2:00 p.m. | Will Perkins  
**Georgia Tech**

**Searching for (sharp) thresholds in random structures: where are we now?**

Phase transitions, hard computational problems, and the emergence of intricate structures in random graphs—how are these phenomena connected and how can we understand them?

3:00 p.m. | Hussein Mourtada  
**Université Paris Cité**

**Hilbert meets Ramanujan: singularity theory and integer partitions**

What can singularities of algebraic varieties say about the various decompositions of a positive integer into a sum of positive integers?

4:00 p.m. | Holly Krieger  
**University of Cambridge**

**Uniformity when arithmetic meets geometry**

Understanding how algebra and geometry provide uniform control over the number of rational points on a curve.

5:00 p.m. | Ravi Vakil  
**Stanford University**

**Passing a curve through n points—solution of a 100-year-old problem**

When can you string a curve through a number of points in space? How two young researchers finally settled an ancient problem.

*This lecture is supported by the Bose, Datta, Mukhopadhyay, and Sarkar Fund.*

Organized by David Eisenbud, University of California, Berkeley
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New Mathematical Center in Ukraine

In summer 2022, a group of mathematicians of Ukrainian origin initiated an ambitious project—a new center for mathematical sciences with a mission to support top-level research in mathematics in Ukraine, with a special emphasis on training younger generations of mathematicians. The goal of the new center is to connect the mathematical community, and to become a catalyst for fundamental changes in Ukraine’s scientific infrastructure.

This letter is to announce the newly formed International Centre for Mathematics in Ukraine (ICMU) to the readers of the Notices. We envision ICMU as a modern facility similar to Mittag-Leffler Institut in Sweden or Isaac Newton Institute in Great Britain, where mathematicians from around the world will collaborate, share new discoveries, and train the next generations of mathematicians. ICMU will carry out educational and outreach activities, and help facilitate interaction of Ukrainian academics with the global mathematical community.

About a year ago, ICMU was registered in Ukraine as a non-governmental organization. The planning of the scientific activities is overseen by the Scientific Board composed of prominent mathematicians. A separate Supervisory Board is tasked with strategic planning of ICMU development. The inaugural chair of the Supervisory Board is the former president of the European Mathematical Society, Jean-Pierre Bourguignon.

The center is proudly supported by XTX Markets, its Founding and Principal Donor. We have begun our fundraising campaign in the US through the KBF. We welcome donations from organizations as well as individual gifts, and no donation is too small. The donations are tax-deductible and currently all are being matched by XTX Markets.

Despite the ongoing war, ICMU has already organized its first scientific event: school and conference “Numbers in the Universe,” which took place on August 7–11, 2023. Its aim was to present the latest breakthroughs in number theory and its applications to the broad public. The event took place in the premises of the Kyiv School of Economics and the Stefan Banach International Mathematical Center in Warsaw with a live connection between the two audiences. The program included lecture series by Vitaly Bergelson, Terence Tao, and Maryna Viazovska. On August 9, a special session of “Numbers in the Universe” dedicated to the opening of the ICMU was held. Government representatives of both Ukraine and Poland were present as well as the presidents of both Academies of Sciences.

The full-scale operation of ICMU will commence once the war ends, and once ICMU’s permanent building is found and renovated. Nevertheless, some programs are already open, and we plan to host them at the premises of partnering institutions such as the Kyiv School of Economics. The enthusiasm of Ukrainian students during the “Numbers in the Universe” school was palpable, and we believe that today the vision of ICMU is motivating students and scientists who are staying in Ukraine, emphasizing the dream of a bright future for the country and contributing to its rebuilding after the war.

More information about ICMU is available on its webpage at https://mathcentre.in.ua/en.

Andrey Gogolev, Professor
The Ohio State University

Volodymyr Nekrashevych, Professor
Texas A&M University

Pavlo Pylyavskyy, Professor
University of Minnesota

Dmytro Savchuk, Associate Professor
University of South Florida

Masha Vlasenko, Associate Professor
IMPAN, Warsaw

Letter to the Editor RE: Roots of Littlewood Polynomials

The image of the roots of Littlewood polynomials in the October 2023 issue of the Notices [Figure 1, page 1495] reminded me of a conversation I once had with Jonathan Borwein (1951–2016), who said that, because of its beauty,
the image “had started a life of its own.” A version of the image appeared on the covers of (at least) two books that Borwein coauthored.

In 1997 Jonathan’s younger brother Peter Borwein (1953–2020), together with Christopher Pinner, now at Kansas State University, published an article that contained the image of zeros of all polynomials with \(\{0, +1, −1\}\) coefficients and degree at most eight and the image of all zeros of all degree twelve polynomials with \(\{+1, −1\}\) coefficients (P. Borwein and C. G. Pinner, Polynomials with \(\{0, +1, −1\}\) coefficients and a root close to a given point, Canadian Journal of Mathematics 49 (1997), 887–915).

Jonathan and Peter Borwein were talented, creative, productive, and highly influential mathematicians. As happens with siblings born only a few years apart, Jonathan and Peter were very close even though they were two completely different characters: Jonathan was an inexhaustible source of energy and ideas while Peter was more of a laid-back thinker.

Their premature deaths are still felt as huge losses for their families, friends, and collaborators.

Veselin Jungic
Department of Mathematics
Simon Fraser University
A WORD FROM...  
Sergei Gelfand, Publisher of the American Mathematical Society

The opinions expressed here are not necessarily those of the Notices or the AMS.

Three years ago, I wrote the “Word from …” column about book publishing at the American Mathematical Society (see Notices of the AMS, vol. 67, issue 7, August 2020). Today I want to address a broader question, which can be asked as follows: why does publishing mathematics books continue to exist as an important and valuable activity? In this column, I’ll try to answer this question. I should add that what I am writing here is coming from my personal experience of working for the Publication Program at the AMS, as well as from talking with my colleagues at the AMS and with many mathematicians and publishers of mathematics. Other people may give different answers to this question.

Why mathematicians write books. Mathematics is and always was a collective endeavor. Mathematicians, probably more often than representatives of other sciences, want to share these discoveries with their immediate colleagues and with other mathematicians. Over many years several tools to do this have been invented. In antiquity and the early Middle Ages mathematics was communicated through personal contacts, such as conversations with pupils and, later, in mathematical letters, which were quite common in Europe in the 17th and 18th centuries (e.g., Euler, members of the Bernoulli family). But there was also an urge to communicate with people beyond their immediate colleagues, and mathematicians used other available methods including clay tablets; writing on papyrus scrolls, on parchment, and on paper; and, finally, printing. As the number of mathematicians grew and technology progressed, printed materials superseded everything else, and, in mathematics, led to printed journals and books. In the 1980s, the internet appeared, bringing with it various new methods of communicating science (electronic books and journals, blogs, videos of talks, online seminars, and much more), and there seems to be no end to new inventions in this area.

Nowadays, whenever a mathematician proves new results, an article (or several) is written and submitted for publication to a journal and/or posted on the arXiv or a similar repository. In many cases the authors view this as “the end of the story” and move to the next challenge. However, from time to time a mathematician (or a group of mathematicians) feels a need to step back, to think about a broader picture, and summarize what was achieved and where the subject should go next. This is when they may decide to write a monograph presenting classical and recent developments in an important area of mathematics. This may also happen when a mathematician believes that the research has reached a peak and wants to organize what was achieved in the area. Often times, writing a book allows the author to discover hidden patterns and simple ideas behind sometimes very long and complicated formal arguments.

For many mathematicians teaching is a significant part of their professional activity. Developing a new or innovative course involves a lot of effort, and a textbook based on such a course offers an avenue for sharing their insight and resources with others. The result might be a textbook on a standard undergraduate or beginning graduate topic (analysis, abstract algebra, topology, probability, etc.) or on a more advanced topic (such as Sobolev spaces, hyperbolic dynamics, analytic number theory, to name a few). The author’s hope when writing a textbook might be that it can help bring more students into mathematics and invite instructors to teach a new important topic. Of course, there

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DOI: https://doi.org/10.1090/noti2856
is a wide gray zone: a research monograph can be used as text for a topics course, and advanced graduate textbook may contain an exposition of recent research.

Recently, a new format of sharing knowledge, which can be called a “collective online initiative,” emerged. This is an endeavor of a group of experts in a certain area of mathematics, where a lot of important information is scattered over hundreds or, sometimes, thousands of research articles in dozens of journals. The goal of such an endeavor is to collect and organize new results and approaches in one place to help others learn them and to facilitate further progress. Often, such collective projects are presented (published) online and may require well developed database-type coding that allows the user to find a particular topic within the area. Examples of such online projects are

- The Stacks Project: https://stacks.math.columbia.edu/
- The L-functions and Modular Forms Database (LMFDB): https://www.lmfdb.org/
- NIST Digital Library of Mathematical Functions: https://dlmf.nist.gov/

To summarize, I would say that mathematicians write books because they want to share their findings and their experience both in mathematical research and in teaching mathematics.

Next, there is a question of why the American Mathematical Society publishes books. The answer to this question is simple: part of the mission of the AMS, as a society of professional mathematicians, is to advance research. Publishing books that present results of mathematical research or aim to help more people enter and master the profession or use mathematics in other areas (e.g., industry, business, public service, etc.,) is among the most direct and important ways to achieve this goal. The books the AMS publishes are chosen with this in mind, and this choice is made by professionals (both the AMS acquisition editors and the members of its book editorial committees). This is one of the main aspects of AMS publishing that distinguishes it from many other publishers of mathematics. I hope that publishing can be viewed as an important part of the AMS’s contribution to advancing research and teaching.

To conclude, I want to repeat what I said in my 2020 column. If you, as a mathematician and an author of a book, share the goal of advancing research and teaching mathematics, then publishing your book with the AMS is a step toward achieving these goals. Many books published by the AMS have a strong positive impact on the profession and on its appreciation by the society. We welcome any author who wants to contribute to our collective vision of mathematics and its role in science, technology, and beyond.
In Search of the Viscosity Operator on Riemannian Manifolds

Magdalena Czubak

Introduction

What should equations modeling fluid flow on Riemannian manifolds look like? As we will see, there are several candidates, and there is no generally agreed upon consensus. We explore this question from the point of view of geometry, analysis, probability, and physics.

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Communicated by Notices Associate Editor Daniela De Silva.

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DOI: https://doi.org/10.1090/noti2840

There are two well-studied fundamental equations of fluid flow. They are the Euler equations and the Navier–Stokes equations. The main difference between the Euler equations and the Navier–Stokes equations is that the Navier–Stokes equations incorporate what is called viscosity. We explain this difference more below. What is important to note now is that there is no ambiguity as to what the Euler equations should be on a general Riemannian manifold. What is less clear is what the Navier–Stokes equations should be. This lack of ambiguity for the Euler equations will become clear as well.
To motivate this, to see that it might not be completely obvious what the Navier–Stokes equations on a manifold are, we start with the Navier–Stokes equations in the Euclidean setting, and see if we can write the equations on a general manifold.

We begin with an incompressible Navier–Stokes equation on \( \mathbb{R}^n \). It is given by

\[
\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0,
\]
\[
\text{div } u = 0. \tag{1}
\]

We explain now the notation and the unknowns. The unknowns are \( u \) and \( p \), where

\[
u : \mathbb{R}^{n+1} \to \mathbb{R}^n,
\]
\[
p : \mathbb{R}^{n+1} \to \mathbb{R}.
\]

So \( u \) is a time-dependent vector field, and \( p \) is a scalar-valued function. Physically, \( u(x,t) = (u^i(x,t), \ldots, u^n(x,t)) \) is the velocity of the fluid going through position \( x \) at time \( t \), and \( p \) is the pressure. The notation in (1) means the following:

- \( \partial_i = \frac{\partial}{\partial x^i} \).
- \( \Delta \) is the Laplacian:

\[
\Delta = \sum_{i=1}^n \partial_i^2, \tag{2}
\]

where \( \partial_i = \partial_{x^i} = \frac{\partial}{\partial x^i} \).

- \( \nabla p \) is the gradient of \( p \):

\[
\nabla p = (\partial_1 p, \ldots, \partial_n p).
\]

Note that since \( \nabla p \) is the vector, and it appears in the equation, this means that all the other terms are also vector valued, so \( \partial_i u \) is interpreted as taking the time derivative of each component \( u^i \), and similarly, \( \Delta u \) means we apply the Laplacian to each \( u^i \).

- \( u \cdot \nabla u \) is written here how it is often denoted, but it is more precise to write it as a dot product of \( u \) with the gradient of each component \( u^i \), \( (u \cdot \nabla u)^i = u^i \cdot \nabla u^i \), or we can think of this term as the directional derivative of \( u \) in the direction of \( u \), so \( u \cdot \nabla u = \nabla_u u \).

- Finally, \( \text{div } u \) is the divergence of \( u \), \( \text{div } u = \partial_i u^i \), where we sum over the repeated indices.

The condition that \( \text{div } u = 0 \) means that the fluid is incompressible. This is an equivalent condition of saying that it is volume preserving: if we take the fluid in any region, measure its volume, and follow it, then at a later time it will have the same volume. We note that incompressibility is an approximation: there is no fluid that is truly incompressible, even water, but it is a good approximation (see, e.g., [Bat99]).

Since the equation is time dependent, we also prescribe initial conditions, and give the fluid the initial velocity:

\[
u(0,x) = u_0(x).
\]

Finally, we remark, that while we have two unknowns \( (u,p) \), the first equation in (1) is an evolution equation only for the velocity vector field \( u \). As a result, one usually solves for \( u \) first, and then later the pressure \( p \) is recovered from \( u \). For example, we can take the divergence of the first equation in (1) and arrive at an elliptic equation for \( p \).

So now, given (1), the most straightforward way to begin to generalize the system to the Riemannian manifolds is to look at each term and find its analog on a manifold.

All the terms, except for one, have a clear analog. Which one does not? First, there is a natural concept of divergence, of gradient, of directional derivative, and since the manifold is fixed in time (we are not looking at relativistic fluids, but see below), we can also just take the time derivative. The only term that is not obvious is the Laplacian.

**Different Laplacians**

When we hear the words manifold and Laplacian, there is a good chance that the first operator that comes to mind is the Laplace–Beltrami operator, the divergence of the gradient,

\[
\text{div } \nabla,
\]

which we can write in coordinates as

\[
\frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j), \tag{3}
\]

where \( \det g \) denotes the determinant of the metric \( g = (g_{ij}) \) written in local coordinates \( \{x^i\} \), \( (g^{ij}) \) is the metric’s inverse, and where we sum over the repeated indices.

For example, if the manifold is the Euclidean space, \( \mathbb{R}^n \), then in the Cartesian coordinates, the metric is \( g_{ij} = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker’s delta. Then, \( \det g = 1, g^{ij} = \delta^{ij} \), and the operator reduces to

\[
\partial_i (\delta^{ij} \partial_j) = \delta^{ij} \partial_i \partial_j = \sum_{i=1}^n \partial_i^2,
\]

which is the Laplacian in (2).

However, the operator in (3) acts on scalar-valued functions, but a solution of the Navier–Stokes equation is a vector field, so this operator cannot be used here. We elaborate this point. It is true that when we discussed the Laplacian in (1) we said we apply (2) to each coordinate. In general, in geometry we look for operators that are independent of the choice of coordinates. We could show that even in \( \mathbb{R}^2 \), if we wrote the vector field in polar coordinates and applied the scalar Laplacian to each coordinate, we would not arrive at an equivalent expression to the one in Cartesian coordinates. So in the introduction, that was the magic of the Cartesian coordinates: they are the most straightforward way to introduce the Navier–Stokes equations on \( \mathbb{R}^n \), and allow us to write the Laplacian of the vector field by applying (2) coordinate-wise. In general, we need a Laplacian operator that acts on vector fields and has an expression that is independent of coordinates.
There are actually several candidates for the choice of the Laplacian that acts on vector fields.

To begin with, if we like to think about the Laplacian as the divergence of the gradient, then the natural generalization of the Laplace–Beltrami operator to vector fields is the Bochner Laplacian (sometimes also still called the Laplace–Beltrami operator)

\[ \operatorname{div} V, \]

where now \( V \) denotes the Levi-Civita connection on \((M,g)\). Recall, the Levi-Civita connection is an operator that allows us to differentiate vector fields on a manifold. It is also a unique operator that satisfies certain properties (compatibility with the metric and the torsion free property).

In local coordinates, we can write

\[ \operatorname{div} V = \nabla^i V_i = g^{ij} \nabla_j V_i, \]

which we observe can also be thought of as taking the trace of \( V^2 \), which is another name for the Bochner Laplacian: the trace Laplacian, or the rough Laplacian (there might be sign conventions involved, too).

To see why this is a natural generalization of (3), we remark that the Levi-Civita connection induces an operator that acts on general tensors, and in particular, on 0-rank tensors, which are just functions. In that case, we could show that the induced operator is a differential of a function, \( d \), and then a computation would show that \( \operatorname{div} d \) is exactly (3).

Next, when \( v \) is a vector field on \( \mathbb{R}^3 \), the Laplacian of \( v \) can be expressed as

\[ (\nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}) v. \quad (4) \]

Now, when \( v \) is a vector field on a Riemannian manifold, then using the metric, we can lower the index, and obtain a unique 1-form \( \alpha \) associated to \( v \). In local coordinates, if \( v = v^i \partial_i \), then \( \alpha = g^{ij} v_j dx_i \).

More precisely, we can use the so-called musical isomorphisms to move between vector fields and 1-forms (raise/lower indices). If \( v \) is a vector field, then \( v^\flat \) is a 1-form, and if \( \alpha \) is a 1-form, \( \alpha^\sharp \) is a vector field. We have the following definitions, which rely on the duality of 1-forms and vector fields: 1-forms act on vector fields: 1-forms act on vector fields.

\[ v^\flat = g(\nu, \cdot), \quad \alpha(\cdot) = g(\alpha^\sharp, \cdot). \]

Then for a 1-form \( \alpha \), \( \alpha = v^\flat \), the analog of (4) is

\[-(d d^* + d^* d) \alpha, \]

which is the Hodge Laplacian acting on \( \alpha \). Here, \( d^* \) is the formal adjoint of the exterior differential operator \( d \) with respect to the inner product given by the metric. The Hodge Laplacian acts on differential forms, and it can also act on functions, and interestingly enough, in that case, just like the Bochner Laplacian, it reduces to (3).

We introduced the Bochner Laplacian as acting on vector fields, but using the musical isomorphisms, we could also have it act on 1-forms, or we could use the musical isomorphisms to define the Hodge Laplacian as acting on vector fields. Either way, we can relate the Bochner Laplacian and the Hodge Laplacian by what is called the Bochner–Weitzenböck formula. The formula is a direct computation using Ricci identities, and it reads

\[-\operatorname{div} V = d d^* + d^* d - \operatorname{Ric}, \quad (5)\]

where \( \operatorname{Ric} \) is the Ricci operator obtained from the Ricci tensor by raising an index, and more precisely, if \( \alpha \) is a 1-form

\[ \operatorname{Ric} \alpha = \operatorname{Ric}(\alpha^\sharp, \cdot). \]

It follows that, in the particular case of \( \mathbb{R}^n \), where \( \operatorname{Ric} \equiv 0 \), we can see that the Bochner Laplacian and the Hodge Laplacian agree. On a manifold that has a constant sectional curvature, \( \lambda \),

\[ \operatorname{Ric} \alpha = (n - 1) \lambda \alpha. \]

In the case of the sphere, \( S^n \),

\[ \operatorname{Ric} \alpha = (n - 1) \alpha, \]

and on a hyperbolic space, \( \mathbb{H}^n \),

\[ \operatorname{Ric} \alpha = -(n - 1) \alpha. \]

There is another operator that can be considered on a Riemannian manifold. The first in-depth study of the Navier–Stokes equations on the Riemannian manifolds was presented by Ebin and Marsden in their seminal 1970 article [EM70]. In their article, Ebin and Marsden indicated that when considering the Navier–Stokes equations on an Einstein manifold, one should use the following operator

\[ 2 \operatorname{div} \operatorname{Def}, \]

where \( \operatorname{Def} \) is the deformation tensor. The deformation tensor can be thought of as a symmetrization of the covariant derivative, and in coordinates we can write it as

\[ (\operatorname{Def} v)_j = \frac{1}{2} (\nabla \nu_j + \nabla_j \nu). \quad (6) \]

A direct computation using (5), and a Ricci identity gives

\[ -2 \operatorname{div} \operatorname{Def} = -\operatorname{div} V + d d^* - \operatorname{Ric} \]

\[ = d d^* + d^* d - \operatorname{Ric} + 2 \operatorname{Ric}. \]

For divergence-free vector fields, we can cancel out \( d d^* \) terms and just write

\[ -2 \operatorname{div} \operatorname{Def} = -\operatorname{div} V - \operatorname{Ric} = d^* d - 2 \operatorname{Ric}. \quad (7) \]

This formula connects all the three operators, and again we can see that they are all the same in case of the Euclidean space.

The article [EM70] states that the “correct” viscosity operator on a manifold should be the operator given by (7), but the article itself uses the Hodge Laplacian, and the
deformation tensor is mentioned only at the end, in the “Note Added in Proof.”

Since 1970 we see mostly the use of either the Hodge Laplacian or the Laplacian coming from the deformation tensor, operator \((7)\). This operator has been called the Ebin–Marsden Laplacian, or just the deformation Laplacian.

Authors working with the Hodge Laplacian are usually working on a compact manifold, including Ebin and Marsden [EM70], so a heuristic notion might be that the Ricci term might not make a difference. However, this is not true, even in the case of compact manifolds as we will discuss when we mention the work in [ST20].

In the rest of this article, we will talk about different perspectives and arguments that we can use to help us decide which operator to work with.

**Continuum Mechanics: Why the Deformation Tensor**

As mathematicians, we are used to seeing the equation as written in (1). That way, we first write the linear term, the heat operator, then the nonlinearity and the pressure. Engineers, on the other hand, often write the equation as

\[
\partial_t u + u \cdot \nabla u = -\nabla p + \mu \Delta u, \quad (8)
\]

or more generally as

\[
\partial_t v + v \cdot \nabla v = f + \text{div} \, T. \quad (9)
\]

The equation \((9)\) emphasizes the parts written in Newton’s 2nd Law (sometimes referred to as conservation of momentum or balance of momentum)

\[
F = ma,
\]

i.e., force equals mass times acceleration. The left-hand side in \((9)\) comes from \(ma\), and the right-hand side denotes all the forces acting on the fluid. These consist of volume (body) forces and surface forces (or long-range and short-range forces). Volume forces act on all elements of the volume of a continuum. An example of a volume force is gravity. Volume forces are usually denoted by a vector valued function \(f\) as in equation \((9)\). We note that in equation \((8)\) we do not have the function \(f\) for simplicity.

The surface forces are what produces the deformation tensor, which in turn gives the Laplacian. First, the surface force acts on a surface element to which we assume we can assign a unit normal \(n\). Then, the surface force, as a force is also a vector, which means it can be written in components. The \(i\)th component can be shown to be \(T_{ij}n_j\), where \(T\) is the stress tensor (see for example [Bat99]). So if we consider a part of a fluid with volume \(V\) and enclosed by a surface \(S\), the total surface force acting on \(S\) is given by

\[
\int_S T_{ij}n_j dS = \int_V \partial_j T_{ij} dV,
\]

where we used the divergence theorem. This is how we get \((9)\) for fluids with constant density.

The question remains: what constitutes the stress tensor \(T\)? For fluids at rest or also for perfect fluids, only normal stresses are exerted, and we have \(T_{ij} = -p\delta_{ij}\), where \(p\) is the pressure, and gives \(\nabla p\) in the equation. For fluids in motion or nonperfect fluids, we also have tangential stresses, which for isotropic fluids give an additional term in \(T_{ij}\), which can be written as [Bat99, p.144]

\[
2\mu D_{ij} - \frac{2}{3}\mu \text{div} v \delta_{ij}, \quad (10)
\]

where \(D_{ij} = (\text{Def} v)_{ij}\) is the deformation tensor, \(\frac{1}{2}(\partial_i v_j + \partial_j v_i)\), and \(\mu\) is the viscosity coefficient. Then, for an incompressible fluid, we are left with

\[
\mu(\partial_i v_i + \partial_j v_j),
\]

and after we take divergence as in \((9)\) we get exactly

\[
\mu \Delta v.
\]

Viscosity is an internal friction of the fluid. In the case of the Euler equations, the fluid is assumed to be inviscid, to have zero viscosity. (Inviscid property, while an approximation, is also another good approximation in certain situations.) For inviscid fluids, there is no Laplacian in the equation, which also means there is no issue what the equation should be on the manifold. Another reason why it is clear what the Euler equations should be on general Riemannian manifolds is due to the existence of the variational principle, which we will briefly explain a little later.

Ebin and Marsden [EM70], when they introduce the deformation Laplacian, refer to Serrin [Ser59]. There, following Stokes, it is assumed that the stress-tensor \(T\) satisfies [Ser59, Section 59]:

1. \(T\) is a continuous function of the deformation tensor \(\text{Def}\), and is independent of all other kinematic quantities.
2. \(T\) does not depend explicitly on the spatial position (spatial homogeneity).
3. There is no preferred direction in space (isotropy).
4. When \(\text{Def} = 0\), \(T\) reduces to \(-pI\), where \(p\) is the pressure and \(I\) the identity matrix.

In case of a general manifold, it is not clear how one would assure properties 2 and 3. In fact, when Serrin discusses the Navier–Stokes equations in curvilinear coordinates (Section 13), Serrin remarks that, on a general manifold, it is not “evident how to formulate the principle of conservation of momentum.” However, right after, he says that “there seems to be no valid objection to taking Eq. (12.3) as a postulate.”

Equation (12.3) involves the divergence of the stress tensor, and it gives the deformation Laplacian if \(T\) satisfies 1 and 4 above and the remaining assumptions used by Serrin.
This connection with continuum mechanics was also pointed out by Taylor in [Tay92]. The connection does seem very natural indeed.

Arguments leading to the appearance of the deformation tensor are based on symmetry, and this is another reason why the deformation Laplacian might be the natural choice, as it involves the symmetric part of the covariant derivative. The Hodge Laplacian, for divergence-free vector fields, reduces to $d^*d$, and $d$ corresponds to the antisymmetric part, and the Bochner Laplacian includes both the symmetric and the antisymmetric pieces.

While these three operators are types of Laplacians on a manifold, the above discussion illustrates that the stress-tensor is what encodes the effects of viscosity, and not the Laplacian. As a result, we should be really searching for the appropriate viscosity operator and not just the appropriate Laplacian.

**Analysis: Argument Against the Hodge Laplacian on $\mathbb{H}^2$**

With Chan and Disconzi [CCD17] we gave a mathematical argument coming from analysis of PDE that was against using the Hodge Laplacian on the hyperbolic plane, $\mathbb{H}^2$.

We showed that if we are working with the Hodge Laplacian, for a forced nonstationary equation on $\mathbb{H}^2$, one cannot establish an energy inequality as part of what is called a weak formulation of the equation. An energy inequality is an important tool in the existence theory of the equations, so the lack of such tool suggests that the Hodge Laplacian might not be the right operator for the Navier–Stokes equation on the hyperbolic plane.

We remark here that since the hyperbolic plane is non-compact, this result does not contradict what was done in [EM70] for compact manifolds with the Hodge Laplacian. Also, the weak formulation assumes data is in $L^2$ and seeks to find weak solutions that exist for all time. In [EM70], the initial data belongs to $H^2$ for $s > \frac{n}{2} + 5$, and the solution is shown to exist for a short time.

Finally, the counterexample in [CCD17] would not apply to the use of the Hodge Laplacian on the sphere $S^2$. This has to do with the equivalence of the $L^2$ norm of the covariant derivative and the exterior derivative $d$, $\|\nabla u\|_{L^2(S^2)}$ and $\|du\|_{L^2(S^2)}$, respectively. One could try to modify the norm in the case of $\mathbb{H}^2$, but there is no obvious modification (either it is not a norm or the same counterexample would apply).

**Physics: Nonrelativistic Limit**

Another way we can investigate the form of the Navier–Stokes equations on Riemannian manifolds is to consider the nonrelativistic limit of the relativistic Navier–Stokes equations. Nevertheless, in the case of relativistic fluids the “correct” form of the equations is not known either.

There are several theories in the literature, and at the time of writing [CCD17], we showed that that they all could lead to the same equations where the Laplacian is the deformation Laplacian.

The main idea of the nonrelativistic limit is to start with the relativistic equation, and take the limit by assuming that the fluid velocities are very small compared to the speed of light. This means that we neglect terms of the order $\frac{|u|}{c}$, where $v$ is the velocity of fluid particles and $c$ the speed of light.

In this case, however, differently than the usual nonrelativistic limit in the general theory of relativity, we considered the metric on $\{t = \text{constant}\}$ hypersurfaces converging to an arbitrary Riemannian metric. Usually, it is considered that the metric converges to the Minkowski metric, so the metric induced on $\{t = \text{constant}\}$ hypersurfaces is the Euclidean one. The manipulations we performed were formal, in a sense, we did not discuss the topology with respect to which the convergence was supposed to occur.

The fact that we arrived at the deformation Laplacian, in a certain sense, might not be a surprise. The relativistic equations are obtained as

$$\nabla^\alpha T_{\alpha\beta} = 0,$$

(11) $\alpha, \beta = 0, \ldots, 3$, where $T_{\alpha\beta}$ is a symmetric two-tensor. If $T_{\alpha\beta}$ depends on first order derivatives of the velocity, it will in general contain the term $\nabla_\alpha u_\beta + \nabla_\beta u_\alpha$, where $u$ is the fluid’s four-velocity (which is the appropriate relativistic notion of velocity). This is indeed the case in all the relativistic theories we considered. Then, by taking the non-relativistic limit, it can be showed that the term $\nabla^\alpha u_\beta + \nabla^\beta u_\alpha$ produces $\nabla(A_\mu v_\lambda + \nabla v_\lambda)$, where $v$ is the (classical) fluid’s velocity (i, $j = 1, 2, 3$). This term gives the deformation Laplacian.

Since [CCD17], Disconzi has been involved in the development of new relativistic theories, which produced what is referred to as the BDNK model, named after Bemfica, Disconzi, Noronha, and Kovtun (see, e.g., [BDN22, Kov19] and references therein). Very recently, Hegade K R, Ripley, and Yunes studied nonrelativistic limit of the BDNK model [HKRRY23], and showed that the limiting equations (on the Minkowski space) depend on the choice of the parameters in the BDNK energy-momentum tensor. This would presumably carry over to Riemannian manifolds, and seems to imply that the nonrelativistic limit might not have the definitive predictive capability or at least, that this approach is more complicated, and more information about the problem the limiting equation is supposed to model might be needed.
Geometry: Gauss Formulas

Another way to look at this problem is to recall the following question: How can directional derivatives be defined on a submanifold embedded in a Euclidean space? One answer is: by taking the extrinsic directional derivative, and then projecting back to the submanifold. What is amazing is that this projected quantity coincides exactly with the intrinsic directional derivative. Moreover, the relationship between the intrinsic and the extrinsic quantities can be observed through the Gauss formula. We make this precise now.

Let \( M \) be an embedded submanifold in a Riemannian manifold \((\bar{M}, \bar{g})\), with the inclusion map, \( i : M \hookrightarrow \bar{M} \). Then the embedding induces a metric \( g \) on \( M \), by \( g = i^* \bar{g} \). For example, in the case of the sphere \( S^2 \) embedded in \( \mathbb{R}^3 \), the induced metric on \( S^2 \) is just the Euclidean metric restricted to the vectors tangent to \( S^2 \), and this is the standard, the so-called round metric on the sphere.

Now, if \( X, Y \) are vector fields on \( M \), and \( p \in M \), we can consider local extensions of \( X, Y \) to a neighborhood of \( p \) in \( \bar{M} \), and still denote them by \( X, Y \). This is done so we can apply the extrinsic Levi-Civita connection on \( \bar{M} \), \( \bar{\nabla} \), to \( X, Y \) and evaluate it at \( p \in M \), i.e., we consider

\[
\bar{\nabla}_X Y|_p, \quad p \in M.
\]

By the definition of a connection, the result is a vector in \( T_p \bar{M} \). We can then decompose it into two parts: a part tangential to \( T_p M \), denoted by \((\bar{\nabla}_X Y)^T \), and a part normal to \( T_p M \), denoted by \((\bar{\nabla}_X Y)^N \). The normal part defines what is called the second fundamental form \( II : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM) \), where \( \mathfrak{X}(M) \) denotes the smooth vector fields on \( M \), and \( \Gamma(NM) \) denotes the smooth sections of the normal bundle of \( M \),

\[
II(X, Y) = (\bar{\nabla}_X Y)^N.
\]

The second fundamental form has the nice properties that it is symmetric in \( X, Y \), bilinear over smooth functions on \( M \), and perhaps the most interesting property is that it is independent of the extensions of \( X, Y \). If \( \nabla \) denotes the Levi-Civita connection on \( M \) corresponding to the induced metric \( g \), then the Gauss formula tells us that the tangential part is \( \nabla_X Y \), so we have

\[
\bar{\nabla}_X Y = \nabla_X Y + II(X, Y), \quad (12)
\]

when evaluated at \( p \in M \).

There is a similar relationship between the intrinsic and extrinsic curvatures. We have

\[
\bar{R}(W, X, Y, Z) = R(W, X, Y, Z)
- \bar{g}(II(W, Z), II(X, Y))
+ \bar{g}(II(W, Y), II(X, Z)), \quad (13)
\]

where \( \bar{R}, R \) are the Riemannian curvature tensors on \( \bar{M} \) and \( M \), respectively, and \( \bar{g} \) is the Riemannian metric on \( \bar{M} \).

To go back to the Navier–Stokes equation, when we discussed in the beginning taking each term in the equation and asking how it would look on a manifold, we can make the following observation. By Nash’s embedding, each Riemannian manifold can be embedded in \( \mathbb{R}^m \) for some \( m \). Then we could take the equation on \( \mathbb{R}^m \) and project it onto the tangent space of the manifold in question. The projection is linear so we could apply it to each term individually. This process coincides with what we expected to get before for the partial in time of the velocity, the gradient of the pressure, and the nonlinear term: for the partial in time and the gradient, this is just a projection, and for the nonlinear term, we can apply the Gauss formula (12). What is left is again the question of the Laplacian. All the Laplacians related in (7) agree on the Euclidean space, so question can be reduced to: is there a Gauss formula for the Laplacian?

We looked into this initially with Chan and Disconzi for the sphere \( S^2 \) in [CCD17] with considering only the tangential part, and not trying to derive a general Gauss formula that includes both tangential and normal pieces. Still, we were excited to arrive at the tangential part being the deformation Laplacian. Later with Chan and Yoneda we wanted to extend this to the case of the ellipsoid. At first, we were surprised that the computations were a lot more involved, and later we realized that the general problem is more involved. In fact, the formula that we eventually derived in [CCY23] showed that the operator obtained by this procedure does depend on the extension of the vector field considered.

For example, in [CCD17] we worked with a vector field in spherical coordinates, and considered the simplest extension/restriction by not having the coordinate functions depend on the radial variable. That extension can be viewed as an extension so its norm grows depending on the distance away from the sphere. If we work with a vector field that has a norm preserved, then one could obtain the Hodge Laplacian instead! For the case of the sphere, this was actually already pointed out in [Ors74, Yam18].

In [CCY23], we extended the prior work to an ellipsoid, and we found a general formula that explicitly depends on the ambient/extrinsic vector field. It also generalizes the work of [CCD17, Ors74, Yam18] as it can be simplified to a formula that covers the case of the sphere.

In [CC22] with Chan, we extended the formula for the projected Laplacian to a general hypersurface embedded in an ambient manifold \( \bar{M} \), and also obtained the analog of the Gauss formula. For simplicity, we present the formula for a Euclidean hypersurface.

**Theorem 1.** [CC22] Let \( n \geq 2 \); let \( M \) denote an embedded hypersurface in \( \mathbb{R}^{n+1} \), \( M \hookrightarrow \mathbb{R}^{n+1}, \dim M = n \); let \( v \) be a vector field on \( M \), \( p \in M \); and let \( v \) be an extension of \( v \) to a
neighborhood of \( p \) in \( \mathbb{R}^{n+1} \), still denoted by \( v \). Then at \( p \)
\[
-\Delta v = -\text{div} \, \nu v - \text{Ric} \, v + nH[N, v]
- \nabla_{\nu} \nabla_{\nu} v + \nabla_{\phi} \nabla_{\phi} v
- \sum_{i} (2x^{i}v_{i}^{j} + v(x^{i})) N,
\]
where
- \( \Delta v \) is the Euclidean Laplacian,
- \( -\text{div} \, \nu v \) is the Bochner Laplacian on \( M \),
- \( \text{Ric} \, v = (\text{Ric}(v, \cdot))^{\delta} \) is the Ricci tensor on \( M \),
- \( H \) is the mean curvature of \( M \),
- \( N \) is the choice of the unit normal on \( M \),
- \( [\cdot, \cdot] \) is the Lie bracket,
- \( \tilde{\nabla} \) is the Euclidean connection,
- \( \kappa^{j} \) are the principal curvatures of \( M \leftrightarrow \mathbb{R}^{n+1} \), and
- \( \kappa^{j} \nabla v_{i}^{j} \) is the divergence of \( v \) weighted by the principal curvatures.

Hence, if we are only interested in the projected piece, we arrive at
\[
(-\Delta v)^{T} = -\text{div} \, \nu v - \text{Ric} \, v + nH[N, v]^{T}
- (\tilde{\nabla}_{\nu} \tilde{\nabla}_{\nu} v^{T} + (\tilde{\nabla}_{\phi} \tilde{\nabla}_{\phi} v)^{T}).
\]
We discuss this formula for the case of the sphere.

We can let \( N = \frac{\partial}{\partial \rho} \), where \( \rho \) is the radial variable in the spherical coordinates, and work with an orthonormal frame obtained from the spherical coordinates
\[
E_{1} = \frac{\partial}{\partial \phi}, \quad E_{2} = \frac{\partial}{\rho \sin \phi}, \quad E_{3} = N = \frac{\partial}{\partial \rho}.
\]

We then let \( v = v^{\alpha} E_{\alpha} \), where we sum \( \alpha \) from 1 to 3, and compute
\[
\nabla_{\nu} N = \nabla_{\nu} \frac{\partial}{\partial \rho} = 0 \quad \text{and} \quad \nabla_{\nu} v = \frac{\partial}{\rho \sin \phi} E_{\alpha}.
\]

So
\[
\tilde{\nabla}_{\nu} \tilde{\nabla}_{\nu} v_{\alpha} = \tilde{\nabla}_{\nu} (\frac{\partial}{\rho \sin \phi} E_{\alpha}) = \frac{\partial}{\rho \sin \phi} E_{\alpha}.
\]

Also
\[
\text{Ric} \, v = v, \quad nH = -2, \quad \text{and} \quad [N, v] = \nabla_{N} v - \nabla_{v} N = \frac{\partial}{\rho \sin \phi} E_{\alpha} - v,
\]
where we use in the last line the shape operator \( s\nu := -\nabla_{N} N \), which is simply \( s\nu = -v \) in the case of the sphere.

Projecting and using formula (15), we have
\[
(\nabla^{*} \nabla v)^{T} = \nabla^{*} \nabla v + v - 2\partial_{\rho} v^{i} E_{i} - \partial_{\rho}^{2} v^{i} E_{i},
\]
where now the Roman indices \( i \) sum from 1 to 2, and \( E_{i} \) is an ON frame on the sphere.

If we wish to arrive at the deformation Laplacian, by (7) we would need
\[
v = v^{i} E_{i} - 2\partial_{\rho} v^{i} E_{i} - \partial_{\rho}^{2} v^{i} E_{i} = -v.
\]

These are the examples we mentioned above. We take an in-depth look at these in an upcoming article with Chan and Fuster.

A reader might be wondering here: but we are looking for solutions of the Navier–Stokes equation, how do we know that these above vector fields solve the Euclidean Navier–Stokes equations? This is a good point, and we look at relating the solutions of the Euclidean Navier–Stokes equations to the ones on an embedded submanifold in the next section.

Analysis and Physics: Thin Shell Limit
Consider the unit sphere, \( S^{2} \), embedded in \( \mathbb{R}^{3} \), and a thin shell around it of \( 2\epsilon \) thickness, where \( \epsilon \) is small. This is a 3D domain with boundary. The boundary has two components, the sphere of radius \( 1 - \epsilon \) and the sphere with radius \( 1 + \epsilon \). We could study the Euclidean Navier–Stokes equations in this domain, or a domain where one boundary is the sphere itself and the other, for example, the sphere of radius \( 1 + \epsilon \), and then take the limit of the solutions when \( \epsilon \rightarrow 0^{+} \).

Such procedures have been performed in elasticity, and also in the case of fluids. For example, in [TZ97], Temam and Ziane take the average of the Euclidean Navier–Stokes solutions and show they converge to the solution of the Navier–Stokes on \( S^{2} \), where the Laplacian is the Hodge Laplacian. In [Miu20], Miura considers a more general domain for a more general surface and takes the limit, and arrives at the Navier–Stokes equation on the surface where the Laplacian is the deformation Laplacian. How can this be?

The answer lies in the choice of the boundary conditions. The thin shell has a boundary, and to solve the Navier–Stokes equation, one needs to impose a boundary condition. Temam and Ziane work with the boundary condition, where it is assumed that \( u \) is tangential on the boundary, and where the cross product of \( \text{curl} \, u \) with the normal is zero. This is sometimes called the Hodge condition. Miura, on the other hand, uses the Navier’s slip condition, which in its simplest form can be written as
\[
((\text{Def} \, u_{i}) n)^{T} = 0,
\]
where we think of the deformation tensor now as the matrix, and we apply it to the normal vector, and then project it onto the boundary.

In an upcoming article with Chan and Fuster, we show that the resulting equation depends not just on the boundary conditions, but also on how we take the average.

Analysis and Geometry: Asymptotic Behavior of the Solutions
Interesting observations have been made by Samavaki and Tuomela [ST20], where the authors consider the equations on compact manifolds. There it is shown that the
difference in operators does make a difference even on compact manifolds, and even in a simple case of the sphere, $S^2$.

The authors investigate qualitative behavior of the solutions for the time-dependent (nonstationary) problem. In particular, the importance of the Killing fields is highlighted. It is shown that for the solutions with the deformation Laplacian, the solution will converge to a Killing field as $t \to \infty$. On the other hand, if the Hodge Laplacian is used, the solutions will tend to a harmonic form. Therefore, as it is pointed out, in the case of the sphere, where there are no nontrivial harmonic forms, the solutions for the equation with the Hodge Laplacian will converge to zero, whereas the solutions with the deformation Laplacian will converge to a Killing field on the sphere, which corresponds to rotations.

**Probability: Stochastic Action Principle and Lagrangian Paths**

The Navier–Stokes equations on compact manifolds with the Hodge Laplacian and the deformation Laplacian have appeared in probabilistic approaches.

In the work of Arnaudon and Cruzeiro [AC12], the Navier–Stokes equation with the Hodge Laplacian is recovered in the connection with a variational principle for stochastic Lagrangian flows. Then Arnaudon, Cruzeiro, and Fang [ACF18] connected the variational principle with the Navier–Stokes equation with the deformation Laplacian.

To put this in context, the seminal work of Arnold [Am66] and Ebin and Marsden [EM70] shows how the solutions to the Euler equation can be realized as minimizers that are geodesics on the infinite-dimensional group of the volume preserving diffeomorphisms. Arnold observed that the group of the volume preserving diffeomorphisms is a Lie group that can be endowed with $L^2$ Riemannian metric, which is the kinetic energy. This approach takes on what is called the Lagrangian point of view. This means we are interested in tracking individual particles and their position. Until now, we were interested in tracking the velocity of the fluid at each point in space, which is called the Eulerian point of view.

In the Lagrangian point of view, the positions are being tracked by the particle paths $\gamma$ satisfying

$$\partial_t \gamma(t, x) = u(t, \gamma(t, x)), \quad \gamma(0, x) = x,$$

where $u$ is the velocity field and satisfies the Euler equation.

The action that the particle paths are minimizing is given by

$$A(\gamma) = \frac{1}{2} \int_0^T \int_M g(\partial_t \gamma, \partial_t \gamma) \, dx \, dt.$$

This approach is called often the variational principle or the least action principle. It is not clear how to extend this principle to the Navier–Stokes equations. The work in [AC12] and [ACF18] establishes the principle in the stochastic setting, but what is important to point out is that the word stochastic only refers to the Lagrangian paths: the velocity solves the deterministic Navier–Stokes equations.

A related approach, also using stochastic Lagrangian paths for the Navier–Stokes equations, is based on the results of Constantin and Iyer [CI08], and it was presented for general compact manifolds by Fang [Fan20]. Here, the main idea of this approach can be related to the Feynman–Kac formula, which connects solutions of linear parabolic PDEs with a stochastic process, which solves a corresponding stochastic differential equation. Constantin and Iyer extend this to the the nonlinear system of the Navier–Stokes equations in the Euclidean setting, and [Fan20] establishes the representation formula for the Navier–Stokes equation on general compact manifold with the Hodge Laplacian.

**Summary**

We have discussed different approaches one can consider when looking at the question of what the Navier–Stokes equations should look like on a general Riemannian manifold. As interesting as all the different approaches are, none of them currently seems to lead to a definite answer. For a while, it was this author’s personal bias that the continuum mechanics argument was the most convincing one. At this point, we believe that the exact form of the equations will depend on the physical problem at hand.

The methods we would like to explore further are the methods related to Gauss formulas as well as the original derivations of the Navier–Stokes equations in the Euclidean setting. The Euclidean equation has an interesting history itself, as it was derived at least five times, starting with Navier, Cauchy, Poisson, Saint-Venant, and Stokes. Revisiting these derivations might lead to interesting insights.

We end by saying that whichever model we choose to work on, ideally we would like to be able to justify it with some method. At the same time, as pure mathematicians, we could study any problem we find interesting, whether for its physical or mathematical motivation.

**References**


The main aim of algebraic geometry is the understanding and classification of algebraic varieties: geometric objects that can be defined by polynomial equations. The Minimal Model Program (MMP) was initiated by the Italian school of algebraic geometers in the dawn of the last century. The MMP proposes a solution for the classification problem using birational geometry. Instead of classifying varieties per se, we first perform some surgeries on them: the so-called birational transformations. Although these surgeries change the object of study, they preserve the main characteristics and nature of the variety. Then, we try to find a canonical element in the birational class of our variety, i.e., an element that is better behaved than others in its class. Finally, we aim to study a canonical model on each birational class to develop a classification.

The Projective Space

The $n$-dimensional projective space $\mathbb{P}^n$ is the playground of algebraic geometers. This space parametrizes the set of lines through the origin in $\mathbb{C}^{n+1}$. We use coordinates

\[
[x_0 : \cdots : x_n] \in \mathbb{P}^n
\]
to represent points in the projective space. The coordinate (1) represents a nonzero point of a line in \( \mathbb{C}^{n+1} \) through the origin, so multiplying a nonzero parameter \( \lambda \) in all the entries of (1) do not change the point in \( \mathbb{P}^n \). This means that the relation

\[
[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]
\]

holds for every \( \lambda \in \mathbb{C}^* \). The projective space admits a description as a disjoint union:

\[
\mathbb{P}^n = \mathbb{C}^n \bigcup \mathbb{P}^{n-1},
\]

where the first set \( \mathbb{C}^n \) is called the \( n \)-dimensional affine space and is described by \( x_n \neq 0 \) in the coordinates (1), while the set \( \mathbb{P}^{n-1} \) is called the hyperplane at infinity and is described by \( x_n = 0 \) in the coordinates (1). The points that lie in the hyperplane at infinity are called points at infinity.

Similar to how the rails of a train track seem to intersect at the horizon, two parallel lines in \( \mathbb{C}^2 \) will intersect at infinity when they are considered as lines in \( \mathbb{P}_2^2 \). This is the advantage that the projective space offers to us compared with the affine space. It allows us to describe phenomena that happen at infinity and so this phenomena cannot be described in the affine space.

The \( n \)-dimensional projective space \( \mathbb{P}^n \) can be covered with \( n+1 \) subsets called affine charts:

\[
H_i := \{(x_0 : \cdots : x_n) | x_i \neq 0\}.
\]

Each of these charts is itself an \( n \)-dimensional affine space, i.e., each \( H_i \) is isomorphic to \( \mathbb{C}^n \) via

\[
[x_0 : \cdots : x_n] \mapsto (x_j/x_i)_{j \neq i}.
\]

Thus, all points in the projective space, even the points that lie at infinity, are endowed with local affine coordinates that allows us to study the geometry around such a point.

**Smooth Projective Varieties**

An affine variety \( X \) is the set of points in \( \mathbb{C}^n \) where a finite set of polynomials vanishes. This means that there are polynomials \( p_1(x_1, \ldots, x_n), \ldots, p_k(x_1, \ldots, x_n) \) such that

\[
X = \{(x_1, \ldots, x_n) | p_1(x_1, \ldots, x_n) = 0 \text{ for } 1 \leq i \leq k\}.
\]

In other words, an affine variety is a subset of \( \mathbb{C}^n \) that can be described only using polynomial equations. For instance, the set \( \{(x, y) | x^2 + y^2 = 0\} \) is an affine variety in \( \mathbb{C}^2 \) while \( \{(x, y) | x = \sin y\} \) is not an affine variety. The variety \( X \) is also called the vanishing locus of the set of polynomials \( p_1, \ldots, p_k \).

Similar to the case of affine spaces and projective spaces, it is natural to consider projective versions of affine varieties, i.e., varieties on which we can study points at infinity. To do so, we consider the vanishing locus in \( \mathbb{P}^n \) of polynomials \( p_1 \) in the variables \( x_0, \ldots, x_n \). Evaluating polynomials on the projective space requires a more careful analysis. If a polynomial \( p \) is zero at the point \( (x_0, \ldots, x_n) \), then we must require that it is also zero when evaluating at every point \( (\lambda x_0, \ldots, \lambda x_n) \) for every \( \lambda \) nonzero. Indeed, the points \([x_0 : \cdots : x_n]\) and \([\lambda x_0 : \cdots : \lambda x_n]\) are the same in \( \mathbb{P}^n \). Polynomials satisfying this property are known as homogeneous polynomials. A projective variety is a subset of \( \mathbb{P}^n \) that can be described using homogeneous polynomials in the variables \( x_0, \ldots, x_n \). For instance, the Fermat curves:

\[
C_k := \{(x_0 : x_1 : x_2) | x_0^k + x_1^k + x_2^k = 0\}
\]

are projective varieties in the projective plane \( \mathbb{P}^2 \).

A projective variety \( X \subset \mathbb{P}^n \) is said to be smooth if around every point \( x \in X \) the variety \( X \) can be approximated with a linear subspace of \( \mathbb{P}^n \) of dimension \( d \). The number \( d \) is known as the dimension of \( X \). Roughly speaking, the dimension \( d \) of a smooth projective variety \( X \) tells us in how many linearly independent directions we can move from a given point \( x \in X \). In the case of a variety \( X \) defined by a single polynomial equation \( p \), as in the case of the Fermat curves, this geometric condition is equivalent to asking that the partial derivatives of \( p \) do not vanish simultaneously in \( X \).

The main aim of algebraic geometry is the understanding and classification of smooth projective varieties. As we explain in the following section, the smoothness condition allows us to define certain functions on \( X \) that are convenient to encode the geometry of \( X \).

**Line Bundles**

When studying a geometric object \( X \) we can study it extrinsically by understanding what kind of functions can be defined on \( X \). Among these functions, line bundles play a special role. A line bundle on a smooth projective variety \( X \) associates to each point \( x \in X \) a line:

\[
\mathcal{L}_x \simeq \mathbb{C}
\]

that vary holomorphically with the given point \( x \in X \). The union of all these lines \( \mathcal{L}_x \) can be put together to form an algebraic variety \( \mathcal{L} \) that is called the total space of the line bundle. The total space of the line bundle is endowed with a projection function \( p_{\mathcal{L}}: \mathcal{L} \to X \) that sends the line \( \mathcal{L}_x \) to the point \( x \). We often identify the line bundle with its total space \( \mathcal{L} \). Every projective variety \( X \) comes with its trivial line bundle \( \mathcal{O}_X \) in which there is no variation of the chosen line.

A global section of a line bundle \( \mathcal{L} \) is a holomorphic function \( s: X \to \mathcal{L} \) for which \( p_{\mathcal{L}} \circ s = \text{id}_X \). For instance, Liouville’s theorem implies that a global section of \( \mathcal{O}_X \) on a projective variety \( X \) must be a constant function \( X \to \mathbb{C} \). However, other line bundles may carry very interesting space of global sections. The space of global sections of a line bundle is denoted by \( \Gamma(\mathcal{L}) \). A set of global sections \( s_0, \ldots, s_n \in \Gamma(\mathcal{L}) \) determine a polynomial function

\[
\phi: X \to \mathbb{P}^n,
\]
given by
\[ x \mapsto [s_0(x) : \cdots : s_n(x)]. \]

The target of this polynomial function is a projective space rather than an affine space to avoid the ambiguity of choosing different isomorphisms in (3). We say that a line bundle \( L \) is very ample if we can find sections \( s_0, \ldots, s_n \in \Gamma(L) \) such that the associated polynomial function \( \phi \) is an embedding. Very ample line bundles are the most important line bundles on a variety as its sections can be used to reconstruct the variety.

The usual operations for vector spaces: dual, tensor, and wedge, can be generalized to the context of line bundles on a variety as its sections can be used to reconstruct the variety. Indeed, line bundles can be put together to construct the tangent plane and wedge, can be generalized to the context of line bundles on a variety as its sections can be used to reconstruct the variety.

The name canonically polarized, Calabi–Yau, or Fano depends on the positivity of the canonical line bundle \( \omega_X \). The cotangent bundle \( \Omega_X \) is defined to be the dual bundle of the tangent bundle, \( \Omega_X := T_X^\vee \). The canonical line bundle of a smooth projective variety is \( \omega_X \), the wedge of the cotangent bundle of the variety, \( \omega_X := \Lambda^n \Omega_X \). It only depends on the isomorphism class of \( X \) and it does not depend on the embedding of \( X \) into an ambient projective space. The name canonical line bundle is assigned as this construction does not depend on any choice, it is intrinsically defined from \( X \).

Every smooth projective variety \( X \) carries a canonical line bundle \( \omega_X \). Once we have constructed this line bundle, we may consider its powers \( \omega_X^m \). These line bundles are called the pluricanonical or anti-pluricanonical line bundles depending on the sign of the integer \( m \). If \( m = 0 \), then we recover the trivial line bundle \( \mathcal{O}_X \). One of the most successful approaches to studying the geometry of \( X \) is via the analysis of the global sections of the pluricanonical line bundles and the anti-pluricanonical line bundles.

The Trichotomy
There are three basic classes of smooth projective varieties depending on the positivity of the canonical line bundle \( \omega_X \). Depending on which class they belong to, either \( \omega_X \cong \mathcal{O}_X \), \( \omega_X^m \) has many global sections for \( m \gg 0 \) or \( \omega_X^m \) has many global sections of \( m \ll 0 \). We say that a smooth projective variety is canonically polarized (resp. Fano) if \( \omega_X^m \) is very ample for some \( m > 0 \) (resp. \( m < 0 \)). We say that a smooth projective variety is Calabi–Yau if \( \omega_X \cong \mathcal{O}_X \).

For a Calabi–Yau variety the only global sections of the pluricanonical or anti-pluricanonical line bundle are constant sections.

Canonically polarized, Calabi–Yau, and Fano varieties are the three building blocks of all smooth algebraic varieties. They are the algebro-geometric versions of the notion of hyperbolic, parabolic, and elliptic geometry either from classical geometry or from differential geometry. Additionally, this analogy becomes a theorem via the theory of Kähler–Einstein metrics. They behave quite differently from almost any perspective: topological, geometrical, or arithmetic. For instance, the following theorem is due to Kobayashi (see [Kob61]).

**Theorem 1.** Let \( X \) be a smooth Fano variety and let \( x \in X \) be a point. Then \( \pi_1(X; x) \cong \{ 1 \} \).

On the other hand, Gromov proved that the fundamental group of a smooth Calabi–Yau variety is almost an abelian group (see [Gro78]).

**Theorem 2.** Let \( X \) be a smooth Calabi–Yau variety of dimension \( n \) and \( x \in X \) be a point. Then \( \pi_1(X; x) \) admits a normal abelian subgroup of rank at most \( 2n \) and finite index.

It is not yet clear how to describe the fundamental group of smooth canonically polarized varieties.

