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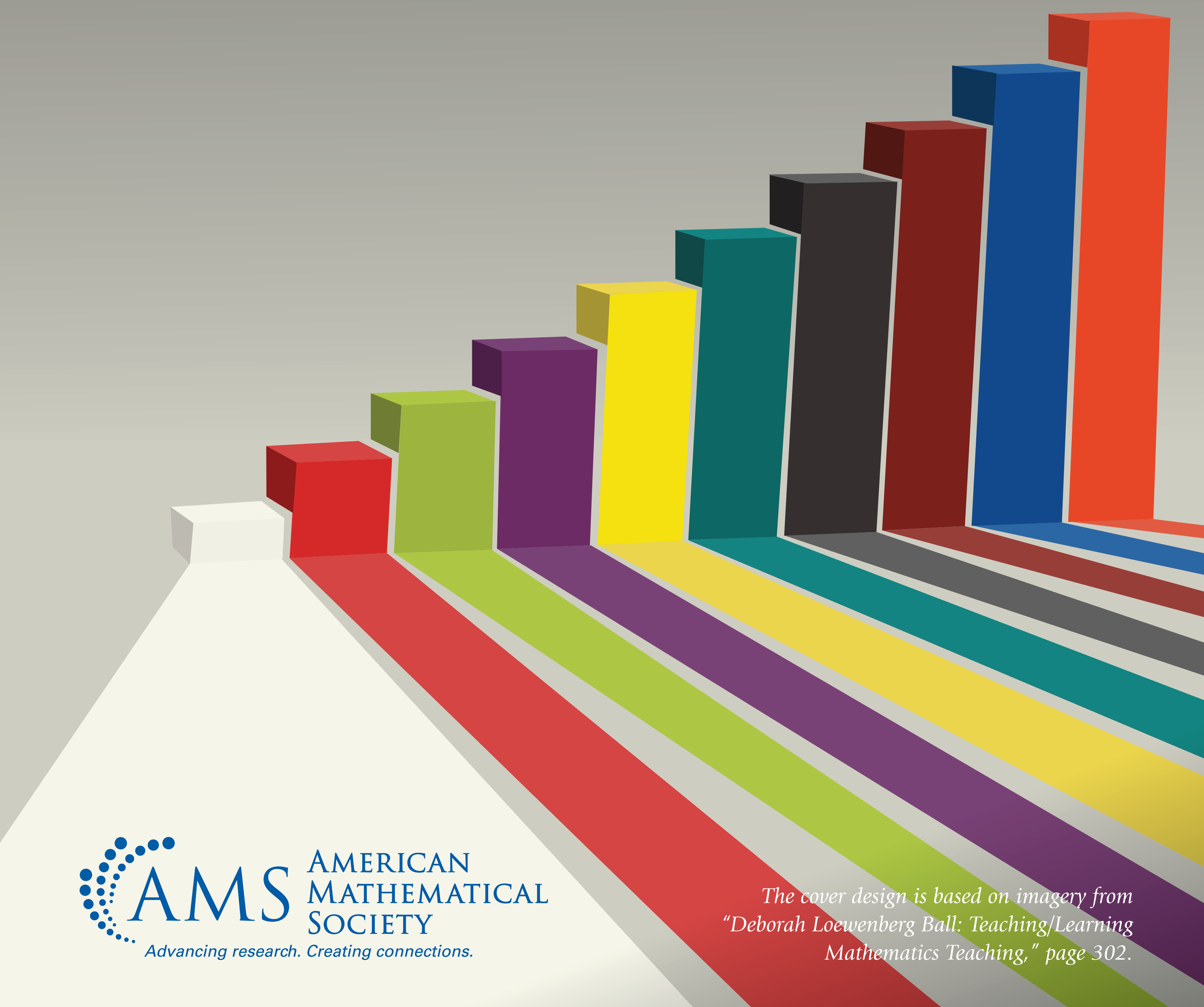

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The cover design is based on imagery from "Deborah Loewenberg Ball: Teaching/Learning Mathematics Teaching," page 302.

2024 AMS Einstein Public Lecture in Mathematics

will be held at the Spring Eastern Sectional Meeting

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FEATURES

FROM THE AMS SECRETARY

INVITATIONS FROM THE AMS

ERRATA. In the December 2023 book review "The Discrete Mathematical Charms of Paul Erdős: A Simple Introduction," there are errors in the citations for the films by George Csicsery. Both films are distributed by Zala Films, and information about both films and how to get DVDs or stream them can be found at `www.zalafilms.com`.

In the February 2024 issue, on p. 254, Andrew Putman's last name is misspelled as "Putnam."

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The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to "advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals," and urge all members to conduct their professional activities with this goal in mind. (*as adopted by the April 2019 Council*)

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A WORD FROM…

Rhonda Hughes, Former President of AWM and Cofounder of EDGE

The Status of Women in Mathematics

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Throughout the 1970s, women were virtually invisible on the national level in mathematics. For most of us coming of age in that decade, the only woman mathematician we had ever heard of was Emmy Noether. Women mathematicians at the time were often underemployed and some received recognition only after achieving fame beyond the mathematics community. Julia Robinson, the most notable example, was finally granted a full-time professorship at Berkeley after she was elected to the National Academy of Sciences. With the founding of the Association for Women in Mathematics (AWM) in 1971, women were somewhat reluctantly offered a seat at the table in professional organizations, and the stories, struggles, and achievements of women became more widely discussed. In the 1980s, women began to be properly recognized for their work.

If we judge purely by the numbers, the status of women in mathematics may not appear to be substantially different from the era when I served as AWM president from 1987–1989. The most quoted percentage is usually that of US citizen women obtaining PhDs in the mathematical sciences. In 1989, that figure was approximately 24% [6], and in recent years ranges from 25%–29%, depending on the source [7, 8].

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AWM: Association for Women in Mathematics, EDGE: Enhancing Diversity in Graduate Education.

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If we look, however, at some other metrics, we do see progress. Some of the standard markers of success in mathematics have been Fields Medals, AMS Colloquium Speakers, and tenured professorships at the leading research universities. As is widely known, two women have now won the Fields Medal, Maryam Mirzakhani (2014) and Maryna Viazovska (2022). Since 1896, when the AMS Colloquium Lectures began, there have been eight women speakers, most in the twenty-first century. The first was Anna Pell Wheeler of Bryn Mawr College in 1927. The second was Julia Robinson, nearly 60 years later, in 1980, followed relatively quickly by Karen Uhlenbeck (1985). There were none until the 2000s, which brought Alice Chang (2004), Alice Guionnet (2013), Dusa McDuff (2014), Ingrid Daubechies (2020) and Karen Smith in (2022). (There is anecdotal evidence here that once a woman is recognized one year, people realize the universe is still in operation, and another is chosen the following year.) In 2019, Karen Uhlenbeck was the first woman to win the Abel Prize [2].

In the 1980s, there were virtually no tenured women in the mathematics departments of the leading research universities. Today, that has changed. With some exceptions, those departments now have tenured professors who are women, and some departments have more than one. However, most mathematicians, regardless of gender, will not attain this level of accomplishment, so these are perhaps unfair measures of success. Instead, we should look more broadly at the progress of women and other underrepresented groups who have historically been marginalized.

One of the areas where we see impressive change is in the professional organizations. One special case worthy of note is the American Statistical Association (ASA), which had its first woman president, Helen Walker, in 1944, and has had a total of nineteen women presidents since its founding in 1939. Among the other professional organizations, none had a woman president until 1979, when Dorothy Bernstein became president of the Mathematical Association of America (MAA). Julia Robinson was the first woman president of the American Mathematical Society (AMS) in 1983, and the Society for Industrial and Applied Mathematics (SIAM) elected Margaret Wright in 1995. It is truly remarkable that the current presidents of AMS, MAA, ASA, the National Association of Mathematicians (NAM), the Society for the Advancement of Chicanos/Hispanics & Native Americans in Science (SACNAS), and the Society of Mathematical Biologists (SMB) are *all* women [3–5]. Moreover, the Association for Women in Mathematics (AWM) had Cora Sadosky, a Hispanic woman, serve as president in 1993 and currently has a Black president and president-elect: Talitha Washington and Raegan Higgins, respectively. Through the work of professional organizations such as SACNAS, AWM, NAM, and Spectra, the Association for LGBTQ+ Mathematicians, we see progress in inclusivity for all groups in the mathematical sciences.

New faces. One of the most welcome changes is the presence of women of color and other historically underrepresented groups at all levels of professional activity. When I became AWM president in 1987, there were few voices that championed the cause of women or people of color in the mathematics community. In 1969, Johnny Houston and Scott Williams organized a gathering that would formally become NAM in 1971. Lee Lorch, the civil rights icon [9], mentored a generation of Black women mathematicians in the short time he was at Fisk University, and Mary Gray, a lifelong champion of women's and human rights, was one of the founders of AWM [10]. Richard Tapia is a visionary under whose leadership the Computational and Applied Mathematics Department at Rice has become a national leader in producing women and underrepresented minority PhDs [11]. The Enhancing Diversity in Graduate Education (EDGE) Program (disclaimer: I am one of its founders, along with Sylvia Bozeman of Spelman College) boasts many prominent women in positions of research and professional leadership among its alumnae. The website Mathematically Gifted and Black [12], which each year features Black mathematicians during Black History Month, has showcased hundreds of Black mathematicians and brought their stories to light.

New voices. While most mathematicians of my generation read the same classic (often excellent) textbooks in graduate school, the internet has provided new and exciting ways of bringing mathematics to a wider audience than could possibly be reached by conventional teaching methods. Innovative ways of explaining mathematics have taken hold. Many of these voices belong to women. Tai-Danae Bradley's Math3ma website [1] offers clear explanations and synthesizes disparate themes in mathematics in an exhilarating way. Lillian Pierce brings her musician's sensibility to bear on her approach to exposition [13]. Her 2022 ICM slides, exceptional in their clarity, invite us to understand her research and give us access to her work [14]. Susan D'Agostino's book, *How to Free Your Inner Mathematician: Notes on Mathematics and Life*, offers a fresh new perspective on how the lessons of mathematics can have bearing on the way we navigate our lives [15]. As the world consumes the content these masterful expositors produce, we can only hope that a new and diverse generation feels more welcome than in the past and embraces all that mathematics has to offer.

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Deborah Loewenberg Ball: Teaching/Learning Mathematics Teaching

Hyman Bass

To its credit, *Notices* has been publishing a series of profiles celebrating the work and careers of several accomplished women in mathematics. Deborah Loewenberg Ball might, at first consideration, seem an unlikely entry into that august company, in that neither her education nor her early life interests exhibited particular mathematical inclinations. Indeed, her first serious engagement with a challenging math problem came after she had graduated college, where she had majored in French. How she addressed that problem is an important part of her story. I write from the perspective of a colleague, friend, and collaborator.

1. Family and Early Education

Let's start from the beginning.

Deborah's family heritage is part of a long line of German Jewish intellectuals. Those who survived Nazi persecution fled to the United States and other countries. Her great-great-uncle, Ernst Cassirer, was a prominent twentieth-century philosopher, who embraced both sides of the Neo-Kantian division between the natural and the human sciences, a sensibility also present in Deborah's work. Her father, Gerhard Loewenberg, was an eminent political scientist, who developed the field of comparative study of legislatures, especially in postwar Europe. He was also a much-appreciated professor and academic leader. Deborah's mother, Ina, is broadly talented—a scholar in philosophy, a certified public accountant, a published poet, an accomplished photographer, and a polyglot. Deborah's one sibling, Michael, is a professor of chemical engineering at Yale. Her husband of more than 50 years, Richard, has also been a teacher, and their three children, Sarah, Joshua, and Jacob, all educators in different contexts, have been central to her life and her learning.

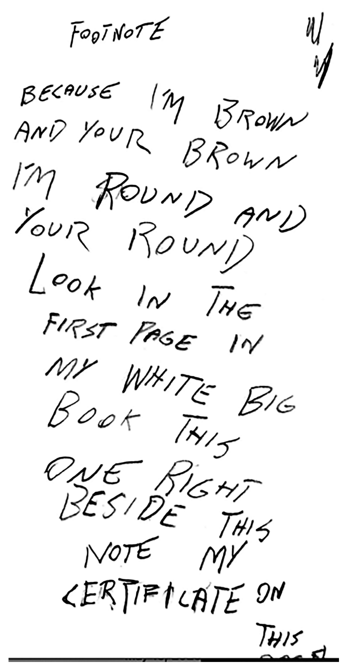

Figure 1. Four-year old Deborah's "proof" that she is related to her father.

Deborah was born in March of 1954 just as her father began his 16 years on the faculty of Mount Holyoke College. Her early education was in South Hadley, Massachusetts, where she was one of only two Jewish children in her elementary school. She read fluently by age four. Her first-grade teacher was unsure how to deal with her and awkwardly arranged for her to read current events materials aloud to the sixth-grade class. Also when she was four, her father teasingly challenged her to prove that the two of them were related. Her "proof" can be seen in Figure 1, in which she even appeals to her birth certificate.

During her father's sabbatical in Germany, to study the evolution of the Bundestag, Deborah, then age 7, attended public school and became fluent in German. In the late 1960s, her father accepted a faculty position at the

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Figure 2. Deborah Ball with her brother Michael, mother Ina, and father Gerhard Loewenberg.

University of Iowa, so Deborah finished high school in Iowa City. While in high school she also spent a term in Zurich on a student exchange program. Her only science-related courses were Algebra I and biology. Instead, she immersed herself in the humanities—English and languages, studying French, Spanish, and German.

Having completed two years of college work in high school, it took Deborah only two more to earn a BA at Michigan State University (MSU) in 1976, majoring in French and elementary education. Because of her love of language, she studied Spanish and German as well as French. Realizing that she didn't want to become a translator, she decided to pursue teaching, choosing elementary education because elementary teachers teach all subjects, and because of Michigan State's excellent teacher education program.

Having always felt herself to be somehow "different," Deborah was deeply interested in people—in how others think and in their experiences and perspectives. In her travels she met people of different identities and cultures, becoming aware both of enormous gaps of experience and opportunity and of the histories and social forces that shaped inequality and injustice. Across time, she had also held a range of jobs: as a cocktail waitress in a bowling alley; as an accounts tabulator at Kmart; and in a bakery, eventually as a cake decorator. Baking became a continuing métier for her, from which her students and friends continue to benefit. In each work environment, she noticed that, though the educational opportunities of her coworkers were often less than hers, they skillfully performed complex work involving rapid mental arithmetic calculations, substantial memory retrieval, complex relational work, and complicated multitasking, both physical and mental. It struck her how accomplished, yet how little noticed or appreciated, was their everyday skilled performance. This deepened a sensitivity to equity that has infused all of her life's work, especially teaching.

2. Early Teaching and Teacher Mathematical Knowledge

For 13 years (1975–1988) Deborah taught public elementary school, grades 1–5 in East Lansing, Michigan. Her husband, Richard Ball, was a middle school teacher. Deborah and Richard found that their experiences as parents infused their work as teachers and reciprocally. Deborah loved teaching and felt generally satisfied with her progress in developing as a teacher, with the exception of one subject: mathematics. She was puzzled and frustrated that things the children seemed to understand on Friday were often forgotten by Monday. She began to suspect that there were important things about teaching mathematics that she did not understand. Inspired by educational philosophers like John Dewey, Joseph Schwab, and Jerome Bruner,[1] she realized that this must begin (but not end) with a more robust knowledge of mathematics itself, something thus far underdeveloped in her education. So, she took and performed well in a four-term calculus sequence, the first two at Lansing Community College, the final two at MSU, plus a number theory course that she loved. This study enabled her to see situations that came up in her teaching in a new light. For example, one day in her first-grade class, she had the children try to measure the area of their hands. They traced their hands on graph paper and then tried to count the grid squares within the outline, with some improvised procedures for squares on the boundary. One child suggested getting the graph paper that the fifth graders used, with smaller squares, so that they might be able to get a more accurate measurement. At that time, Deborah was taking integral calculus, so she recognized that the child was noticing something that was a significant mathematical insight. How should she respond? Not by giving a mini-discourse on calculus, or even on the subtleties in defining area. She responded, "That's a really interesting idea. Why don't you get the fifth-grade graph paper, and let's try it?"

During part of these years of teaching elementary students, Deborah was a doctoral student in the MSU College of Education, majoring in curriculum, teaching, teacher education and professional development, and policy with a minor in mathematics education. For her 1988 doctoral dissertation, titled "Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education," she developed an interview protocol that she used to explore the mathematical understanding, as well as the ideas about learning, teaching, and

[1] *Deborah was inspired by Bruner's famous declaration, "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development."*

students, that preservice teachers held before even beginning their professional training.[2] Here is an item from a survey, used in a study of content knowledge for teaching, that she later developed from an open-ended interview in her dissertation:

> Which of the following story problems can be used to represent the meaning of $1\frac{1}{4}$ divided by $\frac{1}{2}$?

a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?
b) You have \$1.25 and may soon double your money. How much money would you end up with?
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?

Few of the study participants selected the correct answer. Most surprising was that even the mathematics majors were unable to explain the meaning of division by $\frac{1}{2}$.

3. Making Teaching Practice Accessible to Study

Next, as an assistant professor at MSU, Deborah continued to teach elementary school every day, then just as the math teacher in Sylvia Rundquist's third grade class. Together with her colleague, Magdalene Lampert, who was also teaching fifth grade in Thom Dye's class, they set out to conduct a major study of "the work of teaching," with a special focus on mathematics teaching. Often, the scientific study of a phenomenon proceeds first by close and systematic empirical observation of the phenomenon of interest, with detailed data gathering. A characteristic feature of classroom teaching presents an obstacle to this: it is typically private, carried out behind closed doors. A few accomplished teachers are sometimes featured as exemplary, to be emulated, much as listening to operas might equip you to sing arias. This is unlike most other skilled professions—like medicine, nursing, architecture, art, acting, aircraft piloting, plumbing—in which extensive supervised actual or simulated performance is the norm as part of professional training. Teaching, instead, is often positioned as something one can just learn on one's own, through experience. Deborah argued that even more important is that teaching be carefully taught, practiced, and coached. Ball and Lampert set out to design ways for others to engage with the work of teaching, with its complexity and dilemmas, through investigations of primary records of practice.

What Deborah and Magdalene proposed to do was to make their actual practice public, not as a model of exemplary teaching, but to produce a corpus of primary data with which to empirically study the work of teaching. They proposed to the NSF to fully document an entire academic year of their teaching, Deborah of third grade, Magdalene of fifth. The school in which they were both teaching had a diverse student body and teaching faculty, including Black, Latine, and white children, of many different cultures and different home languages. The records they proposed to collect comprised videos with transcripts of every class, student notebooks, teacher journals, homework and quizzes, and interviews with the children. The complexity and scale, innovative use of then available technology, and concomitant expense of such an undertaking were unprecedented. It is a tribute to the vision of the NSF program directors that they decided to fund this. Data were generated during the 1989–90 academic year, and the outcome more than fulfilled its promise. It provided a primary data source for Magdalene Lampert's seminal book, *Teaching Problems and the Problems in Teaching* (2001, Yale University Press) as well as the materials on which Deborah drew as she sought to study and analyze the mathematical work of teaching.

These records of practice were deployed as a kind of empirical "text" for others to study the work of teaching as it relates to various kinds of research questions. Specifically, Deborah wanted to understand the nature and form of mathematical knowledge that teaching mathematics entails. Of course, this would depend in part on the mathematical topic and the level of the students. But there should be some mathematical practices, ways of thinking, and sensibilities that apply across these different contexts. Deborah was aware of the ways that teachers and other educators commonly thought about these questions. But, going back to the influence of Dewey, Schwab, and Bruner, she felt it important to incorporate the perspectives of mathematicians as the disciplinary experts. Moreover, rather than ask for their disengaged opinions and reflections on what mathematical knowledge is needed for teaching children, she was now in a position to invite them to examine primary data (especially video) of teaching, and to ask, more concretely, "What mathematical issues do you see in these classroom episodes?" Of course, the problems occur in contexts that involve more than just the mathematics, as do the solutions. These questions could not be adequately answered with intellectual speculation and reflection, or based on university-level mathematics. But, she believed, mathematicians might see important mathematical aspects of events that might remain invisible to others.

To this end, Deborah enlisted several mathematicians, of whom I was one, to examine these records. Others included, for example, Peter Hilton, Herb Clemens, and Phil Kutzko. I was asked first to look at an interesting

[2] *Her interview protocol was later used by Liping Ma in a comparison of mathematical knowledge of Chinese and American teachers.*

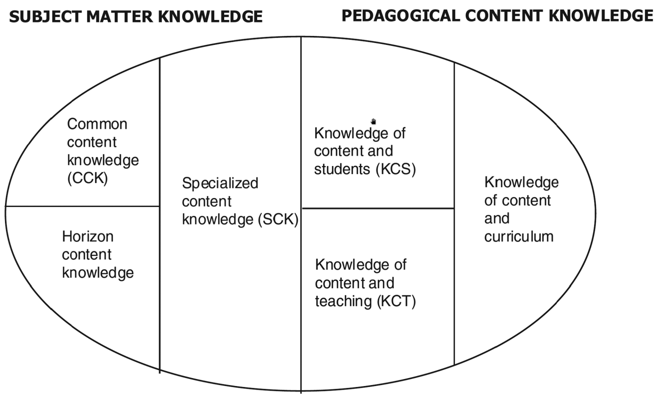

Figure 3. Domains of mathematical knowledge for teaching.

episode from the January data. Recall that the data ranged from September until May, so January was midyear. I was quite excited by the rich mathematical thinking and interactions of the children. I thought that these children showed amazing mathematical curiosity and engagement. I later more fully appreciated the importance of the many subtle teacher moves that supported the children's thinking and discourse. The longitudinality of the data allowed me to also "visit" the first classes, in September, when I saw the mathematical tasks and the teacher's mathematical prompts to which the students were just beginning to respond. It was somehow the teaching that supported these September-children to grow into the January-children, and the comprehensive data allowed me to analyze the pedagogy that supported that transformation.

I continued this study of the 1989–90 data, recorded through annotations of the transcripts, from my office in the Columbia Mathematics Department. Deborah moved from MSU to the University of Michigan (U-M) in 1996, and I spent a spring 1997 sabbatical at U-M to work with Deborah's research group. After returning, I found it difficult to continue this work remotely, away from the research group, so I moved to U-M in 1999, with a joint appointment in the Department of Mathematics and the School of Education.

4. Mathematical Knowledge for Teaching

Over the next several years, Deborah led her group in the development of a theory of "mathematical knowledge for teaching," (MKT). Unlike other theories that claim such a title, the knowledge base here was an in-depth empirical study of teaching practice. The theory was framed as a refinement of Lee Shulman's seminal work on "pedagogical content knowledge" (PCK) and was represented in the "egg" diagram in Figure 3.[3]

[3]*From D. L. Ball, M. H. Thames, and G. Phelps,* Content knowledge for teaching: What makes it special?, *Journal of Teacher Education* 59 *(2008), no. 5, 389–407*, `http://dx.doi.org/10.1177/0022487108324554`.

The right half is PCK, as a kind of hybrid pedagogical and content knowledge developed by Shulman and his colleagues at Stanford (Shulman, 1986; Wilson, Shulman, & Richert, 1987). The left half presents a new decomposition of "pure" content knowledge, which was a unitary component in Shulman's framework. A novel feature is the prominent appearance of "specialized content (in our case, mathematics) knowledge" (SCK). This new construct designated a knowledge of mathematics not involving pedagogy that is needed for teaching but is not typically needed for, or known by, other professionals, including mathematicians. Its discovery posed new questions about mathematics content courses taught in mathematics departments for future teachers. Where, and how, would SCK be taught?

In order for the theory of MKT to have some bearing on instruction and on student learning, it was important to have measures of MKT and of desired outcomes. To this end, Deborah and Heather Hill, together with Mark Hoover, and several talented doctoral students, led the development of measures of MKT, and another of "the mathematical quality of instruction" (MQI). For student learning outcomes standard measures were used. Working with colleagues Brian Rowan and Stephen Schilling who, along with Heather Hill, brought extensive quantitative analytic expertise, the team was able to show that teachers' MKT, as they conceived it, is related to student learning outcomes.

5. Teacher Education: The Work of Teaching

Given this line of research, a new question emerged: Where and how could teachers develop this kind of mathematical knowledge for teaching? The teaching population in the US numbers close to four million. It is America's largest occupation. This scale made clear to Deborah that this could not be an isolated agenda. It had to be integrated into a fundamental transformation of US teacher education and professional development—something perhaps analogous to the reform of US medical education, inspired by the Flexner Report in the early 1900s. Her approach was to ground this in an in-depth and fine-grained analysis of the work of teaching, something that was so far, perhaps unsurprisingly, largely understudied. This launched the next phase of Deborah's work.

Paralleling the work of Lee Shulman, Suzanne Wilson, and Pam Grossman, Deborah noted differences in the preparation of practitioners in other skilled lines of work—for example, surgeons, nurses, airplane pilots, and plumbers. This typically involved decomposition of the work into distinct, more directly learnable practices, and then also learning fluent integrated performance through both close observation of practice (think of the surgical theater) and coached rehearsals and simulations.

During this period Deborah was appointed dean of the University of Michigan School of Education (2005–2016), and the redesign of its teacher education program was a major part of her agenda. To this end, she began by enlisting the faculty in a two-year analysis of the basic everyday practices of teaching. A list of close to 100 such practices of various grain sizes and complexity was identified. This diverse list was clearly impractical as a foundation for teacher education. So, for elementary teacher education, the next phase was a process of distillation and consolidation to arrive at a short list of what they termed "high leverage" teaching practices, and the criteria for choosing them. The resulting, and still evolving, list of nineteen high leverage practices (HLPs)[4] included things like:

(HLP 1) leading group discussions;

(HLP 2) explaining and modeling content;

(HLP 3) eliciting and interpreting individual students' thinking; and

(HLP 11) communicating with families.

Of course, "content" refers traditionally to school subjects like language arts, mathematics, science, social studies, etc. To organize the teacher education program around these high-leverage practices required reconceptualizing the courses. Each content area course would address a small subset of the HLPs, grounded in that specific content.

To more broadly and enduringly amplify this reconceptualization of teacher education, Deborah founded and directs TeachingWorks,[5] an organization housed at U-M that designs resources and practice-based approaches to teacher education that support educators to enact equitable teaching practice that nurtures young people's learning and actively disrupts patterns of injustice.

Though Deborah's innovative scholarship remains prolific, she already received, in 2017, the Felix Klein Medal for lifetime achievement in mathematics education research from the International Commission on Mathematics Instruction (ICMI). This is the highest honor worldwide in mathematics education research. The three most highly cited mathematics education scholars in the world are Deborah Ball (73585); Paul Cobb (51794); and Alan Schoenfeld (45544), each of them also a Klein Medalist. And Deborah co-authored the two most highly cited journal articles in mathematics education research, both related to MKT.

But, as has already been indicated, Deborah's interests in education are broader than mathematics education. In 2017–18 she was president of the (25,000 member) American Educational Research Association (AERA). In her

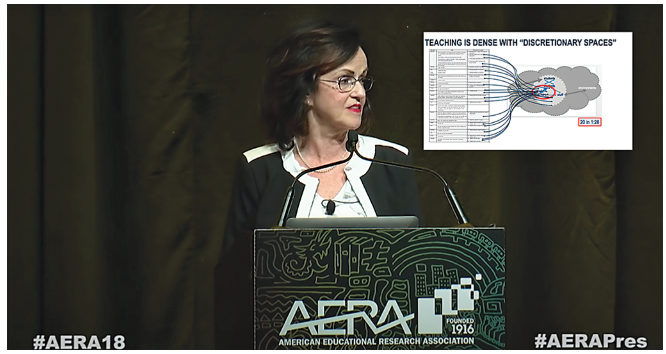

Figure 4. Deborah Loewenberg Ball delivering her AERA presidential address.

presidential address,[6] she introduced and vividly illustrated the notion of "discretionary spaces" in teaching. Drawing on the work of political scientist Michael Lipsky, this idea illuminated how policies and norms, no matter how prescriptive and specified, inevitably leave room for interpretation and (conscious or unconscious) consequential choices by on-the-ground actors that ultimately shape the enactment of the policy or practice. These discretionary actions by teachers can not only profoundly affect student learning, but also either reinforce or disrupt racism and other forms of oppression that infiltrate classrooms from the ambient society, culture, and institutions.

6. Deborah the Teacher

If there is one thing that best characterizes Deborah as a person, a scholar, and a leader, it is that she is quintessentially a teacher. She started with 17 years teaching elementary grades in East Lansing schools, partly during her doctoral studies and postdoctoral work at MSU. She has since taught a range of university courses, such as methods courses for teacher candidates, foundations and policy courses in the graduate program, and courses for students across the university with interests in education. Her courses draw many students and she earns high ratings for her teaching and her support of students. She regularly leads professional development workshops around the country. Further, for two decades now, she has taught 10-year-old children each summer in the Elementary Math Lab (EML). I say more about this below.

Teaching (and learning) for Deborah is a calling, a life practice, her way to interact with the world, to nurture and empower people of all ages, stations, and identities, and to disrupt institutions and practices that oppress them. She assumes that *all* children bring substantial knowledge and skills, and so considers it a core task of the teacher to elicit that and use it as a foundation for what new material the children are ready to learn. This contrasts with the negative orientations of much teaching that makes deficit

[4] `https://marsal.umich.edu/academics-admissions/degrees/bachelors-certification/undergraduate-elementary-teacher-education/high-leverage-practices`

[5] `https://www.teachingworks.org/`

[6] *Video available at* `https://youtu.be/JGzQ7O_SIYY?t=3212`

Figure 5. Deborah teaching in the Elementary Math Lab.

judgments of children's competence, and with testing that tests not what children know, but what they don't.

Deborah as leader: Carla O'Connor, in introducing Deborah's AERA Presidential Address, said:

> Her teacher identity also permeates how she works as a leader and as a public servant. As dean of the U-M School of Education, where I had the privilege of serving as her associate dean for four years, I witnessed how Deborah interpreted every meeting and every convening as an opportunity to cultivate communities of learning and action. She established the goals of the convening, assessing who would be in the room—the identities, knowledge, and resources they brought. And, in cooperation with her cofacilitators, she would determine which instructional tools and strategies could be most powerfully leveraged, when and by whom, to support the stated objective. And all of this was outlined clearly in her written "lesson plan."

I can testify to this. Deborah always honors and takes responsibility for the quality of the time she asks her colleagues/students to meet. Her well-attended faculty meetings were edifying and engaging, on themes from the school budget, to review and reform of our instructional programs, to equitable practices, and more, often enlisting outside expertise.

Deborah and STEM: These skills enriched every environment she entered, even those whose culture and knowledge base were remote from her own. For example, even through the so called "Math Wars" and beyond, Deborah remains one of the most highly respected mathematics educators by the research mathematics community. She participated in the "Reaching for Common Ground..." effort (AMS *Notices*, Oct. 2005). She is a fellow of the AMS. More pointedly, Deborah has twice been appointed to the National Science Board (NSB), a presidential appointment. The NSB is the governing board of the National Science Foundation. It covers the full range of sciences and typically has only one education member (and at most one mathematician). An educator has a limited role in its planned agenda, and often a little-heeded voice otherwise. Deborah's reappointment was welcomed by its members because of the clarity and focus she brought to their deliberations. Even on topics for which she had little technical expertise, she demonstrated great listening, learning, and enabling skills, these being qualities not always highly developed in the other members' environments.

Let me next offer some vignettes of Deborah's classroom teaching.

