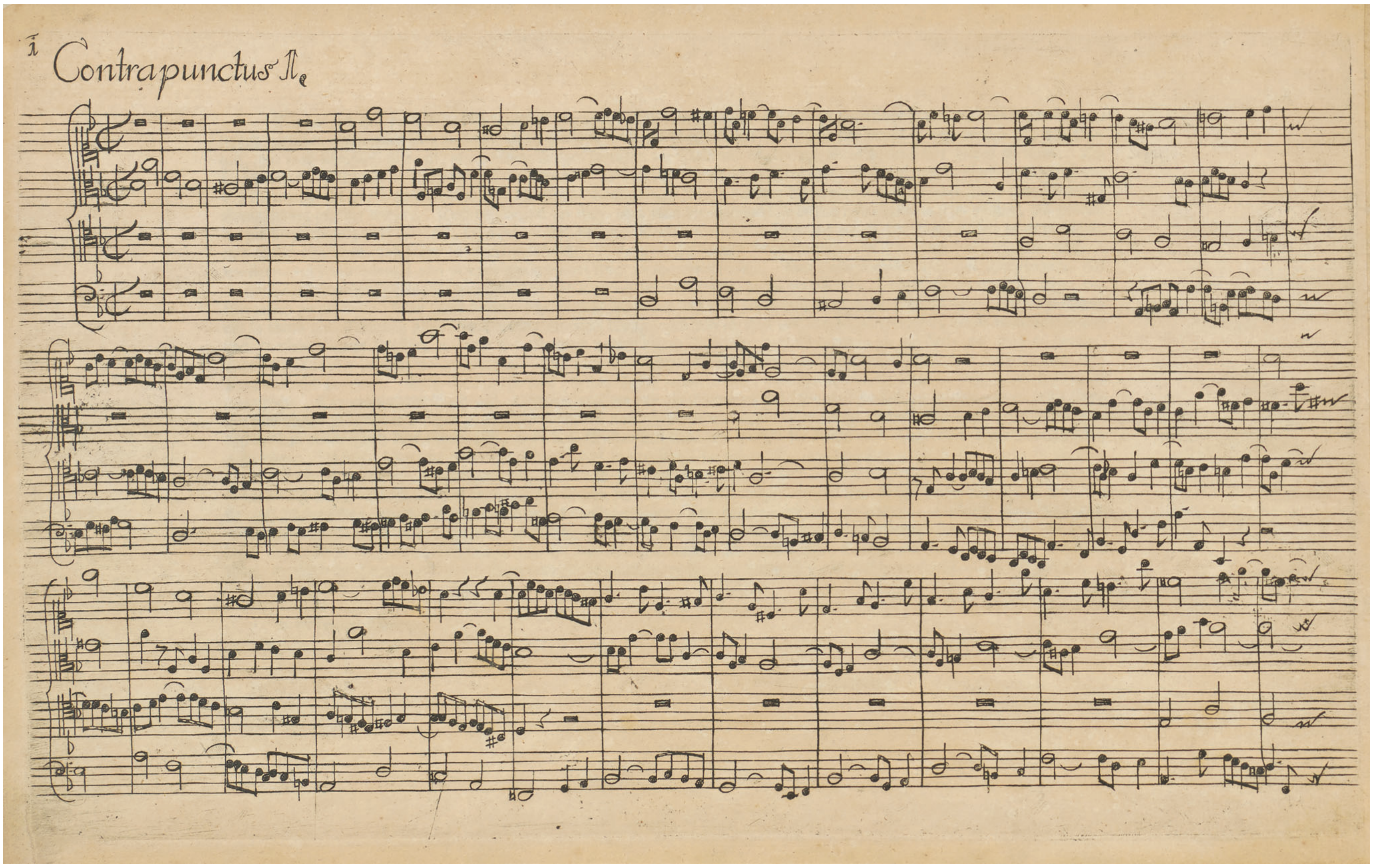
Crossed Modules



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1. Introduction

Modern chatbot software poses a threat to the health of our field. A scholarly article had better pass through the heads of at least two parties, cf. [Huf10, p. 47].[1] Here we undertake the endeavor of writing such an article about crossed modules with an eye toward the past, this past being ignored by the recent activity in this area.

In ancient cultures, *symmetry* arose as repetition of patterns. The human being perceives such symmetry as harmonious and beautiful proportion and balance (music, art, architecture, etc.). The variety of patterns is untold. Symmetry enables us to structure this variety by recognizing repetitions, as Eurynome's dancing structured chaos. The symmetries of an object are encoded in transformations that leave the object invariant. In modern mathematics, abstracting from the formal properties of such transformations led to the idea of a group. A typical example is the group of symmetries of the solutions of an equation, in modern mathematics termed *Galois group*, or each of the 17 plane symmetry groups. The operations that form these groups were already known to the Greeks. Besides in mathematics, symmetry and groups play a major role in physics, chemistry, engineering, materials science, crystallography, meteorology, etc. A group itself admits symmetries: the automorphisms of a group constitute a group. The *crossed*

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Dedicated to the memory of Donald J. Collins.

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[1] *"It appears that the reporter has passed along some words without inquiring what they mean, and you are expected to read them just as uncritically for the happy illusion they give you of having learned something. It is all too reminiscent of an old definition of the lecture method of classroom instruction: a process by which the contents of the textbook of the instructor are transferred to the notebook of the student without passing through the head of either party."*

module concept arises by abstracting from the formal properties which this pair enjoys. For two groups N and Q, the technology of crossed modules allows for a complete classification of the groups E having N as a normal subgroup and quotient group E/N isomorphic to Q. This settles the *extension problem* for groups, raised by Hoelder at the end of the nineteenth century. Schreier explored this problem in terms of factor sets, and Turing implicitly noticed that it admits a solution in terms of crossed modules. Given N and Q as symmetry groups, interpreting an extension E as a symmetry group is an interesting task, as is, given the extension group E as a symmetry group, interpreting N and Q as symmetry groups. Crossed modules occur in mathematics under various circumstances as a means to structure a collection of mathematical patterns. A special example of a crossed module arises from an ordinary module over a group. While the concept of an ordinary module is lingua franca in mathematics, this is not the case of crossed modules.

Beyond some basic algebra and algebraic topology, we assume the reader familiar with some elementary category theory ("categorically thinking" suffices). This article is addressed to the nonexpert. We keep sophisticated technology like group cohomology, homotopy theory, and algebraic number theory at a minimum. We write an injection as $\rightarrowtail$ and a surjection as $\twoheadrightarrow$.

The opener image of this article displays the beginning of the first counterpoint (contrapunctus) of the original printed version of J. S. Bach's "Kunst der Fuge". The order of the counterpoints (of the second part thereof) had been lost and, 100 years ago, Wolfgang Graeser, before enrolling as a mathematics student at Berlin university, restored an order (perhaps the original sequence) by means of symmetry considerations. This order is the nowadays generally accepted performance practice.

2. Definition and Basic Examples

A crossed module arises by abstraction from a structure we are all familiar with when we run into a normal subgroup of a group or into the group of automorphisms of a group: Denote the identity element of a group by e and, for a group G and a G-(operator) group H (a group H together with an action of G on H from the left by automorphisms of H) we write the action as $(x, y) \mapsto {}^{x}y$, for $x \in G$ and $y \in H$. Consider two groups C and G, an action of G on C from the left, view the group G as a G-group with respect to conjugation, and let $\partial : C \to G$ be a homomorphism of G-groups. The triple (C, G, ∂) constitutes a *crossed module* if, furthermore, for every pair (x, y) of members of C,

$$xyx^{-1}({}^{\partial x}y)^{-1} = e, \tag{1}$$

that is, the members xyx^{-1} and ${}^{\partial x}y$ of C coincide. For a crossed module (C, G, ∂), the image $\partial(C)$ of C in G is a normal subgroup, the kernel $\ker(\partial)$ of ∂ is a central subgroup of C, and the G-action on C induces an action of the quotient group $Q = G/(\partial(C))$ on $Z = \ker(\partial)$ turning Z into a module over this group, and it is common to refer to the resulting exact sequence

$$Z \rightarrowtail C \longrightarrow G \twoheadrightarrow Q \tag{2}$$

as a *crossed* 2*-fold extension of* Q *by* Z. A morphism of crossed modules is defined in the obvious way. Thus crossed modules constitute a category. The terminology "crossed module" goes back to [Whi49]. The identities (1) appear in [Pei49] and have come to be known in the literature as *Peiffer identities* [Lyn50] (beware: not "Peiffer identity" as some of the present day literature suggests). There is a prehistory, however, [Bae34, Tur38]; in particular, the Peiffer identities occur already in [Tur38]. The reader may consult [Hue21, Section 3] for details.