**Smooth Projective Curves**
A smooth projective curve is a 1-dimensional smooth projective variety. In this case, every smooth projective curve is either canonically polarized, Calabi–Yau, or Fano. A smooth Fano curve is isomorphic to the projective line \( \mathbb{P}^1 \) that topologically is a 2-dimensional sphere. For instance, the Fermat curves \( C_1 \) and \( C_2 \) are both isomorphic to the projective line. Smooth Fano curves are also called rational curves. A smooth Calabi–Yau curve is isomorphic to a cubic hypersurface in \( \mathbb{P}^2 \), i.e., a smooth curve that is defined by a single homogeneous cubic polynomial in \( x_0, x_1, \) and \( x_2 \). These curves are known as elliptic curves and topologically they are \( S^1 \times S^1 \). For example, the Fermat curve \( C_3 \) is Calabi–Yau. For each \( k \geq 4 \), the Fermat curve \( C_k \) is canonically polarized. Any smooth curve \( C \) is homotopic to a Riemann surface. The genus \( g(C) \) of the curve is the number of handles of the Riemann surface. From the geometric perspective, we can define the genus of a smooth projective curve \( C \) to be
\[ g(C) := \dim_c \Gamma(\mathcal{O}_C), \]
i.e., the genus can be understood as the number of linearly independent global sections of the canonical line bundle. From this perspective, a smooth projective curve \( C \) is Fano (resp. Calabi–Yau or canonically polarized) if and only if \( g(C) = 0 \) (resp. \( g(C) = 1 \) or \( g(C) \geq 2 \)).

The degree-genus formula states that the genus of a smooth projective curve \( C \) defined by a homogeneous
polynomial of degree \(d\) in \(\mathbb{P}^2\) equals
\[
g(C) = \frac{(d-1)(d-2)}{2}.
\]
Thus, a Fermat curve \(C_d\) is canonically polarized if and only if \(d \geq 4\). The most topologically accurate way to draw the Fermat curve of degree 4 is as a sphere with 3 handles attached.

Before we keep discussing the classification problem, we review the concept of divisors.

**Divisors on Curves**

Let \(\mathcal{L} \to C\) be a line bundle over a smooth projective curve. A **meromorphic section** \(s : C \to \mathcal{L}\) is a section that is holomorphic outside finitely many points of \(C\) and has no essential singularities. In other words, the function takes a value on \(\mathcal{L}_c \simeq C\) for each \(c \in C\) while at these special points it **diverges to infinity**. To this meromorphic section \(s\), we can associate a formal sum of points of \(C\) as follows. Around each point \(c \in C\) that the meromorphic function takes value 0 or \(\infty\) it locally behaves as \(z \mapsto z^m\) with \(m > 0\) or \(m < 0\) respectively. This value \(m\) is called the **order** of \(s\) at \(p \in C\) and is usually denoted by \(m_p\). Then to \(s\) we can associate the finite formal sum of points
\[
D(s) := \sum_{p \in C} m_p P. \tag{4}
\]
This finite formal sum of points in a curve is what we call a **divisor** on the curve \(C\). Each divisor on a smooth projective curve \(C\) corresponds uniquely to a line bundle \(\mathcal{L}\) equipped with a meromorphic section \(s\) up to rescaling factor. Thus, we can use finite formal combination of points to encode the information of line bundles and meromorphic sections. For instance, the divisor \(0\), i.e., the formal sum of all points with coefficient zero, corresponds to the trivial line bundle \(O_X\) with the constant section.

The degree of a divisor \(D = \sum_{p \in C} m_p P\) is the sum of the coefficients that appear in the finite formal sum, i.e., \(\deg(D) = \sum_{p \in C} m_p\). The degree of a line bundle \(\mathcal{L}\) on a curve \(C\) is defined to be \(\deg(\mathcal{L}) := \deg(D(s))\), where \(s\) is a meromorphic section of \(\mathcal{L}\) and \(D(s)\) is defined via (4). This number is indeed independent of the meromorphic section. The degree of the canonical line bundle \(\omega_C\) of a smooth projective curve \(C\) is equal to \(2g - 2\). So, the sign of the degree of the canonical line bundle determines whether the curve is Fano, Calabi–Yau, or canonically polarized.

**Divisors on Smooth Projective Varieties**

A divisor \(D\) on a smooth projective variety \(X\) is a finite formal combination \(\sum_{V \subset X} m_V V\), where the \(m_V\) are integers and \(V \subset X\) are subvarieties of codimension 1, i.e., they have dimension exactly one less than the variety \(X\). This definition generalizes the concept of divisors on curves to higher dimensions. The duality between divisors and line bundles with meromorphic functions indeed works on every smooth projective variety. The line bundle associated to a divisor \(D\) is often denoted by \(\mathcal{O}_X(D)\). Two divisors that are associated to the same line bundle are said to be **linearly equivalent**. Linear equivalence forms an equivalence relation between divisors. Linearly equivalent divisors exhibit similar geometric properties. A **canonical divisor** on a variety \(X\), often denoted by \(K_X\), is a divisor that is associated with the canonical line bundle \(\omega_X\), i.e., \(\omega_X \equiv \mathcal{O}_X(K_X)\). For instance, in \(\mathbb{P}^1\) the divisor \(-\{0\} - \{\infty\}\) is a canonical divisor, while the trivial divisor is a canonical divisor on an elliptic curve.

Let \(D\) be a divisor on a smooth projective variety \(X\) and \(C \subset X\) be a curve. We may define the **intersection number** to be
\[
D \cdot C := \deg(\mathcal{O}_X(D)|_C).
\]
This concept generalizes the naive counting of intersection points between \(D\) and \(C\).

Two divisors \(D\) and \(D'\) are said to be **numerically equivalent** if \((D - D') \cdot C = 0\) for every curve \(C \subset X\). Linearly equivalent divisors are numerically equivalent. The **Néron–Severi group** \(\text{NS}(X)\) of a smooth projective variety \(X\) is the group of divisors modulo numerical equivalence. The following theorem is due to Severi.

**Theorem 3.** The Néron–Severi group \(\text{NS}(X)\) of a smooth projective variety is a finitely generated abelian group.

The **Picard rank**, denoted by \(\rho(X)\), is the rank of \(\text{NS}(X)\) and it measures the dimension of the space of divisors on \(X\).

**Smooth Projective Surfaces**

A smooth projective surface is a 2-dimensional smooth projective variety. The study of smooth projective surfaces is much more complicated than the study of smooth projective curves. The main reason for this difficulty is that smooth projective surfaces admit certain **surgeries** that can change the isomorphism class of the surface but leave a dense open subset unchanged. This construction is known as blow-up and we proceed to explain it below.

We can consider the smooth surface
\[
X := \{((x, y), [z : w]) \mid xw - zy = 0\} \subset C^2 \times \mathbb{P}^1,
\]
and let
\[
E := \{((0, 0), [z : w])\} \subset X.
\]
The variety \(E\) is isomorphic to \(\mathbb{P}^1\). The surface \(X\) admits a projection function \(\pi : X \to C^2\). This projection function induces an isomorphism on \(X \setminus E \to C^2 \setminus \{0\}\). Further, the function \(\pi\) restricted to \(E\) is constant and maps the whole curve to the origin. The curve \(E\) is called the **exceptional curve** of the blow-up. The morphism \(\pi\) is what

---

\(1\)The group structure being addition.
we call the blow-up of $\mathbb{C}^2$ at the origin. In the previous example, the variety $X$ is usually denoted by $\text{Bl}_0(\mathbb{C}^2)$. Note that the divisor $E \cong \mathbb{P}^1$ represents the tangent directions of the origin in $\mathbb{C}^2$. In a few words, a blow-up is a surgery on a variety $X$ that cuts a subvariety $Z$ and replaces it with a variety $E$ that represents the tangent directions of $Z$ inside $X$.

A priori, the blow-up construction poses a problem for the classification of smooth projective surfaces. Given any smooth projective surface $X$, for instance the projective space $\mathbb{P}^2$, we may take any finite sequence of points $p_1, \ldots, p_k \in \mathbb{P}^2$ and blow-up the points consecutively. By doing so, we obtain a sequence of surfaces $X_0 := X, X_1, X_2, \ldots, X_k$ with projection functions $X_i \to X_{i-1}$ that are blow-ups. Each surface $X_i$ is not isomorphic to the previous ones, but there is an open dense subset $U_i \subset X_i$ which is isomorphic to an open dense subset of $X$. In this case, we say that $X_i$ is birational to $X$. A morphism from $X_i$ to $X$ that induces an isomorphism on open subsets is called a birational morphism.

Even for $\mathbb{P}^2$ the study of the isomorphism classes of smooth projective varieties that are birational to it is a complicated task.

**Minimal Surfaces**

The previous analysis hints toward the intuitive approach: instead of classifying surfaces up to isomorphism, we aim to study smooth projective surfaces that are not blow-ups of other smooth projective surfaces. This leads to the concept of minimal surface. A minimal surface is, roughly speaking, a surface that is not the blow-up of another smooth projective surface.

Castelnuovo proved the following theorem that characterizes minimal surfaces.

**Theorem 4.** Let $X$ be a smooth projective surface. Let $E \subset X$ be a smooth rational curve with $E^2 = -1$. Then, there exists a projection morphism $\pi : X \to Y$ to a smooth projective surface $Y$ that induces an isomorphism $X \setminus E \cong Y \setminus \pi(E)$ and $\pi(E)$ is a point $y \in Y$.

In the previous theorem the self-intersection $E^2$ is defined as $O_X(E) \cdot E$. Smooth rational curves $E$ with $E^2 = -1$ are known as $(-1)$-curves. Castelnuovo theorem tells us that a surface is minimal if and only if it does not contain $(-1)$-curves. Furthermore, if a smooth projective surface $X$ contains a $(-1)$-curve, then such a curve can be blown-down to a new smooth projective surface $Y$. By doing so, the Picard rank, which is a positive integer, drops by one. This means that $\rho(Y) = \rho(X) - 1$. Thus one cannot blow-down infinitely many $(-1)$-curves; this process is guaranteed to stop with a minimal surface after a finite number of steps. In summary, the Castelnuovo theorem is a very useful technique for the birational classification of surfaces: it postulates that any smooth projective surface can be transformed into a minimal smooth projective surface via blow-downs. Hence, for any smooth projective surface $X$ there is a birational morphism $\pi : X \to X_{\text{min}}$ to a minimal smooth projective surface $X_{\text{min}}$ such that the preimages of $\pi$ are either points or connected union of smooth rational curves. This approach reduces the birational classification of smooth projective surfaces to the classification of minimal smooth projective surfaces. Minimal smooth projective surfaces were classified in the 1950’s by Kodaira and Enriques.

**Cone of Curves**

A 1-cycle on a smooth projective variety $X$ is a finite formal combination of curves $C_0 := \sum_{C \subset X} m_C C$, where the $m_C$ are real numbers. We can define the intersection of a divisor $D$ with a 1-cycle $C_0$ by linearity:

$$D \cdot C_0 = \sum_{C \subset X} m_C (D \cdot C).$$

We say that two 1-cycles $C_0$ and $C_1$ are numerically equivalent, written $C_0 \equiv C_1$, if $D \cdot (C_0 - C_1) = 0$ for every divisor $D$. In other words, numerically equivalent 1-cycles are combinations of curves that cannot be distinguished by intersecting with divisors.

The space of curves, denoted by $N_1(X)$, is the space of 1-cycles modulo numerical equivalence. This $\mathbb{R}$-vector space turns out to be the dual of the space of divisors defined as $N^1(X) = \text{NS}(X) \otimes \mathbb{R}$. The elements of $N^1(X)$ are often called $\mathbb{R}$-line bundles. The intersection product

$$N^1(X) \times N_1(X) \to \mathbb{R} \quad (D, C) \mapsto D \cdot C$$

induces a perfect pairing.

A 1-cycle $C_0 = \sum_{C \subset X} m_C C$ is said to be effective if each $m_C$ is nonnegative. The cone of curves of a smooth projective variety, denoted by $\text{NE}(X)$, is the cone inside $N_1(X)$ spanned by all the numerical classes containing an effective 1-cycle. The cone of curves defines a positive direction in the space of curves. For instance, for $n \geq 2$, we have

$$N_1(\mathbb{P}^n) \cong \mathbb{R} \text{ and } \text{NE}(\mathbb{P}^n) \cong \mathbb{R}_{\geq 0}[\ell]$$

where $[\ell]$ is the class of a straight line $\ell$ in $\mathbb{P}^n$.

If we blow-up points on a surface, then the space of curves increases in dimension and the situation gets more interesting. The space of curves of $\text{Bl}_p(\mathbb{P}^2)$ is 2-dimensional and $\text{NE}(\text{Bl}_p(\mathbb{P}^2))$ is spanned by two curves: the exceptional curve $E$ and any curve $H$ whose image on $\mathbb{P}^2$ is a straight line passing through $p$. The blow-up of $\mathbb{P}^2$ in at most 8 randomly chosen points is a Fano variety and its cone of curves is polyhedral. On the contrary, the cone of curves of the blow-up of $\mathbb{P}^2$ at 9 general points is not polyhedral.
Cone Theorem
An element \( v \) in a cone \( \sigma \) is called extremal if whenever \( v = v_0 + v_1 \) with \( v_0, v_1 \in \sigma \), then \( v, v_0 \), and \( v_1 \) span the same ray in \( \sigma \). A curve \( C \) in a smooth projective variety \( X \) is said to be an extremal curve if it is extremal in the closed cone of curves denoted by \( \overline{\mathcal{N}}E(X) \).\(^2\) For example, \((-1)\)-curves on surfaces are extremal.

For a smooth projective variety \( X \) we can naturally split the closed cone of curves \( \overline{\mathcal{N}}E(X) \) into three pieces: the \( K_X \)-negative curves, the \( K_X \)-trivial curves, and the \( K_X \)-positive curves. The cone theorem states that the \( K_X \)-negative region of the cone of curves carries a pleasant structure (see [KM98]).

Theorem 5. Let \( X \) be an \( n \)-dimensional smooth projective variety. Then, there are countably many rational curves \( C_\ell \), with \( 0 < -K_X \cdot C_\ell \leq n + 1 \) such that
\[
\overline{\mathcal{N}}E(X) = \mathcal{N}E(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i].
\]
Furthermore, the rays \( \mathbb{R}_{\geq 0}[C_i] \) only accumulate to the hyperplane \( K_X^1 \).

In the previous theorem, we write \( \mathcal{N}E(X)_{K_X \geq 0} \) for the set of elements \( [C] \in \overline{\mathcal{N}}E(X) \) with \( K_X \cdot C \geq 0 \).

In the case of a smooth Fano variety \( X \) the cone theorem implies that the closed cone of curves \( \overline{\mathcal{N}}E(X) \) is a polyhedral cone. This is not true for Calabi–Yau varieties or canonically polarized varieties. For instance, if \( E \) is a general elliptic curve, then \( E \times E \) is a smooth Calabi–Yau surface whose closed cone of curves is not polyhedral. Let \( \delta \subset E \times E \) be the diagonal curve, let \( p \in E \) be a closed point, and define \( f_1 := [p \times E] \) and \( f_2 := [E \times p] \). Then, the curves \( f_1, f_2 \), and \( \delta \) generate, over \( \mathbb{R} \), the space of curves. Hence, any \( 1 \)-cycle \( C \) is numerically equivalent to \( xf_1 + yf_2 + z\delta \) for some real parameters \( x, y, \) and \( z \). In these coordinates, the cone of curves is described by
\[
\{(x, y, z) \mid x + y + z \geq 0 \text{ and } xy + xz + yz \geq 0\} \subset N_1(X).
\]
Thus, the closed cone of curves is a circular cone. In this case, by the definition of Calabi–Yau varieties, every \( 1 \)-cycle intersects \( K_X \) trivially.

Contraction Theorem
In the case of a smooth projective surface every \((-1)\)-curve \( E \) satisfies that \( K_X \cdot E = -1 \). Furthermore, every \((-1)\)-curve is extremal. Hence, every \((1)\)-curve appears in the rightmost summand of equality (5). Castelnuovo’s theorem then says that this curve can be contracted to a point to obtain a new smooth projective surface. The following theorem, known as the contraction theorem, states that a similar surgery can be performed on higher-dimensional smooth projective varieties (see [KM98]).

Theorem 6. Let \( X \) be a smooth projective variety. Let \( R \subset \mathcal{N}E(X) \) be an extremal ray that is \( K_X \)-negative. Then, there exists a projective morphism \( \phi_R : X \to Y \) with the following property: the image of a curve \( C \subset X \) on \( Y \) is a point if and only if \([C] \in R \).

Geometrically, the contraction theorem asserts that all the curves that belong to an extremal \( K_X \)-negative ray can be collapsed to points via a polynomial function. The adjective projective in the previous statement roughly means that the preimages are unions of projective varieties. The morphism \( \phi_R \), often called a contraction morphism, may be denoted by \( \phi_C \), where \( C \) is a curve contained in the ray \( R \).

Let \( p \) and \( q \) be two distinct points in \( \mathbb{P}^3 \) and let \( X = \text{Bl}_{p,q}(\mathbb{P}^3) \) be the blow-up of \( \mathbb{P}^3 \) at these two points. Let \( E_p \) and \( E_q \) be the preimages of \( p \) and \( q \) in the blow-up, respectively. Both \( E_p \) and \( E_q \) are surfaces isomorphic to \( \mathbb{P}^2 \). The closed cone of curves \( \overline{\mathcal{N}}E(X) \) is generated by three curves: a straight line \( \ell_p \) in \( E_p \), a straight line \( \ell_q \) in \( E_q \), and the unique line \( \ell_0 \) on \( X \) whose image in \( \mathbb{P}^3 \) is the unique line through \( p \) and \( q \). The canonical divisor \( K_X \) satisfies
\[
K_X \cdot \ell_p = -2, \quad K_X \cdot \ell_q = -2, \quad \text{and } K_X \cdot \ell_0 = 0.
\]

Theorem 6 can be applied to find two projective morphisms \( \phi_{\ell_p} : X \to \text{Bl}_{p,q}(\mathbb{P}^3) \) and \( \phi_{\ell_q} : X \to \text{Bl}_{p,q}(\mathbb{P}^3) \). A variation of Theorem 6 can be applied to find a contraction \( \phi_{\ell_0} : X \to X_0 \) that is an isomorphism in \( X \setminus \ell_0 \) and contracts \( \ell_0 \) to a singular point of \( X_0 \) locally given by the equation \( x y - z w = 0 \): the so-called rational double point.

The Three Types of Contractions
There are three types of contraction morphisms \( \phi_R \) depending on the dimension of the set swept out by the curves \( C \) with \([C] \in R \).

We say that \( \phi_R : X \to Y \) is a Mori fiber space if \( \dim Y < \dim X \). In this case, the curves that belong to the ray \( R \) sweep out the whole variety \( X \), i.e., they cover the whole variety. If \( y \in Y \) is a randomly chosen point of the base, then \( F = \phi_R^{-1}(y) \) is a smooth projective Fano variety. In other words, a Mori fiber space is a way to cover the variety \( X \) using Fano varieties in such a manner that the Fano varieties are compatible with the preimages of a polynomial function. Simple examples of Mori fiber spaces are projections \( X \times F \to X \), where \( F \) is a Fano variety of Picard rank one. A projective bundle over a smooth projective variety is also an example of a Mori fiber space. A Mori fiber space is a generalization of the concept of projective bundle where the projective space is replaced with a Fano variety.

We say that \( \phi_R : X \to Y \) is a divisorial contraction if \( \dim Y = \dim X \) and the locus \( E \) where \( \phi_R \) is not an isomorphism, the so-called exceptional locus, has dimension \( \dim E = \dim X - 1 \). In other words, the set swept out by the curves \([C] \in R \) is a divisor of \( X \). A divisorial

\(^2\)When talking about extremal curves it is more natural to take the closure of the cone. An open cone does not have extremal rays.
contraction is simply the opposite of a blow-up. In this case, we have that \( \rho(Y) = \rho(X) - 1 \) and while \( Y \) can be singular, the singularities of \( Y \) are not bad singularities. For instance, the divisor \( K_Y \) still induces a \( \mathbb{Q} \)-line bundle so intersection theory with \( K_Y \) is well defined. Furthermore, both the cone theorem and contraction theorem still hold for \( Y \).

We say that \( \phi: X \to Y \) is a flipping contraction if \( \dim Y = \dim X \) and the exceptional locus \( E \) has dimension at most \( \dim X - 2 \). Flipping contractions only exist in dimension at least 3. In this case, the singularities of \( Y \) are bad singularities, for instance \( K_Y \) does not induce a \( \mathbb{Q} \)-line bundle. Thus, the cone theorem and contraction theorem do not make sense on \( Y \). In this case, we aim to replace \( Y \) with a different variety. Flipping contractions have a close relative: flopping contractions. The definition is the same but we require \( K_X \cdot R = 0 \). For instance, the contraction \( \phi_0: X \to X_0 \) described above is a flopping contraction.

**Flopes**

Let \( X \) be the blow-up of \( \mathbb{P}^3 \) at two points \( p \) and \( q \). Let \( \ell_0 \) be the unique straight line in \( X \) whose image on \( \mathbb{P}^3 \) is the unique line through \( p \) and \( q \). Using the notation introduced above, we may find a flopping contraction \( \phi_0: X \to X_0 \) that only contracts the curve \( \ell_0 \). The image of \( \ell_0 \) in \( X_0 \) is a singular point, the so-called rational double point \( p_0 \in X_0 \) locally given by the equation \( xy - zw = 0 \). Let \( \tilde{X} := \text{Bl}_{p_0}(X_0) \) be the blow-up of \( X_0 \) at the point \( p_0 \). The exceptional divisor of \( p: \tilde{X} \to X \) is a surface \( E \) that is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). The surface \( E \) in \( \tilde{X} \) can be contracted in two different directions corresponding to the two different projections of \( \mathbb{P}^1 \times \mathbb{P}^1 \). One of the projections \( q: \tilde{X} \to X \) brings us back to the variety \( X \) by collapsing the divisor \( E \) to the curve \( \ell_0 \). The second projection \( q^+: \tilde{X} \to X^+ \) gives a new smooth projective variety \( X^+ \) and a new curve \( \ell^+_0 \) for which \( K_{X^+} \cdot \ell^+_0 = 0 \). There is a natural projective birational map \( \pi: X \to X^+ \), i.e., a projective morphism that is defined everywhere at \( X \) except at \( \ell_0 \). Analogously, the inverse \( \pi^{-1} \) is defined everywhere except at \( \ell^+_0 \). This leads to a commutative diagram

![Diagram](image)

The projective contraction \( \phi^+: X^+ \to X_0 \) is called the flopped contraction while the projective birational map \( \pi: X \to X^+ \) is called a flop. A flop swaps a \( K_X \)-trivial curve with a \( K_{X^+} \)-trivial curve.

In summary, a flop is an algebraic surgery that cuts a curve that intersects the canonical divisor trivially and pastes a new curve that intersects the canonical divisor trivially.

**Flips**

Let \( \phi: X \to Y \) be a flipping contraction. A flip is a birational map \( \pi: X \to X^+ \) together with a projective morphism \( \phi^+: X^+ \to Y \) such that the following properties hold:

1. the divisor \( K_{X^+} \) intersects positively every curve contracted by \( \phi^+ \), and
2. the equality \( \rho(X^+) = \rho(Y) + 1 \) holds.

Thus, we have a commutative diagram:

![Diagram](image)

The morphism \( \phi \) collapses \( K_X \)-negative curves while the morphism \( \phi^+ \) collapses \( K_{X^+} \)-positive curves. Thus, the composition \( \pi = (\phi^+)^{-1} \circ \phi \) contracts \( K_X \)-negative curves and extracts\( K_{X^+} \)-positive curves. This motivates the name of this birational map: it flips the sign of the intersection of the curves with the canonical divisor. It swaps negative curves with positive curves. The morphism \( \phi^+: X^+ \to Y \) is called the flipped contraction.

As explained above, the variety \( Y \) tends to be too singular to work with it. However, if the flip \( \pi \) exists, then \( X^+ \) exhibits similar singularities to the ones in \( X \). In particular, if \( K_X \) induces a \( \mathbb{Q} \)-line bundle, then so does \( K_{X^+} \), so the cone theorem and the contraction theorem may still apply to \( X^+ \).

For a long time, proving the existence of the flip \( \pi \) of a flipping contraction \( \phi \) was one of the most difficult problems on the MMP. This problem was fully settled by the work of Birkar, Cascini, Hacon, McKernan, and Xu, leading to the following theorem (see [BCHM10, Bir12, HX13]).

**Theorem 7.** Let \( X \) be a variety with log canonical singularities and \( \phi: X \to Y \) be a flipping contraction. Then, the flip \( \pi: X \to X^+ \) of \( \phi \) exists.

The concept of log canonical singularities is the largest class of singularities in which we expect the MMP to fully work. It will be explained in the next pages. For now, let us emphasize that both the cone theorem and contraction theorem are valid for varieties with log canonical singularities.

**Log Pairs**

In birational geometry, we often study log pairs instead of algebraic varieties. A log pair is a projective variety \( X \) together with a divisor \( \Delta \) whose coefficients are all nonnegative, such that \( K_X + \Delta \) induces a \( \mathbb{Q} \)-line bundle on \( X \). The divisor \( \Delta \) is called a boundary divisor.

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3These divisors are called effective.
There are several reasons for which we consider pair structures instead of varieties themselves. For instance, an elliptic curve $E$ admits a group structure induced by its universal cover $C$, the group $\mathbb{Z}_2$ acts on $E$ by sending $e \mapsto e^{-1}$. The quotient $E/\mathbb{Z}_2$ is isomorphic to the projective line $\mathbb{P}^1$. However, the quotient $\pi: E \to \mathbb{P}^1$ is not free, there are some ramification points. These points correspond to $0, \frac{1}{2}, \frac{1}{2}i$, and $\frac{1}{2} + \frac{1}{2}i$ in the universal cover $C$ of $E$. Without loss of generality, we may assume that the images of these points in $\mathbb{P}^1$ are $\{0\}, \{1\}, \{\infty\}$ and a fourth point $\{\lambda\}$. Consider the boundary divisor $\Delta_{p1} = \frac{1}{2}(\{0\} + \{1\} + \{\lambda\} + \{\infty\})$. Then, the divisor

$$K_{p1} + \Delta_{p1}$$

induces a Q-line bundle on $\mathbb{P}^1$. Both line bundles and Q-line bundles can be pulled-back via morphisms. For instance, a Q-line bundle $L$ on $\mathbb{P}^1$ induces a Q-line bundle $\pi^*L$ on $E$ that associates the line $L_{\pi(e)}$ to the point $e \in E$. This is called the pull-back of the Q-line bundle. The pull-back to $E$ of the Q-line bundle associated to the divisor (6) equals $\omega_E$. Thus, we write $K_E = \pi^*(K_{p1} + \Delta_{p1})$. Hence, the canonical divisor $K_E$ of an elliptic curve is analogous to studying the log pair

$$(\mathbb{P}^1, \frac{1}{2}(\{0\} + \{1\} + \{\lambda\} + \{\infty\})).$$

This picture generalizes to finite quotients of smooth projective varieties.

The Q-line bundle associated to $K_X + \Delta$ is denoted by $\omega_X(\Delta)$. The concepts of canonically polarized, Calabi–Yau, and Fano varieties extends to log pairs. For instance, a log pair $(X, \Delta)$ is said to be Fano if the sections of $\omega_X^m(m\Delta)$ define an embedding of $X$ into a projective space for some $m < 0$.

## Singularities of Log Pairs

Let $(X, \Delta)$ be a log pair and $\pi: Y \to X$ be a blow-up. Let $E \subseteq Y$ be a prime divisor. The log discrepancy of $(X, \Delta)$ at $E$ is the rational number $1 - \text{coeff}_E(\Delta_Y)$, where $\Delta_Y$ is the unique divisor for which

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta).$$

Roughly speaking, the divisor $E$ in the blow-up represents certain tangent directions on $X$. The log discrepancy $a_E(X, \Delta)$ measures how singular the pair $(X, \Delta)$ along these tangent directions. The larger this number is, the smoother the pair is in this direction.

We say that a log pair $(X, \Delta)$ is log terminal if all its log discrepancies are positive. We say that a log pair $(X, \Delta)$ is log canonical if all its log discrepancies are nonnegative.

For example, the quotient of a smooth projective variety by a finite group has only quotient singularities which are log terminal. Another construction that leads to log terminal singularities is the cone construction. Let $X \hookrightarrow \mathbb{P}^n$ be an embedding of a projective variety. We may take the homogeneous equations $f_1, \ldots, f_k$ defining $X$ inside $\mathbb{P}^n$ and consider them as equations in $\mathbb{C}^{n+1}$. The affine variety that these equations cut out in $\mathbb{C}^{n+1}$ is called the cone over $X$ with respect to the embedding $X \hookrightarrow \mathbb{P}^n$. This affine variety tends to be singular at the origin. If the embedding $X \hookrightarrow \mathbb{P}^n$ is defined by the sections of $\omega_Y^m$ for some $m < 0$, we say that the embedding is the $m$-th anti-pluricanonical embedding. The following theorem allows us to construct several log terminal singularities:

**Theorem 8.** Let $X$ be a smooth Fano variety of dimension $n$. The cone of $X$ with respect to an anti-pluricanonical embedding is a log terminal singularity of dimension $n + 1$.

For instance, the ordinary double point of dimension 2, locally given by the equation $\{x^2 + y^2 + z^2 = 0\}$, is the cone over the second anti-pluricanonical embedding of $\mathbb{P}^1$. The cones over higher anti-pluricanonical embeddings of $\mathbb{P}^1$ require many more equations to be described. Note that a single smooth Fano variety gives a plethora of log terminal singularities.

In a similar vein, any cone over a smooth Calabi–Yau variety gives a log canonical singularity. An elliptic singularity is a cone over an elliptic curve. It is an example of a log canonical singularity that is not log terminal. For this and some other supporting reasons, it is often said that log terminal singularities are the local version of Fano varieties while log canonical singularities are the local version of Calabi–Yau varieties.

### Minimal Model Program

Now we have all the tools to explain what the MMP is. The MMP is both the research topic and the algorithm that we study in this area of research.

The algorithm starts with a log canonical pair $(X, \Delta)$. If $K_X + \Delta$ intersects every curve nonnegatively, we declare $(X, \Delta)$ to be a minimal model. This is the outcome of the algorithm. If there is some $(K_X + \Delta)$-negative curve, then by the Cone theorem 5, there must be some extremal $(K_X + \Delta)$-negative curve $C$. By the Contraction theorem 6, there is a contraction $\phi_C$ associated to this extremal negative curve. There are three possibilities for $\phi_C$: divisorial contraction, flipping contraction, or Mori fiber space. If $\phi_C$ induces a Mori fiber space $X \to Y$, then we again stop the algorithm and declare this to be an outcome. If $\phi_C$ induces a divisorial contraction $X \to Y$, we replace $X$ with $Y$ and $\Delta$ with its image $\Delta_Y$ on $Y$. The pair $(Y, \Delta_Y)$ is again log canonical, so we can go back to the start of the algorithm. In this

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4A prime divisor is a divisor that cannot be written as the sum of two distinct divisors.

5For simplicity, we can think about a smooth projective variety.

6Divisors with this property are known as numerically eventually free or nef for simplicity.
case $\rho(Y) = \rho(X) - 1$, so this step cannot happen infinitely many times in the algorithm. If $\phi_C : X \to Y$ is a flipping contraction, then neither the cone theorem nor contraction theorem apply to $Y$. However, we may apply Theorem 7 to find a flip $\pi : X \to X^\prime$. Then, we replace $X$ with $X^\prime$ and $\Delta$ with its image $\Delta^\prime$ in $X^\prime$. It turns out that $(X^\prime, \Delta^\prime)$ has log canonical singularities. Thus, we may return to the starting point. However, in this case $\rho(X) = \rho(X^\prime)$. So, the Picard rank alone does not rule out the possibility of iterating this step forever. The flow chart explains the steps of the MMP.

The flow chart leads to the following.

**Theorem 9.** Let $(X, \Delta)$ be a log canonical pair. Then, we can run a $(K_X + \Delta)$-MMP.

In summary, the MMP aims to birationally transform a variety either into a minimal variety or a variety that admits a Mori fiber space. In both cases, the variety acquired some feature that is likely to be helpful to understand its geometry.

**Main Conjectures**

Theorem 9 states that we can run a minimal model program for any log canonical pair $(X, \Delta)$ in any dimension. Thus, we can perform the algorithm called the Minimal Model Program (MMP). However, this theorem says nothing about this algorithm stopping in finite time. To achieve this, the main conjecture is the following:

**Conjecture 1 (Termination of flips).** There is no infinite sequence of flips for a log canonical pair $(X, \Delta)$.

The previous conjecture would imply that the algorithm always terminates. On the other hand, we expect that a minimal model has some extra properties.

**Conjecture 2 (Abundance).** Let $(X, \Delta)$ be a log canonical pair with $K_X + \Delta$ nef. Then, there exists some $m > 0$ satisfying the following statement. For every $x \in X$ there exists $s \in \Gamma(\omega_X^m(m\Delta))$ with $s(x) \neq 0$.

The condition described in the previous theorem is known as base point freeness of the line bundle. It implies the existence of a morphism $X \to Y$ such that every global section of $\omega_Y^m(m\Delta)$ is the pull-back of a global section on $Y$. In particular, the variety $Y$ admits the structure of a canonically polarized pair. The variety $Y$ is called an ample model.

In summary, Theorem 9 together with Conjecture 1 and Conjecture 2 imply the following. For any log canonical pair $(X, \Delta)$ there is a birational transformation $(X, \Delta) \to (X', \Delta')$ and a morphism $\phi : (X', \Delta') \to Z$ such that the general fiber of $\phi$ is either: canonically polarized, Calabi–Yau, or Fano. Thus, we would achieve our purpose of reducing the study of varieties to one of the pure classes.

**State of the Art**

The MMP is fully settled in dimension 3 due to the work of many mathematicians: Mori, Shokurov, Reid, Kollár, and Kawamata, among others. The termination of flips for log canonical pairs of dimension 3 is proved in [Sho92], while the abundance for log canonical pairs of dimension 3 is proved in [KMM94].

A pair $(X, \Delta)$ is said to be of log general type if the function

$$m \mapsto \dim_C \Gamma(\omega_X^m(m\Delta))$$

asymptotically behaves as a polynomial of degree $\dim X$. For instance, a canonically polarized log pair is of log general type.

In [BCHM10], the authors define the MMP with scaling that is a particular run of the MMP that tends to terminate faster. They prove that the MMP with scaling terminates
for log terminal pairs of general type. In particular, we have the following.

**Theorem 10.** Let \((X, \Delta)\) be a log terminal pair of log general type. Then, there is some \((K_X + \Delta)\)-MMP that terminates with a minimal model.

On the other hand, if the divisor \(-(K_X + \Delta)\) has some positivity, then in [BCHM10] the authors proved that some MMP terminates with a Mori fiber space.

**Theorem 11.** Let \((X, \Delta)\) be a log terminal pair. Assume that \(\omega_X^m(\Delta)\) admits a global section for some \(m < 0\). Then there is a \((K_X + \Delta)\)-MMP that terminates with a Mori fiber space.

Both, Conjecture 1 and Conjecture 2 are open in dimension at least 4. In the author’s thesis, it is proved that in dimension 4 termination holds provided that \(K_X + \Delta\) satisfies some positivity assumption (see [Mor19, HM20]). In particular, we have the following theorem.

**Theorem 12.** Let \((X, \Delta)\) be a log canonical pair of dimension 4. Assume that \(\omega_X^m(\Delta)\) admits a global section for some \(m > 0\). Then any \((K_X + \Delta)\)-MMP terminates with a minimal model.

**Recent and Future Directions**

It is common in geometry that the study of smooth geometric objects leads to the study of singular geometric objects. For instance, the study of symmetries of manifolds led to the concept of orbifolds and these became a central topic in topology. In a similar vein, the study of smooth projective varieties naturally leads to the study of log terminal varieties, while the study of moduli spaces of smooth projective varieties leads to the study of log canonical varieties. Indeed, the compactification of moduli spaces of smooth projective varieties tends to parametrize varieties with semi-log canonical singularities. Semi-log canonical singularities are possibly nonnormal singularities whose normalizations have log canonical singularities. There has been a trend in the last few decades to generalize theorems for smooth projective varieties to the setting of either log terminal or log canonical singularities. The author expects that every theorem for smooth projective varieties, after possibly some adequate adjustment, can be generalized to the setting of log canonical pairs.

Results related to the MMP usually appear in three flavors: conjectures surrounding the MMP, applications of the MMP, and the MMP on new categories.

Among the results on conjectures surrounding the MMP in the last few years there have been important results on effectiveness of Iitaka fibrations [BZ16], log discrepancies of singularities [Mor21], and termination of flips [HM20].

One substantial application of the MMP is to the construction of moduli spaces. In order to construct moduli spaces one needs to bound the considered varieties. There are important theorems about boundedness of varieties of general type [HMX18], and boundedness of Fano varieties [Bir21b]. The author expects that the tools of the MMP will lead to important boundedness theorems on Calabi–Yau varieties in the next few years. The MMP has played a role in the construction of the moduli spaces of canonically polarized varieties. The compactification of these moduli spaces, that parametrize semi-log canonical varieties, are known as KSBA moduli. On the other hand, the MMP has played a role in the construction of the moduli spaces of K-semistable Fano varieties known as K-moduli. Recently, Kollár published a book that gives a complete treatment of the moduli of canonically polarized varieties [Kol23]. On the other hand, the survey [Xu21] explains several important results on K-stability and the existence of K-moduli spaces.

The MMP has been heavily used to study Fano varieties: global sections of antipluricanonical systems [Bir19], boundedness of Fano varieties [Bir21b], and existence of Kähler–Einstein metrics [Xu21]. In these three directions, the study of log Calabi–Yau structures on Fano varieties has been a central topic.

Finally, the MMP has been generalized to different categories: Kähler complex manifolds [HP16], positive and mixed characteristics [HW23], generalized pairs [Bir21a], and foliated varieties [CS21]. In the upcoming years, many theorems and conjectures of the MMP will be solved in these new categories. When proving the theorems of the MMP in these new categories many new techniques are discovered and tricks introduced. Some of them reprove known theorems in the MMP. The further development of the MMP in these new settings is expected to also shed light on the classic conjectures of the program: termination of flips and abundance.

**References**


Credits
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The Department of Applied Mathematics at the University of Washington is inviting nominations for the annual Frederic & Julia Wan Lecturer Prize.

Nominees are expected to have an outstanding research record in any area of applied mathematics. Nominations are open to all nationalities and institutional affiliations (University of Washington excluded).

Prize recipients are expected to visit the Department of Applied Mathematics at the University of Washington for one week in Spring Quarter of 2024, deliver a set of lectures, and engage with members of the department. All reasonable expenses will be reimbursed and the awardee will receive a $10,000 honorarium.

Nominations consist of a CV and a single 1-page nomination letter, to be sent to wanlecture@uw.edu. Self nominations are welcome. Nominations should be received by January 15, 2024.

JANUARY 2024 NOTICES OF THE AMERICAN MATHEMATICAL SOCIETY 27
How do AMS Graduate Student Chapters support the mathematical community and beyond?

UC Berkeley

This Chapter maintains a Wiki page for its graduate students, with categories such as: Prospective Students, First-Year Students, Upper-Year Students, Qualifying Exams, Applying for Postdocs, and LaTeX Basics. Members also assist with the annual open house for prospective students, manage department excursions, and organize the peer mentor program between graduate students.

University of Texas at El Paso

The UTEP Chapter is one of the newest to join the AMS Graduate Student Chapter Program. According to current president Ebenezer Nkum, the chapter plans to “explore collaborations with the Data Science Section from SIAM (Society for Industrial and Applied Mathematics); influence courses taught in the department; have each member attend (and present at) a conference; and provide training in SQL projects, publications, conferences, presentations, and internships.”

Photos courtesy of their respective Chapters.
University of Toledo

Chapter organizers hold ongoing graduate student seminars virtually and in person, welcoming speakers from University of Toledo and from other schools across the country. They also enjoy coordinating and attending leisure activities including Pi Day and Trivia Night.

The Chapter Program makes meaningful contributions to the professional development of graduate students in the mathematical sciences and connects graduate students with AMS offerings.

www.ams.org/studentchapters
Does One Have to be a Genius to Do Maths?

Terence Tao

This article was initially written for my blog in May 2007, as part of my general advice to my graduate students, and was based on my experience interacting with many such students, postdocs, and colleagues as they went through the process of "learning the ropes" of how to be a research mathematician. It has been one of the more highly viewed and commented upon articles on my blog, perhaps in part due to the unintuitive conclusions drawn.

Better beware of notions like genius and inspiration; they are a sort of magic wand and should be used sparingly by anybody who wants to see things clearly. (José Ortega y Gasset, "Notes on the novel")

Does one have to be a genius to do mathematics? The answer is an emphatic NO. In order to make good and useful contributions to mathematics, one does need to work hard, learn one's field well, learn other fields and tools, ask questions, talk to other mathematicians, and think about the "big picture." And yes, a reasonable amount of intelligence, patience, and maturity is also required. But one does not need some sort of magic "genius gene" that spontaneously generates ex nihilo deep insights.

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DOI: https://doi.org/10.1090/noti2851

unexpected solutions to problems, or other supernatural abilities.

The popular image of the lone (and possibly slightly mad) genius—who ignores the literature and other conventional wisdom and manages by some inexplicable inspiration (enhanced, perhaps, with a liberal dash of suffering) to come up with a breathtakingly original solution to a problem that confounded all the experts—is a charming and romantic image, but also a wildly inaccurate one, at least in the world of modern mathematics. We do have spectacular, deep, and remarkable results and insights in this subject, of course, but they are the hard-won and cumulative achievement of years, decades, or even centuries of steady work and progress of many good and great mathematicians; the advance from one stage of understanding to the next can be highly non-trivial, and sometimes rather unexpected, but still builds upon the foundation of earlier work rather than starting totally anew. (This is for instance the case with Wiles’s work on Fermat’s last theorem, or Perelman’s work on the Poincaré conjecture.)

Actually, I find the reality of mathematical research today—in which progress is obtained naturally and cumulatively as a consequence of hard work, directed by intuition, literature, and a bit of luck—to be far more satisfying than the romantic image that I had as a student of this subject, of course, but they are the hard-won and cumulative achievement of years, decades, or even centuries of steady work and progress of many good and great mathematicians; the advance from one stage of understanding to the next can be highly non-trivial, and sometimes rather unexpected, but still builds upon the foundation of earlier work rather than starting totally anew. (This is for instance the case with Wiles’s work on Fermat’s last theorem, or Perelman’s work on the Poincaré conjecture.)

Of course, even if one dismisses the notion of genius, it is still the case that at any given point in time, some mathematicians are faster, more experienced, more knowledgeable, more efficient, more careful, or more creative than others. This does not imply, though, that only the “best” mathematicians should do mathematics; this is the common error of mistaking absolute advantage for comparative advantage. The number of interesting mathematical research areas and problems to work on is vast—far more than can be covered in detail just by the “best” mathematicians—and sometimes the set of tools or ideas that you have will find something that other good mathematicians have overlooked, especially given that even the greatest mathematicians still have weaknesses in some aspects of mathematical research. As long as you have education, interest, and a reasonable amount of talent, there will be some part of mathematics where you can make a solid and useful contribution. It might not be the most glamorous part of mathematics, but actually this tends to be a healthy thing; in many cases the mundane nuts-and-bolts of a subject turn out to actually be more important than any fancy applications. Also, it is necessary to “cut one’s teeth” on the nonglamorous parts of a field before one really has any chance at all to tackle the famous problems in the area; take a look at the early publications of any of today’s great mathematicians to see what I mean by this.

In some cases, an abundance of raw talent may end up (somewhat perversely) to actually be harmful for one’s long-term mathematical development; if solutions to problems come too easily, for instance, one may not put as much energy into working hard, asking dumb questions, or increasing one’s range, and thus may eventually cause one’s skills to stagnate. Also, if one is accustomed to easy success, one may not develop the patience necessary to deal with truly difficult problems (see also this talk by Peter Norvig for an analogous phenomenon in software engineering, though see this clarification). Talent is important, of course; but how one develops and nurtures it is even more so.

It’s also good to remember that professional mathematics is not a sport (in sharp contrast to mathematics...
Early Career

competitions). The objective in mathematics is not to obtain the highest ranking, the highest “score,” or the highest number of prizes and awards; instead, it is to increase understanding of mathematics (both for yourself, and for your colleagues and students), and to contribute to its development and applications. For these tasks, mathematics needs all the good people it can get.

Further reading:
• “How to be a genius,” David Dobbs, New Scientist, 15 September 2006. [Thanks to Samir Chomsky for this link.]
• “The mundanity of excellence,” Daniel Chambliess, Sociological Theory, Vol. 7, No. 1, (Spring, 1989), 70-86. [Thanks to John Baez for this link.]

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Terence Tao

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Photo of the author is courtesy of Reed Hutchinson/UCLA.

What Does a Mathematician Do?

William Thurston

The following question appeared on mathoverflow.net, and it was answered by the late William Thurston in October of 2010. We republish it here with the permission of Bill’s widow, Julian Muriel Thurston.

Question: I find that mathematics is made by people like Gauss and Euler—while it may be possible to learn their work and understand it, nothing new is created by doing this. One can rewrite their books in modern language and notation or guide others to learn it too, but I never believed this was the significant part of a mathematicians work; which would be the creation of original mathematics. It seems entirely plausible that, with all the tremendously clever people working so hard on mathematics, there is nothing left for someone such as myself (who would be the first to admit they do not have any special talent in the field) to do. Perhaps my value would be to act more like cannon fodder? Since just sending in enough men in will surely break through some barrier.

Answer: It’s not mathematics that you need to contribute to. It’s deeper than that: how might you contribute to humanity, and even deeper, to the well-being of the world, by pursuing mathematics? Such a question is not possible to answer in a purely intellectual way, because the effects of our actions go far beyond our understanding. We are deeply social and deeply instinctual animals, so much that our well-being depends on many things we do that are hard to explain in an intellectual way. That is why you do well to follow your heart and your passion. Bare reason is likely to lead you astray. None of us are smart and wise enough to figure it out intellectually.

The product of mathematics is clarity and understanding. Not theorems, by themselves. Is there, for example any real reason that even such famous results as Fermat’s Last Theorem, or the Poincaré conjecture, really matter? Their real importance is not in their specific statements, but their role in challenging our understanding, presenting challenges that led to mathematical developments that increased our understanding.

The world does not suffer from an oversupply of clarity and understanding (to put it mildly). How and whether specific mathematics might lead to improving the world (whatever that means) is usually impossible to tease out, but mathematics collectively is extremely important.

I think of mathematics as having a large component of psychology, because of its strong dependence on human minds. Dehumanized mathematics would be more like computer code, which is very different. Mathematical ideas, even simple ideas, are often hard to transplant from mind to mind. There are many ideas in mathematics that may be hard to get, but they are easy once you get them. Because of this, mathematical understanding does not expand in a monotone direction. Our understanding frequently deteriorates as well. There are several obvious mechanisms of decay. The experts in a subject retire and die, or simply move on to other subjects and forget. Mathematics is commonly explained and recorded in symbolic and concrete forms that are easy to communicate, rather than in conceptual forms that are easy to understand once communicated. Translation in the direction conceptual -> concrete and symbolic is much easier than translation in the reverse direction, and symbolic forms often replaces the conceptual forms of understanding. And mathematical conventions and taken-for-granted knowledge change, so older texts may become hard to understand.
In short, mathematics only exists in a living community of mathematicians that spreads understanding and breathe life into ideas both old and new. The real satisfaction from mathematics is in learning from others and sharing with others. All of us have clear understanding of a few things and murky concepts of many more. There is no way to run out of ideas in need of clarification. The question of who is the first person to ever set foot on some square meter of land is really secondary. Revolutionary change does matter, but revolutions are few, and they are not self-sustaining---they depend very heavily on the community of mathematicians.

Source: https://mathoverflow.net/questions/43690/whats-a-mathematician-to-do/44213#44213

Summertime at the Mathematics Research Communities (MRCs)

Sarah Bryant

So much energy! That's what I experienced at the 2023 MRCs, starting with the shuttle ride out to Beaver Hollow Conference Center, where I saw joyful reunions between old friends and eager introductions among new colleagues.

Set in the forests of upstate New York, the MRCs offer a summertime retreat from the day-to-day world that allows for immersion in cutting-edge research. Each conference is extremely hands-on and collaborative. Organizers post materials well ahead of the MRC week to ensure that participants are ready to hit the ground running, and they do. Walking around the conference center, I saw whiteboards and flipcharts full of definitions, proof sketches, conjectures, and calculations: usually with a huddle of mathematicians talking through an idea.

But MRCs aren't all math. The setting and structure allow you to walk a trail, sit in nature, jump in a rowboat. There's even an excursion to nearby Niagara Falls. Conversations during these moments are just as valuable as those that happen in working sessions. Graduate students and early-career faculty can discuss careers, work-life balance, coping with setbacks, and other important topics among themselves and with organizers. And in addition to research breakout groups, the MRCs include complementary activities such as career-focused, soft-skills training opportunities.

On my academic journey, I never participated in a program like the MRCs, and I wish I had. As a graduate student, I often felt isolated in my work. I have come to learn that the most enjoyable parts of math, and life, involve collaboration. I am so glad this experience is available now for early-career researchers, as I know it has the potential to change a mathematician's career trajectory positively. Many MRC participants maintain their working collaboration groups and mentoring networks throughout the year and well beyond.

Who should apply? Applicants should be ready to engage in collaborative research and should be “early career”—either expecting to earn a PhD within two years or having completed a PhD within five years of the date of the summer conference. Exceptions to this limit on the career stage of an applicant may be made on a case-by-case basis. Applicants who identify as members of underrepresented groups and gender minorities are especially encouraged to apply.

The following MRC sessions are scheduled for summer 2024. Lead organizers are denoted with an asterisk. To learn more and apply by February 15, 2024, visit http://www.ams.org/programs/research-communities/mrc.

MRC 2024, Week 1: Algebraic Combinatorics
(Organizers: Susanna Fishel, Rebecca Garcia, Pamela Harris*, Rosa Orellana, Catherine H. Yan)

Algebraic combinatorics combines tools and techniques from both algebra and combinatorics to study discrete structures and their properties. This branch of math has strong ties to representation theory, computing, knot theory, mathematical physics, symmetric functions, and invariant theory.

This MRC seeks to advance the frontiers of cutting-edge algebraic combinatorics, including through explicit computations and experimentation, and to strengthen the
research networks of those working in this branch. Postdocs and sufficiently advanced graduate student researchers will work together in small groups on open problems in algebraic combinatorics and closely related areas, including representation theory, special functions, and enumerative combinatorics.

Applicants should have a background in algebra and/or combinatorics. Please mention any programming experience, although it’s not required.

**MRC 2024, Week 2: Mathematics of Adversarial, Interpretable, and Explainable AI** *(Tegan Emerson, Emily King*, Dustin Mixon, Tom Needham, Karamatou Yacoubou Djima)*

Some of the most active areas of research in machine learning today are adversarial artificial intelligence (AI), explainable AI, and interpretable AI. In explainable AI, methods are developed to “open up” black boxes like neural networks, while interpretable AI creates white box methods with possibly lower accuracy. It is possible to “trick” a trained neural network into outputting an error; adversarial AI is the study of such phenomena, which could yield dangerous outcomes. Most progress in these areas has been empirical and rooted in computer science, but there is a growing body of literature that suggests that fresh insights are available in fields that are traditionally considered to be pure mathematics, such as algebra, geometry, topology, and analysis.

The goal of this MRC is to introduce early career mathematicians, coming from a variety of subdisciplines, to these cutting-edge research areas and to show them how they can use their own expertise to make substantial contributions. The establishment of a diverse community of mathematicians working in this research area would be timely, since the role of AI in industry is increasing exponentially, and the demand for theoretical understanding of these algorithms is reflective of this.

**MRC 2024, Week 3a: Climate Science at the Interface of Topological Data Analysis and Dynamical Systems Theory** *(Davide Faranda, Théo Lacombe, Nina Otter, Kristian Strommen)*

A central challenge in climate science is understanding how global warming will affect mid-latitude weather. This MRC will address that question using topological data analysis (TDA), with a broader goal of fostering new collaborations between TDA, climate science, and dynamical systems theory.

The MRC will consider questions both practical—improving algorithms and software—and theoretical, such as analyzing dynamical properties of topological features. We welcome applicants from academia and industry with a wide range of backgrounds. Tutorials on TDA, climate science, and dynamical systems will take place before the research week.

**MRC 2024, Week 3b: Homotopical Combinatorics** *(Andrew Blumberg, Michael Hill, Kyle Ormsby*, Angélica Osorno, Constanze Roitzheim)*

This MRC will bring together early-career researchers with interests in combinatorics, algebraic topology, and abstract homotopy theory. Familiarity with abstract homotopy theory or with modern methods of algebraic topology will allow deeper engagement with some problems, but is not required; much of the subject can be approached purely combinatorially.

Relevant readings will be provided before the workshop, and an online collaboration platform will be used to discuss material and to build community. At the workshop, participants will work in teams on research programs, engage with lectures from senior faculty participants about aspects of homotopical combinatorics, and have open feedback sessions for further discussion.

**Sarah Bryant**

**Credits**

Photo of the author is courtesy of the author.
Engaging Students Through Math Competitions

Béla Bajnok

I was fortunate to grow up in Hungary, a country with a long and distinguished history in mathematical competitions. Like many of my friends, I was keenly involved in these contests. We liked training for them, being exposed to one beautiful problem after another; we enjoyed participating in them, even when we didn’t end up winning; and we loved being in a community that was welcoming yet tight-knit. I have remained involved with mathematical competitions ever since, and thus was excited and honored when, in 2017, I was asked to become the director of the American Mathematics Competitions (AMC) program of the Mathematical Association of America (MAA). While the winners of our competitions—especially those participating in the International Mathematical Olympiad (IMO)—receive much recognition, our goals are to encourage students to engage in mathematics nationwide, and to discover, develop, and nurture talent more broadly.

With the participation of over 300,000 students each year, the AMC is the largest program of the MAA. The competitions start with the AMC 8, the AMC 10, and the AMC 12 exams, open to students in grade 8 or below, grade 10 or below, and grade 12 or below, respectively. Based on their performance on these multiple-choice competitions, approximately 6,000 students are invited to take the American Invitational Mathematics Exam (AIME), a challenging three-hour exam where the answer to each of the 15 problems is a nonnegative integer under 1000. The competition series culminates with the USA Mathematical Olympiad (USAMO) and the USA Junior Mathematical Olympiad (USAJMO), offered to approximately 500 students. These Olympiads follow the style of the IMO: they ask students to provide rigorous proofs for three problems on each of two consecutive days, with an allowed time of four and a half hours each day. We hope that our competitions show students—and their teachers and society at large—the beauty and power of mathematics, usually well beyond what they see in typical mathematics classrooms.

The creation of problems suitable for the AMC is a highly challenging task. Even at the beginning levels where problems stay close to the standard school curriculum, we aim for problems that ask questions in novel and fun ways. Coming up with beautiful—but still elementary—problems at the Olympiad level is particularly challenging. Yet, year after year, collections of ingenious and captivating problems are created; everyone with an interest is encouraged to explore them. Beyond the competitions themselves, we hope that our problems provide learning opportunities for current and future students, their teachers, and anyone with an interest in mathematics. We present three examples here: one from an AMC 8 exam, one from an AIME competition, and one from a USA Mathematical Olympiad.

2022 AMC 8 Problem 25 (Created by Silva Chang)

A cricket hops between four leaves, on each turn hopping to one of the other three leaves with equal probability. After four hops what is the probability that the cricket returns to the leaf where it started?

Figure 1. Four leaves and a cricket.

There is a large variety of different techniques to solve this problem: the official solution guide lists four of them, from case counting to recursions, and additional methods such as dynamic programming or generating functions can be used as well. Problems being suitable for multiple approaches are of course helpful for the contestants at the time, but this subsequently also allows teachers to discuss techniques that are new to their students.

2022 AIME I Problem 1 (Created by David Altizio)

Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients 2 and $-2$, respectively. The graphs of both polynomials pass through the two points $(16,54)$ and $(20,53)$. Find $P(0) + Q(0)$.

Figure 2. Two points and a graph.

This problem can, of course, be solved by finding the two quadratic polynomials that satisfy the conditions of this problem, but this would take a while, and the answers are not particularly nice. The clever approach (perhaps suggested by the question itself) would focus on the polynomial $P(x) + Q(x)$, which is linear; since its graph goes through the points $(16,108)$ and $(20,106)$, it equals $-x/2 + 116$, and thus $P(0) + Q(0) = 116$.