Teaching teaching place value: This is a methods class for intern elementary teachers, in which Deborah is introducing them to the teaching of place value in the early grades. I will describe this in what may seem like excessive detail, but there is a point to this. My description draws from a video recording of one such class that I also observed firsthand.

To simulate a lesson, Deborah as teacher (T) and a dozen of the teacher candidates in the role of children (C) sit cross-legged in a circle on the floor. Another dozen or so teacher candidates sit at tables around them to observe and take notes on the floor dialogue and actions. A box of more than two thousand popsicle sticks is poured into the middle of the circle. T asks, "How many do you think there are?" Various Cs offer (widely varying) guesses. T: "How can we find out?" A few Cs say, hesitantly, "Count them?" T: "Ok, let's do that, ... together." Then the Cs begin eagerly gathering individual sticks and counting them. It quickly becomes apparent that they are running out of room for the sticks and finding it hard to keep track of the counting. Some of the Cs say they could collect the sticks into bundles. But another C asks, "How can we hold the sticks in a bundle together? T says, "We can use rubber bands," and she presents a box of them to distribute to the Cs. At this point, T "exits" from the lesson simulation and asks the teacher candidates on the floor, "What do you think the (real) Cs are going to do when they get the rubber bands?" There is some discussion of the Cs using them as slingshots, and related behavioral issues. They then return to the simulation, and T demonstrates how to use a rubber band to bundle several sticks. But how many sticks should be in a bundle? It becomes clear that the bundles of each C should be of the same size, thinking of eventually combining their collections. But what size? This engenders some lively discussion: Two would be too small (too many bundles); but also we don't want them too big. Five is a good possibility. Then T urges 10, but emphasizes that this is a choice, and it could as well be otherwise. Then the counting resumes, but now with bundling. During this activity, T asks a variety of questions. T holds up a bundle and three sticks and asks, "How many is this?" A C answers, "13." T: "How do you know?' T does not accept "10 + 3 = 13," but asks for a direct proof—unbundling the 10 and counting all the loose sticks. T also asks questions like, "What are different ways to show me 24 sticks?" All of this happens while the counting goes on. After a while, the same crowding problem as before emerges; too many bundles! This leads to "super-bundles." But how many bundles in a super-bundle? Again, there is a *choice* to be made, and T proposes for consistency to make it 10. There ensues a sequence of now more complex representation questions: "How many sticks in two super-bundles, three bundles, and four sticks?" "Show me 302 sticks."

Finally, all the sticks are counted, and the collections of each C are gathered in the center. The assembled individual sticks are consolidated into more bundles, and then the assembled bundles are consolidated into more super-bundles. But now there is an abundance of super-bundles. And, after some discussion, they decide to form "mega-bundles" (comprising 10 super bundles). In the end, they have two mega-bundles, five super-bundles, three bundles, and seven individual sticks. And so, the count of the original pile is captured by these four digits: 2537.

Afterward, the lesson is repeated with the two groups of interns exchanging roles. Two big features of this lesson design are worth noting. Rather than directly presenting the teacher candidates/children with symbolic place value notation (this is the ones place, the tens place, ..., names to be memorized) it creates a context in which (1) they organically construct/invent a place value system, and (2) they are viscerally able to appreciate the amazing power of notational compression so achieved. Moreover, the bundling stick model is a faithful physical representation of the mathematics; bundling and unbundling are just ways of organizing the physically invariant collection. Performing addition and subtraction in this model gives direct meaning to the symbolic processes of "carrying" and "borrowing"—or, more properly, "regrouping," carried out physically with two- and three-digit numbers. This contrasts with the base-ten blocks representation, which involves physical exchange in place of unbundling, with cardinal invariance shown by geometric measurement. Finally, one arrives at symbolic place value notation, with its compactness and computational efficiency, achieved with formal algorithms. The symbolism is abstract and remote from its mathematical meaning, but it arrives with a concrete foundation into which it can be translated when needed.

Deborah's instruction included systematic analysis of these representation systems, which progressed from making the mathematics transparently and directly meaningful but cumbersome, to the powerfully compressed and efficient but opaque in meaning symbolic representation. When children might encounter difficulties in the symbolic system, the teacher candidates were learning to diagnose them and return to a more transparently understood model to help a child understand the meanings of the symbolic manipulations they were making. The teacher candidates were taught to rehearse such explanations, to talk about this and ask questions that could open insights for young learners.

The broken calculator lesson: Deborah was invited to teach a lesson to a third-grade class of mostly Black children in Detroit, whom Deborah had never met. When Deborah showed the regular teacher the lesson plan, she expressed reservations about whether the students could actually handle the lesson.

The class was high energy, with lots of sound and motion. Deborah gave them a picture of a hand calculator and explained that the 2 and the 5 keys were broken. She asked them, "Can we still use this broken calculator?" After eliciting various student opinions, she posed a sequence of addition and subtraction problems, to be performed on the broken calculator. For example, if the 2 and the 5 keys were broken, how might one use the broken calculator to add 32 + 51? The children successfully engaged with these substantial challenges with great enthusiasm.

What mathematics learning was supported by this activity? Given some arithmetic expression to calculate, the children had to construct an equivalent expression (having the same value) that did not involve the digits 2 and 5. For example, 32 + 51 = 33 + 50 = 33 + 40 + 10. More than arithmetic practice, this implicitly involved early algebra skills and equivalence of arithmetic expressions.

With regard to the regular teacher's expectations, Deborah has always, when meeting students she did not know, chosen to *not* look at previous teachers' or others' judgments about the mathematical capabilities or potential behavior issues of the individual students. For her, each child comes with high promise and no presumed deficiencies. This is a crucial stance to take, especially for Black and Brown children who have been historically marginalized and pervasively seen by their white teachers through deficit lenses.

Proving mathematical impossibility: I turn next to a third case: Deborah teaching a summer program class (the EML) of racially and linguistically diverse soon-to-be fifth graders. The goal of the program is growing the children's mathematical identities through a focus on their skill as mathematical thinkers in preparation for fifth grade. The curriculum she developed involved a combination of foundational work on fractions, including their placement on the number line, plus some novel and challenging problem-solving activity. In this regard, the reform mathematics education literature has urged the teaching of practices like reasoning and proving across all grade levels, K–16. But this has proved difficult to achieve even at the

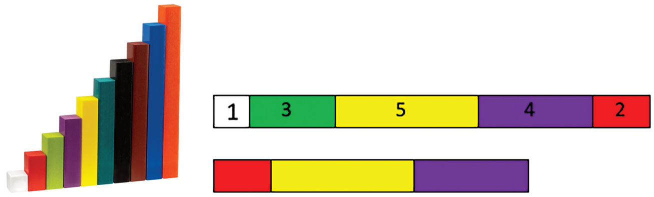

Figure 6. A 5-car train, and a 3-car 11-passenger train.

Figure 7. Children building trains in the EML class.

undergraduate level. Deborah has developed a complex problem called "The Train Problem" on which they work, often in small groups, over several days. It is a nontrivial problem even for most adults.

A plain mathematical statement of the problem goes as follows. Consider the set of numbers, $S = \{1, 2, 3, 4, 5\}$. Note that $1+2+3+4+5 = 15$. The Train Problem has two parts. (I) Can every (whole) number ≤ 15 be expressed as the sum of a subset of S? (II) Is there an ordering $T = (a, b, c, d, e)$ of S such that every number ≤ 15 is the sum of some consecutive terms in T? In what follows, I describe how Deborah stages this problem for the children and, in particular, how she motivates them to seriously engage with it. I draw from one particular year in which I observed the development of the children's work.

The children worked with Cuisenaire rods. These have dimensions (1 cm) x (1 cm) x (n cm) (from n = 1 (white) to n = 10 (orange)). Deborah had them make "trains" (a sequence of rods placed end-to-end).

She told the children that each rod represented a train car with passenger capacity corresponding to its length, and the passenger capacity of a train is the sum of those of its cars. She set the class up as a collective to be called the "EML Train Company," a company that specializes in making trains using only the cars: white, red, light green, purple, and yellow, each one at most once.

Deborah asked, "What's the biggest (most passengers) train you can make?" "The smallest?" After a small calculation, they found 15 to be the largest. For the smallest, the children began to engage in an interesting discussion of whether there could be an "empty train." Deborah then asked if they could make trains with any number of passengers between 1 and 15. Collectively, they constructed trains for each number, and in some cases several trains for some numbers. Incidentally, this activity involved a lot of practice with basic arithmetic as well as the construction of a table to record their findings. At this point, the children felt pretty confident and fluent at train building, and Deborah praised them as being a really smart train company.

She then announced that there was a "customer" who had heard of their high reputation and wanted to hire them to build a special train. This excited the children. In fact, the customer was an adult[7] recruited by Deborah, who appeared before the class and announced, "I've heard that you're a really smart train company. I would like you to build me a special train that uses each of the five different size cars, each one only once. I also want to be able to break apart the special train to make smaller subtrains. But I want these smaller trains to be formed of cars that are next to each other in the special train. And I want to be able to make such a subtrain for every number of passengers between 1 (just the white car) to 15 (all five cars)."

After he left, Deborah worked with the children to unpack the several complex conditions of the problem. Pairs of children constructed various trains to see if they worked. For each train, they had 15 numbers to check, but they soon recognized that 1, 2, 3, 4, 5 (single cars) and 15 (all cars), come for "free," for any train. So, they only had to worry about 6, 7, 8, 9, 10, 11, 12, 13, 14. The children tested these, starting up from 6. For each train, they were able to find, after significant effort, that some numbers could not be made. This work stretched over a couple of days amidst other mathematical work. There was a risk of the children becoming discouraged, especially when the customer visited and asked, "When do you think you'll have my train?"

While this is a finite problem, therefore, in principle answered by examining all cases, it is a BIG problem: There are $5! = 120$ possible trains, and each train requires testing for nine different passenger sizes (6–14 passengers). After some intense work by the group, Deborah contributed a clue. She simulated publicly a phone call from an assistant of the customer, suggesting that it might be important to begin by trying to make trains that would work for the big numbers, such as 13 or 14, instead of just always beginning from the lower end, at 6, 7, etc. She asked whether the children wanted to try this. After further exploration and effort, some of the children realized that the only way to get 14 was to remove 1 (the white car), and that this would be possible only if white is on the end; otherwise, its removal would disconnect what's left. Similar reasoning shows that the only way to get 13 is to remove the red

[7] *Once the client was Roger Howe.*

car. And so, the only possible trains must have red and white on the two ends. Turning the train around, if necessary, one can assume that red is the left end and white the right end. This is a big step in reducing the scope of the possible solution space of the problem. These red-white-ended trains are then determined by the ordering of the three middle cars: green, purple, and yellow.

Prior to working on the train problem, Deborah had given the task of finding all three-digit numbers they could make with the digits 3, 4, and 5 using each one only once. They found all six possibilities but Deborah pressed them to show (prove) that there couldn't be any others. The children had produced a range of convincing arguments for this.

The children were now able to retrieve (transfer) this earlier work to determine the six possible three-car trains using the green, purple, and yellow cars, each one only once. Thus, they had reduced the train problem to a manageable size, examining six (not 120) trains! These reasoning steps were dramatic, challenging, and transformative. Deborah had them make the six possible trains and then formed the class into six groups, each group to test if its train worked. Collectively, they reached the conclusion that none of the six trains worked to meet the customer's order and the constraints of the problem. The problem was impossible to solve!

Mathematicians might judge this to be a mathematical success. But this predicament left some of the children still uncertain. They had failed to make the customer's train and, given past experience, they worried that they were failures. They were disappointed; maybe they were not as smart as the customer had expected. "Will he be mad?" they asked. Deborah assured them that they had done very important and valid mathematical work. One student wondered, "Do you think that someone else might be able to make the customer's train?" After extended discussion, they began to realize what they had accomplished: No one, no matter how "smart," could make the train. In fact, they had done very smart work on this challenging problem.

The customer came to "receive" his train. Deborah discussed with the class how they might respond. They would have to show not only that they could not make his train, but also that it was impossible. And they would show this mathematically. Deborah formed the class into groups to prepare their presentations of parts of the proof and explanations of the various stages of the complex argument.

Before the customer's arrival, Deborah asked the children what they would do if he said, "Well, I know some pretty smart mathematicians, and maybe I'll ask them to make my train." One of the girls responded confidently, "You'd be wasting your time and money."

Figure 8. The children's work "published" on wall posters.

7. The Elementary Math Lab (EML)

The Park City Mathematics Institute (PCMI), sponsored by the Institute for Advanced Study, has a "vertically integrated" structure, in that it assembles several dynamic programs each summer in a common physical environment: one for active mathematics research development involving both graduate students and leading world experts; one for undergraduates and undergraduate faculty; one for mathematical enrichment of high school teachers; etc. In 2004, Herb Clemens, then PCMI director, invited Deborah to contribute an elementary education component to this mathematically rich and intense environment.

This was the birth of what has become the EML, which moved to the University of Michigan in 2007 and is now under the sponsorship there of TeachingWorks. The above discussion of the "Train Problem" is based on EML data. Resonant with her early work with Lampert, Deborah conceived the EML to be a "theater of public teaching" as well as a laboratory in which teaching was a domain of empirical study and experimentation.

In the EML, Deborah teaches a diverse class of about 30 rising fifth graders for two weeks, with an ambitious curriculum that combines foundational topics with high-level mathematical thinking and reasoning. The teaching is directly observed and studied by teachers and educators who come from around the country. The days begin early, when Deborah shares with the observers the lesson plan for the day's (2+ hours) class. She answers questions and invites feedback, including suggested modifications. Then the class is (silently) observed, followed by inspection of the children's notebooks, and then a debriefing with Deborah. This highly organized and multiply purposed structure has some features in common with Japanese Lesson Study, but it is much more complex. All of this activity is professionally recorded and documented. The afternoons offer several professional development options for the observers. The children are given some tutoring sessions by

STEM undergraduates, focused on puzzle solving or other topics, followed by some art-related activity. Beginning in 2019, Deborah was joined by coteacher Darrius Robinson and the two of them have developed a team-teaching approach in working in this special program.

The EML is yet another manifestation of Deborah's essence as a teacher. But it is also an expression both of her institutional creativity and of her continuing commitment to making teaching accessible to primary observation and study.

8. Epilogue

Deborah Loewenberg Ball is an educator—a teacher, a teacher of teachers, and, in particular, a teacher of mathematics. And she is a well-recognized and honored leader, scholar, and practitioner in each of these domains. However, her profile differs from those of other women featured in the *Notices of the American Mathematical Society*.

Mathematicians are frequently critical of school mathematics education in this country, being offended by the perceived distortion of mathematical ideas in various curricula and instruction. And they generally give little attention to mathematics education research. But they do not hesitate to voice strong opinions about what teachers should know and understand to teach mathematics. This typically consists of chosen extractions from the disciplinary canon. These views often find a home in policy and curricular documents.

Deborah made mathematical knowledge for teaching a rigorous research question, grounded in a close empirical study of the work of teaching. The work of her research group on MKT made this into what some have called an area of applied mathematics—that is to say, a complex domain of human practice that makes substantial specialized use of mathematics. It revealed novel kinds of mathematical knowing, thinking, and doing that are essential for teaching mathematics, yet had not previously been clearly identified or measured.

Something mathematicians most appreciate about Deborah is her respect for the integrity of the disciplinary knowledge in whatever domain she teaches, mathematics in particular. Although she has followed a different path in and around mathematics, she reflects a mathematical sensibility and depth of mathematical understanding of all that she teaches, as well as an artistry in pedagogical design that elicits critical thinking and a deep engagement with learning. And she does this with children historically and pervasively marginalized and oppressed in our society.

Finally, I think it is important to say that Deborah is deeply spiritual. A faithfully practicing Jew, she has also joined many of her students of other faith traditions in their religious celebrations. At her core, she considers teaching itself to be a spiritual calling, a challenging practice of cultivating human fulfillment.

Hyman Bass

Credits

Opening image and Figures 1–3 are courtesy of Deborah Loewenberg Ball.

Figure 4 is courtesy of American Educational Research Association.

Figures 5, 7, and 8 are courtesy of TeachingWorks.

Figure 6 is courtesy of Hyman Bass.

Photo of Hyman Bass is courtesy of Hyman Bass.

Marie-Françoise Roy

Saugata Basu

Figure 1. Marie-Françoise in her office in Rennes (c.1997).

1. Brief Biography

Marie-Françoise Roy was born in Paris in 1950. She was educated at Lycée Condorcet, École Normale Supérieure de jeunes filles and Université Paris 7. Married to Michel Coste since 1971, she has two children Denis and Elise and two grandsons Pierre and Alexandre. She started teaching at University of Rennes in 1972 and continued at Université Paris Nord where she received her habilitation in 1980, supervised by Jean Bénabou. In 1981–1983 she spent two years at Abdou Moumouni University in Niger. In 1985 she became a professor of Mathematics at University of Rennes, where she is currently an emerita professor.

She was the president of Société Mathématique de France (SMF) from 2004 to 2007. In 2004, she received an Irène Joliot-Curie Prize and in 2009 she was made a Chevalier of the French Legion of Honour.

2. Mathematical Works

2.1. Background. The major part of Marie-Françoise's work has to do with various aspects of real algebraic geometry. So to put her work in the proper perspective it is good to start with a little bit of history. Historically, real algebraic geometry can be said to have two origins—both of which continues to play an important role as evidenced indeed by the works of Marie-Françoise herself.

Hilbert's 17th problem: Artin's theorem. The origin of real algebraic geometry can be arguably traced back to Artin's solution to Hilbert's 17th problem (in the famous list of 23 problems presented by Hilbert in the first International Congress of Mathematicians in Paris, in 1900 [12]). Hilbert's 17th problem concerns polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ which take nonnegative values at each point in $\mathbb{R}^n$. Obviously any polynomial which is a sum of squares of polynomials has this property. But what about the converse? It is an easy exercise to verify that the converse is also true for polynomials in one variable, i.e., every nonnegative polynomial in $\mathbb{R}[X]$ is a sum of at most two squares (hint. use the "two squares" identity namely, $(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2$, and the fact that every polynomial in $\mathbb{R}[X]$ factors into linear and quadratic factors, where each quadratic factor is a sum of squares). It is also easy to check that any nonnegative polynomial of degree 2 in n variables is a sum of squares of at most n polynomials of degree one (hint. use Sylvester's inertia law). Hilbert also observed that the converse holds in one other case (degree 4 polynomials in two variables) but fails to hold in every other case. He asked nevertheless whether every nonnegative real polynomial is a sum of squares of *rational functions*. Artin [1] resolved this question in his seminal paper by proving Hilbert's statement. In the process he introduced the notion of a *real closed field*.

A real closed field R is an ordered field in which every positive element is a square and which satisfies the *intermediate value property* for polynomials (i.e., for each polynomial $P \in \mathrm{R}[X]$ and $a, b \in \mathrm{R}$ with $a < b$, $P(a)P(b) < 0$ implies that there exists $c \in \mathrm{R}$ with $a < c < b$ and $P(c) = 0$). The field of real numbers, $\mathbb{R}$, as well as well its subfield of real algebraic numbers are familiar examples of real closed fields. These fields satisfy the Archimedean property, but there exist non-Archimedean real closed fields such as the field $\mathrm{R}\langle\langle\varepsilon\rangle\rangle$ of *Puiseux series* in ε with coefficients in a real closed field R. Real closed fields admit a unique ordering (compatible with the field operations), and in this unique order the element $\varepsilon \in \mathrm{R}\langle\langle\varepsilon\rangle\rangle$ is positive but smaller than every positive element of R (ε is often referred to as an *infinitesimal*). The fields of such Puiseux series in one or more "infinitesimals" play an important role in algorithmic real algebraic geometry and they will be mentioned

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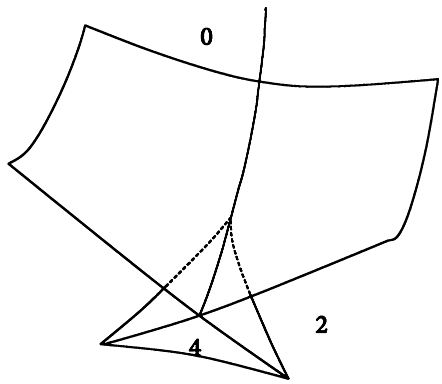

Figure 2. The discriminant hypersurface of the real quartic polynomial in one variable.

several times later in the article. In the rest of this article R will always denote a real closed field.

First order logic: Tarski's theorem. A second root of the subject originates in logic and the work of Tarski [22] who proved that the first order theory of the reals admits quantifier elimination and is decidable.

One usually meets an easy example of this theorem in middle school. The existentially quantified formula

$$(\exists X)X^2 + 2bX + c = 0$$

is equivalent modulo the first-order theory of the reals to the quantifier-free formula

$$b^2 - c \geq 0 \tag{1}$$

(we refrain from defining precisely what we mean by a formula but just say that a formula is built out of atoms of the form $P = 0, P > 0$ where P is a polynomial, logical connectives $\vee, \wedge, \neg$, and existential and universal quantifiers).

While the above example of quantifier elimination may indicate that quantifier elimination in the theory of the reals is a simple problem, this is misleading as one realizes if one tries to eliminate the existential quantifier from the formula

$$(\exists X)X^4 + aX^2 + bX + c = 0. \tag{2}$$

The real hypersurface in $\mathbb{R}^3$ (coordinatized by a, b, c) defined by the *discriminant* of the quartic polynomial $X^4 + aX^2 + bX + c$ is shown in Figure 2. The number of real zeros (counted with multiplicities) can be $0, 2$, or 4. The different connected components of the complement of the discriminant hypersurface in $\mathbb{R}^3$ correspond to real quartics with simple roots and having 0, 2, or 4 real roots, and these are labelled accordingly in Figure 2. A quantifier-free formula equivalent to (2) should describe the union of the closures of the connected components labelled by 2 and 4. Such a formula is considerably more complicated than the formula (1).[1]

Tarski-Seidenberg transfer principle. One important logical consequence of Tarski's theorem is that if ϕ is a sentence (i.e., a formula without free variables) whose atoms are polynomial inequalities with coefficients in a real closed field R, then ϕ is true in the structure R if and only if it is true over any real closed extension $\mathrm{R}' \supset \mathrm{R}$. This is usually referred to as the *Tarski-Seidenberg transfer principle*. As a special case we obtain that if a polynomial inequality $P < 0$ where $P \in \mathrm{R}[X_1, \ldots, X_n]$ has a solution in R'^n (i.e., the sentence $\exists X_1 \cdots \exists X_n \; P(X_1, \ldots, X_n) < 0$ is true over R'), where R' is any real closed extension of R, then it already has a solution in R^n.

Complexity: Of algorithms and certificates. Tarski's proof of quantifier elimination in the theory of the reals is constructive and is based on (a parametrized version of a generalization of) Sturm's theorem for counting real roots of a polynomial.[2] The complexity of this procedure and the size of the quantifier-free formula that is output cannot be bounded by any fixed tower of exponents as a function of the size of the input formula (measured by the number of atomic formulas and the maximum degree of the polynomials appearing in them). However, because of its many applications in different areas of mathematics as well as in computer science, the question of understanding the true complexity of quantifier elimination has been considered a very important problem in real algebraic geometry—a topic on which Marie-Françoise has made significant contributions which we will discuss later.

There is a corresponding facet to Artin's proof as well. Artin's original proof used a delicate specialization argument (now referred to as the Artin-Lang homomorphism theorem [1], see also [3, Theorem 4.1.2]). Abraham Robinson [19, Chapter 6, Section 5] simplified Artin's proof by replacing the use of the Artin-Lang homomorphism theorem by an argument using the Tarski-Seidenberg transfer principle making Artin's proof quite simple to explain. We first sketch this simplified proof below.

If $P \in \mathrm{R}[X_1, \ldots, X_n]$ is not a sum of squares in the field $\mathrm{R}(X_1, \ldots, X_n)$, then the field $\mathrm{R}(X_1, \ldots, X_n)$ admits an ordering $\prec$ (via Zorn's lemma) extending the order in R, in which the evaluation of P at $(X_1, \ldots, X_n) \in \mathrm{R}(X_1, \ldots, X_n)^n$ is negative (with respect to the order $\prec$). The Tarski-Seidenberg transfer principle applied to the *real closure* of the ordered field $(\mathrm{R}(X_1, \ldots, X_n), \prec)$ (i.e., the smallest real closed field containing $(\mathrm{R}(X_1, \ldots, X_n), \prec)$ as an ordered subfield), now implies there already exists $(x_1, \ldots, x_n) \in \mathrm{R}^n$ such that $P(x_1, \ldots, x_n) < 0$.

[1] *See Example 2.6.3 in* [2] *for such a description.*

[2] *A modern account of Tarski's proof appears in* [2, Chapter 2].

It is quite clear from the highly abbreviated sketch of (the simplified version of) Artin's proof given above that it is nonconstructive. Given a nonnegative polynomial P the proof gives no indication of how to write it as a sum of squares of rational functions. Indeed Artin mentions this in his paper [1, page 110].

> Dagegen sind unsere Beweise indirekt und liefern keine explizite Vorschrift für die Zerfallung. Man darf aber wohl erwarten, daß sich die Beweise nach dieser Richtung hin vervollständigen lassen...[3]

One should mention here that Hilbert also asked whether the coefficients appearing in the rational functions could be chosen to belong to the field generated by the coefficients of the given polynomial [12] and Artin's proof being nonconstructive does not answer this question. However, using model theoretic arguments Robinson [20, Theorem 5.1] proved this stronger version. Moreover, Robinson also proved [20, Theorem 8.2] the existence of a uniform bound on the degrees of the rational functions in Hilbert's 17th problem as a function of the degree and the number of variables in the given nonnegative polynomial. But this proof uses the compactness theorem from first-order logic, and thus is nonconstructive. In particular, it does not produce any explicit bound.

To find a constructive proof of Artin's theorem is thus a very natural question by itself. Kreisel provided such a proof (see [8]), with primitive recursive degree bounds. Finding better bounds for Hilbert's 17th problem has taken on added significance in recent times in view of developments in computer science (around sums-of-squares proof systems [11]) and mathematical optimization (semidefinite programming and what is now known as the *Lasserre hierarchy* [13]). These applications make it important to obtain explicit degree bounds on the polynomials appearing in the sum of squares decomposition. We will discuss Marie-Françoise's contribution to this topic later in the article.

Real étale topos and the real spectrum. The theorems of Artin and Tarski belong to the first half of the twentieth century. The subject of algebraic geometry underwent a revolutionary transformation in the second half of the twentieth century with the ideas introduced by Grothendieck (namely, that of schemes and Grothendieck topologies on them). It is in this milieu in Paris that Marie-Françoise started her research career. To describe her work one needs to describe some background.

Sites, sheaves and topos. The fundamental notion of *Grothendieck topology* or *sites* was introduced into algebraic geometry by Grothendieck in the sixties. A site on a category $\mathbf{C}$ is a generalization of the notion of topology on the category sets—where the role of open covers is replaced by collections of morphisms (sieves) satisfying certain axioms. Every topological space gives rise to a site but not vice versa. Moreover, the classical definition of sheaves on topological spaces can be extended to sites.

Another notion introduced by Grothendieck that plays an extremely important role is the notion of *schemes*. Given a finitely generated k-algebra A (for some field k), we denote by $\operatorname{Spec} A$ the set of prime ideals of A. The set $\operatorname{Spec} A$ is topologized by choosing as a basis of open sets the subsets of the form

$$D_a = \{\mathfrak{p} \in \operatorname{Spec} A \mid a \notin \mathfrak{p}\}, a \in A. \tag{3}$$

The corresponding site is referred to as the *Zariski site* and schemes of the form $\operatorname{Spec} A$ are called affine schemes. General schemes are built out of affine schemes by an algebraically defined glueing process.

A *topos* is a category satisfying certain axioms. A prototypical example is the category of sets, but the examples which are more relevant to algebraic geometry are the category of sheaves (of sets) on a topological space or the category of sets with a group action and more generally the category of sheaves on a site.

Topos and logic. Toposes carry an internal logic (which is *intuitionistic*) which makes it possible to interpret logical formulas in an arbitrary topos. This makes it possible to define models of so called *geometric axioms* (involving only conjunctions, disjunctions and existential quantifiers, without negations and universal quantifiers) in arbitrary toposes. A typical example of such axioms is the definition of a ring (commutative and with a unit element) or of a local ring which is a ring where for every element a, either a or $1+a$ is invertible. Thus, one obtains the notion of a ring object in a topos. A classical ring is a ring object in the topos of sets, while the ring object in the topos of sheaves on a topological space is just a sheaf of rings.

Definition of spectrum. Considering objects in arbitrary topos satisfying geometric axioms proves to be very useful in solving certain universal problems which do not admit solutions if restricted to the topos of sets only. Consider for any ring A the problem of finding a homomorphism f from A to a local ring $L(A)$ such that for all such homomorphisms $g : A \to B$, there is a unique local homomorphism $h : L(A) \to B$ such that $g = h \circ f$ (a homomorphism between local rings is local if it reflects invertibility). The solution to this problem unfortunately does not always exist since a ring can have several prime ideals. But now suppose we are allowed to change the topos while looking for the universal homomorphism to a local ring (object) now in a possibly larger topos. So now the universal problem becomes given a pair (A, E) where A is a ring object in a topos E find a pair $(\tilde{A}, \tilde{E})$ and a *geometric morphism*

[3] *This roughly translates "In contrast our proofs are indirect and provide no explicit instructions for the decomposition. One may however expect that the proof can be completed in this direction."*

$f : E \to \tilde{E}$ such that $\tilde{A} = f(A)$, and $\tilde{A}$ is a local ring object in $\tilde{E}$, and the morphism f has the obvious universal property. The pair $(\tilde{A}, \tilde{E})$ is then called the *spectrum* of the pair (A, E).

Zariski and étale spectra. In the case where $E =$ **Sets**, so that A is a classical ring, the spectrum of A happens to be the topos of sheaves on the Zariski topological space $\operatorname{Spec} A$ (i.e., $\tilde{E} = \mathrm{Sh}(\operatorname{Spec} A))$), and the local ring object in $\tilde{E}$ is the structure sheaf $\tilde{A}$ defined on $\operatorname{Spec} A$—namely, which associates to each open set D_f the ring A_f (A localized at f). In this way one recovers the notion of the Zariski spectrum of a ring.

Another example of a spectrum is obtained by considering the axioms of local rings with a separably closed residue field. One obtains this way the topos of sheaves on étale sites on schemes as the étale spectrum of a local ring object. Étale sheaves and their cohomology play a central role in algebraic geometry. They were introduced by Grothendieck as a means to prove the Weil conjectures in number theory. One important point to note is that the étale site on a scheme is in general finer than the Zariski site (i.e., the site induced by the Zariski topology) and that the topos of sheaves on étale sites is not spatial (i.e., not equivalent to the topos of sheaves on some topological space). Indeed the étale spectrum of a field k of characteristic zero is the algebraic closure $\bar{k}$ of k equipped with the action of the Galois group $\mathrm{Gal}(\bar{k}/k)$.

Real spectrum. It is now very natural (from the point of view of real algebraic geometry) to consider the spectrum associated with the axioms of local rings with a real closed residue field (as opposed to being separably closed). The corresponding spectrum (now called the *real spectrum*) was investigated by Marie-Françoise and Michel Coste (in "Topologies for real algebraic geometry," appearing in the book *Topos theoretic methods in geometry*, Various Publications Series, Vol 30, 37-100, 1979).

Unlike the étale spectrum, the real spectrum turns out to be spatial (see Theorem 2 below)—and the underlying topological space is often referred to as the real spectrum. The role of the structure sheaf is now played by a sheaf of functions on this topological space, namely the sheaf of *Nash functions*. Since this marks the starting point of Marie-Françoise's work in real algebraic geometry, we start our description of her work by describing her work on the real spectrum.