For a group Q and a Q-module M, the trivial homomorphism from M to Q is a crossed module structure and, more generally, so is any Q-equivariant homomorphism ϑ from M to Q such that the image $\vartheta(M)$ of M in Q acts trivially on M. Relative to conjugation, the injection of a normal subgroup into the ambient group is manifestly a crossed module structure. Also, it is immediate that the homomorphism $\partial : G \to \mathrm{Aut}(G)$ from a group G to its group $\mathrm{Aut}(G)$ of automorphisms which sends a member of G to the inner automorphism it defines turns $(G, \mathrm{Aut}(G), \partial)$ into a crossed module. It is common to refer to the quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/(\partial(G))$ as the group of *outer automorphisms* of G.

3. Identities Among Relations

An "identity among relations" is for a presentation of a group what a "syzygy among relations", as considered by Hilbert, is for a presentation of a module: Consider a presentation $\langle X; R\rangle$ of a group Q. That is to say, X is a set of generators of Q and R a family of (reduced) words in X and its inverses such that the canonical epimorphism from the free group F on X to Q has the normal closure N_R of (the image of) R in F as its kernel. Then the F-conjugates of the images in F of the members r of R generate N_R, that is, (with a slight abuse of the notation R,) the family R generates N_R as an F-operator group: With the notation ${}^{y}r = yry^{-1}$ ($r \in R$, $y \in F$), we can write any member w of N_R in the form

$$w = \prod_{j=1}^{m} {}^{y_j}r_j^{\varepsilon_j},\ r_j \in R,\ y_j \in F,\ \varepsilon_j = \pm 1,$$

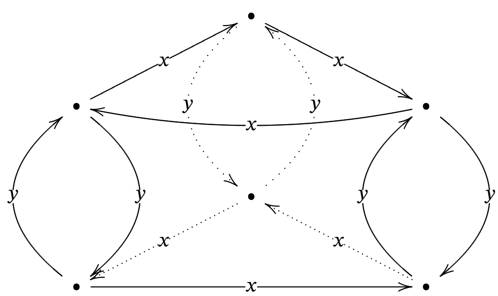
but w does not determine such an expression uniquely. Thus the issue of understanding the structure of N_R as an F-operator group arises. Heuristically, an *identity among relations* is such a specified product that recovers the identity element e of F. For example, consider the presentation

$$\langle x, y; r, s, t\rangle,\ r = x^3, s = y^2, t = xyxy, \tag{3}$$

of the symmetric group S_3 on three letters. Straightforward verification shows that

$$ts^{-1}({}^{x^{-1}}t)({}^{x^{-1}}s^{-1})({}^{x^{-1}y}r^{-1})({}^{x^{-2}}t)({}^{x^{-2}}s^{-1})r^{-1} \tag{4}$$

is an identity among the relations in (3). A standard procedure enables us to read off this identity and others from the following prism-shaped tesselated 2-sphere:



(5)

The reader will notice that reading along the boundaries of the faces recovers the relators in (3). The projection, to a plane, of this prism-shaped 2-sphere with one of the triangles removed is [BH82, Fig. 2 p. 155]. In Section 10 of [BH82], the reader can find precise methods to obtain such identities from "pictures", see in particular [BH82, Fig. 12 p. 194] for the case at hand, and in [CH82] from diagrams etc. The corresponding term in [Pei49] is "Randwegaggregat".

To develop a formal understanding of the situation, let $\widehat{C}_R$ be the free F-operator group on R. The kernel of the canonical epimorphism from the free group on the (disjoint) union $X \cup R$ to F realizes $\widehat{C}_R$. Let $\hat{\partial}_R\colon \widehat{C}_R \to F$ denote the canonical homomorphism. The members of the kernel of $\hat{\partial}_R\colon \widehat{C}_R \to F$ are the *identities among the relations* (or *among the relators*) for the presentation $\langle X; R\rangle$ [Pei49, Rei49, BH82]; Turing refers to them as "relations between the relations" [Tur38, Section 2].

The *Peiffer elements* $xyx^{-1}({}^{\hat{\partial}_R x}y)^{-1}$, as x and y range over $\widehat{C}_R$, are identities that are always present, independently of any particular presentation under discussion. Following [Rei49], let C_R be the quotient group $\widehat{C}_R/P$ of $\widehat{C}_R$ modulo the subgroup P in $\widehat{C}_R$, necessarily normal, which the Peiffer elements generate. The Peiffer subgroup P is an F-subgroup, whence the F-action on $\widehat{C}_R$ passes to an F-action on C_R, and the canonical homorphism $\hat{\partial}_R$ induces a homomorphism $\partial_R\colon C_R \to F$ that turns (C_R, F, ∂_R) into a crossed module. In particular, the kernel $\pi = \ker(\partial_R)$, being central in C_R, is an abelian group, the F-action on π factors through a Q-module structure, and π parametrizes equivalence classes of "essential" or *nontrivial* identities associated to the presentation $\langle X; R\rangle$ of Q [Rei49]. We shall see below that there are interesting cases where π is trivial, that is, the Peiffer identities generate all identities.

Given a group G, a *G-crossed module* is a G-group C together with a homomorphism $\partial\colon C \to G$ of G-groups such that (C, G, ∂) is a crossed module. Given a set R and a set map $\kappa\colon R \to G$ into a group G, the *free crossed G-module on κ* is the crossed module $(C_\kappa, G, \partial_\kappa)$ enjoying the following property: Given a G-crossed module (C, G, ∂) and a set map $\beta\colon R \to C$, there is a unique homomorphism $\beta_\kappa\colon C_\kappa \to C$ of G-groups such that $(\beta_\kappa, \mathrm{Id})\colon (C_\kappa, G, \partial_\kappa) \to (C, G, \partial)$ is a morphism of crossed modules. A standard construction shows that this free G-crossed module always exists. By the universal property, such a free G-crossed module is unique up to isomorphism, whence it is appropriate to use the definite article here. The F-crossed module (C_R, ∂_R) just constructed from a presentation $\langle X; R\rangle$ of a group Q is plainly the free F-crossed module on the injection $R \to F$. The resulting extension $\pi \rightarrowtail C_R \twoheadrightarrow N_R$ of F-operator groups then displays the F-crossed module N_R as the quotient of a free F-crossed module and thereby yields structural insight. It is also common to refer to (C_R, F, ∂_R) as the *free crossed module on $\langle X; R\rangle$*.

4. Group Extensions and Abstract Kernels

Given two groups N and Q, the issue is to parametrize the family of groups G that contain N as a normal subgroup and have quotient G/N isomorphic to Q, or equivalently, in categorical terms, the family of groups G that fit into an exact sequence of groups of the kind

$$N \rightarrowtail G \twoheadrightarrow Q. \tag{6}$$

Given an extension of the kind (6), conjugation in G induces a homomorphism φ from Q to $\mathrm{Out}(N)$. It is common to refer to a triple (N, Q, φ) that consists of two groups N and Q together with a homomorphism $\varphi\colon Q \to \mathrm{Out}(N)$ as an *abstract kernel* or to the pair (N, φ) as an *abstract Q-kernel*. In [Bae34] the terminology is "Kollektivcharakter". We have just seen that a group extension determines an abstract kernel. Given two groups N and Q together with an abstract kernel structure $\varphi\colon Q \to \mathrm{Out}(N)$, the extension problem consists in *realizing* the abstract kernel, that is, in parametrizing the extensions of the kind (6) having φ as its abstract kernel provided such an extension exists, and the abstract kernel is then said to be *extendible*.

When N is abelian, the group of outer automorphisms of N amounts to the group $\mathrm{Aut}(N)$, an abstract Q-kernel structure on N is equivalent to a Q-module structure on N, and the semidirect product group $N \rtimes Q$ shows that this abstract kernel is extendible. When N is nonabelian, not every abstract Q-kernel (N, φ) is extendible, however, [Bae34, p. 415]. We shall shortly see that nonextendible abstract kernels abound in mathematical nature.

For two homomorphisms $G_1 \to Q$ and $G_2 \to Q$, let $G_1 \times_Q G_2$ denote the *pullback* group, that is, the subgroup of $G_1 \times G_2$ that consists of the pairs (x_1, x_2) that have the same image in Q.

For a crossed module (C, G, ∂), the action $G \to \mathrm{Aut}(C)$ of G on C plainly defines an abstract kernel structure $\varphi\colon G/(\partial(C)) \to \mathrm{Out}(C)$. On the other hand, given the abstract kernel (N, Q, φ), the pullback group $G^\varphi = \mathrm{Aut}(N) \times_{\mathrm{Out}(N)} Q$ and the canonical homomorphism $\partial^\varphi\colon N \to G^\varphi$ which the crossed module structure $\partial\colon N \to \mathrm{Aut}(N)$ induces combine to the crossed module $(N, G^\varphi, \partial^\varphi)$, and $\ker(\partial^\varphi)$ coincides with the center of N. In fact, this correspondence is a bijection between abstract kernels and crossed modules (C, G, ∂) having $\ker(\partial)$ as the center of C. The group G^φ occurs in [Bae34] as the "Aufloesung des Kollektivcharakters" φ.