One of the challenges we have when assembling our exams is to be able to select the students who will then move on to subsequent competitions and win awards and prizes, while also providing approachable problems to all...
participants. Three hours may seem long for the AIME competition, but time is an important factor for many participants, so being able to solve the easiest problems quickly is a big advantage.

2021 USAMO Problem 3 (Created by Shaunak Kishore and Alex Zhai)

Let \( n \geq 2 \) be an integer. An \( n \times n \) board is initially empty. Each minute, you may perform one of two moves:

- If there is an L-shaped tromino region of three cells without stones on the board (see Figure 2; rotations not allowed), you may place a stone in each of those cells.

![Figure 2. The tromino region.](image)

- If all cells in a row or column have a stone, you may remove all stones from that row or column.

For which \( n \) is it possible that, after some nonzero number of moves, the board has no stones?

This problem proved to be one of the most challenging ones in the history of the USAMO: only seven students who took the exam were able to solve it. The solution employs what has recently been called the polynomial method, which is based on the simple fact that polynomials have a limited number of roots. (Note that one can prove this fact with some basic algebra.) This technique has seen a lot of interesting applications lately in both mathematics and computer science, and we were excited to feature it in this beautiful problem.

It is not hard to see that when \( n \) is divisible by 3, the empty board is achievable. Divide the board into 3-by-3 sub-boards. In each 3-by-3 sub-board, follow the procedure shown in Figure 3.

![Figure 3. Clearing the board when \( n \) is divisible by 3.](image)

We keep track of our moves as follows: when stones are placed on the board, we add their values, and when stones are removed from the board, we subtract their values. Assume now, indirectly, that a procedure results in an empty board; to be specific, suppose that in this procedure, a tromino with its lower-left corner in position \((i, j)\) was added \(t_{i,j}\) times; the \(i\)th column was cleared \(c_i\) times; and the \(j\)th row was cleared \(r_j\) times. This then means that we must have

\[
T(x, y) - C(x, y) - R(x, y) = 0,
\]

where

\[
T(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2} t_{i,j} x^i y^j (1 + x + y),
\]

\[
C(x, y) = \sum_{i=0}^{3} x^i (1 + y + y^2 + y^3),
\]

\[
R(x, y) = \sum_{j=0}^{3} y^j (1 + x + x^2 + x^3).
\]

Consider now the set \( D = \{-1, i, -i\} \), and observe that \( C(x, y) \) and \( R(x, y) \) both equal zero for every \( x \in D \) and \( y \in D \), but \( 1 + x + y \) is never equal to zero for any such \( x \) and \( y \). We then must have

\[
P(x, y) = \sum_{i=0}^{2} \sum_{j=0}^{2} t_{i,j} x^i y^j = 0
\]

for each \( x \in D \) and \( y \in D \). This is a contradiction, however, as the polynomial \( P(x, y) \) is at most quadratic in both \( x \) and \( y \), and thus it cannot have more than two values of each variable for which it equals zero.
The AMC program would not be possible without our four remarkable editorial boards that create our exams each year: they propose problems, review and rate all submissions, select the problems for the exams, and meticulously edit these problems and their solutions. They also are instrumental in helping us shape general AMC policies and practices. These nearly 180 individuals come from all across the nation and from abroad; include people from academia, high schools, business, and industry; have a wide range of ages represented, from undergraduates to retired mathematicians; and possess an impressive variety of mathematical and cultural expertise. I am immensely grateful to this talented, diverse, and accomplished group of people. Anyone with an interest in joining us is encouraged to contact me.

Béla Bajnok

Credits
Figure 1 is courtesy of Ricardo Conceição.
Figure 2 is courtesy of Shaunak Kishore and Alex Zhai.
Figures 3 and 4 are courtesy of the author.
Photo of Béla Bajnok is courtesy of Steve Berg.

Writing for an MAA Periodical

Della Dumbaugh and Deanna Haunsperger

Introduction
When the Mathematical Association of America (MAA) established its by-laws in 1920, the organization identified its “object” as “to assist in promoting the interests of mathematics in America, especially in the collegiate field...by the publication of mathematical papers, journals, books, monographs and reports...”¹ From the start, then, the MAA has emphasized the publication of mathematics in various forms.

Today, the association maintains communication as a core value and, in particular, aims to advance “creative discoveries in mathematics” and communicate “the role of mathematics in a changing society.”² In alignment with these goals, the MAA provides seven publication venues for writing about mathematics. The different periodicals feature a broad array of articles, from ground-breaking mathematical research to day-to-day tips for improving mathematical education, from interviews with notable mathematicians to puzzles and problems, and from connections between mathematics and art to book reviews.

Overview of MAA Publications
The oldest publication of the MAA, predating the MAA itself, is the American Mathematical Monthly. The Monthly features manuscripts appropriate for a broad audience of professional mathematicians presented in an engaging manner. The main types of submissions to the Monthly include articles (of more than five pages), notes (under five pages in length), filler (very focused articles of less than a page), problem proposals and solutions, and book reviews. (If you have questions or submissions, contact editor Della Dumbaugh at monthly@maa.org.)

The College Mathematics Journal (CMJ) started as a journal for faculty at two-year colleges, but has evolved into a publication that focuses mainly on mathematics encountered during the first two years of college. The CMJ publishes well-written articles (under twelve! pages in length) designed so that a junior or senior in college can read them, classroom capsules (under three pages in length), problems and solutions, book reviews, and media highlights. (If you have questions or submissions, contact editor Tamara Lakins at cmj@maa.org.)

Mathematics Magazine publishes content of all types that would appeal to a broad mathematical audience that includes strong undergraduate students. Submissions may focus on mathematics or math education, math and the humanities, the history of math, mathematics and art, or many other related fields. (If you have questions or submissions, contact editor Jason Rosenhouse and editor-elect Gordon Williams at mathmag@maa.org.)

The MAA’s undergraduate magazine, Math Horizons, targets enthusiastic undergraduate-level readers at any stage of their mathematical careers with a wide variety of articles, interviews, puzzles, book reviews, and career opportunities. Math Horizons is particularly interested in self-contained articles that promote reader engagement. Although the magazine aims to reach an undergraduate audience, the magazine is widely read by professional mathematicians as well. The best submissions have a maximum

² https://www.maa.org/about-maa
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of 2200 words and should include colorful images. (If you have questions or submissions, contact editor Tom Edgar at mh@maa.org.)

MAA FOCUS is the news magazine of the MAA and is provided in print to all nonstudent members. Authors can submit articles with up to 1600 words or write for one of the eight columns (such as Section Happenings, Toolkit, or Spotlight on SIGMAAs) with up to 1200 words. Submissions on innovative pedagogical ideas, inspiring outreach activities, or current issues facing the mathematics community are welcome. FOCUS also accepts submissions of puzzles, cover art, suggestions of MAA members to profile, and more. (If you have questions or submissions, contact editor Allison Henrich at mafocus@maa.org.)

MAA Convergence (https://www.maa.org/press/periodicals/convergence) is a refereed, open-access, online journal on the history of mathematics and its use in teaching subjects typical for grades 8-16. Submissions can include historically based classroom activities, translations of primary sources, expository articles on the history of topics in the grades 8-16 mathematics curriculum, or images for classroom use. (If you have questions or submissions, contact editors Amy Ackerberg-Hastings and Janet Heine Barnett and editor-elect Daniel Otero at convergence@maa.org.)

The MAA’s blog site, Math Values (https://www.mathvalues.org), explores the diverse voices of mathematics and discusses topics related to and affected by mathematics. The blogs are guided by the values of the MAA: Inclusivity, Community, Teaching & Learning, and Communication. Submissions can include thought-provoking essays, inspiring teaching tidbits, stories of the people or communities in mathematics, or ideas for making mathematics more inclusive. Blogs are generally about 1000 words. (If you have questions or submissions, contact editor Kira Hamman at blogs@maa.org.)

Getting Started

The best way to gain familiarity with a particular periodical is to read several recent issues of the publication. This will help you understand the audience, the style of the publication, and the level of mathematics. If you are a member of the MAA, you have access to all of these journals when you log into the MAA website. A university or college library may also subscribe to some or all of these journals. Convergence (https://maa.org/press/periodicals/convergence) and Math Values (https://www.mathvalues.org) are available without an MAA membership.

If you want to learn more about a publication before you submit to it, offer your services as a referee. If you are invited to referee a submission, you will receive a letter with guidelines for reviewing the paper. These instructions will help you learn how to evaluate the paper in your care and guide you in the future if you decide to submit to the journal. Serving as a referee will also improve your own writing and enhance your expository skills.

When you write your article, remember that you are probably writing about ideas your audience has not seen or thought of before. This novelty often creates an exciting manuscript; however, it will only engage your audience if you keep your prospective readers in mind. After you write and edit your article, have your colleagues and friends give you feedback on its clarity, style, and correct use of grammar and punctuation.

Once you determine the most appropriate periodical for your work, read the submission guidelines at https://maa.org/press/periodicals/submissions-to-aa-periodicals. The Monthly, CMJ, and Math Magazine all use a double-blind review process to evaluate submissions so make sure you remove all identifying information from your article before you submit it to one of these journals. Once you submit your article, you should hear back about receipt of your publication within a few weeks.

The Evaluation Process

The Monthly, CMJ, and Math Magazine will send your article out for double-blind review. This process typically takes three to six months, although it can take more or less time depending on referees, exam schedules, holidays, pandemics, etc. The editors and editorial boards of Math Horizons, FOCUS, Convergence, and Math Values all do their own reviewing so a decision is usually forthcoming within a few weeks.

When the editorial board makes a decision, they will send you a message that typically outlines the reason for the evaluation of your manuscript. If you receive a rejection, you may need to set it aside for a few days before you are ready to clearly read the feedback. Once you have put the necessary distance between the referee reports and your (beloved) manuscript, you will often be able to view the comments and insights as guideposts for improving the article and, perhaps, find a better fit for it in another publication. (For the very brave, this can also be a teachable moment for your students to show that not everything comes easily for you, either.) If you receive a decision of “revise,” you will often need to make adjustments or rewrite parts of the article. These recommendations come from dedicated colleagues who have taken on extra work on your behalf to improve the quality of your mathematics and your writing. This process helps ensure that MAA periodicals remain among the leading producers of award-winning exposition. And if you receive a decision of “accept,” well, congratulations!
Publication Begins with Submission

It can be daunting to submit a paper for publication. However, the variety of MAA periodicals offers the opportunity to write about mathematics at many different levels and about many different aspects of the profession. Leverage your strengths, determine where you feel comfortable, and start there. No matter what the outcome, the review process will provide you with helpful feedback that you can use the next time.

Leave your mark on the history of mathematics; publish in an MAA periodical.

Della Dumbaugh
Deanna Haunsperger

Credits
Photo of Della Dumbaugh is courtesy of the University of Richmond.
Photo of Deanna Haunsperger is courtesy of Deanna Haunsperger.

Getting Involved in AMS Governance

Boris Hasselblatt

Having been on the faculty of Tufts University for over a third of a century and having published books and papers along the way, I might (euphemistically?) be referred to as a midcareer mathematician, maybe wizened enough to dispense quaint career advice from the ivory tower. (For the curious, there are several public profile pages.1) Then on February 1, 2021, I became the 11th secretary of the American Mathematical Society. Here, in the form of a Q&A, are some perspectives from this new vantage point.

EC: As the AMS secretary, what sorts of things do you do and how did you get involved with AMS governance?

BH: I had been told to expect this position to entail a work load comparable to being a department chair, and in a number of ways that is about right. There is an ebb and flow across the year, and on any one day something unexpected may need urgent attention.

There are a number of interrelated aspects to this role. The core responsibility, reflected in the title, is to ensure that the decisions taken by the Society are made and recorded in a way that endows them with legal force as needed. This is visible to those in our volunteer governance when I make sure that meetings of our Council, say, are conducted according to the rules of parliamentary procedure. My office has responsibility for keeping and, when needed, checking the records of prior decisions—as well as the preparation of meeting agendas for the Council twice a year.

My office is also central in the management of volunteer governance, which includes well over 100 committees. Each year, new members must be appointed or elected for most of these, and my office assists the president and the Nominating Committee (as well as the Council and the Executive Committee) from the beginnings of their annual work on this through the implementation of their decisions. This entails communication with them as well as every committee member, and by itself this easily involves around 1000 emails per year. A significant number of committees are also directly supported in their work by my office.

Scientific meetings (sectional, national, international) are at the core of the mission of the AMS, and the Secretariat of the AMS is central to this: each of the four Associate Secretaries (for the Eastern, Southeastern, Central, and Western sections of the Society) organizes a meeting in that section each semester and, in rotation, national and international meetings as well. It has been suggested that this part of my role is that of “chief scientific officer,” though it is the Associate Secretaries who provide their judgement and insight.

There are things that I do not have to do. It has long been untenable for the secretary to run the entire AMS organization, and since 1949 there have been executive directors, serving in effect as chief executive officer of the Society with responsibility for the now 200 staff in three locations.

A central part of my role is to enable smart and committed colleagues to make and effect decisions on behalf of the Society. The many interactions are a great part of what makes this role such a pleasure. At different times of the year, this takes different forms, and in all of it I am supported by the most wonderful AMS staff members. Twice a year, the agenda for a Council meeting must be prepared. Including attachments, this can turn out to be a 200-page document. Much of this comes from committees making recommendations or reporting to Council because

1https://math.tufts.edu/people/faculty/boris-hasselblatt
https://mathscinet.ams.org/mathscinet/author?authorId=270790
the deep dives which a committee or its subcommittees can undertake provide the strongest preparation for policy choices. Chief among the committees which support the Council in this way are six policy committees. They meet in person annually, and subcommittee work plus virtual check-in meetings happen in between. These committees are ably guided by senior staff liaisons, and I am a member of each, indeed of over 20 AMS committees altogether. I also often consult with chairs and staff liaisons about committee business. At other times, the staffing of committees comes to the fore. Most committees are formed by appointing members; this is the duty of the president, and together with a Committee on Committees, my office supports this large undertaking. This involves significant communication with those being asked to serve, the president, committee chairs, and so on. At peak, this alone can involve dozens of emails a day. While the creation of slates for election rests mainly with the president and the Nominating Committee, the gathering and preparation of election materials and ballots rests with my office and stretches across the spring and into the middle of the summer. Starting in the spring and through the fall, we are central to the various stages of the awarding of prizes: selection, notification, and the preparation of the awards celebration at the Joint Mathematics Meetings. Also for the JMM, we support the committees which recommend speakers for invited addresses, we communicate with the speakers about their lectures, we work with the Meetings Department on the meetings program. In another analogy with serving as department chair, in any given day, an issue may arise which requires troubleshooting, consultation or other intervention. Some are small, others are such that the initial effort is merely the start of sustained attention.

My first involvement with AMS governance dates back to 2011 when I joined what is now the AMS-ASA-MAA-SIAM Data Committee, but since the 1990s I have also been organizing special sessions at AMS meetings, plus a Mathematics Research Community in 2017 (as well as disciplinary conferences independent of the AMS). Much of what in retrospect has been preparation for this position involved other service as well. At Tufts, I began serving on committees in the department and in the School of Arts and Sciences over 30 years ago, and before long, I chaired several of the school committees. From 2011 I was also involved in university-level committees such as for accreditation and strategic planning. Unbeknownst to me, the latter was an audition: in 2014 I started a term as associate provost after having served as department chair for a second time. Along the way I learned much from senior colleagues as well as provosts and presidents.

**EC**: What advice do you have for those who are interested in serving in a similar position?

**BH**: The preceding gives a sense of what I did prior to filling my present role, but none of that was part of a grand strategy to become secretary of the AMS. My own sense of what makes a fulfilling career has evolved over time, and I suspect few department chairs, deans, or university presidents started their doctoral studies with a view towards attaining their present position. Those who do have a plan for life early on in their careers may appreciate a viral quote from a friend and colleague of mine: The most successful people are those who are good at plan B. There are many ways of serving our community, and service can very much be its own reward, rather than a mere stepping stone. It is a great way of becoming an engaged member of the larger community in a school or university, or in the profession. It also provides a broader perspective on the context in which we work. A faculty position provides freedom and opportunities for engaging professionally in a multitude of ways. Whether this leads somewhere further, and where, depends on circumstances and inclinations. During and after my time as associate provost, search firms made me aware of opportunities for academic administrators, such as searches for deans. Others I know took up such calls, but I decided not to. That my home department is highly collegial and energetic has remained the main attraction in my professional life. But the range of opportunities is wide, and administrative roles can be very fulfilling. Your turn might come up to serve your department as chair, or you might at some point be recruited (or apply) to so serve elsewhere. At the AMS we have a number of staff mathematicians, who bring their expertise to bear in all divisions. The AMS currently seeks a new executive director. Those holding these positions are AMS employees, while I am a volunteer and retain my “day job” (as do those on our many committees, notably on the Council and the Board of Trustees; collectively; they are our volunteer governance). This illustrates the many career opportunities beyond the ones most talked about in graduate school, and the BEGIN (Business, Entrepreneurship, Government, Industry, and Nonprofit) Career Initiative of the AMS makes efforts to provide more connections that help consider such careers. When considering opportunities or ways of seeking them out, think about whether you find them interesting and whether they combine challenge and reward in a way that fits you. And don’t hesitate to tell someone else that they might be good in that role if you think so. I, for one, would have never thought of applying to be an associate provost, and I applied for the position of secretary upon a suggestion to do so. Generally, it is never wrong to tell someone else about their strengths when you recognize them. If someone tells you to think about an opportunity, do so. You may decide that it is not your thing, but it might also be life-changing in a positive way. And,
of course, if ever you get an invitation from me to serve on an AMS committee, you should immediately accept!

This invites a digression into thinking about what a mathematician does and who a mathematician is. I remember being welcomed as a physicist the day I first entered university in Germany (as a physics student), and I since learned that many mathematicians have a rather narrower view of who is a mathematician. The extreme view is that a mathematician must be a faculty member at a research university, hence the rightly despised term “leaving mathematics” for choosing to do something different. I take a broad view of who is a mathematician, though my own experiences are from within universities.

This breadth of perspective relates to mentoring. I aim to develop students to their best selves rather than to my former self. (In that spirit, read this whole text as about my experiences, not about what you should do.) I believe that my doctoral students have each succeeded on their own terms. Yes, that includes holding faculty positions at research universities, but also having a faculty job centered on teaching, starting a hedge fund, and working in insurance. I am proud of them all, and they should be proud of their achievements. Some of the most impressive mathematicians I have known have not gone to a mathematics graduate program, much less an academic career. This does not mean that they are not mathematicians. Some of my favorite mathematicians practice law, medicine, or dental medicine; they do research in banks, insurance, or health care; they build robots or cars; they are directors in a mathematics institute or a nonprofit organization; they raise or teach children; they write about mathematics and science or have served in the halls of Congress. Even those who no longer describe themselves primarily as mathematicians are ambassadors of mathematics. We should celebrate them all, including some who are more famous for something else altogether. Also, talent is not one-dimensional or linearly ordered: being quick, being deep, figuring things out, writing them up, collaborating, picking up ideas, enthusiasm, are distinctive qualities. Mathematics is not a zero-sum game, and you are serving mathematics and science or have served in the halls of Congress. Even those who no longer describe themselves primarily as mathematicians are ambassadors of mathematics. We should celebrate them all, including some who are more famous for something else altogether. Also, talent is not one-dimensional or linearly ordered: being quick, being deep, figuring things out, writing them up, collaborating, picking up ideas, enthusiasm, are distinctive qualities. Mathematics is not a zero-sum game, and you are serving mathematics, the community and yourself well if you recognize, encourage, support, promote, work with, and develop others. I thrive in the company of people who think differently, are smarter than I am, contribute in different ways, and have different backgrounds.

EC: What does a typical day or week look like for you?

BH: A typical week of mine will have some “Tufts days” devoted mostly to my faculty responsibilities, some “AMS days” mainly occupied with work related to the AMS, and some days that combine both. Unusual weeks might be those during part of which I am away at a mathematical conference or to attend AMS governance meetings, the latter most often in one of the three AMS locations, Providence, Washington, and Ann Arbor. I travel to Providence because interacting with staff there is more productive and pleasant in person. The days or parts of days spent on AMS business involve sharing in decision-making through email or meetings as I consult with other governance leaders or AMS staff. Sifting through and writing emails alone is an important daily challenge. Even with aggressive filtering, there can easily be something like 100 new emails in my inbox in a given day. Not all of these are “strictly Tufts” or “strictly AMS.” There are also referee requests, responsibilities as journal editor (and editor in chief), writing reviews for Mathematical Reviews, writing recommendations, reviewing a department or program.

EC: Do you meet with students or conduct mathematical research?

BH: The faculty responsibilities at Tufts include of course teaching—variously service courses, majors courses, or graduate courses. There are service obligations in the department or the school or for the university, and then there is research and time with students. At this time I have four doctoral students, and I meet with each about weekly, and we share time in a working seminar. This is a significant commitment each week, and it is also a research activity, even when the students take the lead. My own research happens on the side. Summers provide prolonged time for attention to projects and collaborations. And unsurprisingly, the most productive periods have involved sabbaticals. These are great opportunities professionally but also for making memories. Right after getting married, I took an unpaid leave for a year, and we spent two weeks honeymooning in Ireland, followed by five months at the IHES outside Paris and another five at the ETH in Zürich. Although it was financially ruinous, we still remember this one-year honeymoon more than three decades later. Later extended visits in Strasbourg, Marseille, Tokyo, and Zürich were exhilarating periods of mathematical work, but I came to realize that for me their value was partly in their exceptional nature. A research-only job would be too one-dimensional for me.

My research and book writing has mainly taken the form of collaborations with colleagues elsewhere, such as during those periods abroad. On one hand, distance has its shortcomings, but on the other hand, when I travel to work with a collaborator, I am aware that this is for a single purpose, which makes those periods quite productive.

Maybe it is good to step back and ask what mathematicians do to produce mathematics. Each in our way, consciously or not, we bring to bear a skill set. We ask questions, figure out problems or strategies, we talk to people, we teach, we write, we organize. I recommend finding ways to hone such skills. I have also found it gratifying to find collaborators whose skills complement my own. Figuring things out is a well-understood core skill, but already
Early Career

with the writing of a research statement, other skills come in, such as, to pose questions and to communicate content and motivation. Good writing is an almost universally useful skill, and it is good to be aware that this includes service writing, such as the writing of letters of recommendation, teaching and research statements, proposals, cover letters, referee reports, tenure cases. Good writing won’t rescue a fatally flawed paper, but it will make a solid publication more impactful. This serves not only the authors, but also the community. Step 1 is of course having the discipline to sit down and do the writing up. Having figured something out is delightful, but writing it up for publication is needed for permanence, credit, validation, and a career. Writing it well is a bonus, and opportunities for practice abound. For instance, consider becoming a reviewer for Mathematical Reviews. This is an important service to the community, it will expose you to mathematical works you might have otherwise missed, and if done attentively, it hones the skill of communicating mathematical ideas and telling a story. I review some 10 papers per year, which is not onerous and almost recreational. Some of us maintain blogs which communicate mathematical ideas in ways that research papers often cannot: a blog can be timely, leisurely, or simply expository. This is valuable practice and service. If you are working on a project with someone who writes well, then this might be another learning opportunity: what is it that makes the writing good, and how might that work for you in other projects?

Writing research statements and grant proposals well is important for your own progress. Writing recommendations, say, may be crucial to the success of someone else, and most of us take this quite seriously. We all should. Indeed, writing emails may not be something we think of as a skill, but it is. Before hitting send, think about what the email says to its reader. Is it correct? Is it clear? Is it kind? Does it pass the New York Times test (would it be OK if it were obtained and published by a national news outlet)? Is anyone copied on it who should not be?

EC: How do you keep up with what's going on in your field?

BH: Browsing the arXiv or getting its daily alerts goes a long way. For me, meetings are crucial for learning about new developments, especially in their early stages and with a focus on ideas rather than the precise published account. Here, I also meet and form connections with colleagues in their early career stages. In the context of working on new problems, contacting some of those I know (likely from conferences) who are working in that field can provide the right information, or hints for needed results, or on whether a question has been answered or is worth pursuing. Access to such expertise is among the rewarding aspects of conference organization, and to make these most impactful, it is critical to ensure that they are not only attended by the usual suspects but broadly represent regions, career stages, populations, and mathematical specialties. It is a skill to approach people at conferences (and to be approachable!) and to make connections between people and between ideas. No small number of joint projects have been started, pursued, or completed at a conference.

BH: Conclusion

I find the position of secretary highly rewarding because of the rich and varied interactions I have with brilliant AMS volunteers and staff, and because of the sense of shared mission centered on mathematics. I am grateful for having had the privilege to write my own script instead of following someone else’s, to arrive on these shores with an education paid for by fellow citizens rather than me, and to have been given this opportunity, and many before it. And we all should appreciate our privilege of having work that is meaningful to us and in which we keep learning. I hope that opportunities large and small will come your way as well as discernment about which of them to take up.

Boris Hasselblatt

Credits

Photo of the author is courtesy of Alonso Nichols/Tufts University.

Math Outreach

Denise Taunton Reid and Sandra Davis Trowell

It is well known that the job of professor includes teaching, research, and service. What is meant by teaching and research may be more obvious, but the service area can involve more than working on various committees. Beyond committee work, community outreach can be an important aspect of service. It also allows one to share their

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DOI: https://doi.org/10.1090/noti2850
passion for mathematics and a desire to learn. This is a satisfying type of service which is often overlooked.

Why Outreach

Outreach is worthwhile for many reasons. First, it increases the public awareness of the possibilities and opportunities available in mathematics; both at your institution and in the world in general. Secondly, it builds and improves relationships with local schools and communities. This relationship can lead to more students enrolled in mathematics and more specifically in your mathematics program. It enables one to connect with students, parents, and peers. In addition, it is an investment in the future of STEM and of your department and university.

Our Outreach History

An outreach program at our institution has focused mainly on female high school students by hosting an on-campus mathematics day. The goal is to mentor the students and encourage them to continue their studies in mathematics. Our university has hosted yearly events since 1996, excluding 2 years due to Covid-19. This event has been codirected for most of those years while some years it was a one-person project. This day includes hands-on mathematics workshops, career speakers, and a mathematics contest. Our goal continues to be to encourage the students in their mathematical studies. We want to show the participants that math is more than computations, memorization, and algebra. We also mentor the students and expose them to various math related careers.

Other outreach activities that we have been involved with include outreach programs with younger children. One such activity was Camp Invention, a STEM Camp through National Inventor’s Hall of Fame. In addition, our university hosts Science Saturdays for the community with a focus upon middle school students. We have also participated in local STEM and STEAM nights at regional schools at a variety of levels. If such events are available in your area, then choose the ones that interest you and participate. If no such events exist, then consider creating one.

Some More Ideas

Over the years we have done many activities and workshops. Some topics that we have covered during workshops with middle and high schoolers include, Fibonacci numbers, cryptography, tessellations, the Mobius strip, origami, and graph theory. Graph theory topics can vary from the seven bridges problem to scheduling conflicts to map coloring. When working with elementary aged students, topics have included pentominoes, making bracelets using binary code, making pictures and solving puzzles with tangrams, and the Tower of Hanoi puzzle. It is imperative to keep the participants engaged. They do not want to hear a talk or only see a speaker do things. The most successful outreach activities are ones that allow the student to participate. For example, participants always enjoy origami activities. Similarly, younger students enjoy being able to play with the Tower of Hanoi in order to explore the solutions. It is relatively easy to make several of these towers using items from the dollar store. Having hands-on activities builds the interest of the students. After explaining and demonstrating how graph theory can be used for conflict resolution or map coloring, give the participants a problem to solve. Make sure that you have paper and colored pencils for them to explore ideas and allow them to work together. When using this approach at a high school outreach event we often have teachers requesting the materials to explore further with their students.

Things to Consider

When planning an outreach event, begin by having a goal for the event and understanding who your audience will be. The goals may be community relations, recruitment for your program, or something else. Once the goal is known, the audience will become apparent. These goals will determine how the event is publicized and what activities will need to be developed.

Another consideration is speakers and activities for the event. These will be determined by the goal of the event. Activities may include workshops, rotation between activity tables, career speakers, contests, panel discussions, or demonstrations. Volunteers to assist with these can come from faculty, students, alumni, or community volunteers. Participants enjoy being involved with an activity and the opportunity to share with their peers throughout the day. Career speakers have also been an important part of our event as students have an opportunity to see how mathematics is used outside the mathematics classroom.

Once the audience and goal are set, a date and location can be picked. You will need to review the calendars of local schools to avoid conflicts. If this is to be an annual event, then be consistent from year to year. For our mathematics day event, the third Thursday in April has worked well. After you decide on the date, make your room reservation(s). Ensure that the size and setup of the venue is appropriate for your needs. We like to use round tables for the participants and extra tables for registration, food, and door prizes. Remember to make sure that you request all needed items such as microphones, computers, projectors, and extra trash cans.

To run an event, you will need to make a budget and obtain funding. Budget items to include are supplies such as folders, paper and pencils, food, travel for speakers, and contest or door prizes. Speakers and presenters may also need additional items for their activity or...
Early Career

presentation. Possible sources for funding are grants, local businesses, and campus funds such as departmental and college level funds. Oftentimes, local businesses will donate door prizes, food items, or in-kind donations. In-kind donations are donations that are matched by another source such as our department. Another helpful hint would be to buy inexpensive plastic buckets to use at each table. This allows you to easily place needed supplies at each table.

While the focus of the day may be on mathematics, it is important to take care of the physical things and actions that make it possible for participants to be there. A few things to keep in mind: (1) contact your facilities department to have an appropriate place for your event, (2) contact parking ahead of the event in order to share parking details with participants, (3) require preregistration to ensure that you have an appropriate sized space and enough supplies and are adequately prepared, and (4) have follow-up questionnaires to obtain feedback on the event so your next event can be improved.

The information from follow-up questionnaires has allowed us to change our presentations and workshops. It has reinforced the importance of engagement and hands-on opportunities. One question we include is what other activities or speakers would you like to see. This gives us more insight into our audience so that we can better prepare for future events. Our participants often ask for visits to our planetarium or for more workshops related to engineering or computing fields.

One thing we have done is take pictures at each event and maintain a scrapbook of the event through the years. This has been a great way to document the event. Participants enjoy looking through these to see pictures of their teachers from years past. We’ve actually reached a point where they are now looking for pictures of their parents.

Closing Remarks

Outreach is a way for one to give back to the community and to the discipline. It allows connections to be made which aid in recruiting potential students. After over a quarter of a century of hosting high school mathematics days, it has been very rewarding to see many of the participants transition into college math majors and beyond. Some of these students became math teachers who bring their students to our yearly event. It has been very fulfilling to see the event come full circle.

Dear Early Career

I am just starting my fourth year of grad school. Whom should I ask for letters of recommendation when applying for postdocs?

—Organized

Dear Organized,

Letters of recommendation are a very important part of the application for postdoctoral positions, and you are certainly right to be giving it some thought. Every application and university is different, and our advice is based on experience in both applying, and being on hiring committees, for postdoctoral positions which are research focused at universities based in the US.

Typically, one asks for a letter from their PhD advisor and a teaching letter from their home institution. The letter from your advisor is considered important as they are likely to have seen your research process up-close. The teaching letter is also important. Many US mathematics departments rely on postdocs to teach a large portion of their service courses, and they want to make sure that new postdocs will be able to handle the demands of teaching. If you have been a TA in large classes or an instructor in a large coordinated course, then you shouldn’t worry if your teaching letter is quite short and matter of fact. If you haven’t taught during your graduate studies, then you may wish to be creative and arrange an activity which shows that you will take the job of teaching seriously in your postdoc (e.g., run a
learning seminar on a topic and ask a professor to drop-in and then report on it as the teaching letter).

Applications for US universities usually ask for three (or four) letters of recommendation total, so the additional letter(s) would come from researchers other than your advisor that are familiar with your work. If you do obtain four letters of recommendation, then it’s fine to provide all of them even if the position only asks for three (but make sure you read the job advertisement). If your current institution has a research group in your area with several faculty members, then a letter from one (or more) of them would certainly be appropriate. However, in our experience on committees, letters describing your research from researchers outside of your home institution are very beneficial—the phrase “but all of their letters are from the same institution” has been uttered at many a committee meeting when the final ordering of postdoc offers is being decided. External letters provide some evidence that your work has made an impact outside of your research group. It is a good time to try and build connections with people outside of your university. If you haven’t had the opportunity to give a talk outside your home institution, perhaps you could talk to your advisor about arranging a research visit, and certainly it would be a great idea to attend conferences in your area, and talk to the speakers both about your work and your interest in their work. It is also quite common for conferences to give early career researchers the opportunity to contribute a short contributed talk or a poster. Networking may be intimidating, but we are in a small industry and people want to get to know you.

In terms of practical advice, the most important thing is to make sure you ask your letter writers early. This is especially true if you do not know them as well. If you are applying for postdoctoral jobs in the US, the deadlines begin in October and then most are in November and December (and it seems the shift is for the process to be done earlier). In order to allow someone the time to fit writing a good letter into their schedule at the beginning of a busy semester, giving at least one month’s notice is preferable. Good luck!

—Early Career editors

Have a question that you think would fit into our Dear Early Career column? Submit it to Taylor.2952@osu.edu or bjaye3@gatech.edu with the subject Early Career.

DOI: https://doi.org/10.1090/noti2847

*If you are in the situation where your advisor will not be one of your letters of recommendation, this role can be replaced by another professor at your home institution, or a professor outside of your university, but the situation should probably be brought up in one of the letters.*

*This is meant in the broad sense—it could be teaching as an instructor, TA, tutor, etc.*
John Coates (1945–2022): Celebration of a Life

Barry Mazur, Stephen Lichtenbaum, Andrew Wiles, Leila Schneps, Sujatha, and Mahesh Kakde

1. Thoughts About John Coates

Barry Mazur

The broad span of John Coates’s mathematical work, including his intellectual delights and his talent for inspiring and mentoring generations of mathematicians, was a gift to all of us. His energy and his generosity of thought, his appreciation of ideas, and his friendship were evident from the earliest days that I knew him. In 1969, he came to Harvard as a Benjamin Peirce Assistant Professor. There, as he wrote in his memorial to John Tate, the “very cramped quarters,” its “physical smallness,” and “tiny coffee room or corridors between the offices” offered him an environment that made it easy to interact with people. He was introduced by Tate to aspects of mathematics that would lead him to the area in which he would make some of his later great contributions.

As Steve Lichtenbaum recounts (§2), John Coates had started in the analytic number theoretic terrain of Kurt Mahler, algebraic approximation, and $p$-adic Thue, Roth and Baker methods, moving to the scheme theoretic terrain of Grothendieck’s Langage des Schémas, before connecting with his true passion, the core of algebraic number theory and the construction of bridges between analysis and arithmetic: the Birch and Swinnerton-Dyer conjecture, and Iwasawa theory.

The Coates–Wiles theorem (1977) is such a bridge and was a breakthrough moment in our understanding of the connection between purely arithmetic concepts—such as whether a CM elliptic curve $E$ defined over the field of rational numbers has infinitely many rational points—and “corresponding” analytic concepts—such as whether its $L$-function $L(E, s)$ vanishes at the value $s = 1$. It follows from the Birch and Swinnerton-Dyer conjecture that if the first arithmetic event occurs, so does the second analytic event; this is the Coates–Wiles theorem (§3).

Iwasawa theory—a companion to the Birch and Swinnerton-Dyer conjecture—is anchored in the structure of cyclotomic fields (a subject that Serge Lang had called “the backbone of number theory”). For any prime $p$ (say, $p > 2$) Iwasawa theory starts with that “backbone,” the $p$-cyclotomic tower

$$
K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots \subset L := \bigcup_{n=1}^{\infty} K_n
$$

of field extensions where $K_n$ is the field generated over $\mathbb{Q}$ by the $p^n$-th roots of unity. The Galois group of that $p$-cyclotomic tower (i.e., of the field extension $L/K$) has an easily described topological generator: namely, the automorphism $L \to L$ induced by the rule that sends any $p^n$-th root of unity $\zeta$ to $\zeta^{1+p}$. Such a topological generator
γ plays the role in Iwasawa theory of a fundamental operator: it operates naturally on various arithmetic structures defined over the base field $K$ when such a structure is “base-changed” to the extension field $L$. The operator also acts, for example, on the projective limit of the $p$-primary parts of the ideal class groups of the rungs $K_n$ of the $p$-cyclotomic tower—this projective limit producing a finite dimensional vector space over the field of $p$-adic numbers, with $γ$ operating as a linear automorphism on it.

Iwasawa had thought of this structure—i.e., an operator acting on a vector space defined using concepts in one branch of mathematics as having the (conjectured) property that its characteristic polynomial ties in with an object in another branch of mathematics—as metaphorically linked to André Weil’s vision of the Frobenius operator (in the absolute Galois group of a finite field) acting on (then-conjectured, now-known) cohomology groups of varieties over those finite fields with the property that the resulting characteristic polynomials give the relevant $L$-functions—or, going further back,—as metaphorically linked to what is often referred to as the Hilbert–Polya dream: to express the Riemann zeta-function as the characteristic series of a Hermitian operator on some (perhaps: Hilbert) space.

The “main conjecture” of Kenkichi Iwasawa was that the characteristic polynomial of this operator $γ$ gives detailed information about relevant classical (and $p$-adic) $L$-functions. This connection between analytic number theory ($L$-functions) and arithmetic has resonance with the classical Dirichlet class number formula.

Iwasawa theory is now sometimes referred to as classical Iwasawa theory for even more extensive companion theories are being evolved (e.g., in the context of automorphic forms and even more recently in derived categories). John was one of the great contributors to that evolution,

- from his interest in motivic $p$-adic $L$-functions—thus studying—as did Ralph Greenberg—the arithmetic of general algebraic varieties as one ascends the rungs of a cyclotomic tower,
- to his shaping with collaborators the impressive beautiful noncommutative version of Iwasawa theory—where the role of a single operator is replaced by a more general $p$-adic Lie group arising from a Galois representation.

It was thrilling to hear John talk about his ideas about the Birch and Swinnerton-Dyer conjecture, his vision of Iwasawa theory, and about his joint work with Andrew Wiles (§3). I had that opportunity especially when he was based in Orsay (§4) and I was at the IHES, since the two of us would jog around the bassin in Bures-sur-Yvette—this is a reservoir, a catch-basin, for the overflow of water (usually dry). We would chat as we jogged. He would explain his latest mathematical thought, we would talk about our young children, and he would tell me about his other passion: Japanese and Chinese poetry and illuminating critical works about them.

What enormous influence he had in the development of mathematics, and as a teacher; the list of his students is incredibly impressive, both in terms of the great contributions to mathematics that they made, but also in view of the range of different interests (albeit within number theory) that they have. This is truly a gift to all of us.
MEMORIAL TRIBUTE

chapter described as “notes by I. Coates and O. Jussila.” It was quite remarkable that Coates was able to do this, and he did not find it a very pleasant experience. In any case, after a year in Paris, Coates decided that he was in the wrong place, and so he wrote to J. W. S. Cassels in Cambridge asking if he could be admitted to graduate study there, and Cassels gladly assented.

At Cambridge, Coates worked with Alan Baker, who would soon receive the Fields Medal. Under Baker’s guidance, Coates wrote his doctoral thesis on “The effective solution of some Diophantine equations.” He and Baker then wrote a paper together entitled “Integral points on curves of genus one” which appeared in Inventiones. Carl Ludwig Siegel had proved that any elliptic curve over a number field had only finitely many integral points, but his proof was not effective. Coates and Baker succeeded in giving a (rather enormous) bound for the size of the solutions in terms of the coefficients of the curve.

After Coates received his PhD, he left Cambridge to take up a three-year position as Benjamin Peirce Assistant Professor at Harvard. There he came under the influence of John Tate, and became part of the modern world of abstract mathematics that he had found so difficult to enter in Paris.

In the nineteenth century, Richard Dedekind had defined a zeta-function \( \zeta_F(s) \) for any algebraic number field \( F \), which yields the Riemann zeta-function \( \zeta(s) \) when \( F = \mathbb{Q} \). The Dedekind zeta-function has many of the same properties as the Riemann zeta-function. For example, it can be defined by a power series which converges for \( \text{Re}(s) > 1 \) and then continued to a meromorphic function on the entire plane which is analytic except for a simple pole at \( s = 1 \), and satisfies a functional equation relating \( \zeta_F(s) \) to \( \zeta_F(1 - s) \). It was known by work of Siegel that \( \zeta_F(-1) \) is a rational number which is nonzero if and only if \( F \) is totally real: we have \( \zeta_F(-1) = -1/12 \). The zeta-function can also be defined for varieties over finite fields, where it can be completely described in terms of cohomology.

It is a very challenging, and so far completely unsolved, problem to give such a description for Dedekind zeta-functions, so any relation between Dedekind zeta-functions and cohomology is extremely interesting. Together with Bryan Birch of Oxford, Tate had made a cohomological conjecture about the value of \( \zeta_F(-1) \), where \( F \) is a totally real number field. Let \( \mathcal{O}_F \) denote the ring of integers in the number field \( F \). The Birch–Tate conjecture (slightly modernized) says that if \( F \) is a totally real number field then the absolute value of the rational number \( \zeta_F(-1) \) should be equal to \( |K_2(\mathcal{O}_F)|/w_2(F) \), where \( K_2(\mathcal{O}_F) \) is a certain group arising from algebraic K-theory and \( w_2(F) \) is defined to be the number of roots of unity contained in the compositum of all quadratic extensions of \( F \). Tate had shown that \( |K_2(\mathbb{Z})| = 2 \), and it is easy to see that \( w_2(\mathbb{Q}) = 24 \), so the Birch–Tate conjecture is true for \( F = \mathbb{Q} \).

Projective nonsingular curves \( X \) over finite fields are often thought of as the geometric analog of rings of integers in algebraic number fields, and they also have their own zeta-functions, and in fact it is possible to state the Birch–Tate conjecture in such a way that it makes sense for this situation as well. We understand the zeta-functions of varieties over finite fields much better than we do the zeta-functions of algebraic number fields, and Tate was able to prove the Birch–Tate conjecture in that case.

Motivated by Tate’s proof, Coates started working on the Birch–Tate conjecture in the original number field case. He had been studying the notes which Kenkichi Iwasawa had sent him for a course Iwasawa had given at the Institute for Advanced Study on \( p \)-adic L-functions. He then realized that a very natural conjecture describing the \( p \)-adic \( L \)-function as a characteristic polynomial could lead to a version of the Birch–Tate conjecture where \( K_2(\mathcal{O}_F) \) could be described in terms of Galois cohomology. As it happens, I had been working on trying to understand \( \zeta_E(-1) \), and I had attended Iwasawa’s course on \( p \)-adic \( L \)-functions at the Institute. When I wrote to Tate telling him that I had a conjecture relating \( \zeta_F(-1) \) to the orders of étale cohomology groups, which are fancy versions of Galois cohomology groups, he set to work trying to compare the two conjectures and to the work of Coates. Tate then succeeded in describing \( K_2(\mathcal{O}_F) \) in terms of Galois cohomology, and so Coates’s work showed that the conjecture on \( p \)-adic \( L \)-functions implied the Birch–Tate conjecture in the number field case. Tate also suggested that Coates and I ought to get to know each other, and so I invited Coates to give a talk at Cornell.

John drove from Harvard to Ithaca, bringing his wife Julie and his young son David with him. We each had already submitted a paper on \( K_2 \) and \( p \)-adic \( L \)-functions to the Annals, so it was too late to combine them as might have been desirable, but we realized that our work on special values of zeta-functions could be extended to \( L \)-functions, and in addition our earlier results could be improved. We then started to work together on these problems. In order to further this collaboration John invited me to visit him at Harvard, and I was glad to do so. I also brought my family, which at that time consisted of a wife and three small children. When we arrived, John told us that we should come to his apartment for a dinner which Julie had prepared for us before going to the hospital to have her second child. We were stunned.

Our collaboration eventually resulted in a third paper appearing in the Annals, and in our families becoming close. A few years later we spent a semester together at the Institut des Hautes Études in Bures-sur-Yvette, and then we...
saw each other every summer. I was visiting the IHES and John was living nearby because he had become a professor at the University of Paris at Orsay. Eventually these summertime meetings sadly came to an end as John left to go the Australian National University for one year and then went on to accept a well-deserved chair at Cambridge.


Andrew Wiles

Moving to Europe as a graduate student from his native Australia, and after a year spent in the Grothendieck seminar in Paris, John went to Cambridge to study under Alan Baker. This was his first brush with elliptic curves and together they proved a result bounding the size of integral points on cubic equations. There is a finality to this result which is very appealing, but already it was clear that the questions about rational points were much more exciting. There can be finitely many or there can be infinitely many. How do you tell which and how do you find them when they do exist?

After his thesis, John went to the US but returned to Cambridge in 1975, the same year that I started my thesis under his supervision. John only spent two years at Cambridge on this visit and it was part of a whirlwind tour which saw him taking permanent positions in Stanford (1972–75), Cambridge (1975–77), Canberra (1977–78), and Paris (1978–86) within the space of a few years. Nevertheless it was a very productive and exciting period in terms of research. A few months after John’s arrival we started working together on elliptic curves and John would not leave this beloved topic for the rest of his career. I will try here to explain the fascination.

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For our purposes we can view an elliptic curve (defined over the rationals) as a cubic equation of the form

\[ E = \{(x, y) : y^2 = x^3 + Ax + B \} \cup \{\infty\}, \quad A, B \in \mathbb{Z} \]

with

\[ \Delta_E := 4A^3 + 27B^2 \neq 0. \tag{1} \]

The last condition ensures that the cubic is irreducible and that the curve has genus 1. (If it does not hold then the curve has genus 0 and the question of rational points had been settled by the Greeks.) A point at \( \infty \) is included with the curve to make it projective. Surprisingly perhaps, there is then an abelian group structure on the curve which has the property that the sum of any three points of the curve which lie on a line is zero. This holds over any field, for example over the rationals or over the complex numbers. In 1922, Mordell proved that over the rationals, this group (which we denote by \( E(\mathbb{Q}) \)) is finitely generated. So as a group,

\[ E(\mathbb{Q}) \cong \mathbb{Z}^g \oplus T(E), \tag{2} \]

where \( T(E) \) is a finite (abelian) group. The big questions are (i) how do we find \( g \) and (ii) how do we find generators for the group of points?

The form an answer should probably take had been given ten years earlier by Birch and Swinnerton-Dyer working in Cambridge just prior to the time that John was at Cambridge as a graduate student. It involves the \( L \)-function which we can define as follows. Let \( N_p \) denote the number of points on \( E(F_p) \), in other words the number of solutions mod \( p \) of the equation (1) including the point at \( \infty \). Then set \( a_p := 1 + p - N_p \) and define an \( L \)-series by

\[ L(E,s) := \prod_{p \nmid \Delta_E} (1 - a_p p^{-s} + p^{1-2s})^{-1}. \tag{3} \]

Birch and Swinnerton-Dyer found heuristic reasons why the products \( \prod_{p<N} N_p/p \) should tend to \( \infty \) (as \( N \to \infty \)) if
and only if $E(\mathbb{Q})$ is infinite, and then formulated the conjecture:

Conjecture.  
\[ L(E, 1) \neq 0 \iff E(\mathbb{Q}) \text{ is finite.} \tag{4} \]

Now $L(E, s)$ is not absolutely convergent at $s = 1$ so this does not make any obvious sense. To overcome this we should also assume that $L(E, s)$ has an analytic continuation. The more refined version of their conjecture then gave the rank $g$ as the order of vanishing of $L(E, s)$ at $s = 1$, and moreover a precise formulation of the leading term involved a regulator made from the generators of the group of points.

Since the work of Mordell, much of the arithmetic study of elliptic curves had made use of class field theory. The first step is to use that the complex points of $E$ are given by $E(\mathbb{C})$. The influence of Iwasawa theory on elliptic curves. From then on it would guide his approach to elliptic curves. Let us assume that we are in the case of an imaginary quadratic field. In the work of Kronecker and Weber, later completed by Fueter and Takagi, all the abelian extensions of an imaginary quadratic field were constructed explicitly in this way. As an example the curves $y^2 = x^3 + Ax$ (for any $A$) have complex multiplication by $\mathbb{Z}[i]$. This is because there is an automorphism of $E$ given by $x \mapsto -x$, $y \mapsto iy$. The theory then gives an explicit construction of the abelian extensions of $\mathbb{Q}(i)$ using the torsion points of the elliptic curve.

The main result of our collaboration was the following, which we proved in the summer of 1976.

Theorem 1 ([8, Theorem 1]). Suppose that $E$ is an elliptic curve over the rationals with complex multiplication. Then

\[ L(E, 1) \neq 0 \Rightarrow E(\mathbb{Q}) \text{ is finite.} \tag{6} \]

This was the first general result on the Birch and Swinnerton-Dyer conjecture. The first thing to note is that the hypothesis of complex multiplication ensured that $L(E, s)$ has an analytic continuation by a theorem of Deuring. This theorem is now known without the hypothesis of complex multiplication but the reverse direction of (6) is still unknown.

The approach we took to proving this theorem was the reverse of the nineteenth-century programme in that we would use their explicit class field theory to study the elliptic curve. Let us assume that we are in the case of $y^2 = x^3 + Ax$ for simplicity. Let $K = \mathbb{Q}(i)$ and pick a prime $p$ that splits in $K$, $p = \pi \overline{\pi}$. Let $E_\pi = \{ Q \in E(\mathbb{C}) : \pi Q = 0 \}$ be the $\pi$-torsion points. Then by the theory of complex multiplication $E_\pi$ is defined over an abelian extension of $K$ which we write $K_0 = K(E_\pi)$. Suppose that $P$ is a point of infinite order in $E(\mathbb{Q})$ and consider the point $\frac{1}{\pi}P \in E(\mathbb{C})$. This point lies in an abelian extension of $K_0$ and as always class field theory describes such abelian extensions of $K_0$ in terms of data coming from the field $K_0$ itself.

Now we need to see how $L(E, 1)$ is related to all this. In the case of complex multiplication there is a canonical period on the elliptic curve denoted $\Omega$ which has the property that $L(E, 1)/\Omega$ is rational. It turned out that we could relate this value to the explicit construction of units in the ring of integers of the field $K(E_\pi)$. Using this and class field theory we showed that

\[ \frac{L(E, 1)}{\Omega} \neq 0 \mod p \Rightarrow \text{Gal}(K_0/K) \text{ is unramified.} \tag{7} \]

We suspected that the conclusion was false for most $\pi$ but we could not prove it. However we realised after a while that using Iwasawa theory, a similar claim with $\pi^n$ would also be true and in this case we could prove that the extension would be ramified for sufficiently large $n$. So given the existence of $P$ we found that $\frac{L(E, 1)}{\Omega} \equiv 0 \mod p$ for infinitely many $p$, and hence was zero. This contradicts the hypothesis, so there was no point $P$ of infinite order.

The influence of Iwasawa theory in the proof sketched above is only apparent at the end, but in fact it motivated the whole approach. John had used the Iwasawa theory of cyclotomic fields prominently with Lichtenbaum in his previous work on zeta values and Euler characteristics. From then on it would guide his approach to elliptic curves.

The idea in Iwasawa theory is to study not just $K_0 = K(E_\pi)$ but also the whole tower of fields $K_n = \bigcup_{\pi \in \pi^n} K_n$, where $K_n = K(E_{\pi^n+1})$. Let $M_n$ denote the maximal $p$-power abelian extension of $K_n$ which is unramified outside the prime above $\pi$ ($\pi$ turns out to be totally ramified in $K_n$ so this prime is unique), and set $M_\infty = \bigcup_{n=1}^{\infty} M_n$. Then set

\[ X_\infty = \text{Gal}(M_\infty/K_\infty). \tag{8} \]

There is an action of $\text{Gal}(K_\infty/K_0)$ on this given by $\sigma: x \mapsto \sigma x \sigma^{-1}$. Now $\text{Gal}(K_\infty/K_0)$ decomposes as

\[ \text{Gal}(K_\infty/K_0) \simeq \text{Gal}(K_\infty/K_0) \times \text{Gal}(K_0/K). \tag{9} \]

We write $\Gamma = \text{Gal}(K_\infty/K_0) \simeq \mathbb{Z}/p$ and $\Delta = \text{Gal}(K_0/K) \simeq (\mathbb{Z}/(p-1)\mathbb{Z})$. Then we decompose $X_\infty$ into eigenspaces for
the tame action of $\Delta$

$$X_\infty = \bigoplus X^{(x)}, \tag{10}$$

where $x : \Delta \to \mathbb{Z}_p^*$ runs through the characters of $\Delta$. Then each $X^{(x)}_\infty$ is a $\Gamma$-module. Now such $\Gamma$-modules can be described in a more familiar way as $\mathbb{Z}_p[[T]]$-modules where the action of $\gamma$ is the same as that of $1 + T$. Using results on the units in the fields $K_n$ it can be shown that there is a homomorphism

$$X^{(x)}_\infty \to \bigoplus_{i=1}^r \mathbb{Z}_p[[T]]/(f_i, x) \tag{11}$$

with finite kernel and cokernel. We set $(f_\chi) = (\prod f_i, \chi)$ and call this the characteristic power series of $X^{(x)}_\infty$. It is an invariant of the module.

At first encounter, this construction may seem artificial but it was suggested by Weil (in the cyclotomic case) and taken up by Iwasawa as an analog of a construction in the function field case of a module where the characteristic polynomial was related to the zeta function. So what corresponds to the zeta function in our case? This question is most interesting when $\chi = \chi_0$ is the character giving the action on $E_F$, i.e., where

$$\chi_0 Q = \zeta Q \text{ for } Q \in E_F, \quad \zeta \in \mu_{p-1} \subseteq \mathbb{Z}_p. \tag{12}$$

(Here $\zeta \equiv a \bmod p$ for some $a \in \mathbb{Z}$ and $\zeta Q = a Q$.) Similarly let $u = \chi(y) \in \mathbb{Z}_p$, where $\chi$ gives the action of $\Gamma$ on $E_F$. In this case the main conjecture (now a theorem due to Rubin) predicted

$$(f_\chi(T)) = (G_\chi(T)), \tag{13}$$

where

$$G_\chi(u^{k-1}) = \Omega_p^{-k} \mu_k(1 - \frac{\psi(p)}{p})L(k, \psi^{-k}), \tag{14}$$

where $\Omega_p \in C_p$ is a $p$-adic period, $\psi$ is the Grossencharacter associated to $E$, and

$$\mu_k = 12(-1)^{k-1}(k-1)! (\Omega/f)^{-k},$$

$f$ being a generator of the conductor of $\psi$. For our purposes $\psi$ is multiplicative on prime ideals of $\mathbb{Z}[i]$ and satisfies $\psi(\lambda) = \lambda$ for some choice of generator $\lambda$ in a prime ideal $(\lambda)$. One checks that $L(s, \psi) = L(s, E)$. The function on the right in (13) is called a $p$-adic $L$-function.

Although we were unable to prove the conjecture, we proved a related result that constructed the $p$-adic $L$-function from elliptic units. Let $U_n$ denote the local units of the field $K_n$ at the prime above $\pi$. We considered the elliptic units $\xi_n$ as a subgroup of $U_n$ and let $\bar{\xi}_n$ be the $p$-adic closure in $U_n$. We set

$$Y^{(x)}_\infty = \lim\limits_{\longrightarrow} U_n^{(x)}/(\lim\limits_{\longleftarrow} \bar{\xi}_n^{(x)}). \tag{15}$$

Then we proved the following theorem.

**Theorem 3 ([9]).**

$$Y^{(x)}_\infty \simeq \mathbb{Z}_p[[T]]/(G_{\chi_0}(T)). \tag{16}$$

Now $Z^{(x)}_\infty$ and $Y^{(x)}_\infty$ have enough in common that we could give a slightly different proof of Theorem 1 using just one prime (see also [3]). But perhaps the more important part was the proof which involved attaching a canonical power series to elements of $\lim\limits_{\longleftarrow} U_n$ and then applying logarithmic derivatives. This was motivated by an explicit reciprocity law of Iwasawa. These power series were studied and generalised by Coleman and became important in the study of local fields.

The study of Iwasawa theory and in particular of $p$-adic $L$-functions of this kind were the main focus of John’s work for many years after this. While elliptic curves were always his main interest he also tried to develop a theory of $p$-adic $L$-functions for motives (e.g., [2]) and studied particular cases such as the symmetric square of an elliptic curve (see [5]).