2.2. The topos of real étale sheaves.

Let R be a real closed field and $V(\mathrm{R})$ denote the R-points of a variety V defined by a finite set of polynomial equations $P_1 = \cdots = P_m = 0$, where $P_i \in \mathrm{R}[X_1, \dots, X_n]$.

Definition (Real étale site). [6] The real étale site on V is the site generated by collections of étale morphisms $(W_i \to W \subset V)_{i \in I}$ such that $(W_i(\mathrm{R}) \to W(\mathrm{R}))_{i \in I}$ is a surjective family.

There is another site defined on V (called the semi-algebraic topology [6]) whose coverings are generated by covers of $V(\mathrm{R})$ by open semi-algebraic subsets of $V(\mathrm{R})$ (i.e., finite unions of subsets of $V(\mathrm{R})$ defined by finite conjunctions of strict inequalities). Note that despite its name it is not really a classical topology—but only a Grothendieck topology.

With Michel Coste, Marie-Françoise proved the following two fundamental results clarifying the main properties of real étale sheaves thereby answering questions raised previously in the works of Brumfiel, Knebusch, and Delfs.

Theorem 1 ([6]). *The topos of sheaves with respect to the real étale site on V is isomorphic to the topos of sheaves with respect to the semi-algebraic topology site.*

Theorem 2. *The topos of sheaves with respect to the real étale site on V (and so using Theorem 1 also the topos of sheaves with respect to the semi-algebraic topology on V) is spatial (i.e., isomorphic to the topos of sheaves on a topological space).*

(Note that the underlying topological space of the spatial topos in Theorem 2 is the real spectrum of the ring $\mathrm{R}[V]$ described below.)

The algebraic definition of the real spectrum is as follows. Let A be a ring (commutative with a unit element). A subset $\alpha \subset A$ is called a *prime cone* if it satisfies the properties:

(i) $\alpha + \alpha \subset \alpha$,
(ii) $\alpha \cdot \alpha \subset \alpha$,
(iii) $A^2 \subset \alpha$,
(iv) $-1 \notin \alpha$, and
(v) $a \cdot b \in \alpha \Longrightarrow a \in \alpha$ or $-b \in \alpha$.

Definition (Real spectrum). The real spectrum, $\operatorname{Sper} A$, of A is the set of prime cones of A. The set $\operatorname{Sper} A$ is topologized by choosing as a basis of open sets the subsets

$$D_a = \{\mathfrak{p} \in \operatorname{Sper} A \mid a \notin \mathfrak{p}\}, a \in A \tag{4}$$

(compare with equation (3)).

Example. The real spectrum of a ring can be identified with its set of preorderings. The real spectrum of a field is the set of its total orderings. The real spectrum of the ring $A = \mathrm{R}[X]$ can be described as follows

$$\operatorname{Sper} A = \{\pm\infty\} \cup \{\alpha, \alpha_-, \alpha_+ \mid \alpha \in \mathrm{R}\},$$

where α (resp. α_-, α_+) is the cone of elements of A which are nonnegative at α (resp. immediately to the left of α, immediately to the right of α), and $-\infty$ (resp. $+\infty$) is the set of elements of A which are positive at negative (resp. positive) infinity.

The real spectrum $\operatorname{Sper} A$ shares some of the well-known properties of $\operatorname{Spec} A$ (for example, it is quasi-compact).

There is a canonical injection of $V(\mathrm{R})$ into $\operatorname{Sper} A$, and a bijection between open semi-algebraic subsets of $V(\mathrm{R})$

and the compact open subsets of $\operatorname{Sper} A$. This last bijection gives a translation between geometric properties of V and algebraic properties of A. For example, the local (semi-algebraic) dimension of $V(\mathrm{R})$ at a point $x \in V(\mathrm{R})$ is equal to the maximal length of a chain of prime cones terminating at x. We refer the reader to the book [3, Chapter 7] for a detailed study of the real spectrum and its various applications.

The fact that the topos of sheaves on the real étale site of a real variety is spatial and isomorphic to the topos of sheaves on the real spectrum makes the study of these sheaves easier especially from the point of view of proving various kinds of comparison theorems between different cohomology theories on real varieties. This is exploited by Scheiderer in [21] who amongst other things gave an alternative proof of Theorem 2 (avoiding the use of categorical logic).

Like most other important notions in mathematics the notion of real spectrum arose independently from several directions such as in the work of Lou van den Dries in model theory. It is also interesting to note that one consequence of Theorems 1 and 2 is that abstract topos theory from which the idea of real spectrum originated perhaps becomes less relevant in real algebraic geometry—since the real spectrum and its constructible subsets can be studied using geometric tools without referring to Grothendieck topologies etc. Nevertheless, as we shall see next, topos theory (and intuitionistic logic that goes with it) seems to have influenced many of Marie-Françoise's works on topics that are a priori far from logic and topos theory.

2.3. Algorithms in real algebraic geometry.

A significant part of Marie-Françoise's work has been in the area of algorithms in real algebraic geometry. This switch from abstract topos theory to more algorithmic aspects of real algebraic geometry was probably inspired by new developments in the then-new and extremely active field of computer algebra in the late eighties. This is exemplified by the biannual conference MEGA (Effective Methods in Algebraic Geometry) which started in 1990, with Marie-Françoise in its initial committee and has continued from then.

The algorithmic problems addressed in Marie-Françoise's work include some of the fundamental algorithmic problems in real algebraic geometry.[4] The gamut of her work in this area extends from the decision problem and more generally quantifier elimination in the theory of the reals mentioned before, to problems with more topological flavor (deciding connectivity of semi-algebraic sets, computing higher Betti numbers and Euler-Poincaré characteristics, dimension etc.) These algorithmic problems arise in many applications—in discrete and computational geometry, mathematical optimization, theoretical computer science amongst others. Designing better algorithms for such problems is clearly of wide interest. A second (perhaps less well-known) aspect is that the mathematical results underlying the design of these algorithms and often their complexity analysis yield quantitative results in real algebraic geometry. Indeed, the fact that these two aspects are very intertwined is very explicit in Marie-Françoise's work. I mention some examples later.

Symbolic algorithms and their complexity. We first note that by the word "algorithm" in this section we mean algorithms which are exact, symbolic algorithms. This means that the algorithms take as input polynomials with coefficients in some ordered domain $\mathrm{D} \subset \mathrm{R}$, use only rational arithmetic and sign determinations on elements of D, and terminate after a finite number of steps with the correct output. By "complexity" of such an algorithm we mean the number of arithmetic operations and sign determinations. If $\mathrm{D} = \mathbb{Z}$, then the number of bit-operations is called the bit-complexity.

Algorithms come with upper bounds on their complexity. These upper bounds are in terms of the size of the input—and this is measured by the number of polynomials (denoted by s), an upper bound on the degrees of the input polynomials (denoted by d) and the number of variables k (and the bit-lengths of the coefficients of the input polynomials in case $\mathrm{D} = \mathbb{Z}$).

Doubly vs. singly exponential. Several important problems in algorithmic real algebraic geometry can be solved using a technique called *cylindrical algebraic decomposition*. Given any semi-algebraic subset $S \subset \mathrm{R}^k$, a cylindrical algebraic decomposition of R^k adapted to S, is a partition of R^k into "cylindrical cells" (each semi-algebraically homeomorphic to $(0,1)^\ell, 0 \leq \ell \leq k$), such that for each cell C of the decomposition $C \cap S = C$ or empty. If S is closed and bounded such a cylindrical decomposition can be refined to a semi-algebraic triangulation of S. This technique was already familiar to geometers, in particular Lojasiewicz [14]. This was made algorithmic by Collins [5] using *subresultants* of pairs of polynomials and became a widely known algorithm. However, cylindrical algebraic decomposition is a big hammer. Having a cylindrical decomposition at hand allows one to solve all the algorithmic problems listed previously. However, since computation of a cylindrical decomposition involves iterated projection in which the degrees and the number of polynomials (roughly) square in each step—the size of a cylindrical decomposition (as well as complexity of computing it) is necessarily *doubly exponential* (of the form $(sd)^{2^{O(k)}}$).

Critical point method. A major focus of research in algorithmic real algebraic geometry has been in obtaining

[4] *A unified treatment of a major part of this work appears in the book* Algorithms in Real Algebraic Geometry [2] *(coauthored with Richard Pollack and the author).*

algorithms with *singly exponential* complexity. Even though singly exponential complexity might already seem very expensive from a practical point of view it should be remembered that each of the problems mentioned previously is conjecturally very hard from the computational complexity theory point of view (**NP**-hard or even **PSPACE**-hard), and that often the output itself has a singly exponential size in the worst case.

The key is to use more sophisticated ideas inspired by Morse theory (often called the critical point method), eliminating variables by blocks instead of eliminating variables one at a time. Even though the critical point method has been used by several researchers (in particular Grigoriev and Vorobjov), it is fair to say that Marie-Françoise is a pioneer in its application in a wide variety of settings achieving nearly optimal bounds in many cases.

Thom encoding. The key to the critical point method is to compute the set of critical points of a function restricted to certain real algebraic subsets of R^k. Using an initial deformation depending on one or more infinitesimals one ensures that the set of critical points is finite. But there still remains the problem of representing the coordinates of these points (which are algebraic over the ring generated by the coefficients of the input polynomials). A very elegant and also very general way of doing so is by using *Thom's lemma*—which was introduced to the area of symbolic computation by Marie-Françoise in joint work with Michel Coste ("Thom's lemma, the coding of real algebraic numbers and the computation of the topology of semi-algebraic sets," *Journal of Symbolic Computation*, Vol 5, 121–129, 1988).

One consequence of Thom's lemma is that each real root $\alpha \in \mathrm{R}$ of a polynomial $f \in \mathrm{R}[T]$ is characterized by the signs of the various derivatives of f at α. (So the root $\sqrt{2}$ of the polynomial X^2-2 is distinguished from the root $-\sqrt{2}$ by the signs of the derivative $2X$ at these two roots.) The tuple

$$\left(\mathbf{sign}(f^{(i)}(\alpha))\right)_{1\leq i\leq \deg(f)} \in \{0,1,-1\}^{\deg(f)}$$

is now known as the *Thom encoding* of the root α of f. Moreover the sign determination algorithm, computing the realizable sign conditions on a finite set of polynomials at the roots of f (see for example [2]), can be used to determine the Thom encoding of the roots of f.

A point in R^k can be described by a k-tuple of rational functions

$$u = \left(\frac{g_1(T)}{g_0(T)}, \dots, \frac{g_k(T)}{g_0(T)}\right)$$

evaluated at a real root α of another polynomial f specified by its Thom encoding σ. The tuple $(f, g_0, \dots, g_k) \in \mathrm{R}[T]^{k+2}$ and the Thom encoding σ, specifies the point

$$\left(\frac{g_1(\alpha)}{g_0(\alpha)}, \dots, \frac{g_k(\alpha)}{g_0(\alpha)}\right).$$

This method of representing real points, which works over arbitrary real closed fields (even non-Archimedean ones), is called *real univariate representation* in [2], and was introduced and used by Marie-Françoise in a series of papers. Parametrized versions of the same representations also play an important role in algorithmic real algebraic geometry in order to represent semi-algebraic curves (for example, in algorithms for computing *roadmaps* of semi-algebraic sets discussed below).

Sample points algorithm. The first application of the critical points method is in designing an algorithm that given a finite set $\mathcal{P}$ of polynomials in $\mathrm{R}[X_1, \dots, X_k]$ as input, computes a finite set of "sample points" guaranteed to intersect every semi-algebraically connected component of the realizations of every realizable sign condition on $\mathcal{P}$. The coordinates of the points are represented by rational functions evaluated at a real root of a univariate polynomial—the real root specified by a Thom encoding. The main idea is computing these points as critical points of a function restricted to certain algebraic sets obtained by making infinitesimal perturbations of the polynomials in $\mathcal{P}$, and so technically they belong to some real closed extension of R (field of algebraic Puiseux series with coefficients in R). This algorithm which appears in the paper "On computing a set of points meeting every cell defined by a family of polynomials on a variety," *Journal of Complexity*, Vol 13, No. 1, 28–37, 1997 (coauthored with Richard Pollack and the author), and whose complexity is bounded by $s^{k+1}d^{O(k)}$, is a crucial ingredient in many subsequent papers. Moreover, the degrees of the (univariate) polynomials appearing in the output are bounded by $O(d)^k$ (independent of s).

Quantitative curve selection lemma. A refinement of the degree bound from the last paragraph has recently been exploited by Marie-Françoise and the author to prove quantitative upper bounds for the *curve selection lemma* in semi-algebraic geometry ("Quantitative curve selection lemma," *Mathematische Zeitschrift*, Vol 300, No. 3, 2349–2361, 2022). The curve selection lemma is a key result in semi-algebraic geometry which states the following: for every semi-algebraic set S and $x \in \overline{S}$ (the closure of S in the Euclidean topology) there exists a semi-algebraic curve $\varphi : [0,1) \to \overline{S}$, such that $\varphi(0) = x, \varphi((0,1)) \subset S$. A quantitative version of this lemma asks for a bound on the degree of the Zariski closure of the image of φ in terms of the parameters of the formula defining S.

In what follows it is useful to begin with the following definition.

Definition. Let $\mathcal{P} \subset \mathrm{R}[X_1, \dots, X_k]$. We will call a quantifier-free first-order formula (in the theory of the reals) with atoms $P = 0, P > 0, P < 0, P \in \mathcal{P}$ to be a *$\mathcal{P}$-formula* and the set defined by it a $\mathcal{P}$-semi-algebraic set.

We denote by $\mathrm{R}[X_1,\dots,X_k]_{\le d}$ the subset of polynomials in $\mathrm{R}[X_1,\dots,X_k]$ with degrees $\le d$. The following result was proved by Marie-Françoise and the author.

Theorem 3 (Quantitative curve selection). *Let $\mathcal{P} \subset \mathrm{R}[X_1,\dots,X_k]_{\le d}$ be a finite set, S a $\mathcal{P}$-semi-algebraic set, and $x \in \overline{S}$. Then, there exist $t_0 \in \mathrm{R}, t_0 > 0$, a semi-algebraic path $\varphi : [0,t_0) \to \mathrm{R}^k$ with*

$$\varphi(0) = x,\ \varphi((0,t_0)) \subset S,$$

such that the degree of the Zariski closure of the image of φ is bounded by

$$(O(d))^{4k+3}.$$

Notice that the bound on the degree of the image of the curve φ in the above theorem has no combinatorial part, i.e., there is no dependence on the cardinality of $\mathcal{P}$.

The key ingredient in the proof of Theorem 3 is an accurate analysis of the degrees of the polynomials output in the sample points algorithm mentioned before, along with the identification of the field of algebraic Puiseux series with coefficients in R, with germs of semi-algebraic functions $(0,1) \to \mathrm{R}$. This is a good illustration how accurate complexity analysis of symbolic algorithms can lead to quantitative mathematical results.

Quantifier elimination algorithm. The critical point method can be used to eliminate whole blocks of quantifiers at the same time, leading to improvement in complexity. The following theorem, proved by Marie-Françoise with Richard Pollack and the author has been applied in many contexts ("On the combinatorial and algebraic complexity of quantifier elimination," *Journal of the ACM*, vol 43, No. 6, 1002–1045, 1996).

Theorem 4. *Let $\mathcal{P} \subset \mathrm{R}[X_{[1]},\dots,X_{[\omega]},Y]_{\le d}$ be a finite set of s polynomials, where $X_{[i]}$ is a block of k_i variables, and Y is a block of ℓ variables. Let*

$$\Phi(Y) = (Q_1 X_{[1]}) \cdots (Q_\omega X_{[\omega]}) \Psi(X_{[1]},\dots,X_{[\omega]},Y)$$

be a quantified-formula, with $Q_i \in \{\exists,\forall\}$ and Ψ a quantifier-free formulas with atoms $P=0, P>0, P<0, P \in \mathcal{P}$.

Then there exists a quantifier-free formula

$$\Psi(Y) = \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} \Big(\bigvee_{n=1}^{N_{ij}} \mathbf{sign}(P_{ijn}(Y)) = \sigma_{ijn}\Big),$$

where $P_{ijn}(Y)$ are polynomials in the variables Y, $\sigma_{ijn} \in \{0,1,-1\}$,

$$\begin{aligned} I &\le s^{(k_\omega+1)\cdots(k_1+1)(\ell+1)} d^{O(k_\omega)\cdots O(k_1) O(\ell)}, \\ J_i &\le s^{(k_\omega+1)\cdots(k_1+1)} d^{O(k_\omega)\cdots O(k_1)}, \\ N_{ij} &\le d^{O(k_\omega)\cdots O(k_1)}, \end{aligned}$$

equivalent to Φ, and the degrees of the polynomials P_{ijk} are bounded by $d^{O(k_\omega)\cdots O(k_1)}$.

The proof of Theorem 4 is effective, in that an algorithm is described to obtain the quantifier-free formula Ψ given the formula Φ as input, whose proof of correctness and complexity analysis yield Theorem 4. The bounds on the size of the quantifier-free formula Ψ in Theorem 4 and on the degrees of the polynomials appearing in Ψ, are doubly exponential in ω which is the number of alternations in the blocks of quantifier (this is unavoidable) but is singly exponential if ω is fixed. The improvement comes from the critical point method. Several quantifier elimination algorithms with doubly exponential complexity in the number of blocks exist [10, 18], but Theorem 4 is more precise by treating differently the number of polynomials and their degrees.

Combinatorial vs. algebraic complexity. Notice the different roles played by the "combinatorial" parameter $s = \mathrm{card}(\mathcal{P})$ and the "algebraic" parameter d (a bound on the degrees of the polynomials in $\mathcal{P}$). This separation of the "combinatorial part" from the "algebraic part" in the complexity upper bounds in algorithms as well as in other quantitative bounds in real algebraic geometry is an important distinguishing feature in many of Marie-Françoise's papers on quantitative and algorithmic aspects of real algebraic geometry. For example, the fact that the degrees of the polynomials in the quantifier-free formula Ψ in Theorem 4 can be bounded only in terms of the algebraic parameter d (independent of s) has many applications (for example, it plays a key role in several results in discrete and computational geometry).

Algorithmic vs. proof complexity. The critical point method produces a "better" algorithm than that using cylindrical algebraic decomposition in the sense of algorithmic complexity—singly exponential as opposed to doubly exponential. However, the proof of the correctness of this algorithm is much more complicated since it depends on connectivity results. Thus, if one is interested in converting an instance of a run of this algorithm into a formal mathematical proof (starting from the axioms of real closed fields) of the equivalence of the output and the input formula—then an algorithm using the critical point method is less suitable. In recent work (joint with Daniel Perrucci) Marie-Françoise has given an algorithm with elementary recursive (fixed tower of exponents) complexity algorithm for quantifier-elimination ("Elementary recursive quantifier elimination based on Thom encoding and sign determination," *Annals of Pure and Applied Logic*, Vol 168, No. 8, 1588–1604, 2017). Though from the point of view of algorithmic complexity this might seem much worse than the algorithm in Theorem 4, or even than the cylindrical algebraic decomposition algorithm, it is better from the point of view of *proof theory*, since its proof of correctness is purely algebraic (and does not require arguments involving connectivity which can be quite complicated from the point of view of formal logic).

An algebraic proof of the fundamental theorem of algebra. This is a natural place to mention the paper "Quantitative fundamental theorem of algebra," *The Quarterly Journal of Mathematics*, Vol 70, No. 3, 1099–1037, 2019, (with the same coauthor) which gives a new algebraic proof of one of the oldest theorems of algebra—namely, the fundamental theorem of algebra which states that the field $\mathrm{R}[i]$ is algebraically closed (assuming that R is real closed). The proof is based on a previous proof the same theorem by Michael Eisermann [9] which uses an algebraic definition of the winding number. The important new property of the proof in the paper under discussion is that in order to prove that every polynomial in $\mathrm{R}[i][T]$ of degree d has a root in $\mathrm{R}[i]$, the intermediate value property for polynomials in $\mathrm{R}[T]$ is needed only for polynomials of degree at most d^2, using subresultant polynomials which makes remainder sequences more efficient (subresultants are ubiquitous in Marie-Françoise's work). The classical proof due to Laplace that one meets in many textbooks of abstract algebra requires the use of the intermediate value property for real polynomials of exponential degree.

Roadmaps and connectivity. There is another class of algorithmic problems in real algebraic geometry that goes beyond the logical realm—namely, computing topological invariants of semi-algebraic sets. While initially motivated by problems of motion planning in robotics in the pioneering works of Jack Schwartz and Micha Sharir as well as John Canny, it has developed into a very active area of research in which Marie-Françoise has left an indelible mark. One of the first problems to be investigated in this area was the problem of counting the number of (semi-algebraically) connected components of a given semi-algebraic set. An important construction—namely, a semi-algebraic subset of a given semi-algebraic set of dimension at most one (also called a roadmap) was introduced by Canny in [4] to solve this problem within a singly exponential complexity bound. Once a roadmap of a semi-algebraic set has been computed, the problem of counting the number of connected components simplifies to a combinatorial problem of counting the number of connected components of a graph for which efficient algorithms are known. The history of the development of algorithms for computing roadmaps is quite long with several key contributions along the way (including contributions due to Marie-Françoise as well as Canny, Gournay, Grigoriev, Heintz, Pollack, Risler, Solerno, and Vorobjov amongst others.

We mention here two fundamental contributions due to Marie-Françoise. In the paper "Computing roadmaps of semi-algebraic sets on a variety," *Journal of the American Mathematical Society*, Vol 13, No. 1, 55–82, 2000, Marie-Françoise (with Richard Pollack and the author) gave a deterministic algorithm for computing the roadmap of a semi-algebraic set contained in a variety of dimension k' whose complexity is bounded by $s^{k'+1}d^{O(k^2)}$.

The underlying geometric idea behind algorithms for computing roadmaps has stayed the same over the years. This is roughly as follows. Suppose the goal is to compute the roadmap of a closed and bounded algebraic hypersurface $V \subset \mathrm{R}^k$. One first computes descriptions of two semi-algebraic subsets $V^0, V^1 \subset V$, where $V^1 = \pi^{-1}(M) \cap V$, where $\pi : \mathrm{R}^k \to \mathrm{R}^\ell$ is a linear projection map and $M \subset \mathrm{R}^\ell$ is a certain well-chosen finite subset, and V^0 is a certain polar subvariety of V of dimension ℓ. Then, $\dim(V^0) = \ell$, and $\dim V^1 = k - \ell - 1$. One then proves that $V^0 \cup V^1$ has a good connectivity property with respect to V—namely, that the intersection of $V^0 \cup V^1$ with each semi-algebraically connected component of V is nonempty and semi-algebraically connected. The algorithm then makes recursive calls on V^0 and V^1 taking advantage of the fact that the dimensions of V^0, V^1 are strictly smaller than $\dim V$.

The algorithm mentioned above (and in fact in all prior algorithms for computing roadmaps) used $\ell = 1$ in the definition of V^0 and V^1, and in this case the quadratic dependence on k in the exponent of the complexity is unavoidable (there are too many recursive calls). For a decade afterwards, this remained the algorithm with the best complexity bound for the problem and it was thought that the quadratic dependence on k was unavoidable. In a series of two papers ("A baby step–giant step roadmap algorithm for general algebraic sets," *Foundations of Computational Mathematics*, Vol 14, No. 6, 1117–1172, 2014, with Mohab Safey-el-Din, Éric Schost, and the author, and "Divide and conquer roadmap for algebraic sets," *Discrete & Computational Geometry*, Vol 52, No. 2, 278–343, 2014, with the author), Marie-Françoise improved the exponent (in the case of algebraic sets) to $O(k^{3/2})$ and then to $\tilde{O}(k)$ (suppressing poly-log factors). The latter is the best exponent currently known for the complexity of computing roadmaps at the moment. The mathematical results that make the advances in the above mentioned papers possible are new connectivity results similar to a result proved in an earlier paper by Safey-el-Din and Schost. The distinguishing feature of the new connectivity results as opposed to that in the prior work of Safey-el-Din and Schost is that no assumptions (such as genericity) are needed on V.

In another direction, it is natural to ask about the complexity of computing the higher Betti numbers of semi-algebraic sets (the number of connected components being the zeroth Betti number). Another contribution of Marie-Françoise (with Richard Pollack and the author) is the first singly exponential complexity algorithm for computing the first Betti number ("Computing the first Betti number of a semi-algebraic set," *Foundations of Computational Mathematics*, Vol 8, No. 1, 97–136, 2008). The key new ingredient is an algorithm with a singly exponential complexity for computing covers of semi-algebraic sets by closed contractible semi-algebraic subsets. This construction which is also based on the roadmap algorithm is a fundamental ingredient for more recent works on computing higher Betti numbers of semi-algebraic sets.

2.4. **Quantitative real algebraic geometry.** Real algebraic geometry has important connections with the field of discrete geometry which has blossomed in recent years—partly because of the injection of algebraic methods into incidence combinatorics due to Larry Guth and Nets Katz. Marie-Françoise was an early pioneer. It is in this work that she started her long collaboration with Richard Pollack which led to many of the works mentioned above (I was fortunate to be a part of some of them). A basic ingredient from real algebraic geometry is in proving upper bounds on the number of combinatorially distinct geometric configurations of various kinds—for example, the maximum number of *order types* that can be realized by n distinct points in $\mathbb{R}^k$ (the order type of a set S of n points in $\mathbb{R}^k$ is an element of $\{0,1,-1\}^{\binom{S}{k+1}}$, recording the orientation of each $(k+1)$-tuple of points of S). Questions of this type often reduce to bounding the number of realized sign conditions of certain finite sets of real polynomials restricted to some real variety. Such an upper bound follows from a bound on the number of connected components (the zeroth Betti number) of the realizations of every realizable sign condition of the set of polynomials.

The problem of proving upper bounds on the Betti numbers of real varieties has a long history. An upper bound on the sum of the Betti numbers of a real variety $V \subset \mathrm{R}^k$ defined by polynomials of degrees bounded by d was proved by Petrovskiĭ and Oleĭnik and later rediscovered by Milnor and Thom. They proved:

Theorem.

$$\sum_i b_i(V) \leq d(2d-1)^{k-1}.$$

An asymptotically tight upper bound on the number of connected components (the zeroth Betti number) of the realizations of all realizable sign conditions for finite sets of polynomials in $\mathrm{R}[X_1,\dots,X_k]_{\leq d}$ was proved by Marie-Françoise in a joint paper with Richard Pollack in ("On the number of cells defined by a set of polynomials," *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, Vol 316, No. 6, 573–577, 1993) and extended to sign conditions restricted to varieties in ("On the number of cells defined by a family of polynomials on a variety," *Mathematika*, Vol 43, No. 1, 1201–26, 1996, with Richard Pollack and the author). It is in these papers that the formal techniques of introducing infinitesimals, extending the given real closed fields to the field of algebraic Puiseux series in certain infinitesimals, and considering neighborhoods of various algebraic sets using different infinitesimals, were introduced, and these have proved to be the standard techniques in quantitative study of real algebraic geometry. A culmination of this line of work is the following theorem due to Marie-Françoise (with Richard Pollack and the author) which gives a bound on the sum of the Betti numbers (in any fixed dimension not just 0) of the realizations of all realizable sign conditions of a finite set of polynomials of bounded degree restricted to a variety ("On the Betti numbers of sign conditions," *Proceedings of the American Mathematical Society*, Vol 133, No. 4, 965–974, 2005). An extra topological ingredient needed in this semi-algebraic situation (compared to the Petrovskiĭ-Oleĭnik upper bound) is certain inequalities coming from the Mayer-Vietoris exact sequence.

Theorem 5. *The sum of the i-th Betti numbers of the realizations of all realizable sign conditions of a set of s polynomials in* $\mathrm{R}[X_1,\dots,X_k]_{\leq d}$ *restricted to a variety* $V \subset \mathrm{R}^k$ *of dimension* $\leq k'$ *defined by polynomials of degree at* d *is bounded by:*

$$\sum_{1\leq j\leq k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

This theorem recovers prior bounds on the number of connected components of sign conditions by substituting i by 0.

It is to be noted that the techniques introduced in the paper mentioned above have had an impact beyond real algebraic geometry. They are crucial ingredients in quantitative results on Betti numbers in more general structures—such as in o-minimal geometry and even in the theory of algebraically closed valued fields of arbitrary characteristics.

2.5. **Constructive Positivstellensatz.**

The language of "certificates" and analogy with Hilbert's Nullstellensatz. One suggestive way of viewing Artin's theorem is that it produces an *algebraic certificate* for the nonnegativity of a real polynomial $P \in \mathbb{R}[X_1,\dots,X_n]$ (or equivalently the unrealizability of the formula $P<0$). A generalization of this theorem which produces an algebraic certificate for the unrealizability of more general formulas of the form

$$\bigwedge_{i\in I}(P_i \neq 0) \wedge \bigwedge_{j\in J}(Q_j \geq 0) \wedge \bigwedge_{k\in K}(R_k = 0) \qquad (5)$$

was proved by Krivine and independently by Stengle. The following formulation is due to Stengle.

Theorem. *The formula in equation* (5) *is unrealizable in* R^n *if and only if there exists P belonging to the monoid generated by the polynomials* P_i^2, *Q belonging to the nonnegative cone generated by the polynomials* Q_j, *and R belonging to the ideal* *generated by the polynomials R_k, such that*

$$P + Q + R = 0. \tag{6}$$

The equality (6) is called an *algebraic certificate* of the unrealizability of the formula in (5).

This is known as the *Positivstellensatz* in analogy with the case of algebraically closed fields where, as is well known, Hilbert's Nullstellensatz produces such an algebraic certificate for the unrealizability of polynomial equations—namely, the emptiness of an algebraic set defined by polynomial equations $P_1 = \cdots = P_s = 0$ in C^n where C is an algebraically closed field, can always be certified by polynomials $Q_1, \dots, Q_s$ satisfying

$$1 = P_1 Q_1 + \cdots P_s Q_s. \tag{7}$$

Moreover, due to the work of Brownawell, Kollár, and Jelonek, very tight (singly exponential) upper bounds are known on the maximum degrees of the polynomials necessary in such a certificate in terms of the maximum degrees of the polynomials P_i. The following theorem proved by Marie-Françoise, in joint work with Henri Lombardi and Daniel Perrucci [16] provides the first elementary recursive upper bound on the algebraic certificate in Hilbert's seventeenth problem.

Theorem 6. *Let $P \subset \mathrm{R}[X_1, \dots, X_k]_{\leq d}$ be a nonnegative polynomial. Then P can be written as a sum of squares of rational functions, and the degrees of the numerators and denominators of these rational functions are bounded by*

$$2^{2^{2^{d^{4^k}}}}.$$

A similar bound (tower of five exponents) is also proved for the algebraic certificate for the Positivstellensatz in the same paper [16, Theorem 1.3.2].

We explain below some of the ideas that go into the proof of Theorem 6.

Constructive proofs of Postivstellensatz. In joint work with Henri Lombardi and Michel Coste [7], Marie-Francoise introduced a very general method for producing constructive proofs of theorems that guarantee the existence of algebraic certificates (for example, Nullstellensatz for algebraically closed fields, Positivstellensatz for real closed fields, and even a Positivstellensatz for algebraically closed valued fields).

We restrict to the case of real closed fields in the following.

One starts with an algorithm for quantifier elimination in the theory of the reals. Such an algorithm with input the formula in (5) preceded by a block of existential quantifiers will produce the output 'FALSE' if and only if the semi-algebraic set defined by the formula in [16] is empty. The steps taken by the algorithm can be thought of as a tree with branchings depending on the signs of certain elements of R computed by the algorithm.