We will now explain the solution of the extension problem: Consider an abstract kernel (N, Q, φ), and let $(N, G^\varphi, \partial^\varphi)$ be its associated crossed module. Let $\langle X; R\rangle$ be a presentation of the group Q and, exploiting the freeness of F, choose the set maps $\alpha\colon X \to G^\varphi$ and $\beta\colon R \to N$ in such a way that (i) the composite $F \to Q$ of the induced homomorphism $\alpha_F\colon F \to G^\varphi$ with the epimorphism from G^φ to Q coincides with the epimorphism from F to Q and (ii) the composite $R \to G^\varphi$ of α_F with the injection from R to F coincides with the composite of ∂ with β. In view of the universal property of the free F-crossed module (C_R, ∂_R), the set map β induces a morphism $(\beta_R, \alpha_F)\colon (C_R, F, \partial_R) \to (N, G^\varphi, \partial^\varphi)$ of crossed modules defined on the free crossed module (C_R, F, ∂_R) on $\langle X; R\rangle$. Let $\pi = \ker(\partial_R)$ and display the resulting morphism of crossed modules as

$$\begin{tikzcd} \pi \arrow[r, tail] \arrow[d] & C_R \arrow[r, "\partial_R"] \arrow[d, "\beta_R"'] & F \arrow[r, two heads] \arrow[d, "\alpha_F"] & Q \arrow[d, equal] \\ Z \arrow[r, tail] & N \arrow[r, "\partial_\varphi"'] & G^\varphi \arrow[r, two heads] & Q. \end{tikzcd} \tag{7}$$

(This is Diagram 8 in [Hue21, Section 3]). The following is a version of [Tur38, Theorem 4 p. 356] in modern language and terminology; see [Hue21, Section 3] for details. Crossed modules, in particular free ones, exact sequences, commutative diagrams, pullbacks, etc. were not at Turing's disposal, however, and he expressed his ideas in other ways. [2]

Theorem. *The abstract kernel (N, Q, φ) is extendible if and only if, once α has been chosen, the set map β can be chosen in such a way that, in Diagram 7, the restriction of β_R to π is zero.*

[2] *A. Turing and J.H.C. Whitehead worked as WWII codebreakers at Bletchley Park (UK) but we shall never know whether they then discussed the idea of a crossed module.*

Proof. It is immediate that the condition is necessary. To establish the converse, suppose, α having been chosen, there is a choice of β with the asserted property. Let $N_R = \partial_R(C_R) \subseteq F$ denote the normal closure of R in F. From (7), we deduce the morphism

$$\begin{tikzcd} N_R \arrow[r, tail, "\iota"] \arrow[d, "\widetilde{\beta}"'] & F \arrow[r] \arrow[d, "\alpha_F"] & Q \arrow[d, equal] \\ N \arrow[r, "\partial_\varphi"'] & G^\varphi \arrow[r] & Q \end{tikzcd} \tag{8}$$

of crossed modules. Since (8) is a morphism of crossed modules, in the semidirect product group $N \rtimes F$, the members of the kind $(\widetilde{\beta}(y), \iota(y^{-1}))$ as y ranges over N_R constitute a subgroup, necessarily normal. The quotient group E of $N \rtimes F$ by this normal subgroup yields the group extension E of Q by N we seek. ☐

This modern proof of the theorem is in [Hue80a, Section 10] (phrased in terms of the coequalizer of $\widetilde{\beta}$ and ι). The diligent reader will notice that the proof is complete, i.e., no detail is left to the reader. It is an instance of a common observation to the effect that *mathematics consists in continuously improving notation and terminology*. Even nowadays there are textbooks that struggle to explain the extension problem and its solution in terms of lengthy and unilluminating cocycle calculations, and it is hard to find the above theorem, only recently dug out as a result of Turing's [Hue21].[3] Eilenberg–MacLane (quoting [Tur38] but apparently not understanding Turing's reasoning) developed the obstruction for an abstract kernel to be extendible in terms of the vanishing of the class of a group cohomology 3-cocycle of Q with values in Z [EM47, Theorem 8.1]. The unlabeled arrow $\pi \to Z$ in Diagram 7 recovers this 3-cocycle, and the condition in the theorem says that the class of this 3-cocycle in the group cohomology group $\mathrm{H}^3(Q, Z)$ is zero. Indeed, once a choice of α and β has been made, the vanishing of that 3-cocycle is equivalent to the unlabeled Q-module morphism $\pi \to Z$ in (7) admitting an extension to a homomorphism $\gamma\colon C_R \to Z$ of F-groups, and setting $\widetilde{\beta}_R(y) = \beta_R(y)\gamma(y)^{-1}$ and substituting $\widetilde{\beta}_R$ for β_R, we obtain a diagram of the kind (7) having $\pi \to Z$ zero.

To put flesh on the bones of the last remark, recall the *augmentation* map $\varepsilon\colon \mathbb{Z}\Gamma \to \mathbb{Z}$ of a group Γ defined on the integral group ring $\mathbb{Z}\Gamma$ of Γ by $\varepsilon(x) = 1$ for $x \in \Gamma$ and write the *augmentation ideal* $\ker(\varepsilon)$ as $I\Gamma$. The F-action on C_R induces a Q-action on the abelianized group $C_{R,\mathrm{ab}} = C_R/[C_R, C_R]$ in such a way that the set R constitutes a set of free generators; thus $C_{R,\mathrm{ab}}$ is canically isomorphic to the free Q-module $\mathbb{Z}Q[R]$ freely generated by R (with a slight abuse of the notation R). The induced morphism

[3] *Turing would have slipped through an evaluation system based on bibliographic metrics.*

$\pi \to C_{R,\mathrm{ab}}$ of Q-modules is still injective. For $r \in R$ and $x \in X$, using the fact that $\{x-1 \in \mathrm{I}F; x \in X\}$ is a family of free F-generators of $\mathrm{I}F$, define $r_x \in \mathbb{Z}F$ by the identity $r - 1 = \sum r_x(x-1) \in \mathrm{I}F$, and let $[r_x] \in \mathbb{Z}Q[X]$ denote the image. Defining d_2 by $d_2(r) = \sum [r_x]x \in \mathbb{Z}Q[X]$ where r ranges over R and x over X, and d_1 by $d_1(x) = [x]-1 \in \mathbb{Z}Q$ where $[x] \in \mathbb{Z}Q$ denotes the image of x, as x ranges over X, renders the sequence

$$\pi \rightarrowtail \mathbb{Z}Q[R] \xrightarrow{d_2} \mathbb{Z}Q[X] \xrightarrow{d_1} \mathbb{Z}Q \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \tag{9}$$

exact, and the three middle terms thereof together with the corresponding arrows constitute the beginning of a free resolution of $\mathbb{Z}$ in the category of Q-modules. When we take $\langle X; R\rangle$ to be the *standard* presentation having X the set of all members of Q distinct from e, we obtain the first three terms of the *standard resolution*, and the composite of the Q-module epimorphism $B_3 \to \pi$ defined on the fourth term B_3 of the standard resolution with the above unlabeled arrow $\pi \to Z$ verbatim recovers the 3-cocycle in [EM47, Theorem 8.1].

A *congruence* between two group extensions of a group Q by a group N is a commutative diagram

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, equal] & E_1 \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ N \arrow[r, tail] & E_2 \arrow[r, two heads] & Q \end{tikzcd} \tag{10}$$

with exact rows in the category of groups. For a group N with center Z, the multiplication map restricts to a homomorphism $\mu\colon N \times Z \to N$.

Complement. *Let (N, Q, φ) be an extendible abstract kernel and realize it by the group extension $N \overset{j_1}{\rightarrowtail} E_1 \twoheadrightarrow Q$. The assignment to an extension $Z \overset{j}{\rightarrowtail} E \twoheadrightarrow Q$ of Q by the center Z of N realizing the induced Q-module structure on Z of the group extension $N \overset{j_2}{\rightarrowtail} E_2 \twoheadrightarrow Q$ which the commutative diagram*

$$\begin{tikzcd} N \times Z \arrow[r, tail, "(j_1,j)"] \arrow[d, "\mu"'] & E_1 \times_Q E \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ N \arrow[r, tail] & E_2 \arrow[r, two heads, "j_2"] & Q \end{tikzcd} \tag{11}$$

with exact rows in the category of groups characterizes induces a faithful and transitive action of $\mathrm{H}^2(Q, Z)$ on the congruence classes of extensions of Q by N realizing the abstract kernel (N, Q, φ).