The study of $p$-adic $L$-functions in the context of elliptic curves had already begun with work of Mazur and Swinnerton-Dyer a few years before this, and $p$-adic versions of the Birch and Swinnerton-Dyer conjecture were formulated. It was hoped that these might be more tractable than the original complex version. However, the most important advances in the following decades were the work of Gross and Zagier on the analytic side and Kolyvagin on the algebraic side. This seemed to definitely move the focus of work on the conjecture to the study of modular forms, since that was the context of the seminal work of Gross and Zagier. Surprisingly, no one has been able to extend Theorem 1 even to the case of abelian surfaces with complex multiplication. On the other hand, the jury is still out on whether the $p$-adic versions are easier or harder than the original complex ones and whether the further study of the former will help in the study of the latter.

Leila Schneps

When John Coates came to Paris in 1978, it was not his first stay; as mentioned in the previous sections, he had already spent a year there as a graduate student, attending the famous SGA seminar of Alexander Grothendieck and trying to get started on a thesis before renouncing the idea due to a radical difference in style. Grothendieck speaks of Coates in his memoirs with a certain mea culpa:

... it happened that after a few weeks or months [he] found that my style didn’t suit [him]. Actually, it seems to me now that it was a case of a mental block, which I too quickly interpreted as a sign of unsuitability for mathematical work. Today I would be much more prudent in making such a prediction. I had no hesitation in telling [him] about my impression, and advising [him] not to continue in a career which did not appear to me to correspond to [his] natural abilities. But I learned later that I was completely wrong—this young researcher went on to become well-known thanks to his work in very difficult subjects at the frontier of algebraic geometry and number theory.

As was said earlier, Coates left Paris in 1966 for Cambridge where he completed his PhD with Alan Baker, followed by stints at Harvard (§2), then Stanford, then Cambridge again (§3), then Canberra. But in 1978, Georges Poitou of the École Normale Supérieure (ENS) in Paris organized a job offer for him at the University of Paris in Orsay. Poitou was gearing up to become director of the ENS, and with this aim in mind he wanted to wind down his activities of directing research, and thought that Coates would be just the person to fill the ensuing gap in Paris number theory.

Orsay is a small town located very near the IHES where Grothendieck’s seminar had taken place in the 1960s, and the university there was and still is one of the most prestigious of the Paris math departments. Poitou’s offer was interesting enough to make Coates decide once again to uproot himself. Naturally, the first thing he wanted to do when he came was to gather a group of graduate students to study questions arising from his recent work on the Birch and Swinnerton-Dyer conjecture with Wiles (§3), centered around $p$-adic $L$ functions and Iwasawa theory. To start with, Rod Yager, a student of Coates from Australia, came to join him in Paris, and Poitou sent him three of his own students who had completed their third cycle theses and were about to embark on their thèse d’état\(^1\). In the spring of 1979, Coates taught a course in Orsay on elliptic curves with complex multiplication, to spread the word and to recruit more students. After that, he continued to teach research-level courses on recent work each year, generally gaining a further new graduate student or two each time.

The topics that John Coates gave his students mainly concerned generalizations of the recent results proven with Wiles in the case where $E$ is an elliptic curve defined over a quadratic imaginary field $K$ with complex multiplication by the ring of integers $\mathcal{O}_K$ (see §3). The more general situations he considered in Paris were firstly the case of CM elliptic curves defined over a finite extension $F$ of $K$, and secondly the 2-variable $p$-adic $L$-functions, first defined by Katz in the case $F=K$, interpolating the values $L(k,\psi^k)$ instead of just $L(k,\psi)$. In each case, he proposed (a) showing that the values to be interpolated were algebraic, (b) constructing the $p$-adic $L$-function that interpolated them, (c) using it to state generalized versions of a number of conjectures, above all the “main conjecture” first given in his work with Wiles (see (13) of §3), and then various consequences of the main conjecture, considered as separate conjectures in their own right, (d) giving numerical evidence for the conjectures, and (e) proving any parts that could be proven directly. He viewed his advisees also as collaborators in this research program, and distributed problems among them in a purposeful and coherent way.

In the very first years of his stay in Paris, he worked with Catherine Goldstein on the case where $E$ is defined over a finite extension $F$ of $K$, and $p$ a prime that splits as $p = \pi\bar{\pi}$ in $K$ ([4]). They constructed the $p$-adic $L$-function $G_{\pi}(T)$ such that $G_{\pi}(u^k - 1)$ interpolates the values $L(k,\psi^k)$ for $k \equiv 1 \mod p - 1$ (suitably corrected for algebraicity as usual), formulated the main conjecture in that situation, and drew several consequences, for example:

A) if $E(F)$ contains a point of infinite order then $L(s,\psi)$ vanishes at $s = 1$,

B) if the subgroup $\text{III}_\pi$ of elements of the Tate–Shafarevitch group annihilated by a power of $\pi$ contains a nontrivial divisible subgroup (so in particular is not finite), then $L(s,\psi)$ vanishes at $s = 1$,

C) under certain hypotheses on $\pi$, if $L(s,\psi)$ vanishes at $s = 1$, then either $E(F)$ contains a point of infinite order or $\text{III}_\pi$ contains a specific divisible subgroup, namely a copy of $K_\pi/\mathcal{O}_\pi$.

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\(^1\)At that time, the French system had two theses following the master’s: the third cycle thesis that was expected to take about two years and produce a student’s first original publication, and the thèse d’état that gave confirmed researchers the right to have students of their own.

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when most students still wrote up their work by hand and John’s student Bernadette Perrin-Riou, who defined a $p$-adic pairing to the elliptic curve whose characteristic power series was related to Katz’s power series, stating and proving a 2-variable analog of Theorem 3 of §3 ([16]). Jacques Tilouine proved that if the normalized special value of the 2-variable $L$-function is $\not\equiv 0 \mod p$, then the rank of the Mordell–Weil group is zero over the maximal $\mathbb{Z}_p$-extension of $K$ ([14]). Later, Coates had me work together with Pierre Colmez on constructing the 2-variable $p$-adic $L$-function when $E$ is defined over a finite extension $F$ of $K$ ([11]).

As time passed, Coates’s style in choosing research topics changed. Being constantly in touch with researchers around the world and up-to-date on all the most recent results, he began to draw inspiration from interesting ideas that came his way—a conjecture stated in a lecture, or a preprint that landed on his desk. When Pierre Colmez first came to him in 1982 to do a master’s thesis, he asked him to give a complete proof of Zagier’s part of the yet-unpublished Gross–Zagier theorem based on a set of lecture notes he had taken. At the same time, he gave Bernadette Perrin-Riou the ambitious project of developing a $p$-adic analog of the Gross–Zagier theorem, which she did successfully in the case of an elliptic curve defined over the rationals with CM by a quadratic field $K$ (under the condition that the discriminant of $K$ is prime to the conductor $N$ of the curve, and that all primes dividing $N$ split in $K$), giving a striking formula valid for primes $p$ where $E$ has good ordinary reduction relating the first derivative of the $p$-adic $L$-function of $E$ to the $p$-adic height of the Heegner point for $K$. As for me, when I arrived in the fall of 1983—his very last graduate student in Paris, as it turned out—John immediately handed me a preprint by Warren Sinnott, a former student of his at Stanford, giving a new proof of the Ferrero–Washington theorem, and asked if I could adapt the new argument to find a proof that the $\mu$-invariant (the infimum of the $p$-adic valuations of the coefficients) of the $p$-adic $L$-function attached to a CM curve $E$ defined over $K$ was equal to zero ([13]). By that time, research in the whole subject was accelerating rapidly all over the world, and would lead in a few more years to the proof of the main conjecture by Karl Rubin ([19]).

Ken Ribet describes John as having a mission “to elevate young mathematicians around him—to promote their work and to invisibly improve it in various ways. He became an editor of Inventiones very early on in his career...”

In Dominique Bernardi, John was delighted to find a student who was also one of the rare mathematicians in France at that time who could handle a computer (back when most students still wrote up their work by hand and slipped it under their adviser’s door...), and he grouped Dominique and Catherine together with a visiting former student of Bryan Birch’s to work on numerical verifications of the conjectures for elliptic curves of the form $y^2 = x^3 − Dx$ ([11]). A former student of Coates at Stanford, Nicole Arthaud, gave a proof of consequence A) in the special case where $F$ is an abelian extension of $K$ and $E$ is defined over $K$ but has a point of infinite order defined over $F$. A particularly striking contribution was made by John’s student Bernadette Perrin-Riou, who defined a $p$-adic height on $E(F)$ and a $p$-adic pairing ($p$-adic versions of the canonical height and the Néron–Tate pairing), and under certain hypotheses proved that the zero of $L_p(s,E)$ at $s = 1$ has multiplicity greater than or equal to the rank of $E(F)$ over $\mathcal{O}_K$, and that equality holds if and only if the $\pi$-primary component of the Tate–Shafarevitch group is finite and the pairing is nondegenerate ([12]). To Pierre Colmez, John proposed an algebraicity conjecture for special values of $L$-functions associated to Hecke characters not necessarily the Grossencharakter, in the case of a CM curve $E$ defined over $F$ ([10]).

He also had several students working on the 2-variable $p$-adic $L$-functions first defined by Katz in the case where $E$ is defined over $K$. Rod Yager constructed an Iwasawa module associated to the elliptic curve whose characteristic power series was related to Katz’s power series, stating and proving a 2-variable analog of Theorem 3 of §3 ([16]).

Figure 3. Left to right: John Coates, Kazuya Kato, Sujatha, Otmar Venjakob, Takako Fukaya.
(in the late 1970s, I think) and would devote hours on end to rewriting and improving manuscripts that he wanted to publish but that he thought needed polishing. This was well before the era of laptops or even word processors; everything that he did was handwritten with a fountain pen.” Certainly, John had a real flair for picking out topics that were simultaneously cutting-edge and approachable by students, and virtually all of our work ended up as published articles, to the quality of whose writing up he devoted particular attention.

When I first met him (in December 1982) to talk about pursuing my graduate studies in Paris, he was affable and welcoming, as he was to every student who wanted to work with him, but he warned me—perhaps from the memory of his own first unsuccessful experience there—that for foreigners, “while Paris was one huge math seminar, there was no actual campus and no student life, and it was easy to feel lost.” Of course he was quite right, and of course I paid no attention—who would have? I came to Paris and plunged immediately into an intense and exciting period, filled with a perfect stream of new ideas and research results, lectures and courses on new papers, and summer conferences in Luminy in the South of France, where we would go hiking among the seaside cliffs and climb down to swim in the turquoise-blue calanques. There was also a constant, lively flux of short- and long-term visitors such as Norbert Schappacher, Gudrun Brattström, Peter Schneider, Leslie Federer, Nelson Stephens, Ralph Greenberg, Dick Gross, Barry Mazur, Andrew Wiles, Ken Ribet, John Tate, and many other younger and older researchers who came to give courses and lectures and sometimes to collaborate with his students—all people whose daily language spoke of elliptic curves, complex multiplication, Iwasawa theory, \( p \)-adic \( L \)-functions, Leopoldt transforms, Heegner points, Selmer groups and so on, while I was still struggling with French.

When John left Paris in 1986 to return to Cambridge, our tight-knit group scattered, and those who had not yet defended a thèse d’etat or even completed their third cycle thesis were compelled to find substitute advisers (Michel Raynaud, Guy Henniart, and Jean-Marc Fontaine all stepped up to the plate). But the subject he had started continued to flower. By launching the arithmetic of elliptic curves and Iwasawa theory in Paris, John Coates left an unforgettable mark.

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5. Iwasawa Theory in a Noncommutative Setting

**Mahesh Kakde and Sujatha**

5.1. Genesis of noncommutative Iwasawa theory. In the early 1990s, John Coates started exploring the formulation of the main conjecture of noncommutative Iwasawa theory. To systematically study the Iwasawa theory of elliptic curves and \( p \)-descent, it is natural to consider the extension of a number field obtained by adjoining all \( p \)-power torsion points of an elliptic curve. The algebraic study of such extensions was first taken up in the doctoral thesis of Michael Harris.

In the mid 1990s, John visited Ohio State University, where Sujatha was doing a postdoc. Their meeting led to John’s visiting the Tata Institute of Fundamental Research (TIFR), where he gave some lectures on Iwasawa theory; following the TIFR tradition, these were subsequently expanded in a TIFR Lecture Series books entitled *Galois Cohomology of Elliptic Curves*. The meeting with Sujatha gave rise to a fruitful collaboration lasting several years.

By the time that Mahesh became a graduate student of John’s at Cambridge in the early 2000s, he had developed an interest in the extension of Iwasawa theory to a noncommutative version, in which the main objects connected to the main conjecture needed to be defined.

For simplicity of exposition, we will only consider an elliptic curve \( E \) defined over the rational numbers \( \mathbb{Q} \). Furthermore, we fix a prime number \( p \geq 5 \). Assume that \( E \) has good ordinary reduction at \( p \). When \( E \) has complex multiplication by an order in an imaginary quadratic field \( K \), we have already seen that the field generated by all \( p \)-power torsion points of \( E \) is an abelian extension of \( K \). On

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the other hand, in the case when $E$ does not have complex multiplication, a celebrated theorem of Serre says that the extension of $\mathbb{Q}$ generated by all $p$-power torsion points of $E$ is a Galois extension of $\mathbb{Q}$ with Galois group given by an open subgroup of $GL_2(\mathbb{Z}_p)$.

Following the work of Venjakob on a good definition of pseudo-null modules over Auslander regular rings [15], Coates, Schneider, and Sujatha proved a structure theorem for finitely generated torsion modules over the Iwasawa algebra of a compact, $p$-valued, $p$-adic Lie group, much along the lines of the classical result [7]. However, it is unclear if this result can be used to define a characteristic element or even a characteristic ideal.

An important consequence of the main conjecture in classical Iwasawa theory is the formula for an Euler characteristic in terms of values of $p$-adic $L$-functions. The study of Euler characteristics of Selmer groups of elliptic curves without complex multiplication was undertaken by Coates with his graduate student Susan Howson. A noncommutative main conjecture would relate the Euler characteristic $\chi(G, Sel(E/\mathbb{Q}))$ with special values of the $L$-function of $E$, which would yield the $p$-adic Birch and Swinnerton-Dyer conjecture. Coates and Sujatha defined this refined $G$-Euler characteristic which was used to study the consequences of a noncommutative main conjecture. The extension $\mathbb{Q}_\infty = \mathbb{Q}(E[p^\infty])$ contains the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{cyc}$ of $\mathbb{Q}$. Set $H = Gal(\mathbb{Q}_\infty/\mathbb{Q}_{cyc})$ and $\Gamma = Gal(\mathbb{Q}_{cyc}/\mathbb{Q}) \cong G/H$. One of the major results of Coates in the joint work with Schneider and Sujatha [6] is the definition of the “Akashi series” (named after the Akashi chapter in The Tale of Genji, a favorite of John’s). Let $M$ be a $\Lambda(G)$-module that is finitely generated over $\Lambda(H)$. It is known that the homology groups

$$H_i(H, M) \quad (i \geq 0)$$

are finitely generated torsion $\Lambda(\Gamma)$-modules and are 0 for $i \gg 0$. Let $g_i(M)$ be a characteristic element of $H_i(H, M)$. The Akashi series for $M$ is defined by

$$f(M) = \prod_{i \geq 0} g_i(M)^{(-1)^i}.$$

These insights directly led to the definition of characteristic elements for an interesting class of modules over $\Lambda(G)$ in the habilitation thesis of Venjakob.

Venjakob observed that the right category to work with is $\mathfrak{M}_H(G)$, the category of finitely generated $\Lambda(G)$-modules $X$ such that $X/X(p)$ is a finitely generated $\Lambda(H)$-module. This category was first defined in the paper of Coates-Schneider-Sujatha.

The $\mathfrak{M}_H(G)$ conjecture of Coates and Sujatha says that the Pontryagin dual of the Selmer group of $E$ over $\mathbb{Q}(E[p^\infty])$ is in the category $\mathfrak{M}_H(G)$.

For simplicity, we make the additional assumption that $G$ has no $p$-torsion. This ensures that all finitely generated $\Lambda(G)$-modules have a finite projective resolution. Then we can define a characteristic element for every module in $\mathfrak{M}_H(G)$. It is an element in a certain $K_1$-group.

The main conjecture for $E$ can now be formulated as follows: There is a unique element $\mathcal{L}(E)$ such that

(i) $\mathcal{L}(E)$ is a characteristic element of the Pontryagin dual of the Selmer group of $E$.
(ii) at every Artin representation of $G$, the element $\mathcal{L}(E)$ interpolates the value at $s = 1$ of the $L$-function of $E$ twisted by the representation (appropriately normalized).

The insight of relating noncommutative Iwasawa theory to commutative Iwasawa theory turns out to be useful in tackling all known cases of the noncommutative main conjecture. This study was initiated by Kato. Calculations of $K_1$-groups imply that a necessary condition for the existence of a noncommutative $p$-adic $L$-function is a congruence between commutative $p$-adic $L$-functions over extensions corresponding to abelian sub-quotients of $G$. These congruences are between $p$-adic $L$-functions over different number fields and seem very different from earlier congruences between $L$-functions (for example Kummer congruences).

John placed a great deal of faith in these two predictions coming from noncommutative Iwasawa theory—the congruences between $p$-adic $L$-functions and the $\mathfrak{M}_H(G)$ conjecture. Both of these remain wide open. What progress is made on these conjectures and what role they play in the arithmetic of elliptic curves, only time will tell.

References


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Metacognition, a popular area in educational research, sparked an interesting panel discussion at the Joint Mathematics Meetings in Boston last January when it was applied to the teaching of math in the classroom. Annalisa Crannell from Franklin & Marshall College, a member of the Joint Committee on Women in the Mathematical Sciences (JCW) had suggested this topic and the JCW eagerly adopted it to organize a contribution to the meeting in Boston and to promote the JCW.

Founded in 1971, the JCW forms an umbrella organization for the various mathematical societies in our country: American Mathematical Association of Two-Year Colleges (AMATYC), American Mathematical Society (AMS), American Statistical Association (ASA), Association for Women in Mathematics (AWM), Institute of Mathematical Statistics (IMS), Mathematical Association of America (MAA), National Association of Mathematicians (NAM), National Council of Teachers of Mathematics (NCTM), and Society for Industrial and Applied Mathematics (SIAM).

The committee serves primarily as a forum for communication among member organizations about the ways in which each organization enhances opportunities for women in the mathematical and statistical sciences. The committee collects data, disseminates information, and facilitates discussion with a view toward developing best practices. (See https://jcwmath.wordpress.com.) Nancy Sattler (Terra Community College) and Jennifer Schultens (University of California, Davis) have been co-chairs of the JCW since 2020.

The term “metacognition” lacks the precise meaning we have come to expect in mathematics. Broadly speaking, the term refers to monitoring one’s thought processes. For example, in solving a mathematical problem, one might try different approaches, evaluate the likelihood of success of a certain strategy, spur oneself on. In a 2018 report by the National Academies How People Learn II: Learners, Contexts, and Cultures metacognition is described as “the ability to monitor and regulate one’s own cognitive processes and to consciously regulate behavior, including affective behavior” (p. 70, [7]). There are variations in the usage of the term metacognition. Most importantly, in the context of education, the term is used to describe scaffolding activities intended to promote metacognition, e.g., exit tickets, exam wrappers, keeping a learning journal. Teachers can design learning environments that include monitoring each student’s performance or learning environments in which students monitor each other’s performance. These types of environments are sometimes called metacognitive. It is worth noting that the thought processes of students in such environments are not, strictly speaking, metacognitive, but the hope is that they later become so.

While largely unfamiliar with research concerning metacognition, the members of the JCW eagerly adopted
this theme for the panel discussion at the 2023 Joint Meetings of the Mathematical Societies because it evoked so many moments of awareness, and the promise of navigating around stumbling blocks both in mathematical research and in career trajectories. As working mathematicians we are all familiar with the frustrations of facing difficult problems and the exhilaration of solving them. As members of marginalized communities, we more often face these stumbling blocks in isolation. Metacognition can help pinpoint inequality. Even without being experts in psychology, or mathematics education, and perhaps even without thinking in terms of concepts as refined as metacognition, members of the JCW immediately perceived the relevance of the concept and the benefits of a panel discussion on the topic.

As readers of the Notices are aware, the discipline of mathematics holds a central role in education by virtue of its unique combination of practical applicability (to fields as disparate as medicine, construction, financial planning) and training in rational thinking, distinguishing between true and false, and searching for universal truth. This central role goes hand in hand with a broader pedagogical function in society, as a potentially stabilizing force in public discussion. Yet to appreciate and develop mathematical ways of thinking, we must encourage students to reflect on how they think as well as what it means to engage in mathematics. The panel on Metacognition in the Mathematics Classroom examined teaching practices that do so.

Participants in the panel discussion were Jo Boaler (Stanford University), Lakeshia Legette Jones (Clark Atlanta University), Yvonne Lai (University of Nebraska-Lincoln), and John Nardo (Oglethorpe University). Jennifer Schultens moderated. After the panel discussion on Metacognition in the Mathematics Classroom, panelists were asked to summarize their contributions. Some summaries included recollections from the questions and answers portion of the panel discussion. We now provide these summaries.

1. Metacognition

Jo Boaler

In 1979, Stanford professor of psychology John Flavell created the theory of metacognition, and researchers have been investigating its impact ever since then. The word “meta” comes from the Greek prefix, meaning beyond, and metacognition regards the important processes that go beyond thinking, such as planning, tracking, and assessing. Flavell describes metacognition as including knowledge of ourselves, knowledge of the task at hand, and knowledge of strategies (Moritz & Lysaker 2018, see [6]), so it is no surprise that it boosts problem solving, and enhances mathematics achievement (Wilson & Conyers, 2016, see [8]).

In classrooms I find it easy to spot people who have learned metacognitive strategies and others who have not. I see some learners who are discouraged when they are given difficult challenges, assume that they cannot do well, and give up in the face of roadblocks. By contrast I see learners who are inquisitive and curious, they are eager to learn, and they appreciate diverse viewpoints. If they are stuck in a problem, they may circle back and think about what they know and need to know, or they may choose from other different strategies they have learned. Importantly, they enjoy the process of problem solving and learning. This complex combination of high-level problem solving, mindset, and planning that occurs when people are metacognitive, takes place in the anterior prefrontal cortex of our brains (Fleming, 2014, see [5]).

I see the potential of taking a metacognitive approach in three different areas of teaching and learning. First, the area most often associated with metacognition, is the self-awareness we have of our own learning and interacting. At youcubed we have developed a mindset rubric to help people engage in this important self-reflection.1 A second aspect of metacognition involves different ways of focusing on the task at hand, being willing and able to unpack it and think about what is involved. A metacognitive person will think in important ways—possibly going back to the question, considering what information is needed, thinking out loud, drawing the problem, or taking a smaller case. Someone who has developed and reflected on different strategies can choose among them, or try a few different approaches. When we taught 82 middle school students in a youcubed summer camp we taught the students these strategies as they worked on open tasks. At the end of the four-week camp the students had increased their achievement on standardized tests by the equivalent of 2.8 years (Boaler et al, 2021, see [3]).

The third part of metacognition involves assessment, and being able to track one’s own progress and reflect on what is needed to achieve goals. This is where teachers and parents play a critical role in setting out for students where they should be going, and ways to get there. The education leaders Paul Black and Dylan Wiliam, who proposed the approach, which they called “assessment for learning” defined it in these ways—communicating to students where they are now, where they need to be, and ways to close the gap between the two (Black & Wiliam, 1998, see [1]). One of my favorite strategies for assessing in this way is to give students a rubric, that sets out their mathematical

journey, and use the rubric to share feedback on where stu-
dents are, and where they need to be, and ways they can
get there. A K–8 school using this approach is shared on
youcubed, with some example rubrics.

The most mathematically empowered people in the
world take an approach to learning math that is different
from those who are less successful (Boaler, 2024, see [2]).
It is not typically the case that they achieve highly because
they were born with special advantages, but because they
have been given access to important approaches to learn-
ing. This selection of articles shares many of them.

2. Metacognition as a Soft Skill

Lakeshia Legette Jones

Unknowingly, I have practiced metacognition in the class-
room for quite some time. I have always been intentional
about my teaching methods but didn’t realize there is a
name for it. Metacognition was formally introduced to
me by my son’s first-grade teacher after noticing similari-
ties in our teaching styles. At that point, I began reading
and researching the benefits, best practices, and strategies
for implementation. I was pleased to find that a number
of the strategies are already a part of my practice.

For example, each class I recite the narrative as a form
of review, and to help students prepare for conceptual as-
essment questions. To help students understand the big
picture, I discuss the purpose of the class and the over-
all expected outcomes. I also outline the details. I reit-
erate the types of problems we can expect, the general pro-
cess toward solving such problems and the questions we
should ask ourselves in the process. I explain how the cur-
rent content connects to the previous and future content. I
also ask students to explain their work, provide reasoning
for their approaches and describe their thought processes.
In a class such as advanced calculus or other proof-based
courses, I ask students to communicate the best method
of proof, which definitions are needed, or what previous
results might be necessary.

Although it was great to learn that my teaching practices
are generally in a good ballpark, what I also learned is that
my strategies for promoting metacognition were largely be-
ing conducted orally. I was not allowing adequate oppor-
tunity for students to think about their processes or ques-
tion their reasons. They were not journaling or creating a
document that would grow and develop with them. Essen-
tially, my classes were lacking in formal, structured time
for reflection. Understanding that part of my responsibil-
ity is to help students discover how they learn best and
gain an accurate account of their strengths and areas of
improvement, I immediately committed to incorporating
time for written reflections. Now, among other low-stakes
summative assessments, I require students to keep a learn-
ing journal, participate in exit tickets and complete exam
wrappers.

What I notice are what I believe some of the greatest ben-
fits to strengthening students’ metacognitive ability. That
is, it also increases their agency and makes them proud to
accept responsibility for their learning. Students become
more aware of what learning strategies are most effective
and when more or less time is needed for grasping a con-
cept. Students are honest in their assessments and will ad-
mit to not studying enough when they knew more time
was required. They can more intelligently articulate what
they don’t understand and express when there are gaps in
their learning. They respect the notion that sometimes
struggle (without notes or other resources) is necessary.
There is an overall improvement in work ethic.

Although introducing metacognitive strategies to our
students is one of the greatest lifelong gifts we can share, I
think it is imperative that instructors understand that de-
veloping strong metacognitive abilities requires a level of
vulnerability from students. Therefore, care must be taken
to create a safe space and welcoming environment inside
the classroom. Students will be more likely to operate in
honesty and with an eye toward self-improvement when
they know there is sincerity and true care and concern for
their success.

Audience Question: Are the strategies different for ma-
jors versus nonmajors?

Answer: In my opinion, metacognition may be viewed
as the umbrella term that encompasses many of the other
soft skills we desire for our students, including commu-
nication, organization, leadership, work ethic, integrity,
time and stress management, and collaboration. Strength-
ening metacognition will consequently also positively af-
flect each of our soft skills in some way. With this under-
standing, it is clear that course content is less of a factor
in metacognition. We are contending with the self and
not necessarily the textbook, or course content. For this
reason, in a class filled with nonmajors, my strategies for
promoting metacognition are largely unaffected.

Audience Question: How do you know whether your
strategies are effective?

Answer: Ask the students! This is a great way to al-
low individual, as well as the collective student voice to
be heard, included, and considered. It is an explicit show
of inclusivity and creates buy-in from the students. They
appreciate any opportunity to contribute feedback that
will ultimately lead to a better learning experience. Stu-
dents find comfort in knowing their instructor is will-
ing to accommodate their most reasonable requests. The

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solicitation of feedback can be achieved in several different ways. For example, securing instantaneous feedback through Mentimenter, Poll Everywhere, iClicker, or similar platforms. There is also the option of an exit question or journal prompt.

After each major assessment I ask my students, not if it was easy, but if it was fair. They recognize and appreciate the difference in those two questions. The former speaks to their responsibility and the latter speaks to my responsibility. Students are refreshingly reflective in their responses to my question and 100% of the time state that the assessments are fair and within bounds. However, they also trust that if the answer is “no,” then I am willing to make adjustments to get it right. At multiple points of the semester, I also check in with students with a “temperature check” survey. It is interesting to notice the changes in responses as the semester progresses. Students become aware of strategies that no longer serve them well and realize others that are more helpful. They will earnestly respond to the effectiveness of a given strategy. I find solace in this because it lets me know they understand we are all playing for the same team.

3. Metacognition for Learning—and Teaching

Yvonne Lai

As mathematics faculty, we may hope for our students to know beauty through mathematics, and joy through mathematical community. In proof-based courses, we may hope for proofs to be a vehicle for mathematical challenge and fulfillment. We may also hope for students to begin to understand and embrace the reasons for why proof is essential to knowledge building in mathematics. Yet proof-based courses are a barrier to too many students, including prospective middle and high school teachers. These courses may also be disproportionately a barrier to women and students of color.

I am not a researcher in metacognition. However, I find the concept useful for teaching. I will argue here that developing students’ metacognition in mathematics learning requires developing our own metacognition in mathematics teaching.

Broadly speaking, to my understanding, metacognition can be thought of as knowledge and regulation. That is, mathematical metacognition involves knowledge about mathematical processes and one’s own thinking processes as they relate to mathematics. Moreover, mathematical metacognition includes how one might regulate these processes. I think about metacognition when I try to find ways to understand why mathematical proof and reasoning is so hard and frightening for students, and to find inroads to helping students to embrace proof.

Here is one example of a routine I use to open the proof and reasoning doors to future high school teachers and math majors. It is not my own; I learned it from Sameer Shah. I will then say my view of what metacognition has to do with undergraduate students’ learning and also instructors’ teaching.

Attacks and Counterattacks

Sameer Shah designed a brilliant extension of the idea of asking students to come up with the definitions of key terms.

Prior to reading Shah’s work, the way that I understood this idea was that students generated drafts of definitions and, akin to the fictional students of Lakatos’ though perhaps with more informal language, the students found examples to motivate reworking drafts. After enough drafts, a satisfactory definition would emerge. I’ve never been happy with this model. In practice, I’ve found that it takes too long for the result. Moreover, it seemed pedagogically inconsistent to ask students to go through this process, but then use a textbook or other source that had its own definition. Shah’s extension retains the advantages of students making definitions while also engaging them with the language of their own mathematical textbook.

For Shah, generating drafts of definitions (the “attacks”) and coming up with examples that satisfy the drafts but are not actually the desired object to be defined (the “counterattacks”) is only Part One. He proposes a Part Two and Part Three. In Part Two, you show students the definition from the textbook you are using (or another one), and ask them to reflect on the language chosen. In Part Three, you take these textbook definitions, cross out a condition of the definition, and ask them to produce a counterattack for the amended definition. (For instance, if defining triangle, you might cross out the condition that the vertices must be non-collinear.)

I have now used Shah’s materials (which ask students to define circle, triangle, and polygon) and adapted his routine to multiple other terms (including vector, inverse vector, angle measure, isometry, among others). My personal reflection is that through this routine, students reconfigure their relationship with mathematical language from receiving to doing. Part Two, in combination with Part One, helps them understand why mathematics is written the way it is. I find that their ability to read and understand
technical mathematical language, even outside of this routine, is changed. They are more patient with the stilted language that mathematics sometimes requires. They also see counterattacking as a way of unpacking the meaning of a definition, even when they know they are reading a textbook definition. It used to be that only the students with the most proof-based courses in their background would find mathematical errors; now I find that more students spot and are willing to bring up potential mathematical errors.

There may not be time in every mathematics course to do this routine for every definition. That said, I think it is worth doing at least once with every proof-based course. It is a way for students to get to know each other mathematically, and for you to see all students have the potential to see everyone—including themselves—contribute meaningfully to the classroom discourse. This routine is also a way for students to build for themselves an understanding of why mathematics might sound different from natural language. It is an invitation to talk about and ultimately participate in mathematical language.

**Of Horses and Zebras**

I read recently a New Yorker article referencing the medical school adage, “When you hear hooves, think horses, not zebras.” It means that when you are a doctor, and you observe particular symptoms in a patient, you should first think of the most common explanation for these symptoms rather than the most exotic one. I think that there is a lot of work that we as mathematics faculty can do to improve how we see “horses” and “zebras” in student thinking, especially if we use activities such as Shah’s Attacks or Counterattacks, or any other routine. I’ll use myself as an example.

When I first began teaching, the “horses” in my mind were diagnoses such as, “made a careless mistake,” “doesn’t know what they are doing,” or “isn’t reading the question.” While these diagnoses may be true in some sense, they are also unhelpful for taking action, because I’m describing what students are not doing rather than what they are doing, and also because this is like describing “horses” as “some animals of some sort that are not doing what I think they should be doing.” Now, when I teach introduction to proof or abstract algebra, some typical “horses” might be “needs support structuring a proof,” “has worked out examples to show the theorem and may not know how to generalize,” or “may think that ‘unit’ only refers to ‘1’ rather than any element with a multiplicative inverse.” In general, I have tried to shift my own way of observing towards describing both what I see students doing as well as what they might move toward doing, rather than only the latter. In this way, I can begin building on what they are doing rather than only imposing my own ideas of where they “should” be.

When it comes to “zebras,” I have in my early years thought that a student’s question alluded to open or difficult problems, like thinking that a student was asking about Fermat’s Last Theorem when they were only asking about a detail of the Pythagorean Theorem. Although it does happen that students ask about mathematics years beyond what we are formally studying in the course, we should first check whether it is horse. And only after we are sure that it is not a horse, should we hypothesize that it is a zebra.

When we teach proof-based courses, we have the opportunity to shape—for better or worse—the mathematical experiences of a future generation. I also try to remember that students include future parents and teachers. In finding ways to tune my own metacognition for teaching, I try to think about how my own ways of interpreting student talk and thinking can help me be more responsive to what students need for a more joyful, beautiful, and community-oriented experience.

4. **How to Begin Using Metacognition to Help Your Students Thrive**

**John Nardo**

As a professor at a small liberal arts college, I proudly view myself as a teacher-scholar, and I approach this work on metacognition in mathematics firmly through an applied lens. My focus on metacognition is central to my career-long goal: to help students learn and grow. By thinking and writing about their learning, my students have gained both confidence and skills; this metacognitive work has also helped shift their view of mathematics from a collection of algorithms and processes into a richer, integrated view of our field.

I am privileged to have a close cohort of colleagues who are experimental in the classroom, and we share our successes and challenges frankly with each other. We visit each other’s classrooms often. I routinely adopt best practices of others and implement them in my own classes; many of my ideas have been improved after talking them over with my peers or hearing how they implemented them differently in their classrooms. I have benefitted as both generator and receiver of ideas.
Even after decades in the profession, I struggle with perfectionism in my work. Often, I have wanted to make sure that an activity or assignment is “just right” before deploying it to students. I have lately realized the trap in this type of thinking: perfectionism can make me timid and hinder my actions to innovate in the classroom.

I have learned that students can be potent allies in course reform. You do not have to wait until official course evaluations at the end of the semester: you can use your own informal “check in” surveys. Students are not shy in telling you how class is functioning. Fully engaging with student feedback, by mentioning (anonymously) things they have shared or modifying a class strategy based on their insights, can strengthen their sense of belonging.

To get started, it helps to think small. The commitment to sustain multiple experimental approaches over a semester can be daunting to the point of inaction. Pick a single activity or assignment for experimentation and see how it goes. Here are some experiments that have enriched my students’ learning.

**General education.** In Oglethorpe’s “Mathematics and Human Nature” course, writing and reflection are central. We start the course with a tried-and-true activity: writing mathematical autobiographies. This fortifies students’ identities as mathematicians and exercises demons from past interactions with our discipline. This often cathartic experience is revisited multiple times through discussions and revisions; thus, it helps them take agency in learning mathematics. When students share their biographies in small groups, they see common experiences and build community. They realize how quickly they attribute their successes or failures to other people instead of taking ownership of their education. Rather than a “one and done,” professor-validated assignment, this essay becomes a living, breathing document, and students become agents for change in our beloved discipline.

**Transition to the major course.** To encourage metacognition for majors, I use a summative portfolio assignment instead of a final. The portfolio showcases the proof techniques learned in class on a subset of the battery of problems given at the beginning of the semester. At regular checkpoints, students practice communicating formal mathematics by submitting solutions for feedback. Discussing how to tackle problems improves both grades and proofs; it helps to instill pride in their increasing sophistication. These reflections and discussions are routinely praised in course evaluations. Moreover, the portfolio is a resource for their future success. When struggling in an upper-level course on a proof technique, why go to a textbook or the internet for an example? They can refer to their own work in the portfolio to refresh their familiarity with the proof technique and to jump-start their thinking.

**Introductory/intermediate-level courses.** I adopted a colleagues’ suggestion to allow variable point allocations on take-home tests: students pick point values from target ranges. Students pick a grading scheme that aligns with their conceptual status, thereby celebrating their successes and minimizing the fallout from their challenges; it has also allowed them to have agency in even their high-stakes work. To reduce anxiety and encourage reflection over the entire course, I have also changed the format of final exams. Each final is explicitly structured with sections covering the material from each unit test along with a section for the material since the last test. The final is scored on its own merit as a whole, but students are allowed to pick a section to replace the corresponding test, if it is better. They know about this opportunity ahead of time. In making that selection, they must balance the desire to target the section corresponding to their lowest test grade with an assessment of which section on the final will have the highest success and percentage score when tallied by itself.

These are just a few experiments that I have tried in my teaching, and my students have encouraged me to keep experimenting. Students generally respond with grace when experiments fall short, knowing that we will improve things together (or in the next iteration of the class).

**Question:** How can you be supported to do this metacognitive work if you have a very traditional department or lack a group of supportive peers?

**Answer:** My close professional interactions keep me energized, especially in the wake of challenges posed to education/educators during the COVID-19 pandemic. However, not everyone has a supportive department; traditionally oriented departments make it difficult for innovative work by contingent faculty or beginning tenure-track colleagues. If you are not fortunate to have a close department or one open to experimentation, then you can grow an external network online through the video technologies normalized during the pandemic. Surround yourself with people who will both challenge and nurture you.

The discussion proved both lively and informative. We invite you to look out for our next panel discussion, to be held on Thursday, January 4, 2024, 3:00–4:30 p.m., at the Joint Mathematics Meetings in San Francisco. If you have a topic you would like to suggest, please write to us at jcs@math.ucdavis.edu or jcw-comm@ams.org.

**References**


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**Credits**

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WHAT IS...?

a Skew Brace?

Leandro Vendramin

A skew brace is a fascinating mathematical structure that involves a pair of compatible groups sharing the same underlying set. The notion of skew braces was inspired by a class of rings introduced by Jacobson in 1945, which we will discuss in more detail later. As algebraic structures, skew braces exhibit similarities to both groups and rings.

What makes skew braces particularly interesting is their ability to serve as an algebraic framework for exploring combinatorial solutions to the Yang–Baxter equation. By delving into the world of skew braces, we can uncover new insights and approaches to understanding this fundamental equation.

Let us begin by understanding what we mean by combinatorial solutions to the Yang–Baxter equation.

We are interested in pairs $(X, r)$, where $X$ is just a set and $r : X \times X \to X \times X$ is a bijective map that satisfies a specific equation in $X \times X \times X$, called the Yang–Baxter equation:

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

This equation may look a bit abstract, but there is a nice way to think about it:

![Figure 1. The Yang–Baxter (or braid) equation.](image)

The arrangements of strings, representing the Yang–Baxter equation, should be read from top to bottom, with the crossing symbolizing the application of the map $r$ and the straight line representing the identity mapping. The picture itself is self-explanatory.

Because of this braiding-like behavior, the equation is also known as the braid equation.

The identity map on $X \times X$ satisfies the Yang–Baxter equation. However, without additional assumptions, the task of finding solutions becomes highly unpredictable. Therefore, we will focus on solutions that meet specific extra assumptions. Given the combinatorial nature of the problem and our intention to utilize group theory, it is compelling to investigate the following intriguing class of solutions. We say that a solution $(X, r)$, where

$$r(x, y) = (\sigma_x(y), \tau_y(x)),$$

is nondegenerate if the maps $\sigma_x : X \to X$ and $\tau_y : X \to X$ are bijective, for all $x, y \in X$.

With the nondegeneracy assumption, exploring solutions to the Yang–Baxter equation becomes even more intriguing as we can now leverage groups that inherently act on our solutions. This opens up new avenues for investigation. By incorporating group actions, we gain a richer understanding of the equation’s behavior and its connections to various mathematical structures.

We can find many examples of solutions:

(a) If $\sigma : X \to X$ and $\tau : X \to X$ are commuting bijections, then $r(x, y) = (\sigma(y), \tau(x))$ is a solution. In particular, the flip map $r(x, y) = (y, x)$ is a solution.

(b) If $X$ is a group, then $r(x, y) = (y, y^{-1}xy)$ is a solution.

Definition 1. A skew brace is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are (not necessarily abelian) groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in A$. The groups $(A, +)$ and $(A, \circ)$ are respectively the additive and multiplicative group of the skew brace $A$. 
In one of the groups, we embrace the use of additive notation, even when our group is not necessarily assumed to be abelian.

Radical rings have their roots in Jacobson’s work. Subsequently, Rump unearthed an algebraic structure encompassing Jacobson radical rings as examples and coined the term *braces*. This structure proposed by Rump was further expanded upon in [3], leading to what we currently know as *skew braces*.

Many familiar mathematical objects have skew brace structures. For example, groups trivially produce skew braces. Example 2. If $G$ is a group, the operations $x + y = xy$ and $x \circ y = xy$ define a skew brace structure on $G$.

Now let us explore Jacobson radical rings. Imagine we have a ring $R$. In this ring, we can define a new operation—called the Jacobson circle operation—that takes two elements, let us say $x$ and $y$, and maps them to

$$x \circ y = x + xy + y.$$  

What is surprising is that this operation is always associative with the zero of the ring being its neutral element. When $(R, \circ)$ is a group, we say that $R$ is a radical ring. For instance, nilpotent rings, such as rings of strictly upper triangular matrices, are Jacobson radical rings.

Example 3. The subset

$$\left\{ \frac{2x}{2y + 1} : x, y \in \mathbb{Z} \right\}$$

of the rational numbers is a radial ring with the usual addition of rational numbers and circle operation

$$u \circ v = u + uv + v.$$  

Inverses of elements with respect to the circle operation are given by

$$\left( \frac{2x}{2y + 1} \right)’ = \frac{-2x}{2(x + y) + 1}.$$  

Now, here is the exciting part discovered by Rump: Jacobson radical rings are also examples of skew braces. When considering a radical ring, the combination of its addition with the Jacobson circle operation transforms the ring into a skew brace.

It is time to unveil the fascinating connection between skew braces and solutions to the Yang–Baxter equation.

The reader is encouraged to explore the solutions obtained by applying Theorem 4 to the skew braces described in Examples 2 and 3.

For a solution $(X, r)$, we define the structure group of $(X, r)$ as the group $G(X, r)$ with generators $X$ and relations

$$xy = uv$$

whenever $r(x, y) = (u, v)$.

In the upcoming example, we will express permutations as products of disjoint cycles. For instance, the symbol $(123)$ denotes the bijective mapping from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, where $1$ is mapped to $2$, $2$ is mapped to $3$, and $3$ is mapped to $1$.

Example 5. Let $X = \{1, 2, 3, 4\}$ and

$$r(x, y) = (\sigma(x), \tau(y)), \text{ where}$$

$$\sigma_1 = (12), \sigma_2 = (1324), \sigma_3 = (34), \sigma_4 = (1423), \tau_1 = (14), \tau_2 = (1243), \tau_3 = (23), \tau_4 = (1342).$$

Then $(X, r)$ is a solution. The group $G(X, r)$ has generators $x_1, x_2, x_3, x_4$ and relations

$$x_1^2 = x_2 x_4, \quad x_1 x_3 = x_3 x_1, \quad x_1 x_4 = x_4 x_3,$$

$$x_2 x_1 = x_3 x_2, \quad x_2^2 = x_4^2, \quad x_3^2 = x_4 x_2.$$  

This group admits the following faithful linear representation:

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$x_3 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_4 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

Each matrix’s first principal $4 \times 4$ block contains a permutation matrix. For instance, in the matrix associated with $x_1$, the block contains the permutation matrix corresponding to the permutation $\sigma_1$.

The linear representation we observed in the previous example was first found by Etingof, Schedler, and Soloviev and can be now explained by the theory of skew braces.

Theorem 6. Let $(X, r)$ be a solution. Then there exists a unique skew brace structure over $G(X, r)$ such that $r_{G(X, r)}(x \times i) = (i \times i)r$, where $i : X \to G(X, r)$ is the canonical map. If $r^2 = id_{X \times X}$, then the additive group of $G(X, r)$ is abelian and the map $i$ is injective.

The previous theorem reveals a profound connection between solutions and skew braces. It uncovers a hidden bridge that connects the world of combinatorial properties of solutions to the algebraic properties of skew braces and vice versa.
Example 7. Let us revisit Example 5. The map from $X$ to $\mathbb{Z}^4$,

\[
1 \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad 2 \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad 3 \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad 4 \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]

can be extended to a bijection between $G(X, r)$ and $\mathbb{Z}^4$. This bijection is a 1-cocycle, with the action of $G(X, r)$ on $\mathbb{Z}^4$ induced by the permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. The additive group structure of $G(X, r)$ is isomorphic to that of $\mathbb{Z}^4$.

It is remarkable how seemingly different realms of mathematics are intertwined, providing us with new opportunities to explore the intricate relationship between combinatorial and algebraic concepts.

Skew braces offer an appealing framework to explore various mathematical problems that may not initially seem connected to the solutions of the Yang–Baxter equation. Examples are the links between skew braces and Lie theory or Hopf–Galois structures. It is amazing how these apparently unrelated concepts can come together and inspire new insights in mathematics.

References


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1. If $\lambda : G \to \text{Aut}(A)$ is a group homomorphism, a 1-cocycle is a map $\pi : G \to A$ such that $\pi(xy) = \pi(x) + \lambda(x)(\pi(y))$ for all $x, y \in G$. 

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With great curiosity and summer enthusiasm, I hold the book in my hands. It is relatively light for a hardcover publication, excellent for reading during a casual vacation or other travels. Just a brief glimpse inside displays numerous pictures and photos. To my surprise, the book is in the grayscale; with growing curiosity I want to read it and see how the authors handle explanations of the history of the map-coloring problem without using colors. The foreword, written by Gary Chartrand, explains that the motivation for the book came from a question of one of his doctoral students, who inquired about the origins of math concepts and the way they developed. I appreciate this conscientious frame of mind, and I found myself wishing I used a more comprehensive and historical approach to mathematical concepts regularly with my students. In the Introduction, the authors clarify that the book is based on a doctoral dissertation of David J. Parks (the third author) written under the supervision of Robin Wilson (the first author) but gives no hints of the involvement of John J. Watkins (the second author), leaving the reader curious about the process of writing and publishing the book. The Introduction is followed by a list of Featured Papers, i.e., papers summarized later in the text. These summaries stand out in the text due to their placement on a darker background. The next part of the book is the Chronology of Events, which gives not only the years of the mathematical events featured in the book (between 1876 and 1976) but also provides additional historical brackets as early as 1636 and as late as 2021. At first impression, the book appears very much like a history book, but a deeper dive reveals the presence of rigorous mathematical proofs and a very neat mathematical formalism. Thus, it could make a good textbook for a course about the history of graph theory or a supporting textbook for a course in graph theory or discrete mathematics. Compared to literature in the area, the idea of presenting graph theory with significant historical background somehow resembles Biggs, et al. (1976) which covers the history of graph theory from 1736 to 1936. This book provides significant updates for recent history and includes many more nonmathematical aspects. In comparison to a standard textbook, written for example by Gary Chartrand himself (Chartrand, Zhang (2012)), this book does not offer as much variety in math topics, examples, and proofs. However, it is unique because it harmoniously weaves together threads of various natures: politics, society, colorful characters, wars, economics, and most importantly, mathematics.

The book is framed in American mathematics but includes necessary aspects of European episodes. It seems the authors were intentional in their placement so as not to disturb the logical and chronological flow. These European chapters are entitled “Interlude A” and “Interlude B.”
Here, the European and American threads intertwine in a way that resembles my own life lived on two continents simultaneously. In this review, I will highlight not only the historical journey of a proof that spans centuries, but also my summer trip to Europe, which overlaps with my journey through the book.

**The Development of the Four-Color Theorem and Other Historical Elements**

The story begins with early American colleges of the seventeenth and eighteenth centuries; the oldest among these is the College of William and Mary in Williamsburg, Virginia. It is well-known that early American colleges were created based on traditional English schools and oriented toward teaching. It was not until Johns Hopkins University hired J.J. Sylvester that encouragement and facilities for research, especially mathematics research, began. The book provides a brief description about the funding of the institution and then a longer depiction of the personality of Sylvester. It highlights his writing style, which tightly and skillfully combined solid academic statements with human emotions, feelings, and impressions. Here is a quote from Sylvester’s paper published in 1878 displaying his eloquent style and a highly integrated mind:

> In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry, the idea enveloped in matter, depending in part on manual processes and the resource of art for its due manifestation.

Sylvester’s drawings of chemical diagrams presented in the book motivated work in algebra and graph theory. While this is a book about graph theory, it also includes tangential topics incorporated without chaos. Truly, after reading the first chapters it is already evident that only a highly integrated mind, in this case, three integrated minds, could create such a highly integrated book, without cacophony or even the slightest dissonance.

Another interesting feature of the book is the display of the first pages of historic articles and journals. For instance, early in the book we see the opening page of the first issue of the *American Journal of Mathematics*, published in 1878. Here names of well-known editors with colorful personalities whose complicated human interactions (also described in the book) are displayed in an excessive number of fonts. The book also touches upon the first appearance of *Mathematical Reviews* in 1940 by including its front page and the announcement. In opposition to the *Zentralblatt für Matematik*, the *Reviews* were beyond the political and antisemitic disturbances of Europe, and their goal was to publish reviews of every mathematical research publication.

Nearly mundane, practical applications of actual map coloring performed by Alfred Bray Kempe motivated the problem of the coloring of maps with just four colors so that no two neighboring countries are colored the same. Amazingly, the authors reflect on the entire process of proving the coloring theorem, including the incomplete proof by Kempe in 1879. Reading about the process of creating the proof across centuries felt very educational; upon reflection, I am surprised this is often omitted from the college curriculum. Major threads of the proof are presented in detail, including Kempe’s patching process and further notes on that topic by W.E. Story. Human thinking has its meanders, and a generalization might illuminate a particular case. Percy John Headwood began considering the coloring problem on surfaces with different genus. Another generalization by George D. Birkhoff (in 1912) expresses in terms of chromatic polynomials the number of ways a given map can be colored with a given number of colors. Sister Mary Petronia Van Straten (whose photo is on page 163) is the only female noted in the book. Her contributions are related to the topology of graphs.

As World War I begins, the questions of mathematicians reflect the real-life experiences that brings more technology and different motivations to math problems (for example, in airplane designs). It was the war that delayed the publication of Oswald Veblen’s presentation *Analysis Situs* for Colloquium Lectures in Cambridge with his newly baked algebraic ideas that were later transformed by J.B. Listing to algebraic topology (topological complexes) and by G.R. Kirkhoff to electrical networks. At the same time, in another part of Europe, due to the outbreak of World War I, Kazimierz Kuratowski was not able to return from Poland (where I visited last summer) to his engineering studies in Scotland. Instead, as soon as the University of Warsaw reopened, he enrolled as one of the first mathematics students. Later, his characterization of planar graphs was published nearly at the same time as a similar proof by Orrin Fink and Paul A. Smith. As some of us know, simultaneous discoveries are a painful feature of mathematical research.

Partway through the text, the authors present a list of topics discussed in the book until this point. This roots the reader in the multitude of topics related to the four-color theorem and other milestones in the development of graph theory.

Multiple quotations enrich the text, giving a flavor of the style of writing and thinking of the mathematicians discussed. For instance, Hassler Whitney explains why he could not become a physicist by saying, “So I soon decided that since physics required learning and remembering facts, which I could not do, I would move into mathematics” (p. 123).

I was inspired by this quote as it could be inspirational to students and hopefully direct some of them to study mathematics. Among other contributions, Whitney
improved the work of Birkhoff, his doctoral advisor, by applying to the chromatic polynomial the principle of inclusion and exclusion, which is now part of an introductory combinatorics course. In addition, he related graph theory to linear algebra, which is explained in this book with a list of correspondences.

The Overlap with My European Journey
This book is not constrained to only historical content; mathematical chapters are shuffled with historical chapters about World Wars I and II. These are complemented by descriptions of the human aspects of research such as human error and pride and social aspects such as lack of funding or social pressures. When I myself was leaving Hamburg, Germany, last summer, I read another surprising and lesser-known fact: Emil Artin was a refugee from Hamburg. Being married to Natalie Jasny, who was half Jewish, Artin had no tolerance for the growing antisemitism in Germany. He came to the US when he was forced to leave his job due to the famous paragraph 6 of the Act to Restore the Professional Civil Service issued by the Nazi government. I promised myself to search for traces of Artin in Hamburg when I visit next time.

The problem of flows in networks was motivated by a secret report on the Soviet Union and Eastern Europe performed by Ted Harris and was formulated as follows:

Consider a rail network connecting two cities by way of a number of intermediate cities. If each link of the network has a number assigned to it, representing its capacity, find a maximal flow from one city to the other.

Seeing in the book a map of Eastern Europe illustrating the problem made me feel sentimental. While on a ferry from Poland to Sweden last summer, I realized that all of the countries I visited on my vacation could be located on this map. As an application of graph theory, I could apply a search algorithm to determine if my trip had been planned efficiently. It seemed very fortuitous to me that the map Ted Harris posed in his paper inspired me to consider the problem of efficient routes from one city to another.

Graph theory got a serious boost from the introduction of computers and programming. For early computers, the question of complexity of algorithms was particularly crucial. The input size influences the running time and could force researchers to wait for their results until the end of the universe if they did not consider the complexity of the algorithms. In 1971 Stephen Cook introduced the satisfiability problem which can be solved in NP time and has the property that a vast number of other NP problems can be converted to it in polynomial time. The list of NP problems is long and contains, for example, the travelling salesman problem, determining the existence of a Hamiltonian cycle in a graph, and the question of whether two graphs are isomorphic.

Again, my travels aligned with the book when I was reading about Heinrich Heesch receiving his doctorate from the University of Zurich and his work on the four-color theorem. Heesch combined the ideas of unavoidable sets of configurations and reducible configurations into the idea of unavoidable sets of reducible configurations. He was hoping to prove the four-color theorem by checking ten thousand configurations, literally testing one per day. Wolfgang Haken had the idea to use computers to check the configurations, though he ran into the problem of prolonged waiting time for large sets. Once back in New York City and having returned to my academic work at CUNY, I read that Kenneth Appel, a CUNY graduate from Queens College, offered his expertise in programming and his contributions allowed much faster progress. Page 238 contains a sample of reducible configurations with original computer printout. The next page contains 35 drawings of reducible configurations from the paper which announced the proof that every planar graph is four-colorable. This, however, does not end the book. The chapter entitled “Aftermath” summarizes the consequences of the growth of graph theory. Multiple conferences and proceedings were followed by numerous books on the four-color theorem and more generally, graph theory. In the explosion of graph theory discourse, new research directions and new connections were developed.

Conclusion
The book concludes with a few closing sections. An alphabetical Glossary of mathematical terms, algorithms, and theorems allows a reader to quickly find a forgotten term. Notes, References, and Further Reading contain a list of articles and books organized by chapter. Acknowledgements and Picture Credits (somehow placed together) attribute shared efforts of multiple institutions and individuals from America and Europe. Pictures of mathematicians, their hand drawings, and first pages from their papers are listed here with helpful attributions. The book ends with an Index of terms used in the book.

I know that it was not the intention of the authors but in my humble opinion, adding some open problems for the reader to consider would add a lot to this already rich book. I wish that possible future editions of this book would accommodate colorful pictures. I hope that the authors will continue writing and publishing in this style.

To summarize, the book is an excellent display of the detailed mathematical history of the map coloring problem and other accompanying problems, such as graph planarity and P- or NP-completeness. It is a blend of mathematical and historical facts that make an accessible read for interested students at a high school or college level. I would say that anyone with any background in
mathematics would benefit from reading this book. I enjoyed being immersed in the type of exposition which highlights not only the topic itself but places it in the historical and biographical context across the centuries and countries, and I believe many college instructors could incorporate it into their teaching. I am hoping that new generations of mathematicians will grow more aware of the historical and epistemological aspects of mathematics, as I did after reading this book. I wish for more publications like this.