This tree can be converted into a formal mathematical "proof" having a special shape (referred to as a dynamical proof in [7]). (It is interesting to mention here that the dynamical theories and proofs have very close connections with Grothendieck toposes as explained in [7, Section 1.1].)

As an illustration, at a certain step of the proof one might want to infer the conclusion $P(u) > 0 \vee P(u) < 0$ from the hypothesis $P(u) \neq 0$ (where u is a tuple of indeterminates and $P \in \mathrm{R}[u]$). More generally, such an inference will be usually needed in a "context" where the signs of some other polynomials in $v = (u, u')$ are fixed.

A key notion defined first in a paper by Lombardi [15] is *weak inference* and more generally *weak existence*.

Definition (Weak existence, weak inference)**.** We follow the same notation introduced above. A weak existence

$$(\exists t_0)\mathcal{F}(u, t_0) \vdash (\exists t_1)\mathcal{F}_1(u, t_1) \vee \cdots \vee (\exists t_m)\mathcal{F}_m(u, t_m)$$

is a construction which produces, given sign condition $\mathcal{H}(u, u') = (\mathcal{H}_{\neq 0}, \mathcal{H}_{\geq 0}, \mathcal{H}_{=0})$ and algebraic certificates for the unrealizability of $\mathcal{F}_i \wedge \mathcal{H}$, $i = 1, \dots, m$ (these are called *initial incompatibilities* in [16]), an algebraic certificate for the unrealizability of $\mathcal{F} \wedge \mathcal{H}$ (called the *final incompatibility*).

A weak inference is defined similarly but without the existential quantifiers.

Since the final incompatibility is given by an explicit construction taking as input the initial incompatibilities, the degrees of the polynomials in the final incompatibility can be bounded explicitly in terms of the degrees of the initial incompatibilities.

Going back to the preceding illustrative example, the weak inference version says the following. We consider a context given by sign conditions $\mathcal{H} = (\mathcal{H}_{\neq 0}, \mathcal{H}_{\geq 0}, \mathcal{H}_{\geq 0})$ and start from the two initial incompatibilities,

$$P_1 + Q_1 + R_1 = 0,$$

$$P_2 + Q_2 + R_2 = 0,$$

where P_1, P_2 belong to the monoid generated by the polynomials $H_{\neq 0} \cup \{P^2\}$, Q_1 is in the cone generated by $H_{\neq 0} \cup \{P\}$ (resp. Q_2 is in the cone generated by $H_{\geq 0} \cup \{-P\}$ and R_1, R_2 are in the ideal generated by $H_{=0}$.

The final incompatibility is constructed as follows: multiply both sides of

$$P_1 = -Q_1 - R_1,$$

$$P_2 = -Q_2 - R_2,$$

to obtain

$$P_1 P_2 = -Q_3 - R_3,$$

where $P_1 P_2$ belongs to the monoid generated by the polynomials $H_{\neq 0} \cup \{P^2\}$, Q_3 is in the cone generated by $H_{\geq 0}$ and

R_3 is in the ideal generated by $H_{=0}$. Thus, we obtain the final incompatibility

$$P_1 P_2 + Q_3 + R_3 = 0.$$

The construction described above is quite simple and the bounds on the degrees easy to obtain.

Here is a more complicated weak inference that is one of the many (!) key steps in the proof of Theorem 6.

Weak inference version of the intermediate value theorem.

Theorem 7. [16, Theorem 3.1.3] *Let* $P = \sum_{0 \le h \le p} C_h \cdot y^h \in \mathrm{R}[u][y]$. *Then,*

$$\exists (t_1, t_2)[C_p \neq 0 \wedge P(t_1)P(t_2) \le 0] \vdash \exists t P(t = 0).$$

The degree of the monoid part of the final incompatibility is bounded by a function which is doubly exponential in the degree p *of* P *in* y *(see* [16] *for a much more precise statement).*

The proof of Theorem 7 (including the estimate on the degree) is not straightforward and is based an inductive argument on the degree of P, and is an adaptation of the proof by Artin [1] that if a field is real (i.e., in which -1 is not a sum of squares), then its extension by an irreducible polynomial of odd degree is also real.

In summary, the idea behind the proof of Theorem 6 is the following. Start from a quantifier elimination algorithm applied to the given sign condition (with empty realization). Convert the steps of the algorithm into a proof each of whose steps are logical deductions of a certain type. Prove weak inference/existence versions of these steps and make a careful accounting of how the degrees are growing in each step. As one can imagine this is a formidable task and includes as substeps giving new constructive proofs of very classical theorems—like the Laplace's algebraic proof of the fundamental theorem of algebra, Hermite's theorem for counting real roots using signatures of quadratic forms, the intermediate value theorem amongst others—all the time keeping track of the degrees appearing in the algebraic certificates.

2.6. **Works not covered.** I hope that in this article I have been able to give a snapshot of Marie-Françoise's work and some of the beautiful mathematics behind them. Unfortunately, because of its breadth and large volume, as well as constraints on the length of this article, it was not possible to discuss many very important aspects of her work. In particular, just to mention a *few*, I have not discussed her work with Diatta, Diatta, Rouillier, and Sagraloff on efficient algorithms for computing topology of curves, with Aviva Szpirglas on Sylvester double sums, with Dima Pasechnik and the author on the topology of semi-algebraic sets defined by "partly" quadratic polynomials, with Fatima Boudaoud and Fabrizio Caruso on certificates of positivity using the Bernstein basis, with Thomas Lickteig on Sylvester-Habicht sequences and fast Cauchy index computation, and with Nicolai Vorobjov on complexification and degree of semi-algebraic sets.

Figure 3. Louis Mahé, Marie-Françoise Roy and Michel Coste in Rennes, 2011.

2.7. **Impact.** While the topic of real algebraic geometry is now firmly rooted as a subdiscipline of mathematics worthy of study—it was certainly not the case when Marie-Françoise began her career. Indeed it is fair to say that the book *Géométrie algébrique réelle* (with Jacek Bochnak and Michel Coste) [3] published in 1987, and the once-a-decade series of conferences in Rennes (1981, 1991, 2001, and 2011) with published proceedings played a major role in establishing the topic as an important area of research in mathematics (one with many connections to both pure and applied aspects of mathematics).

Within the community of real algebraic geometry (as I hope it is clear from this article) Marie-Françoise has been involved in a very wide spectrum of research—from the very abstract, to constructive and computational. She has had an unusually large number of collaborators some of whom are from outside the area of real algebraic geometry. I think what made this possible is Marie-Françoise's rather rare ability to grasp and communicate key ideas very lucidly—even to mathematicians not versed in real algebra. This led to building bridges between different areas—such as between real algebraic geometry and discrete and computational geometry as well as to the area of symbolic computation. Indeed it was such a collaboration (with my Phd advisor Richard Pollack) that brought me into contact with her for which I am grateful. From her I learned that one does not really understand a proof (even one's own) unless one is able to "see" it and the vital importance of proper notation in writing and communicating mathematics. Many ideas that have been mentioned above (use of infinitesimals in algorithms, dynamical proofs, subresultants etc.) existed prior to Marie-Françoise's work.

However, it is Marie-Françoise (in collaboration with various coauthors) who clarified and sharpened these ideas leading to new advances. I would be remiss if I don't mention one other quality. Some of Marie-Françoise's projects have taken a long time to bring to fruition. This is indicative of a particularly obstinate trait in her character—of not giving up even when the technical obstacles to carrying through a particular program might seem impossible to overcome. In this she is a role model for all young mathematicians.

Over her career Marie-Françoise has mentored many mathematicians from many parts of the world (including the author) and she (along with Michel Coste and Louis Mahé) made Rennes a leading center of research in real algebraic geometry with a constant stream of visitors and weekly seminars.

3. Women in Mathematics

Marie-Françoise has been very active in promoting the cause of women in mathematics in various national and international forums. Marie-Françoise was one of the founders of European Women in Mathematics (EWM), and was the convenor of EWM between 2009 and 2013. In 1987 she cofounded the French organization for women in mathematics, Femmes et Mathématiques, and became the organization's first president and was one of the founding members of the African Women in Mathematics Association (founded in 2013). She was the first chair of the IMU Committee for Women in Mathematics (CWM) between 2015 and 2022 and led the "Gender gap in science"[5] project (funded by the International Science Council [ISC] in cooperation with IUPAC). A book[6] as well as a booklet[7] containing the summary of the results of the project and a full list of its recommendations in several languages have been published.

4. Work in Africa and Niger

Marie-Françoise spent two years of her professional career (1981–1983) at Abdou Moumouni University in Niger. She has continued to be very deeply involved in mathematical and social projects in Africa and in particular in Niger. She was the scientific officer for Sub-Saharan Africa in Centre International de Mathématiques Pures et Appliquées, CIMPA] (2007–2013) and is the president of Association d'Echanges Culturels Cesson Dankassari (Tarbiyya-Tatali), an organization working to support the sustainable development of the commune of Dankassari in Niger through the solidarity of the French commune Cesson-Sévigné where she lives. Her book [17] (coauthored with Nicole Moulin, Boubé Namaiwa, and Bori Zamo) is a sociopolitical work describing the history of a village called Lougou in Niger and of its queen Saraouniya, its encounter with colonialism and its aftermath.

[5] https://gender-gap-in-science.org

[6] https://zenodo.org/record/3882609

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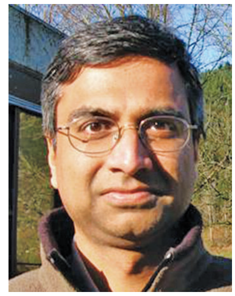

Saugata Basu

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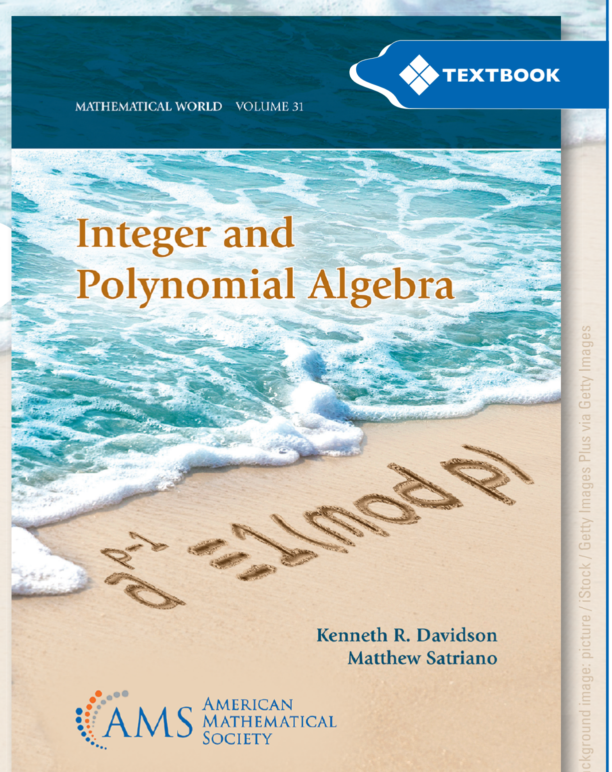

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The Contributions of Chuu-Lian Terng to Geometry

Karen Uhlenbeck

1. Introduction

Chuu-Lian Terng is a geometer who has made important contributions to both submanifold geometry and integrable systems. My collaboration with her over many years has been rewarding and profitable. This article is an attempt to share my appreciation of her work with a wide audience. I am indebted to her for decades of profitable collaboration, a warm personal relationship and help with writing this article.

The article is incomplete, as space constraints do not allow me to discuss all of her research. I chose to emphasize personal details and her work on integrable systems. Many schools of mathematics contributed to the development of integrable systems and I cannot include them all. The outline given in Section 6 is as I have come to see the subject. This is close, but not identical to Terng's view, and will differ from other views. I hope to give a window opening onto her work, not to provide a definitive treatise. The ideas belong to many people and errors are mine. The reader will surely join me in thanking the referees for helpful comments and important corrections.

2. Background and Education

Chuu-Lian Terng was born in 1949 in Hualian, Taiwan. Her family moved to Taipei when she was three. Her father was in the army under Chiang Kai-shek and had moved from mainland China to Taiwan when the Communists under Mao took over China in 1948. She was the eldest, with three younger brothers, and her family was very poor. She was warned that she had to do well in school, lest she end up working as a maid.

Chuu-Lian was a good student, and graduated the Taipei First Girls' High School. Her comment: "Mathematics was easy. I liked it." When it came time to go to college, she was automatically admitted to the National Taiwan University on the basis of her grades. This was not free, unlike the Normal University, but graduates of the Normal University were expected to teach for ten years in middle and high school after graduation. Her sights were set higher. So she tutored middle school and high school students for ten hours a week to earn tuition, books and money to give her family. And she of course lived at home.

Chuu-Lian's class had 8 women and 24 men. At first it was a big shock for the women to study for the first time with men. In the boys school many students had already studied calculus and read English mathematics books on their own. English was not Chuu-Lian's strong suit. (More about that later!) Her recollection is that the first year there were two math courses: calculus and "What is Mathematics" from Courant and Robbins. However, the students got together, ran a seminar and studied extra material. There was a lot of homework, and the exams were hard. She recalls that in her senior year she just missed passing an algebra course with a 55 when 60 was passing. Only one student had a higher grade. "The atmosphere was good. The students worked together and presented material, teaching themselves. I remember when I once got the highest grades in my freshmen year, and the men called me 'brother Terng.'" She graduated at the top of her class.

At that time there were no PhD programs in Taiwan. The top students routinely went to the United States for graduate study and most of them did get PhD's. They often went to the top graduate schools and did well there. But Chuu-Lian was faced with two problems. The first was her lack of proficiency in English. It was her worst subject and she hated it. All the Taiwanese students had to take the TOEFL exams, and almost all graduate schools in the

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US required at least a score of 550. Chuu-Lian's score was 520. She found three schools that only required 500, applied and was accepted into all three. She elected to go to Brandeis.

The second problem was that her family had no money and she had been giving them most of her tutoring earnings. So she delayed for a year, working as a TA and taking a few more courses. This time she kept her earnings for her plane fare and a nest egg until settled at Brandeis. She again sent money home during her graduate student years.

"I loved my courses at Brandeis. I also found out how good my undergraduate training had been." She still speaks with enthusiasm of Ed Brown teaching topology, Maurice Auslander who gave lot of homework and no lectures and Dick Palais's course on pseudo-differential operators on the circle. And she remembers what many of us discovered: that we loved complex variables, but that the charm does not necessarily carry over to several complex variables. Her subjects were geometry and topology. No PDE and bad memories of a course in harmonic analysis. At the end of the first year she started working with Dick Palais. Her first readings were his book on the Atiyah-Singer index theorem and Spivak's *Differential Topology volume I*. Her thesis was on the classifications of natural vector bundles and natural differential operators.

3. Early Career

Terng's first postdoctoral position was at the University of California at Berkeley, where she went to study differential geometry with S. S. Chern. When she went to ask him for a problem, his reply was "I don't give problems. Go to seminars, library, talk to people and find your own problems." Then Chern became interested in solitons and Bäcklund transformations.

3.1. The sine Gordon equation and Bäcklund transformations. The sine Gordon equation is the equation for $\phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\phi_{uv} = \sin(\phi).$$

We easily see that a trivial solution is given by $\phi = 0$. We use the notation that subscripts denote partial derivatives.

Theorem 3.1.1. *If ϕ is a solution of the sine-Gordon equation and*

$$\begin{cases} \xi_u = \phi_u + 2a\sin\frac{\xi+\phi}{2}, \\ \xi_v = \phi_v + 2a^{-1}\sin\frac{\xi-\phi}{2}, \end{cases}$$

for any constant a, then ξ is also a solution of the sine Gordon equation.

In fact, we see that the two equations for ξ are compatible only when ϕ solves the sine Gordon equation. Just as a test, if $\phi = 0$ is the trivial solution, then, we get

$$\begin{cases} \xi_u = 2a\sin\frac{\xi}{2}, \\ \xi_v = 2a^{-1}\sin\frac{\xi}{2}. \end{cases}$$

Changing variables to $t = a(u+v)$ and $s = 1a^{-1}(u-v)$ gives equations $\xi_s = 0$ and $\xi_t = 2\sin(\xi/2)$, yielding a one-parameter family of solutions. The process can be iterated, giving a chain of solutions known as solitons. After two steps there are relations between solutions known as permutability formulas. We have found a chain of special solutions to the partial differential equation by solving systems of ordinary differential equations. This is the purpose of Bäcklund transformations.

The three classical "integrable systems" with Bäcklund transformations, scattering and inverse scattering theory, tau functions and Virasoro actions are:

(i) sine Gordon equation (SGE)

$$\phi_{uv} = \sin(\phi),$$

(ii) nonlinear Schrödinger equation (NLS)

$$u_t = 2i(u_{ss} + |u|^2 u),$$

(iii) Korteweg-de Vries equation (KdV)

$$u_t = u_{sss} - (u^3)_s.$$

These three equations have similar properties but very different origins. The sine Gordon equation arose in 1862 in the investigation by Edmund Bour into constant Gaussian curvature -1 surfaces in $\mathbb{R}^3$. The angle between the asymptotic lines satisfies the sine Gordon equation. Bäcklund discovered the transformations which bear his name using line congruences. The Korteweg-de Vries equation dates back to 1895 as a description of shallow water waves. The more recent nonlinear Schrödinger equation describes a plethora of phenomena including the propagation of signals in fiber optic cables. Terng and I wrote an expository article for the *Notices* describing more of the history [TU00]. Chern was, of course, interested in the geometry described by the first of these equations. He wanted to extend Bäcklund transformations to affine minimal surfaces in $\mathbb{R}^3$. Chuu-Lian read Chern's notes on affine geometry, and they obtained results very fast. However, they only obtained a single transformation without a parameter. After they obtained the results, the senior author said "You write it up" [CT80]. It is in stories like this that we get hints of how we might have been better thesis advisors and mentors.

3.2. Higher-dimensional versions. Chern then put Chuu-Lian in touch with Keti Tenenblat, a Turkish-Brazilian mathematician visiting Chern. The two women collaborated on the problem, suggested by Chern, of generalizing the classical Bäcklund transformations to hyperbolic n-dimensional submanifolds in $\mathbb{R}^{2n-1}$. Later

Ablowitz, Beals and Tenenblat found a Lax pair and did the scattering and inverse scattering. Chuu-Lian found a loop algebra given by involutions which have a suitable splitting. There was a lot of interest in these equations at the time. I recall driving Chuu-Lian to the University of Chicago where she had been invited to talk on this work to the theoretical physics group.

3.3. Chinese women mathematians. Chuu-Lian Terng was not an isolated Chinese woman mathematician. As we noted before, students from the National Taiwan University routinely went on to PhD's in the United States. The year before Chuu-Lian, a number of women whose names you may recognize graduated: Alice Chang, Fan Chung, Winnie Li, Gloria Wu. The success of this group of Chinese women mathematicians is not part of a general trend, and cannot be easily explained. Chuu-Lian did know about the other four, and discovered only recently that all of them have similar backgrounds. They all credit their mothers and an excellent education system which was open to women. She took her teachers' questionings "Did your father help you with this?", "Did your boyfriend help you study?" and "You don't like to look up the answers in the back of the book?" as praise. As we attempt to diversify the pool of individuals succeeding in research mathematics, we might pay attention to what worked here. I recommend a forthcoming article by Allyn Jackson about this group of Chinese women mathematicians.

4. Midcareer

After her postdoc at Berkeley, Terng moved on to an assistant professorship at Princeton. Princeton had only started admitting women students in 1969, and admission had become gender blind in 1974. Chuu-Lian came to Princeton in 1978 as the first female assistant professor in the mathematics department. She was one of ten assistant professors who used their time at Princeton to get as much research done and move on to a "real" tenure track or tenured position elsewhere. Inconveniences, such as having an office on the tenth floor when the only women's bathroom was on the third floor, happened all too often in those days. But in general, it was a matter of being ignored rather than treated badly.

When I came as a member to the Institute for Advanced Study for the academic year 1979–1980, we tried to work together for the first time. I recall that we had offices next to each other in Building C, and at least once Professor Langland slammed the door of the office we were working in. Our husbands have also noted that we are not quiet when we work together. Perhaps we were expected "to be seen but not heard"? I had been learning Teichmüller theory and was attending Bill Thurston's course at the university. We learned about hyperbolic three manifolds which fiber over the circle topologically. No natural geometric fibration was known. With my background in minimal surfaces and Chuu-Lian's command of tools in geometry, the problem seemed made for us.

Alas, it was not to be. The problem is still unsolved. When Bill Thurston heard of our project, he suggested that these manifolds might be fibered by minimal surfaces. We thought it more likely that there would be a fibration by constant mean curvature surfaces. In 1979 the problem was esoteric, but it is now known that every compact hyperbolic 3-fold has a cover that topologically fibers over a circle; these are now essential examples in the study of hyperbolic 3-folds.

I return to this problem periodically, but Chuu-Lian's next project explored polar actions and isoparametric submanifolds. Due to the length limit of this article, we will refer the readers to a survey article by G. Thorbergsson [Th] and the book written by Terng and Palais [PT88] on these topics. This research was carried out during Terng's years at Northeastern University. These were not easy years with heavy teaching load, an unpleasant commute, and the kind of mistreatment and outright sexism that was all too common (which none of us enjoy revisiting). On the good side, she had good interactions with Boston area mathematicians and has fond memories of weekly joint seminar with her colleagues in topology, PDE, and geometry. She also benefited from lectures on integrable systems by Mark Adler and Pierre van Moerbeke at Brandeis. She moved from Northeastern University to University of California at Irvine in 2004 and enjoyed very much the differential geometry group, the graduate students, a house on campus, and the wonderful climate.

5. Soliton Equations in Geometry

My road into integrable systems was from the opposite direction from Chuu-Lian's introduction via Chern. Through one of my PhD students Louis Crane, I learned about the interest of the physics community in "loop groups," and read the papers of Louis Dolan on the sigma model. I had kept up with Chuu-Lian and her husband, only partly because I had relatives in the Boston area. After I noted that Dolan's loop group actions were dressing actions and the classical Bäcklund transformations for sine Gordon were given by actions of some rational loops, we started to work together again. This section is intended to connect examples of integrable equations with equations familiar to geometers.

5.1. Finite dimensions. In finite dimensions, there is a definition of completely integrable.

Definition 5.1.1. Let (M, ω) be a symplectic manifold. The Hamiltonian system given by $H : M \to \mathbb{R}$ is

$$\frac{d\gamma}{dt} = X_H(\gamma),$$

where X_H is the dual of dH with respect to ω, i.e.,

$$\omega(X_H, v) = d_\gamma H(v) \quad \text{for all tangent vector } v.$$

Definition 5.1.2. $H_j : M \to R$ is a conservation law for the Hamiltonian system given by H if for every solution γ, $\frac{d}{dt}H_j(\gamma) = 0$, or equivalently,

$$\{H, H_j\} := \omega(X_H, X_{H_j}) = dH(X_{H_j}) = 0.$$

A Hamiltonian system given by H on a manifold of dimension $2m$ is *completely integrable* if there are m conservation laws $H = H_1, H_2, \dots, H_m$, which are in involution with each other, i.e., $\{H_j, H_k\} = 0$.

If two Hamiltonians are in involution with each other, their flows commute. This is the picture in finite dimensions. However, in infinite dimensions, the integrable systems in this article always have an infinite number of conservation laws whose flows commute. Only in very rare cases is there a theory which indicates "completeness." The example I know involves finding action angle coordinates for a special case of KdV. Nevertheless, we shall see that the infinite-dimensional theory is much richer than the finite-dimensional theory.

The symplectic manifolds in this section are the Grassmanians $N = G(k,n)$, which are adjoint orbits in the Lie algebra $u(n)$ of $U(n)$,

$$N = G_{(k,n)} = \left\{g^{-1}Jg \mid J = \frac{i}{2}\operatorname{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})\right\}.$$

Geometric information 5.1.3. For $h \in N$,

(i) $T_hN = \{[a,h] \mid a \in u(n)\}$.

(ii) $\langle a, b\rangle = -\operatorname{tr}(ab)$.

(iii) The orthogonal projection a_h of a on $T_h(N)$ is

$$a_h = -[h,[h,a]] = -(\operatorname{ad} h)^2(a).$$

(iv) The symplectic form ω is

$$\omega_h(a,b) = \operatorname{tr}([h,a]b).$$

(v) The associated complex structure in the tangent space is

$$i(h)a = [h,a].$$

It is important to know at least one example.

Example 5.1.4. Let $a \in u(n)$ and let $H_a(h) = \operatorname{tr}(ah)$. Then the hamiltonian flow for H_a is

$$\frac{dh}{dt} = -[a,h].$$

Note that if $a, b \in u(n)$ commute, then

$$\{H_a, H_b\} = dH_a(X_{H_b}) = \operatorname{tr}(a[h,b]) = \operatorname{tr}(h[b,a]) = 0.$$

However, when $n > 2$, $[a, b_j] = 0$, does not imply that $[b_j, b_k] = 0$. So the integrals for the Hamiltonian flow $\frac{dh}{dt} = -[a,h]$ are not in involution. In this case, the symmetries are Poisson and we can expect to see such symmetries for equations with target $N = G_{(k,n)}$ with $n > 2$.

5.2. Equations of global analysis.

We now switch gears. Geometers are familiar with the nonlinear elliptic equation of harmonic maps between Riemannian manifolds and associated parabolic heat and hyperbolic wave map equations. However, in the case that the image manifold is symplectic, we have a geometric nonlinear Schrödinger (GNLS) equation.

As a general principle, the image negatively curved symmetric spaces correspond to defocusing equations and those with positive curvature correspond to focusing. Formally they appear similar, but technically and geometrically they are very different. Terng and I considered the case of the positively curved symmetric space $N = G_{(k,n)}$ described above.

We briefly describe the GNLS on $\mathbb{R} \times \mathbb{R}^{n-1}$ with image $G_{(k,n)}$. We consider

$$C_J(\mathbb{R}^{n-1}, N) = \{h : \mathbb{R}^{n-1} \to N \mid h_{x_i} \in S(\mathbb{R}^{n-1}, u(n)\},$$

where $S(\mathbb{R}^{n-1}, u(n))$ is the Schwartz space and $\lim_{x\to-\infty} h(x) = J$. Corresponding to the geometry listed in 5.1.3, we have the following geometry in $C_J(R,N)$.

5.2.1. *Geometric information in $C_J(\mathbb{R}^{n-1}, N)$.*

For $h \in C_J(\mathbb{R}^{n-1}, N)$,

(i) $T_hC_J(\mathbb{R}^{n-1}, N) = \{[h,A] \mid A \in S(\mathbb{R}^{n-1}, N)\}$.

(ii) The formal inner product on the tangent space is

$$\langle A, B\rangle = -\int_{\mathbb{R}^{n-1}} \operatorname{tr}(AB)(dx)^{n-1}.$$

(iii) The orthogonal projection A_h of $A \in S(\mathbb{R}^{n-1}, u(n))$ on $T_hC_J(\mathbb{R}^{n-1}, N)$ is

$$A_h = -\frac{1}{4}(ad(h))^2 A.$$

(iv) The symplectic structure on $C_J(\mathbb{R}^{n-1}, N)$ is

$$\Omega(h)(A,B) = -\int \operatorname{tr}([h,A]B)(dx^{n-1}).$$

(v) The associated complex structure in the tangent space to $C_J(\mathbb{R}^{n-1}, N)$ at h is

$$i(h)A = [h,A].$$

Theorem 5.2.2. *The Hamiltonian flow for $H = \frac{1}{2}\langle dh, dh\rangle$ is the GNLS equation*

$$\frac{\partial h}{\partial t} = [h, \triangle h].$$

We will explain in Section 6 why this equation is integrable for $n = 2$.

5.3. Ward harmonic maps and space-time monopoles.

The 2- and 1+1-dimensional examples of harmonic maps and wave maps into $\mathrm{SU}(n)$ are examples of equations which have many of the usual properties of soliton equations. However, there is a variant of the wave map equation in $1 + 2$ into $\mathrm{SU}(n)$ due to Ward which has solitons and scattering and inverse scattering theories. We refer to

Ward's equation as the modified wave map. The wave map equation for $g : \mathbb{R}\times\mathbb{R}^2 \to \mathrm{SU}(2)$ is

$$\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}g\right)g^{-1}\right) - \frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right) = 0.$$

Ward's equation is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}g\right)g^{-1}\right) - \frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right)\\ = \left[\left(\frac{\partial}{\partial x}g\right)g^{-1}, \left(\frac{\partial}{\partial t}g\right)g^{-1}\right].\end{aligned}$$

The Lax pair for this equation is more easily seen if we transform using the variables $\xi = \frac{1}{2}(t+x)$ and $\eta = \frac{1}{2}(t-x)$. Then the modified wave map equation simplifies to

$$\frac{\partial}{\partial \xi}\left(\left(\frac{\partial}{\partial_\eta}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right) = 0.$$

Proposition 5.3.1. *There is a Lax pair for the modified wave map of the form* $[L(\lambda), \tilde{L}(\lambda)] = 0$, *where*

$$\begin{aligned}L(\lambda) &= \lambda\frac{\partial}{\partial\xi} - \left(\frac{\partial}{\partial y} - \left(\frac{\partial}{\partial y}g\right)g^{-1}\right),\\ \tilde{L}(\lambda) &= \lambda\frac{\partial}{\partial y} - \left(\frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \eta}g\right)g^{-1}\right).\end{aligned}$$

The frame E_λ is useful for constructing a large number of examples, which are not known for the wave map. The modified wave map equation can be transferred (via a gauge transformation similar to the Hasimoto transformation) to the space-time monopole equation, which is a reduction of anti-self-dual Yang-Mills with signature $(2,2)$ to a monopole equation in $\mathbb{R}^{1,2}$.

Proposition 5.3.2 ([DTU06]). *Solutions to the Ward equation can be identified with special solutions to the space-time monopole equation in* $\mathbb{R}^{1,2}$,

$$*D_A\Phi = F_A.$$

Here $D_A = d + A, F_A = \{D_A, D_A\}$ *is the curvature 2-form,* Φ *is the (scalar) Higgs field in* $su(n)$ *and* $*$ *is the map from 1-forms to 2-forms induced by the indefinite metric on* $\mathbb{R}^{1,2}$.

The analytic properties of the wave map problem are hard theorems in geometric analysis and it is difficult to find explicit examples. Description of Euclidean monopoles are only accurate at large spacings. However there is both a scattering theory for the monopole equation and Bäcklund transformations for Ward's equation which are gauge equivalent to the Lax pair for the monopole equation. This is due to the existence of Lax pairs for both variants of the equation which are gauge equivalent. For smooth decaying initial data, the scattering theory shows that solutions exist for all time. Terng's papers [DT07] and [DTU06] contain a wealth of information about these solutions.

6. The Drinfel′d-Sokolov Construction for the MNLS Hierarchy

In this section, we outline a very general procedure developed by many mathematicians of different backgrounds for generating soliton equations and their properties. There is a long list of contributors to this project, starting with Zhakarov and Shabat in 1976. In the last and final section we will come back to Terng's ideas for classifying them. Much of the work of Terng is based on a fundamental paper of Drinfel′d and Sokolov [DdS84]; hence I have called the general scheme a Drinfel′d-Sokolov (DS) construction. The flows are traditionally in the variables $(x = t_1, t = t_2, t_3)$. However, certain pieces of the structure can be applied with only part of the structure available. For example, Bäcklund transformations can be applied when there is only a single flow if there is a Lax pair description. In perhaps most examples, the needed factorizations for scattering theory and Bäcklund transformations are only local. The matrix nonlinear Schrödinger equation (MNLS) is a generalization of the scalar NLS, and our choice is motivated by the fact that the factorizations are well-understood in this case. Also note that there may be multiple descriptions of the same flows. The computations do not need to be done in any fixed order.