As for the proof we note that, as before, since the left-hand and middle vertical arrows in (11) constitute a morphism of crossed modules, in the group $N \rtimes (E_1 \times_Q E)$, the triples $(y_1y_2, j_1(y_1)^{-1}, j(y_2)^{-1})$, as y_1 ranges over N and y_2 over Z, form a normal subgroup, and the group E_2 arises as the quotient of $N \rtimes (E_1 \times_Q E)$ by this normal subgroup. When N is abelian, it coincides with its center, Diagram 11 displays the operation of "Baer sum" of two extensions with abelian kernel, and the complement recovers the fact that the congruence classes of abelian extensions of Q by N constitute an abelian group, the group structure being induced by the operation of Baer sum [Bae34], indeed, the group cohomology group $\mathrm{H}^2(Q, N)$.

5. Combinatorial Group Theory and Low-dimensional Topology

Consider a space Y and a subspace X (satisfying suitable local properties, Y being a CW complex and X a subcomplex would suffice), and let o be a base point in X. The standard action of the fundamental group $\pi_1(X, o)$ (based homotopy classes of continuous maps from a circle to X with a suitably defined composition law) on the second relative homotopy group $\pi_2(Y, X, o)$ (based homotopy classes of continuous maps from a disk to Y such that the boundary circle maps to X with a suitably defined composition law) and the boundary map $\partial\colon \pi_2(Y, X, o) \to \pi_1(X, o)$ turn $(\pi_2(Y, X, o), \partial)$ into a crossed $\pi_1(X, o)$-module [Whi41]. See [BHS11] for a leisurely introduction to homotopy groups building on an algebra of composition of cubes. Suffice it to mention that the higher homotopy groups of a space acquire an action of the fundamental group. In [Whi41], J.H.C. Whitehead in particular proved that, when the space Y arises from X by attaching 2-cells, the crossed $\pi_1(X, o)$-module $(\pi_2(Y, X, o), \partial)$ is free on the homotopy classes in $\pi_1(X, o)$ of the attaching maps of the 2-cells.

The *Cayley graph* of a group Q with respect to a family X of generators is the directed graph having Q as its set of vertices and, for each pair $(x, y) \in X \times Q$, an oriented edge joining y to xy. The oriented graph that underlies (5) is the Cayley graph of the group S_3 with respect to the generators x and y in (3). The *geometric realization* $K = K\langle X; R\rangle$ of a presentation $\langle X; R\rangle$ of a group Q is a 2-dimensional CW complex with a single zero cell o, having 1-cells in bijection with X and 2-cells in bijection with R in such a way that the fundamental group $\pi_1(K^1, o)$ of the 1-skeleton amounts to the free group on X and that the attaching maps of the 2-cells define, via the boundary map $\partial\colon \pi_2(K, K^1, o) \to \pi_1(K^1, o)$, the members of R. By construction, the fundamental group $\pi_1(K, o)$ of K is isomorphic to Q, the fundamental group $\pi_1(K^1, o)$ is canonically isomorphic to the free group F on the generators X and, by Whitehead's theorem, the $\pi_1(K^1, o)$-crossed module $(\pi_2(K, K^1, o), \partial)$ is free on the attaching maps of the 2-cells and hence canonically isomorphic to the $\pi_1(K^1, o)$-crossed module written above as (C_R, ∂_R). The 1-skeleton $\overline{K}^1$ of the universal covering space $\widetilde{K}$ of K then has fundamental group isomorphic to the normal closure N_R of the relators in F and, together with the appropriate orientation of its edges, then amounts to the Cayley graph of Q

with respect to X. Every 2-dimensional CW complex with a single 0-cell is of this kind. Since the higher homotopy groups of K^1 are zero, the long exact homotopy sequence of the pair (K, K^1) reduces to the crossed 2-fold extension

$$\pi_2(K,o) \rightarrowtail \pi_2(K,K^1,o) \xrightarrow{\partial} \pi_1(K,o) \twoheadrightarrow Q. \qquad (12)$$

Thus the group of "essential identities" among the relations R amounts to the second homotopy group $\pi_2(K,o)$ of K (based homotopy classes of continuous maps from a 2-sphere to K, with a suitably defined group structure). For illustration, consider the 1-skeleton of the prism-shaped tesselated 2-sphere (5). For each 2-gon and each 4-gon, attach two copies of the disk bounding it and, likewise, for each triangle, attach three copies of the disk bounding it. Thus we obtain a prism-shaped 2-complex having, beyond the six vertices and twelve edges, six faces bounded by 2-gons, six faces bounded by 4-gons, and six faces bounded by triangles, and this 2-complex realizes the universal covering space $\widetilde{K}$ of the geometric realization K of (3). By its very construction, it has eleven "chambers" (combinatorial 2-spheres). Since $\widetilde{K}$ is simply connected, we conclude its second homotopy group is free abelian of rank eleven. For the inexperienced reader we note that choosing a maximal tree in $\overline{K}^1 = \widetilde{K}^1$ and deforming this tree to a point is a homotopy equivalence, and the result is a bunch of eleven 2-spheres, necessarily having second homology group free abelian of rank eleven. Since $\widetilde{K}$ is simply connected, the Hurewicz map from $\pi_2(\widetilde{K},o)$ to this homology group is an isomorphism, and so is the homomorphism $\pi_2(\widetilde{K},o) \to \pi_2(K,o)$ which the covering projection induces. Hence $\pi_2(K,o)$ is free abelian of rank eleven. Under the isomorphism $\pi_2(K,K^1,o) \to C_R$ of crossed F-modules, the based homotopy class of the innermost chamber goes to the class of the identity (4) in C_R, necessarily nontrivial, as are the identities that correspond to the other chambers. Another interesting piece of information we extract from (5) is that $\pi_1(\overline{K}^1,o) \cong N_R \subseteq F$, being the fundamental group of the Cayley graph $\overline{K}^1$ of S_3 is, as a group, freely generated by any seven among the eight constituents of (4).

The paper [Pei49] arose out of a combinatorial study of 3-manifolds: The present discussion applies to the 2-skeleton M^2 of a cell decomposition of a 3-manifold M, and attaching 3-cells to build the 3-manifold under discussion "kills" some of the essential identities associated with M^2: The commutative diagram

$$\begin{tikzcd}
\pi_2(M^2,o) \arrow[r, tail] \arrow[d, two heads] & \pi_2(M^2,M^1,o) \arrow[r, "\partial"] \arrow[d, two heads] & \pi_1(M^1,o) \arrow[r, two heads] \arrow[d, equal] & \pi_1(M,o) \arrow[d, equal] \\
\pi_2(M,o) \arrow[r, tail] & \pi_2(M,M^1,o) \arrow[r, "\partial_M"'] & \pi_1(M^1,o) \arrow[r, two heads] & \pi_1(M,o)
\end{tikzcd} \qquad (13)$$

with exact rows displays how the $\pi_1(M^1,o)$-crossed module $(\pi_2(M,M^1,o), \partial_M)$ arises from the free $\pi_1(M^1,o)$-crossed module $(\pi_2(M^2,M^1,o), \partial)$. For these and related issues, see, e.g., [BH82] and the literature there. For illustration, consider a finite subgroup Q of $\mathrm{SU}(2)$ such as, e.g., $Q = C_n$, the cyclic group of order $n \geq 2$, or Q the quaternion group of order eight. The orbit space $M = \mathrm{SU}(2)/Q$, a *lens space* when $Q = C_n$, is a 3-manifold having $\pi_2(M^2,o)$ nontrivial but $\pi_2(M,o)$ trivial since the 3-sphere has trivial second homotopy group. Via Papakyriakopoulos's *sphere theorem*, the second homotopy group of a general 3-manifold being nontrivial is equivalent to the manifold being geometrically splittable, similarly to what we hint at for tame links below. Thus a 3-manifold is geometrically splittable if and only if attaching the 3-cells of a cell decomposition does not kill all the essential identities arising from its 2-skeleton.