References


Małgorzata Marciniak

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Super Courses
The Future of Teaching and Learning

The concern in some classrooms seems to be that students are rarely motivated to learn. Sure, they may be motivated to earn a good grade to keep their GPA up to be eligible for a good job [insert desirable career field here]. The classic carrot-and-stick approach to learning and grades has been, well, the bread and butter of education. Food clichés aside, Super Courses examines deeper questions behind student motivation; it asks, “What can we do with classrooms of disengaged students? How can we motivate students to learn?”

A “super course,” as Ken Bain explains, does not rely solely on innovative grading structures, active learning techniques, or long lists of course objectives. Rather, it fosters natural critical learning, which is an environment that promotes collaboration, productive failure, and an investment in a goal larger than the course itself, among other traits. The other super courses in the book are from a variety of disciplines. Honestly, forming or teaching a super course seems daunting, and yet, I’m certain there are mathematicians with the drive to create such a course.

This book issues a challenge to those excellent teachers among us: how can we create the most effective learning environment for our students and give them skills that will be the most useful to them in their future careers? Yes, they may need to know practical mathematics as well as mathematics that will be used in other STEM courses. In addition to that, how can our math courses promote the type of critical thinking and problem solving that will be necessary in a future society we can hardly imagine? If you’re intrigued, let me know what you think of this book. And I’d love to hear your ideas for a super course as well.

The Norton Guide to Equity-Minded Teaching

The generation of students we see in our classrooms in 2024 is notably different from 5 or 10 years ago. Societies are constantly changing and evolving, and with a changing student body comes a need to shift higher education. Students today seek connection, not only with the material in their courses but also with their peers and instructors. To be an effective teacher of today’s students, it is vital to nurture connection, be transparent about the purpose of activities or assignments, and welcome all students to your class. If you want help achieving these goals, look no further than this guide. Each unit in The Norton Guide to Equity-Minded Teaching starts with what the research says about a pedagogical topic. Then, the authors offer suggestions for how you can put the research to work. Many of the suggestions are small, doable ideas. One that I have incorporated into a course is to have students assist in creating a rubric for one of their project-based assignments. The guide includes tips for in-person, hybrid, and fully online courses. If you are seeking larger changes to your course, there are examples to help you change your assessment strategies, write a welcoming syllabus, and add structure to existing activities. Even if you feel confined within a coordinated course with very little freedom, this guide has suggestions for you. In short, this book has four powerhouse authors, three sections, and countless ideas to improve your classroom. I read this guide as part of an interdisciplinary faculty book club, and I enjoyed engaging with educational issues both within and outside of mathematics. This book is for any faculty member or administrator in higher education. I hope it inspires you to try something new in your classes, as it did for me.
Teaching and Learning with Primary Source Projects
Real Analysis, Topology, and Complex Variables

Teaching undergraduate real analysis can be a delicate balancing act between intuition and rigor. On the one hand, the whole point of the course is to erect the formal machinery underlying the calculus in a rigorous way. On the other hand, if you operate solely in a formal way, it is notoriously difficult for the average undergraduate math major to appreciate what you are doing and why you are doing it. You need to include intuition and motivation and explain the need for the formality. And the details of the rigor can seem to students to have been handed down from Olympus, where did the $\varepsilon - \delta$ definition of continuity come from? Who ever thought that up, and how? It can be tempting to motivate the material by presenting it in its historical context. The problem is that most of us don’t know the details of the historical development and they can be difficult for nonexperts to master.

Enter the elaborately anagrammatical NSF-funded TRIUMPHS project: TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources. The project, organized by a collective of mathematicians and historians of mathematics, developed a collection of Primary Source Projects (PSP) designed to be inserted into a course that develops a topic with historical motivation and context. Every PSP contains an essay providing context, an excerpt or excerpts from one or several historical documents, and a skillfully scaffolded series of student exercises to help the student unpack the excerpt and relate it to the modern understanding of the topic. Importantly, the point is not to teach the history, the point is to teach the relevant mathematical idea in a rich historical context that provides motivation. Every PSP has been rigorously edited and classroom-tested in multiple institutions. Each also includes several pages of teaching notes and recommendations for instructors along with helpful suggestions for further reading.

This volume contains two dozen PSPs for use in real analysis, topology, and complex analysis classes. For example, in one PSP we can read D’Alembert’s early attempt to define the limit concept. This is entirely verbal, illustrated by the example of polygons circumscribing and inscribing a circle, and it explicitly requires approaching the limit from a single side. There are eight exercises presented after this excerpt focused on making D’Alembert’s description precise by our standards and contrasting it to the modern definition of a sequential limit. In another PSP, we read Abel’s letter to Holmboe in which he describes divergent series as “devilish” and challenges Holmboe to explain what goes wrong with the Fourier sine series for $\frac{x^2}{2}$ at $\pi$. In yet another, students are asked to unpack the Darboux–Hoüel correspondence in which Darboux goads Hoüel to reconcile the latter’s explanation of a derivative with the behavior of $x^2 \sin(\frac{1}{x})$ at zero. There are, as noted, two dozen of these, but you would be cheating your students if you never showed them Euler’s derivation of $e$ presented here. He starts by asking us to consider the expression $a^\omega = 1 + k \omega$ for “infinitely small” $\omega$. After proving that when $a = 10$, then $k = \ln(10)$, he asks us to consider what choice of $a$ will give $k = 1$. It is full of beautiful Eulerian algebraic prestidigitation and the student tasks are to explain and unpack all the wizardry.

Use of these modules enables a nonhistorian to present, in a historically accurate way, the development of core ideas in analysis, topology, and complex analysis. The student tasks required here are deep, rich, and challenging. This is inquiry-based learning through a historical lens with an extraordinary level of care given to effective pedagogy.
Abel Interview 2023: Luis Ángel Caffarelli

Bjørn Ian Dundas and Christian F. Skau

Bjørn Ian Dundas/Christian F. Skau: Professor Caffarelli, firstly we want to congratulate you on being awarded the Abel Prize for 2023 for your “seminal contributions to regularity theory for nonlinear partial differential equations, including free boundary problems and the Monge–Ampère equation.” You will receive the prize tomorrow from His Majesty the King of Norway.

We will return to your mathematics, but first it might be a good idea to know something about your background. You were born in Buenos Aires in 1948. How would you describe your childhood?

Luis Ángel Caffarelli: We lived in a nice middle-class area of Buenos Aires. At the time when I was a kid I would play soccer a lot with my friends. Another game we enjoyed a lot was to throw a ball, or something else, and see who got it closest to a wall or to a line.

The place where I lived was what I would characterize as an engineering area. My father was a mechanical engineer who worked in the shipping industry, assembling and repairing vessels for navigation in the Río de la Plata bay. When I was 16 years I joined him to assemble a ship engine. This is one of my warmest memories from my adolescence.

[BID/CFS]: So, your father was an inspiration to you?

LÁC: Yes, he was. He pushed me, with a lot of love, to do some serious things like engineering or science, or something like that. And I followed his advice. I was admitted to a renowned secondary school, Colegio Nacional de Buenos Aires, run by the University of Buenos Aires. There my scientific interests were channeled by inspiring teachers toward physics and mathematics. I graduated from high school in 1966, and I joined the University of Buenos Aires in March 1967, majoring in physics and mathematics. I received my undergraduate degree in 1970.

As a student I was heavily influenced and inspired by Luis Santaló [1911–2001], Manuel Balanzat [1912–1994] and Carlos Segovia [1937–2007]. Santaló and Balanzat were both Spanish mathematicians who moved to Argentina as a consequence of the Spanish Civil War. Santaló made important contributions to integral geometry and geometric probability, while Balanzat worked in functional analysis. They built, jointly with Rey Pastor [1888–1962] and Pi Calleja [1907–1986], a superb...
undergraduate and graduate mathematics program at the University of Buenos Aires, generating a very strong group in analysis, geometry and algebraic geometry.

The harmonic analyst Segovia was a prominent graduate from Universidad de Buenos Aires who did his PhD at the University of Chicago in 1967 with Alberto Calderón [1920–1988]. While closer to me in age than Santaló and Balanzat, Segovia was always a strong support.

[BID/CFS]: You then embarked on your graduate studies, receiving your PhD in mathematics in 1972. Your PhD advisor was Calixto Calderón [1939–], the stepbrother of the much older and very famous mathematician Alberto Calderón you mention. Tell us about your graduate work.

LÁC: Calixto Calderón steered my imagination towards special function theory, which was a vibrant subject associated with finding optimal representations to solutions of partial differential equations (PDEs) by analytic methods consisting of series representations of special polynomial forms. A key point of my thesis involves Abel summability techniques, which I find fascinating considering I am being honored by receiving the Abel Prize! In fact, one of the papers I published with Calixto Calderón in 1974, in the wake of my thesis work, is titled: “On Abel summability of multiple Jacobi series.”

[BID/CFS]: May we interject a remark here which, at least tangentially, is related to this? In 2007 you published, together with your former PhD student Luis Silvestre [1977–], the enormously influential paper “An extension problem related to the fractional Laplacian.” Historically, Abel [1802–1829] was the first to introduce fractional derivation and integration (he effectively used it to solve the generalized isochrone problem—a problem that goes back to Christiaan Huygens [1629–1695]). So broadly speaking, some key notions in your work can be traced back to Abel.

LÁC: Well, that’s why we are here!

Early Career and Free Boundary Problems

[BID/CFS]: After obtaining your PhD at the University of Buenos Aires you went to the University of Minnesota as a postdoc, arriving there in January 1973. What made you choose to go there?

LÁC: There was a connection established earlier between the mathematics departments at the University of Buenos Aires and the University of Minnesota, so it was quite natural for me to go there as a postdoc student. Also, my PhD advisor Calixto Calderón had arrived there already in 1972, having been offered a permanent position. However, he left the University of Minnesota in the fall of 1974 to take up a tenured position at the University of Illinois at Chicago.

I found my colleagues at the mathematics department at the University of Minnesota to be friendly, extremely generous and dedicated, and they taught me much of what I know. They shared their ideas and gave me guidance as I began my research program.

[BID/CFS]: Was there some person in particular that was important for your mathematical research while at the University of Minnesota?

LÁC: Yes, upon arrival I met Hans Lewy [1904–1988]. He was an extraordinary analyst working on nonlinear PDEs and minimal surfaces. I attended a lecture series on harmonic analysis given by Lewy. He gave me two problems, which I succeeded in solving in a few months. Having his support was a game changer for me, influencing strongly my career path. One of the problems Lewy suggested was the obstacle problem, which is an example of what is known as a free boundary problem.

[BID/CFS]: In 1977 you published a paper titled “The regularity of free boundaries in higher dimensions” in the prestigious journal Acta Mathematica. You surprised the mathematical world with this paper, a work whose novelty and brilliance was the basis of your future fame. You were the first mathematician to really understand the free boundary problem in more than one dimension. Furthermore, the methods that you introduced have been extremely powerful and are still being used in many other problems. Could you elaborate on all this?

LÁC: Melting ice is an example of a free boundary problem, the free boundary being the surface between ice and water. The surface shifts as ice melts.
Another example of this type of a problem would be a balloon inside a box (or a drop inside a cavity). If the balloon is suspended in the air without constraints, a first approximation to its shape is given by a prescribed mean curvature equation—a mildly nonlinear PDE that we can deduce from the fact that the balloon tries to minimize the energy of the configuration. If constrained to lie inside the box, the surface of the balloon will behave differently depending on whether it presses against the wall or not, giving rise to a strongly nonlinear PDE. The separation curve between the different regions is called the free boundary. In this area I have investigated extensively the mathematical problems associated with solid-liquid interfaces, jet and cavitational flows, and gas and liquid flow in porous media.

### The Navier–Stokes and Monge–Ampère Equations

[BID/CFS]: Let us put things in perspective. The early 1960s saw great developments in the theory of linear PDEs, where a PDE is called linear if it is linear in the unknown and its derivatives. Many people contributed to this, but the deepest and most significant results were due to Lars Hörmander [1931–2012], according to the citation when he received the Fields Medal in 1962. It is worth noting, though, that some results obtained in the 1950s by Alberto Calderón were very important for this development.

The upshot of all this is that for linear PDEs there exists a theory. This contrasts with what is the case for nonlinear PDEs, where it is usually acknowledged that there is no "general" theory, with specialist knowledge being somewhat divided between several essentially distinct subfields. PDEs, and in particular, nonlinear PDEs, are ubiquitous in physics, ranging from gravitation to fluid dynamics. They also are important in mathematics, and have been used to solve problems such as the Poincaré conjecture and the Calabi conjecture.

Many of the fundamental PDEs in physics, such as the Einstein equations of general relativity and the Navier–Stokes equations, are quasilinear (meaning that the highest-order derivatives appear only as linear terms, but with coefficients possibly functions of the unknown and lower-order derivatives). On the other hand, the Monge–Ampère equation, which we shall return to and which arises in differential geometry, is fully nonlinear, meaning that it possesses nonlinearity in one or more of the highest-order derivatives.

Among the many open questions are the existence and regularity of solutions to the Navier–Stokes equations, named as one of the Millennium Prize Problems of the Clay Foundation in 2000. Tell us about your involvement with the Navier–Stokes equations.

LAC: In 1980 Louis Nirenberg [1926–2020], Abel Prize recipient in 2015 (a prize he shared with John Nash, Jr. [1928–2015]) invited me to join the Courant Institute at NYU as a full professor. This experience was a game changer with respect to my academic research. Nirenberg steered my interest towards fluid dynamics and fully nonlinear equations. Walking one day in Chinatown with Nirenberg and Robert Kohn [1953–], we decided to work together on a paper about the Navier–Stokes equations, a set of nonlinear PDEs that models the evolution of viscous incompressible fluid flows (in dimension three).

The result of this “CKN” collaboration was the 1982 paper titled, "Partial regularity of suitable weak solutions of the Navier–Stokes equations." We showed that the flow had singularities at most on a set of zero one-dimensional measure (i.e., less than a curve) in space and time. This is a nearly optimal result according to examples given by Vladimir Scheffer [1950–2023].
A more technical way to state the CKN theorem is the following: Let \( u \) be a weak solution of the Navier–Stokes equations for incompressible fluids, satisfying suitable growth conditions. (To arrive at the idea of a weak solution of a PDE, one integrates the equation against a test function, and then integrates by parts (formally) to apply the derivatives on the test function.) The result says that \( u \) is regular away from a closed set whose one-dimensional parabolic Hausdorff measure is zero.

This is the best partial regularity theorem known so far for the Navier–Stokes equations. It appears to be very hard to go further; we need some deep, new ideas.

[BID/CFS]: Nirenberg has said that you had a “fantastic intuition,” which made it hard for collaborators to keep up with you: “He somehow immediately sees things other people don’t see, but he has trouble explaining them,” Nirenberg said.

Let us move on to the Monge–Ampère equation, on which you have made seminal contributions, especially on their regularity properties, i.e., high order differentiability. The Monge–Ampère equation is a fully nonlinear PDE that frequently arises in differential geometry; for example, it is used to construct surfaces of prescribed Gaussian curvature. How did you get started to work on the Monge–Ampère equation and, more generally, on fully nonlinear equations?

[LAC]: While at Courant, I listened to a talk of Pierre-Louis Lions [1956–], who left an open question on this problem. I was able to solve it by using ideas from free boundary theory, thus finding a connection between two different parts of PDEs. Then I started to work with Nirenberg and Joel Spruck [1946–], and we published several papers together on the Monge–Ampère equation, including one paper on the complex Monge–Ampère equation (together with Kohn).

These papers, published between 1984 and 1986, were among the first to develop a general theory of second-order elliptic differential equations which are fully nonlinear, with a regularity theory that extends to the boundary. It should be remarked that the paper from 1985, published in Acta Mathematica and titled "The Dirichlet problem for nonlinear second-order elliptic equations III. Functions of the eigenvalues of the Hessian" has been particularly influential in the field of geometric analysis, since many geometric PDEs are amenable to its methods.

[BID/CFS]: Over several decades you wrote many papers, often with coauthors, which have "Monge–Ampère equation" in their titles. It is especially noteworthy that your regularity theorems from the 1990s represented a major breakthrough in the understanding of the Monge–Ampère equation. In particular, according to the Abel Prize citation you "closed the gap in the understanding of singularities by proving that the explicitly known examples of singular solutions are the only ones."

The Fractional Laplacian

Now let’s move on to a more recent paper of yours from 2007 (co-authored with Luis Silvestre), a paper we alluded to above, and which is titled "An extension problem related to the fractional Laplacian." The paper is 15 pages long, but had an enormous impact in many different subfields of PDEs. It is also your most-cited paper, with more than 1600 citations on MathSciNet! Tell us about this paper.

[LAC]: Without getting too technical, let me try to explain the main idea, as well as some results, of this paper. The Laplacian is the second-order differential operation \( \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \). On \( \mathbb{R}^n \) the fractional Laplacian \((-\Delta)^s\), for \( s \) a real number between 0 and 1, is perhaps easiest understood by means of its Fourier transform, where differentiating corresponds to multiplication with the norm of the variable. So, a fractional Laplacian is mirrored in the Fourier transform by multiplication by the norm of the variable to the appropriate power. Another way to define the fractional Laplacian \((-\Delta)^s\) is via the formula

\[
(-\Delta)^s f(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} \, d\xi,
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a function and \( C_{n,s} \) is a normalizing constant.

The main idea of the paper is proving a result which relates a nonlocal problem associated to the fractional power of the Laplacian to a local degenerate elliptic problem. This allows us to prove several regularity results concerning the solutions of problems involving the Laplacian by exploiting purely local techniques. In particular, we prove both a Harnack inequality and a boundary Harnack inequality.
Work Style, Students, and More
Recent Research

[BID/CFS]: As a mathematician, you are extraordinarily prolific—and extraordinarily sociable. You have published more than 320 papers, you have cowritten papers with more than 130 people and advised more than 30 PhD students. Do you have any comments on this?

LÁC: Through the years, I have had the opportunity to belong to wonderful institutions, starting with the University of Buenos Aires, and then the University of Minnesota, the Courant Institute, NYU, the University of Chicago, the Institute for Advanced Study, Princeton, and for the last 26 years, the University of Texas at Austin.

This gave me the opportunity to befriend and collaborate with extraordinary scientists all over the world. It also led to further opportunities to mentor very talented young people who have invigorated my research with new ideas. I have moved between topics within the wider field of PDEs. There are people who do wonderful things in very concentrated areas. But science is more like a global evolution. It requires the exchange of ideas, looking at things from different angles, slowly improving whatever can be improved.

[BID/CFS]: Are you more of a problem solver than a theory builder?

LÁC: Yes, definitely. Broadly speaking, there are two kinds of mathematicians. There are those who develop theories and those who are primarily problem solvers. I belong to the latter group.

[BID/CFS]: As you told us, you have spent the last 26 years at the University of Texas at Austin. Your wife Irene Martínez Gamba, who is also a mathematician, is professor of computational engineering and sciences at the same university. You have had more than 20 graduate students while at the University of Texas. Tell us about your research activity there.

LÁC: My research expanded in an immense way due to the fact that I had graduate students again; many of them I still have long-lasting collaborations with. I am indebted to all of them. Of specific research topics I worked on, I would list three:

(i) Free boundaries in (Lipschitz) surface detection.
(ii) Problems with fractional diffusion; a completely new theory that filled my agenda for many years.
(iii) I kept on working on the Monge–Ampère equation and its regularity theory in fully nonlinear configurations.

I would also mention that I interacted with a group of engineers and natural scientists. I enjoyed enormously having discussions with them, as they somehow relied on some of my ideas.

[BID/CFS]: We always end these interviews by asking about interests outside of mathematics. Do you have any special interests or hobbies?

LÁC: I like to cook, but my wife is a better chef than I am. I also like to play some soccer if I have time. I play the piano. I prefer mostly classical music.

[BID/CFS]: On behalf of the Norwegian Mathematical Society, the European Mathematical Society and the two of us, we would like to thank you for this interesting interview.

LÁC: Thank you very much.

Credits
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Figure 4 is courtesy of Alf Simensen - NTB/The Abel Prize.
Figure 5 is courtesy of Peter Badge/The Abel Prize.
A **proof** is one of the most important concepts of mathematics. However, there is a striking difference between how a proof is defined in theory and how it is used in practice. This puts the unique status of mathematics as exact science into peril. Now may be the time to reconcile theory and practice, i.e., precision and intuition, through the advent of **computer proof assistants**. This used to be a topic for experts in specialized communities. However, mathematical proofs have become increasingly sophisticated, stretching the boundaries of what is humanly comprehensible, so that leading mathematicians have asked for formal verification of their proofs. At the same time, major theorems in mathematics have recently been computer-verified by people from outside of these communities, even by beginning students. This article investigates the different definitions of a proof, the gap between them, and possibilities to build bridges. It is written as a **polemic** or a **colleague** by different members of the communities in mathematics and computer science at different stages of their careers, challenging well-known preconceptions and exploring new perspectives.

**1. What is a Mathematical Proof?**

**Dierk Schleicher**

Mathematics often prides itself as the most fundamental of all sciences: a mathematical truth, once established, will be true forever. But what exactly is a mathematical proof? Here is one possible answer in terms of the other fundamental concept in mathematics, a definition:

**Definition** (Proof: formal definition). A mathematical proof is a sequence of deductions that are based on a given set of axioms and formally deduce consequences following formal rules of deduction.

This might be the first, “idealistic,” definition of proof that one encounters, and it is not too hard to prove some simple results in this formal way, perhaps in combinatorics, elementary number theory, or on Euclidean triangles. But sooner rather than later, one discovers that this definition is too clumsy and impractical for any profound result in mathematics. In practice, mathematicians use a very different definition of proof.

**Definition** (Proof: practical definition). A mathematical proof is a sequence of arguments that convinces an educated reader.

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DOI: https://doi.org/10.1090/noti2860
So one might object that mathematicians routinely make one of the most fundamental mistakes: define things in one way and then use them in a different way. But does this not give away all the fundamental virtues of mathematics?

Many mathematicians are quick to point out that a typical proof, given as a convincing sequence of arguments, can be elaborated in more detail: in principle, for every step one may ask why this is so, and one can insert additional steps of explanation. Each of these steps, in turn, may be expanded if need be, until eventually one arrives at the axioms. One would expect that any of these expansion steps can be performed by the author of a proof, until one arrives at sufficiently fundamental levels when one reaches statements that are already established. It is a common understanding among mathematicians that most current mathematical knowledge can be thus justified—in principle. One may thus reinterpret the practical definition of a proof in the sense that a proof is a sequence of arguments that convinces an educated reader that a formal proof in the sense of the first definition “can be constructed.”

But it is true that mathematical “proofs” written by humans occasionally do have errors. Many of them are of a trivial kind when some missing steps can be inserted, or an overlooked special case can be treated in the same way as the rest of the proof. Sooner or later, one is often told, any mistakes would be found by the scrutiny of the mathematical community.

But now and again there are theorems that were thought to be proven where much later the proof is recognized to be false or incomplete, and where perhaps even the result itself is recognized as wrong. How would you know that for a given theorem the “sooner or later” might not happen tomorrow; when a possibly fundamental flaw is discovered? How can one then be certain of the mathematical correctness of a proof?

A rather prominent classical example is the four-color theorem: this theorem was thought to be proved already in the late 19th century, when not one but two proofs were given—that were both found to be incorrect eleven years later.

A recent example from dynamical systems. Proofs that are later discovered to be flawed are not just a theoretical possibility, or an anecdote from the early times of mathematics (when perhaps standards of mathematical rigor were not so high). This is illustrated in a very current example from dynamical systems. The claim is stated quite easily:

Let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function but not a polynomial (i.e., a transcendental entire function). Then \( f \) can have at most one maximal completely invariant domain: that is a domain \( U \) with \( f(U) = U = f^{-1}(U) \), and such that \( \mathbb{C} \setminus U \) contains at least two points; maximality means that \( U \) is not contained in a strictly larger domain \( U' \) with the same properties.

Until very recently, this result would be stated as a theorem, proved by Noel Baker, an eminent pioneer in the field, in 1970 [1]. However, Duval observed about a half century later that the proof is flawed, and Rempe and Sixsmith [19] showed that it cannot be fixed by the methods stated. In particular, a key step in the proof is the statement that if \( U, V \subset \mathbb{C} \) are disjoint simply connected domains, then one of \( f^{-1}(U) \) and \( f^{-1}(V) \) must be disconnected. This statement, however, is false; even as simple a function as \( f(z) = e^z + z \) provides a counterexample, as shown by Rempe and Sixsmith: it has infinitely many disjoint simply connected domains with connected preimages. Baker’s proof thus cannot be repaired along the lines of its original version. The main result, as of today, is an open question—half a century after it was accepted as a correctly proved theorem.

One might wonder how relevant this result is—perhaps the error went unnoticed because nobody cared? Unfortunately, this is not so. Rempe and Sixsmith [19, Section 9] give a shocking list of results that depend on the flawed paper in a variety of ways. One list contains several publications by a variety of authors that all depend either on the flawed paper, or even on other papers that used the flawed result, but where the main results can be fixed by other methods developed in [19]. Another list contains several publications, some of them 35 years old, that use the flawed paper, or for which the main results must now be considered open once again. Yet another list contains two survey papers, between 25 and 30 years old, that refer to the result in [1]. And perhaps most alarmingly, there are several publications, some of which meanwhile classical and much-cited, where the flawed method of proof was found so useful that it was adapted, further developed, and generalized (without recognizing the flaw). As a result, a whole area of mathematics has had to sort out how its theory was affected by a rather “convincing” mistake made half a century earlier.

This is just a recent example, rather worrisome for the mathematical community, that shows that problems can be discovered at any time even in supposedly well-known results. However, it may well be that mathematics is now developing toward the point when, finally, the two definitions of “proof” can be reconciled. How this may come about is one of the key topics of this paper.

Another development in current mathematics, leading to similar conclusions, is that proofs tend to become longer and more complex, possibly so much so that they can no longer be stored as a whole in human memory, nor be verified by referees. For instance, Peter Scholze has recently asked that one of his key results on condensed
mathematics be formally verified, a project called the Liquid Tensor Experiment that has attracted broad attention and was successfully completed in 2022. A number of years earlier, Vladimir Voevodsky had similarly been worried about the correctness of results in his field leading to the development of Homotopy Type Theory.

One of the inspirations for this paper was a workshop at the Heidelberg Laureate Forum in 2018, titled “The future of mathematical proof,” where many of the coauthors of this paper were present, and where some of the aspects treated here were discussed and developed.

This paper should be seen as a kaleidoscope of many aspects of mathematical proofs and computer proof assistants, contributed by various researchers with diverse backgrounds. In Section 2, Yuri Matiyasevich describes his vision that, soon, mathematical publications need to be supported by formal proofs, and that this may well be accomplished by the coming generation. In Section 3, Efim Zelmanov argues in favor of the meaning and explanations of mathematics, rather than formal verification. In Section 4, Leslie Lamport introduces a way of presenting mathematical proofs that “should make it much harder to publish false proofs.” In Section 5, Christoph Benzmüller discusses several major accomplishments of proof assistants and derives a vision for integrated formal and traditional proofs. Then, in Section 6, Jonas Bayer, Marco David, and Benedikt Stock, three undergraduate students at the time, describe how they learned to work with proof assistants from early on, and completed one of the first major formalizations done by the coming generation. In Section 7, Kevin Buzzard explains how he makes proof assistants “sexy” to mathematicians and recounts how he incorporated them into classroom teaching. In Section 8, Lawrence Paulson outlines open challenges and a perspective for the future.

Our contributors express different points of view in this text, so it is natural that upon reading, you might not agree with all of their claims. In fact, this also applies to contributors and referees: reading some of the exchanged messages led one of us to remark, “is this still mathematics or already real-life soap opera?” We hope that the kaleidoscope in this text will provide inspiration and help readers develop their own point of view.

2. Why Formalize Mathematical Results?
And Why Hilbert’s Tenth Problem?

Yuri Matiyasevich

A decade ago, the St. Petersboung mathematical society held a meeting titled “Mathematical proof: yesterday, today, tomorrow.” Being one of the three spokesmen, I completely disagreed with the previous speaker, and dared [15] to publicly announce the following:

PREDICTION

In 25 years mathematical journals (if they are destined to survive so long) won’t take into consideration any paper unless it is accompanied by proofs which can be verified by a computer.

Already at that time at least one (and only one?) journal existed in which proofs passed preliminary computer verification. This was Formalized Mathematics founded in 1990. Over the past 10 years, mathematical journals continued to multiply in number but I do not know of any new journal requiring formalized proofs. Nevertheless, I dare to repeat my prediction verbatim et litteratim, that is meaning “In 25 years from now....”

In fact, the progress in computer verification of proofs is very impressive, both in the software development and in the number and the significance of actually verified theorems. There exists a compendium [23] of such achievements; the list of proofs is regularly updated but initially restricted by a selection of “Top 100” theorems.

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But what could be the goal of computer verification of results already proven by human beings? One obvious answer is as follows: to get additional trust in the correctness of a proof (there are many examples of important and widely used theorems with flaws in their proofs which remained unrevealed for decades). However, some people believe that computers themselves are not sufficiently reliable because of possible errors in their design/software/runs. In my opinion, “in 25 years” the progress in this area will remove such objections completely.

Formalized proofs are vital for the very ambitious project of a World digital mathematics library [11]. The ultimate goal of this project is to transfer all mathematical knowledge (not just axioms/definitions/theorems/proofs) to computers. This would require tremendous efforts but who would care to spend her/his precious time on the meticulous presentation of known results? Mathematicians prefer to produce new ideas (definitions, hypotheses, theorems) and do not appreciate the hard work of writing proofs with all minute details (luckily, computer scientists do appreciate such occupation).

The following remedy was proposed a long time ago: senior, “retired” mathematicians, who are no longer capable of generating new brilliant mathematical ideas, could devote the rest of their lives to “teaching mathematics to computers.” But there is an opposed option: this could be done by the young people who just start to master mathematics. In this way they can get an acute feeling of what mathematical rigor is. However, the following remains uninvestigated: how would involvement in such an activity influence the ability to create new mathematics?

So when I heard that a group of students from Jacobs University in Bremen had studied, under supervision of Professor Dierk Schleicher, the proof of the undecidability of Hilbert’s tenth problem, I suggested to them to demonstrate their understanding of the whole construction by producing a fully formalized proof. My role in the project was very restricted: I supplied the students with a sufficiently (for human beings) detailed proof, and I was responsible for the choice of Isabelle as the verifier. I was happy to see that the students became very enthusiastic, maybe because they (and I too) at that moment underestimated the amount of required work.

Hilbert’s tenth problem is not very difficult for formalization. It is a bit strange that at first we were waiting for it for half a century but then four independent verifications in Coq, Isabelle/HOL, Lean and Mizar emerged in a short time ([17], [16], [21], [14]).

3. On Proof and Progress in Mathematics: From the Perspective of a Research Mathematician

Efim Zelmanov

I will add my 5 cents to the wonderful discussion of computer proof verification, organized by students. For more than 40 years, I’ve lived in the world of proofs and, sometimes, complicated proofs.

If I were told that a proof is correct because a computer program says so, but I don’t see big ideas “turning the wheels,” then probably I would continue thinking about the problem, as if a computer blessing did not exist.

The purpose of a proof is understanding. For mathematicians it is not enough to know if this or that statement is correct or not. They want to know why it is correct or not. Often this understanding comes as a link to some big ideas that come into play now and then in different contexts. I am not able to say it better than W. Thurston in his beautiful paper on proofs (see [22]).

In my opinion, a credible computer verification of a proof is an amazing achievement in AI. It is valuable and interesting for its own sake, leaving alone proofs. It may also find other applications. I have to admit that as far as a straightforward computation is concerned, I trust computers more than humans. Difficult and important proofs are often written at the edge of human intellectual abilities (think of the proof of G. Perelman of the Poincaré conjecture). Should we expect the authors to present it in a computer-friendly form? I am afraid that it is too much to ask.

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Modern mathematics created a new challenge: proofs of enormous complexity. I know proofs of some far-reaching statements that have been around for quite a while and still nobody (except the authors) can say that they understand all the details. The only hope is that good proofs are like live organisms. They evolve with time and go through a natural selection. New ideas appear and bring new understanding. And new proofs may appear.

I will end with the controversial statement that in practice a proof is what is considered to be a proof by all mathematicians.

Efim Zelmanov

4. Making Math More Rigorous

Leslie Lamport

Mathematics, as practiced by most mathematicians, is not very rigorous. There is evidence that about 1/3 of all published, refereed math papers contain significant errors—incorrect proofs or theorems that their authors believed to be correct. (I have presented evidence elsewhere [12].) Math can be made more rigorous, and mathematicians can make fewer errors, by replacing archaic customs with more sensible practices. Here is how.

Formulas. A few hundred years ago, formulas were written in prose. Today, mathematicians recognize the advantages of writing formulas in mathematical notation: they’re shorter, easier to understand, and easier to manipulate. Replacing prose by mathematics must have reduced errors.

Mathematicians think they’ve stopped using prose to write formulas. They’re wrong. They’ve replaced only some of the prose in their formulas by math. Consider this definition of what it means for \( \lim_{x \to a} f(x) \) to equal \( b \).

For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for all \( y \),
if \( 0 < |y - x| < \delta \) then \( |b - f(y)| < \varepsilon \). (1)

A mathematician would find (1) perfectly normal, even though it’s a mathematical formula written with many words. Here is that formula written without words:

\[
\forall \varepsilon > 0 : \exists \delta > 0 : \forall y : (0 < |y - x| < \delta) \implies (|b - f(y)| < \varepsilon).
\]

(2)

I believe most mathematicians would find (2) harder to understand and uglier than (1). I expect mathematicians a few hundred years ago would have found \( 0 < |y - x| < \delta \) hard to understand and ugly.

Why write (2) rather than (1)? For the same reason we don’t write \( 0 \) is less than the absolute value of \( \ldots \) It’s shorter, easier to understand (when you become comfortable with the notation), and easier to manipulate. And it will reduce errors. Show elementary calculus students the definition (1) and ask them to write what it means for \( \lim_{x \to a} f(x) \) equals \( b \) to be false. I doubt if many of them would get it right. Teach them a little elementary logic and they could easily compute the negation of (2). The most obvious use of words in formulas is to express logical operations; but they are also used in other ways, such as describing sets and functions.

Formulas written without words can now be manipulated by computer programs. Programs can easily compute the negation of (2). They can’t compute the negation of (1).\(^1\) Those programs can help students become comfortable with mathematical concepts, if the concepts are described with math rather than prose.

Mathematicians think it’s difficult to write formulas completely mathematically, without words. I have asked a number of mathematicians how long a completely rigorous, wordless definition of the Riemann integral would be—assuming definitions of the set of real numbers and its arithmetic operations, as well as simple set theory. I’ve received answers ranging from 50 lines to 50 pages. They’re wrong.

I’ve developed a language called TLA\(^+\) that engineers use to write completely formal mathematical descriptions of computer systems. It has tools for checking the correctness of their mathematics. The Riemann integral can be defined in TLA\(^+\) in about a dozen lines.

Proofs. A few hundred years ago, proofs were written in prose. They still are. Mathematicians haven’t even begun to change the way they write proofs. They think their proofs express rigorous logical reasoning. They’re wrong. Their prose proofs are written in a literary style that obscures the logic of the proof. Consider the following opening sentence of a proof from an elementary calculus book:

1Restricted, unnatural languages have been proposed for writing formulas approximately like (1) so they can be understood by a computer program. Such languages are of little or no use to people not afraid of mathematics.
by Michael Spivak [20, page 170]—a book that is considered to be very rigorous.

Let $a$ and $b$ be two points in the interval with $a < b$.

It is obviously wrong because the interval in question could consist of a single point, so it might be impossible to choose $a$ and $b$. That sentence is actually part of a correct proof, but the reader must discover for herself the proof hidden inside Spivak’s prose.

Writing proofs with prose leads to errors. How can those errors be avoided? Most mathematicians and computer scientists believe that the only way is to write machine-checked proofs. This requires writing formulas in a formal language. TLA$^+$ is simple enough that mathematically unsophisticated engineers can use it, and it is enough like everyday math that mathematicians should find it fairly natural. But it's too simple to be adequate for writing the kinds of proofs found in most math journal articles. Formalizing such proofs requires a language too complicated for most engineers to learn—one that I believe most mathematicians would find quite obscure. Few mathematicians would go to the effort of learning such a language unless it made writing their proofs significantly easier. Today, it makes writing most proofs much more difficult. Routine machine-checked proofs are now practical in just a few situations, including some safety-critical applications. I don't expect this to change in the next couple of decades.

Fortunately, there is a simple method that anyone can use now to write proofs with fewer errors. It can’t eliminate all errors, but it can make them much less likely to occur. Its basic idea is to replace the linear order of ordinary prose by a hierarchical structure, and to name hypotheses and proved facts so they can be referred to later in the proof. Here is a brief explanation of the method; a complete description has appeared elsewhere [13].

A theorem consists of a statement together with its proof. A proof is either a short paragraph or a sequence of statements and their proofs. At each point in a proof, there is a current goal and a set of usable facts that can be assumed in proving that goal. Statements can be written in prose or in math. When written in math, the logical structure of the statement often determines the hierarchical decomposition of its proof. Figure 1 shows the structure of part of a proof containing the statement $A \land B \implies C$, in which $C$ is proved by first proving statements $D$ and $E$. Usually, those two statements would easily imply $C$, making the proof of qed step (3)4 simple. The number (2)3 indicates that it is the third statement in the level-2 proof of a level-1 statement.

This is a straightforward proof, and presented in this way there seems no reason to structure it. But suppose it were a small part of a large proof, and the proofs of $D$ and $E$ were each half a page long. If the proof were written as prose, how could the reader keep track of where the scope of the hypotheses $A$ and $B$ ended, and where it was no longer valid to use $D$? Mathematicians try to handle complexity by using lemmas; but that just provides one level of hierarchy, which doesn’t get you very far.

Making a proof more rigorous requires filling in all the gaps that could conceal errors. This means making it longer. Making a prose proof longer makes it harder to read. But with hierarchical structure, the extra length makes the proof easier to read. The additional explanation appears at lower levels of the hierarchy, so it doesn’t obscure the structure of the proof. This will be especially true when mathematicians stop producing pictures of print on dead trees and start using hypertext, so lower-levels of the proof can be hidden when not being read. Avoiding errors requires more detailed proofs than are currently found in journals. Until journals use hypertext, this means writing a detailed proof to catch errors, then shortening it for publication. That's easy to do with structured proofs: you just replace the lower levels of the hierarchy with short proof sketches. (One can even write $\LaTeX$ macros so a single file can produce either version by changing a few characters.)

Students can learn to write structured proofs by teaching them to write very simple machine-checked proofs in some field. Any good proof system should allow hierarchical structuring of proofs. The language for writing theorems should be simple—not the kind of complicated language needed for serious math. TLA$^+$ and its proof system

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**Figure 1.** A statement and its structured proof.

\[ \langle 2 \rangle 3. \quad A \land B \implies C \]

Current goal set to $A \land B \implies C$

\[ \langle 3 \rangle 1. \quad \text{SUFFICES ASSUME } A, B \]

\[ \text{PROVE } C \]

Proof: By simple logic. Trivial proof that assuming $A$ and $B$, then proving $C$, proves the current goal.

Current goal set to $C$; and $A$ and $B$ added to usable facts.

\[ \langle 3 \rangle 2. \quad D \]

Proof of $D$

$D$ added to usable facts.

\[ \langle 3 \rangle 3. \quad E \]

Proof of $E$

$E$ added to usable facts.

\[ \langle 3 \rangle 4. \quad \text{QED} \]

Proof of $C$

Current goal and usable facts same as before (2)3 except with fact $A \land B \implies C$ added.

\[ \langle 2 \rangle 4. \ldots \]
are not ideal, but they could be used if nothing better is available.

Students should understand that the facts they learn in their math classes can, in principle, be formally proved from simple axioms and proof rules. In practice, we only carry proofs down to the level where we believe the reader will find the steps to be obviously true. That level rises with education and experience. We also sometimes take shortcuts by writing formulas with words. But students and mathematicians should have the confidence that they could make their proofs completely rigorous and carry them down as close as they want to basic axioms.

Writing hierarchically structured proofs can help you avoid errors; it can’t guarantee that you will. You have to be honest with yourself about what’s obvious and what should be proved. My advice is to write the proof down to a level at which you think everything is obvious, and then go one level deeper. But if you don’t care whether your proofs are correct, nothing short of having to write a machine-checked proof will keep you from making errors.

**What should you do now?** If you agree that writing formulas with words or that writing proofs with prose is silly, just stop doing it. You don’t have to wait for others to change.

**Formulas.** You needn’t remove all the words from your formulas. Start by using the quantifiers ∀ and ∃. Then try eliminating “…””, which is not a mathematical operator. The sequence $x_1, …, x_n$ is just a function $x$ with domain $1..n$ that maps each $i$ in its domain to $x_i$. Often, though not always, the math becomes simpler and more elegant if you eliminate the “…” and instead use the function $x$. Give it a try. Be aware of when you’re using words and sloppy notation instead of being rigorous. If you’re open to change, you will find that the mathematically rigorous approach is often the simplest. If you’re a teacher, your students should have learned, or should be learning, the basic math needed to write formulas with fewer words than they now use. Help them to become more comfortable with proper mathematical notation by using it in your classes.

**Proofs.** There is no reason not to start writing structured proofs now. It takes only a sentence or two to explain to readers how to read them. I’ve been doing it for about 30 years, and no editor or referee has complained about my proofs. Start by reading how I write structured proofs, but feel free to modify my style as you see fit. There are just two features that should be preserved: hierarchical structuring and the ability to name and refer to hypotheses and already proved statements.

If you’re a professor, teach your students to structure their proofs the way you do. They’re not yet set in their ways, and they’ll appreciate how the structure makes your proofs easier to understand. Encourage them to write structured proofs in all their courses. Other professors are unlikely to complain that the proofs are too rigorous; and they might even be inspired to write them themselves.

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**5. What is a Proof? What Should It Be?**

**Christoph Benzmüller**

Does the notion of a mathematical proof refer to the rigorous but typically rather nonintuitive formal derivation of a new “truth” from its premises using accurately defined rules of inference? Or is it an artful communication act in which the beautiful structures underlying a new mathematical insight are revealed to peers in such a way that they can easily see and accept it, and even gain further inspiration?

The former notion of a formal proof is primarily concerned with logical rigor and soundness. Intuition and beauty is a secondary concern, if at all. Formal proofs have recently attained increased, albeit quite controversial, attention in mathematics, triggered, e.g., by successful applications of modern automated theorem proving technology to challenging mathematical verification and reasoning tasks. Using theorem provers, hard problems were solved that humans could not manage on their own. Some examples include:

(i) *The four-color theorem:* this notorious challenge was solved already in 1977 using automated theorem technology by Appel & Haken, and an interactive formal

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proof was more recently developed by Gonthier in the proof assistant Coq. (ii) The Kepler conjecture (about best sphere packings in Euclidean 3-space): a board of experts had gone to great lengths to verify the solution to the problem submitted by Hales to the Annals of Mathematics, but in the end could not fully verify his contribution. Eventually, Hales and his team mastered it in interaction with HOL Light. As the main result, a formal proof is now available that is independently verifiable—by humans and computer programs. (iii) The Pythagorean triple problem (whether all positive numbers can be colored by two colors so that there is no monochromatic Pythagorean triple): this challenge problem was solved by Heule, Kullmann, and Marek [9] using automated SAT solving technology (boolean SATisfiability), and the formal proof that was generated by the computer program is of enormous size (about 200TB); it is still independently verifiable though (at least by machines). This line of research has been continued by an automatic solutions for Schur number five [8] and Keller's conjecture [5]. (iv) My own current work with colleagues focuses on higher-order metalogical reasoning technologies [3] that have enabled us to detect and explain errors and problems in peer-reviewed publications in mathematics, computational metaphysics, and machine ethics. These include the discovery of an unnoticed inconsistency in Gödel's modern variant of the ontological argument for the existence of God, the discovery and clarification of a deeply rooted paradox in Zalta's *Principia Logico-Metaphysica*, and the revelation of some minor problems in a well-known textbook on category theory [4].

There are many other works that could be mentioned here, e.g., the machine-checked proof on the odd order theorem.

It is to be noted that the formal reconstruction of mathematical work is generally a very resource- and time-consuming task, for example due to the lack of large and easily reusable libraries of formalized mathematics, or software that does not pose additional challenges along the way. Conversely, some technical mathematical results, such as those mentioned in (iii), may not support intuitive and insightful mathematical proofs. In general, mathematics is confronted with increasingly complex problems whose solution and subsequent evaluation (e.g., by peer review) require techniques that go beyond traditional practice. Examples (i)–(iv) above are only early evidence of this kind. When technologies are available that can help detect errors in publications, they should certainly be used to optimize scientific quality. Formal proofs should therefore take a more central role, in maths and beyond.

In fact, I see it as a societal duty to take up this challenge. Clinging to the traditional notion of mathematical proof alone is not an option in an increasingly technological world. Think of areas such as “program verification in computer science” or “trusted AI,” where ideally we want formal guarantees that implemented, complex solutions are mathematically correct, but where intuitive, traditional proofs might be lacking.

Nonetheless, formal proofs alone are of limited interest and they should ideally always be coupled to intuitive proofs. Explainability, transparency, and intuition must remain virtues of the highest priority, not only in mathematics. In the long run, the increased trustworthiness and beauty of a combined approach, where both notions are coupled pari passu, will justify the additional resources that must now be invested. Publishing errors (even minor ones) will be prevented by formal proofs, and designing ill-conceived and inaccessible theories will be impeded by the demand for mathematical intuition and beauty.

So, what should a proof be? In conclusion, it should ideally be both a human-oriented traditional proof and a machine-oriented formal proof. Traditional mathematical proofs are made by, and consumed by, humans, while formal proofs are predominantly generated with, and consumed by, machines. Yet, by pairing both, the antipodes will increasingly engage in a harmonious dance.

One step further, beyond pairings, one may dream of the integration of mathematical and formal proofs into one object. Modern proof assistants such as Isabelle/HOL provide increasingly intuitive languages for constructing and representing proofs, but significant scientific progress is still needed to achieve this ambitious goal. Finding a comprehensive solution that integrates both notions of proof can even be understood as a challenge for AI, as it requires a seamless semantic integration of natural language, diagrams, formula language, etc., up to the exchange of meaningful arguments between humans and theorem provers. Recent work by Marco David, Benedikt Stock, Jonas Bayer, and their fellow students gives reason for hope. Their verification project of Hilbert's tenth problem is significantly smaller in scope than some projects mentioned above, but differs in that the mathematics students began their formalization project, encouraged by Matiyasevich, with no prior knowledge of proof assistant technology. Nevertheless, they rose to the challenge and made great progress by working independently with the proof assistant. This is particularly notable because it is another example (in addition to, e.g., [6]) of an impressive formalization project carried out by people from outside the community of formalized mathematics, and it is another
excellent demonstration of the maturity that modern proof assistant technology has now reached.

Christoph Benzmüller


Jonas Bayer, Marco David, and Benedikt Stock

Little did we know at the time that Yuri had made a “grand plan” to establish his vision on the future of mathematical proof when we first met him. He visited our university back in 2017 and presented the idea to formalize the DPRM theorem: this is the key result in his proof of the undecidability of Diophantine equations, providing a negative solution to Hilbert’s tenth problem. Little did we know that we played the role of “guinea pigs” who, in a carefully set-up experiment, should demonstrate the idea that mathematical proof verification by computers is feasible even for young and inexperienced university students. Today, we are very grateful for the role we have been given.

The following paragraphs elaborate what we have learned in the years since then, now that we are ourselves supervising another project by a fresh set of students that had never worked on computer-verified proofs before.

Yuri set up the scene swiftly: he presented his theorem to us and invited us to work on its computer verification. Neither he nor we had any idea how much effort this would turn out to be. At the time, it seemed to us that, once the proof of the DPRM theorem was understood, we merely needed to “translate” its arguments in such a way that Isabelle could verify them. Anyone who has ever worked with an interactive theorem prover, however, knows that the word “translate” does not do much justice to the process. In reality, this includes filling the possible gaps that are frequently found in mathematical papers. Common phrases such as “it is easy to see that...” needed to be brought to logical life during this process. Soon, we realized that the challenges often lay in lemmas that looked rather innocent. In fact, we found ourselves five times reconsidering the formalization of the mere concept of a register machine before our formal definition proved useful.

Despite the challenges we encountered, we insist that learning how to work with an interactive theorem prover is fully feasible, although not yet easy. In a previous paper [2], we have reflected upon our own mistakes and the challenges we encountered. A key realization is that making progress is almost impossible without an expert at hand to answer questions. Hence, to popularize proof assistants among the next generation, we are now proactive ourselves in mentoring a new group of students learning how to formalize mathematics. The current need to pass on experience person-to-person distinguishes proof assistants from most programming languages or computer algebra systems like Mathematica. Here, online Q&A forums such as Stack Overflow provide a freely accessible and searchable database of almost any imaginable question about these tools, and hence provide the means to make them accessible to the broad public. Interactive theorem provers and their currently secluded communities must follow suit to become useful for the average mathematician!

Beyond the technical skills and results, this project prompted a paradigm shift in the way we now view ordinary mathematics: In our university courses, we no longer ask if a proof is convincing to us, but instead wonder if we would be able to formalize it in Isabelle. As the computer often exhibits assumptions or edge cases that humans gloss over, this thought pattern leads to a more rigorous approach to the (informal) argument. Our new, sharpened perspective was largely forged through the interaction with the computer and its unique manner of reasoning. It illustrates how to reconcile the two conflicting definitions of “proof” in the next generation of mathematicians.

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3Named after Martin Davis, Hilary Putnam, Julia Robinson, and Yuri Matiyasevich, the DPRM theorem states that every recursively enumerable set is Diophantine.
7. Some Thoughts on the Formalization of Mathematics

Kevin Buzzard

Making formalization sexy. Mathematics has fashions. The Langlands philosophy seems to have been a fashionable area for many decades now. Conjectures get proved, new conjectures get made, there was the classical theory, and then a mod $p$ theory (crucial for Fermat's Last Theorem), and now a bewildering $p$-adic theory as well as geometric and categorified versions. I will unashamedly confess that my work with Johan Commelin and Patrick Massot where we translated the line “Let $X$ be a perfectoid space” into Lean’s language was an attempt to market theorem proving software to mathematicians. We wanted to show them that theorem provers are now ready to handle fashionable modern mathematics. No, we cannot do anything spectacular yet like come up with an incomprehensible billion-line long proof of the Riemann hypothesis (indeed, computers generating proofs of hard mathematical conjectures of mainstream interest is science fiction right now, and may well remain so for decades to come). However, let’s think about this. If we stick with the status quo (i.e., essentially nobody types sexy mathematics into proof assistants), then proof assistants will never be able to do sexy mathematics, because they will simply never learn it! Computers certainly can’t directly read the crap we write in our papers (and humans often can’t read them either). Humans need to do the translation by hand, and the sooner the better. So, whose job is it to type in the statements of the global Langlands conjectures into these systems? Who will do this, and give humanity the chance to make computer teaching and proof mining resources, and large language model training data, for students learning and working in the Langlands program? Surely it is our job as mathematicians. Nobody else is going to do it – we cannot expect computer scientists to become technical experts in the Langlands philosophy; it is much easier for mathematicians to learn a programming language. If the maths staff won’t do the translation, let’s get the maths PhD students doing it. If you’ve recently graduated with a PhD from a pure mathematics department, can you, or even can humanity, state in Lean the main theorems you proved in your thesis? The practical answers with current technology are still highly dependent on the nature of the area of the thesis. For my MSc and PhD students I am 100% certain that we can not only state the main results in Lean, but also prove them.

Looking to the future, what if mathematicians start flocking to formalization, and all of a sudden we have got perfectoid spaces and the statements of the Langlands conjectures in Lean and also in Arend (a HoTT prover) and Isabelle/HOL (a simple type theory prover) and MetaMath (a set theory prover) and Coq and HOL 4 and HOL Light and Mizar and cubical Agda and... What then? I have already said that, in my opinion, it is science fiction to expect that these systems will start proving the Langlands philosophy.

But the following is near-reality. Software such as Lean can be used to power an interactive resource for PhD students learning algebraic geometry or some other dense area. Once we have a database of statements of many theorems corresponding to tags in the Stacks Project or Kerodon (online databases of algebraic geometry and other sexy mathematics), we can let the computer scientists take over. They have tools known as hammers, which can attempt to build proofs or counterexamples to propositions fed to the system, using the database we mathematicians constructed. Note that such a tool does not need any formalized proofs and so even a huge database like the the statements of the theorems in the Stacks project could be constructed manually over a period of several person-years, ideally by young algebraic geometers interested in trying new ways of learning the area. Of course, formalising the proofs is the fun part, so these will no doubt follow.
Another possibility is training large language models such as ChatGPT to write Lean code. This area is still in its infancy but may turn out to be decisive.

**Making formalization fun.** The examples in the previous paragraph show that the mathematical community might well come to benefit from having serious mathematics formalized in a theorem prover. Experience suggests that it is mostly young people who formalize. Thus teaching young people about formalization is important; however it is a different task to teaching young people about mathematics. The Natural Number Game is a browser-based Lean game that came out of many hours at Imperial College London just writing random undergraduate-level mathematics puzzles in Lean and then watching students solving them. The idea of building up facts about natural numbers from first principles was a big hit, and it went on from there. Students say to me “I’ve finished the natural number game, what next?” The correct answer to that is “Install Lean following the instructions on the community website”. But after they’ve done that, what next? This depends on their mathematical interests and abilities. I often encourage an enthusiastic student to formalize some mathematics they already know well for their first Lean project. The rule is: if it compiles, you won. After this we can start talking about how to write code which is of a high standard.

For PhD students and more mathematically mature people who are interested in seeing what’s going on, it’s always worthwhile pointing out current mathematical projects which are run via Zulip at Leanprover.zulipchat.com, the platform behind the Lean community chat. We are always looking for new people to help out with various projects, and are happy to “onboard” newcomers. Right now some of the main active projects on the site are: a formalization of the proof of Fermat’s Last Theorem for regular primes, being led by Riccardo Brasca, and a formalization of some of the main results in the theory of condensed mathematics, being led by Adam Topaz. A future project, starting in 2024, led by me and funded by the EPSRC, is to formalise a full proof of Fermat’s Last Theorem. Past projects which ran on the site include the sphere eversion project, led by Patrick Massot, which formalised a modern proof that one can evert a sphere in three dimensions, and “the liquid tensor experiment”, led by Johan Commelin, formalising the proof of a 2019 theorem of Clausen and Scholze. In 2020 Scholze challenged the formalisation community to verify a crucial theorem he had announced with Clausen, and the Lean community rose to the challenge. The back story is interesting; Scholze suggested that perhaps the current refereeing process that we have in mathematics would not dive deeply into a specific part of the work, and Scholze was interested to see whether a theorem prover could do this instead. It turns out that this was indeed possible. Scholze was an advisor throughout the project, which took 18 months to complete; on the way the community developed many other theories (for example a theory of homological algebra), some of which are now being used in other projects.

**Teaching formalization skills.** Teaching students how to formalize mathematics means teaching them how to translate mathematical ideas between English and Lean. Just like learning a foreign human language, you begin by translating basic stuff between the two languages, and you ask if you don’t understand something. In the Lean course I teach in Imperial College London’s mathematics department, we work through human mathematics that the students have already been taught, and we learn to speak Lean’s language. Later on I introduce some new mathematics, when the students have already learnt some Lean. Athina Thoma and Paola Iannone taught me that teaching the first years equivalence relations and Lean at the same time would usually not end well. However to a student who knows both, formalising the fact that equivalence relations biject with partitions would be an excellent Lean learning experience. The art is to find the right project, and the correct project typically depends on the student.

**Fixing issues in modern research.** In an interview with Wired I was once quoted as suggesting that all maths was wrong. This is something I know not to be true—some maths is definitely correct, and (at least in classical logic) these statements are opposite to one another. However, I did say that perhaps some of our castles were built on sand, and I now think even this is a bit naive. Having had conversations with David Rabouin, a historian and philosopher of mathematics, I now understand that cutting edge maths always looks like this. There are bits which are not quite checked but everyone knows it will usually work out fine, or at least well enough to make the main theorem go through.