6.1. Choice of a triple group.

Most flows are based on a triple loop group $(\mathcal{G}, \mathcal{G}^+, \mathcal{G}^-)$. Here $\mathcal{G}^\pm$ are subgroups of $\mathcal{G}$ and elements of $\mathcal{G}$ factor into products in both orders on an open, dense set. The loop parameter plays the role of a spectral parameter λ, and the group factoring is done via Birkhoff factorization. These groups dependent on a spectral or loop parameter are often called "loop groups," especially in the physics literature. There exists an open dense set Ω of $\mathcal{G}$ (*big cell*) such that for $g \in \Omega$, we can factor

$$g = g_+g_- = h_-h_+, \quad g_+, h_+ \in \mathcal{G}_+, g_-, h_- \in \mathcal{G}_-.$$

In our example:

$$\begin{aligned}\mathcal{G} &= \{g(\lambda) \in \mathrm{GL}(n,\mathbb{C}) \,|\, g \text{ defined and holomorphic in}\\ &\qquad \{\lambda : k < |\lambda| < \infty\}, (g(\bar\lambda))^\dagger g(\lambda) = \mathrm{I}\},\\ \mathcal{G}^+ &= \{g \in \mathcal{G} \,|\, g \text{ extends holomorphically to } \mathbb{C}\},\\ \mathcal{G}^- &= \{g \in \mathcal{G} \,|\, g(\infty) = \mathrm{I}, g \text{ holomorphic in a}\\ &\qquad \text{neighborhood of } \infty\}.\end{aligned}$$

At the Lie algebra level $(\mathfrak{G}, \mathfrak{G}_+, \mathfrak{G}_-)$ we have $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$. We can decompose a Laurent series converging in $\{\lambda : k < |\lambda| < \infty\}$ into negative powers of λ and nonnegative powers of λ. At the Lie group level, this is called Birkhoff factorization and involves writing a power series as the products of series that converge in the two regions. It fails on the group, but is valid on a dense open subset Ω of what we call we call "the big cell."

6.2. Choice of a vacuum frame. A vacuum frame $V_0 \in C(\mathbb{R}^s, \mathcal{G}^+)$ determines the flows. Different flows result from different choices. For our example

$$V_0 = \exp\left(\sum_{j=1}^{n_0} t_j \lambda^j J\right),$$

$J = \frac{1}{2}\,\mathrm{diag}(\pm i)$ with k +'s and $n-k$ minuses. Here the upper limit n_0 is arbitrarily large. We never use infinite sums.

6.3. The dependent variable in the flow. Our first step is completely formal. Choose an arbitrary element $f \in \mathcal{G}^-$. Factor

$$V_0 f^{-1} = M^{-1} E$$

for $M^{-1} \in \mathcal{G}^-$, $E \in \mathcal{G}^+$.

Note that the scattering data f does not depend on the flow variables $\vec{t} = (x = t_1, t = t_2, t_3)$. Since V_0 does, M and E do as well. We call E the frame and M the reduced frame (also called the Baker function). The frame E has positive powers of λ and the reduced frame M has negative powers:

$$\partial_x M\, M^{-1} + M \partial_x V_0 M^{-1} = \partial_x E E^{-1},$$
$$\partial_x E\, E^{-1} + [M \partial_x V_0 M^{-1}]_+ = \lambda J + [M_{-1}, J].$$

Then $u = [M_{-1}, J]$ (or technically $\partial_x + u$) will be our dependent variable in the flow, where M_{-1} is the coefficient of λ^{-1} the power series of M. Once the theory is sufficiently developed, ∂_x plays the role of the dual of the element defining a central extension of $\mathcal{G}$. Note that u is in $[J, u(n)]$ and the flows generated by this framework are equations for maps $(x = t_1, t = t_2, \dots, t_\ell)$.

If $f = I$, u is identically 0. This is the "vacuum" solution.

6.4. Lax pairs. Let

$$Q = MJM^{-1} = \sum_{j=0} \lambda^{-j} Q_j = J + \lambda^{-1}[M_{-1}, J] + \sum_{j>1} \lambda^{-j} Q_j$$

$$\left(\frac{\partial}{\partial t_\ell} M\right) M^{-1} = [M \lambda^\ell J M^{-1}]_-.$$

The dependent variable for the flow is $u = Q_1 = [M_{-1}, J]$. To get $\frac{\partial}{\partial t_\ell} u$, we compare the coefficients of λ^{-1} to get $\frac{\partial}{\partial t_\ell} M_{-1} = Q_{\ell+1}$, or

$$\frac{\partial}{\partial t_\ell} u = [J, Q_{\ell+1}] = [\partial_x + u, Q_\ell],$$

the *ℓ-th flow in the MNLS hierarchy*.

It is not necessarily true that the Q_j are determined algebraically, which was first proved by Sattinger for our example by using the following two equations.

$$Q^2 = J^2 = -\frac{1}{4} I,$$
$$\partial_x Q = [(\lambda J + u), Q].$$

Lemma 6.4.1. *The Q_j are polynomials in u and its derivatives in x of order up to j.*

The Lax pair formulation is now

$$L = \partial_x + \lambda J + u,$$
$$\tilde{L}_k = \partial_{t_k} + J\lambda^k + \sum_{j=0}^{k-1} \lambda^j Q_{(k-j)}$$

and $[L, L_k] = 0$. For $\vec{t} = (t_1, \dots, t_{n_0})$, we call $E(\lambda, \vec{t})$ a frame for the MNLS hierarchy if $E(\lambda, \vec{t})$ is holomorphic for $\lambda \in \mathbb{C}$ and $L_k E = 0$ or equivalently

$$E^{-1} E_{t_k} = J\lambda^k + \sum_{j=0}^{k-1} \lambda^j Q_{(k-j)}$$

for all $1 \le k \le n_0$. In particular, for $\lambda = 0$, we get $[\partial_x + u, \partial_{t_k} + Q_k(u)] = 0$, or equivalently, u is a solution of the k-th flow $u_{t_k} = [\partial_x + u, Q_k(u)]$ if and only if the system $g^{-1} g_x = u$, $g^{-1} g_t = Q_k(u)$ is solvable for $g : \mathbb{R}^2 \to U(n)$. It also follows that there exists an open subset Ω_- of the negative group $\mathcal{G}^-$ such that we can factor

$$V_0(\lambda, \vec{t}) f^{-1}(\lambda) = M^{-1}(\lambda, \vec{t}) E(\lambda, \vec{t}),$$
$$M(\cdot, \vec{t}) \in \mathcal{G}^-, E(\cdot, \vec{t}) \in \mathcal{G}^+.$$

Then

$$u_f = [M_{-1}, J]$$

is a solution of the MNLS hierarchy and $E(\lambda, \vec{t})$ is the frame for u_f with $E(\lambda, \vec{0}) = \mathrm{I}$.

We compute a few terms:

$$Q_1(u) = u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad Q_2(u) = \begin{pmatrix} -iq^* q & iq_x \\ iq_x & iq^* q \end{pmatrix},$$

$$Q_3(u) = \begin{pmatrix} -qq_x^* + q_x q^* & -q_{xx} + 2qq^* q \\ q_{xx}^* + 2q^* qq^* & q_x^* q - q^* q_x \end{pmatrix}.$$

6.5. Bäcklund transformations and dressing actions. Since factorizations can be carried out on an open dense subset (the big cell) Ω of $\mathcal{G}$, dressing actions are locally defined.

Proposition 6.5.1. *If $E(\lambda, \vec{t})$ is a frame of a solution $u(\vec{t})$ for $\vec{t}$ in a compact domain $\mathcal{O}$, then there exists an open subset Ω_- of $\mathcal{G}^-$ of the identity such that for $g \in \Omega_-$, we can factor $Eg^{-1} = \tilde{M}^{-1}\tilde{E}$ with $\tilde{M} \in \mathcal{G}^-$ and $\tilde{E} \in \mathcal{G}^+$, and*

$$g\sharp E := \tilde{E} = \tilde{M} E g^{-1}$$

is the frame for a new solution in $\mathcal{O}$. In particular, if $u = u_f$ for some $f \in \mathcal{G}^-$, i.e., $E = MV_0 f^{-1} = f\sharp V_0$, then $g\sharp E$ is the frame of u_{gf} and $g\sharp E = \tilde{M} M V_0 (gf)^{-1}$.

There is a fair amount of analysis needed to understand more than this in the case of continuous scattering data. The original results for our example are due to Beals and Coifman, but the proofs and results vary by example.

When $g \in \mathcal{G}^-$ in the above Proposition is meromorphic with a simple pole then the factorization $Eg^{-1} = \tilde{M}^{-1}\tilde{E}$ can be written down explicitly, which gives a Bäcklund transformations. We identify particularly simple meromorphic examples in $\mathcal{G}^-$. For the group we have chosen, with the reality condition $g(\bar{\lambda})^\dagger g(\lambda) = I$, the examples with simple poles are

$$f_{a,\pi}(\lambda) = \pi + \left(\frac{\lambda - \bar{a}}{\lambda - a}\right)\pi^\perp.$$

Here π is the Hermitian projection on a subspace, $\pi^\perp = I - \pi$ is the projection on the orthogonal subspace, and a is a complex number whose imaginary part is nonzero. Note that

$$f_{a,\pi}^{-1}(\lambda) = \pi + \frac{\lambda - \alpha}{\lambda - \bar{\alpha}}\pi^\perp.$$

Given a frame $E = E(\lambda, x, t_2)$, we can find a new $f_{a,\pi}\#E$ for a new solution.

Proposition 6.5.2. *Given a simple element $f_{a,\pi}$ and the frame E of a solution u of the flow, then $\tilde{u} = u + (a - \bar{a})[J, \tilde{\pi}]$ is a new solution with frame*

$$f_{a,\pi}\#E = f_{a,\tilde{\pi}(\vec{t})} E f_{a,\pi}^{-1},$$

where $\tilde{\pi}(x,t)$ is the Hermitian projection onto $E(a,\vec{t})^{-1}(\Im\pi)$.

Proof. Note that the residues of the right-hand side of $f_{\alpha,\pi}\#E$ are zero at $\lambda = \alpha$ and $\lambda = \bar{\alpha}$ if and only if

$$\tilde{\pi}(\vec{t}\,)E(a,\vec{t}\,)\pi^\perp = 0,$$
$$\tilde{\pi}(\vec{t}\,)^\perp E(\bar{a},\vec{t}\,)\pi = 0.$$

These are compatible. To see this, apply the complex transpose to the second equation to see that $\pi E(a,\vec{t})^{-1}\tilde{\pi}^\perp = 0$. But both equations are equivalent to $\pi(x,t)$ being the projection on the subspace orthogonal to $E(a,\vec{t})^{-1}(\Im\pi)$. □

The permutability formulas follow easily from the fact that a product of two such f's with poles at a and b can be factored with either pole first. This yields a rather complicated identity between applications of two Bäcklund transformations which is known as the permutability formula. When Bäcklund transformations are generated by an alternative argument, these permutability formulae are mysterious. Solitons are generated by applying Bäcklund transformations to the vacuum frame V_0 of the trivial zero solution.

6.6. **Symplectic structure and Hamiltonians.** We have generated flows using Lax pairs without mentioning either a symplectic structure or Hamiltonians. These are, however, already embedded in our computations. For $u_1, u_2 \in S(\mathbb{R}, \mathcal{P})$, let

$$\langle u_1, u_2 \rangle = -\int_{-\infty}^{\infty} \mathrm{tr}(u_1 u_2)dx.$$

We start with a basic symplectic structure (more about this later)

$$\theta(u_1, u_2) = \langle [J, u_1], u_2 \rangle = -\langle [J, u_2], u_1 \rangle.$$

If we compute the gradient ∇H of a Hamiltonian H as

$$dH(\delta u) = \langle \nabla H(u), \delta u \rangle,$$

then the Hamiltonian flow for H is

$$u_t = [J, \nabla H].$$

Proposition 6.6.1 ([Ter97]). *The j-th flow is the Hamiltonian flow for*

$$H_j(u) = -\frac{1}{j+1}\int_{-\infty}^{\infty} \mathrm{tr}(Q_{j+2}(u)J)dx.$$

6.7. **Sequences of symplectic structures.** Terng tells me she first noticed this bi-Hamiltonian structure in a very general context in Drinfel'd and Sokolov [DdS84]: not only can we formulate the flows as Hamiltonian flows of a sequence of Hamiltonians, we can fix the Hamiltonian and vary the symplectic structure ([Ter97]).

Recall that our soliton equations have a phase space of rapidly decaying smooth maps $u \in \mathcal{S}(R, [J, u(n)])$, which can be identified as a coadjoint orbit. Coadjoint orbits come equipped with an orbit symplectic form, and the symplectic form we used in Section 5 was exactly this form. New symplectic structures are found by finding embeddings of the phase space in another coadjoint orbit (cf. [Ter97]).

6.8. **Relation between MNLS and GNLS.** It was proved in [TU06] that given $\gamma : \mathbb{R} \to N = G_{(k,n)}$ there exists $g : \mathbb{R} \to U(n)$ such that

$$\gamma = gJg^{-1}, \quad u := g^{-1}g_x \in [J, u(n)].$$

We call such g an *adjoint frame* along γ, and u the *adjoint curvature* defined by g. Moreover, if g_1 is also an adjoint frame along γ, then there exists a constant $c \in K = U(k)\times U(n-k)$ such that $g_1(x) = g(x)c$. Hence the adjoint curvature u_1 defined by g_1 is $u_1 = c^{-1}uc$.

Given $\gamma : \mathbb{R} \to N = G_{(k,n)}$ and $\xi : \mathbb{R} \to u(n)$ such that $\xi(x)$ is tangent to N at $\gamma(x)$, we define

$$\begin{aligned}\nabla\xi = &\ \text{the orthogonal projection of } \xi_x \text{ onto } T_\gamma N\\ = &-\mathrm{ad}(\gamma)^2(\xi_x).\end{aligned}$$

A direct computation implies that $\gamma_x = g[u, a]g^{-1}$ and $\nabla^{(j)}\gamma_x = g[u_x^{(k)}, a]g^{-1}$. Hence we have

$$gu_x^{(j)}g^{-1} = \mathrm{ad}(\gamma)(\nabla^j\gamma_x).$$

We have seen that the adjoint curvatures u_1, u defined by adjoint frames g and g_1 along γ are related by $u_1 = c^{-1}uc$ for some constant $c \in K$. Hence $g[Q_j(u), J]g^{-1} = g_1[Q_j(u_1), J]g_1^{-1}$. This implies that

$$\gamma_{t_j} = g[Q_j(u), J]g^{-1} \qquad ((*)_j)$$

defines a flow on $C^\infty(\mathbb{R}, N)$ (independent to the choice of adjoint frame) and is the j-th flow in the GNLS hierarchy.

Use the formulas for $Q_j(u)$ given in Section 6 to see that the first three flows are

$$\gamma_{t_1} = \gamma_x,$$
$$\gamma_{t_2} = [\gamma, \gamma_{xx}],$$
$$\gamma_{t_3} = -\nabla^2\gamma_x - (\gamma_x)^3.$$

Notice that the second flow is the GNLS.

Theorem 6.8.1 ([TU06]). *If $\gamma(x,t)$ is a solution of $((*)_j)$, then there exists $g : \mathbb{R}^2 \to U(n)$ such that $g(\cdot,t)$ is an adjoint frame along $\gamma(\cdot,t)$ such that $u := g^{-1}g_x$ is a solution of the j-th flow $u_t = [Q_{j+1}(u), J]$ in the MNLS hierarchy. Conversely, given a solution $u(x,t)$ of the j-th flow, let $g : \mathbb{R}^2 \to U(n)$ be a solution of $g^{-1}g_x = u$ and $g^{-1}g_t = Q_j(u)$. Then $\gamma(x,t) = g(x,t)Jg(x,t)^{-1}$ is a solution of $((*)_j)$. Hence the Hamiltonian theory of the MNLS can be translated to that of the GNLS.*

6.9. **A sketch of the construction of τ functions.** Integrable systems (of the type of KdV) appear in conformal quantum field theory, and τ functions are the partition functions. A τ function is a function associated to a solution $u(\vec{t}\,)$. Terng found a definition due to Wilson which applies to all the systems under discussion. The second derivatives of u can be derived from τ; however, she also discovered that in the general case, the map from $u \to \tau$ is not injective. The definition of τ using second derivatives of u appears in the physics literature [AvdL03].

The general definition of τ for a system depends on a choice of central extension of the Lie group $\mathcal{G}$, which in turn depends on a skew symmetric bilinear form w on its Lie algebra $\mathfrak{G}$ which is compatible with the splitting $\mathcal{G} = \mathcal{G}^+\mathcal{G}^-$. The choice of the bilinear form for our example is

$$w(A,B) = \left(\frac{\partial}{\partial\lambda}A, B\right) = \sum_j j\,\mathrm{tr}(A_j, B_{-j}).$$

Here both $\mathfrak{G}_\pm$ are isotropic subspaces for w.

The central extension $\hat{\mathcal{G}}$ is the principal $\mathbb{C}\setminus 0$ bundle over $\mathcal{L}$ with first Chern class w. Because $\mathcal{G}^\pm$ are isotropic subspaces for w, the central extension is canonically trivial over $\mathcal{G}^\pm$, so there are canonical liftings of $\mathcal{G}^\pm$ to $\hat{\mathcal{G}}$. We use this to define Wilson's μ functional over the big cell Ω in which factorizations occur: For $c \in \Omega$, factor $c = f_+f_- = g_-g_+$ with $f_\pm, g_\pm \in \mathcal{G}^\pm$. Then $\tilde{f}_+\tilde{f}_-$ and $\tilde{g}_-$ and $\tilde{g}_+$ lies in the same fiber, where $\tilde{f}_\pm$ and $\tilde{g}_\pm$ are natural lifts of $f_\pm, g_\pm$. Hence they differ by a nonzero complex number.

Definition 6.9.1. For $c \in C$, define $\mu(c)$ as the difference between the canonical lifts given by two factorizations.

Definition 6.9.2 (Wilson). Let $E = MV_0f^{-1}$ represent a (partial) solution to an integrable system. Then the tau function defined by $f \in \mathcal{G}^-$ is

$$\tau_f(\vec{t}\,) = \mu(V_0f^{-1}).$$

The collective papers of Terng and myself work out many of the details. For example, we prove in [TU16] (i) the solution u_f is determined by the second derivatives of $\ln(\tau_f)$ for the the Gelfand-Dickey n-KdV hierarchy, and (ii) the second derivatives of $\ln\tau_f$ determines the solution u_f up to a conjugation by a constant in $U(k)\times U(n-k)$.

6.10. **The Virasoro action.** We found in the previous section that the τ function was in fact a function on the scattering data. The same is true for the Virasoro action. Again, we only give the prototype for the example we are using.

Definition 6.10.1. The Virasoro algebra is the real Lie algebra $\mathcal{V}$ with generators ξ_j and relations

$$[\xi_j, \xi_k] = (k-j)\xi_{k+j}.$$

The positive algebra $\mathcal{V}_+$ is the subalgebra generated by $j \geqq -1$.

It is an important fact that the Lie algebra of the conformal group of S^2 is generated by $\{\xi_{-1}, \xi_0, \xi_1\}$. It is best to think of $\mathcal{V}_+$ as generated by the operators

$$\xi_j = \lambda^{j+1}\frac{\partial}{\partial\lambda}, \quad j \geq -1.$$

Again, we only give the action on our particular choice of splitting.

Lemma 6.10.2. *The formulae for a representation of $\mathcal{V}_+$ on f is*

$$\delta_j f(\lambda)f^{-1}(\lambda) = \textit{negative powers of } (\lambda^j f'(\lambda)f^{-1}(\lambda)).$$

Note that we will be able to compute the induced representation on the flow variables from

$$E = MV_0f^{-1},$$
$$\delta_jE = \delta_jMM^{-1}E - E\delta_jff^{-1}.$$

Theorem 6.10.3 ([TU16]). *The action of the positive half of the Virasoro algebra on the scattering data f induces an action on the flows. In many cases this can be shown to be via partial differential operators.*

7. Terng's Contributions

The previous section was an attempt to convey the ideas and techniques used in investigating integrable systems. Explaining in detail which piece is due to which mathematician would not make for interesting or enticing reading, although I have attempted to cite the major references. I have also cited a number of Terng's publications. Many of those are joint with me, but anybody familiar with me will reinforce my assertion that the bulk of the geometry is due to Chuu-Lian. We cite here some of her most important results.

Chuu-Lian served as AWM president [Ter22] from 1995–1997 and a co-organizer of the Women and Math Program at the Institute for Advanced Study in Princeton. Her influence on mathematics is evident in many ways.

The observation that the gauge transformation between the matrix nonlinear Schrödinger equation and the geometric nonlinear Schrödinger equation used by Fadeev-Tahkajen and Hasimoto is an application of moving frames and the details surrounding this is due to her, as is the worked out relationship between the modified wave map and the space-time monopole equation. In order to get geometric realization of Drinfel′d-Sokolov KdV-type systems associated to a noncompact simple Lie group G, she constructed curve flows on flat space with the symmetry group G. For example, affine curve flows on $\mathbb{R}^n$ with symmetry $\mathrm{SL}(n, \mathbb{R})$ are related to the n-KdV system ([TZ19]).

Terng made many contributions to the scheme of the DS flow described in the previous section. The details of applications to the various symmetric spaces are all hers. But the novel contribution is the extension of the 2×2 KdV flow to $n \times n$ KdV flows using the structure of the Lie group $\mathrm{SL}(n, \mathbb{C})$. We describe briefly the 2×2 KdV flow.

Here the major change is in the first step, the choice of the Lie groups. Instead of restricting the flows to the subgroups in which $g(\bar{\lambda})^\dagger g(\lambda) = I$, we introduce a reality condition on g:

$$G(g, \lambda) = \Phi(\lambda) g(\lambda) \Phi(\lambda)^{-1} G(g, \lambda) = G(g, -\lambda),$$

where $\Phi = \mathrm{diag}(1, -1)\lambda + e_{2,1}$.

Now the flows are generated by the vacuum sequence

$$V_0 = \exp\Big(\sum_{j \geq 0} J^{2j+1} t_j\Big),$$

where $J = \mathrm{diag}(1, -1)\lambda + e_{1,2}$.

The extension to $\mathrm{sl}(n, \mathbb{C})$ turns out to generate the Gelfand-Dickey or n-KdV flows. If we let $\alpha = e^{(2\pi i/n)}$, we similarly define a $G(g, \lambda)$ with a different choice of Φ, and require

$$G(g, \lambda) = G(g, \alpha\lambda).$$

Now the vacuum sequence will be

$$V_0 = \exp\Big(\sum_{j \neq 0 \bmod n} J^j t_j\Big),$$

with $J = \mathrm{diag}(1, \alpha, \ldots, \alpha^{n-1})\lambda + \sum_{k=1}^{n-1} e_{(k,k+1)}$.

The definition of Φ is very complicated involving the roots of $\mathrm{sl}(n, \mathbb{C})$. See Theorem 2.1 of [TU11]. However, one can see from that description that it can be extended to loop groups constructed from other simple Lie algebras.

Theorem 7.0.1. *With the appropriate choice of Φ, the DS flows correspond to the Gelfand Dickey n-KdV integrable systems. In particular, the τ function and Virasoro actions correspond to the constructions of Segal-Wilson* [SW85].

This latter result is most satisfactory. The Segal-Wilson construction uses pseudo-differential operators on the line, and does not involve Birkhoff factorization. Note in these examples involving complicated splittings, the map from flows to τ functions is injective, meaning the τ functions determine the flow. As just commented, with some work, this construction may generalize to other Lie groups and root systems.

8. Interesting Questions and Open Problems

The approaches of Terng and her collaborators to integrable systems, when they are rigorous, involve rapidly decaying fields on the line (or $\mathbb{R}^2$). Any question involving periodic boundary conditions is intimately tied up with algebraic geometry. The formal construction of the flows is the same. However, the factorization

$$V_0 f^{-1} = M^{-1} E$$

cannot be made global so easily, even in specially chosen examples. This is not an open problem, but a problem which is addressed by a separate literature.

As we comment at the end of the previous section, the Drinfel′d-Sokolov scheme of Section 6 appears to have a good many abstract applications in geometry and an internal consistent structure derived from loop groups and Birkhoff factorization. The problems we discuss now are less abstract and have more to do with physical applications. The approach of integrable systems, with rapidly decaying initial data, has very little overlap with questions of analytic well-posedness. The goal of well-posedness is to prove existence and smooth dependence on initial data of the flows for small times t_j, $j > 1$ in Sobolev spaces of minimal regularity and decay, and then to separately investigate finite time blow-up and the behavior at infinity. To study well-posedness, it is necessary to choose a suitable gauge. An integrable system such as the space-time monopoles equation can look very different in different gauges [Czu10], [HY18]. There is also extensive work by authors such as Deift [Dei19], [DZ02] on numerics for Riemann Hilbert problems which is less familiar to me. It should be possible to combine these approaches into an organized scheme which takes advantages of what I might call the geometric, the analytic and the algebraic approaches to integrable systems.

It should not be impossible to approximate other nonlinear equations (i.e., harmonic maps in 1 + 2) by integrable ones (i.e., Ward wave maps) and derive new approximation schemes for certain nonlinear equations using integrable equations. Much as we study nonlinear equations using their linearizations. However, I am not myself aware of any references.

My own favorite problem at the moment is the question of integrability for the Gross-Pitaevskii hierarchy, an infinite hierarchy of coupled linear inhomogeneous PDE appearing in the derivation of nonlinear Schrödinger from quantum many-particle systems. In fact, the entire hierarchy of flows appears in describing special solutions. Recently Mendelson, Nahmod, Pavlovic and Staffilini [MNPS19] have shown the existence of an infinite number of conservation laws, but as mentioned in the very beginning of Section 5, this is not equivalent to showing integrability. And in fact, it is not clear what integrability is. The key tool in their analysis is the existence of an invariant measure (due to Finetti). From the point of view of the integrable systems in this article, the measure itself would be the key ingredient to describing the geometric structure of the GP hierarchy. While the problem has interested many analysts, it cries out for the geometric interpretation present in Terng's work.

We return to a point addressed in Section 5. What is an integrable system? Certainly any system of the Drinfel'd-Sokolov type addressed in Section 6 is integrable. The modified wave map, monopoles and self-dual Yang-Mills are all equations with Lax pairs and posses at least some of the same properties, as do equations in higher dimensions which fail to be "algebraic" in the sense of algebraic geometry, but which also have special rigidity properties. Perhaps we should think of integrable as referring to the circle of ideas presented in this article, rather than having a precise definition (we mathematicians do love precision).

The final point is a philosophic one. Integrable systems are very specialized algebraically rigid equations. Why do they appear in so many subjects of mathematics? The systems considered by Terng and collaborators can most easily be described by the geometry connected with finite- or infinite-dimensional group theory. However, KdV arises in the study of shallow water waves, nonlinear Schrödinger arises in the description of waves in optical fibers and the equations of 2-dimensional gravity are integrable. Even more confusing, space-time monopoles and Ward wave maps are gauge equivalent. How should we regard the subject?

If we were describing fundamental particles in physics, we wouldn't be surprised if they corresponded to equations determined by the geometry of loop groups. And certainly there was a short period of time in which this was a hope. So I am not surprised that integrable systems are important in conformal field theory. But these are somewhat esoteric equations appearing in completely different areas of mathematics.

There are two wildly different guesses as to why this is. Maybe, in some Platonic sense, these equations are all that is available to our human consciousness. So in modeling phenomena, we mathematicians can only use what is there. The opposite view would be that these equations represent fundamental phenomena in the world we are trying to describe. Hence we will use them in many different applications to describe this same behavior.

The subject of integrable systems and Terng's work in it can be appreciated without knowing the answer.

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Karen Uhlenbeck

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Catherine Goldstein: The Question of Long-Term Histories and Mathematical Identity

Jenny Boucard and Jemma Lorenat

1. Introduction: Configurations in the History of Mathematics

"Mathematics is the art of giving the same name to different things." With this quote from Henri Poincaré's *The Future of Mathematics* (1908), Catherine Goldstein opened the plenary conference on the history of mathematics for the 2018 International Congress of Mathematicians.[1] It underlines two important points. First, at the beginning of the twentieth century, this question of absorbing mathematical objects under the same name—and the same general concept—was central to the gradual establishment of a theory of structures. Second, this point of view also played a important part in the construction of a retrospective history of mathematics, by looking to the (sometimes very distant) past for traces of the objects of current mathematics [Gol19a]. Goldstein is one of several historians who, particularly since the 1980s, have been working on a new history of mathematics, integrating developments in the history of science and technology while taking serious account of the specific nature of mathematics. Two interrelated motifs run through Goldstein's research: the identity of mathematics over time and the implications of long-term histories.

Currently research director (*directrice de recherche*) at the Institut de mathématiques de Jussieu-Paris Rive Gauche in Paris, Goldstein began her research career in pure mathematics. She was one of the first French students to complete a thesis in number theory under the supervision of John Coates at Université Paris-Sud.[2] She defended her doctoral thesis on p-adic L-functions in 1981, just as she obtained a tenured position as a researcher in the Centre national de la recherche scientifique at the Mathematics Laboratory of the Université Paris-Sud. For historians of science, the 1980s "crackled with debate"—as historian Lorraine Daston recounts—invigorated by ethnographic and sociological approaches, feminist theory, and political movements [Das09, 803]. Interested in the humanities and social sciences, Goldstein seized the opportunity to attend a series of lectures on the history, sociology, and philosophy of science organized by the philosopher Michel Serres, along with other academics such as Bernadette Bensaude-Vincent, Bruno Latour, and Isabelle Stengers. This seminar became a collective book, *A History of Scientific Thought: Elements of a History of Science* (published in 1989 in French, translated into English in 1995), for which Goldstein wrote two chapters. This experience was a turning point for her.

The scholars involved in this collective project desired to go beyond the so-called internalist ("that of concepts

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[1]*Both authors are former doctoral students of Catherine Goldstein. We would like to thank Catherine for the opportunity to write this article and her support in its research. Thank you also to Laura Turner for suggesting this piece and for her editorial guidance.*

[2]*Coates, who had studied at the École Normale Supérieure, held positions in the United States and Australia prior to joining the faculty at Paris-Sud.*

and results" [Gol19b, 1]) versus externalist ("that of institutions or scientific politics") approach to history by showing the inextricable interweaving of "internal contents which are exclusively scientific and external conditions which are exclusively social" [Ser95, 14] in the history of science and technology. Goldstein, along with other historians of mathematics, was convinced that the specific nature of *mathematics* (as compared to the natural sciences) called for the use of appropriate historical methods. The aim of this paper is to present some of the methods that Goldstein developed to obtain new results in the history of mathematics.

In her introduction to the second volume of Ernest Coumet's *Collected Works*, Goldstein recalls:

> For many of us, it seemed that it was time to move on [from the internalist/externalist debates]: that it was appropriate to approach the texts of the past as closely as possible, because it was in their technical details, in their organisation, both material and intellectual, at the bend of an unexpected word or the unexpected meaning of a word, an imported notation, an approximation of classifications or formulations, that the traces of a wider culture, of a hitherto unsuspected collective, could be caught; and, in the opposite direction, that it was necessary to examine how these traces were imprinted in symbolism, in a particular type of exercise, in priorities selecting the direction of the development of mathematics. Everything here remained to be done. [Gol19a, p. 11, to verif.]