A 2-complex K is said to be *aspherical* when its degree ≥ 2 homotopy groups are trivial. This is equivalent to the second homotopy group being trivial. Lyndon's identity theorem [Lyn50] implies that, when K has a single 2-cell, the 2-complex K is aspherical if and only if the relator r arising as the boundary of the 2-cell is not a proper power in $F = \pi_1(K,o)$. Thus the F-crossed module structure of the normal closure $N_{\{r\}}$ of $\{r\}$ in F is free in this case. For example, a closed surface distinct from the 2-sphere is aspherical, but this is an immediate consequence of the universal covering being the 2-plane. In the same vein, the exterior of a tame link in the 3-sphere is homotopy equivalent to a 2-complex—the operation of "squeezing" the 3-cells of a cell decomposition achieves this—and the link is geometrically unsplittable if and only if the exterior and hence the 2-complex is aspherical. Here "tame" means that the link belongs to a cellular subdivision. Consequently, for the Wirtinger presentation $\langle X; R\rangle$ [BH82, Fig. 3 Section 9 p. 183] of the link group, as a crossed F-module, the normal closure N_R of the relators R in F is free if and only if, by the already quoted result of Papakyriakopoulos, the link is geometrically unsplittable [BH82, Theorem (P) Section 9 p. 183]. In particular, a tame knot is a geometrically unsplittable link. In [Whi41], Whitehead raised the issue, still unsettled, whether any subcomplex of an aspherical 2-complex is itself aspherical. This is equivalent to asking whether, for a presentation $\langle X; R\rangle$ of a group, when the normal closure N_R of R in F is free as an F-crossed module, this is still true of the normal closure $N_{\widetilde{R}}$ of a subset $\widetilde{R}$ of R. See [BH82, Section 9 p. 181 ff.] for more details and literature on partial results. In [Whi41], in the proof of the freeness of the crossed module arising from attaching 2-cells to a space, Whitehead interpreted the Peiffer identities as Wirtinger relations associated to the link arising from a null homotopy; see [BH82, Section 10 p. 187 ff.] for details and more references.

6. Interpretation of the Third Group Cohomology Group

The notion of congruence for group extensions extends to crossed 2-fold extensions, and congruence classes of crossed 2-fold extensions of the kind (2), together with a suitably defined operation of composition arising from a generalized Baer sum, constitute the third group cohomology group $\mathrm{H}^3(Q,Z)$. Under this interpretation, the crossed 2-fold extension associated to the crossed module $(Z,Q,0)$ represents the identity element. See [Mac79] for the history of this interpretation; it parallels the result in [EM47] saying that suitably defined equivalence classes of abstract Q-kernels having Z as its center are in bijection with the members of $\mathrm{H}^3(Q,Z)$. Under this interpretation, the zero class corresponds to the extendible Q-kernels. This is essentially Turing's theorem in a new guise. Thus the crossed 2-fold extension (2) associated to a G-crossed module (C,∂) defines a *characteristic class* in $\mathrm{H}^3(Q,Z)$ with $Q = G/\partial(C)$. In particular, such a G-crossed module having $\ker(\partial)$ equal to the center Z of C defines an abstract Q-kernel, and this Q-kernel is extendible if and only if its characteristic class is zero. The characteristic class in $\mathrm{H}^3(\pi_1(K,o),\pi_2(K,o))$ of the $\pi_1(K^1,o)$-crossed module $(\pi_2(K,K^1,o),\partial)$ associated with a CW complex K recovers the (first) k- (or *Postnikov*) invariant; when K is 2-dimensional, $\pi_1(K,o)$, $\pi_2(K,o)$ and this k-invariant determine the homotopy type of K. A CW complex with nontrivial fundamental group and nontrivial second homotopy group typically has nonzero k-invariant but, to understand the present discussion, there is no need to know anything about k-invariants beyond the fact that the crossed 2-fold extension associated with the corresponding crossed module represents it. The crossed module associated with the geometric realization of (3) provides an explicit example of a nontrivial k-invariant; see below. Here is an even more elementary example, with explicit verification of the nontriviality of its k-invariant: The free crossed module $(C_{\{r\}},C_x,\partial)$ associated with the presentation $\langle x,r\rangle$ with $r = x^n$ of the finite cyclic group C_n of order $n \geq 2$ has C_x the free cyclic group generated by x and $C_{\{r\}}$ the free abelian C_n-group which r generates, equivalently, the free $\mathbb{Z}C_n$-module which r generates when we use additive notation, the action of C_x on $C_{\{r\}}$ is the composite of the projection $C_x \to C_n$ with the action coming from the C_n-group structure on $C_{\{r\}}$, and ∂ sends ${}^{x^k}r$, $0 \leq k \leq n-1$, to $x^n \in C_x$. The n identities

$$i_1 = {}^{x}r r^{-1}, i_2 = {}^{x}i_1 = {}^{x^2}r({}^{x}r^{-1}), \dots, i_n = {}^{x^{n-1}}i_1 = r({}^{x^{n-1}}r^{-1}),$$

generate the kernel $\pi = \ker(\partial)$ as an abelian group, subject to the relation $i_1 i_2 \dots i_n = 1$. Thus i_1 generates π as an abelian C_n-group, equivalently, as a C_n-module when we write π additively. The experienced reader will recognize π as being, as a C_n-module, isomorphic to the augmentation ideal $\mathrm{I}C_n$ of C_n. Sending each i_j, for $1 \leq j \leq n$, to the generator of (a copy of) C_n defines an epimorphism $\pi \to C_n$; indeed, this is the epimorphism that arises by dividing out the C_n-action on π. Let $(\widehat{C}_{\{r\}},\hat{\partial})$ denote the C_x-crossed module which requiring the diagram

$$\begin{tikzcd} \pi \arrow[r, tail] \arrow[d] & C_{\{r\}} \arrow[r, "\partial"] \arrow[d] & C_x \arrow[r, two heads] \arrow[d, equal] & C_n \arrow[d, equal] \\ C_n \arrow[r, tail] & \widehat{C}_{\{r\}} \arrow[r, "\hat{\partial}"'] & C_x \arrow[r, two heads] & C_n \end{tikzcd} \tag{14}$$

with exact rows to be commutative characterizes. Let $C_r \subseteq C_x$ denote the free cyclic subgroup which $r = x^n$ generates. As an abelian group, $\widehat{C}_{\{r\}} \cong C_n \times C_r$ and, with the notation u for the generator of the copy of C_n, the rules ${}^{x}r = ur$ and ${}^{x}u = u$ characterize the C_x-group structure. Let v denote a generator of the cyclic group C_{n^2} of order n^2. With respect to the trivial actions, the homomorphism $\cdot^n : C_{n^2} \to C_{n^2}$ which sends v to v^n defines a C_{n^2}-crossed module structure on C_{n^2}. Sending u to v^n, r to v and x to v we obtain a congruence morphism

$$\begin{tikzcd} C_n \arrow[r, tail] \arrow[d, equal] & \widehat{C}_{\{r\}} \arrow[r, "\hat{\partial}"] \arrow[d] & C_x \arrow[r, two heads] \arrow[d] & C_n \arrow[d, equal] \\ C_n \arrow[r, tail] & C_{n^2} \arrow[r, "\cdot^n"'] & C_{n^2} \arrow[r, two heads] & C_n \end{tikzcd} \tag{15}$$

of crossed 2-fold extensions. The upper row of (14) represents the k-invariant in $\mathrm{H}^3(C_n,\pi)$ of the geometric realization of the presentation $\langle x;r\rangle$ of the group C_n while the lower row of (15) represents a generator of $\mathrm{H}^3(C_n,C_n) \cong C_n$. Diagram (14) says that, under the induced map $\mathrm{H}^3(C_n,\pi) \to \mathrm{H}^3(C_n,C_n)$, that k-invariant goes to a generator of a cyclic group of order $n \geq 2$. Hence that k-invariant is necessarily nontrivial. It is also worthwhile noting that the left-hand copy of C_n in (15) amounts to the third group homology group $\mathrm{H}_3(C_n)$ of C_n and that suitably exploiting the *Yoneda* interpretation of the traditional definition of $\mathrm{H}^3(C_n,C_n)$ as $\mathrm{Ext}^3_{C_n}(\mathbb{Z},C_n)$ identifies the class of the bottom row in (15) with the corresponding member of $\mathrm{Ext}^3_{C_n}(\mathbb{Z},C_n)$. Restricting the crossed 2-fold extension associated with the geometric realization of (3) to any of the cyclic subgroups of S_3 and playing a bit with the data, one can also show that the crossed module associated with the geometric realization of (3) has nonzero k-invariant. Thus crossed modules having nonzero characteristic class and in particular nonextendible abstract kernels abound.