Sometimes maths does go wrong though. This century, in my area alone (number theory), prestigious mathematicians have announced proofs of Leopoldt’s conjecture and the ABC conjecture; the latter work was even published in a reputable mathematical journal. Yet our community does not seem to accept the proofs as rigorous. Unfortunately, Lean will not deliver us from these problems, at least not yet. Right now, Lean proves theorems by having humans translate those theorems from the English, and if nobody is prepared to take on the monumental task of translating the Mochuzuki proof of the ABC conjecture into a theorem prover (and why should anybody? This is not how the mathematical community has treated published proofs in the past), then I don’t see any way past the impasse.
Some proofs have already gone beyond the one brain barrier—no one human understands all of the ideas in it. Areas which were formerly fashionable can die out if the big conjectures driving them forwards are resolved. Anything not properly documented actually runs the risk of being lost. One can hope that new and simpler proofs will come along. But history shows that this is not always true: sometimes things are complicated, and stay complicated. Whether or not we choose to use theorem provers to do it, mathematicians need to start thinking more carefully than ever about precisely documenting what we think we already know, so we can answer technical questions from future generations of mathematicians.

Kevin Buzzard

8. When Will Computer-Assisted Proof Become Part of Everyday Mathematics?

Lawrence Paulson

Introduction and background. The idea of applying technology to mathematical reasoning began to be realized in the 1960s. N. G. de Bruijn’s AUTOMATH was a type theory for expressing mathematical definitions and proofs. Trybulec’s Mizar system included a human-readable formal language for abstract mathematics. Both were professional mathematicians and intended their work to be beneficial to mathematics itself, but the technology was not ready. Nevertheless, they made valuable progress. AUTOMATH led to the dominant type theory today, the Calculus of Inductive Constructions. Mizar accumulated a substantial and wide-ranging library of formalized mathematics, while its readable structured language is still the best there is.

Interactive theorem provers, or proof assistants, emerged in the 1970s. The earliest is arguably Mizar, but the most influential was undoubtedly Edinburgh LCF. It implemented a Logic for Computable Functions that promptly became obsolete, but it also introduced an architecture that would be adopted by most successor systems, including Coq, HOL, Isabelle, and Nuprl [7]. These tools were intended for verification in computer science. HOL (for higher-order logic) had been chosen in order to implement a certain style of hardware verification. These verification tasks seldom required any mathematics beyond the integers.

The 1994 discovery of a bug in the floating-point unit of Intel’s Pentium dramatically focused the verification community on the real numbers. John Harrison formally proved the correctness of an algorithm for computing the exponential function, taking account of all the peculiarities of floating-point arithmetic. He went on to play a major role in the Flyspeck project: the formal confirmation of Thomas Hales’s proof of the Kepler conjecture. He formalized many landmark results in mathematics, such as the prime number theorem.

Gonthier’s formalization of the four-color theorem had already demonstrated that an interactive theorem prover (Coq in this case) could help settle a genuine question in mathematics. This task was similar to Flyspeck in that it involved formally checking a large number of computations. Isabelle. Isabelle [18] emerged from the LCF tradition with the aim of supporting a multiplicity of formalisms, including set theory. However, with higher-order logic dominating the verification world, Isabelle/HOL became the dominant instance of Isabelle. Its formalism extended that of the various HOLs with axiomatic type classes, allowing the systematic reuse of formal material sharing the same axiomatic basis [10]. Isabelle adopted a structured proof language, Isar, based on Mizar’s mathematical language. Isar expresses proofs with a nested structure where milestones—intermediate claims and proofs—are explicitly written out. Isabelle provides automation for proof (powerful external provers can be invoked through sledgehammer) and also for disproof in the form of counterexample search. The user interface is a unique interactive development environment for editing live proof documents.

There are powerful synergies between these features. Structured proof text is easy to reuse. Copied into a new development, it will instantly be checked and any errors flagged. Structured proofs also work well with sledgehammer: if a given statement is too difficult to be proved automatically, the user may propose an intermediate statement that might be easy enough to prove automatically and lead eventually to a proof of the original statement.

This powerful automation has greatly relieved the tedium that accompanied formal proof in the early days.

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It has also helped encourage the growth of substantial libraries of formalized mathematics.

**Proof assistants: Ready for prime time?** Today we have a wide variety of systems: the HOL family, for higher-order logic; Isabelle/HOL, for higher-order logic with axiomatic type classes; Coq and Lean, implementing the Calculus of Constructions. These choices are motivated by conflicting priorities such as the simplicity of the implementation and its semantics as opposed to the expressiveness of the logical calculus. Competition between research groups was spurred by Freek Wiedijk's online list detailing which of the "top 100 theorems" [23] had been proved in various systems. Only one of the hundred theorems remains unproved, and that is Fermat's Last Theorem!

Much of modern mathematics falls within the scope of existing verification tools. There are now huge libraries of formalized mathematics. Isabelle's Archive of Formal Proofs[^3] contains over 650 entries including much core mathematics—linear algebra, multivariate analysis, probability, complex analysis, topology—and over three million lines of proofs. Lean's mathlib[^4] is an immense and rapidly-growing corpus of material from every branch of mathematics.[^4]

So are these systems finally ready to support mathematicians? There are still many obstacles:

- **Formal syntax looks artificial and often is barely legible.** To see the difficulties presented by traditional notation, contrast $x^2$, $\nabla^2 f$, $\sin^2 \theta$, $f^2(x)$. In group theory, $G$ denotes a group but also a set; $ab$ is the product of $a$ and $b$ but $HK$ is quite a different thing, simply because we used a different part of the alphabet. In set theory, $\lambda < \aleph_1$ has a different meaning from $\lambda < \omega_1$ even though $\aleph_1 = \omega_1$.
- **The libraries of formalized mathematics still have many gaps, and what they do have is difficult to find.** The names of theorems are frequently ambiguous (what is Roth's theorem?), as are concepts such as limits (which could refer to analysis, topology or even set theory). But one should be able to search for "limit theorems" and get something relevant. Better still, the system might make suggestions unprompted.
- **Obvious statements are often too hard to prove.** Examples include showing a set to be finite, showing a function to be continuous, calculating a derivative and evaluating a limit. In some cases, skilful use of the existing automation can be effective. In others, specialized decision procedures must be coded. Best known are decision procedures for linear arithmetic; a recent development is Manuel Eberl's limit solver.

**Toward the future.** Recently, with funding from the European Research Council,[^7] my colleagues and I have been exploring and stretching the limits of today's technology. We have formalized relatively recent (post 1970) and deep results.[^6] The latter include additive combinatorics, extremal graph theory and combinatorial design theory. We have even formalized some sophisticated definitions, notably Grothendieck schemes—necessary for advanced work in algebraic geometry—which had hitherto been thought to lie outside the scope of Isabelle/HOL's simple type theory.

Mathematicians also need help navigating our huge library of formalized mathematics. It can be hard to know whether a desired result has been formalized: many theorems go by various names, or conversely, one name (e.g., Young's inequality) may be applied to a family of distinct results. For every well-known result, there may be dozens of technical lemmas, mostly obvious and yet needed in order to prove anything on today's systems. My colleagues have been building an experimental search engine, called SERAPIS[^9], that makes it possible to search the entire library with the help of a huge dictionary of mathematical concepts.

Also attractive is the idea that the proofs in our library might be used to generate new proofs automatically. This is another way the computer can help the user get value out of our three million lines of formal proofs. Promising results are just starting to appear from a number of research groups.

For the further future, we may hope to mechanize mathematical intuition. This is the knowledge that tells us that a given function is surely continuous or that a certain formula cannot generate only prime numbers. That's how we know that a particular claim cannot be true as stated or proved using the methods advertised. Formalization has demonstrated time and again that while published proofs often contain errors, theorem statements are generally correct. Mathematicians can perceive the truth if they can't always write down the correct argument. Giving such intuition to a computer would transform our field. This task will remain open for the next generation or two.

[^3]: http://www.isa-afp.org
[^4]: https://leanprover-community.github.io/mathlib-overview.html
[^6]: https://www.cl.cam.ac.uk/~lp15/Grants/Alexandria/
[^7]: Project ALEXANDRIA, GA 742178
[^9]: https://behemoth.cl.cam.ac.uk/search/
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Using Generative AI for Literature Searches and Scholarly Writing: Is the Integrity of the Scientific Discourse in Jeopardy?

Paul G. Schmidt and Amnon J. Meir

1. Introduction

With the public release of ChatGPT on November 30, 2022, AI has gone mainstream. We are aware of colleagues who are using ChatGPT or other generative AI tools to assist with literature searches and the writing of research papers, proposals, and reviews. The practice is likely to spread rapidly. While the advent of broadly available generative AI tools like ChatGPT has spawned exciting new possibilities in numerous fields, including scholarly writing, there is also a huge potential for abuse. Ever since the public release of ChatGPT, serious concerns have been raised about the impact and potentially dire consequences of the widespread use of generative AI tools in scientific writing and publishing.\textsuperscript{17–21}

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DOI: https://doi.org/10.1090/noti2838

\textsuperscript{*}Since Notices of the AMS does not allow extensive bibliographies, most of the citations in this article, marked by numerical superscripts, refer to the bibliography of [1], an online collection of supplementary material for this article, available at SMU Scholar. The bibliography of [1] contains all our references, including the sixteen listed in this article (with the exception of [1]), the ones cited but not listed in this article, and additional titles cited only in [1].

In a recent issue of the journal \textit{ACS Energy Letters} \textsuperscript{2}, published by the American Chemical Society (ACS), Grimaldi and Ehrler write that “a text-generation system combining speed of implementation with eloquent and structured language could enable a leap forward for the serialized production of scientific-looking papers devoid of scientific content, increasing the throughput of paper factories and making detection of fake research more time-consuming.” Observing that ChatGPT and other generative AI tools can write papers that might well pass peer review for a prospective article, the authors point out that “this ability makes the urgent need for a code of conduct for the use of AI-generated text in scientific literature abundantly clear.”

Within a month after the public release of ChatGPT, the editors of flagship journals like \textit{Science} and \textit{Nature}, numerous publishers of scientific literature, and the boards of e-print repositories such as arXiv, ChemRxiv, bioRxiv, and medRxiv were discussing policies and guidelines regarding the use of AI in scholarly writing \textsuperscript{3}. On January 26, 2023, \textit{Science} published an editorial by editor-in-chief H. Holden Thorp \textsuperscript{4}, explaining the rationale behind a recent change to the editorial policies for the \textit{Science} group of journals \textsuperscript{5}: “Text generated from AI, machine learning, or similar algorithmic tools cannot be used in papers published in science journals, nor can the accompanying figures, images, or graphics be the products of such tools,
without explicit permission from the editors. In addition, an AI program cannot be an author of a science journal paper. A violation of this policy constitutes scientific misconduct.

On January 24, 2023, Nature published an editorial [6], postulating that “tools such as ChatGPT threaten transparent science” and suggesting that “researchers should ask themselves how the transparency and trust-worthiness that the process of generating knowledge relies on can be maintained if they or their colleagues use software that works in a fundamentally opaque manner.”

To address the threat, Nature, along with all the Springer Nature journals, adopted the following ground rules [6] for the use of large language models (LLMs): “First, no LLM tool will be accepted as a credited author on a research paper. That is because any attribution of authorship carries with it accountability for the work, and AI tools cannot take such responsibility. Second, researchers using LLM tools should document this use in the methods or acknowledgements sections. If a paper does not include these sections, the introduction or another appropriate section can be used to document the use of the LLM.” Note that Nature’s policy on AI use in authoring is much less encompassing and much more permissive than Science’s.

On February 13, 2023, the Committee on Publication Ethics (COPE) published a position statement on AI and authorship [7], closely mirroring the two principles formulated in the Nature editorial. Numerous editors, publishers, research institutes, corporations, universities, and professional organizations across all fields of academia are members of COPE and adhere to its code of conduct; current membership exceeds 14,000 entities. Many major publishers have subsequently issued detailed guidelines for authors, reviewers, and editors governing the use of generative AI tools.

Different from the American Physical Society (APS), the ACS, and many other professional societies in the sciences, neither SIAM nor AMS are members of COPE. As far as we are aware, and as of this writing, SIAM has not published any policy or guidelines regarding the use of AI tools in scholarly writing. We should know, as one of us serves on the editorial board of the SIAM Journal on Numerical Analysis (SINUM).

A SIAM web page entitled “Information for Journal Authors” includes a section on “Authorial Integrity in Scientific Publication,” at the end of which a subsection on “Other Resources” contains a link to the COPE homepage, where the diligent reader may find COPE’s position statement on AI use in scholarly writing [7]. Another link, unfortunately broken, should lead to the whitepaper “Recommendations for Promoting Integrity in Scientific Journal Publications,” issued by the Council of Scientific Editors (CSE). Section 2.1.15 and part of Section 2.2.2 of the pdf version of this white paper address the use of AI in scholarly writing. The relevant paragraphs were added on April 25, 2023, and are still missing in the html version of the whitepaper. On June 22, 2023, CSE hosted a webinar on “Updates to CSE’s Editorial Policy Recommendations”; the first item on the agenda was to “explore the background of the recent guidance on artificial intelligence and chatbots and how it affects scholarly publishing.”

Only recently, some time this past June, the AMS amended its “Ethical Guidelines and Journal Policies” with a section on the use of artificial intelligence in authoring. “The following statement, issued by the Committee on Publication Ethics in February 2023, has been adopted by the AMS Committee on Publications.” What follows is the three-paragraph COPE position statement referred to earlier, declaring that AI tools cannot be authors and that authors using AI tools must fully disclose any such usage.

Adopting the COPE position statement appears to be little more than a stopgap measure, pending the drafting of more specific AMS policies and guidelines for authors, referees, and editors. In fact, AMS has recently established an “Advisory Group on Artificial Intelligence and the Mathematical Community,” authorized by AMS president Bryna Kra on July 10, 2023, which is charged with discussing “the role of mathematics in the development and deployment of artificial intelligence, the impact of artificial intelligence on research in mathematics, the use of AI in publications, education, and research, and the impact of AI on our community.”

We are fairly certain that SIAM and other professional organizations in our field have also, by now, formed advisory committees charged with the drafting of policies and guidelines regarding the use of generative AI in scientific writing, reviewing, and publishing; yet they appear to be bidding their time, while time is running out.

Professional societies in other fields are much further along. The “Editorial Policies and Practices” of journals published by the American Physical Society (APS) contain a statement on the “Appropriate Use of AI-Based Writing Tools” that addresses both authors and referees. Already on February 27, 2023, the ACS issued “best practices for using AI when writing scientific manuscripts,” including recommendations for scientists using AI tools at any stage of their research and, in particular, when drafting manuscripts. On April 20, 2023, the Publications Board of the Association for Computing Machinery (ACM) approved a new “Policy on Authorship” that includes guidance for the use of generative AI tools. On May 31, 2023, the World Association of Medical Editors
(WAME) published “Recommendations on Chatbots and Generative Artificial Intelligence in Relation to Scholarly Publications.” Similar recommendations were issued by the International Committee of Medical Journal Editors (ICMJE). Also the American Medical Association (AMA) has published clear rules for the use of AI in manuscripts submitted to its journals. All these policies are in line with and expand upon the COPE position.

Also research funding agencies are slowly catching up. In a notice released on June 23, 2023, the Office of Extramural Research of the National Institutes of Health (NIH) informed the community that “the NIH prohibits NIH scientific peer reviewers from using natural language processors, large language models, or other generative Artificial Intelligence (AI) technologies for analyzing and formulating peer review critiques for grant applications and R&D contract proposals.” The rationale: “Uploading or sharing content or original concepts from an NIH grant application, contract proposal, or critique to online generative AI tools violates the NIH peer review confidentiality and integrity requirements.”

In a policy paper published on July 7, 2023, the Australian Research Council (ARC), a peer institution of the National Science Foundation (NSF) in Australia, banned the use of AI tools by grant assessors (i.e., grant proposal reviewers), postulating that “the use of generative AI may compromise the integrity of the ARC’s peer review process by, for example, producing text that contains inappropriate content, such as generic comments and restatements of the application.” Moreover, “when information is entered into generative AI tools, it enters the public domain and can be accessed by unspecified third parties.” Thus, the “release of material into generative AI tools constitutes a breach of confidentiality,” and ARC peer reviewers “must not use generative AI as part of their assessment activities.”

Remarkably, the ARC policy goes well beyond advice and guidance: “In cases where the use of generative AI by assessors is suspected, the ARC will remove that assessment from its assessment process. If, following an investigation, an assessor is found to have breached the Code during ARC assessment, the ARC may impose consequential actions in addition to any imposed by the employing institution.” Here “the Code” refers to the Australian Code for the Responsible Conduct of Research. As reported by Research Professional News, at least one suspect assessor report was actually removed from the peer review process, after a complaint had been lodged. Quoting the Twitter account ARC Tracker, an article in Times Higher Education recounts that multiple applicants to the ARC Discovery Projects program have publicly complained that assessor reports they received were “generic regurgitations of their applications with little evidence of critique, insight or assessment” and appeared to be written by AI or, specifically, ChatGPT.

Different from its peer institutions NIH and ARC, the NSF has been slow in responding to a development widely recognized as having a profound impact on scholarly activity. The 203-page Proposal and Award Policies and Procedures Guide (PAPPG), effective January 30, 2023, contains the string “ai” only as part of words like email, training, or failure; “AI” appears only in RAISE (Research Advanced by Interdisciplinary Science and Engineering); the string “A.I.” does not occur at all, and neither does “Artificial Intelligence.” The same holds for the “Instructions for Proposal Review” found on Fastlane, which were last modified in September 2017. At the time of this writing, a search of the NSF and Fastlane webpages for policies or guidelines regarding AI use in proposal preparation, proposal review, and progress reports does not yield anything relevant. Given that AI research is a major focus of current NSF programs, we find this surprising, to say the least.

Our objective in writing this article is threefold: to call attention to the fact that AI-induced contamination of the scientific literature is not only a threat, but already a reality; to alert the community to the potential pitfalls of using generative AI tools as aids in scholarly writing; and to call upon leaders in our field to issue adequate policies and guidelines as soon as possible. In fact, this article started out as an open letter that we sent to the leadership of SIAM, AMS, and NSF-MPS. We thank AMS president Bryna Kra for her suggestion to turn the letter into a paper suitable for publication in the Notices of the AMS. We are grateful to several colleagues, who saw the original letter or early versions of the paper, for their feedback and encouragement. Guillaume Cabanac, who has been at the forefront of the battle against AI-induced contamination of the scientific literature for several years, was instrumental in opening our eyes and helping us uncover evidence of existing contamination. Finally, we thank the anonymous referees for their constructive criticism and helpful suggestions.

The remainder of this article is organized as follows. In Section 2 we present evidence supporting our claim that contamination of the scientific literature by way of generative AI tools is occurring on a surprisingly large scale, and has been occurring for some time. For a much more detailed discussion, quoting chapter and verse, we refer the reader to the supplementary material available online, at SMU Scholar. In Section 3 we describe our own experiments with ChatGPT and the concerns they triggered. We show how the naïve use of ChatGPT and other generative AI tools may pollute the literature with pseudo-science and potentially serious yet hard to detect falsities and fabrications. Our observations are based on two long conversations with ChatGPT; summaries and verbatim
transcripts are provided in [1]. Finally, in Section 4 we complement our largely negative assessment of ChatGPT’s potential as a research assistant with a discussion of some useful features and sensible applications of the tool in scholarly writing.

2. Generative AI and Scholarship: A Reality Check

ChatGPT was publicly released on November 30, 2022. By the end of January 2023, it had reached 100 million active users, making it the fastest-growing consumer software application in history. TikTok needed about nine months after its global launch to achieve this feat, and Instagram two-and-a-half years.46

ChatGPT is a marvel of generative artificial intelligence, and its successor ChatGPT Plus, powered by OpenAI’s latest and most advanced large language model GPT-4 and available by subscription, is even more versatile. Yet perfect they are not. As amply documented in news reports [12] and technical papers [13], they continue to be plagued by “hallucinations,” a concept introduced by Google AI researchers several years ago. Technically, the term “hallucination” refers to semantically or syntactically plausible, yet factually incorrect or nonsensical output.47

Opinions are split, among AI experts, as to the severity and permanence of this phenomenon.48–50 In any case, it is a fact that ChatGPT and even today’s most advanced chatbots routinely fabricate content and present this fictitious content authoritatively and convincingly. The following are quotes from OpenAI’s “GPT-4 System Card” [13, pp. 59/60]:

“Despite GPT-4’s capabilities, it maintains a tendency to make up facts, to double-down on incorrect information, and to perform tasks incorrectly. Further, it often exhibits these tendencies in ways that are more convincing and believable than earlier GPT models (e.g., due to authoritative tone or to being presented in the context of highly detailed information that is accurate), increasing the risk of overreliance.”

“Overreliance occurs when users excessively trust and depend on the model, potentially leading to unnoticed mistakes and inadequate oversight.”

“As mistakes become harder for the average human user to detect and general trust in the model grows, users are less likely to challenge or verify the model’s responses.”

Nota bene: the creators of ChatGPT are explicitly warning us not to trust the tool in any circumstance where veracity is of the essence.51

Will such warnings deter people from employing chatbots for “truth-sensitive” tasks? Will such warnings at least instill a sense of caution? — A New York lawyer recently filed a ten-page brief in Federal District Court, citing more than half a dozen judicial precedents relevant to the case he was presenting.52 The brief was complete with summaries of court decisions, including direct quotes and internal citations, naming courts and judges, listing dates and docket numbers. Alas, neither the opposing lawyers nor the presiding judge were able to verify these citations. As it turns out, ChatGPT had done the legal research and made it all up. “I did not comprehend that ChatGPT could fabricate cases,” the lawyer told the judge in a subsequent hearing.53

This is what happens when human stupidity meets artificial intelligence. Scientists, of course, and mathematicians in particular, are immune to the disease. Or are they? — In the introduction we reported on the case of a peer review of a research proposal submitted to the ARC that was rejected by the agency after the applicant complained that it was clearly written by ChatGPT.59,43 The smoking gun?

— Apparently, the review contained, out of place and devoid of meaning, the phrase “regenerate response.” This is the label of a button below ChatGPT’s output window. If you copy a response issued by ChatGPT, and you do so in a sufficiently sloppy way, you will capture the label as part of the response. If you then paste the response into a piece of your own writing, and you do so without any editing or even reading, the telltale phrase will end up in your paper or review.

Of course, no professional scientist or mathematician would ever fall into this trap — the case of the ARC referee is surely an anomaly. Maybe they were under a lot of pressure and needed to complete the job really fast. It happened, but it will likely never happen again. Or will it?

PubPeer54 is an online platform for the post-publication peer review of scientific papers, operated by the PubPeer Foundation, a California-registered public-benefit corporation with nonprofit status. Since May 2023, at least 28 papers, published in supposedly reputable and peer-reviewed journals, or posted on the arXiv or other e-print repositories, were flagged on PubPeer for the out-of-place occurrence of the phrase “regenerate response”; at least seven more were flagged for containing the phrase “as an AI language model,” which ChatGPT frequently includes in its responses (usually as part of a caveat, warning the user not to blindly trust its output). At the time of this writing, additional papers are being flagged on an almost daily basis.

We decided to refrain from directly quoting any of these problematic papers in the present article. However, we furnish a detailed analysis of a representative sample, quoting chapter and verse, in the supplementary material available online, at SMU Scholar [1].

One particularly striking example came to our attention just a few days ago: a paper presented at a May 2023 conference on emerging technologies and published in the
IEEE Xplore digital library by the Institute of Electrical and Electronics Engineers (IEEE). The paper was flagged on PubPeer in July, for including the unexpected phrase “regenerate response” at the end of the abstract. The introduction of the paper describes related prior work, referencing Saponas et al. (2009), Wang et al. (2015), Jin et al. (2016), Kang et al. (2019), and Moussa et al. (2020), albeit without providing any further bibliographical details. In early August, a commenter on PubPeer noted that none of these five “references” appears in the bibliography at the end of the paper and that all of them are very likely ChatGPT hallucinations. We searched both Scopus and Dimensions AI for 2009 publications with first author Saponas (a Microsoft researcher) and found two; however, neither one matches the description in the introduction of the suspect paper. The other four “references” are harder to unmask, since the names of the first authors are very common; but we are convinced they are hallucinations as well.

In fact, we believe that besides the abstract, also the entire introduction of the paper is AI-generated. The style is vintage ChatGPT, including the vague and imprecise way of describing and referencing prior work — we have seen this many times in our own experiments. Had the human authors prodded ChatGPT to provide additional bibliographical information about those five “references,” it would have happily obliged, producing precise and complete, if completely bogus, references. Too bad that this thought did not occur to them. Most likely, they didn’t even notice that ChatGPT’s introduction included incomplete references not included in their bibliography...

A number of AI tools are available to detect artificially generated text. One of them is OpenAI’s GPT-2 Output Detector, which estimates the probability of a given text being AI-generated. The source code is publicly available, an easy-to-use implementation can be found online. According to OpenAI, the tool “is able to detect 1.5 billion parameter GPT-2-generated text with approximately 95% accuracy.” The phrase “1.5 billion parameter GPT-2-generated text” refers to output produced by the largest GPT-2 model; the claim about the detector tool’s accuracy is cryptic at best, and we have been unable to obtain more precise information. However, according to Cabanac et al. [15], and confirmed by our own experiments, the tool is quite effective in detecting not only GPT-2 output, but also other AI-generated content. As a caveat, we note that AI detection tools have recently come under a lot of fire. In January 2023, OpenAI launched a new detection tool, which it promptly retracted less than six months later, “due to its low rate of accuracy.”

For whatever it’s worth, we fed both the abstract and the introduction of the afore-mentioned paper into the GPT-2 output detector. Both garnered whopping scores of 99.98% and 99.94%, respectively. Again, the detector score represents the probability of the respective text being AI-generated. Interestingly, if the unexpected phrase “regenerate response” is included, the abstract’s detector score is merely 11.71% — apparently the detector considers this phrase so weird and out of place that it must be human-generated...

The papers referred to above all feature telltale signs of ChatGPT’s unacknowledged involvement in the writing, which typically constitutes a violation of the publisher’s code of conduct, but does not necessarily qualify as contamination of the scientific literature.” While the detection of AI-generated content can be automated, parsing content for evidence of plagiarism, factual errors, fabricated reviews, and fictitious references still takes a human expert.

Typically, issues of this kind are uncovered by people who are directly affected, like the afore-mentioned ARC grant applicants who received AI-generated referee reports. Another example, recently reported in Times Higher Education, is the case of Robin Bauwens, a social scientist at Tilburg University in the Netherlands, whose submission to an Emerald Publishing journal was rejected by a referee. In the report, the referee suggested that Bauwens familiarize himself with several relevant literature reviews in his field, all allegedly authored by Dutch academics unbeknownst to him. References and authors turned out to be fictitious, most likely fabricated by ChatGPT.

Emerald Publishing told Times Higher Education that it had recently updated its guidelines for authors and referees, stating in particular that “ChatGPT and other AI tools should not be utilized by reviewers of papers submitted to journals published by Emerald” and that “AI tools/LLMs should not replace the peer review process that relies on human subject matter expertise and critical appraisal.” The mere thought that the latter should warrant emphasizing is cringe-inducing, at least to some of us...

Nobody outside the corporate headquarters of OpenAI knows exactly what data ChatGPT was trained on. There is anecdotal evidence that the MEDLINE database of references and abstracts on life sciences and biomedical topics was included in its training data. In fact, ChatGPT did fairly well in a recent study [16] published in the journal Cureus (part of Springer Nature Group). The experiment started with the following prompt: “Suggest 50 novel medical research topics that can be performed by undergraduate medical students in India. The topics must be feasible, interesting, novel, ethical, and relevant” (the so-called FINER criteria). ChatGPT readily obliged and was subsequently instructed to write an elaborate research protocol for each of the 50 topics, including a proper...
introduction, sections on objectives, methodology, and implications, and a list of references. The protocols were then evaluated by a panel of experts and generally found to be feasible.

However, of the 178 references provided by ChatGPT, 69 did not have a valid DOI, while 28 could not be located at all and are presumed to be fictitious. The authors conclude that “researchers using ChatGPT should exercise caution in relying solely on the references generated by the AI chatbot.” We promise to keep that in mind, especially since the reference hallucination rate we observed in our own experiments (see Section 3) was much closer to 100% than the measly 28/178 ≈ 16% found in the Cureus study.

AI-induced contamination of the scientific literature predates the public release of ChatGPT by several years. In [14] the computer scientists Guillaume Cabanac and Cyril Labbé discussed the prevalence in the scientific literature of “research papers” devoid of meaningful results and sometimes plainly nonsensical. They suspected that such papers are algorithmically generated by programs like SCIgen or MATHgen and proposed methods of detection and elimination.

In [15] Cabanac, Labbé, and Magazinov coined the term tortured phrases: unexpected strange phrases in lieu of established ones, such as “counterfeit consciousness” in lieu of “artificial intelligence.” A typical tortured phrase emerges when word-by-word synonymic substitution is applied to a multi-word technical phrase. In this way, “high-performance computing” may morph into “elite figuring,” “data warehouse” into “information stockroom,” “network attack” into “organization ambush.” Nontechnical articles summarizing and discussing the results of [14, 15] appeared in Nature and the Bulletin of the Atomic Scientists.

Cabanac et al. made a list of 30 tortured phrases they had encountered in the scientific literature, most involving computer-science parlance. Using the Dimensions AI academic search engine, they exposed more than 800 papers that included at least one of the phrases; 31 of those appeared in a single journal published by Elsevier, Microprocessors and Microsystems. “Preliminary probes show that several thousands of papers with tortured phrases are indexed in major databases,” they write, pointing out that their study focussed on tortured phrases in computer science and that “tortured phrases related to the concepts of other scientific fields are yet to be exposed.”

Text containing tortured phrases typically scores high when run through an AI output detection tool, which estimates the probability of the text being AI-generated. Cabanac et al. [15] used OpenAI’s GPT-2 Output Detector to analyze the abstracts of tens of thousands of papers across the scientific literature. They retrieved the abstracts of all articles published in Volumes 80–83 (2019–2021) of Microprocessors and Microsystems (MPMS) that were processed in less than 30 days (from date received to date accepted), a total of 389 articles. This set of abstracts (Set 1) was tested against several control sets, including the abstracts of 50 articles recently published in MPMS with processing times exceeding 40 days (Set 2), the abstracts of 50 computer-science related articles recently published in SIAM journals (Set 3), and the abstracts of a sample of 139,236 articles published in 2021 by Elsevier (Set 4).

Of the abstracts in Set 1 (MPMS with fast processing), 81% scored at least 0.9 (90% probability of being AI-generated), while only 8.5% scored below 0.1 (10% probability of being AI-generated). Of the abstracts in Set 2 (MPMS with not so fast processing), 16% scored at least 0.9, while 78% scored below 0.1. Of the abstracts in Set 3 (SIAM journals), none scored above 0.7, while 90% scored below 0.1. Of the abstracts in Set 4 (Elsevier 2021), 3% scored at least 0.9, while 89.9% scored below 0.1.

Elsevier journals that published at least 25 articles in 2021 with GPT-2 output detector scores of at least 0.7 include the Journal of Computational and Applied Mathematics (39 articles, representing 13.6% of all articles tested), the Journal of Differential Equations (27, 11.2%), and the Journal of Mathematical Analysis and Applications (25, 7.5%).

Let this sink in for a moment: the Journal of Differential Equations, the gold standard in the field, published at least 27 articles in 2021 whose abstracts were, with a probability of at least 70%, AI-generated. We take comfort in the thought that Joseph LaSalle and Jack Hale did not live to witness this moment…

Since the study by Cabanac et al. predates the public release of ChatGPT by more than a year, the culprits are earlier generative AI tools. GPT-2 was released in 2019, GPT-3 in 2020; but more likely, the authors of the suspect abstracts employed automatic translation or paraphrasing tools such as SpinBot, a free, automatic article spinner that will rewrite human readable text into additional, intelligent, readable text” that has been around for more than a decade. Authors resorting to automatic translation may be doing so with the perfectly honest objective of improving their writing in English. Authors employing paraphrasing apps such as SpinBot most likely do so to conceal plagiarism.

As noted before, the mere fact that a research paper was written with the aid of an AI tool does not render it fraudulent or otherwise objectionable, at least not if the fact is acknowledged (which it usually and unfortunately isn’t). Obviously, Cabanac et al. were not able to fact-check every one of the suspect papers they discovered; they did, however, perform a detailed analysis of a sample of the papers in MPMS that had been flagged as most likely...
containing AI-generated content. Beyond evidence of plagiarism (paraphrasing of existing text or recycling of images without acknowledgement), they found references to nonexistent literature, references to nonexistent internal features (such as labeled theorems or formulæ), and a lot of tortured phrases and meaningless gibberish. Several of the papers were found to be so similar in style and structure that they are likely traceable to the same source, presumably a paper mill that produces scholarly articles on demand and for hard dollars.68

Sometimes a suspect paper is retracted by the publisher after being flagged on PubPeer. A typical example is a 2019 paper in the *Journal of Physics: Conference Series*,69 flagged on PubPeer in January 2023 for containing numerous “tortured phrases.” IOP Publishing retracted the paper in March 2023, citing “concerns this article may have been created, manipulated, and/or sold by a commercial entity” and noting a lack of “evidence that reliable peer review was conducted on this article, despite the clear standards expected of and communicated to conference organisers.” According to Dimensions AI, the paper was cited at least five times before it was retracted.

Based on what we have seen, the retraction of a suspect paper is still a rare event. Most of these monstrosities continue to lurk in the literature and garner citations — by authors who use AI tools to find references relevant to their work or choose their references by title only. It will take a concerted effort by all stakeholders in the business of scientific publishing to design and implement procedures to stem the flood of fake papers and the proliferation of pseudo-science.

While Cabanac et al. focussed on computer science papers and many others reported on AI infection of the medical literature,19–21,70–72 our own field has not received much attention. However, as illustrated by the aforementioned findings about possibly infected articles in three prominent mathematics journals published by Elsevier, we cannot assume to be immune against the virus.

To furnish an example of an infected math paper, published in a supposedly reputable math journal, we analyzed a marvelously tortured article73 that appeared in a 2021 issue of *Partial Differential Equations in Applied Mathematics*, an open-access journal published by Elsevier with a clear and rigorous peer-review policy.

Supposedly a survey of spectral methods in the numerical analysis of nonlinear fluid flow problems, the article brims with tortured phrases, mangled sentences, and incomprehensible gobbledegook. Examples of tortured phrases include “conventional differential conditions” (ordinary differential equations?), “liquid stream models” (fluid flow models?), “limited distinction techniques” (finite-difference methods?), and “limited volume techniques” (finite-volume methods?). In the introduction, the authors reference a paper entitled “Highly Accurate Solutions of the Blasius and Falkner-Skan Boundary Layer Equations via Convergence Acceleration” (emphasis ours), creatively describing it as concerned with “profoundly precise arrangements of Blasius and Falkner-Skan limit layer conditions through combination quickening.”

The literature referred to in the introduction of the paper is correctly referenced in the bibliography and appears to be real. Despite all the gibberish, the introduction reaps a GPT-2 output detector score of 42.81%, indicating that it is more likely than not of human origin or has at least undergone some human editing. That said, we found numerous passages throughout the paper that are completely incomprehensible and garner GPT-2 output detector scores exceeding 99%. The paper was processed in less than 30 days (received March 4, received in revised form April 1, accepted April 2, 2021), and it has been cited at least 18 times in subsequent publications (according to Dimensions AI).

We have been labelled alarmists for asking the question in the title of this paper; our concerns about the possible contamination of the scientific literature with AI-generated pseudoscience, fabrications, and falsities have been called “empty fear-mongering.” This is quite understandable. Had we been told about this cesspool four months ago, we would have been incredulous ourselves. How can it be that a professional scientist or mathematician enlists ChatGPT to write parts of a paper, copying and pasting its responses without any editing, not even eliminating telltale phrases such as “as an AI language model” or mistakenly copied “regenerate response” labels? How can it be that professional scientists or mathematicians employ automatic translation tools or paraphrasing software to write papers brimming with tortured phrases, mangled sentences, and incomprehensible gobbledegook? How is it possible for such papers to pass peer review as well as editorial scrutiny and be published in supposedly reputable journals? Well, dear colleagues, it is not only possible, but happening on a large scale. Time for a reality check!

Let’s say it one more time: clear-cut policies must be put in place, and urgently so. First, to discourage abuse of generative AI tools, any use of such tools in drafting a manuscript must be formally acknowledged, specifying exactly which tools were used and for what purpose. Any deliberate violation, if detected, should result in automatic retraction. Further, employing generative AI tools in refereeing manuscripts or research proposals is unacceptable. In fact, since chatbot conversations are stored and monitored, sharing a manuscript or proposal under peer review with a chatbot constitutes a violation of confidentiality akin to posting the material on the internet. More importantly, while chatbots may be able to summarize a research paper...
or proposal, they cannot assess its intellectual merit or discuss its place in the literature — this is still the province of human experts. Any peer review found to violate these principles should be automatically rejected.

Such policies would be largely in line with those published by the Committee for Publication Ethics [7], the Council of Science Editors,31,32 many major publishers,24–28 the World Association of Medical Editors,36 the International Committee of Medical Journal Editors,37 the American Medical Society,38 the American Physical Society,33 the American Chemical Society,34 the Association for Computing Machinery,35 the National Institutes of Health [10], and the Australian Research Council [11]. We urge SIAM, AMS, and other professional organizations in our field to follow suit; we call on NSF and other agencies that provide research funding in our field to follow the lead of NIH and ARC.

Beyond clear-cut policies as proposed above, we urgently need detailed guidelines for authors, reviewers, and editors, addressing what does or does not constitute legitimate use of generative AI in scholarly writing, reviewing, and publishing.

Given that this article will appear in the Notices of the AMS, let us consider, for example, the role of generative AI as it relates to Mathematical Reviews and MathSciNet. Is it acceptable for a MathSciNet reviewer to have ChatGPT write a review of a paper they have been assigned? Probably not. That said, is it acceptable for the reviewer to have ChatGPT summarize the paper, then evaluate the paper based on this summary rather than the full text? Maybe yes. Yes or no, the temptation to do so is almost irresistible! Why go through the trouble of reading a paper and summarizing its content, given that ChatGPT can generate a decent summary in mere seconds? Given that most MathSciNet reviews are anyway little more than summaries, should we just give up on human reviewers and let ChatGPT do the job? That would save a lot of us a lot of time and would most certainly improve the overall quality of the English! — We are not proposing this in earnest; but we do wish to provoke some serious discussion in the community.

Clearly, there are legitimate applications of generative AI in scholarly writing. Specifically, ChatGPT can help authors improve their writing, in terms of spelling, grammar, syntax, choice of words, structure of the narrative, and overall presentation. It is also quite a capable translator, which may be useful for authors who wish or need to write in English, but lack proficiency in the language. It can read, write, and improve LaTeX code. It can summarize articles, which may be a great help when searching the literature. ChatGPT Plus, with suitable plugins enabled, can search the internet for open-access literature and help with reference management. It can also facilitate computation, data analysis, and visualization. In Section 4 we describe these capabilities in more detail.

As long as sensible applications of generative AI in scholarly writing are formally acknowledged, and as long as authors take care in checking and double-checking any AI-generated content for soundness and accuracy, we don’t see a problem. That said, we have encountered numerous cases, where authors used generative AI without acknowledging the fact and without as much as diligently reading the output, let alone checking for soundness and accuracy. Worse, generative AI has been used to produce pseudo-scientific papers devoid of any scholarly value. Worse yet, it has been used to conceal plagiarism: simply lift a text from someone else’s work, ask ChatGPT to rewrite it in a different style, with different wording, then copy and paste the output into your own work. Unless you are sloppy enough to copy and paste also the “regenerate response” label, nobody might ever notice. Of course, it doesn’t take artificial intelligence to generate this kind of concealed plagiarism — human dishonesty is all it takes. However, AI makes the job so much easier that AI-assisted plagiarism is a serious concern.

In an earlier version of this article we wrote that, unless the mathematics and wider science community takes appropriate action, “it is not only perceivable, but unavoidable, that wide-spread use of ChatGPT and other chatbots for purposes of scholarly writing will contaminate the scientific literature with potentially serious yet hard to detect errors and fabrications, from erroneous DOIs to phantom references, allusions to imaginary scholars with fantasy biographies, and factually wrong descriptions of prior research.” By now we know that such contamination has been occurring for some time, and on a fairly large scale. Given the universal availability of generative AI tools such as ChatGPT, the problem is presumably growing exponentially by now.

This is a watershed moment. The integrity of the scientific discourse is on the line.

3. Experiments with ChatGPT

About half a year ago, we decided to experiment with the free version of ChatGPT, publicly released by OpenAI on November 30, 2022. Initially, we thought of it as entertainment rather than research. We would quiz ChatGPT on questions of calculus or linear algebra. We would marvel at its abysmal performance and the fact that it reacts much like a subpar college student, with superior factual knowledge, yes, but lacking the most basic algebra and reasoning skills and making all the standard mistakes that any teacher of entry-level college mathematics has witnessed time and again.
Some of these deficiencies of ChatGPT and other chatbots will likely disappear soon enough. Already ChatGPT Plus, in the most advanced version powered by GPT-4, can call external APIs such as Wolfram Alpha, a plugin that greatly enhances its algebraic and computational capabilities. A recent paper by Microsoft scientists, reporting on early experiments with GPT-4, describes impressive mathematical reasoning capabilities, many persistent deficiencies notwithstanding. Further improvements can be expected in the near future.

While quizzing ChatGPT on calculus is entertaining rather than disturbing, we soon discovered some very serious issues. As discussed earlier, ChatGPT and other chatbots are prone to “hallucinations”: they routinely fabricate content and present this fictitious content authoritatively and convincingly. In the case of ChatGPT, the frequency and depth of these fabrications are nothing short of astounding.

Specifically, when questioned about research-level mathematical problems and related literature, ChatGPT typically responds in a coherent and seemingly competent manner, explaining the problem and its context, producing complete references to what appear to be relevant publications, and concisely summarizing their content. Typically, the titles and summaries of these references will closely mirror the language and wording employed by the user in their prompts (a feature technically known as “sympathy”): ChatGPT is talking your language and finding exactly what you are looking for! However, the user who then tries to locate the references identified by ChatGPT will soon realize that practically all of them are fake: they simply do not exist. They are fabrications, concocted to please the user. (Of course, ChatGPT is not actually capable of “concocting content” or “pleasing the user”; but its algorithms are clearly designed to ensure a positive user experience.)

In [1, Section 2] we illustrate these disturbing features in the context of one specific conversation with ChatGPT, which started as a genuine inquiry rather than a test. A verbatim transcript of this rather long and meandering conversation is also provided.

Subsequently, we performed more systematic experiments, asking ChatGPT to summarize research papers, or to retrieve abstracts or reviews thereof.

When provided solely with the URL or DOI or arXiv identifier of a given paper, ChatGPT will readily provide bibliographical information and a concise summary of what it claims is the paper in question; alas, usually it is not. Many times, the “paper” that ChatGPT references and summarizes simply does not exist; also the alleged authors may be fictional.

When provided with the full title of the paper in question, ChatGPT’s summary will usually seem plausible, employing language matching the title and terminology commonly used in the field, but it may well include factually wrong statements; also bibliographical details, such as authorship and publication data, may be wrong. Even when provided with full bibliographical information about the paper in question, ChatGPT’s summary, while seemingly plausible, may describe nonexistent features and fall far short of capturing the essence of the paper.

When asked to retrieve the DOI of a given paper, ChatGPT will usually oblige, but the DOI it returns is almost always wrong, either identifying a different paper or simply nonexistent. When asked to retrieve the abstract of a paper, ChatGPT will furnish the alleged abstract, usually rendered as a direct quote, enclosed in quotation marks. Alas, the quote rarely agrees with the actual abstract of the paper, neither in wording nor in substance. This behavior seems to be independent of what information the user provides about the paper in question, be it a URL, a DOI, an arXiv identifier, partial or full bibliographical details, or any combination thereof.

When asked to retrieve the MathSciNet review of a paper, ChatGPT will occasionally admit lack of access to MathSciNet; more often, however, it will oblige and produce a fake review, complete with a bogus MR number and not bearing any resemblance to the actual review.

Of course it is naïve and unreasonable to ask ChatGPT, which lacks real-time access to the internet, to summarize a paper available at a given link, be it publicly accessible or behind a paywall, or to retrieve the MathSciNet review of an article. It is deeply troubling, though, that ChatGPT will readily indulge such prompts and fabricate the requested summary or review, instead of informing the user that it cannot retrieve the paper in question or access MathSciNet.

Also, while ChatGPT was trained on huge amounts of text purloined from the internet, including scientific papers and books, it can obviously not be expected to have encountered paywall-protected material, including the content of most professional journals or databases such as MathSciNet (even though it will occasionally claim that OpenAI has the resources to pay, and does actually pay, for content available by subscription only). That said, the DOI registry and the arXiv are public-access repositories, and it is not unreasonable to assume that their content, as of September 2021, the cutoff date for ChatGPT’s initial training, were included in its training data. If so, ChatGPT’s abysmal performance in our experiments seems hard to explain.

One might suspect that ChatGPT is simply not designed to retrieve verbatim text that it encountered during training; but this is not the case. Asked, for example, to recite the poem “The Second Coming” by William Butler Yeats, ChatGPT will return a complete and correct
quotation. Likewise, it will correctly quote the opening lines of the novel Cat’s Cradle by Kurt Vonnegut or Rutger Hauer’s “Tears in Rain Monologue” from the movie Blade Runner. It is not clear why ChatGPT would be able to retrieve verbatim text from works of fiction, but not the abstract of a paper posted on the arXiv that it encountered during training.

In [1, Section 3], we illustrate our observations regarding ChatGPT’s information retrieval capabilities (or lack thereof) in the context of a specific conversation with the chatbot; a verbatim transcript is also provided.

We note that there exist numerous AI-powered apps, such as Elicit, Scite, Scholarcy, or Humata, which are specifically designed to assist with literature searches or to summarize articles, and which may well perform better at these tasks than ChatGPT. We have not tested any of these, nor have we experimented with Microsoft’s Bing (powered by OpenAI’s recently released GPT-4), Google’s Bard (powered by the company’s own language model LaMDA), or Meta’s BlenderBot (powered by the company’s open-source OPT-175B language model).

4. Sensible Uses of ChatGPT as an Aid in Scholarly Writing

After all the vitriol we have spilled about the pitfalls of using ChatGPT for purposes of scholarly writing, the reader may have concluded that the only way to avoid those pitfalls is to refrain from using the tool at all; however, this conclusion would be premature. As with every technology, the key is to use it judiciously: to be aware of its capabilities, but also of its limitations; to put its capabilities to work, while not asking for the impossible.

A peculiar issue with ChatGPT is that it will happily indulge most any user request, be it sensible or not. If it cannot fill your request, instead of telling you so, it will likely fudge a response, and typically, the response will sound perfectly reasonable and authoritative, even though it may be completely bogus. We have seen multiple examples of this behavior in our conversations with the chatbot [1, Sections 2 and 3]. The obvious remedy is to avoid unreasonable requests. This, however, is easier said than done. OpenAI is not very transparent about ChatGPT’s capabilities and limitations, and ChatGPT itself has no idea what it can or cannot do. When quizzed about its capabilities and limitations, it tends to get entangled in contradictions.

We are aware of a few articles on good practices for scientific writing with ChatGPT. Compiling a more comprehensive practical guide might be a worthwhile endeavor, but is clearly beyond the scope of this paper. Here, we will only briefly discuss a few capabilities of ChatGPT that are potentially useful for scholars. Some of these capabilities are accessible only via ChatGPT Plus, the subscription version of ChatGPT, which is powered by the most recent incarnation of OpenAI’s large-language model, GPT-4, and can be enabled to use external APIs or plugins, including several that provide real-time access to the internet.

Most of the jobs addressed below require ChatGPT to process text it has been fed by the user. Plain text is the preferred input format, and pdf documents should be converted to plain text before input; but also LaTeX source code is acceptable and may be preferable in case of a mathematical manuscript.

There is a limit to the length of text that ChatGPT is able to process in a single prompt-and-response interaction: input and output combined should not exceed 4096 “tokens” (words, punctuation marks, special characters, symbols), very roughly the equivalent of three to six pages of a journal article or book. To leave enough space for the response, the input text should, in fact, be shorter than this.

Depending on the version of ChatGPT you are using, it may or may not allow you to input text that exceeds the 4096-token limit. If so, ChatGPT will process only a chunk of it. Be aware that, unless prodded, ChatGPT won’t tell you that it didn’t arrive at the end of the text! If you inquire, however, it will tell you whether it processed all or just part of the input. If it processed only a chunk, it may be able to go back and process additional chunks. Even when technically possible, inputting text in excess of the 4096-token limit is best avoided.

4.1. Translating text. ChatGPT is quite a capable translator, at least between the languages we are familiar with, English, German, Italian, and, to a lesser degree, Hebrew. According to a recent news article, it is also quite proficient in Chinese, albeit somewhat less so than Baidu’s recently released chatbot Earnie. While we have not performed systematic experiments, circumstantial evidence suggests that ChatGPT is better than other translation apps that we have used in the past. Obviously, this capability is potentially useful, for example, to authors who need or wish to write in English, but lack proficiency in the language. Practical advise: don’t input much more than a page at a time!

4.2. Improving poor writing. You can feed ChatGPT a text and ask it to suggest improvements, in terms of spelling, grammar, syntax, choice of words, structure of the narrative, and overall presentation. You can also ask it to rewrite the text on its own, in accordance with the rules of penmanship that it has been taught. This may be useful to authors who don’t consider themselves distinguished wordsmiths. For advice on formatting and admissible text length, refer to the comments preceding Section 4.1.

4.3. Improving LaTeX code. ChatGPT can read and write LaTeX code. You can feed it a messy LaTeX source file and
ask it to suggest improvements or to rewrite and streamline the code. You can even ask it to modify the code so as to match the style of a particular publisher or journal; but the process is so cumbersome that you may prefer to do the job yourself, at least if you are using the standard version of ChatGPT. Obviously, GPT-4 with internet browsing enabled is more helpful, as it can retrieve style files from publishers’ websites. Practical advice: input your LaTeX code in snippets rather than large chunks!

4.4. Summarizing text. ChatGPT is quite adept at summarizing text that it has been fed or that it retrieved from the internet. To obtain a summary of a paper in your possession, feed ChatGPT a plain-text version (or the LaTeX source file, if available). There are a number of tools to convert PDF to text, but not all are created equal. Adobe Acrobat Reader offers a “save as text” option, which is convenient and works reasonably well for standard text, but is not adequate for converting displayed math or other non-textual elements of the original PDF document.

Whether you input plain text or LaTeX code, unless it’s a very short paper, you will need to chop it into sensible chunks, for example, the sections or subsections of the paper. ChatGPT can summarize each chunk separately or generate a summary of the entire paper after it has processed all the chunks. You can ask for a brief (“high-level”) summary or a detailed description, including technical details or avoiding technical jargon.

If the paper is publicly available on the internet, say, on the arXiv, ChatGPT Plus can usually retrieve it, using, e.g., a plugin called “ScholarAI.” Unless it’s a very short paper, ScholarAI will retrieve it in chunks, suitable for processing by ChatGPT. One drawback: ScholarAI will not actually retrieve the full text of the paper, but drop all non-textual elements such as displayed math, tables, and figures.

Another plugin capable of retrieving documents from public-access direct-download links is “Link Reader.” Compared to ScholarAI, Link Reader is better at handling nontextual elements such as displayed math, but does not support “retrieval in chunks” and hence delivers usually a severely truncated version of the requested document.

In any case, you need to stay alert! Don’t trust ChatGPT blindly, as it will not inform you of its limitations. For instance, when asked to summarize a paper available online, it will happily furnish a summary, based on whatever ScholarAI or Link Reader delivered, even if it was just a snippet of the actual paper. You can look at the output produced by ScholarAI or Link Reader by clicking on a link just above ChatGPT’s response. However, this output is not meant for human consumption and thus very hard to read. As it was aptly put in the title of a paper we cited earlier: “ChatGPT is a Remarkable Tool — For Experts.”

4.5. Searching the literature. ScholarAI can search the open-access scientific literature for abstracts matching two to six keywords, specified by the user or chosen by ChatGPT in accordance with the user’s prompt. The search can be narrowed by publication years; results can be sorted by relevance or other criteria. Once a matching abstract has been found, ScholarAI is usually able to retrieve the full text of the paper or at least provide a download link. Relevant citations can be saved to a reference manager.

While all of this sounds great in theory, the practice is much less appealing. For one, it is not clear at all what sources ScholarAI draws upon, and frequently the search results seem rather random. We faked a search, having some particular references in mind and providing exactly matching keywords. ScholarAI dug up a number of loosely related works, but not the much more relevant ones we had in mind, not even after we specified the exact publication year, and not even after we provided the names of the authors. When we suggested that preprints or postprints of relevant papers might be available on the arXiv, ChatGPT invoked the Link Reader plugin, asking it to search www.arxiv.org for work by the authors whose names we had provided. Link Reader succeeded, listing the references we were after as its top search results.

Obviously, neither ScholarAI nor Link Reader can access content behind a paywall or otherwise restricted. But even sites that are freely accessible to humans may be off-limits for these apps, as the sites may be blocking “web-crawling” or any access attempts by bots. We suspect, for example, that the abstracts of papers published in SIAM journals, while freely available to humans, are not accessible to apps such as ScholarAI and Link Reader.

4.6. Computation, data analysis, and visualization. ChatGPT Plus can be enabled to access Wolfram Alpha and, thereby, a plethora of computational, analytical, and graphical capabilities. As we have yet to perform systematic experiments, we defer a detailed discussion to a future article.

4.7. Words of caution. At this point, we feel the need to put a damper on the reader’s possibly growing enthusiasm about all these exciting capabilities. To begin with, let’s recall that plugins such as ScholarAI, Link Reader, and Wolfram Alpha, are presently still beta features, for good reasons, and things are changing on a daily basis. As of this writing, one of the latest features, “Browse with Bing,” is temporarily disabled, because it could apparently bypass paywalls and gain access to protected content.

Moreover, ChatGPT’s interaction with active plugins is quite unstable, to say the least. For one, it tends to overuse them, invoking them frequently when doing so is useless or even detrimental. During a routine task, such as summarizing a given input text, in chunks, as necessitated by
ChatGPT listed as author on research papers: many scientists disapprove, Nature 613 (2023), 620–621. https://doi.org/10.1038/d41586-023-00107-z

H. H. Thorp, ChatGPT is fun, but not an author, Science 379 (2023), no. 6630, 313. https://doi.org/10.1126/science.adg7879


On the Importance of Illustration for Mathematical Research

Rémi Coulon, Gabriel Dorfsman-Hopkins, Edmund Harriss, Martin Skrodzki, Katherine E. Stange, and Glen Whitney

In the last decade, it has become increasingly possible for researchers to augment their experience of abstract mathematics with sensory exploration: 3D-printed or CNC-milled models, the ability to walk through seemingly impossible physical spaces with virtual reality, and the potential to explore high-dimensional mathematical spaces through computer visualization all provide such opportunities, among other media. Now much more than simply an aid to understanding, these tools have reached a level of sophistication that makes them indispensable to many frontiers of mathematical research. To preview one particular case recounted below, the tantalizing structure visible in Figure 1 (and many others like it) led to conjectures and proofs that would likely otherwise have been inaccessible.

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DOI: https://doi.org/10.1090/noti2839

The list of examples of research driven by illustration is rapidly expanding. We use the term illustration to encompass any way one might bring a mathematical idea into physical form or experience, including handmade diagrams or models, computer visualization, 3D printing, or virtual reality, among many others. We will discuss instances of this interplay in fields ranging from representation theory to geometry and many others. Many readers will also be aware of the recent and celebrated solution of the Einstein problem with the hat monotile and its chiral version, the spectre \[ SMKGS23 \]. These solutions came after many years of physical exploration by Dave Smith.
Illustration is beginning to find a home at programs like the Special Semester in Illustrating Mathematics in Fall 2019 at the Institute for Computational and Experimental Research in Mathematics (ICERM), and the Institute for Advanced Study (IAS)/Park City Math Institute (PCMI) virtual program in Summer 2021, and a community is forming around many modern tools. Of course, the importance of illustration to research is not new: abstraction was linked to plane diagrams in the work of the ancients, including Euclid’s *Elements* or the Chinese treatise *The Nine Chapters on the Mathematical Art* (九章算術). Precise three-dimensional models were produced by skilled artisans in the nineteenth century, notable examples of which remain in the collections at the Institute Henri Poincaré or Göttingen University, among many other institutions. When computer visualization first became widely available in the 1980s, the Geometry Center was founded at the University of Minnesota, with a mission to exploit these new tools on behalf of mathematics. But we are now at another cusp: modern technological tools have recently made 3D models and virtual reality widely available, and computation and computer visualization is more accessible and more powerful than ever. We can now collect huge mathematical datasets and examples, and it has become urgent to develop intuitive ways to interact with this data.