This program is already apparent in the two chapters written by Goldstein for *Elements of a History of Science*. The first, titled "Stories of the Circle," reveals the unexpected heterogeneity of seemingly uncontroversial pieces of mathematics. Here, the plural of "stories" is essential: it underlines the multiplicity of historical lines, as opposed to the idea that there is *one* single, linear history—the true history—of mathematical concepts. In this chapter, Goldstein proceeds from Mesopotamia and Egypt to Hilbert's *Foundations of Geometry* identifying multiple precise ways that the circle has manifested within mathematics. The circle in the *Śulvasūtra* (first millennium BCE, India) is introduced with the constructive and numerical procedure of converting a square into a circle of equal area, the circle in Euclid's *Elements* (third century BCE, Greece) is a plane figure like the straight line without numerical values, while the algebraic approach to the circle (for instance, in the work of Descartes in the seventeenth century) identifies it "with the conics, whose equations are of the same type" [Gol95a, pp. 182–184]. Such reorganization has profound consequences. To take one among many, the circle as a conic section leads to the introduction of circular points at infinity.

The second chapter deals with the question of "Working with Numbers in the Seventeenth and Nineteenth Centuries," going from what Goldstein calls *Homo ludens* to *Homo faber*. Starting from the "same" mathematical statement in the seventeenth and nineteenth centuries—what is now known as Fermat's last theorem, namely the impossibility of the equation $x^n + y^n = z^n$ in nonzero integers—Goldstein shows how different were the epistemological, mathematical, and social contexts involved. Number theory is a particularly good case for historical investigation:

> The choice of mathematics and more particularly number theory as a field of study is in this respect more interesting for being an extreme case: among the sciences, mathematics, and the theory of numbers in particular, retains, rightly or wrongly, a reputation for splendid and unchanging isolation which discourages over-hasty explanation of its professional development. [Gol95c, p.345]

Number theory, both as an "extreme case" and as a familiar field, has been Goldstein's topic of historical investigation *par excellence* since the 1980s, even as her research has also encompassed other themes in the history of mathematics. The two chapters published in [Ser95], investigations of a mathematical object and a mathematical domain over a long period of time, also reflect two central questions in Goldstein's work: how to construct long-term histories of mathematics and to how to analyze the identity of mathematical objects. These issues are also fundamental in her keynote address for the 2018 International Congress of Mathematicians, titled "Long-Term History and Ephemeral Configurations." Goldstein motivates her choice of the history of Hermitian forms in the second half of the nineteenth century as "a minimal example: the concrete case of a rather technical and apparently stable concept" [Gol19b, 493].[3] She based her presentation on the sociological concept of configuration, which she first borrowed and adapted from Norbert Elias in her early work in the 1990s. Elias introduced the concept of *configurations* in the early twentieth century to study interdependent networks between individuals. Goldstein first adapted configurations of "knowledge, objectives, and priorities" to inform a social history of texts [Gol95b, p. 9]. She describes a *configuration* as the interaction of texts and persons in which "mathematics weaves together objects, techniques, signs of various kinds, justifications, professional lifestyles, epistemic ideas" [Gol19b, 491]. Marking these configurations as ephemeral does not restrict the history of mathematics to local studies but rather is a call

[3] *Goldstein defines Hermitian forms in this address as "simply an expression of the type* $\sum_{i,j=1}^{n} a_{ij} x_i \bar{x_j}$*, with coefficients* a_{ij} *in* $\mathbb{C}$*, such that* $a_{ij} = \bar{a_{ij}}$ *(here, the bar designates the complex conjugation); in particular, the diagonal coefficients* a_{ii} *are real numbers."*

to investigate "the various ways in which they [local studies] are, or not, connected" [Gol19b, 511] thereby forming long-term histories of mathematics.

While the details are crucial to fully appreciating the value of this scholarship, here we focus on three examples of Goldstein's research to provide a sense of how methodological frameworks can address questions of long-term histories and identity through methods and concepts—such as configurations—adapted from the human and social sciences. For a deeper investigation, the reader is recommended to read the original texts as cited in the bibliography.

2. Mathematical Versus Historical Readings: One Theorem and Its Multiple Identities

In the conclusion to *Un théorème de Fermat et ses lecteurs* (*A Fermat Theorem and Its Readers*), Goldstein traces the history of her book back to 1988 and "a question from a colleague, Norbert Schappacher, about the identity of Fermat's and Frenicle's proofs" [Gol95b, 177]. The theorem in question—that the area of a right triangle with rational sides is not a square—was proved by Pierre de Fermat and Frenicle de Bessy at around the same time in the seventeenth century. At first Goldstein supposed that the proofs were "similar": Fermat and Frenicle had similar number-theoretic knowledge, they wrote without algebraic symbolism, used the method now known as infinite descent. But then she began to have doubts, not just about the texts, but about the stability of similarity.

To address this notion of stability and answer the central question of her book—what a "historical description of a mathematical theorem" can be—Goldstein identifies two categories of readings: "mathematical readings aim to find problems to solve, methods to use, in brief, a source of inspiration to produce new mathematics"; whereas historical readings (perhaps by the same people!) "search in the text for information about the activity of past mathematics and its development in order to produce historical reflections and commentaries" [Gol95b, 6–7]. She is then able to "observe at close hand the concrete reading and working practices" of mathematicians and historians [Gol95b, 16]. In presenting many mathematical and historical readings of the famous theorem, Goldstein demonstrates how perceptions of mathematics are shaped by prior mathematical knowledge. She directs attention to the slippery question of identity (which becomes a problem to be studied rather than a self-evident given), the plurality of contexts, the inadequacy of historiographical dichotomies.

In spite of its geometrical appearance and apparent specificity, the theorem, considered by Fermat and Frenicle as about numbers, can be seen as a "cornerstone (...) of the study of quadratic forms (...and) of the arithmetic of elliptic curves" [Gol95b, 5]. This position has important effects for both types of reading. For example, Goldstein analyzes how the rewriting of Fermat's and Frenicle's proofs in algebraic symbolism erases specifities of each text. An algebraic rewriting requires the clarification of initially ambiguous points, such as the relationships between elements introduced as a reasoning progresses, and standardizes initially differentiated terminologies [Gol95b, 88–89].

Goldstein compares the two texts "point by point" [Gol95b, p.76] and shows how a nonalgebraic reading reveals the differences in Fermat's and Frenicle's approaches. Both proofs rely on a method of infinite descent, but each author had different conceptions of the process. Frenicle considered the decrease in size of the triangles, the next is smaller than the last "diminishing always" until forced to stop at 1. Fermat, by contrast, first asserted that there can only be a finite number of whole numbers smaller than a given number without reference to the triangles.[4] For the study of right triangles, Frenicle's approach is "more direct, shorter, more economical" while that of Fermat is "richer in varied connections with other problems" [Gol95b, 80].

Goldstein also discusses a historical reading of Fermat's theorem based on elliptic curves, showing how this approach "gives a unified meaning to a corpus of apparently heterogeneous statements" [Gol95b, 100]. Goldstein notes that this second kind of reading is more controversial because the concepts involved are a priori more distant than the content of the original texts. But she highlights that all readings she studied are "strictly speaking anachronistic" [Gol95b, 106]. More subtle instances of anachronism underscore the fragility of identity of mathematics. As a case in point, neither Frenicle nor later historical readings provide any justification that "if a product of two prime numbers together is a square, each of the two numbers must be so." Does this assertion point to the mathematical knowledge of the seventeenth century or is it a feature of the style of his *Traité des Triangles Rectangles en Nombres*? Goldstein asserts that ignoring the lack of justification or replacing the result would be to tacitly replace "the mathematical knowledge of the original readership by another" [Gol95b, 59].

Evaluating a text depends on the configuration in which it is read. That is, "the similarity between two mathematical texts, whether contemporary or distant in time, has no absolute significance" [Gol95b, p.178]. These configurations may articulate "a general conception of the history of mathematics and of the subject to be dealt with (...); technical knowledge; a vision of the state of the mathematical field concerned, in the past and in the present, in particular problems or texts whose connection with the one

[4] *As a further example of multiplicities of readings, Goldstein notes that Fermat's use of infinite descent has been read as anticipating the modern height function* [Gol95b, 11].

studied may be relevant; an individual reading of the texts; a chronology" [Gol95b, 108]. In particular, the construction of a long-term historical narrative depends on the situation of the historian and their choices accordingly. It is in this sense that "the so-called long-term histories of mathematics are local histories" [Gol95b, 111].

3. Men, Letters, and Numbers: A Micro-Social Analysis to Solve Fermat's Paradox

> You sent me 360, whose aliquots[5] are the same number as 9 to 4, and I'm sending you 2016, which has the same property. —Letter from Fermat to Mersenne, February 20, 1639

In his history of mathematics "from Ancient to Modern Times," Morris Kline states that "Fermat's work in the theory of numbers determined the direction of the work in this area until Gauss made his contributions." He then claims that Fermat "had great intuition, but it is unlikely that he had proofs for all of his affirmations" [Kli72, 274–275]. More generally, contrasting biographical accounts on Fermat bring to light an apparent paradox: sometimes presented as a theorist, his correspondence on number theory reveals a number of particular statements, without proofs and where the methods are very rarely made explicit. This combination led one of his biographers, Michael S. Mahoney, to describe him as a "problem-solver."[6] The quotation in the epigraph to this section is in fact a typical example of Fermat's exchanges on number theory in his letters.

For Goldstein, two pitfalls explain the paradox that seems to punctuate Fermat's biography: a too narrow focus on Fermat and neglect of the fact that certain characteristics of his work are not specific to him but are, on the contrary, characteristic of the environment in which he evolved. Our current knowledge of Fermat's work on number theory is based on the version annotated by Fermat of Bachet's edition of Diophantus's *Arithmetica*, published by Fermat's son in 1670 as well as a collection of letters, handwritten and partially published between the seventeenth and twentieth centuries, and mostly exchanged in the context of Marin Mersenne's network. Fermat was then a wealthy magistrate who worked in Toulouse and Castres. He was in contact with the learned circles of Bordeaux and Paris, and corresponded with Mersenne, René Descartes, Bernard Frenicle de Bessy, Blaise, Pascal, and Gilles Personne de Roberval, among others.[7] Here again, studying the *configuration* in which Fermat's texts were produced allows the resolution of these apparent biographical contradictions. Goldstein engages in a micro-analysis of Mersenne's correspondence as a "specific social space for mathematics" [Gol09, 41], studying letters as "both social links and texts" [Gol13, 254] and thereby underlying the interdependance between mathematical content and "paratextual and social features."

Several important results are obtained by this socio-historical approach of mathematical texts. The specific form of exchanges on number theory in Mersenne's correspondence—where sharing challenges and particular problems prevail—can be explained: first, the different participants in letter networks do not usually know each other and this allows the exchange of mathematics without disclosing methods before a relationship of trust is established. It also has to be recalled that most of the correspondents were amateurs in mathematics, in the sense that they earn their living from another profession, which left them little time to work on questions of numbers—numerous correspondents, including Fermat, Descartes, and De Beaune, explicitly complained about a lack of time. Secondly, this form enabled assessing the difficulty of a question and the value of the solution method without making it explicit. A lexicographical analysis of the network of letters (in which the word "method" appears very frequently—"several dozen occurrences in Fermat's correspondence" [Gol95b, 135]) demonstrates the strategic intention in the statement of problems, which might appear disparate at first glance.

A global study of Mersenne's correspondence does not only make it possible to catch the collective rules in this epistolary academy; it also makes it possible to distinguish between what is specific to an individual and what comes under the norms and values constituting a collective. First, the fact that Fermat exchanged particular number problems confirms that he had a thorough understanding of the collective norms of Mersenne's correspondence. Then, Fermat occupies a singular place. He was the only one who mastered algebra and arithmetic and he was very familiar with the workings of the Mersenne academy. He transformed his statements so that they were in the desired form and adapted his questions to his readership: if he sent a problem to an algebraist like Descartes, for

[5] *The aliquot parts of a number are the divisors of that number, except itself. In the case of Fermat's example here, the aliquot parts of 360 are: 1, 2, 180, 3, 120, 4, 90, 5, 72, 6, 60, 8, 45, 9, 40, 10, 36, 12, 30, 15, 24, 18, 20. Their sum is 810. We then have:* $\frac{810}{360} = \frac{9}{4}$.

[6] *See* [Gol09, 25–27]. *Goldstein introduces her paper with a paradoxical fictional biogaphy of Fermat constructed from pieces taken from a dozen biographies and writings by the mathematician to underline this apparent paradoxal aspect of Fermat's historical persona, among others. Note that what follows concerns Fermat's number theory, but many of these results are also valid for his work in geometry and probability.*

[7] *Mersenne organized, between 1617 and his death in 1648, a network of mathematicians, first with meetings in Paris, then in the form of a network of correspondents linking mathematical amateurs in French and European cities. At least 2,000 letters were exchanged between 180 correspondents between 1635 and 1648 in this "all-mathematical academy." At a time when mathematical journals did not yet exist, this network was fundamental to mathematical communication, especially for those who did not live in Paris or near other practitioners in capital cities.*

example, he made sure that the question could not be solved by algebra, whereas his questions for an arithmetician like Frenicle could only be solved by algebra. This gave him a mathematical superiority even if he was in a marginal geographical position and it allowed him to prove (to others but also to himself) that his methods were more efficient than algebra or the arithmetical methods used by Frenicle. This kind of social history of mathematics therefore shows Fermat "occupying a special position in a social configuration of knowledge practices, where he dominated the operation and specific features" [Gol09, 55] and at the same time resolves apparent contradictions in historical images of Fermat.

4. From Great Men to Large Numbers? Capturing the Collective in Nineteenth-Century Number Theory through Citation Networks

> Our main goal in this part has been to reshape the global representation of the history of the D[isquisitiones] A[rithmeticae] [...]. We were not satisfied with the usual summary—a period of latency and awe, then a succession of a small number of brillant contributors, one or two per generation, who were deeply involved with the book, and finally the blossoming of algebraic number theory at the turn of the twentieth century. What we wanted was to pay more attention to the relations among mathematicians (and among their results), and to the actual mechanisms of knowledge transfer, in particular from one generation to another, that is, to understand some part of the dynamics in the changing role of the D. A. and of number theory. [GS07b, 97–98]

This quotation appears in the conclusion of the first two long chapters by Goldstein and Schappacher introducing the collective book *The Shaping of Arithmetic after C. F. Gauss's Disquisitiones Arithmeticae* (2007). In understanding the dynamics of number theory publications in the nineteenth century, Goldstein pursues a double ambition: to go beyond the "great number-theorists," mostly German, making up most of the stories about number theory[8] and to develop and test suitable methods for the case under study, namely, identifying and analyzing a corpus of nineteenth-century number-theoretic texts. From the systematic exploration of research journals containing mathematics—which were booming throughout the period—and several review journals created in the second half of the nineteenth century such as the *Jahrbuch über die Fortschritte der Mathematik*, she has studied number theory over the long nineteenth century from several corpora of texts: first, notes in the weekly proceedings of the Paris Academy of Sciences [Gol94], then 3,500 articles and books reviewed in the *Jahrbuch*, [Gol99], and finally references to Gauss's *Disquisitiones arithmeticae* in major journals and complete works of identified mathematicians published from 1801 to 1914 [GS07a, GS07b].

In purely quantitative terms, the initial results of these surveys already contrast strikingly with the standard image mentioned above: a handful of German mathematicians are replaced by dozens of authors of various nationalities working on a wide variety of number-theoretic subjects. However, despite an interest in quantitative methods for the study of collective phenomena, Goldstein underlines the extent to which their application, relatively common in social history, poses particular problems for nineteenth-century mathematics, for example due to the difficulty of "gathering (...) homogeneous series of sufficient size" [Gol99, 195]. Questions arise at every stage of such a historical investigation. How do we select the relevant texts, or in other words, how do we decide what is and isn't number theory? The "actors' point of view" is heterogeneous over the period under consideration, with classification categories depending on the journals considered and evolving over time. Another example: purely quantitative approaches put the texts in the corpus under study on the same level, whether they are extensive academic works, a brief research note (the classic format of the weekly proceedings of the Paris Academy of Sciences) or a few lines of response to a question posed in a journal aimed at mathematics amateurs and teachers (such as the *Educational Times*). All these categories of text provide relevant information, provided that their form and links with other texts are taken into account. To create a social analysis of texts, Goldstein applied a form of network analysis to these corpora. However, the heterogeneity of the texts identified and the disparate citation practices in the nineteenth century preclude any automated processing of such a corpus. On the contrary, it is necessary to work on the basis of qualified links (i.e. differentiating between quotations of a result or a method, of homage or opposition, cf. [Gol99, 203–204]), taking into account explicit and implicit quotations (such as the vague mention of a result). It is then possible to identify clusters of texts, "characterized by strong intercitation" [Gol99, 205], by combining computerized sorting and search capabilities with manual groupings.

This approach, based on the analysis of qualified quotations, renews our understanding of "the shaping of arithmetic." First of all, the status of number theory, its objects

[8] *From this point of view, the description given by André Weil and taken up by Goldstein* [Gol99, 189] *as an example of this kind of heroic story is particularly representative: "The great number-theorists of the last century are a small and select group of men. The names of Gauß, Jacobi, Dirichlet, Kummer, Hermite, Eisenstein, Kronecker, Dedekind, Minkowski, Hilbert spring to mind at once. To these one may add a few more, such as the universal Cauchy, H. Smith, H. Weber, Frobenius, Hurwitz"* [Wei75, p. 1].

and methods, are far from stable. Goldstein and Schappacher distinguish two main periods in the reception of Gauss's *Disquisitiones arithmeticae*. The very early reception of the work was very timid and rather French. Attempting to expand algebra (in the sense of equation theory), mathematicians mobilized the arithmetic tools of Gauss's section VII, which presents an algebraic method for solving binomial equations, also providing the conditions of constructibility of regular polygons with ruler and compass. In the 1820s, new readers of Gauss's work became familiar with *Disquisitiones arithmeticae* as part of their mathematical training, including Jacobi, Dirichlet, Galois, and Libri. The numerous research projects undertaken at the time can be grouped under a single, international *research field* in the sense of the sociologist Pierre Bourdieu [GS07a, 52], that Goldstein and Schappacher called *Arithmetic Algebraic Analysis*. Contemporary research in analysis, for example, on complex numbers, Fourier analysis, and elliptic functions, is used to address a number of arithmetical questions, such as those developed by Gauss in his work on residues, reciprocity laws, and the theory of quadratic forms. Here, the field of research is not organized around a mathematical object per se, but is delimited by a set of tools and methods shared by the various players. However, their practices are not uniform, and their objectives are often different. Note that many of them develop a common discourse on the unity of mathematics, which gives legitimacy to number theory: the unity here comes from the possibility of being able to build links between different mathematical objects and different proofs, of being able to transfer methods from one domain to another.

Then, for the period 1870–1914, three main text clusters are identified, none of them linked directly to algebraic number theory. As for the previous period, they do not fall within strictly national frameworks. The first cluster, named "L–G cluster" (Legendre and Gauss), attracted authors from a wide range of social positions (academics, teachers, engineers, military personnel, amateurs, etc.), mostly British, French, and Italian. The texts referred to both Legendre and Gauss, and avoided any recourse to analysis. This network enabled nonacademic authors to participate and constituted a gateway into circles where advanced number theory had no place. Nevertheless, most of the authors had no students, and the number of papers of articles in this cluster declined rapidly. At the end of the nineteenth century, they turned to media not taken into account in the network analysis conducted so far, such as the congress reports of the *Association française pour l'avancement des sciences*, whose aim was to promote useful and entertaining science. The authors and arithmetic practices identified in this cluster have been the subject of a number of recent studies in the history of teaching and the history of amateurs in sciences. A good example is the highschool teacher Édouard Lucas (1842–1891), who developed a "fabric geometry" that applies number theory to textile issues, around the figure of the chessboard. Along with other mathematicians, teachers, and engineers in particular, he promoted mathematical recreations and a visual number theory, not well represented in academic circles, but adapted to pedagogical issues and the desire to popularize mathematics at the time [Déc07]. The second main cluster, the "D-cluster" (Dirichlet), groups together texts whose authors adopted a "complex-analytic approach, inherited from Riemann and centering around Dirichlet series" [GS07b, 74]. Ernesto Cesáro, Pafnuti Tchebycheff, and Rudolf Lipschitz are among the regular authors of this cluster. The third, "H–K cluster" (Hermite–Kronecker), focuses on the arithmetic theory of forms, to which authors such as Émile Picard and Luigi Bianchi contribute. In addition to these three main clusters, there are several smaller ones, one of which includes the texts that engaged with Kummer's number ideals, Dedekind's ideals and algebraic number theory in the form promoted by Hilbert at the turn of the twentieth century. This cluster is surprisingly small, as measured by the number of texts, compared with its important place in the standard historiography of number theory.

Indeed, the authors from the H–K cluster who studied decomposable forms considered them as an alternative to ideals. Some, such as the Germans Eduard Selling and Paul Bachmann, even began by generalizing Kummer's work on ideal numbers before turning to the theory of ternary forms. A systematic study of number-theoretical publications then shows the existence of "alternative paths of development" [GS07b, 76], quantitatively and qualitatively important, to algebraic number theory and these diverse groupings are a far cry from an exclusively German number theory centered on algebraic number theory.

5. Conclusion

In a conference given in a seminar on science policy, Goldstein remarked that her family had no connection whatsoever with academia. When she entered the mathematical community, she was struck by the contradiction between, on the one hand, her mathematician colleagues' sense of belonging to a strong organization with collective values of integrity, rigor, care for others, etc., and, on the other, an individualistic presentation of self, embedded in a history spanning several centuries built around a few "great" white Western men [Gol21]. One of her goals upon entering the history of mathematics, was to analyze this contradiction by studying the craft or the art of mathematicans. The "various ways" in which local configurations are, or are not, connected is a way of understanding how mathematics actually works, which is also important for understanding the nature of the work of mathematicians today.

Construed broadly, mathematical practices are at once technical and political, as the examples presented above illustrate. Goldstein's individual and collective research contributions extend well beyond the themes treated in this paper.[9] Two recent projects highlight other historical configurations that circumscribe the politics of mathematics.

First, politics can occupy center stage, as in the recent collaborative project around the history of mathematics in World War I. Until recently, this war was understood primarily as a site of loss in which mathematics was disrupted. *The War of Guns and Mathematics* shows how the practices and social position of mathematics were also impacted. Pulling together international expertise and archival access, the work draws on individual experiences of mathematicians "in specific *places* through the war" demonstrating disparate experiences among similarly situated actors as well as symmetries across national boundaries [AG14, p. 43]. Second, Goldstein's own political activity has included leadership in the formation and advancement of *Femmes et mathématiques*.[10] Goldstein served on the council until 2002, was elected president in 1991, and has continually brought her historical expertise to bear in connected scholarly contributions, such as interrogating the assumptions that underlie the popular, but poorly defined question of whether there exists a particularly feminine kind of mathematics [Gol94]. In her historical work on gender and science, Goldstein again draws attention to the pitfalls in assuming a "univocal, long-term historical process," in particular, that of "increasing estrangement between domestic women and public science" [Gol00, p. 3]. Such research corrects contemporary assumptions around women as eternally domestic that ignores both "subtler mechanisms of exclusion" and, in the other direction, opportunities, ideals, positive aspirations, and the actual possibilities of life [Gol00, pp. 8, 27].

Moreover, in her position at the Institut de Mathématiques de Jussieu, Goldstein has supervised doctoral research in the history of mathematics, from seventeenth-century controversies over indivisibles to the geometry of numbers in the twentieth century. This breadth of time and mathematical discipline is undergirded by sustained attention to and documentation of specific methodological approaches, such as network analysis, social history of texts, and microhistory. Like in mathematical research, Goldstein has trained her students—and exemplified through her talks and papers—the value of clearly defining methodological choices, what counts as evidence, and the process of securing precise historical results.

[9] *See* `http://webusers.imj-prg.fr/~catherine.goldstein` *for a more complete documentation.*

[10] *The association began in 1987 with the objective of increasing the participation of girls and women in mathematical fields, from creating early educational programming to maintaining awareness of gender equality among scientific professionals.*

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Credits

The opening photo is courtesy of ICM2018.

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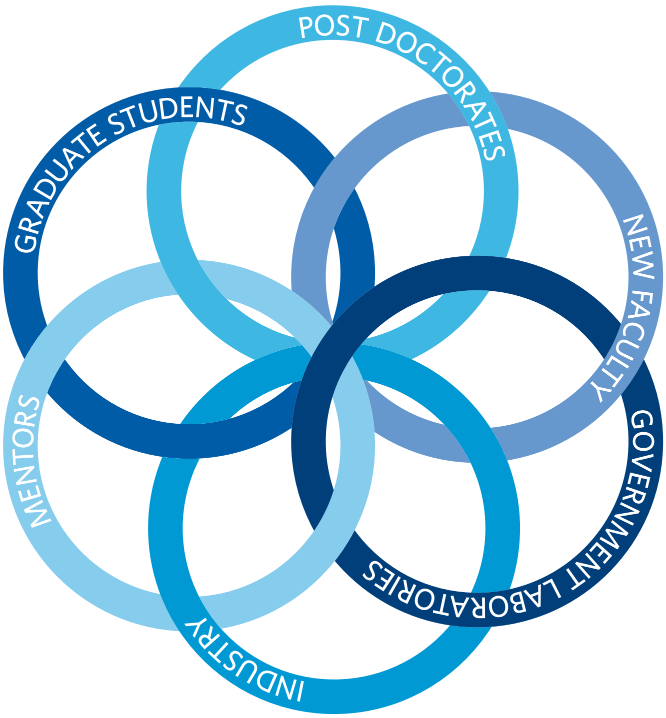
EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Math Instruction.



Women's History Month

What Would It Mean for Students to Bring Their Entire Selves to the Mathematics Classroom?

Ksenija Simić-Muller

In his influential essay "On proof and progress in mathematics" William Thurston [8] poses "How do mathematicians advance human understanding of mathematics?" as a fundamental question. We do not just do mathematics for our own satisfaction, we also wish to share the joy of doing mathematics with others. Thurston writes, "We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems." Those of us who love teaching love to share that joy with our students, to help them solve their mathematical problems. So why is it that so many of our students do not feel this joy? Why is their mathematical experience dehumanized?

To rehumanize our classrooms, we have to make different decisions than we have in the past. By rehumanizing, here I specifically refer to the recent conceptualization by Rochelle Gutiérrez. Gutiérrez [3] writes about the dehumanizing nature of mathematics classrooms: students are asked to leave their identities at the door and are taught mathematics as an exercise in compliance; and mathematics is decoupled from joy and play, and from authentic connections with the real world. While Gutiérrez primarily focuses on K–12 classrooms, her writing easily applies to college classrooms. She issues a call for rehumanizing mathematics (rehumanizing rather than humanizing to acknowledge ways people have long used mathematics in humanizing ways) and, much like Su [7], advocates for mathematics classrooms that focus on play and exploration; where students' identities are welcomed, honored, and centered in the curriculum; and where mathematics is used not just to solve abstract problems, but to make sense of the world, including issues of power and oppression.

Gutiérrez [3, 4–5] identifies eight central traits of rehumanizing mathematics. All are important, and some are traits shared with active learning classrooms, for example, "Students collaborate and see each other as resources and authorities; the instructor is no longer the sole authority in the classroom," or "Mathematics is a creative practice rather than consisting simply of rule following." I have written about a few others in [6]. Here I would like to focus on one trait that I think is particularly salient for sharing the joy of doing mathematics with all students: "*Emotions have a place in a mathematics classroom; students are able to bring their entire selves to the classroom; and they should be able to feel joy when doing mathematics.*" I am especially interested in the question of what it would look like for students to bring their entire selves to the classroom, particularly for students who are typically marginalized in mathematics classes, primarily Black, Indigenous, and People of

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Color (BIPOC) students, LGBTQ+ students, and students with disabilities.

I teach at a medium-sized private liberal arts university in the Pacific Northwest. I trained as a mathematician and retrained as a mathematics educator, and I teach a wide range of courses, from liberal arts mathematics to capstone, for majors and nonmajors alike. I am a white cisgender, heterosexual, able-bodied woman who immigrated to the United States from Eastern Europe in the 1990s. These identities are all relevant to how I show up in the classroom, engage with students, and work to rehumanize the mathematical spaces I occupy. Below are some strategies that I have found helpful in working toward the goal of cocreating with students classroom spaces where they feel safe bringing their whole selves.

Invest time in building a classroom community. Even though it seems that it is taking away from time spent on learning content, which is the primary purpose of a math class, time spent on creating community pays back generously in student engagement and commitment to the course. In my courses, we check in at the beginning of each class. I share relevant campus events, and sometimes students share events they are involved with. We almost always do warm-up activities. These are typically low-stakes, the types of activities where the focus is on mathematical discourse rather than correct answers. I especially like noticing and wondering, and especially about real-world issues [6]. Students sit at tables in groups of 3-4, which has been one of the simplest and most effective ways I have been able to build community in the courses I teach.

Appreciate the students exactly as they are. Typically, we save our best interactions for students who are already doing well in our classes. We encourage them to take more classes, to major in math, to be a grader, or to apply for a scholarship. This seems like common sense. But we may find that the students who are doing well are those who look like us, who went to the same schools that we did, or who had opportunities to take more challenging math courses. We may tell ourselves we don't have the time to catch up the student who is behind, don't have the skills to work with the student with disabilities, or don't need to accommodate the student who is working multiple jobs. And so we never know what potential we lost in the students we chose not to encourage. I yearn to give every student, regardless of how much math they know, or what grade they got in their previous class, the same amount of encouragement, faith, and respect. Even though I have a long way to go, I am most successful when I remember to enjoy the students as human beings, in all their quirkiness, insecurity, and confusion; when I connect with them as human beings first and mathematics learners second.

Provide opportunities for hands-on learning, regardless of the course content. During the past academic year, which was the first year since 2020 that felt almost normal, but not quite, I found hands-on learning especially beneficial for reestablishing some engagement skills we seem to have lost during the pandemic. When learning about symmetry in a course for future teachers, students made *papel picado*, a traditional Mexican craft made from sheets of tissue paper with designs (often symmetric) cut out of them. Then, for the remainder of the semester, I put out construction paper and scissors every day so that students could make other constructions, including paper snowflakes. We hung the art pieces on the classroom wall, where they stayed until the last day of classes, thus celebrating their creations and making the classroom more personable. In discrete mathematics and proofs classes, we began class once a week with a team building activity with interlocking cubes, where each student in a group gets a clue card, and all clue cards are needed to build a required object [1]. This prompted groups to bond: there was often laughter heard, in addition to lively discussions about strategies for solving the problem. Most days I left the cubes on the tables after the warm-up activities. I never knew what elaborate structure I would find at the end of class. One day a whole family of ninja turtles appeared, on another day a family of ducks.

Allow different kinds of feelings in the classroom. In a perfect world, there would always be joy in all my classes, but this is not possible. My students are allowed to hate math, as many who come to my classes for nonmajors do, and with good reason, considering the histories they have with the subject; my job is to show them that they can do well in math and enjoy it, but not to force them to like it. They are allowed to be frustrated and to dislike whatever activity we are working on, though if this happens it is a clear message to me that I need to understand what it is about the activity that is not working and revise it as needed. Everyone is allowed to have bad days. If it is a bad day in the news, I acknowledge it and provide space for conversation. The world is always challenging for young people, but perhaps especially now in the era of climate disasters and attacks on rights and liberties that, at least for many of my students, are personal. Sometimes students really want to talk about issues. Sometimes they just want to work on math with their friends, and sometimes they want to work quietly on their own and listen to music. These are all valid, and I have learned to read the mood of the room and ask students what they need, then adapt.