7. Higher Group Cohomology Groups

Suitably extended, the interpretation in terms of crossed 2-fold extensions leads, for $n \geq 1$, to an interpretation of the group cohomology group $\mathrm{H}^{n+1}(Q,Z)$ in terms of "crossed

n-fold extensions". See [Mac79] for the history of this interpretation. For example, the crossed n-fold extension associated with a cell decomposition of an n-dimensional CW-complex X with nontrivial fundamental group and trivial homotopy groups $\pi_j(X)$ for $2 \leq j < n$ represents the first nonzero k-invariant in $\mathrm{H}^{n+1}(X, \pi_n(X))$ of X [Hue80b]. For illustration, as in Section 5, consider the orbit space $M = \mathrm{SU}(2)/Q$ for a (nontrivial) finite subgroup Q of SU(2). We will now use, without further explanation, some classical material which the reader can find in standard textbooks. The 2-skeleton of a suitable cell decomposition of M yields the geometric realization of the presentation of Q resulting from the cell decomposition, the geometric realization of the presentation $\langle x; r\rangle$ of C_n, with $r = x^n$, when $Q = C_n$. For a general finite subgroup Q of SU(2), the exact homotopy sequence of the pair (M, M^2), necessarily one of Q-modules, takes the form

$$\begin{aligned} \ldots \pi_3(M^2, o) \to \pi_3(M, o) \to \pi_3(M, M^2, o) \\ \to \pi_2(M^2, o) \twoheadrightarrow \pi_2(M, o). \end{aligned} \tag{16}$$

With respect to the epimorphism from $\pi_1(M^1, o)$ to Q, (16) becomes a sequence of $\pi_1(M^1, o)$-modules. The composite

$$\pi_3(M, M^2, o) \to \pi_2(M^2, M^1, o) \to \pi_2(M^2, M^1, o)_{\mathrm{ab}} \tag{17}$$

of $\pi_3(M, M^2, o) \to \pi_2(M^2, o)$ with the injection $\pi_2(M^2, o) \to \pi_2(M^2, M^1, o)$ and, thereafter, with abelianization, amounts to the boundary operator $C_3(S^3) \to C_2(S^3)$ of the Q-equivariant cellular chain complex $C_*(S^3)$ of the (cellularly decomposed) 3-sphere S^3 that underlies SU(2), and this boundary operator has the homology group $\mathrm{H}_3(S^3)$ as its kernel. Using the Hurewicz isomorphism $\pi_3(S^3, \hat{o}) \to \mathrm{H}_3(S^3)$ and the covering projection isomorphism $\pi_3(S^3, \hat{o}) \to \pi_3(M, o)$ (with the notation $\hat{o}$ for a preimage in S^3 of the base point o of M), we deduce that the Q-morphism $\pi_3(M, o) \to \pi_3(M, M^2, o)$ in (16) is injective. (Beware: We must be circumspect at this point since $\pi_3(M^2, o)$ is nontrivial when Q is not the trival group, and we cannot naively deduce that injectivity from the exactness of (16).) Since the second homotopy group of the 3-sphere S^3 is trivial, so is $\pi_2(M, o)$. Consequently the Q-morphism $\pi_3(M, M^2, o) \to \pi_2(M^2, o)$ in (16) is surjective. Hence splicing (12), with M^2 substituted for K, and (16) yields the crossed 3-fold extension

$$\pi_3(M, o) \rightarrowtail \pi_3(M, M^2, o) \to \pi_2(M^2, M^1, o) \xrightarrow{\partial} \pi_1(M^1, o) \twoheadrightarrow Q \tag{18}$$

of Q by the free cyclic group $\pi_3(M, o) \cong \mathrm{H}_3(S^3)$ (as Q-modules), and the Q-action on $\pi_3(M, o)$ is trivial since this action amounts to the induced Q-action on $\mathrm{H}_3(S^3)$, necessarily trivial. The group $\mathrm{H}^4(Q, \pi_3(M, o))$ is cyclic of order $|Q|$ (the number of elements of Q), and the crossed 3-fold extension (18) represents a generator of $\mathrm{H}^4(Q, \pi_3(M, o))$ and thence the first nonzero k-invariant of M. This k-invariant, the fundamental group $\pi_1(M, o) \cong Q$, and the third homotopy group $\pi_3(M, o)$ determine the homotopy type of M. Furthermore, the group Q has periodic cohomology, of period 2 when Q is cyclic and of period 4 otherwise. Thus, the operation of cup product with the class of (18) induces isomorphisms $\mathrm{H}^s(Q, \cdot) \to \mathrm{H}^{s+4}(Q, \cdot)$, for $s \geq 1$, for any integer s when we interpret the notation H as *Tate* cohomology. When Q is a cyclic group C_n of order $n > 1$, the extension $\pi_3(M, o) \rightarrowtail \pi_3(M, M^2, o) \twoheadrightarrow \pi = \ker(\partial)$ of C_n-modules amounts to the familiar extension $\mathbb{Z} \rightarrowtail \mathbb{Z}C_n \twoheadrightarrow \mathrm{I}C_n$ of C_n-modules arising from the standard small free resolution of $\mathbb{Z}$ in the category of C_n-modules, and it is immediate that the C_n-module structure on $\pi_3(M, o)$ is trivial.

8. Generalization

The description of group cohomology in terms of crossed n-fold extensions ($n \geq 1$) is susceptible to generalizations where cocycles are not necessarily available. For example, for an extension of a topological group G by a continuous G-module whose underlying bundle is nontrivial, (global) continuous cocycles are not available. See [Hue21, Section 3] and the literature there for more situations where this happens.

9. Lie Algebra Crossed Modules

The axiom (1) makes perfect sense for Lie algebras, and the interpretation of the $(n+1)$th Lie algebra cohomology group in terms of crossed n-fold extensions ($n \geq 1$) is available. Also the abstract kernel concept extends to Lie algebras in an obvious manner, as does the equivalence between abstract kernels and crossed modules with the central kernel constraint explained above, and there is an analogue of Turing's theorem. Indeed, much of the above material carries over to Lie algebras. See [Hue21, Section 5] and the literature there for details.

10. Normality of a Noncommutative Algebra Over Its Center

Crossed modules arise in Galois theory. We will now briefly delve into this. See [Hue21, Section 4] for the history and [Hue18b] for a more complete account and references:

Let S be a commutative ring and A an S-algebra having S as its center. Let Q be a group of operators on S. The development of a Galois theory for such algebras leads to the following question: *Does every automorphism in Q extend to an automorphism of A*? The algebra A is said to be *Q-normal* when this happens to be the case. We formalize the situation as follows:

Denote by $\mathrm{Aut}(A)$ the group of ring automorphisms of A and by $\mathrm{U}(A)$ its group of units. The obvious homomorphism $\partial\colon \mathrm{U}(A) \to \mathrm{Aut}(A)$ assigns to a unit of A the associated inner automorphism of A, the obvious action of $\mathrm{Aut}(A)$ on $\mathrm{U}(A)$ turns the triple $(\mathrm{U}(A), \mathrm{Aut}(A), \partial)$ into a crossed module, and $\ker(\partial) = \mathrm{U}(S)$, the group of units of S. Write $\mathrm{Out}(A) = \mathrm{Aut}(A)/(\partial\mathrm{U}(A))$. Each inner automorphism of A leaves S elementwise fixed whence the restriction map $\mathrm{Aut}(A) \to \mathrm{Aut}(S)$ induces a homomorphism $\mathrm{res}\colon \mathrm{Out}(A) \to \mathrm{Aut}(S)$. Let Q be a group and $\kappa\colon Q \to \mathrm{Aut}(S)$ an action of Q on S by ring automorphisms. Define a *Q-normal structure* on the central S-algebra A *relative to the given action* $\kappa\colon Q \to \mathrm{Aut}(S)$ *of Q on S* to be a homomorphism $\sigma\colon Q \to \mathrm{Out}(A)$ that lifts the action $\kappa\colon Q \to \mathrm{Aut}(S)$ of Q on S in the sense that the composite of σ with $\mathrm{res}\colon \mathrm{Out}(A) \to \mathrm{Aut}(S)$ coincides with κ. A *Q-normal S-algebra* is, then, a central S-algebra A together with a Q-normal structure $\sigma\colon Q \to \mathrm{Out}(A)$.