Making full use of modern tools is not without its challenges: beyond the obvious technical challenges and software learning curves, there are important questions about how an illustration, much like a statistic or an experiment, can subtly mislead the researcher, or miss the essential mathematical pattern sought. Researchers often individually reinvent the necessary skill sets as they seek to advance their own projects, and these projects are pushing the boundaries of the possible. But by building a discipline around this enterprise, we can develop its full potential to advance mathematical research.

**Some Highlights from the History of Mathematical Illustration**

Illustration of mathematics goes back as far as mathematical ideas themselves. In fact, some of the earliest evidence we have for abstract thinking comes from human-made designs, for example the cross-hatched carvings in Blombos Caves in South Africa, potentially from 73,000 years ago. A little more recently, the Middle Eastern tradition of geometry presented in Euclid’s *Elements* provides a structural link between statements deduced from axioms, and figures made with straight edge and compass. These two tools provide a physical realization of the two key objects (straight lines and circles) described by the axioms. Euclid’s diagrams give a map to help follow (or discover!) the chain of deduction in a proof. Conversely, the proof validates the image (which could otherwise mislead by error or the selection of a nongeneric example). This approach leads at the conclusion of Book 1 to a proof of the Pythagorean theorem; see Figure 3.

In Chinese mathematics, this theorem is the 勾股 (Gougu) theorem. In the classic *Nine Chapters on the Mathematical Art* (九章算術), it plays a key role in applying the arithmetical mathematics of the text to geometric problems, for example in measuring altitude. The Chinese tradition also gives an elegant visual proof of the result by rearranging triangles, as in Figure 4.

Although the Chinese proof is not considered rigorous by modern standards, Euclid was also criticized

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1. See [http://illustratingmath.org/] for links to these two programs, along with many other resources.
2. [https://patrimoine.ihp.fr/](https://patrimoine.ihp.fr/)
3. [https://sammlungen.uni-goettingen.de/sammlung/slg_1017/](https://sammlungen.uni-goettingen.de/sammlung/slg_1017/)
by Bertrand Russell when he wrote “A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid’s earlier proofs fail before this test.” [Rus02]. This criticism reveals one of the challenges of mathematical illustration. A powerful example comes from a well-known "proof," apparently originally due to W. W. Rouse Ball, that all triangles are equilateral. In this case a subtly misleading diagram leads to a false conclusion following an otherwise entirely correct argument. Briefly, starting with no constraints on the triangle $ABC$, Ball produces points $O$, $E$, and $F$ by a construction depicted in Figure 5. He establishes (without error) that $AF = AE$ and $FB = EC$. The diagram then makes it "clear" that $AB = AF - FB$ must equal $AC = AE - EC$. Hence the triangle is isosceles, and since the labels of $A$, $B$, and $C$ were arbitrary, it "must" in fact be equilateral!

Disallowing these particular subtle errors requires axioms capturing the meaning of "between," that took considerable work by David Hilbert to formulate. Armed with these axioms, one can show that, except when the triangle is isosceles, Ball’s construction perfectly executed will produce either $E$ within segment $AC$ or $F$ in segment $AB$ (but not both), destroying the final subtraction step.

A related pitfall—when a good illustration, overused, can become a pair of blinders—is illustrated by the following example. In the Elements, the concept of number is based on the concept of length. So the squares in the Pythagorean theorem are actual squares (the area of which are equal), not squared numbers. In the eleventh-century algebra treatise of Omar Khayyam, although he gives solutions to equations with higher powers than three, he also states: “Square-square, which, to the algebraists, is the product of the square by itself, has no meaning in continuous objects. This is because how can one multiply a square, which is a surface, by itself? Since the square is a two-dimensional object ... and two-dimensional is a four-dimensional object. But solids cannot have more than three dimensions.” [KK72]. The relation between number and length was also an important factor in the European reluctance to consider negative numbers. A line, after all, cannot have negative length. In contrast, negative quantities are used freely in the Nine Chapters, where arithmetic is the foundational idea, with geometry built from it. In Europe, the development of the number line, starting with John Wallis, gave an alternative illustration of number (see Figure 6) with the capacity to include negative quantities as numbers in their own right.

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4 For a modern presentation of this fallacy and a lively discussion of its flaws, see Joel Hamkins’ essay at [https://jdh.hamkins.org/all-triangles-are-isosceles/](https://jdh.hamkins.org/all-triangles-are-isosceles/).
Powers and negative numbers are but two examples of a productive pattern of mathematics developing from the tension between illustration and symbolic idea. The study of complex numbers advanced significantly with the concept of the complex plane, and then allowed a new algebraic approach to the geometry of the plane. Both quaternions and matrices were developed to try to extend that understanding to higher dimensions. In the case of real numbers, although the symbolic ideas would refine the illustrations needed, it was not until the late nineteenth century when fully symbolic definitions were developed, such as Dedekind cuts and Cauchy sequences. At that point the need for illustrations as foundational objects was removed, although the potential for developing intuition and challenging what might be done with the concepts remained.

Projective geometry, first developed (as perspective) by artists as a tool to create realistic images, provided one such challenge. These ideas were explored mathematically by Johannes Kepler, Gérard Desargues, and Blaise Pascal. In the early nineteenth century, perspective was developed by Gaspard Monge into “descriptive geometry” for the training of engineers in constructing forts and later developed and axiomatized in the foundational work by Jean-Victor Poncelet. In turn this work would be key in establishing models for non-Euclidean geometry, explored axiomatically by Nikolai I. Lobachevsky and János Bolyai. In this case it was such models, themselves illustrations, that convinced mathematicians of the existence and interest of the non-Euclidean geometries.

Projective geometry also spurred the study of algebraic geometry. In the late nineteenth century an industry emerged to reveal the surfaces constructed in this field and their properties, such as cone singularities and embedded straight lines. One pioneer was Alexander Brill, a student of Alfred Clebsch with a degree in architecture. Following the work of Peter Henrici (another student of Clebsch), Brill made sliceform paper models of surfaces. He later worked with Felix Klein in Munich to set up a laboratory for the design and production of mathematical objects. This lab grew into a company that, when it was taken over by Martin Schilling in 1911, had a catalogue of over 400 models. His work combined deep mathematical understanding with a knowledge of printing and construction from his family business [PB07].

The need to combine mathematical knowledge with fabrication techniques is also highlighted by a story of missed opportunity: how to make physical patches of hyperbolic planes. In addition to his disk model (often called the Poincaré disk model), Eugenio Beltrami also attached together strips of paper to approximate the surface. Other examples used paper polygons connected to make a sort of hyperbolic “soccer ball.” These paper models are often fragile, and the rigidity of the paper means that it cannot change its local geometry; thus such models are crude approximations. Roughly a century later, Daina Taimiņa realised that crocheting could produce far more resilient surfaces, with local stretching that meant the negative curvature was more smoothly distributed [Tai09]. An example of this medium of representation is shown in Figure 7. In fact, similar techniques had been used to create ruffles in scarves and skirts for decades. If the methods of fiber arts had earlier been considered seriously and not dismissed as “work for women,” researchers could have had the opportunity to handle robust hyperbolic planes far sooner.

The Incredible Potential for Mathematical Illustration

Turning to recent developments, the work of Lionel Levine, Wesley Pegden, and Charles K. Smart provides an excellent example of the value of illustration as a research tool. Their *Annals of Mathematics* paper *The Apollonian structure of integer superharmonic matrices* [LPS17] was motivated by the study of Abelian sandpiles on $\mathbb{Z}^2$: place a large number $N$ of sand grains at the origin, and allow any position with at least four grains to distribute those grains equally to its four neighbors. The stable configuration that results from this simple system displays impressive large-scale structure that can be discovered through computer visualization (see Figure 1). Especially striking is the vivid visual impression that the structure continues to refine at larger $N$ toward a continuum scaling limit, which was proven earlier by Pegden and Smart. To describe the PDE governing this process, the individual periodic tilings in the regions of the limit must be understood. They are
each governed by an integer superharmonic matrix. Levine, Pegden, and Smart generated a picture of the set of integer superharmonic matrices, and were astonished to see the familiar fractal structure of an Apollonian circle packing (Figure 8). Each circle of the packing was associated to a periodic pattern appearing in the scaling limit. Through extensive computer investigation, the authors were able to determine the intricate recursive relationships between the patterns for circles generated from one another (“ancestors” overlap and merge to form “descendent” patterns according to complicated rules). These recursions led to a difficult inductive proof that the set did indeed have the Apollonian structure evident in experiments. The development of these results provide a perfect example of the role illustration can play in the cycle of conjecture, theorem, and proof. Without the data available through large-scale computer experimentation and the ability to explore it visually, the question of the scaling limit may not have been raised at all, and the recursive proof of their main result would likely not have been discovered.

Another area where research is intertwined with illustration is in the study of William Thurston’s geometrization conjecture, proved by Grigori Perelman. This key tool in our understanding of 3-manifolds implies, for instance, the famous Poincaré conjecture. Geometrization states that any compact topological 3-manifold can be cut into finitely many pieces, each of which carries a geometric structure. There are eight possible such structures, known as Thurston geometries. Some of them are rather familiar to mathematicians, such as the 3-dimensional Euclidean and hyperbolic spaces or the 3-sphere.

Despite the fact that Thurston’s geometries have been intensively studied, the more exotic geometries such as Nil and Sol still defy our “Euclidean-grown” spatial intuition. Keeping in mind the well-established power of our physical and visual intuition to aid geometrical research, Rémi Coulon, Elisabetta Matsumoto, Henry Segerman, and Steve Trettel developed virtual reality software to immerse the user in any of the eight Thurston geometries [CMST22] (see Figure 9). Besides building much-needed intuition for these spaces, the development of the software itself raised mathematical questions. To build their virtually rendered Thurston geometries, these researchers use raymarching techniques which require computation of distances between objects. But, for example, there is no closed formula for the distance in Nil or Sol! Thus, the development of the algorithms themselves becomes a mathematical result in its own right.

Work on Thurston’s geometries has very often been closely tied with illustration. For example, the study of Spheres in Sol by Matei P. Coiculescu and Rich Schwartz in Geometry and Topology (positively) answers an old open question, whether metric spheres in Sol are homeomorphic to $S^2$ [CS22]. Each step of the proof was found after numerous graphical experiments, and 3D printing brings yet another perspective (see Figure 10).

For an example at the intersection of algebraic geometry and number theory, a few key illustrations have helped drive developments in the field of $p$-adic analytic geometry. At the same time, illustrating the $p$-adic analogs of complex analytic manifolds presents unique challenges, not the least of which is the fact that the $p$-adic numbers themselves are topologically a Cantor set. Nevertheless, clever and meaningful illustrations of $p$-adic analogs to the complex upper half-plane and complex unit disk have proved incredibly fruitful. An illustration of Vladimir

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**Figure 8.** Set of integer superharmonic matrices in the space of all symmetric real matrices. Image by Stange.

**Figure 9.** In-space view of a finite-volume hyperbolic 3-manifold lit by a single white light. Image by Coulon, Matsumoto, Segerman and Trettel [CMST22].
Drinfeld’s $p$-adic upper half plane as tubular neighborhoods of Bruhat-Tits trees (Figure 11) clarified the behavior of the action of $\text{GL}_2(\mathbb{Q}_p)$ by Möbius transformations. Understanding this action was instrumental in the construction of $p$-adic analytic uniformization of elliptic curves (reflecting the famous complex analytic uniformization of elliptic curves as quotients of the complex upper half plane). Similarly, Peter Scholze’s illustrations of the adic unit ball (Figure 12) provide access to the foundational geometric construction in his theory of perfectoid spaces [Sch12]. The act of illustrating the central geometric objects of $p$-adic analysis has proven both beneficial and uniquely challenging, demanding a systematic and critical approach.

An example arising somewhat further afield of geometry is the work of Allen Knutson, Terence Tao, and Christopher Woodward in representation theory [KT01]. Knutson and Tao introduced the notion of honeycombs (subsets of the plane as in Figure 13) to solve a longstanding open problem: Alfred Horn’s conjectured shape of the polyhedral cone (sometimes called the Littlewood–Richardson cone) of triples of eigenvalue spectra $(\lambda, \mu, \nu)$ for Hermitian matrices $A, B, C$ which satisfy $A + B + C = 0$.

This sum-of-Hermitian-matrices problem has applications to perturbation theory, quantum measurement theory, and the spectral theory of self-adjoint operators. Knutson and Tao were able to show that there exist such Hermitian matrices with the specified spectra if and only if there exist honeycombs with a specified boundary. They used this correspondence to prove Horn’s conjecture. The honeycomb formalism also led naturally to a polynomial time algorithm to decide whether a triple of spectra can be realized by Hermitian matrices. In a follow-up, Knutson, Tao, and Woodward extended the study of honeycombs to define puzzles (Figure 14), which they described as replacing the Schubert calculus in past approaches to the Hermitian matrices problem, and used geometric arguments to give a complete characterization of the facets of the polyhedral cone.
the cone [K Tw04]. Puzzles and honeycombs provide an example of the power of rephrasing an algebraic problem as one about visual objects, where we can draw on other types of intuition. In what circumstances can we expect these sort of insightful geometric versions to exist for algebraic problems? When a geometric analog exists, it naturally exhibits additional features—can we then find new corresponding objects in the original problem? For example, what do the vertices of a honeycomb actually represent?

There are, of course, many more examples. Among these, the most famous may be the computer exploration of the Mandelbrot set and fractal geometry in the 1980s (Figure 15). In the 1990s, Jeffrey Weeks created SnapPea (which now exists as SnapPy under the guidance of Marc Culler and Nathan Dunfield) as part of his doctoral thesis, to explore the cusp structures of hyperbolic 3-manifolds. Its use inspired David Gabai, Robert Meyerhoff, and Peter Milley to invent mom structures to answer questions of the volumes of hyperbolic 3-manifolds [GMM10]. In the same decade, the Geometry Center founded by Al Marden was focused on the use of computer visualization in mathematics. It hosted mathematicians such as Eugenio Calabi, John Horton Conway, Donald Knuth, Mumford, and Thurston, among others, and produced the GeomView software used to create some famous early computer visualizations, including the sphere eversion and illustrations for knot theory. Illustration has shown its importance in virtually all areas of mathematics, from random tilings in combinatorics, to diagrammatic approaches to algebra, to Apollonian circle packings and Schmidt arrangements in number theory, and their higher-dimensional analogs, to mention just a few.

The examples above focus on pure mathematics, which is poised to join a great many other areas of scientific endeavour embracing illustration. In applied mathematics, illustration has already made great strides. Consider for instance the process of Alan H. Schoen, when describing the gyroid decades before it was mathematically proven to be a minimal surface. He worked with both a sculpture of the surface and various models in Computer-Aided Design / Modelling (CAD/CAM), which ultimately led to the structure being found in various lipid and liquid crystalline systems [Sch12a]. Other fields, like mathematical geometry processing rely equally on quantitative measures and qualitative visualizations for judging the quality of their results. Still, a back-and-forth between the development of mathematical procedures and their application to real-world data yields results that are well-grounded in mathematical quality guarantees, yet efficient and relevant for their applications. In the field of exploratory data analysis, visualizations even form the main tool for finding research results. Here, large, possibly high-dimensional, data sets are investigated for patterns by embedding them, e.g., as 2D scatter plots that can then be inspected by domain experts. With this technique, in 2020, a novel type of antitumor cell was discovered [dVvUI+20]. None of these research results would have been possible without the utilization of illustrations. Furthermore, this last example utilized non-linear dimensionality reduction techniques for the visualization of high-dimensional data. These techniques were
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themselves the result of research driven by the desire for better illustrations.

The very closely allied field of computation in mathematics is a little ahead of illustration in its maturity as a tool for mathematical research. To give just one significant example in number theory, much recent activity has centered around the multi-million-dollar Simons Collaboration on Arithmetic Geometry, Number Theory, and Computation, whose mission states:

> Our common perspective is that advances in computational techniques accelerate research in arithmetic geometry and number theory, both as a source of data and examples, and as an impetus for effective results. The dynamic interplay between experiment, theory, and computation has historically played a pivotal role in the development of number theory.

The work supported by the collaboration is rapidly expanding the L-Functions and Modular Forms Database, an online database of mathematical objects (including visualizations) that is at the center of much modern progress in number theory. The discipline of mathematical computation is supported by a number of journals and has engendered areas of research in their own right, such as computational geometry. Illustration appears to be following a similar trajectory. As it becomes more accessible and pervasive it demands rigorous and careful study, leading to the development of mathematical illustration as a discipline in its own right.

**Illustration as a Discipline**

Thurston once said, “mathematicians usually have fewer and poorer figures in their papers and books than in their heads” [Thu94]. Although the power of good illustrations to advance mathematical knowledge is clear, they are not simple to produce.

The challenges to creating powerful and trustworthy illustrations come on many levels. On the one hand, some challenges are technical and concern rather practical questions regarding the production of mathematical illustrations. Especially with newer technologies like virtual reality or 3D modeling, the learning curves are steep and while there are general tutorials available, just a handful target issues specific to the illustration of mathematics. Consider for instance [LST20] for a nice discussion of some of the challenges of 3D printing for mathematical illustration.

On the other hand, there are challenges within the mathematics itself. The objects to be illustrated do not necessarily come with a description that lends itself to a suitable illustration. Thus, a necessary initial step is the translation of the underlying mathematical object into a form that allows illustration in the first place. However, this transformation is usually not enough by itself. Subsequent steps aim at making the illustration effective, which can entail bridging the gap between the theoretical and the computational, crafting a responsive and immersive experience, or ensuring the illustration actually imparts the desired aspects of the mathematical object. In particular the last part implies important theoretical considerations: What exactly do we want to illustrate? And how do we do so faithfully, i.e., without creating wrong impressions of the mathematical object illustrated?

Mathematics is not the first field of research to tackle these difficulties. There are parallels to be found in the development of the scientific method and statistical methods for the natural sciences: Which experimental designs and statistics can be relied upon for developing conjectures and conclusions? Cornerstones of the scientific methods were laid down, such as the important notion of falsifiability of a scientific theory. Similarly, statistical methods amplified their usefulness and trustworthiness when expanded from pure descriptive statistics to inference statistics and statistical tests to assert the validity of results. So in fact, all scientific fields have progressed by examining head-on some of the questions raised by their methodologies.

The question of illustrating well has been asked in statistics and data visualization, as explored in Darrell Huff’s best-selling book *How to Lie with Statistics*, which became a standard college text. The pioneering and richly illustrated books of Edward Tufte and Tamara Munzner on data visualization established that field in its own right. Every year, new research in data visualization is discussed at various venues, such as the Institute for Electrical and Electronics Engineers (IEEE) VIS meeting or the EuroVis conference, and published in outlets like the *IEEE Transactions on Visualization and Computer Graphics*. As it matures, the data visualization community addresses meta-questions on its research, such as where “the value of visualization” lies [VW05] or “Are we making progress in visualization research?” [Cor22].

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9. [https://simonscollab.icerm.brown.edu/](https://simonscollab.icerm.brown.edu/)
10. [http://www.lmfdb.org](http://www.lmfdb.org)
11. See the extensive list of publications arising from the collaboration: [https://simonscollab.icerm.brown.edu/publications/](https://simonscollab.icerm.brown.edu/publications/).
13. A noteworthy example for introductory material, aimed at illustration of mathematics, is the Processing tutorial of Roger Antonsen, to be found online: [http://rant.codes/pcmi/](http://rant.codes/pcmi/).
Thus, the example of data visualization provides a pattern of development that the field of mathematical illustration might follow. However, in comparison, mathematical illustration is just taking the first steps on its journey toward being a research field. It is still facing basic challenges with regard to creating and evaluating the illustrations it produces.

As an example of these challenges, consider the images in Figure 16 showing polynomial roots near $i$ in the complex plane. Figure 16(a) is an image of all roots of polynomials of degree 3 with integer coefficients between $N$ and $-N$, where here $N = 10$ [HST22]. Figure 16(c) is an image of all roots of polynomials with coefficients from $\{-1, 1\}$ and degree no more than $D$, where in this case $D = 13$. In both, in the region around $i$, there appears to be a hole shaped like two ellipses overlapping at right angles.

How to interpret this shape? It turns out that in Figure 16(a) it is very much an artifact of the algorithm for creating these images. If you consider the picture as an approximation of all cubic roots (by allowing $N$ to tend to infinity), there are infinitely many such polynomials. By limiting $N$, we are looping through them in a growing hypercube in the coefficient space. The corners of this cube are the corners jutting in toward $i$, and as the cube expands in the coefficient space, this hole will get filled in. If instead of looping through coefficient space in a growing cube, we choose a different ordering, the apparent shape of the void changes. This phenomenon is shown in Figure 16(b).

In Figure 16(c), however, the limiting shape turns out to be refined but not substantially changed or filled in as $D$ increases from 13 to infinity. So this hole “really exists” in the picture! The shapes one sees at the boundaries of the limiting set of roots are explained in terms of fractal geometry and certain symmetries of this set.\textsuperscript{14}

As another example of the challenges discussed above, the virtual reality versions of Thurston’s geometries of [CMST22] are a profound way to experience these spaces, but can feel overwhelming and nearly psychedelic, as our brains struggle to make sense of what we are seeing. As an alternative, for several of the geometries, it is possible to place the geodesics of the geometry into familiar Euclidean space as curves (see Figure 17). The interplay between these two methods of illustration can be much more enlightening than either one alone. The mathematical arguments that are developed to explain how one view can predict the other can end up as the basis of a mathematical proof. Conjectures and mathematical arguments about the space can quickly be evaluated by predicting their effect on these illustrations.

\textsuperscript{14}These features are beautifully described by John Baez on his personal website: https://math.ucr.edu/home/baez/roots/.
Looking Forward
Illustrations have been used both historically and in recent state-of-the-art research projects to expand the boundaries of knowledge in pure mathematics. Other fields of research, such as statistics and microbiology, have systematized visualization, and studied it in its own right.

However, as our gallery of examples shows, the quality of illustrations in pure mathematics varies, and there is no common framework to create, discuss, or evaluate them. To further the possibilities that illustrations provide, there needs to be a dedicated community to tackle the next important problems. These include, among others:

1. How to identify illustrations that have rich potential to provide insight?
2. How to identify (and mitigate) the ways that illustrations can mislead and distract?
3. How to measure the fidelity of an illustration; are perceived patterns a result of its construction or the underlying mathematics?
4. How can we harness the processing power and pattern-recognition capabilities of the human visual system?
5. How can we empower a next generation of mathematical illustrators to create and leverage sophisticated illustrations?
6. And how do we increase professional recognition of the illustration of mathematics?

Exploring these questions will lay the foundation of a discipline built around the illustration of mathematics, providing powerful tools for the advancement of mathematical research.

ACKNOWLEDGMENT. The authors would like to thank the two referees for their valuable feedback and suggestions. The illustrating mathematics group also provided insightful discussions on the topics presented here. Finally, we want to acknowledge the many authors and sources that we unfortunately did not have space to cite here; more references may be found in the arXiv companion to this article.

References


Credits

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FAN CHINA EXCHANGE PROGRAM

• Gives eminent mathematicians from the US and Canada an opportunity to travel to China and interact with fellow researchers in the mathematical sciences community.

• Allows Chinese scientists in the early stages of their careers to come to the US and Canada for collaborative opportunities.

Applications received before March 31 will be considered for the following academic year.

For more information on the Fan China Exchange Program and application process see www.ams.org/china-exchange or contact the AMS Programs Department:

TELEPHONE: 800.321.4267, ext. 4189 (US & Canada)
401.455.4060 (worldwide)

EMAIL: chinaexchange@ams.org
A crochet row of chain stitches is formed by repeatedly pulling a folded over loop of yarn (a "bight") through the previous loop. Each bight encircles the waist of the bight in front, like a row of elephants, each holding the tail of the next in its trunk.

Figure 1. A chain stitch row of jute.

So that the whole row does not come loose, we must “tie off” the left-hand side with some sort of knot. In a more schematic picture (Figure 2) this is accomplished by adding an arc in the lower left to create a trivalent vertex:

Figure 2. An abstracted crochet line.

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DOI: https://doi.org/10.1090/noti2854

Taking the fundamental group of the complement translates topology into algebra; if $x$ denotes a meridian loop around the initial loop (in red), and $y$ denotes a meridian loop around the free end (in green), then a meridian around the first bight is the conjugate $y^x := xyx^{-1}$, and meridians around successive bights are the conjugates of $y$ by the meridian around the previous bight; i.e., they form a sequence

$$x, y^x := xyx^{-1}, y(y^x) := (xyx^{-1})y(xy^{-1}x^{-1}),...$$

The meridian around the last bight (in blue) represents the result of $n$-fold iterated conjugation, where $n$ is the number of chains in the row.

If one cuts Figure 2 along the base and rejoins it as in Figure 3 one obtains an unlink, so that the short meridian loops (in red) become algebraically independent free variables $x_1, x_2, ...$, and now the meridian around the last bight (in blue) represents the result of free iterated conjugation; i.e., the element

$$x^n_{x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}...} := x_1x_2x_3^{-1}x_4^{-1}x_5^{-1}x_6^{-1}x_7^{-1}...$$

Figure 3. An interesting embedding of an unlink.
The exponential growth of the word length of this element is at the heart of a classical puzzle, known (amongst other names) as a baguenaudier (Figure 4) — literally “time waster.” The red hook in the figure (which by implication extends indefinitely far to the left) is made of some rigid material and must be extricated from the rings, by sliding it left or right (when the rings are not in the way) and by moving the rings on their side through the hook to remove them or put them back.

![Figure 4. Can you remove the red hook from the rings?](image)

After some experimentation, one learns that it is possible to perform the following two moves (or their inverses):

1. one can always take the rightmost ring off the red hook (if it is on); or
2. if the kth and k + 1st rings are on the red hook but no l for l > k + 1, then one can take off the kth ring.

Either move is accomplished physically by tipping the ring on its side and slipping it between the parallel bars of the red hook. The second move is illustrated in Figure 5.

![Figure 5. Can you see how to perform this move?](image)

If one thinks of the configuration of the puzzle as a binary string (with 1 denoting on and 0 denoting off) the initial configuration is 111 ⋯ 1, the target configuration (the hook removed from the rings) is 000 ⋯ 0, and the two legal moves are of the form *1 ↔ *0 and *110 ⋯ 0 ↔ *010 ⋯ 0 where * denotes an arbitrary initial string of 0s and 1s. Using these operations one may solve the puzzle by induction.

In complexity terms, this puzzle may be unentangled by a procedure that takes up resources that are linear in space, but exponential in time (this fact is not obvious, but a rigorous argument is beyond the scope of this article). In an appendix to a paper of Hastings, Freedman shows that this phenomenon may be used to give simple examples of quantum systems with simply-connected configuration space for which the numerical quantum Monte Carlo algorithm fails to find the ground state of the system in polynomial time.

Some closely related objects appear in knot theory, specifically in the theory of Habiro claspers.

Two knots or links in the 3-sphere are said to be related by a \( C_n \)-move if they differ as indicated in Figure 6 (the figure shows \( n = 7 \); the convention in knot theory to illustrate a “move” is to show a fragment of two link diagrams, with the implication that the two diagrams are identical outside the fragments shown). They are \( C_n \)-equivalent if they are related by a finite sequence of \( C_n \)-moves.

![Figure 6. A \( C_7 \)-move relating two knots or links.](image)

These equivalence relations become successively finer as \( n \) increases: \( C_{n+1} \)-equivalence implies \( C_n \)-equivalence for any \( n \). This is easy to see — Figure 7 shows how a \( C_7 \) move may be obtained as the composition of two \( C_6 \) moves and an isotopy.

![Figure 7. A \( C_7 \) move as the composition of two \( C_6 \) moves and an isotopy.](image)

Habiro showed that knots in the 3-sphere are \( C_n \) equivalent if and only if they have the same finite type (i.e., Vassiliev) invariants of degree < \( n \). This latter condition is known to have several geometric avatars:

1. by Stanford, it is identical to the relation on knots obtained by a finite sequence of moves of the following form: grab a family of \( m \) parallel strands, and replace it by a pure braid in the \( k \)th term of the lower central series of the pure braid group on \( m \) strands for some \( k < n \);
2. by Conant–Teichner, it is equivalent to the relation on knots of capped group cobordism of class \( n \).

A major open problem in knot theory is whether the entirety of all finite type invariants together is sufficient to distinguish knots — i.e., whether it is the case that if two knots \( K \) and \( K' \) have the same finite type invariants of all degrees, they must necessarily be isotopic. Groves of infinite order (with geometric control on the ends) may be thickened (in four dimensions) into Casson handles, wild geometric...
beasts that turn out to be homeomorphic — though typically not diffeomorphic! — to ordinary 2-handles, and play a key role in Freedman’s proof of the (topological) 4-dimensional Poincaré conjecture. Might grope cobordisms of infinite order between knots with the same finite type invariants provide a 4-dimensional approach to this 3-dimensional open problem?

You may choose to work on this problem; successful or not, I hope you do not consider the effort to be time wasted.

References


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All images are courtesy of Danny Calegari.
AMS Prizes and Awards

I. Martin Isaacs Prize for Excellence in Mathematical Writing

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

About this Prize
The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in Communications of the AMS, Journal of the AMS, Mathematics of Computation, Memoirs, Proceedings of the AMS, or Transactions of the AMS, and is therefore not a prize selection criterion.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups—that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.

Next Prize: January 2025

Nomination Period: The deadline is March 31, 2024.

Nomination Procedure: www.ams.org/isaacs-prize

Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination, and a complete bibliographic citation for the article being nominated.

Joint Prizes and Awards

2024 MOS–AMS Fulkerson Prize

The Fulkerson Prize Committee invites nominations for the Delbert Ray Fulkerson Prize, sponsored jointly by the Mathematical Optimization Society (MOS) and the American Mathematical Society (AMS). Up to three awards of US$1,500 each are presented at each (triennial) International Symposium of the MOS. The Fulkerson Prize is for outstanding papers in the area of discrete mathematics. The prize will be awarded at the 25th International Symposium on Mathematical Programming to be held in Montreal, Canada, in the summer of 2024.

Eligible papers should represent the final publication of the main result(s) and should have been published in a recognized journal or in a comparable, well-refereed volume intended to publish final publications only, during the six calendar years preceding the year of the Symposium (thus, from January 2018 through December 2023). The prizes will be given for single papers, not series of papers or books, and in the event of joint authorship the prize will be divided.

The term “discrete mathematics” is interpreted broadly and is intended to include graph theory, networks, mathematical programming, applied combinatorics, applications of discrete mathematics to computer science, and related subjects. While research work in these areas is usually not far removed from practical applications, the judging of papers will be based only on their mathematical quality and significance.

Previous winners of the Fulkerson Prize are listed here: www.mathopt.org/?nav=fulkerson#winners.

Further information about the Fulkerson Prize can be found at www.mathopt.org/?nav=fulkerson and https://www.ams.org/fulkerson-prize.
The Fulkerson Prize Committee consists of
• Julia Böttcher (London School of Economics),
  MOS Representative
• Rosa Orellana (Dartmouth College), AMS Repre-
  sentative
• Dan Spielman (Yale University), Chair and MOS
  Representative

Please send your nominations (including reference to the
nominated article and an evaluation of the work) by Feb-
ruary 15, 2024 to the chair of the committee:
Professor Daniel Spielman
Email: daniel.spielman@yale.edu

American Mathematical Society

Policy on a Welcoming Environment
(as adopted by the January 2015 AMS Council
and modified by the January 2019 AMS Council)

The AMS strives to ensure that participants in its
activities enjoy a welcoming environment. In all
its activities, the AMS seeks to foster an atmo-
sphere that encourages the free expression and
exchange of ideas. The AMS supports equality
of opportunity and treatment for all partici-
pants, regardless of gender, gender identity or
expression, race, color, national or ethnic origin,
religion or religious belief, age, marital status,
sexual orientation, disabilities, veteran status, or
immigration status.

Harassment is a form of misconduct that
undermines the integrity of AMS activities and
mission.

The AMS will make every effort to maintain an
environment that is free of harassment, even
though it does not control the behavior of third
parties. A commitment to a welcoming envi-
ronment is expected of all attendees at AMS
activities, including mathematicians, students,
guests, staff, contractors and exhibitors, and par-
ticipants in scientific sessions and social events.
To this end, the AMS will include a statement
concerning its expectations towards maintain-
ing a welcoming environment in registration
materials for all its meetings, and has put in
place a mechanism for reporting violations.
Violations may be reported confidentially
and anonymously to 855.282.5703 or at
www.mathsociety.ethicspoint.com. The report-
ing mechanism ensures the respect of privacy
while alerting the AMS to the situation.

For AMS policy statements concerning
discrimination and harassment, see the
AMS Anti-Harassment Policy.

Questions about this welcoming environment
policy should be directed to the AMS Secretary.
AMS Updates

Math Camp Directors: Apply to AMS Young Scholars Program

January 15, 2024 is the deadline by which mathematics summer-camp directors may apply to the American Mathematical Society’s Young Scholars Program.

Since 2000, the AMS Epsilon Fund has made annual grants of $2,500–$15,000 to existing math summer programs for talented and highly motivated high-school students.

Applications are accepted on MathPrograms.org from program directors only (not from students or parents). More than a dozen awards are made each year to summer camps that focus on problem-solving or research in any area of mathematics. Award announcements will be made in late February.

See [http://www.ams.org/prizes-awards/paview.cgi?parent_id=3](http://www.ams.org/prizes-awards/paview.cgi?parent_id=3) for more information about the AMS Young Scholars Program, to read the list of previous grant awardees, or to learn about the AMS Epsilon Fund.

—AMS Communications

AMS Hosts Graduate Education Mini-Conference

On September 28, 2023, representatives from the American Mathematical Society Committee on Education, the National Association of Mathematicians (NAM), the Math Alliance, and ParaDIGMS hosted the annual education mini-conference on enhancing graduate programs in the mathematical sciences. Approximately 50 people attended the mini-conference online and in person in Washington DC, according to event organizer Tyler Kloefkorn, AMS associate director for government relations.

The event focused on three areas: federal resources for graduate programs, supporting nonacademic career paths for mathematicians, and making connections between graduate programs and programs at primarily undergraduate institutions.

The federal government offers funding, fellowships, and data to graduate programs, as described by a panel of representatives from the National Institutes for Health (NIH), the National Institute for Standards and Technology (NIST), the Department of Energy (DoE) Office of Science, and the National Science Foundation (NSF) National Center for Science and Engineering Statistics. Two examples: The NIH Data and Technology Advancement (DATA) National Service Scholar Program invites mathematical scientists to join NIH for one to two years to tackle challenging biomedical data problems, and the DoE’s Computational Science Graduate Fellowship Program.

DOI: https://doi.org/10.1090/noti2862
covers tuition and provides stipends to awardees in graduate school.

Talitha Washington (Clark Atlanta University / Atlanta University Center) delivered the keynote address, “How the Workforce is Shaping Mathematics Graduate Education.”

Washington challenged the attendees and organizers to think about how higher education programs in the mathematical sciences should respond to changing demands in the US workforce,” Kloefkorn said. “She highlighted the work of current and former students in the Atlanta University Center Data Science Initiative, noting the critical role of data science in STEM.”

A panel of PhD mathematical scientists representing government, industry, and consulting/nonprofits discussed how graduate programs can improve and expand pathways to nonacademic careers. They cited internships, supportive mentoring, access to nonacademic networks, and a willingness to explore different career paths. “Even with the large and growing number of PhD mathematicians who end up in BEGIN careers, several panelists noted that there are cultural challenges to exploring and pursuing nonacademic jobs in graduate school,” Kloefkorn noted.

A town hall discussion on demystifying the graduate school process concluded the event.

—AMS Communications

Departments Coordinate Job Offer Deadlines

For the past twenty-three years, the American Mathematical Society has led the effort to gain broad endorsement for the following proposal:

That mathematics departments and institutes agree not to require a response prior to a certain date (usually around February 1 of a given year) to an offer of a postdoctoral position that begins in the fall of that year.

This proposal is linked to an agreement made by the National Science Foundation (NSF) that the recipients of the NSF Mathematical Sciences Postdoctoral Fellowships would be notified of their awards, at the latest, by the end of January.

This agreement ensures that our young colleagues entering the postdoctoral job market have as much information as possible about their options before making a decision. It also allows departmental hiring committees adequate time to review application files and make informed decisions. From our perspective, this agreement has worked well and has made the process more orderly. There have been very few negative comments. Last year, more than 185 mathematics and applied mathematics departments and institutes endorsed the agreement.

Therefore, we propose that mathematics departments again collectively enter into the same agreement for the upcoming cycle of recruiting, with the deadline set for Monday, February 5, 2024. The NSF's Division of Mathematical Sciences has already agreed that it will complete its review of applications and notify all applicants no later than Friday, January 26, 2024.

The American Mathematical Society facilitated the process by sending an email message to all doctoral-granting mathematics and applied mathematics departments and mathematics institutes. The list of departments and institutes endorsing this agreement will be widely announced on the AMS website and updated weekly through January.

We ask that you view a proposed updated version of last year's formal agreement at http://www.ams.org/employment/postdoc-offers.html along with this year's list of adhering departments.

Important: To streamline this year’s process for all involved, we ask that you notify the AMS (postdoc-deadline@ams.org) if and only if:

(1) your department is not listed and you would like to be listed as part of the agreement; or

(2) your department is listed and you would like to withdraw from the agreement and be removed from the list.

Please feel free to email us with questions and concerns. Thank you for consideration of the proposal.

—Sarah Bryant, AMS Director of Programs
Lucy R. Maddock, Interim AMS Executive Director

Deaths of AMS Members

Daniel Henry Gottlieb, of Marina del Rey, California, died on January 19, 2022. Born on December 7, 1937, he was a member of the Society for 62 years.

Erwin Kleinfeld, of Reno, Nevada, died on January 14, 2022. Born on April 19, 1927, he was a member of the Society for 72 years.

Clinton M. Petty, of San Marcos, California, died on December 16, 2021. Born on June 4, 1923, he was a member of the Society for 68 years.

Emma Previato, of Boston, Massachusetts, died on June 29, 2022. Born on November 29, 1952, she was a member of the Society for 42 years.

Jean-Marc Terrier, of Canada, died on January 3, 2023. Born on July 11, 1935, he was a member of the Society for 55 years.
Mathematics People

Doolittle Receives 2023 Adrien Pouliot Award

The Canadian Mathematical Society (CMS) presented Edward Doolittle with the 2023 Adrien Pouliot Award in recognition of his outstanding contributions to mathematics education.

Doolittle is Kanyen’kehake, a member of the Lower Mohawk band of Six Nations. He earned a PhD in pure mathematics from the University of Toronto in 1997 with his thesis on partial differential equations. Since 2001, Doolittle has been first a faculty member at First Nations University of Canada (formerly the Saskatchewan Indian Federation College). His duties there include teaching, research, working with Elders, service to the university, and service to Indigenous communities.

Doolittle is an internationally recognized leader on indigenous mathematics and related concepts such as indigenizing mathematics, traditional mathematics, and ethnomathematics. For two decades he has worked tirelessly to introduce insights around the ways in which mathematics as a field of study intersects with indigenous knowledge systems, and the educational possibilities afforded by those different views of mathematics.

—Canadian Mathematical Society

Zhang Awarded SASTRA Ramanujan Prize

The 2023 SASTRA Ramanujan Prize has been awarded to Ruixiang Zhang of the University of California, Berkeley. “In summary, Zhang is an original and highly skilled mathematician who has had a large impact in a wide range of areas,” the prize citation noted. “He is a rising world leader in the field of harmonic analysis and in its striking applications, and amply merits this award.”

Zhang received his BS degree in mathematics from Peking University (2012) and his PhD in mathematics under the supervision of Peter Sarnak at Princeton University (2017). He has held positions as a visiting member at the Institute for Advanced Study (2017–2018 and 2020–2021) and as Van Vleck Visiting Assistant Professor at the University of Wisconsin (2018–2021). Since 2021, he has been an assistant professor at the University of California, Berkeley. Awards for Zhang include the Gold Medal at the 2008 International Mathematics Olympiad and a Silver Medal for his doctoral thesis at the New World Mathematics Awards. He currently holds a Sloan Fellowship (2022–2024) and an NSF CAREER award (2022–2027).

The annual prize is for outstanding contributions by individuals not exceeding the age of 32 in areas of mathematics influenced by Srinivasa Ramanujan in a broad sense. “The age limit has been set at 32 because Ramanujan achieved so much in his brief life of 32 years,” according to the citation. The prize is awarded at an annual international conference in number theory at SASTRA University in Kumbakonam, India (Ramanujan’s hometown).

Pozharska Wins Traub Young Researcher Award

Kateryna Pozharska, Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv, and Chemnitz University of Technology, is the winner of the 2023 Joseph F. Traub Information-Based Complexity Young Researcher Award. The honor is awarded by the Journal of Complexity for significant contributions to information-based complexity by a young researcher who has not reached their 35th birthday by September 30 in the year of the award. The award consists of $1000 and is sponsored by Elsevier. The award committee consisted of the former winners, David Krieg (2020), Michaela Szölgyenyi (2021) and Mathias Sonnleitner (2022), and the editors Henryk Woźniakowski and Erich Novak.

—Journal of Complexity

DOI: https://doi.org/10.1090/noti2861
Tardos Awarded Knuth Prize

Éva Tardos, the Jacob Gould Schurman Professor of Computer Science and department chair in Cornell University’s Ann S. Bowers College of Computing and Information Science, has been awarded the Donald E. Knuth Prize, which recognizes visionaries in computer science whose work has had foundational, long-term impact.

Tardos is considered a pioneer who has shaped multiple areas of algorithms, including foundational work in combinatorial algorithms, approximation algorithms, and algorithmic game theory. She was honored for “her extensive research contributions and field leadership, namely co-authoring an influential textbook, Algorithm Design, co-editing the Handbook of Game Theory, serving as editor-in-chief of the Journal of the ACM and the Society for Industrial and Applied Mathematics (SIAM) Journal on Computing and chairing program committees for several leading field conferences,” according to the prize citation.

“As a theoretician, I’m honored to be recognized with the Knuth Prize by my original home community,” Tardos said. “I truly appreciate it.”

—Cornell University

Schilling to Deliver Noether Lecture

Anne Schilling of the University of California Davis will deliver the 2024 AWM-AMS Noether Lecture at JMM 2024 in San Francisco. These one-hour expository lectures are presented at the Joint Mathematics Meetings each January.

Schilling is professor and chair of the Mathematics Department at UC Davis, which she joined in 2000. Previously she held a Moore Instructorship at MIT and was a postdoc at the Institute for Theoretical Physics at the University of Amsterdam. Schilling has been awarded a Humboldt Research Fellowship (2002), a Simons Fellowship (2012–2013), and was named an AMS Fellow (2019).

The Association for Women in Mathematics established the Emmy Noether Lectures in 1980 to honor women who have made fundamental and sustained contributions to the mathematical sciences. In April 2013 the lecture was renamed “AWM-AMS Noether Lecture” and in 2015 was jointly sponsored by the AWM and the American Mathematical Society (AMS).

The lectures honor Emmy Noether (1882–1935), one of the great mathematicians of her time, whose life and work remain a tremendous inspiration.

—Association for Women in Mathematics
Assistant Professor/Associate Professor/Professor — Quantum Information Science, Mathematics

The College of Science at Northeastern University (Boston Campus) invites applications for positions at all ranks (Assistant Professor, Associate Professor, or Professor), beginning in academic year 2024–2025 in the field of Quantum Information Science.

The successful candidate will ideally complement existing research strengths and will have the opportunity to collaborate with, and help develop, cross-disciplinary teams across the University. The primary appointment will be in Mathematics, with possible joint appointments in other departments including Physics, Computer Science, Electrical and Computer Engineering, and Chemistry. Cross-disciplinary research programs and appointments are strongly encouraged within the College, and across the University.

At Northeastern University, we embrace a culture of respect and inclusion, where each person is valued and empowered with equitable opportunities and access to resources.

The potential hires are expected to develop vigorous research programs cross cutting the fields of Physics, Computer Science, Electrical and Computer Engineering, and Chemistry. Faculty members at Northeastern are expected to develop independent research programs that attract external funding; teach courses at the graduate and undergraduate level; supervise students and postdocs in their area of research; and participate in service to the department, university, and discipline. Qualified candidates must have excellence in, or a demonstrated commitment to, working with diverse student populations and/or in a culturally diverse work and educational environment.

Applicants must have a PhD in Mathematics or a closely related field by the appointment start date. We encourage applicants from a wide range of backgrounds, including academia and industry. All applicants should have a strong record of scholarly accomplishment that demonstrates the ability to build a strong research program. Candidates seeking appointment at the Associate or Full Professor level should have substantial research productivity and an established history of grant support and academic service. Academic rank at the Associate Professor and Professor levels will be commensurate with experience and qualifications reflecting a record of demonstrated teaching and scholarly excellence. Research excellence is the top-most priority.

Interested candidates should apply with a curriculum vita that includes a list of publications, statements addressing: Research, Teaching, Equity, and names and contact information for at least three professional references from whom letters of reference can be solicited as needed. Applications will be reviewed beginning on October 30, 2023. To apply and see the complete advertisement with all of the details please visit: [https://northeastern.wd1.myworkdayjobs.com/en-US/careers/details/Open-Rank--Assistant-Professor--Associate-Professor--Professor---Quantum-Information-Science--Mathematics_R120037?q=quantum](https://northeastern.wd1.myworkdayjobs.com/en-US/careers/details/Open-Rank--Assistant-Professor--Associate-Professor--Professor---Quantum-Information-Science--Mathematics_R120037?q=quantum).
Northeastern University is an equal opportunity employer, seeking to recruit and support a broadly diverse community of faculty and staff. Northeastern values and celebrates diversity in all its forms and strives to foster an inclusive culture built on respect that affirms inter-group relations and builds cohesion. All qualified applicants are encouraged to apply and will receive consideration for employment without regard to race, religion, color, national origin, age, sex, sexual orientation, disability status, or any other characteristic protected by applicable law. To learn more about Northeastern University’s commitment and support of diversity and inclusion, please see: www.northeastern.edu/diversity.

Membership is an act of service to mathematics—thank you. Investing in membership strengthens services that support the entire math community.

We have updated our dues rates to help make membership easy to access. Choose the membership that best fits your circumstances.

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QIUZHEN COLLEGE, TSINGHUA UNIVERSITY

ANNOUNCES NEW PHD PROGRAM IN MATHEMATICS

WORLD-RENOVED MATHEMATICANS
Graduate students at Qiuwen College will be mentored by a first-class international faculty which includes two Fields Medalists as well as many dynamic early career mathematicians. The dean of Qiuwen College is 1982 Fields Medalist Shing-Tung Yau. Other distinguished mathematicians who have joined us include Caucher Birkar, a 2018 Fields Medalist for his work in algebraic geometry; Nicolai Reshetikhin, a Fellow of the American Mathematical Society known for his contributions to representation theory and low-dimensional topology. Kenji Fukaya, an expert in symplectic geometry and Riemannian geometry, will join in September 2024. Six main research fields are represented in Qiuwen College: analysis and differential equations, algebra and number theory, geometry and topology, probability and statistics, theoretical physics, and applied and computational mathematics.

FINANCIAL SUPPORT
Qiuwen College proudly offers a three-tier scholarship system, designed to fully cover a student's tuition and standard living expenses. Our premier scholarship offering stands on par with those granted to doctoral students at globally renowned universities, including institutions such as Harvard and Princeton.

EXCHANGE OPPORTUNITIES
Qiuwen College has established exchange programs with Harvard, UC Berkeley, the University of Chicago, and Oxford, among other institutions. Students in the graduate program are encouraged to partake in short-term funded visits as part of their academic experiences.

FOR MORE ABOUT THE COLLEGE: https://qzc.tsinghua.edu.cn/en
FOR INFORMATION SPECIFIC TO THE PHD PROGRAM: https://qzc.tsinghua.edu.cn/en/Admissions/graduate/Application1.htm
New Books Offered by the AMS

New in Memoirs of the AMS

Algebra and Algebraic Geometry

**Cluster Algebra Structures on Poisson Nilpotent Algebras**
K. R Goodearl, *University of California, Santa Barbara, CA*, and M. T. Yakimov, *Northeastern University, Boston, MA*

Memoirs of the American Mathematical Society, Volume 290, Number 1445

[bookstore.ams.org/memo-290-1445](bookstore.ams.org/memo-290-1445)

**Tate Duality in Positive Dimension over Function Fields**
Zev Rosengarten, *Stanford University, CA*

Memoirs of the American Mathematical Society, Volume 290, Number 1444

[bookstore.ams.org/memo-290-1444](bookstore.ams.org/memo-290-1444)

**Sur un Problème de Compatibilité Local-Global Localement Analytique**
Christophe Breuil, *Université Paris-Saclay, France*, and Yiwen Ding, *Peking University, Beijing, China*

This item will also be of interest to those working in number theory.

Memoirs of the American Mathematical Society, Volume 290, Number 1442

[bookstore.ams.org/memo-290-1442](bookstore.ams.org/memo-290-1442)

Analysis

**The Space of Spaces: Curvature Bounds and Gradient Flows on the Space of Metric Measure Spaces**
Karl-Theodor Sturm, *University of Bonn, Germany*

This item will also be of interest to those working in geometry and topology.

Memoirs of the American Mathematical Society, Volume 290, Number 1443

[bookstore.ams.org/memo-290-1443](bookstore.ams.org/memo-290-1443)
NEW BOOKS

Differential Equations

Global Existence of Small Amplitude Solutions for a Model Quadratic Quasilinear Coupled Wave-Klein-Gordon System in Two Space Dimension, with Mildly Decaying Cauchy Data
A. Stingo, École Polytechnique, Palaiseau, France

Memoirs of the American Mathematical Society, Volume 290, Number 1441

New AMS-Distributed Publications

Analysis

Sur les Ensembles de Rotation des Homéomorphismes de Surface en Genre ≥2
G. Lellouch, Institut de mathématiques de Jussieu-Paris Rive Gauche, France


Mathematical Physics

On Einstein’s Effective Viscosity Formula
Mitia Duerinckx, Université Libre de Bruxelles, Belgium, and Université Paris-Saclay, France, and Antoine Gloria, Sorbonne Université, France, Institut Universitaire de France, and Université Libre de Bruxelles, Belgium

In his PhD thesis, Einstein derived an explicit first-order expansion for the effective viscosity of a Stokes fluid with a suspension of small rigid particles at low density. His formal derivation relied on two implicit assumptions:
(i) There is a scale separation between the size of the particles and the observation scale.
(ii) At first order, dilute particles do not interact with one another.

In mathematical terms, the first assumption amounts to the validity of a homogenization result defining the effective viscosity tensor, which is now well understood. The second assumption allowed Einstein to approximate this effective viscosity at low density by considering particles as being isolated. The rigorous justification is, in fact, quite subtle as the effective viscosity is a nonlinear nonlocal function of the ensemble of particles and as hydrodynamic interactions have borderline integrability.

In this memoir, the authors establish Einstein’s effective viscosity formula in the most general setting. In addition, they pursue the low-density expansion to arbitrary order in form of a cluster expansion, where the summation of hydrodynamic interactions crucially requires suitable renormalizations. In particular, they justify a celebrated result by Batchelor and Green on the second-order correction and explicitly describe all higher-order renormalizations for the first time.

In some specific settings, the authors further address the summability of the whole cluster expansion. The authors’ approach relies on a combination of combinatorial arguments, variational analysis, elliptic regularity, probability theory, and diagrammatic integration methods.

This item will also be of interest to those working in differential equations.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Memoirs of the European Mathematical Society, Volume 7
October 2023, 196 pages, Softcover, ISBN: 978-3-98547-055-6, 2020 Mathematics Subject Classification: 76T20; 35R60, 76M50, 35Q35, 76D03, 76D07, List US$75, AMS members US$60, Order code EMSMEM/7

bookstore.ams.org/emsmem-7
The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: stovall@math.wisc.edu; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: mamanes@hawaii.edu; telephone: (808) 956-4679.

Meetings in this Issue

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The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.
Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

San Francisco, California
Moscone North/South, Moscone Center

January 3–6, 2024
Wednesday – Saturday

Meeting #1192
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/national.html.

Joint Invited Addresses
Natalie E. Dean, Emory University, Tales From the Front Lines of Pandemic Communications (JPBM Communications Award Lecture).
Anne Schilling, University of California, Davis, The Ubiquity of Crystal Bases (AWM-AMS Noether Lecture).
Peter M Winkler, Dartmouth College, Permutons (AAAS-AMS Invited Address).
Kamuela E. Yong, University of Hawaii West Oahu, When Mathematicians Don’t Count (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).

JOINT Other Events
AMS - PME Undergraduate Student Poster Session, Chad Awtrey and Frank Patane, Samford University.
Awards Celebration, Boris Hasselblatt, Tufts University.
Awards Celebration Prize Winner Meet & Greet, Boris Hasselblatt, Tufts University.
Ribbon Cutting Ceremony, Penny Pina, American Mathematical Society.

AMS Invited Addresses
Ruth Charney, Brandeis University, From Braid Groups to Artin Groups (AMS Retiring Presidential Address).
Daniel Erman, University of Hawaii, From Hilbert to Mirror Symmetry.
Suzanne Marie Lenhart, University of Tennessee, Knoxville, Natural System Management: A Mathematician’s Perspective (AMS Josiah Willard Gibbs Lecture).
Ankur Moitra, Massachusetts Institute of Technology, Learning From Dynamics (von Neumann Lecture).
Kimberly Sellers, North Carolina State University, Dispersed Methods for Handling Dispersed Count Data.
Terence Tao, UCLA, Machine Assisted Proof (AMS Colloquium Lecture I - Terence Tao, University of California, Los Angeles).
Terence Tao, UCLA, Translational Tilings of Euclidean Space (AMS Colloquium Lecture II - Terence Tao, University of California, Los Angeles).
John Urschel, MIT, From Moments to Matrices (AMS Erdős Lecture for Students).
Suzanne L Weekes, SIAM, Mathematics in (and for) the Real World (AMS Lecture on Education).

AMS Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.
Advances in Analysis, PDE’s and Related Applications, Tepper L. Gill, Howard University, E. Kwessi, Trinity University, and Henok Mawi, Howard University (Washington, DC, US).
Advances in Coding Theory, Emily McMillon, Virginia Tech, Christine Ann Kelley and Tefjol Pllaha, University of Nebraska - Lincoln, and Mary Wootters, Stanford.
Algebraic Approaches to Mathematical Biology, Nicolette Meshkat, Santa Clara University, Cash Bortner, California State University, Stanislaus, and Anne Shiu, Texas A&M University.
Algebraic Structures in Knot Theory, V Sam Nelson, Claremont McKenna College, and Neslihan Gugumcu, Izmir Institute of Technology in Turkey.
AMS-AWM Special Session for Women and Gender Minorities in Symplectic and Contact Geometry and Topology, Sarah Blackwell, Max Planck Institute for Mathematics, Luya Wang, University of California, Berkeley, and Nicole Magill, Cornell University (Joint).
Analysis and Differential Equations at Undergraduate Institutions, Evan Daniel Randles, Colby College, and Lisa Naples, Fairfield University.
Applications of Extremal Graph Theory to Network Design, Kelly Isham, Colgate University, and Laura Monroe, Los Alamos National Laboratory.
Applications of Hypercomplex Analysis, Mihaela B. Vajiac, Chapman University, Orange, CA, Daniel Alpay, Chapman University, and Paula Cerejeiras, University of Aveiro, Portugal.
Applied Topology Beyond Persistence Diagrams, Nikolas Schonsheck, University of Delaware, Lori Ziegelmeier, Macalester College, Gregory Henselman-Petrusek, University of Oxford, and Chad Giusti, Oregon State University.
Applied Topology: Theory, Algorithms, and Applications, Wooyin Kim, Duke University, Johnathan Bush, University of Florida, Alex McQuaery, Ohio State University, Sarah Percival, Michigan State University, and Iris H. R. Yoon, University of Delaware.
Arithmetic Geometry with a View toward Computation, David Lowry-Duda, ICERM & Brown University, Barinder Banwait, Boston University, Shiva Chidambaram, Massachusetts Institute of Technology, Juanita Duque-Rosero, Boston University, Brendan Hassett, ICERM/Brown University, and Ciaran Schembri, Dartmouth College.
Coding Theory for Modern Applications, Rafael D’Oliveira, Clemson University, Hiram H. Lopez, Cleveland State University, and Allison Beemer, University of Wisconsin-Eau Claire.
MEETINGS & CONFERENCES

Combinatorial Insights into Algebraic Geometry, Javier Gonzalez Anaya, Harvey Mudd College.
Combinatorics for Science, Stephen J Young, Bill Kay, and Sinan Aksoy, Pacific Northwest National Laboratory.