Do not be afraid of difficult conversations, and learn how to facilitate them. I have difficult conversations with students, even though (or especially because) they are difficult. If there is a current issue that students want to discuss, I have learned to hold space for despair and for hope, and to find a balance between the two. I also make difficult conversations part of my curriculum. We use mathematics to analyze systemic racism, talk about the limitations of Western mathematics, and analyze the myth of the brilliant and troubled mathematician, among other things. There are now great curriculum resources available to address social justice-related topics [4, 5], though I still often create my own. I have learned to have conversations about complex issues, related to the curriculum or not, from readings, professional development sessions, and conference presentations, as facilitating difficult discussions is not something most of us learn in graduate school. I have especially benefitted from participating in book clubs with colleagues.

Be more flexible. This is a complicated suggestion, as I know that faculty who are BIPOC, women, and/or trans, may not automatically get respect from students. However, in my experience, my willingness to bend allows students to be more comfortable being themselves, and fosters learning. Does letting students make animals from cubes during a proofs class negatively impact their learning? I have actually found that some students are able to focus better. And last semester, the student who built the most elaborate structures participated more in classroom discussions the more he was allowed to build with cubes in class. I have also learned that no pedagogical approach needs to be followed exactly. Even though learning is most meaningful in groups, sometimes students need to work on their own; even though research suggests that groups should be changed often and randomly, sometimes students just need to sit with their friends all semester; even though I think homework is essential, sometimes students have good arguments for why less should be assigned, and I listen to them (though grudgingly).

Do not sacrifice high expectations while being more flexible. While there are times when mathematics learning can briefly be put aside to deal with a timely issue, enjoy a side conversation, or just take a short break, learning always comes first. Nor is there a distinction in my classes between "fun" warm-up activities at the beginning of class and "serious work" afterward; all work should be equally joyful, challenging, fun, and frustrating. It is always a balancing of setting high expectations and not taking our classes too seriously. In my classes we play games, do a lot of hands-on activities, laugh, and listen to music; but I also remove myself from the front of the classroom so students can have more ownership, give honest feedback, keep pushing, and never let a student say they are not good at math.

Remember that you will never arrive at the final destination. I recently received grade data from the 15 years I have been at my institution. I was troubled but not surprised to learn that I do not give nearly as many As or Bs to Black students as I do to students of other races. It was humbling to be reminded that all the professional development I participate in and all the efforts I make to include all students are not sufficient to undo my biases and the impacts of living in a system that ranks and sorts by race, class, gender, sexual orientation, or ability (among others), as well as by perceived mathematical ability. It is easy to become discouraged, but we can also become curious, dig deeper into the data, pay more attention to our actions and student feedback, and keep learning.

Oftentimes we argue that we need to include previously excluded students in mathematics (particularly BIPOC students) because mathematics will benefit them, but we fail to recognize how mathematics itself will benefit from their perspectives. Even though written in 2002, this quote by Gutiérrez still resonates today:

> The assumption is that certain people will gain from having mathematics in their lives, as opposed to the field of mathematics will gain from having these people in its field. In other words, most equity research currently assumes the deficit lies within the students who need mathematics as opposed to, or in addition to, lying within mathematics, which needs different people. Such programs seem to imply that the people being served by the programs need to improve but that the mathematics does not [2, p. 147].

I do not think that mathematicians can advance human understanding of mathematics if they are not willing to let humans rehumanize (and therefore advance) mathematics. Our courses are better and we are better when every student has the opportunity to contribute in meaningful and authentic ways. I still do not know what it would look like for all my students to be able to bring their entire selves to my classroom, especially as this is not something I can determine on my own without student input; but I think I get occasional glimpses. Like when a future teacher and a "Swiftie" writes a fantastic lesson about the inequities involved in buying tickets for Taylor Swift concerts. When a student solves a problem while bobbing his head to a tune in his head, confident in his knowledge, and content to be working on math with his group members. Or when a student who was never successful in a math class excels in a probability unit partly due to her being an experienced Dungeons & Dragons player.

Whatever work we choose to do to create more equitable classrooms, departments, committees, research teams, and universities is indefinitely ongoing. It is also full of joy even when/because it is incredibly hard. Our humanity is intertwined in the humanity of others, and the project of rehumanizing mathematics necessarily includes rehumanizing ourselves.

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Credits

Photo of Ksenija Simić-Muller is courtesy of John Froschauer.

The Infinite Possibilities Conference: Creating Moments of Belonging

Lily S. Khadjavi, Tanya Moore, and Kimberly Weems

Sometimes remarkable creations come from a moment of gratitude. In 2003, on an uncharacteristically misty afternoon in Los Angeles, Spelman alumnae found themselves serving as members of the audience for a dozen middle-school-aged girls. All shades of brown skin reflected beautifully against the white lab coats the students proudly wore, as they explained their science experiments with big smiles. Ostensibly, the Spelman graduates were to serve as role models for the girls. They fulfilled their designated role . . . and much more. The experience inspired them to reflect on their time at Spelman as mathematics majors, the uniqueness of their college experience, and the critical role of professors who served as role models that balanced high expectations for academic performance with support and encouragement. Although they did not realize it at the time, those Spelman faculty offered them their first glimpse of a mathematical community.

The two alumnae continued to talk with each other about their college peers with whom they had worked collaboratively on problem sets and exam preparation. They remarked on how many of the Spelman math majors had gone on to graduate school, successfully completing master's and doctoral programs in mathematics, statistics, and related fields. The conversation then veered to the recent passing of Dr. Etta Falconer, a Spelman giant who was instrumental in increasing opportunities that supported Black women mathematicians and scientists. In appreciation of the Spelman experience that launched so many women into math careers, they thought out loud about how great it would be to have a reunion or celebration to bring everyone back together, to honor Dr. Falconer, and to acknowledge the contributions of the math faculty

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in shaping their paths. Then Marlisa Johnson turned to Tanya Moore and said, "You should do that." At that moment, the Infinite Possibilities Conference (IPC) was born.

The first IPC. IPC was created with the mission to support, encourage, and celebrate underrepresented minority (URM) women[1] in the mathematical sciences. The first IPC was organized under the leadership of Leona Harris, Tanya Moore, and Nagambal Shah and took place at Spelman College in 2005. In grassroots fashion, they assembled an organizing committee of URM women mathematicians who, along with faculty and staff of Spelman College, thought that IPC could form part of a response to the problem of underrepresentation.

Even as of 2018, only two percent of doctoral degrees in mathematics were awarded to African-American, Latina, and Native American women in the US [3, 6], while those same groups comprise approximately seventeen percent of the US population. Even earlier in the pipeline, fewer than six percent of bachelor's degrees were awarded to URM women in the mathematical sciences [6]. The largest share of bachelor's and doctoral degrees in the mathematical sciences was awarded to white males, 32% and 51% respectively [6]. While the overall number of doctoral degrees in mathematics has increased between 2006 and 2018, the number of URM women receiving those degrees has stayed relatively flat. In tandem, the persistence of stereotypes of the typical mathematician has held strong. In most minds, the image of a mathematician or scientist conjures up a white male, and the data regarding who receives a degree in mathematics at the undergraduate and graduate level support that association.[2] The lack of diversity in the mathematical sciences becomes a self-fulfilling prophecy—a cycle of acceptance around traditional notions of who rightly belongs in the mathematical community.

Armed with personal testimonies and their awareness of the lack of representation of URM women in the broader math community, the organizing committee set out to design the first IPC. The initial planning discussions provided an opportunity to envision a conference they would want to attend. The organizing committee posed exciting and empowering questions: What kind of workshops and speakers should be featured? Should the conference target only Black women, or all women? What were the key elements of environments, programs, and relationships that supported success in completing graduate school? What needs to be known and shared with other women interested in math?

It was ultimately decided that there was a unifying experience of African-American, Latina, and Native American women in math, in that their identity stood in direct contrast to the stereotypical image of a mathematician. So IPC would be created intentionally for them, by them. There was also a desire to create a platform to showcase different career paths chosen by women who love math. IPC could expand awareness beyond the perceived limited professional opportunities available with a math degree as well as combat conventional beliefs of who could do math; hence, the organizers chose the name "Infinite Possibilities."

As the organizers planned a combination of plenaries, panels, and smaller parallel sessions, they wanted to address and share the multi-faceted factors contributing to retention in the field, such as the importance of being seen and valued, feeling connected, and finding the appropriate level of challenge to excel. As the only (or one of a few) URM women in their graduate program, many of the organizers had begun to question whether they belonged in mathematics and if they could be part of the broader math community without sacrificing their identity. If the conference could provide an opportunity for URM women to know that they were not alone and that relatable models of success existed, then perhaps it could provide encouragement for those in attendance to persist.

Many of the Spelman graduates on the organizing committee had experienced the privilege of mentorship from Spelman faculty that extended into their time at graduate school. Was there something different about mentoring women of color in mathematics that should be explored? The organizing committee also aimed to have all aspects of the pipeline represented, from students to professionals in the field, in order to provide opportunities to connect to role models and encourage networking. These are activities that have been identified as effective mentoring strategies, especially for women of color in the mathematical sciences [5]. The first conference would include a panel discussion on mentoring women in mathematics and a session that proactively facilitated dialogues on mentoring among women at different stages in their journey. Linking high school students to undergraduates to graduate students to professionals could provide connection points to "near peers," with the belief that near peers can often provide the most useful advice and insight for the next step in

[1] *Consistent with language which has been employed by academic institutions, we use the terms "underrepresented" or "minority" (or URM) to refer to someone who self-identifies as Black/African-American, Hispanic/Latino(a), Native American or Alaska Native, Native Hawaiian, or other Pacific Islander. We also use "women of color" to refer to racial or ethnic groups traditionally marginalized in the United States and to those who self-identify with the term woman. The term BIPOC, or Black, Indigenous, People of Color, recognizes the complexity of marginalization. While some scholars have pointed out that terminology can implicitly suggest white as neutral and in effect essentialize racial groups, these more recent umbrella terms are embraced by many activists and have the advantage of moving away from majority/minority designations.*

[2] *In a striking "Draw a scientist" study which originated in the late 1960s and has been repeated in more recent eras, children drew male images far more often than female. In the study's first iteration, out of over 4800 drawings of scientists, only 28 were of women (all drawn by girls). As of 2008, children still drew twice as many male figures as female, whereas for professions such as teacher, only a quarter of the sketches were of men* [9].

the journey. Moreover, the intent was for the conference to create a community of mentors of various roles, such as advocates, role models, peer mentors, and coaches, who could augment existing mentoring structures at their home institution or industry [2].

Figure 1. IPC Group Photo at UCLA, 2010.

Acknowledging that the gold standard for success in academia is often publications, the organizers felt compelled to support one another along the research path by providing a supportive environment for sharing, receiving feedback, and dialoguing about research, necessary steps in the process of becoming a high-caliber mathematician and researcher. It was important to include opportunities for IPC attendees to communicate their results during the conference through Research Roundtables, concurrent sessions of oral presentations that showcased the participants' work in applied mathematics, mathematical biology, mathematics education, pure mathematics, and statistics. Students could also make poster presentations. These sessions would provide a supportive environment to discuss and receive feedback on theses, dissertations, works-in-progress, and research reports.

The participation of Spelman alumnae and faculty in the planning of the first IPC influenced another key decision. To honor the legacy of Etta Falconer, the IPC Steering Committee, Spelman College Mathematics Department, and the Falconer family established the Dr. Etta Z. Falconer Award for Mentoring and Commitment to Diversity. The award recognizes the importance and value of individuals who make significant efforts toward building connections and community. For her dedication to increasing the number of women and African-American students in mathematics, Janis Oldham of North Carolina A&T State University became the first Falconer recipient. As noted by her nominators, Oldham embodies the spirit of Falconer by having high expectations of her students and a willingness to invest generous amounts of her time to nurture their mathematical development.

Initially, the organizers planned to have just one conference. But after the closing banquet of IPC 2005—which opened with a drum ceremony, gave tribute to Falconer, presented an overview of the history of URM women in mathematics by Sylvia Bozeman, and honored Oldham with the Falconer award—the room was filled with tears and full hearts. Initial feedback on evaluations from the first conference further validated the organizers' intuition about the importance of providing a space for women of color to feel supported and more connected within the math community. As a result, planning for the next conference began.

The power of reflection and critical mass. We can all understand the power of images to influence our beliefs. Many of the organizers for that first IPC were newly minted PhDs and had their own personal experiences in graduate school fresh on their minds. They shared their need to form their own communities outside of mathematics or to seek therapy in order to cope with the stress, doubt, and sense of isolation while in school. For those who attended a Historically Black College or University (HBCU) as undergraduates, their first introduction to a math community had included supportive faculty and peers. Professors and classmates served as living proof of the possibility to obtain educational degrees and achieve professional success. The question of gender or race as linked with skill in mathematics was implicitly removed from consideration. While this knowledge fortified them, as they pursued graduate-level education in environments that were radically different from HBCUs, the sense of isolation and the experience of not fitting in could still be painful and challenging to navigate.

The numerical data for URM women in math imply that most are the only (or one of very few) women or URMs at their respective institutions. One of the most common themes from conference evaluations by attendees focused on how IPC made them feel less isolated by being in the presence of so many women with whom they identified. Attending the conference encouraged them to continue along their paths in mathematics.

> Mostly it made me feel less alone, which has made me a little more comfortable pursuing the PhD. (IPC attendee)

> The conference is one of the most encouraging conferences I have been to. I found inspiration and made connections with women who 'looked' like me or had the same 'walk.' I am more determined to finish my PhD and to continue to build my village. I sincerely will recommend this conference to all mathematicians and scientists (females). Let me know how I can be more involved. (IPC attendee)

Impostor syndrome is a common experience among many students and professionals in the academic environment, regardless of background or identity. This feeling challenges one's belief that success in math is possible. The suspicion that one has been let in "by mistake" is further exacerbated by the experience of not fitting in. Additionally, when faced with microaggressions or more explicit comments regarding doubt in ability, impostor syndrome begins to feel like a credible belief. IPC provided a brief but impactful respite from the questions and doubts of belonging by bringing in URM women *en masse*. When we think about the fact that there are on average twenty URM women that complete doctorates in math each year in the US, being in a room with 100–200 URM women all at one time takes on another level of significance (Table 1). With this moment, there is an opportunity for seeing to translate into believing that belonging in math is possible as a URM woman.

> While attending IPC I was very doubtful if I should continue striving toward a PhD because I had one more qualifying exam to pass that I had taken four times already. But after attending IPC and talking to many of the very inspiring women who were just like me, it made me see that I am not alone, and if they could do it, I could too. So that June I took the test, passed it and even received a fellowship for the next year. I am extremely grateful to have been a part of IPC; it is a conference that was both inspirational and educational to me. It is an experience I will never forget. (IPC attendee)

> My first conference was a pivotal moment in helping me see my personal fear in pursuing a doctorate degree (feeling like "I can't do it") and address this fear along with seeing my true desire to earn a degree and follow my passion. (IPC attendee)

> It made me believe that I can be one of the professors in a future meeting. (IPC attendee)

	2005[3]	2007	2010	2012	2015	2018
Total Registrants	148	206	191	288	203	135[4]
% Female	95%	94%	96%	90%	86%	96%
% URM[5]	87%	70%	81%	70%	53%	78%
% URM Women	85%	67%	79%	67%	49%	76%

Table 1. IPC registrant totals and demographics between 2005 and 2018.

Wherever possible, IPC sought to put notable URM women in math at the forefront of the conference. Critically, the organizers themselves reflected the intended audience and the conference showcased the accomplishments of women of color who were featured as plenary speakers and panelists. Photographs of attendees were included throughout the conference proceedings. The reflection of URM women in every aspect of the conference was very purposeful and important.

For some of the organizers, the relationships with former undergraduate professors continued throughout graduate school, helping them to contend with a sense of isolation and a new environment. The conference became a vehicle to express appreciation for the dedication of these mentors. Mentors can have an especially significant role in supporting long-term success for women of color in STEM fields pursuing graduate degrees [7]. Yet, the work of mentoring is often not as visibly acknowledged as other aspects of professional activities. To honor this important work, the culminating event of each IPC has been the presentation of the Dr. Etta Z. Falconer Award for Mentoring and Commitment to Diversity, recognizing individuals that were nominated by their mentees and colleagues and selected by a panel of reviewers. Past recipients of the Dr. Etta Z. Falconer Award include Janis Oldham, Sylvia Bozeman, Ivelisse Rubio, Roselyn Williams, Genevieve Madeline Knight, and Javier Rojo.

Addressing intersections of identity.

> Your conference fulfills a unique role in the mathematical community. You inspire and encourage young and minority women to excel in mathematics and address the concerns of being a professional and leading a satisfying life. Your holistic approach has produced two exciting conferences that are to my knowledge completely new to the math world. (IPC attendee)

From the outset, the Infinite Possibilities Conference was implicitly designed to address the intersections of identity. Noted legal scholar Kimberlé Crenshaw introduced the idea of intersectionality in the late 1980s as a legal theory in order to shine a light on the inadequacy of the law in addressing racism and sexism, and indeed other forms of discrimination, when it failed to take into account the "intersection" of individuals' identities. The notion of intersection is of course fundamental to mathematicians. An intersectional perspective in this context requires us, as a mathematical community, to take into account how, for example, a woman of color might experience gender bias differently from a white woman. People at the intersections can even inadvertently be displaced, and there is a need to address their experiences. While valuable mathematical programming has been created to support women or to gather students and researchers of color, few initiatives are aimed specifically at those in the intersections. (To put another mathematical spin on it, the many labels for race/ethnicity, gender, sexual orientation, marital status, religion, and so on, are not mutually exclusive.) How, then, can we create a mathematical community where we don't have to check some part of our identity at the door in order to enter?

[3] *Estimate based on partial demographic data.*

[4] *Attendance intentionally capped because of space constraints at host institution.*

[5] *URM includes African American, Hispanic/Latino, Native American, Pacific Islander/Native Hawaiian and Bi- or Multi-Racial.*

I was able to connect with other women of color who are in [the] mathematics field and learn how they navigated the academi[c] world. (IPC attendee)

While mathematics has been a consistent theme at each IPC with parallel research talks and poster sessions, over the years the programming committee has always made space for participants to directly discuss the intersections of their identities and their academic experiences. Sometimes these conversations have taken place in plenary panels and sometimes through small-group discussions. All spaces created a comfortable venue to share experiences and advice.

Sessions on race and gender issues elicited stories that illustrated the common experiences of isolation, tokenism, and more. Even seemingly incidental examples reported in the very first Proceedings of IPC, paraphrased here, captured these sentiments: One student expressed frustration at feeling like a token as an undergraduate when a faculty member said to her, "Do you know how good it would make us feel to have a black female go to graduate school?" Discouragement could come from fellow students, as in, "Many had teaching assignments and when a student heard I [a Latina female] had a fellowship [that year], he said, 'Oh, if I changed my name and gender, I'd have one too."'

These encounters had not created a sense of belonging in the programs where they occurred. Importantly, IPC was an opportunity to air these frustrations and larger challenges, at every level of participants' careers, in a safe space. Attendees shared suggestions for building support.

Later IPC programs have included sessions on the many intersections of our lives: sessions on balancing career and personal life, challenges of motherhood, being LGBTQ in mathematics, and more. Parallel sessions allowed us to create smaller group discussions that also spoke to participants at their specific stage in education and career. While college students could attend a session on applying to and thriving in graduate school, those later in their studies or career could discuss negotiation skills and fighting gender and racial stereotypes in the workplace. Specifically targeting the retention and advancement of faculty, Kerry Ann Rockquemore, founder of the National Center for Faculty Development and Diversity, led a workshop on "Solo Success: How to Thrive in the Academy When You're the Only ______ in the Department."

I have multiple identities: by race/ethnicity, gender, as a math nerd, by my [sexual] orientation ... IPC has been a space where I can find community. I feel a sense of belonging in the mathematics world here, with others who don't need to match me in every characteristic. (IPC attendee)

Living history and *Hidden Figures*.

It really gave me more push and motivation to finish my degree. I was surrounded by women who were great mentors and inspiration for me. (IPC attendee)

I got to see influential women in the mathematical sciences who looked like me but also who had gone through struggles and overcome similar obstacles like myself in order to pursue education. (IPC attendee)

The first IPC featured a special keynote speaker, Evelyn Boyd Granville, one of the first African-American women to earn a PhD in mathematics. Granville's very presence gave conference attendees a powerful connection with their history. Her rendition of the debate regarding the first African-American woman—Euphemia Lofton Haynes, Marjorie Lee Browne, or Granville—to receive a PhD in mathematics served as a priceless testimony celebrating this groundbreaking achievement as a whole rather than as an individual accomplishment [1]. Even more, Granville's mentor Lee Lorch, himself a civil rights activist and mathematician, also attended the event. To round out the weight of the moment, after Granville spoke, she posed for a picture with Tasha Inniss, Sherry Scott and Kimberly Weems, the first three African-American women to receive doctorates in math from the University of Maryland in the same year. Subsequent IPCs have featured trailblazers including Freda Porter, an entrepreneur and one of the few Native American women with a PhD in mathematics, and Ruth Gonzalez, recognized as the first US-born Mexican-American woman to earn a PhD in mathematics [4].

Figure 2. (From left) Kimberly Weems, Tasha Innis, Evelyn Boyd Granville, Lee Lorch, and Sherry Scott. IPC at Spelman College, 2005.

So often, when history is discussed in mathematics courses, the names mentioned are connected to the theorems and equations that fill textbooks. We know the names of Euler, Descartes, and Pythagoras, but we are less familiar with the names of Granville, Gonzalez, or

Porter. These pioneers and role models are important to inspire future generations of URM women. Not only have they paved a way forward, but they also serve as examples that it is possible to complete advanced studies in mathematics. Plenary speakers discussed a range of mathematical subjects, often sharing their personal journey in mathematics—both highs and lows. The IPC provided an opportunity to demonstrate that URM women have made and are making meaningful contributions to the field of mathematics.

Upon learning about the upcoming release of *Hidden Figures*, the book and movie that showcase African-American women's contributions to mathematics, computing, and space exploration, IPC co-founder Tanya Moore conceived the idea of creating some type of IPC and *Hidden Figures* collaboration. The connection was evident, with IPC striving to promote URM women in mathematics and *Hidden Figures* shining a spotlight on unsung "shero" mathematicians behind aeronautical breakthroughs. The timing could not have been more perfect as the film's screen release date was in late December 2016—just weeks prior to the largest gathering of mathematicians in the country.

Thus, at the January 2017 Joint Mathematics Meetings in Atlanta, IPC—along with its umbrella organization, Building Diversity in Science—partnered with the American Mathematical Society, Association for Women in Mathematics, Enhancing Diversity in Graduate Education, and the National Association of Mathematicians, to sponsor a panel on "The Mathematics and Mathematicians behind *Hidden Figures*." Moderated by Moore, the panel featured Margot Lee Shetterly, author of the book on which the movie is based; Christine Darden, retired NASA "human computer" whose contributions are discussed in the book; and Ulrica Wilson, Associate Professor of Mathematics at Morehouse College, who presented the work of another hidden figure, Dorothy Hoover.

This panel broke down walls—literally and figuratively. Originally, the event was assigned to a room with space for about 200; at the last minute, staff removed room dividers to increase the capacity and accommodate the massive crowd. This panel marked the first time IPC programming was introduced to the broader, mainstream, (inter)national mathematics community. To the delight of the organizers, positive response to this panel was overwhelming, and it is our hope that after hearing about the struggles and accomplishments of hidden figures, attendees who may have felt like outsiders began to experience a sense of belonging in mathematics.

The power of learning this history is captured in inspiring terms by Shetterly in the prologue to her book: "What I wanted was for them to have the grand, sweeping narrative that they deserved, the kind of American history that belongs to the Wright Brothers and the astronauts, to Alexander Hamilton and Martin Luther King Jr. Not told as a separate history, but as a part of the story we all know. Not at the margins, but at the very center, the protagonists of the drama. And not just because they are Black, or because they are women, but because they are part of the American epic. [8]."

Figure 3. (From left) Tanya Moore, Margot Lee Shetterly, Christine Darden, and Ulrica Wilson. The Mathematics and Mathematicians behind *Hidden Figures* Panel, Joint Math Meetings, Atlanta, Georgia, 2017.

Looking towards the future. Each IPC has been planned by a group of women from around the country who volunteered their time to make IPC a reality. Membership in the organizing committee intentionally changed for each conference in order to continue to bring fresh ideas and diverse perspectives to the program. Meanwhile, at each host institution, a local committee supported the conference activities. A consistent group of women provided leadership from year to year as members of an IPC Advisory Board. Members of the IPC Advisory Board have included Erika Camacho, Leona Harris, Lily Khadjavi, Tanya Moore, Nagambal Shah, and Kimberly Weems, who shared their perspectives on sites for the conference, key themes to incorporate, and suggestions for organizing committee members.

Though the first IPC was a grassroots effort, the leadership realized the need to form collaborations for sustainability of the initiative. After the first conference, IPC formally joined a non-profit, *Building Diversity in Science*, whose mission to inspire, empower, and support underrepresented groups in the pursuit of STEM careers aligned with the goals of IPC. Early in the history came a partnership with the NSF Math Institutes. These collaborations resulted in increased staff support for essential tasks such as registration, travel reimbursements, and advertising, and allowed the organizing committee to focus more of its energy on creative programming for the conference. Primary and consistent conference support was obtained from the National Science Foundation and, in various years, the National Security Agency, host institutions, and other corporate funders. A significant portion of the funding went towards providing student travel scholarships and for underwriting the majority of conference expenses in order to keep registration fees low. Partnerships have been critical to IPC's ability to endure over the last fifteen years.

> The conference really lifted my spirits and helped me to move forward. As a first year graduate student, I needed to hear the stories of women who look just like me who have also struggled with racism, sexism, and favoritism in their respective institutions but through faith, help, and support from family and peers, and, most importantly, with strong determination were able to overcome these obstacles and succeed. After the conference I came back to my school rejuvenated and ready to take on the world. (IPC attendee)

As the only program of its kind, IPC has aimed, since 2005, to sustain the spark for women who have an interest in math. IPC creates a space where personal identity doesn't have to be separated from identity as a mathematician. Shared cultural experiences help establish a community in mathematics so that participants are not alone in their academic and professional pursuits. We have learned that a two-to-three-day conference, as one piece of a web/network of activities, can make a difference. The founders set out to create a conference to honor their *alma mater* and share the best of what they received with other women like them. They wanted a conference that created community, embraced the full identity of participants, and highlighted relatable models of success for students and professionals. They came to see it as creating belonging.

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Credits

Figure 1 is courtesy of David Weisbart.
Figure 2 is courtesy of Frederick Moore.
Figure 3 is courtesy of Edray Herber Goins.

A Source List to Support DEI/EDI Work in Mathematical Sciences

Deborah Kent, Emilie Aebischer, and Stuart Neave

Many mathematicians have recently become interested in considering matters of diversity, equity, and inclusion (DEI/EDI) in their professional work. For some, this is prompted by documents such as the Mathematics Framework[1] proposed in 2021 by the California Department of Education, or the Quality Assurance Agency (QAA) 2022 revised subject benchmark for Mathematics, Statistics, and Operational Research[2] (MSOR) in the United Kingdom, both of which stress the importance of considering DEI/EDI work for the discipline. The QAA benchmark states that "Equality, diversity and inclusion (EDI) is essential for the health of MSOR, and it is important

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[1] `https://www.cde.ca.gov/ci/ma/cf/`

[2] `https://www.qaa.ac.uk/the-quality-code/subject-benchmark-statements/subject-benchmark-statement-mathematics-statistics-and-operational-research`

that the discipline encourages inclusivity and access to ensure learners are attracted from diverse backgrounds, that the curriculum and environment enable them to succeed in their studies, and that the subject is enriched by input from diverse practitioners."

The source list presented in this article is available in full and linked here to arXiv,[3] is designed as a resource to support work in the education and practice of mathematics that involves DEI/EDI. The project developed alongside a UK network funded by the Issac Newton Institute to create an environment for discussing the place of DEI/EDI in mathematics curricula in higher education.

The associated source list provides a variety of starting points for people interested in a range of concerns related to gender, race, ethnicity, (dis)ability, class, or sexuality that currently impact the study and practice of mathematical sciences at the postsecondary level. It makes no claims of completeness—many individual sources listed here could generate entire bibliographies of other relevant material—but is intended to provide a collection of references with a variety of perspectives that can serve as an introduction to relevant literature for mathematical practitioners. There is no explicit or implicit instruction to read these in any particular order, or to read every source on the list. Indeed, interested individuals will be in different institutional contexts with a varied range of interests and priorities related to DEI/EDI initiatives. This document is intended as a resource for anyone looking to deepen their own understanding of societal oppression, discrimination, inequity; how these things could impact the teaching and learning of mathematics; and ways instructors might redress the situation. It will necessarily involve individual efforts to implement resulting insights for a particular audience or approach. Each section of the reading list is introduced by a brief overview to assist readers with identifying materials of interest to them.

The selected sources are organized by scale, starting with general interest works that interrogate gender, race, ethnicity, (dis)ability, class, or sexuality in contemporary society. This is followed by articles and books that present research results about how matters of inequity, exclusion, and homogeneity surface in STEM educational contexts. Then come sources focused specifically on the past and present of interplay between these themes and the study and teaching of mathematics. Each section includes a range of material organized alphabetically by author and not further categorized. Taken all together, these sources indicate some contours of current challenges that face mathematicians engaged in efforts to create diverse, equitable, and inclusive classroom environments.

The second part of the list looks at visions for and approaches to creating better environments for all students of mathematics in higher education. It begins with a collection of sources that have a conceptual or theoretical orientation toward the challenges of creating more inclusive mathematics classrooms. The list then moves to consider approaches that specifically involve the history of mathematics as a tool to address the challenges faced by the community of mathematical practitioners. There are two primary threads of related research: one focuses on using primary source material to teach mathematical content, while the other draws on historical research to increase representation in mathematical sciences.

General Social/Cultural Context

These general-interest books introduce a broad range of issues related to inequity and discrimination in contemporary society. One core theme is a call to recognize normative conditions that create discriminatory perceptions of "nonconforming" people. Both Lennard J. Davis in the introduction to *The Disability Studies Reader* and Fiona Kumari Campbell in a more technical tome, *Contours of Ableism*, interrogate normativity in relation to perceptions of people with disabilities. Similar considerations of exclusionary norms appear in Reni Eddo-Lodge's *Why I'm No Longer Talking to White People About Race* and the introduction to Nikesh Shukla's *The Good Immigrant*. Another theme in this section is the recovery of neglected histories. David Olusoga's *Black and British: A forgotten history* counters the narrative that excludes Black people from British history before Windrush. In *Racism, Class, and the Racialized Other*, Satnam Virdee presents a social history of Britain where race is a vital factor in the development of a working class. Selina Todd's *Snakes and Ladders: The Great British Social Mobility Myth* is another historical work that provides social context relevant to the matter of higher education in mathematics and other subjects. While most of these works aim at a general audience, there is variation in approach and tone. Some, like *The Routledge Handbook of Epistemic Injustice* are considerably more technical than public-facing titles. On the more personal-development end of the spectrum, Layla Saad's *Me and white supremacy* encourages the reader to engage in self-reflective journalling alongside the text. Other works, like Ibram X. Kendi's *How to be an Antiracist* are more autobiographical. Similarly, Eileen Pollack's *The Only Woman in the Room* relies on her own experience as a mathematics student at Yale to illuminate past and present biases in the discipline.