Let (A, σ) be a Q-normal S-algebra, let $G^\sigma = \mathrm{Aut}(A) \times_{\mathrm{Out}(A)} Q$, and let G^σ act on $\mathrm{U}(A)$ via the canonical homomorphism from G^σ to $\mathrm{Aut}(A)$. The homomorphism $\partial^\sigma\colon \mathrm{U}(A) \to G^\sigma$ which $\partial\colon \mathrm{U}(A) \to \mathrm{Aut}(A)$ induces turns $(\mathrm{U}(A), G^\sigma, \partial^\sigma)$ into a crossed module, and the crossed 2-fold extension

$$\mathrm{e}_{(A,\sigma)}\colon \mathrm{U}(S) \rightarrowtail \mathrm{U}(A) \xrightarrow{\partial^\sigma} G^\sigma \twoheadrightarrow Q \tag{19}$$

represents a class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$, the *Teichmueller class* of (A, σ). For the special case where S is a field, a cocycle description of this class (independently of any crossed module) is in [Tei40].[4]

Define a *Q-equivariant* S-algebra to be a central S-algebra A together with a homomorphism $\rho\colon Q \to \mathrm{Aut}(A)$ that induces the Q-action κ on S. For example, let $R = S^Q$, the subring of Q-invariants in S; the S-algebra $A = B \otimes_R S$ for some central R-algebra B plainly admits a canonical Q-equivariant structure. Consider a Q-normal S-algebra (A, σ). We can then ask, as did Teichmueller in the situation he considered, whether the Q-action on S lifts to a Q-equivariant structure. When such a lift exists, it induces a congruence between $\mathrm{e}_{(A,\sigma)}$ and the crossed 2-fold extension arising from the crossed module $(Z, Q, 0)$, and hence the class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$ is zero. As for the converse, let $\mathrm{M}_I(A)$ denote the $(I \times I)$ matrix algebra over A for an index family I; when I is not finite, we interpret $\mathrm{M}_I(A)$ as being the endomorphism ring of $\oplus_I A^{\mathrm{op}}$. The algebra $\mathrm{M}_I(A)$ is again a central S-algebra. It is obvious that an automorphism of A yields one of $\mathrm{M}_I(A)$ in a unique way, and the obvious map $A \to \mathrm{M}_I(A)$ is a ring homomorphism. Hence a Q-normal structure $\sigma\colon Q \to \mathrm{Out}(A)$ on A determines one on $\mathrm{M}_I(A)$, and we denote this structure by $\sigma_I\colon Q \to \mathrm{Out}(\mathrm{M}_I(A))$. By [Hue18a, Theorem 6.1], the Teichmüller class of a Q-normal S-algebra (A, σ) is zero if and only if, for $I = Q$, the Q-normal structure σ_I on the matrix algebra $\mathrm{M}_I(A)$ comes from an equivariant one. Thus the class $[\mathrm{e}_{(A,\sigma)}] \in \mathrm{H}^3(Q, \mathrm{U}(S))$ is the obstruction for the Q-normal algebra (A, σ) to be equivalent to a Q-equivariant one in the sense just explained. In general, we cannot have (A, σ) itself to be equivariant. See (22) below for a special case.

Suppose S is a field. Then we are running into ordinary Galois theory. Here is a family of explicit examples of a Q-normal algebra having nontrivial Teichmueller class: Consider a field K and let $L = K(\zeta)$ be a normal extension having Galois group N cyclic of order $n \geq 2$ (say). Let τ denote a generator of N, let $\eta \in \mathrm{U}(K)$, and consider the cyclic central simple K-algebra $D(\tau, \eta)$ generated by $L = K(\zeta)$ and some (indeterminate) u subject to the relations

$$u\lambda = {}^\tau\lambda u,\ u^n = \eta,\ \lambda \in L = K(\zeta). \tag{20}$$

In $D(\tau, \eta)$, the member u is a unit having inverse $u^{-1} = u^{n-1}\eta^{-1}$ and, for $\lambda \in L$, we get $u\lambda u^{-1} = {}^\tau\lambda$, that is, the action of the Galois group N on L extends to the inner automorphism of $D(\tau, \eta)$ which u determines. The algebra $D(\tau, \eta)$ is a crossed product of N with L relative to the $\mathrm{U}(L)$-valued 2-cocycle of N determined by η but this fact need not concern us here. The field L is a maximal commutative subalgebra of $D(\tau, \eta)$. The capital D serves as a mnemonic for the fact that Dickson explored such algebras.

Distinct choices of $\eta \in \mathrm{U}(K)$ may lead to the "same" algebra of the kind $D(\tau, \eta)$: The assignment to $\vartheta \in \mathrm{U}(L)$ of $\prod_{j=0}^{n-1} {}^{\tau^j}\vartheta$ defines the classical *norm map* $\nu\colon \mathrm{U}(L) \to \mathrm{U}(K)$. Let $\vartheta \in \mathrm{U}(L)$. Then $u \mapsto u_\vartheta = \vartheta u$ induces an isomorphism $\alpha_\vartheta\colon D(\tau, \eta) \to D(\tau, \nu(\vartheta)\eta)$ which restricts to the identity of L, an automorphism α_ϑ of $D(\tau, \eta)$ if and only if $\nu(\vartheta) = 1$. Furthermore, for $\eta = \nu(\vartheta)$ with $\vartheta \in \mathrm{U}(L)$, the algebra $D(\tau, \eta)$ comes down to the algebra of $(n \times n)$-matrices over K. Thus the cokernel $\mathrm{coker}(\nu)$ of the norm map $\nu\colon \mathrm{U}(L) \to \mathrm{U}(K)$ parametrizes classes of algebras of the kind $D(\tau, \eta)$ such that $D(\tau, \eta_1)$ and $D(\tau, \eta_2)$ belong to the same class if and only if an isomorphism which restricts to the identity of L carries $D(\tau, \eta_1)$ to $D(\tau, \eta_2)$. The expert will recognize that $\mathrm{coker}(\nu)$ amounts to $\mathrm{H}^2(N, \mathrm{U}(L))$ and $\ker(\nu)$ to the group of multiplicatively written $\mathrm{U}(L)$-valued 1-cocycles of N.

Let $\mathrm{Aut}_L(D(\tau, \eta))$ denote the group of automorphisms of $D(\tau, \eta)$ that restrict to an automorphism of $L|\mathfrak{k}$. For $\chi \in \mathrm{U}(L)$, the member $\vartheta_\chi = {}^\tau\chi\chi^{-1}$ of $\mathrm{U}(L)$ lies in the kernel of ν. *The assignment to χ of $\alpha_{\vartheta_\chi} \in \mathrm{Aut}_L(D(\tau, \eta))$ induces an embedding of $\mathrm{U}(L)/\mathrm{U}(K)$ into $\mathrm{Aut}_L(D(\tau, \eta))$ onto the subgroup of automorphisms that restrict to the identity of L.* Indeed, every automorphism α of $D(\tau, \eta)$ that restricts to the identity of L necessarily satisfies the identity

$$\begin{aligned}\alpha(u)u^{-1}\lambda u\alpha(u)^{-1} &= \alpha(u)({}^{\tau^{-1}}\lambda)\alpha(u)^{-1}\\ &= \alpha(u({}^{\tau^{-1}}\lambda)u^{-1}) = \alpha(\lambda) = \lambda\end{aligned}$$

[4] *Teichmueller worked as WWII codebreaker for the high command of the German army.*

for every $\lambda \in L$, and this implies $\alpha(u)u^{-1} \in \mathrm{U}(L)$ since L is a maximal commutative subalgebra of $D(\tau,\eta)$; then $\vartheta = \alpha(u)u^{-1}$ belongs to the kernel of ν. By Hilbert's "Satz 90", every member ϑ of $\ker(\nu)$ is of the kind ϑ_χ, for some $\chi \in \mathrm{U}(L)$.

Let Q be a finite group of operators on K and let $\mathfrak{k} = K^Q$, so that $K|\mathfrak{k}$ is a Galois extension. Suppose that, furthermore, $L|\mathfrak{k}$ is a Galois extension, let $G = \mathrm{Gal}(L|\mathfrak{k})$, and suppose that the resulting group extension of Q by N is central. The Q-action on $\mathrm{U}(K)$ passes to an action of Q on $\mathrm{coker}(\nu)$. In terms of classes of algebras of the kind $D(\tau,\eta)$, the assignment to $D(\tau,\eta)$ of $D(\tau,{}^x\eta)$, as x ranges over Q, induces this action.

A little thought reveals that the following are equivalent: (i) *The algebra $D(\tau,\eta)$ is Q-normal*; (ii) *the restriction $\mathrm{Aut}_L(D(\tau,\eta)) \to G$ is an epimorphism*; (iii) *for $x \in Q = \mathrm{Gal}(K|\mathfrak{k})$, there is a unit $\vartheta \in \mathrm{U}(L)$ such that $\nu(\vartheta) = {}^x\eta\eta^{-1}$. Furthermore, under the circumstances of* (iii), *the unit ϑ is unique up to multiplication by a unit in K.*

Condition (iii) plainly characterizes the members $[\eta]$ of the subgroup $\mathrm{coker}(\nu)^Q$ of Q-invariants of the cokernel of the norm map $\nu\colon \mathrm{U}(L) \to \mathrm{U}(K)$. Let $\sigma\colon Q \to \mathrm{Aut}(K)$ denote the Galois action. The assignment to the class $[\eta] \in \mathrm{coker}(\nu)^Q$ of the crossed 2-fold extension $\mathrm{e}_{(D(\tau,\eta),\sigma)}$ of Q by $\mathrm{U}(K)$ associated to a chosen representative η defines a map $t\colon \mathrm{coker}(\nu)^Q \to \mathrm{H}^3(Q,\mathrm{U}(K))$, the *Teichmueller map* associated with the data. Since for $\eta_1,\eta_2 \in \mathrm{U}(K)$, the tensor product algebra $D(\tau,\eta_1) \otimes_K D(\tau,\eta_2)$ is the algebra of $(n\times n)$-matrices over $D(\tau,\eta_1\eta_2)$, the Teichmueller map t is a homomorphism of abelian groups.