Commutative Algebra and Algebraic Geometry (associated with Invited Address by Daniel Erman), Daniel Erman, University of Hawaii, and Aleksandra C Sobieska, University of Wisconsin - Madison.
Complex Analysis, Operator Theory, and Real Algebraic Geometry, J. E. Pascoe, Drexel University, Kelly Bickel, Bucknell University, and Ryan K. Tully-Doyle, Cal Poly SLO.
Complex Social Systems (a Mathematics Research Communities session) I, Ekaterina Landgren, University of Colorado, Boulder, Cara Sulyok, Lewis University, Casey Lynn Johnson, UCLA, Molly Lynch, Hollins University, and Rebecca Hardenbrook, Dartmouth College.

Computable Mathematics: A Special Session Dedicated to Martin D. Davis, Valentina S Harizanov, George Washington University, Alexandra Shlapentokh, East Carolina University, and Wesley Calvert, Southern Illinois University.
Computational Biomedicine: Methods - Models - Applications, Nektarios A. Valous, Center for Quantitative Analysis of Molecular and Cellular Biosystems (Bioquant), Heidelberg University, Im Neuenheimer Feld 267, 69120, Heidelberg, Germany, Anna Konstorum, Center for Computing Sciences, Institute for Defense Analyses, 17100 Science Drive, Bowie, MD, 20715, USA, Heiko Enderling, Department of Integrated Mathematical Oncology, H. Lee Moffitt Cancer Center & Research Institute, Tampa, FL, 33647, USA, and Dirk Jäger, Department of Medical Oncology, National Center for Tumor Diseases (NCT), University Hospital Heidelberg (UKHD), Im Neuenheimer Feld 460, 69120, Heidelberg, Germany.

Computational Techniques to Study the Geometry of the Shape Space, Shira Faigenbaum-Golovin, Duke University, Shan Shan, University of Southern Denmark, and Ingrid Daubechies, Duke University.
Covering Systems of the Integers and Their Applications, Joshua Harrington, Cedar Crest College, Tony Wing Hong Wong, Kutztown University of Pennsylvania, and Matthew Litman, UC Davis.
Cryptography and Related Fields, Ryan Cartor, Clemson University, Angela Robinson, NIST, and Daniel Everett Martin, Clemson University.
Derived Categories, Arithmetic, and Geometry (a Mathematics Research Communities session) I, Anirban Bhaduri, University of South Carolina, Gabriel Dorfman-Hopkins, St. Lawrence University, Patrick Lank, University of South Carolina, and Peter McDonald, University of Utah.
Development Students’ Technical Communication Skills through Mathematics Courses, Michelle L. Ghrist, Gonzaga University, Timothy P Chartier, Davidson College, Maila B. Hallare, US Air Force Academy, USAFA CO USA, and Denise Taunton Reid, Valdosta State University.
Diffusive Systems in the Natural Sciences, Francesca Bernardi, Worcester Polytechnic Institute, and Owen L Lewis, University of New Mexico.
Discrete Homotopy Theory, Krzysztof R. Kapulkin, University of Western Ontario, Anton Dochtermann, Texas State University, and Antonio Rieser, CONACYT-CIMAT.

Dynamical Systems Modeling for Biological and Social Systems, Daniel Brendan Cooney, University of Pennsylvania, Chadi M Saad-Roy, University of California, Berkeley, and Chris M. Heggerud, University of California, Davis.
Dynamics and Management in Disease or Ecological Models (associated with Gibbs Lecture by Suzanne Lenhart), Suzanne Lenhart, University of Tennessee, Knoxville, Christina Edholm, Scripps College, and Wandi Ding, Middle Tennessee State University.
Dynamics and Regularity of PDEs, Zongyuan Li, Rutgers University, Zhiyuan Zhang, Northeastern University, Xueying Yu, Oregon State University, and Weinan Wang, University of Oklahoma.

Epistemologies of the South and the Mathematics of Indigenous Peoples, María Del Carmen Bonilla Tumialán, National University of Education Enrique Guzman y Valle, Wilfredo Vidal Alangui, College of Science, University of the Philippines Baguio, and Domingo Yocjcom ROCHE, Center for Scientific and Cultural Research.

Ergodic Theory, Symbolic Dynamics, and Related Topics, Andrew T Dykstra, Hamilton College, and Shrey Sanadhya, Ben Gurion University of the Negev, Israel.
Ethics in the Mathematics Classroom, Victor Piercey, Ferris State University, and Catherine Buell, Fitchburg State University.
Extremal and Probabilistic Combinatorics, Sam Spiro, Rutgers University, and Corrine Yap, Georgia Institute of Technology.

Geometric Analysis in Several Complex Variables, Ming Xiao, University of California, San Diego, Bernhard Lamel, Texas A&M University At Qatar, and Nordine Mir, Texas A&M University at Qatar.
Meetings & Conferences

Geometric Group Theory (Associated with the AMS Retiring Presidential Address), Kasia Jankiewicz, University of California Santa Cruz, Edgar A. Bering, San José State University, Marion Campisi, San José State University, and Tim Hsu and Giang Le, San José State University.

Geometry and Symmetry in Differential Equations, Control, and Applications, Taylor Joseph Klotz and George Wilkens, University of Hawaii.

Geometry and Topology of High-Dimensional Biomedical Data, Smita Krishnaswamy, Yale, Dhananjay Bhaskar, Yale University, Bastian Rieck, Technical University of Munich, and Guy Wolf, Université de Montréal.

Group Actions in Commutative Algebra, Alessandra Costantini, Oklahoma State University, Alexandra Seceleanu, University of Nebraska-Lincoln, and Andras Cristian Lorincz, University of Oklahoma.

Hamiltonian Systems and Celestial Mechanics, Zhifu Xie, The University of Southern Mississippi, and Ernesto Perez-Chavela, ITAM.

Harmonic Analysis, Geometry Measure Theory, and Fractals, Kyle Hambrook, San Jose State University, Chun-Kit Lai, San Francisco State University, and Caleb Z Marshall, University of British Columbia.

History of Mathematics, Adrian Rice, Randolph-Macon College, Sloan Evans Despeaux, Western Carolina University, Deborah Kent, University of St. Andrews, and Jemma Lorenat, Sloan Evans Despeaux, Western Carolina University.

Homological Techniques in Noncommutative Algebra, Robert Won, George Washington University, Ellen E Kirkman, Wake Forest University, and James J. Zhang, University of Washington.

Homotopy Theory, Krzysztof R. Kapulkin, University of Western Ontario, Daniel K. Dugger, University of Oregon, Jonathan Beardsley, University of Nevada, Reno, and Thomas Brazelton, University of Pennsylvania.

Ideal and Factorization Theory in Rings and Semigroups, Scott Chapman, Sam Houston State University, and Alfred Geroldinger, University of Graz.

Informal Learning, Identity, and Attitudes in Mathematics, Sergey Grigorian, Mayra Ortiz, Xiaohui Wang, and Aaron T Wilson, University of Texas Rio Grande Valley.

Integer Partitions, Arc Spaces and Vertex Operators, Hussein Mourtada, Université Paris Cité, and Andrew R. Linshaw, University of Denver.

Interplay Between Matrix Theory and Markov Systems: Applications to Queueing Systems and of Duality Theory, Alan Krinik and Randall J. Swift, California State Polytechnic University, Pomona.

Issues, Challenges and Innovations in Instruction of Linear Algebra, Feroz Siddique, University of Wisconsin-Eau Claire, and Ashish K. Srivastava, Saint Louis University.

Knots, Skein Modules, and Categorification, Rhea Palak Bakshi, ETH Institute for Theoretical Studies, Zurich, Sujey Mukherjee, University of Denver, and Jozef Henryk Przytycki, George Washington University.

Large Random Permutations (affiliated with AAAS-AMS Invited Address by Peter Winkler), Peter M Winkler, Dartmouth College, and Jacopo Borga, Stanford University.

Loeb Measure after 50 Years, Yeneng Sun, National University of Singapore, Robert M Anderson, UC Berkeley, and Matt Insall, Missouri University of Science and Technology.

Looking Forward and Back: Common Core State Standards in Mathematics (CCSSM), 12 Years Later, Younhee Lee, Southern Connecticut State University, James Alvarez, University of Texas Arlington, Ekaterina Fuchs, City College of San Francisco, Tyler Kloefkorn, American Mathematical Society, Yvonne Lai, University of Nebraska-Lincoln, and Carl Olimb, Augustana University.

Mathematical Modeling and Simulation of Biomolecular Systems, Zhen Chao, University of Michigan-Ann Arbor, and Jiahui Chen, University of Arkansas.

Mathematical Modeling of Nucleic Acid Structures, Pengyu Liu, University of California, Davis, Van Pham, University of South Florida, and Svetlana Poznanovic, Clemson University.

Mathematical Physics and Future Directions, Shanna Dobson, University of California, Riverside, Tepper L. Gill, Howard University, Michael Anthony Maroun, University of California, Riverside, CA, and Lance Nielsen, Creighton University.

Mathematics and Philosophy, Tom Morley, Georgia Tech, and Bonnie Gold, Monmouth University.

Mathematics and Quantum, Kaifeng Bu and Arthur M. Jaffe, Harvard, Sui Tang, UCSB, and Jonathan Weitsman, Northeastern University.

Mathematics and the Arts, Karl M Kattchee, University of Wisconsin-La Crosse, Doug Norton, Villanova University, and Anil Venkatesh, Adelphi University.

Mathematics of Computer Vision, Timothy Duff and Max Lieblich, University of Washington.

Mathematics of DNA and RNA, Marek Kimmel, Rice University, Chris McCarthy, BMCC, City University of New York, and Johannes Hamilton, Borough of Manhattan Community College, CUNY.

Metric Dimension of Graphs and Related Topics, Briana Foster-Greenwood, Cal Poly Pomona, and Christine Uhl, St. Bonaventure University.
Metric Geometry and Topology, Christine M. Escher, Oregon State University, and Catherine Searle, Wichita State University.

Mock Modular forms, Physics, and Applications, Amanda Folsom, Amherst College, Terry Gannon, University of Alberta, and Larry Rolen, Vanderbilt University.


Modelling with Copulas: Discrete vs Continuous Dependent Data, Martial Longla, University of Mississippi, and Isidore Seraphin Ngongo, University of Yaounde I.

Modern Developments in the Theory of Configuration Spaces, Christin Bibby, Louisiana State University, and Nir Gadish, University of Michigan.

Modular Tensor Categories and TQFTs beyond the Finite and Semisimple, Colleen Delaney, UC Berkeley, and Nathan Geer, Utah State University.

New Faces in Operator Theory and Function Theory, Michael R Pilla, Ball State University, and William Thomas Ross, University of Richmond.

Nonlinear Dynamics in Human Systems: Insights from Social and Biological Perspectives, Armando Roldan, University of Central Florida, and Thomas Dombrowski, Moffitt Cancer Center.

Number Theory in Memory of Kevin James, Jim L. Brown, Occidental College, and Felice Manganiello, Clemson University.

Numerical Analysis, Spectral Graph Theory, Orthogonal Polynomials, and Quantum Algorithms, Anastasia Minenkova, University of Hartford, and Gamal Mograby, University of Maryland.

Partition Theory and q-Series, William Jonathan Keith, Michigan Technological University, Brandt Kronholm, University of Texas Rio Grande Valley, and Dennis Eichhorn, University of California, Irvine.

Polymath Jr REU Student Research, Steven Joel Miller, Williams College, and Alexandra Seceleanu, University of Nebraska-Lincoln.

Principles, Spatial Reasoning, and Science in First-Year Calculus, Yat Sun Poon and Catherine Lussier, University of California, Riverside, and Bryan Carrillo, Saddleback College.

Quantitative Justice, Ron Buckmire, Occidental College, Omayra Ortega, Sonoma State University, and Robin Wilson, California State Polytechnic University, Pomona (NAM-SIAM-AMS).

Quaternions, Chris McCarthy, BMCC, City University of New York, Johannes Hamilton, Borough of Manhattan Community College, CUNY, and Terrence Richard Blackman, Medgar Evers Community College, CUNY.

Recent Advances in Mathematical Models of Diseases: Analysis and Computation, Najat Ziyadi and Jamel S Mohammed-Awel, Department of Mathematics, Morgan State University.

Recent Advances in Stochastic Differential Equation Theory and its Applications in Modeling Biological Systems, Tuan A. Phan, IMCI, University of Idaho, Nhu N. Nguyen, University of Rhode Island, and Jianjun P. Tian, New Mexico State University.

Recent Developments in Commutative Algebra, Austyn Simpson and Alapan Mukhopadhyay, University of Michigan, and Thomas Marion Polstra, University of Virginia.

Recent Developments in Numerical Methods for PDEs and Applications, Chunmei Wang, University of Florida, Long Chen, UC Irvine, Shuhao Cao, University of Missouri-Kansas City, and Haizhao Yang, University of Maryland College Park.

Recent Developments on Markoff Triples, Elena Fuchs, UC Davis, and Daniel Everett Martin, Clemson University.

Recent Progress in Inference and Sampling (Associated with AMS Invited Address by Ankur Moitra), Ankur Moitra, Massachusetts Institute of Technology, and Sitran Chen, Harvard University.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, Darren A. Narayan, Rochester Institute of Technology, John C. Wierman, Johns Hopkins University, Mark Daniel Ward, Purdue University, Khang Duc Tran, California State University, Fresno, and Christopher O’Neill, San Diego State University.

Research Presentations by Math Alliance Scholar Doctorates, Theresa Martines, University of Texas, Austin, and David Goldberg, Math Alliance/Purdue University.

Ricci Curvatures of Graphs and Applications to Data Science (a Mathematics Research Communities session) I, Aleyah Dawkins, George Mason University, Xavier Ramos Olive, Smith College, Zhaiming Shen, University of Georgia, David Harry Richman, University of Washington, and Michael G Rawson, PNNL.

Roots of Unity - Mathematics from Graduate Students in the Roots of Unity Program, Allechar Serrano Lopez, Harvard University, and Patricia Klein, Texas A&M University.

Serious Recreational Mathematics, Erik Demaine, Massachusetts Institute of Technology, Robert A. Hearn, Gathering 4 Gardner, and Tomas Rokicki, California.

Solvable Lattice Models and their Applications Associated with the Noether Lecture, Anne Schilling, University of California, Davis, Amol Aggarwal, Columbia, Benjamin Brubaker, University of Minnesota - Twin Cities, Daniel Bump, Stanford, Andrew Hardt, Stanford University, Slava Naprienko, University of North Carolina at Chapel Hill, Leonid Petrov, University of Virginia, and Anne Schilling, University of California, Davis (AMS-AWM).

Spectral Methods in Quantum Systems, Matthew Powell, Georgia Institute of Technology, and Wencai Liu, Texas A&M University.


The EDGE (Enhancing Diversity in Graduate Education) Program: Pure and Applied Talks by Women Math Warriors, Quiyana Murphy, Virginia Tech, Sofia Rose Rose Martinez Alberga, Purdue University, Kelly Buch, Austin Peay State University, and Alexis Hardesty, Texas Tech University.


Theoretical and Numerical Aspects of Nonlocal Models, Nicole Buczkowski, Worcester Polytechnic Institute, Christian Alexander Glusa, Sandia National Laboratories, and Animesh Biswas, University of Nebraska Lincoln.

Theta Correspondence, Edmund Karasiwicz and Petar Bakic, University of Utah.

The Teaching and Learning of Undergraduate Ordinary Differential Equations, Viktoria Savatorova, Central Connecticut State University, Chris Goodrich, The University of New South Wales, Itai Seggev, Wolfram Research, Beverly H West, Cornell University, and Maia B. Hallare, US Air Force Academy, USAFA CO USA.

Thresholds in Random Structures, Will Perkins, Georgia Tech.

Topics in Combinatorics and Graph Theory, Cory Palmer and Anastasia Halfpap, University of Montana, and Neal Bushaw, Virginia Commonwealth University.

Topics in Equivariant Algebra, Ben Spitz, University of California Los Angeles, and Christy Hazel and Michael A. Hill, UCLA.


Water Waves, Anastassiya Semenova and Bernard Deconinck, University of Washington, John D Carter, Seattle University, and Eleanor Devin Byrnes, University of Washington.

Invited Addresses of Other JMM Partners


Henri Darmon, McGill University, Fourier Coefficients of Modular Forms (CRM-PIMS-AARMS Invited Address - Henri Darmon, McGill University).

Ranthony A C Edmonds, The Ohio State University, Quantitative Justice: Intersections of Mathematics and Society (NAM Cox-Talbot Address).

Katherine Ensor, Rice University, Celebrating Statistical Foundations Driving 21st -Century Innovation (ASA Invited Address- Kathy Ensor, Rice University).


Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland; past president of American Physical Society, National Medal of Science, What Challenges Does Data Science Present to Mathematics Education? (TPSE Invited Address - Sylvester James Gates, Jr, Clark Leadership Chair in Science, University of Maryland).

Matthew Harrison-Trainor, University of Illinois Chicago, The Complexity of Classifying Topological Spaces (ASL Invited Address).

Åsa Hirvonen, University of Helsinki, Games for Measuring Distances Between Metric Structures (ASL Invited Address).

Trachette Jackson, University of Michigan, Mobilizing Mathematics for the Fight Against Cancer (PME Invited Address).

Shelly M Jones, Central Connecticut State University, Choosing Hope: Teaching Culturally Relevant Mathematics as a Human Endeavor (NAM Claytor-Woodard Lecture).

Yvonne Lai, University of Nebraska-Lincoln, (Why) To Build Bridges in Mathematics Education (MAA Lecture on Teaching and Learning).
Francois Loeser, Institut Universitaire de France, Sorbonne, Model Theory and Non-Archimedean Geometry (ASL Invited Address).

Toby Meadows, University of California, Irvine, A Modest Foundational Argument for the Generic Multiverse (ASL Invited Address).

Dima Svetosla Sinapova, Rutgers University, Combinatorial Principles at Successors of Singular Cardinals (ASL Invited Address).

Slawomir Solecki, Cornell University, Descriptive Set Theory and Generic Measure Preserving Transformations (ASL Invited Address).

Joni Teräväinen, University of Turku, Uniformity of the Möbius Function in Short Intervals (AIM Alexanderson Award Lecture - Joni Teräväinen).

Mariel Vazquez, University of California, Davis, Topological Considerations in Genome Biology (SIAM Invited Address).

Mariana Vicaria, University of California, Los Angeles, Model Theory of Valued Fields (ASL Invited Address).

AIM Special Sessions

Equivariant Techniques in Stable Homotopy Theory, Michael A. Hill, UCLA, and Anna Marie Bohmann, Vanderbilt University.

Graphs and Matrices, Mary Flagg, University of St. Thomas, and Bryan A Curtis, Iowa State University.

Little School Dynamics: Cool Research by Researchers at PUIs, Kimberly Ayers, California State University, San Marcos, Ami Radunskaya, Pomona College, Andy Parrish, Eastern Illinois University, David M. McClendon, Ferris State University, and Han Li, Wesleyan University.

Math Circle Activities as a Gateway Into Research, Jeffrey Musyt, Slippery Rock University, Lauren L Rose, Bard College.

Tom G. Stojasavljevic, Beloit College, Nick Rauh, Julia Robinson Math Festivals, Edward Charles Keppelmann, University of Nevada Reno, Allison Henrich, Seattle University, Violeta Vasilevska, Utah Valley University, and Gabriella A. Pinter, University of Wisconsin, Milwaukee.

Multiplicative Number Theory and Additive Combinatorics, Joni Teräväinen, University of Turku, Terence Tao, UCLA, Kasia Matomäki, University of Turku, Maksym Radziwill, Northwestern University, and Tamar Ziegler, Hebrew University.

ASL Special Sessions

Descriptive Methods in Dynamics, Combinatorics, and Large Scale Geometry, Jenna Zomback, University of Maryland, College Park, and Forte Shinko, UCLA.

AWM Special Sessions

EvenQuads Live and in person: The honorees and the games, sarah-marie belcastro, Mathematical Staircase, Inc., Sherli Koshy-Chenthittayil, Touro University Nevada, Oscar Vega, California State University, Fresno, Monica D. Morales-Hernandez, Adelphi University, Linda McGuire, Muhlenberg College, and Denise A. Rangel Tracy, Fairleigh Dickinson University.

Mathematics in the Literary Arts and Pedagogy in Creative Settings, Shanna Dobson, University of California, Riverside, and Claudia Maria Schmidt, California State University.

Recent Developments in Harmonic Analysis, Betsy Stovall, University of Wisconsin-Madison, and Sarah E Tammen, UW-Madison.

Women in Mathematical Biology, Christina Edholm, Scripps College, Lihong Zhao, University of California, Merced, and Lale Asik, University of the Incarnate Word.

COMAP Special Sessions

Math Modeling Contests: What They Are, How They Benefit, What They Did – Discussions with the Students and Advisors, Kayla Blyman, Saint Martin’s University.

ILAS Special Sessions

Generalized Numerical Ranges and Related Topics, Tin-Yau Tam and Pan-Shun Lau, University of Nevada, Reno.

Graphs and Matrices, Jane Breen, Ontario Tech University, and Stephen Kirkland, University of Manitoba.

Innovative and Effective Ways to Teach Linear Algebra, David M. Strong, Pepperdine University, Sepideh Stewart, University of Oklahoma, Gil Strang, MIT, and Megan Wawro, Virginia Tech.

Linear Algebra, Matrix theory, and its Applications, Stephan Ramon Garcia and Konrad Aguilar, Pomona College.

Sign-pattern Matrices and Their Applications, Bryan L Shader, University of Wyoming, and Minerva Catral, Xavier University.

**MAA Special Sessions**

Navigating the Benefits and Challenges of Mentoring Students in Data-Driven Undergraduate Research Projects, Vinodh Kumar Chellamuthu, Utah Tech University, and Xiaoxia Xie, Idaho State University.

Undergraduate Research Activities in Mathematical and Computational Biology, Timothy D Comar, Benedictine University, and Anne E. Yust, University of Pittsburgh.

**PMA Special Sessions**

BSM Special Session: Mathematical Research in Budapest for Students and Faculty, Kristina Cole Garrett, St. Olaf College.

**SIAM Minisymposium**

SIAM ED Session on Artificial Intelligence and its Uses in Mathematical Education, Research, and Automation in the Industry, Alvaro Alfredo Ortiz Lugo, University of Cincinnati, Kathleen Kavanagh, Clarkson University, and Sergio Molina, University of Cincinnati.

SIAM Minisymposium on Computational Mathematics and the Power Grid, Todd Munson, Argonne National Laboratory.

SIAM Minisymposium on Recent Developments in the Analysis and Control of Partial Differential Equations Arising in Fluid and Fluid-Structure Interactive Dynamics, George Avalos, University of Nebraska-Lincoln, and Pelin Guven Geredeli, Clemson University.


SIAM-USNCTAM Minisymposium on Mathematical Modeling of Complex Materials Systems, Maria G Emelianenko, George Mason University, and Dmitry Golovaty, The University of Akron.

**SLMATH Special Sessions**

African Diaspora Joint Mathematics Working Groups (ADJOINT), Caleb Ashley, Boston College, and Anisah Nabila Nu’Man, Spelman College.


*The MSRI Undergraduate Program (MSRI-UP), Maria Mercedes Franco, Queensborough Community College-CUNY.*

**SPECTRA Special Sessions**

Research by LGBTQ+ Mathematicians, Devavrat Dabke, Princeton University, Joseph Nakao, Swarthmore College, and Michael A. Hill, UCLA.

**Invited Addresses of Other Organizations**

Arezoo Islami, San Francisco State University, The Unreasonable Effectiveness of Mathematics: Dissolving Wigner’s Applicability Problem (Special Interest Group of the MAA on the Philosophy of Mathematics Guest Lecture and Discussion).

**Other Special Sessions**

Exploring Funding Opportunities in the Division of Mathematical Sciences, Elizabeth Wilmer and Junping Wang, National Science Foundation.

Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences I, Michael Ferrara, Division of Undergraduate Education, National Science Foundation.
AMS Contributed Paper Sessions

AMS Contributed Paper Session on Algebraic Topology and Manifolds, Michelle Ann Manes, American Institute of Mathematics.
AMS Contributed Paper Session on Complex Variables, Michelle Ann Manes, American Institute of Mathematics.
AMS Contributed Paper Session on Control Theory, Quantum Theory, and Related Topics, Michelle Ann Manes, American Institute of Mathematics.
AMS Contributed Paper Session on Geometry, Michelle Ann Manes, American Institute of Mathematics.
AMS Contributed Paper Session on Mathematical Biology, Michelle Ann Manes, American Institute of Mathematics.
AMS Contributed Paper Session on Numerical Analysis, Michelle Ann Manes, American Institute of Mathematics.

ASL Contributed Paper Sessions

ASL Contributed Paper Session, David Reed Solomon, University of Connecticut.

NAM Contributed Paper Sessions

NAM Haynes-Granville-Broune Session of Presentations by Recent Doctoral Recipients, Aris Winger, Georgia Gwinnett College, Torina D. Lewis, National Association of Mathematicians, and Omayra Ortega, Sonoma State University.

PME Contributed Paper Sessions

PME Contributed Session on Research by Undergraduates, Thomas Philip Wakefield, Youngstown State University, and Jennifer Beineke, Western New England University.

TPSE Contributed Paper Sessions


AMS Other Events

AMS Career Fair, Sarah Klyberg, American Mathematical Society.
AMS Committee on Education Panel Discussion, I - “Mathematics online: PDFs and issues regarding accessibility”, Terrence Richard Blackman, Medgar Evers Community College, CUNY, Michael Dorff, TPSE Math, William Yslas Velez, University of Arizona, and Erica Walker, Ontario Institute for Studies in Education.
AMS Committee on Equity, Diversity and Inclusion Panel Discussion: Successful Programs that Support Equity, Diversity and Inclusion, Sarah J. Greenwald, Appalachian State University, Lily Khadjavi, Loyola Marymount University, and Dennis Davenport, Howard University.
AMS Committee on the Profession Panel Discussion: Building a Successful Research Career in Mathematics, Edray Herber Goins, Pomona College, and Pamela Harris, Williams College.
AMS Current Events Bulletin, David Eisenbud, MSRI.
AMS DC-Based Policy & Communications Opportunities, Karen Saxe, American Mathematical Society.
AMS Department Chairs and Leaders Workshop, Timothy Flood, Pittsburg State University, Emille Davie Lawrence, University of San Francisco, and Charles N. Moore, Washington State University.
AMS Directors of Graduate Studies Focus Group, Sarah Bryant, American Mathematical Society.
AMS Directors of Undergraduate Studies Focus Group, Sarah Bryant, American Mathematical Society.
AMS Grad School Fair, Rosalynde Vas Dias, American Mathematical Society.

ASL Other Events
Association for Symbolic Logic Tutorial: Large Cardinals, Determinacy, and Inner Models, David Reed Solomon, University of Connecticut, and John Steel, UC Berkeley.

AWM Other Events
AWM Panel: Celebrating Academic Pivots in Mathematics, Kate Petersen, University of Minnesota Duluth, and Kelly McKinnie, University of Montana.
AWM Workshop Poster Presentations, Radmila Sazdanovic, NC State University.
AWM Workshop: Women in Operator Theory, Catherine Anne Beneteau, University of South Florida, and Asuman Aksoy, Claremont McKenna College.

COMAP Other Events
COMAP Panel on Math Modeling Contests: Trends, Topics, and Tips, Kayla Blyman, Saint Martin’s University, Keith Erickson, Georgia Gwinnett College, and Catherine Roberts, COMAP.

JMM Other Events
JMM Networking Center II, Penny Pina, American Mathematical Society.
JMM Networking Center I sponsored by Maplesoft, Penny Pina, American Mathematical Society.
JMM Panel: Cal-Bridge: Building Bridges and Diversifying Mathematics, Suzanne Sindi, University of California, Merced, and Oscar Vega, California State University, Fresno.
JMM Panel: Decolonizing Mathematics, Tarik Aougab, Haverford College, Marissa Kawehi Loving, University of Wisconsin Madison, and Brandis Whitfield, Temple University.
JMM Panel: Regional Math Alliances: Activities and Formation of Regional Groups to Support the Goals of the National Math Alliance, Theresa Martines, University of Texas, Austin, and David Goldberg, Math Alliance/Purdue University.

JMM Workshop on Teaching Student-Centered Mathematics: Active Learning & the Learning Assistant Model, Katherine V Johnson, Florida Gulf Coast University, and Brittannney Adelmann, Florida Atlantic University.

Joint Committee on Women Panel: Financial Empowerment for Mathematicians, Jennifer Schultens, University of California Davis.

Professional Enhancement Program (PEP) 2: GitHub for Mathematicians, Steven Craig Clontz, University of South Alabama, and Francesca Gandini, St. Olaf College.

Professional Enhancement Program (PEP) 3: Changing Math Department Culture: Embracing Servingness, Ben Ford, Sonoma State University, Rochelle Gutiérrez, University of Illinois, Brigitte Lahme, Sonoma State University, Luis Antonio Leyva, Vanderbilt-Peabody College, Omarya Ortega, Sonoma State University, and Aris Winger, Georgia Gwinnett College.


Professional Enhancement Program (PEP) 5: Development of Mathematics Programs for Workforce Preparation, Rick Cleary, Babson College, and Chris Malone, Winona State University.

Professional Enhancement Program (PEP) 7: Effective Technical Advocacy: How to Talk About Mathematics so Policymakers will Hear you, Audrey Malagon, Virginia Wesleyan University, and Stephanie Singer, Hatfield School of Government, Portland State University and Campaign Scientific.

Professional Enhancement Program (PEP) 8: Bringing Ethics and Justice to the Mathematics Classroom Through Historical Case Studies, Jemma Lorenat, Pitzer College, and Deborah Kent, University of St. Andrews.


Special Interest Group of the MAA on Mathematics and the Arts Professional Enhancement Program (PEP) 1: Visualizing Projective Geometry Through Photographs and Perspective Drawings, Annalisa Crannell, Franklin & Marshall College, and Fumiko Futamura, Southwestern University.

JMM Other Events

Julia Robinson Math Festival, Daniel Kline, Julia Robinson Mathematics Festival.

MAA Other Events

MAA Project NExT: Active Learning Strategies for a Large Class, Hannah Burson, University of Minnesota, Paul Herstedt, Macalester College, and Richard Wong, UCLA.

MAA Project NExT: Classrooms Meet the Future: How Modern Technology Is Enhancing the Classroom Experience of Mathematics, Keegan Kang, Bucknell University, Rachel Perrier, Francisian University of Steubenville, and Shuyi Weng, Purdue University.

MAA Project NExT Fostering a Growth Mindset in the Classroom, Shuler Hopkins, The University of the South, Camille Schuetz, University of Wisconsin - Platteville, and Adam Yassine, Pomona College.

MAA Project NExT: Making Student Thinking Visible with Team-Based Inquiry Learning, Christina Duron, Pepperdine University, Erin Ellefsen, Earlham College, and Aaron Osgood-Zimmerman, Bucknell University.

MAA Project NExT Panel Discussion on Diversity, Equity, and Inclusion Practices in an Undergraduate Math Class, Maria Amarakristi Onyido, Northern Illinois University, and George Nasr, Augustana University.

MAA Project NExT: Setting a New Standard: Implementing Standards-Based Grading, Daniel Graybill, Fort Lewis College, Alexis Hardesty, Texas Woman’s University, and Margaret Regan, College of the Holy Cross.

OTH Other Events

Addressing Unfinished Learning and Improving STEM Access with ALEKS PPL, Courtney Cozzy, McGraw Hill.


Focus Group: Developing Calculus Content for Today’s STEM Students, Courtney Cozzy, McGraw Hill.

Mathematically Bent Theater, Colin Adams, Williams College.

Top Hat Focus Group: Aktiv Math, Andrew Noble, Top Hat.

PME Other Events

PME Panel: What Every Student Should Know about the JMM, Stephanie Edwards, Hope College, Jennifer Beineke, Western New England University, and Thomas Philip Wakefield, Youngstown State University.
MEETINGS & CONFERENCES

SIAM Other Events

SPECTRA Other Events
Spectra Workshop: Creating an Inclusive Undergraduate Mathematics Curriculum, Devavrat Dabke, Princeton University, and Michael A. Hill, UCLA.

TPSE Other Events
TPSE Panel on Developing Innovative Upper Division Pathways in Mathematics: Strategies for Enrollment and Inclusion, Oscar Vega, California State University, Fresno, and Padmanabhan Seshaiyer, George Mason University.
TPSE Panel on Grading for Active Learning & Department Change, Katherine F Stevenson, CSU Northridge, Rachel Weir, Allegheny College, Scott Andrew Wolpert, University of Maryland and TPSE Math, and Stan Yoshinobu, University of Toronto.

Tallahassee, Florida

Florida State University

March 23–24, 2024  
Saturday – Sunday

Meeting #1193  
Southeastern Section  
Associate Secretary for the AMS:

Program first available on AMS website: To be announced  
Issue of Abstracts: Volume 45, Issue 2

Deadlines  
For organizers: To be announced  
For abstracts: January 23, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.html.

Invited Addresses
Wenjing Liao, Georgia Institute of Technology, Exploiting low-dimensional data structures in deep learning.
Olivia Prosper, University of Tennessee, Knoxville, Modeling Malaria at Multiple Scales.
Jared Speck, Vanderbilt University, Singularity Formation for the Equations of Einstein and Euler.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Numerical Methods for Partial Differential Equations and Their Applications (Code: SS 1A), Seonghee Jeong, Louisiana State University, Sanghyun Lee, Florida State University, and Seulip Lee, University of Georgia.

Advances in Financial Mathematics (Code: SS 2A), Qi Feng, Alec N Kercheval, and Lingjiong Zhu, Florida State University.

Advances in Shape and Topological Data Analysis (Code: SS 3A), Emmanuel I Hartman, Eric Klassen, and Ethan Semrad, Florida State University.

Algebraic Groups and Local-Global Principles (Code: SS 4A), Suresh Venapally, Emory University, and Daniel Reuben Krashen, University of Pennsylvania.

Bases and Frames in Hilbert spaces (Code: SS 5A), Laura De Carli, Florida International University, and Azita Mayeli, City University of New York.

Combinatorics in Geometry of Polynomials (Code: SS 6A), Papri Dey, Georgia Institute of Technology.

Control, Inverse Problems and Long Time Dynamics of Evolution Systems (Code: SS 7A), Shitao Liu, Clemson University, and Louis Tebou, Florida International University.

Data Integration and Identifiability in Ecological and Epidemiological Models (Code: SS 8A), Omar Saucedo, Virginia Tech, and Olivia Prosper, University of Tennessee/Knoxville.

Diversity in Mathematical Biology (Code: SS 9A), Daniel Alejandro Cruz and Skylar Grey, University of Florida.

Fluids: Analysis, Applications, and Beyond (Code: SS 10A), Aseel Farhat and Anuj Kumar, Florida State University.
Geometry and Symmetry in Data Science (Code: SS 12A), Dustin G. Mixon, The Ohio State University, and Thomas Needham, Florida State University.
Homotopy Theory and Category Theory in Interaction (Code: SS 13A), Ettore Aldrovandi and Brandon Doherty, Florida State University, and Philip John Hackney, University of Louisiana at Lafayette.
Human Behavior and Infectious Disease Dynamics (Code: SS 14A), Bryce Morsky, Florida State University.
Mathematical Advances in Scientific Machine Learning (Code: SS 15A), Wenjing Liao, Georgia Institute of Technology, and Feng Bao and Zecheng Zhang, Florida State University.
Mathematical Modeling and Simulation in Fluid Dynamics (Code: SS 16A), Pejman Sanaei, Florida State University.
Mathematical Models for Population and Methods for Parameter Estimation in Epidemiology (Code: SS 17A), Yang Li, Georgia State University, and Guihong Fan, Columbus State University.
Moduli Spaces in Algebraic Geometry (Code: SS 18A), Jeremy Usatine, Florida State University, Hulya Arguz and Pierrick Bousseau, University of Georgia, and Pierrick Bousseau, University of Arizona.
Recent Advances in Geometry and Topology (Code: SS 22A), Thang Nguyen, Samuel Aaron Ballas, Philip L. Bowers, and Sergio Fenley, Florida State University.
Recent Advances in Inverse Problems for Partial Differential Equations and Their Applications (Code: SS 23A), Anh-Khoa Vo, Florida A&M University, and Thuy T. Le, North Carolina State University.
Recent Development in Deterministic and Stochastic PDEs (Code: SS 24A), Quyuan Lin, Clemson University, and Xin Liu, Texas A&M University.
Recent Developments in Numerical Methods for Evolution Partial Differential Equations (Code: SS 25A), Thi-Thao-Phuong Hoang, Yanzhao Cao, and Hans-Werner Van Wyk, Auburn University.
Stochastic Analysis and Applications (Code: SS 28A), Hakima Bessaih, Florida International University, and Oussama Landoulsi, FLORIDA INTERNATIONAL UNIVERSITY.
Stochastic Differential Equations: Modeling, Estimation, and Applications (Code: SS 29A), Sher B Chhetri, University of South Carolina Sumter, Hongwei Long, Florida Atlantic University, and Olusegun M. Otunuga, Augusta University.
Theory of Nonlinear Waves (Code: SS 30A), Nicholas James Ossi and Ziad H Musllimani, Florida State University.
Topics in Graph Theory (Code: SS 31A), Songling Shan, Auburn University, and Guantao Chen, Georgia State University.
Topics in Stochastic Analysis/Rough Paths/SPDE and Applications in Machine Learning (Code: SS 32A), Cheng Ouyang, University of Illinois at Chicago, Fabrice Baudoin, University of Connecticut, and Qi Feng, Florida State University.
Topological Interactions of Contact and Symplectic Manifolds (Code: SS 34A), Angela Wu, University College of London and Louisiana State University, and Austin Christian, Georgia Institute of Technology.

Washington, District of Columbia
Howard University

April 6–7, 2024
Saturday – Sunday
Meeting #1194
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 2

Deadlines
For organizers: Expired
For abstracts: February 13, 2024
The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Ryan Charles Hynd, University of Pennsylvania, Title to be announced.


Jian Song, Rutgers, State University of New Jersey, Geometric Analysis on Singular Complex Spaces.

Talitha M Washington, Clark Atlanta University & Atlanta University Center, The Data Revolution (Einstein Public Lecture in Mathematics).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Mathematical Methods in Naval Engineering Research (Code: SS 1A), Michael Traweek, Office of Naval Research, and Anthony Ruffa, Emeritus Naval Undersea Warfare Center.

Algebraic and Enumerative Combinatorics (Code: SS 2A), Samuel Francis Hopkins, Howard University, Joel Brewster Lewis, George Washington University, and Peter R. W McNamara, Bucknell University.

Analysis of PDE in Inverse Problems and Control Theory (Code: SS 3A), Matthias Eller, Georgetown University, and Justin Thomas Webster, University of Maryland, Baltimore County.

Artificial Intelligence Emergent From Mathematics and Physics (Code: SS 4A), Bourama Toni, Howard University, and Artan Sheshmani, MIT IAiFi.

Automorphic Forms and Langlands Program (Code: SS 5A), Baiying Liu and Freydoon Shahidi, Purdue University.

Automorphic Forms and Trace Formulae (Code: SS 6A), Yiannis Sakellaridis, Johns Hopkins University, Bao Chau Ngo, University of Chicago, and Spencer Leslie, Boston College.

Coding Theory & Applications (Code: SS 7A), Emily McMillon, Eduardo Camps, and Hiram H. Lopez, Virginia Tech.

Commutative Algebra and its Applications (Code: SS 8A), Hugh Geller, West Virginia University, and Rebecca R.G., George Mason University.


Computational and Machine Learning Methods for Modeling Biological Systems (Code: SS 11A), Christopher Kim, Vipul Periwal, Manu Aggarwal, and Xiaoyu Duan, National Institutes of Health.

Control of Partial Differential Equations (Code: SS 12A), Gisele Adelie Mophou, Université des Antilles en Guadeloupe, and Mahamadi Warma, George Mason University.

Culturally Responsive Mathematical Education in Minority Serving Institutions (Code: SS 13A), Lucretia Glover, Lifoma Salaam, and Julie Lang, Howard University.


Interactions Between Analysis, Geometric Measure Theory, and Probability in Non-Smooth Spaces (Code: SS 17A), Luca Capogna, Smith College, Jeremy Tyson, University of Illinois at Urbana-Champaign, and Nageswar Shanmugalingam, University of Cincinnati.

Mathematical Modeling, Computation, and Data Analysis in Biological and Biomedical Applications (Code: SS 18A), Maria G Emelianenko and Daniel M Anderson, George Mason University.

Mathematical Modeling of Climate-Biosphere Interactions (Code: SS 19A), Ivan Sudakov, Department of Mathematics, Howard University.

Mathematical Modeling of Type 2 Diabetes and Its Clinical Studies (Code: SS 20A), Joon Ha, Howard University.
MEETINGS & CONFERENCES


Moduli Spaces in Geometry and Physics (Code: SS 23A), Artan Sheshmani, MIT IAiFi.


Nonlinear Hamiltonian PDEs (Code: SS 25A), Benjamin Harrop-Griffiths, Georgetown University, and Maria Ntekoume, Concordia University.

Optimization, Machine Learning, and Digital Twins (Code: SS 26A), Harbir Antil, Rohit Khandelwal, and Sean Carney, George Mason University.

Permutation Patterns (Code: SS 27A), Juan B Gil, Penn State Altoona, and Alexander I. Burstein, Howard University.

Post-Quantum Cryptography (Code: SS 28A), Jason LeGrow, Virginia Tech, Veronika Kuchta, Florida Atlantic University, Travis Morrison, Virginia Tech, and Edoardo Persichetti, Florida Atlantic University.

Qualitative Dynamics in Finite and Infinite Dynamical Systems (Code: SS 29A), Roberto De Leo, Howard University, and Jim A Yorke, University of Maryland.


Recent Advances in Harmonic Analysis and Their Applications to Partial Differential Equations (Code: SS 31A), Guher Camliyurt and Jose Ramon Madrid Padilla, Virginia Polytechnic Institute and State University.

Recent Advances in Optimal Transport and Applications (Code: SS 32A), Henok Mawi, Howard University (Washington, DC, US), and Farhan Abedin, Lafayette College.

Recent Advances on Machine Learning Methods for Forward and Inverse Problems (Code: SS 33A), Haizhao Yang, University of Maryland College Park, and Ke Chen, University of Maryland, College Park.

Recent Developments in Geometric Analysis (Code: SS 34A), Yueh-Ju Lin, Wichita State University, Samuel Perez-Ayala, Princeton University, and Ayush Khaitan, Rutgers University.

Recent Developments in Noncommutative Algebra and Tensor Categories (Code: SS 35A), Kent B. Vashaw, Massachusetts Institute of Technology, Van C. Nguyen, U.S. Naval Academy, Xingting Wang, Louisiana State University, and Robert Won, George Washington University.

Recent Developments in Nonlinear and Computational Dynamics (Code: SS 36A), Emmanuel Fleurantin and Christopher K. R. T. Jones, University of North Carolina.

Recent Advances in the Study of Free Boundary Problems in Fluid Mechanics (Code: SS 41A), Huy Q. Nguyen, University of Maryland, and Ian Rice, Carnegie Mellon University.

Recent Progress on Model-Based and Data-Driven Methods in Inverse Problems and Imaging (Code: SS 37A), Yimin Zhong, Auburn University, Yang Yang, Michigan State University, and Junshan Lin, Auburn University.

Recent Trends in Graph Theory (Code: SS 38A), Katherine Perry, Soka University of America, and Adam Blumenthal, Westminster College.

Riordan Arrays (Code: SS 39A), Dennis Davenport and Lou Shapiro, Howard University, and Leon Woodson, SPIRAL REU At Georgetown.

Skein Modules in Low Dimensional Topology (Code: SS 40A), Jozef Henryk Przytycki, George Washington University.


Tensor Algebra & Networks (Code: SS 43A), Giuseppe Cotardo, Gretchen Matthews, and Pedro Soto, Virginia Tech.

Variational Problems with Lack of Compactness (Code: SS 44A), Cheikh Birahim Ndiaye, Howard University, and Ali Maalaoui, Clark University.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), Steven H Weintraub, Lehigh University.
Milwaukee, Wisconsin
University of Wisconsin-Milwaukee

April 20–21, 2024
Saturday – Sunday

Meeting #1195
Central Section
Associate Secretary for the AMS: Betsy Stovall

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Mihaela Iftim, University of Wisconsin-Madison, The small data global well-posedness conjecture for 1D defocusing dispersive flows.
Lin Lin, University of California, Berkeley, Title To Be Announced.
Kevin Schreve, LSU, Homological growth of groups and aspherical manifolds.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic methods in graph theory and applications I (Code: SS 1A), Tung T. Nguyen, University of Chicago/ Western University, Sunil K. Chebolu, Illinios State University, and Jan Minac, Western University.
Algorithms, Number Theory, and Cryptography I (Code: SS 3A), Jonathan P Sorenson, Butler University, Eric Bach, University of Wisconsin at Madison, and Jonathan Webster, Butler University.
Applications of Algebra and Geometry I (Code: SS 8A), Thomas Yahl, University of Wisconsin - Madison, and Jose Israel Rodriguez, University of Wisconsin Madison.
Applications of Numerical Algebraic Geometry I (Code: SS 14A), Emma R Cobian, University of Notre Dame.
Artificial Intelligence in Mathematics I (Code: SS 9A), Tony Shaska, Oakland University, Alessandro Arsie, The University of Toledo, Elira Curri, Oakland University, Rochester Hills, MI, 48126, and Mee Seong Im, United States Naval Academy.
Automorphisms of Riemann Surfaces and Related Topics I (Code: SS 4A), Aaron D. Wootton, University of Portland, Jennifer Paulhus, Grinnell College, Sean Allen Broughton, Rose-Hulman Institute of Technology (emeritus), and Tony Shaska, Research Institute of Science and Technology.
Cluster algebras, Hall algebras and representation theory I (Code: SS 5A), Xueqing Chen, University of Wisconsin, Whitewater, and Yiqiang Li, SUNY At Buffalo.
Combinatorial and geometric themes in representation theory I (Code: SS 23A), Jeb F. Willenbring, UIW-Milwaukee, and Pamela E. Harris, University of Wisconsin, Milwaukee.
Complex Dynamics and Related Areas I (Code: SS 16A), James Waterman, Stony Brook University, and Alastair N Fletcher, Northern Illinois University.
Computability Theory I (Code: SS 25A), Matthew Harrison-Trainor, University of Illinois Chicago, and Steffen Lempp, University of Wisconsin-Madison.
Connections between Commutative Algebra and Algebraic Combinatorics I (Code: SS 10A), Alessandra Costantini, Oklahoma State University, Matthew James Weaver, University of Notre Dame, and Alexander T Yong, University of Illinois at Urbana-Champaign.
Developments in hyperbolic-like geometry and dynamics I (Code: SS 11A), Jonah Gaster, University of Wisconsin-Milwaukee, Andrew Zimmer, University of Wisconsin-Madison, and Chenxi Wu, University of Wisconsin At Madison.
Geometric group theory I (Code: SS 28A), G Christopher Hruska, University of Wisconsin-Milwaukee, and Emily Stark, Wesleyan University.
Geometric Methods in Representation Theory I (Code: SS 2A), Daniele Rosso, Indiana University Northwest, and Joshua Mundinger, University of Wisconsin - Madison.
Harmonic Analysis and Incidence Geometry I (Code: SS 17A), Sarah E Tammen and Terence L. J Harris, UW Madison, and Shengwen Gan, Massachusetts Institute of Technology.

Mathematical aspects of cryptography and cybersecurity I (Code: SS 24A), Lubjana Beshaj, Army Cyber Institute.

Model Theory I (Code: SS 15A), Uri Andrews, University of Wisconsin-Madison, and James Freitag, University of Illinois Chicago.

New research and open problems in combinatorics I (Code: SS 12A), Pamela Estephania Harris, University of Wisconsin, Milwaukee, Erik Insko, Central College, and Mohamed Omar, York University.

Nonlinear waves I (Code: SS 22A), Mihaela Ifrim, University of Wisconsin-Madison, and Daniel I Tataru, UC Berkeley.

Nonstandard and Multigraded Commutative Algebra I (Code: SS 13A), Mahrud Sayrafi, University of Minnesota, Twin Cities, and Maya Banks and Aleksandra C Sobieska, University of Wisconsin - Madison.

Panorama of Holomorphic Dynamics I (Code: SS 21A), Suzanne Lynch Boyd, University of Wisconsin Milwaukee, and Rodrigo Perez and Roland Roeder, Indiana University - Purdue University Indianapolis.

Posets in algebraic and geometric combinatorics I (Code: SS 26A), Martha Yip, University of Kentucky, and Rafael S. González D’León, Loyola University Chicago.

Ramification in Algebraic and Arithmetic Geometry I (Code: SS 29A), Charlotte Ure, Illinois State University, and Nick Rekuski, Wayne State University.

Recent Advances in Nonlinear PDEs and Their Applications I (Code: SS 27A), Xiang Wan, Loyola University Chicago, Rasika Mahawattege, University of Maryland, Baltimore County, and Madhumita Roy, Graduate Student, University of Memphis.

Recent Advances in Numerical PDE Solvers by Deep Learning I (Code: SS 7A), Dexuan Xie, University of Wisconsin-Milwaukee, and Zhen Chao, University of Michigan-Ann Arbor.

Recent Developments in Harmonic Analysis I (Code: SS 6A), Naga Manasa Vempati, Louisiana State University, Nathan A. Wagner, Brown University, and Bingyang Hu, Auburn University.

Recent trends in nonlinear PDE I (Code: SS 19A), Fernando Charro and Catherine Lebiedzik, Wayne State University, and Md Nurul Raihen, Fontbonne University.


San Francisco, California

San Francisco State University

May 4–5, 2024
Saturday – Sunday

Meeting #1196
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 3

Deadlines
For organizers: To be announced
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Julia Yael Plavnik, Indiana University, Title to be announced.
Mandi A. Schaeffer Fry, University of Denver, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Commutative and Noncommutative Algebra, Together at Last (Code: SS 1A), Pablo S. Ocal, University of California, Los Angeles, Benjamin Briggs, University of Copenhagen, and Janina C Letz, Bielefeld University.
Diagrammatic Algebras in Representation Theory and Beyond (Code: SS 2A), Jieru Zhu, Université Catholique de Louvain, Mee Seong Im, United States Naval Academy, Liron Speyer, Okinawa Institute of Science and Technology, and Arik Wilbert, University of Georgia.

Extremal Combinatorics and Connections (Code: SS 3A), Sam Spiro, Rutgers University, and Van Magnan, University of Montana.

Geometry and Topology of Quantum Phases of Matter (Code: SS 4A), Ralph Martin Kaufmann, Purdue University, and Markus J Pflaum, University of Colorado.

Groups and Representations (associated with Invited Address by Mandi Schaeffer Fry) (Code: SS 6A), Nathaniel Thiem, University of Colorado, Mandi A. Schaeffer Fry, University of Denver, and Klaus Lux, University of Arizona.

Geometry, Integrability, Symmetry and Physics (Code: SS 5A), Birgit Kaufmann and Sasha Tsymbaliuk, Purdue University, and Nathaniel Thiem, University of Colorado, Mandi A. Schaeffer Fry, University of Denver, and Klaus Lux, University of Arizona.

Homological Methods in Commutative Algebra & Algebraic Geometry (Code: SS 7A), Ritvik Ramkumar, Cornell University, Michael Perlman, University of Minnesota, and Aleksandra C Sobieska, University of Wisconsin - Madison.

Inverse Problems (Code: SS 8A), Hanna E. Makaruk, Los Alamos National Laboratory, Los Alamos, NM, and Robert M. Owczarek, University of New Mexico.

Mathematical Fluid Dynamics (Code: SS 9A), Igor Kukavica and Juhi Jang, University of Southern California, and Wojciech S. Ozanski, Florida State University.

Mathematical Modeling of Complex Ecological and Social Systems (Code: SS 10A), Daniel Brendan Cooney, University of Illinois at Urbana-Champaign, Mari Kawakatsu, University of Pennsylvania, and Chadi M Saad-Roy, University of California, Berkeley.

Partial Differential Equations and Convexity (Code: SS 11A), Ben Weinkove, Northwestern University, Stefan Steinerberger, University of Washington, Seattle, and Albert Chau, University of British Columbia.


Probability Theory and Related Fields (Code: SS 13A), Terry Soo and Codina Cotar, University College London.

Recent Advances in Differential Geometry (Code: SS 15A), Lihan Wang, California State University, Long Beach, Zhiqin Lu, UC Irvine, and Shoo Seto and Bogdan D. Suceavă, California State University, Fullerton.

Recent Developments in Commutative Algebra (Code: SS 16A), Arvind Kumar, Louiza Fouli, and Michael DiPasquale, New Mexico State University.

Research in Combinatorics by Early Career Mathematicians (Code: SS 18A), Nicholas Mayers, North Carolina State University, and Laura Colmenarejo, NCState.

Special Session in Celebration of Bruce Reznick's Retirement (Code: SS 19A), Katie Anders, University of Texas at Tyler, Simone Sisneros-Thiry, California State University- East Bay, and Dana Neidmann, Centre College.

Tensor Categories and Noncommutative Algebras, I (associated with invited address by Julia Plavnik) (Code: SS 20A), Ellen E Kirkman, Wake Forest University, and Julia Yael Plavnik, Indiana University, Bloomington.

Contributed Paper Sessions
AMS Contributed Paper Session (Code: CP 1A), Michelle Ann Manes, University of Hawaii.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Deadlines

Issue of Abstracts: To be announced

For organizers: To be announced

For abstracts: To be announced
San Antonio, Texas
University of Texas, San Antonio

**September 14–15, 2024**
Saturday – Sunday

**Meeting #1198**
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 3

**Deadlines**
For organizers: February 13, 2024
For abstracts: July 23, 2024

*The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).*

**Invited Addresses**
- James A. M. Álvarez, University of Texas, Arlington, *Title to be announced.*
- Jason Schweinsberg, University of California, San Diego, *Title to be announced.*
- Anne J. Shiu, Texas A&M University, *Title to be announced.*

Savannah, Georgia
Georgia Southern University, Savannah

**October 5–6, 2024**
Saturday – Sunday

**Meeting #1199**
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

**Deadlines**
For organizers: March 5, 2024
For abstracts: August 13, 2024

Albany, New York
University at Albany

**October 19–20, 2024**
Saturday – Sunday

**Meeting #1200**
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

**Deadlines**
For organizers: March 19, 2024
For abstracts: August 27, 2024

Riverside, California
University of California, Riverside

**October 26–27, 2024**
Saturday – Sunday

**Meeting #1201**
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of Abstracts: Volume 45, Issue 4

**Deadlines**
For organizers: March 26, 2024
For abstracts: September 3, 2024
Auckland, New Zealand

December 9–13, 2024
Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025
Wednesday – Saturday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Lawrence, Kansas

The University of Kansas

March 29–30, 2025
Saturday – Sunday
Central Section
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Hartford, Connecticut

Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

April 5–6, 2025
Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026
Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
Bridges and Wheels, Tricycles and Squares

Tucked underneath the Cody Dock Rolling Bridge in London, you'll find an unusual curved track. It swoops down, then up, then down again in a smooth wave. Resting atop the track is the bridge itself — a flat base, flanked by two large squares of steel and concrete through which pedestrians and cyclists enter and exit.

Wait around for a boat to come by, and you’ll learn the purpose of this strange setup. Powered by a hand crank, the bridge rolls upside down along the track, balancing on the top edges of those two square portals. With the base of the bridge now sitting high above the water, there is room for a boat to float below.

The Cody Dock Rolling Bridge is a recent and ambitious example of a “square wheel” that rolls without any slipping or jarring bumps. The curved track enables the smooth ride by keeping the center of the bridge level as it rolls. To do this, it follows a shape called a catenary, described by the mathematical formula:

$$\frac{e^x + e^{-x}}{2}$$

Square wheels also live west of the Atlantic. The Exploratorium in San Francisco hosts a square wheel exhibit. Inspired by this, the mathematician Stan Wagon enlisted his neighbor Loren Kellen in building a square-wheeled tricycle and catenary track in 1997. For years, you could ride the tricycle at Macalester College in St. Paul, Minnesota. The National Museum of Mathematics in New York also has square-wheeled tricycles that can be ridden around a circular track.

References:

Watch an interview with an expert!
Explore recent conference proceedings from the AMS.

Staying up to date!