Up to this stage the discussion is elementary except, perhaps, the quote of Hilbert's "Satz 90". Now we borrow some classical algebra: Recall the Brauer group $\mathrm{B}(K)$ of K consists of classes of central simple K-algebras, two such algebras being equivalent when they are matrix algebras over the same division algebra, the inverse being induced by the assignment to an algebra of its opposite algebra. A field $L|K$ *splits* the central simple K-algebra A when $A \otimes_K L$ is L-isomorphic to a matrix algebra over L. It is common to denote by $\mathrm{B}(L|K)$ the subgroup of Brauer classes that are split by L. For rings more general than fields, the appropriate equivalence relation is Morita equivalence. The properties of being Q-normal and Q-equivariant are properties of the Brauer classes, the Q-action on K induces an action of Q on $\mathrm{Br}(K)$, and the Q-invariants $\mathrm{Br}(K)^Q$ constitute the subgroup of Brauer classes of Q-normal central simple K-algebras. In terms of the canonical isomorphisms $\mathrm{H}^2(Q,\mathrm{U}(K)) \to \mathrm{Br}(K|\mathfrak{k})$, $\mathrm{H}^2(G,\mathrm{U}(L)) \to \mathrm{Br}(L|\mathfrak{k})$ and $\mathrm{H}^2(N,\mathrm{U}(L)) \to \mathrm{Br}(L|K)$, with the notation sc for "scalar extension", the classical five-term exact sequence in the cohomology of the (central) group extension of Q by N with coefficients in $\mathrm{U}(L)$ takes the form

$$\mathrm{Br}(K|\mathfrak{k}) \rightarrowtail \mathrm{Br}(L|\mathfrak{k}) \xrightarrow{\mathrm{sc}} \mathrm{Br}(L|K)^Q \xrightarrow{t} \mathrm{H}^3(Q,\mathrm{U}(K)) \xrightarrow{\mathrm{inf}} \mathrm{H}^3(G,\mathrm{U}(L)). \tag{21}$$

To reconcile this sequence with the above remarks about the vanishing of the Teichmueller class of a Q-equivariant central S-algebra we note that, by "Galois descent", every Q-equivariant central simple K-algebra arises by scalar extension from a central simple $\mathfrak{k}$-algebra.

To arrive at explicit examples, let K be an algebraic number field (a finite-dimensional extension of the field $\mathbb{Q}$ of rational numbers) and, as before, let Q be a finite group of operators on K and $\mathfrak{k} = K^Q$. Let m denote the l.c.m. of the local degrees $[K_{\mathcal{P}} : \mathfrak{k}_p]$ as p ranges over the primes of $\mathfrak{k}$ and $\mathcal{P}$ over extensions thereof to K. By [Mac48, Theorem 3], the cokernel of scalar extension $\mathrm{sc}\colon \mathrm{Br}(\mathfrak{k}) \to \mathrm{Br}(K)^Q$ is a finite cyclic group of order $s = \frac{[K:\mathfrak{k}]}{m}$. Let $L = K(\zeta)$ be a cyclotomic extension having Galois group N cyclic of order $n \geq 2$ (say), that is, ζ is a primitive ℓth root of unity for some ℓ prime to n, and the Galois group N of order n acts faithfully and transitively on the primitive ℓth roots of unity. The field $L|\mathfrak{k}$ coincides with the composite field $\mathfrak{k}(\zeta)K$ in L, and the canonical action of the pullback group $\mathrm{Gal}(\mathfrak{k}(\zeta)|\mathfrak{k}) \times_{\mathrm{Gal}(\mathfrak{k}(\zeta)\cap K|\mathfrak{k})} Q$ on $\mathfrak{k}(\zeta)K$ identifies this group with a finite group of operators on $L = \mathfrak{k}(\zeta)K$ having $\mathfrak{k}$ as its fixed field. Hence $L|\mathfrak{k}$ is a Galois extension having Galois group G canonially isomorphic to the pullback group $\mathrm{Gal}(\mathfrak{k}(\zeta)|\mathfrak{k}) \times_{\mathrm{Gal}(\mathfrak{k}(\zeta)\cap K|\mathfrak{k})} Q$, and G is a central extension of Q by the cyclic group $N = \mathrm{Gal}(K(\zeta)|K) \cong \mathrm{Gal}(\mathfrak{k}(\zeta)|(\mathfrak{k}(\zeta)\cap K))$ of order n, a split extension if and only if $\mathfrak{k}(\zeta)\cap K = \mathfrak{k}$.

Every class in $\mathrm{Br}(K)$ has a cyclic cyclotomic splitting field but, beware, this only says that, for a central simple K-algebra A, some cyclic cyclotomic field splits a matrix algebra over A. On the other hand, it implies that every class in $\mathrm{Br}(K)$ and in particular in $\mathrm{Br}(K)^Q$ has a representative of the kind $D(\tau,\eta)$. Thus we may choose a member η of K and, for some $n \geq 2$, a cyclic degree n cyclotomic Galois extension $L = K(\zeta)$ of K such that the image $t[D(\tau,\eta] \in \mathrm{H}^3(Q,\mathrm{U}(K))$ of the class $[D(\tau,\eta)] \in \mathrm{Br}(L|K)^Q$ of the Q-normal central K-algebra $D(\tau,\eta)$ generates the group $\mathrm{H}^3(Q,\mathrm{U}(K))$; such a member η is unique up to multiplication by some $\lambda \in \mathfrak{k}$ and by $\nu(\vartheta)$ for some $\vartheta \in L$. From the exactness of (21) we deduce that the Teichmueller map t fits into the exact sequence

$$\mathrm{Br}(\mathfrak{k}) \xrightarrow{\mathrm{sc}} \mathrm{Br}(K)^Q \overset{t}{\twoheadrightarrow} \mathrm{H}^3(Q,\mathrm{U}(K)) \cong \mathbb{Z}/s \tag{22}$$

but beware, exactness only implies that the class of a Q-normal algebra having trivial Teichmueller class arises by scalar extension, not necessarily the algebra itself. An example of such an algebra that does not arise by scalar extension while its class does is in [Tei40]. The images $t[D(\tau,\eta)], t[D(\tau,\eta^2)], \dots, t[D(\tau,\eta^{n-1})] \in \mathrm{H}^3(Q,\mathrm{U}(K))$

exhaust the nontrivial members of the group $H^3(Q, U(K))$, and the algebras $D(\tau, \eta^j\lambda)$, for $1 \leq j \leq n-1$ and $\lambda \in \mathfrak{k}$, cover all Brauer classes of Q-normal algebras split by L with nontrivial Teichmueller class and, when we let L vary, we obtain all Brauer classes of Q-normal algebras with nontrivial Teichmueller class. Thus, to get examples, all we need is a Galois extension $K|\mathfrak{k}$ having $s > 1$. While, in view of the Hilbert–Speiser theorem, this is impossible when the Galois group Q is cyclic, for example, the fields $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$ or $K = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ have as Galois group the four group and $s = 2$.

Again we see that crossed modules having nonzero characteristic class and in particular nonextendible abstract kernels abound.

11. Outlook

A topological group is a group in the category of topological spaces. Groups in the category Cat of small categories—equivalently, categories internal to groups—constitute a category, that of *2-groups*, and there is an equivalence of categories between crossed modules and groups in Cat, observed by the Grothendieck school in the mid 1960s (unpublished), see [BHS11, Section I.1.8, p. 29; Section 2.7, p. 58]. It is an interesting exercise to see how the Peiffer identities fall out from this equivalence.

Crossed modules, variants, and generalizations thereof are nowadays very lively in mathematics; see [BHS11], [Hue21, Section 3] and the references there. The equivalence of crossed modules (in the category of groups) and 2-groups is relevant in string theory, see [Hue21, Section 3] for references. Suffice it to mention that when we pass from particles to strings, we add an extra dimension, and replacing groups by groups in Cat reflects this adding an extra dimension.

For a leisurely introduction and survey of the state of the art at the time consider [BH82]. A particularly important work is [BHS11], with its special emphasis on foundational issues.

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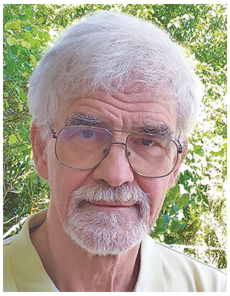

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