In STEM Education

Inequity and discrimination based on gender, race, ethnicity, (dis)ability, class, and sexuality observed and experienced in society is also documented to exist in educational contexts in both the United States and the United Kingdom. In 2015, the Council for Mathematical Sciences

[3] https://arxiv.org/archive/math

report on The Mathematical Sciences People Pipeline found that in the UK, female students make up about 40% of undergraduate students in the mathematical sciences, but this number drops to 33% in postgraduate students in the mathematical sciences. This drops again for mathematics postgraduate students, of which 28% identify as female. Black students make up 3% of undergraduate students in the mathematical sciences, which is lower than the 6% of the general undergraduate student population. The proportion of students with a reported disability is lower for postgraduate students (5%) than undergraduate students in the mathematical sciences (8%) and both of these are lower than in the general student population. Beyond university students, women made up only 15% of all active authors in math, physics and computer science in the UK in 2010 (see Huang J., Gates A. J., Sinatra R., Barabási A.-L. [2020]).

The US Equal Employment Opportunity Commission published its Annual Report: Women in STEM (FY2019) which states "Overall, women accounted for 29.3 percent of STEM federal workers. Science occupations had the most (49,546), while math occupations in the federal sector had the fewest number of women (6,469). There were significantly fewer women in Technology and Engineering than expected." A common metaphor to describe the phenomenon of more and more women dropping out of STEM as they advance through education and their career (at a higher rate than males), is that of a "leaky pipeline." The linked Catalyst[4] website has extensive related data and information.

It is generally acknowledged that the lack of diversity and inclusion stems less from overt exclusion happening at the level of student acceptance and recruitment, and more from deeper-seated societal pressures combined with implicit stereotypes and expectations. The longstanding Draw-a-Scientist[5] project both reveals and questions existing assumptions about the identity of a scientist as white, able, and male. One suggested explanation for this is that scientific epistemology represents an inherently white, masculine world view. Other existing research posits other possibilities, including: stereotypes of intellectual brilliance and implicit bias; absence of role models; greater accessibility of science curricula to men than women; lack of belonging; sexism; sexual harassment; and a notion that "doing mathematics is doing masculinity." More generally, there is are questions of who can be a mathematical practitioner and what doing mathematics entails.

[4]https://www.catalyst.org/research/women-in-science-technology-engineering-and-mathematics-stem/

[5]https://doi.org/10.1111/j.1949-8594.2002.tb18217.x

Past and Present in Mathematics

While the sources above explore questions of equity, diversity, and inclusion in STEM educational contexts, this section focuses more narrowly on mathematics. References here vary widely in both intention and scope as they explore both historical development and contemporary factors that have contributed to persistent, systemic inequities and discrimination in the field.

June Barrow-Green's "The Historical Context of the Gender Gap in Mathematics," analyzes cultural attitudes to women involved in mathematics from the eighteenth century onward. Similarly, Claire G. Jones' *Femininity, Mathematics and Science, 1880–1914* tracks the mechanisms of inclusions and exclusions in science and mathematics navigated by women at the turn of the twentieth century. In *Masculinity and Science in Britain, 1831–1918*, Heather Ellis investigates ways in which mathematics and science gained a masculine identity. These three sources—and others on the list—highlight matters related to gender in mathematics. There are many additional factors that impact inequities and exclusions in the field. Texts by Joseph Dauben and Kapil Raj scrutinize approaches to the history of mathematics through an exclusively western lens. Others, including Theodore M. Porter and Ruth Schwartz Cowan, highlight how mathematical reasoning created and perpetuated individual categories of people. In another sense, the sources included here encourage the reader to consider the scope of mathematics, and to recognize its involvement particularly in the physical and invisible world around us.

Mathematical and technical tools have at times facilitated the construction of an exploitative reality. For example, Langdon Winner's "Do Artifacts have Politics" and subsequent book *The Whale and the Reactor* explain how technological machines and constructions can contain and exercise power on society. More recent works invite an examination of everyday algorithms and how they might codify systemic inequities. Cathy O'Neil's *Weapons of Math Destruction* is at the forefront of this work, showcasing how mathematical algorithms perpetuate existing biases. Studies about the deployment of algorithms by police departments has been especially insightful; as seen in works by Virginia Eubanks and Sarah Brayne. Meanwhile, Joanna Radin's "Digital Natives" shows how demographic data samples can be taken and reused in radically different contexts, and can perpetuate patterns of "settler colonialism" in the process. As the influences of big data and algorithms grow, so too does the potential for them to perpetuate oppression. Above is a sample of the themes represented in this section of the reading list. We encourage the reader to dive in wherever their interest takes them.

Alternative Visions and Approaches

Given the above articulations of various ways mathematical education faces challenges in this social context, this section contains materials with suggestions on how one might approach mathematics and mathematics education differently. Central to such efforts is the realization that mathematics is a social practice, which has been shaped by (and reciprocally also has shaped) historical and social trends: it is not acontextual or independent from human experience. Rather than something existing independently from humans, mathematical practice is continuously constructed and reconstructed by humans. Glas (2006) investigates how mathematics can be both objective and a historically contingent practice.

The acknowledgment of mathematics as dependent on human context is essential within a discussion of DEI/EDI because it counters the pervasive notion that mathematics is based on intuition, possessed by some and not others. This myth serves to justify arguments which posit only certain people as capable of doing/understanding mathematics. Further, once mathematics is considered as a social practice, it becomes possible to acknowledge that different societies have different ways of doing mathematics, without positing a hierarchy between them. This insight was central to the development of ethnomathematics, a field pioneered by Ubiratan D'Ambrosio, who critiqued the separation of mathematics from questions of social justice and cultural issues. Many of the articles included in this section thus argue that mathematics education should address questions of justice and ethics, and aim to include students' backgrounds in the learning of mathematics.

From such a point-of-view, the history of mathematics is not only essential to the realization that mathematics has been constantly changing through societal interaction and that different ways of doing mathematics have existed, but it also becomes a central feature of mathematics education. While some argue that the inclusion of history in mathematics education serves to deepen the knowledge/understanding of students and offers different ways of learning and approaching mathematics (for such approaches see the section on using primary sources for mathematics education), others see it as a valuable end in itself. Indeed, a study of the history of mathematics confronts students with different ways of doing mathematics. This can both interrogate existing assumptions about mathematical knowledge production and invite students to position themselves not as passive recipients, but as meaning-makers in the classroom who are navigating their identity in relation to mathematics.

There are a number of websites designed to increase visibility of diverse role models in mathematics. The website that is now Mathematicians of the African Diaspora[6] was started in 1997 by Scott Williams to promote the contributions to mathematical research from members of the African diaspora. The name of the Mathematically Gifted & Black[7] website comes from a song sung by Nina Simone and cowritten by Weldon Irvine. It was created in 2016 and features the accomplishments of Black scholars in the mathematical sciences. Marie A. Vitulli created The Women in Math Project[8] in 1997 to make accessible some resources and information for and about women in mathematics. It is still hosted by the University of Oregon. As part of an ongoing project hosted by Agnes Scott College, Biographies of Women mathematicians[9] spotlights achievements in mathematics from both contemporary and historical women. The SACNAS[10] biography project has an online archive of stories by and about Chicano/Hispanic and Native American scientists with advanced degrees. Spectra, The Association for LBGTQ+ Mathematicians hosts Out Lists[11] to provide resources and support for interested mathematicians.

Additional sources suggest other ways that careful and thoughtful implementation of research in the history of mathematics can contribute to the discipline developing in a direction of greater inclusivity. Work in the history of mathematics can provide a platform from which to view the discipline critically and many of these works aim to alter our view of mathematical knowledge, its production and people. They have expanded the discussion to include previously ignored contributions, to take a global view of mathematical practice, and to challenge existing notions about the mathematical professions. Such resources can assist in highlighting past incidents of oppression and discrimination, while uncovering some roots and causes of persistent legacies today. Research in and resources from the history of mathematics can also be effective tools to help build a more inclusive discipline.

Additionally, there exists a body of work about using primary historical sources to teach technical mathematical content. This can facilitate differently accessible learning and also create connections beyond the standard curricular content.

Conclusion

One purpose of this source list is to facilitate an aim articulated in the recent QAA Benchmark Statement for MSOR: "values of EDI should permeate the curriculum and every aspect of the learning experience to ensure the diverse nature of society in all its forms is evident." Although the collection of sources listed here is only a sample of available resources, the list has been assembled as an aid to

[6] `https://www.mathad.com/home`

[7] `https://mathematicallygiftedandblack.com/`

[8] `https://pages.uoregon.edu/wmnmath/biographies.html`

[9] `https://mathwomen.agnesscott.org/women/women.htm`

[10] `https://www.sacnas.org/sacnas-biography-project`

[11] `http://lgbtmath.org/People.html`

make this work less daunting. It is not possible to provide a single document that will target every particular concern or priority given the wide range of contexts in which mathematical practitioners work. Specific implementation in a particular context will necessarily involve personal effort and perhaps attempts and revisions. This curated collection of resources nonetheless aims to give a broad range of documents that can support DEI/EDI efforts. Particularly important to this resource is the inclusion of work from the history of mathematics alongside texts that document DEI/EDI issues in broad cultural contexts as well as more specific educational contexts. The history of mathematics is especially well positioned to facilitate a move within the discipline of mathematics toward greater inclusivity. Historical studies can not only illuminate roots and causes of oppressive practices, but also challenge dominant narratives of who mathematics includes. The QAA statement states "it is highly desirable that students encounter a wide range of role models within higher education." While the community works to elevate an array of identifiable role models, sources listed here can perhaps help ameliorate challenging circumstances. In these ways and others, historical scholarship provides resources to help set a brighter course for the future of the mathematical sciences.


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Credits

Photo of Deborah Kent is courtesy of Radina Droumeva.
Photo of Emilie Aebischer is courtesy of Emilie Aebischer.
Photo of Stuart Neave is courtesy of Morag Neave.

Challenges and Rewards Facing Foreign-Born Female Mathematicians in the US

Bhamini M. P. Nayar

My story begins with my father, who was a renowned educator and a well-known poet in Malayalam, the language of Kerala State, India. I grew up watching his interactions with colleagues and former students who often came to see him and would discuss poetry and literature. I observed how he cherished his students and how much he was admired. He devoted his life to teaching and learning and left a lasting legacy of a true scholar. His ability to listen, explain things without making others feel intimidated, and his directness in conveying ideas were some of his inimitable qualities. His students often commented about his calm explanations and I observed that many sought the opportunity to be his students. He instilled in me a courage based on humility and purpose. My father encouraged me to study any subject I wanted but insisted that I study it well. I wanted to follow his career choice as an educator and chose to become a mathematician.

When I was in the tenth standard, my father wrote four lines of a poem just for me, which translates as the following:

> *My daughter, victory is at hand for actions of an unwavering and incorruptible mind; darkness envelopes the world with vengeance, but sun is sure to rise and rise it will.* [1]

The message of those four lines, were prophetic later when I faced all of the challenges mentioned below. They kept me going knowing that the sun's rays were sure to rise to chase away the darkness. The source of those rays came in the form of dedicated mentors, colleagues, friends, and especially students.

All my education was in India. I started my career in the United States soon after I received my PhD in mathematics from the University of Delhi, India. When I arrived, I was full of anticipation, especially since I was starting my career in a new country with a different culture. I was excited to be an educator and to be responsible for helping and preparing young students for their future careers. I knew it was going to be challenging because I was unfamiliar with the new environment. In retrospect, it was rewarding as well as challenging.

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Dedicated mentors who stand by us when situations become exceptionally challenging are invaluable. I was extremely fortunate to have had such help. While I was doing my PhD in India, I had the opportunity to read several articles by Professor James. E. Joseph. When I arrived in the United States, one of the first things I did was to ask to meet with Professor Joseph. He was generous with his time and during our first meeting he showed great interest in my work [1]. To quote from what I said at the time of Professor Joseph's retirement, "I went to see Professor Joseph that day with a very apprehensive mind of a nervous student going to discuss his/her ongoing research with a well-known and well established researcher in the field and I came home with the relief of finding someone with whom I felt comfortable to discuss my doubts, even those I thought I ought to know. That has been my experience with Professor Joseph, ever since" [2]. Although I had offers for positions from other universities, I accepted a position in his department to work with and learn from him [1]. He has been a mentor to me in the US until his recent passing.

When I started teaching, I was not familiar with the educational system in the US. As a young woman and fresh PhD, I was very apprehensive despite being passionate about teaching and research. I made every effort to prepare any topic I was going to teach well in advance so that I could teach my classes with confidence. I knew that a good student could recognize an instructor who was not confident about the subject and who was not in control of the class. During one of my first semesters teaching Calculus, I found that the students did not do well on the first test I gave. With the graded tests in hand, I asked the class what the difficulty was on the test. One student was bold enough to tell me, "well, we all speak English" and went on to say something more. For someone who had all her higher education in English, I had no doubt of my ability to explain mathematics in English, but of course I had the accent of someone who recently came from India. I did not notice the undertone of that student's comment until after the class when several students came to apologize to me for their classmate's statement.

I was fortunate to have had excellent teachers throughout my education. I graduated with an MSc in Mathematics from Union Christian College (UCC), Kerala, India. My success as an educator is significantly due to the early training I received there. Two of my professors, M. Madhavankutty (MM) and N. S. Neelakantan (NSN) taught me for seven years (two years of predegree courses, three years of BSc courses, and two years of MSc courses). MM's ability to control the class with a winning smile, never being agitated and NSN's ability to describe a problem in meticulous detail are phenomenal. I learned from each of these exemplary educators and I try to emulate them in my interaction with students, both in and out of the classroom [1]. The challenges we face in the classroom are generally manageable with dedication to the profession and prior preparation, with the understanding that the instructors and students are in this journey together, and that without challenge there is no excellence.

I recall that one of the questions I was asked by the vice president for academic affairs during the interview before I was hired was how I would handle a class in which students have different levels of prior preparation. My response was that I would look at the students directly while teaching so that if anyone was feeling lost, I would stop and explain what was not clear. From that question, it was evident to me that he valued an instructor's ability to communicate well with the students. I kept that in mind throughout my career as an instructor. Later when one of my graduate students commented that I move with students in their journey of learning, I felt satisfied to note that I was not unsuccessful in that attempt.

Every day brought new teaching challenges, primarily because my educational background was from a different system. In particular, the challenges were in navigating the cultural and training differences of students in the US and meeting the expectations of the profession without compromising my integrity. The rewards came in seeing my students succeed as a result of my teaching and molding their thinking about mathematics, as well as when I got to know the students individually and was able to help them navigate whatever difficult situations they were struggling with. Also, I felt rewarded when students took the time to let me know that all the hard work, I had expected of them was worth it, though they did not realize at the time. Anyone who has had similar experiences is bound to feel the same immense satisfaction.

Outside of the classroom, I was originally extremely happy to be among seemingly friendly colleagues and students seeking me out for help. I did not mind my office being crowded with students. When many of my colleagues were happy and eager to attend my wedding reception, and when the baby shower organized for me by the department was attended by many of my colleagues, I thought these were indications that I was accepted. Later, when it came time for recognition and rewards and I saw the role that power dynamics played, I understood the emptiness of that acceptance. One of my trusted senior colleagues had warned me (often indirectly) not to believe every statement of appreciation, yet I only realized later what a valuable lesson this was.

After that, I understood that I might have to face intimidating situations, both in the classroom and among colleagues. This was confirmed by my experiences of being constantly tested, questioned, and doubted. When I was asked to do a task for the department, I would complete it with professionalism and dedication, expecting to be appreciated as a valuable team member. To my dismay, my work would often be forgotten. For someone who has done above and beyond their peers in similar situations, such blows would make me lose balance and create self-doubt, which affected my overall well-being. When it came to career advancement, I was required to meet higher expectations than what was asked of my male peers. This made me wonder whether women must be better than their male counterparts in order to receive the same level of recognition. During each of these periods, my students were always the shining light. Working with them, seeing their appreciation of my dedication, and watching their progress always brought joy to my life.

The early training I received from my teachers, and how they listened to my doubts and guided me with respect, molded my approach to my profession. As my father expressed in his poem, the "actions of unwavering and incorruptible mind" chase away the vicious darkness. For me, what chases away the vicious darkness are the hardworking students, dedicated mentors of incorruptible integrity, and professional colleagues and friends who make the workplace atmosphere a supportive family environment.

Having gone through this journey as a mathematician, I wonder whether those who are starting their careers know how to navigate a fulfilling professional journey, without being battered along the path as I was. I think that this question is more pronounced for female mathematicians than for their male counterparts, especially those who come from a different country and culture. I pondered throughout my career how to mentor my students so that I could look back later in life with the satisfaction of having helped them achieve productive and satisfying careers. So, when I was invited to write an article for the Early Career section, I accepted the offer with the hope that sharing my experiences might be of some help to others who are in similar situations.

One lesson I have learned is that female mathematicians originally from another country who choose to be mathematics educators in the US may face the following challenges:

1. In the beginning, we are naive and just happy to be in the profession of our choice, so we might not be aware of judgements about us. But they can hit us hard later.
2. We are often subject to a disparity in career advancement in terms of tenure, promotion, salary and compensation, recognition, and workplace atmosphere, which may be presented as "keeping standards up."

The most important advice that I want to pass on is the importance of finding a good mentor with experience and integrity and following their guidance to navigate the rough waters you may face in your professional journey.

I shall conclude this essay with another quote from the article I wrote to Professor Joseph on the occasion of his retirement.

> I am fortunate to have had some great teachers and I remember them quite affectionately. Even though I was not fortunate to be one of your formal students, among my teachers, you are in the forefront. There was not a single instance when I had a conversation with you and I had not learned something, let it be mathematics or otherwise. You taught me to continue to be a student to be a better teacher; you taught me how to care for the students, while being insistent about their learning; you taught me not to be intimidated by pretentious individuals; you taught me not to lose self-respect while facing obnoxious and condescending opposition; you taught me to respect an opposite point of view; you taught me not to lose confidence in the right when faced with unfair opposition, and I am still learning from you not to be vengeful, but to concentrate on the task at hand. Yes, I was looking for a friend like you, a friend whom I can always count on, a friend to whom I can confide without being afraid of being judged, a friend who will invariably try to make me feel better when I feel sad, a friend who understands what I say without reading between the sentences, a friend who does not hesitate to oppose me and correct me when I am wrong and a friend whose silence is also eloquent. Yes, I was also looking for a mathematician whose mathematical abilities did not reach to arrogance. I thank God that I found in you not only such a great mathematician, but also a teacher, a friend and above all a genuine integral human being. [2]

I believe the understanding without judgement and support without being asked that I received from Professor Joseph are the type of mentoring that we all need. I know very well how it prepared me to be strong and courageous, and to speak the truth when necessary while maintaining my professional integrity. I strive to be the kind of mentor to my students and my junior colleagues that he was to me.

I hope that each person, especially a foreign-born female who embarks on a rewarding journey as an educator and mathematician, will find a mentor who will see her inner strength and mold that strength into an unbreakable spine to stand tall. I hope she will have supportive colleagues to work with who value her contributions. We all need that support to survive, flourish, and never give up, even when faced with intimidating situations or pretentious opposition. As we stand tall with our professionalism and integrity, we create lasting images for generations of students to follow. That is our legacy.

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Bhamini M. P. Nayar

Credits

Photo of Bhamini M. P. Nayar is courtesy of Bhamini M. P. Nayar.

Dear Early Career

Is it beneficial to pause the tenure clock after the arrival of a baby?

—Assistant Professor

Dear Assistant Professor,

I am a tenured associate professor in mathematics at the Ohio State University and mother of two young children, and I chose not to pause the tenure clock. However, one of my colleagues in a similar position decided to pause the clock twice on account of the birth of her two children. So, why do some people decide to pause the clock while others do not?

I don't think there is an easy answer to this question, and it depends on personal circumstances and the culture of your department, but I'll share the information that I have gathered on the topic. Ultimately, I hope this article acts as a catalyst for further dialog.

The tenure evaluation process for assistant professors takes place after a fixed period of time that is usually specified when the candidate is hired. During this time, the candidate is expected to produce a portfolio that reflects their productivity and contributions, their "brand" really, as an academic and member of the department. If you stop doing research while taking care of a new baby, it may make it difficult to meet research expectations for tenure. Pausing the clock is a way of "erasing" this inactive period from your CV when you go up for tenure. The idea being that no research is expected during this time.

There are primarily two types of tenure clock stopping policies. Gender-neutral policies offer equal benefits to both new mothers and fathers, while female-only policies are only available to women. For a discussion of the pros and cons of the two different types of policies see [1]. Note that pausing the tenure clock is distinct from taking leave. If your university offers paid leave or a semester course release, then take advantage of this.

My first son was born shortly before I was eligible to go up for tenure, and I felt that my file was already in good shape for promotion. I did not want to spend another year worrying about it, and so I chose not to pause the clock. If I had my children at the beginning of my tenure-track position, I may have chosen differently.

My aforementioned female colleague, however, had both of her children in the early years of her assistant professorship. From talking with her, I got the impression that stopping the clock was a necessity. When asked about her decision, she said, *"it was imperative that I extend the clock as I did not get much of a break to rebound back health/work wise quickly.... I would encourage others to extend the clock since it ends up being difficult to get back into the swing of research after having a child (although, the time and effort of being a parent never ceases even after the year of giving a birth). And by doing so, this allows the option to become more of a norm for other women without stigmatizing them."*

I strongly encourage you to talk with other parents at your university to hear about their experiences. Be assertive and don't hesitate to ask for advice about how and what to ask for when discussing your situation with your chair or dean. Did they get a course release? Did they take paid leave? Also, if possible, find someone in your department who already paused the clock and ask them if it was worth it. You may also talk to your human resources department to learn about your university's offerings to support the birth or adoption of a child.

Deciding to pause the tenure clock or not depends largely on your individual circumstances, including the culture of your department and the timing of the arrival of your baby. Stopping the clock will mean you get tenure later, but you will have more time to bolster your record. On the other hand, if you are confident that your academic file is already in good shape for tenure, then you may decide not to stop the clock. This comes with the benefit of being promoted earlier and having the task of getting your file together for tenure off your plate. For ideas on keeping the momentum going while on parental leave, see this piece by Yumeng Ou [2]. Congratulations on the addition to your family, and best of luck in your research endeavors.

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—*Early Career editors*

Have a question that you think would fit into our Dear Early Career column? Submit it to `Taylor.2952@osu.edu` or `bjaye3@gatech.edu` with the subject Early Career.

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Celebrating WOMEN'S

The Joan and Joseph Birman Fellowship for Women Scholars awards exceptionally talented women extra research support during their mid-career years.

Photo courtesy of Greta Panova

Greta Panova

Greta Panova, Gabilan Distinguished Professor of Science and Engineering and professor of mathematics at the *University of Southern California*, received the 2024–2025 Birman Fellowship for her research in Algebraic Combinatorics with connections to Representation Theory, Computational Complexity Theory within Theoretical Computer Science and with Probability and Statistical Mechanics. Separately, she works with a team of molecular biologists on modeling DNA repair dynamics.

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2021–2022 **Helen Wong**

2020–2021 **Karin Melnick**

2019–2020 **Lilian Pierce**

2018–2019 **Margaret Beck**

Joan S. Birman earned her Ph.D. in mathematics at 41 years of age and went on to become a leading scholar of braid theory and knot theory. Read about her career and her reasons for creating this fellowship on **www.ams.org/giving/giving_op/birman-fellowship-fund**.

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The 2023 Ruth Lyttle Satter Prize is awarded to **Panagiota Daskalopoulos** of *Columbia University* and **Nataša Šešum** of *Rutgers University* for their groundbreaking work in the study of ancient solutions to geometric evolution equations.

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The Unfailing Optimism of Dr. Gloria Ford Gilmer (1928–2021)

Tanya Moore and Josh Levy

Introduction

Figure 1. Gloria Ford Gilmer, undated.

Chatting with a friend at a wedding reception one day, a guest's braided hair caught Gloria Gilmer's eye.[1] Her friend, unimpressed, turned to her and said, "I can't stand that hairstyle!" "But look at his scalp," Gilmer replied. "It's completely tessellated with hexagons!" Her friend laughed. Gloria Ford Gilmer was known for seeing mathematics in places others did not. But she was also known for understanding the power mathematics could have in building a more just and equitable world. So the tessellations stayed with her, and within a few years became the basis for a project that aimed to bring hair braiding into the math classroom, asking not only how their geometry could help students build skills but how math could help stylists outside the classroom build community wealth. Over the course of a long career, Gilmer brought a unique vision and drive to her classrooms, to the professional spaces of the mathematics community, and to the growing international ethnomathematics movement.

In 2022, Gloria Gilmer became the first Black woman mathematician to have her papers archived in the Library of Congress. Her career blended a lifelong effort to empower students to learn math for themselves with a spectrum of broader civil rights concerns, linking what she sometimes called "mathematical power" to a deep commitment to equity and justice. Rooted in the deep mentorship she received in the classrooms and communities of segregated Baltimore, including at the historically Black Morgan State College, Gilmer developed a pedagogy grounded in research and focused on empowerment and care. Already a leader focused on educational policy at a national level, in the early 1980s she encountered the ethnomathematics movement, and thereafter her focus turned increasingly toward global intersections of math and culture. Always, however, she remained most urgently concerned with equalizing access to mathematical resources and skills. In many ways, her work remains an innovative response to issues that remain urgent, her interventions still capable of generating enthusiasm.

The Gloria Ford Gilmer Papers at the Library of Congress, now open for research, provide a rich new resource for understanding Gilmer's life and career, the history of the ethnomathematics movement in the United States, and its intersections with organizations like the American Mathematical Society and Mathematical Association of America. The collection runs to tens of thousands of items, ranging from meticulous daily logs that record Gilmer's prolific networking to her lesson plans, correspondence,

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[1] *See* https://www.intersectinglines.us/blog/dr-gloria-ford-gilmer *for a version of this article with citations included.*

research files, and extensive documentation of her committee work. Foreshadowing more recent discussions related to racial and gender inequities within the mathematics profession, the collection especially documents Gilmer's unique ability to merge an unstoppable activism with an unfailing optimism.

Early Life and Education

Born in 1928, Gloria Ford grew up in Baltimore, Maryland. Her father, James Ford, was an immigrant from Barbados and, by the 1930s, the owner of Ford's Grocery Store on Baltimore's George Street. Her mother, Mittie Ford (née Hall), had been a teacher in her native Georgia, but left home to study business in New York. An obituary of Mittie Hall Ford in a 1992 issue of the *Baltimore Sun* shares that in New York the two had met, having two daughters in Chicago before coming to Maryland where Gloria was born, the youngest of three. The 1940 census shows the Fords living in a rowhouse in the city's Harlem Park neighborhood, a transitioning area of West Baltimore where White residents had, just a few years earlier, finally failed in their efforts to keep out African American homebuyers, and which had become a center of the city's Black middle class. James Ford reported working 90 hours per week in the family grocery that year, and Mittie 35. At just four blocks from home, the children spent much of their time there as well.

Gilmer later remembered Ford's Grocery as the place she first became comfortable with math. "There," she recalled," I was always counting, weighing (measuring), locating items, and reasoning with customers." Later in life, she would identify those four activities, along with playing and designing, as the core practices from which mathematics derives, and which she argued were common to all of the world's cultural groups. In school, she encountered teachers whose mentorship extended outside of the classroom. There was Mrs. Henderson, a first-grade teacher whose simple act of placing a caring arm on her shoulder still resonated into adulthood. Miss Young, who recognized a fourth-grade Gloria's talent and asked her to arrive early to school to tutor classmates and to help grade arithmetic papers. And teachers from Mrs. Henderson, to a high school algebra instructor, to a college calculus professor who all did double duty in her church, as her Sunday school teachers.

Gilmer's early education, fueled by community support and peer learning, was probably as much a product of her own ambition as of the city's segregated, crumbling public schools. While the city's Black teachers were, on average, better credentialed than its White teachers—about 25% more had college degrees—Black children attended overcrowded schools in substandard facilities. A 1940 report by the *Baltimore Afro-American* newspaper found students in dilapidated portable buildings infested with rats and filth, an elementary school equipped with only chemical toilets, and several schools operating in long-condemned buildings. A newly revitalized local NAACP chapter lobbied urgently for improvements, and Baltimorean Thurgood Marshall began his career as the organization's lead attorney with a lawsuit demanding a new high school for the Black students of Baltimore County. What historians call the "long Civil Right Movement" was unfolding in Gloria's own backyard, her classrooms a site of political activism long before she reached adulthood.

After high school Gloria enrolled at Morgan State College, a historically Black school in northeast Baltimore, and pursued a degree in math. There, in her junior year, she encountered Clarence Stephens, whom she later credited with teaching her "how to learn mathematics."[2] Some of the staples of his pedagogy, flexibility with teaching methods, avoiding lecturing, and a conviction that, as Stephens later put it, "I could teach mathematics effectively to most of my students only if I were successful in protecting and strengthening their self-esteem," later emerged as core elements of Gilmer's teaching as well. She graduated Morgan in 1949 with a BS degree, high honors, and a math prize, placing fifth in her class.

The addresses at Gilmer's commencement foreshadowed a certain willingness to do the impossible. As a 13-year-old, Gloria had written to the *Afro-American* in 1941 to share her "goofy ambition" of sitting on a rainbow while painting it. Now she heard Methodist Bishop Alexander Preston Shaw exhort the graduates to perfection, and historian Merze Tate demand they not allow race to become a barrier to their betterment of a world plunged into a new atomic age, and in urgent need of their help.[3] She seemed prepared for a challenge.

Out Into the World

In the summer of 1949, the *Afro-American* reported that Gilmer's application to the University of Maryland at College Park had been ignored by admissions officers, along with those of six other African American applicants. College Park, her dream school, had never before admitted a Black mathematics student. Several of the applicants chose to sue, and NAACP lawyers took up their defense. Under cross-examination by Thurgood Marshall the following year, university president Harry Clifton "Curley" Byrd admitted that, outside of its law school, the university had

[2] *Clarence Stephens was noted for his innovative teaching methods and was the ninth African American to earn a PhD in mathematics.*

[3] *Remarks to the graduates were highlighted in the June 7, 1949 issue of* Baltimore Afro-American, *an African American newspaper established in the 1890s.*

refused admission to all qualified Black applicants to the College Park campus as a matter of policy, either diverting them to institutions like Morgan State or offering them scholarships to schools outside Maryland. In 1951 the university bowed to legal pressure and enrolled one of those students, Hiram Whittle, as the campus's first Black undergraduate. By then, Gilmer was already wrapping up a mathematics master's program at University of Pennsylvania, paid in part through a scholarship funded by Maryland's segregationist legislature.

At Penn, she found herself without familiar support systems, in classes taught by professors like number theorist Hans Rademacher, from whom she received her first C in a math course. That environment, Gilmer later recalled, caused her to doubt her abilities and "the quality of my former education in an all-black setting." But she graduated with good marks, and in 1952 began a position with the US Army, calculating exterior ballistics and bombing trajectories at the Aberdeen Proving Ground back in Maryland. Finding the work tedious, she remained only for the summer, launching into a peripatetic teaching career that had already included a stint at Hampton Institute, and now brought her to Virginia State College and then back to Morgan State. At Morgan, she coauthored two peer-reviewed mathematics journal articles with Luna Mishoe, the first to be published by an African American woman, and found encouragement to pursue further study.[4]

In 1956, Gilmer began a PhD program at the University of Wisconsin-Madison, where she soon felt so isolated she began to seek out friends by chatting up other Black people she ran into on the street. By the end of 1957, she had married Jay Gilmer, the son of a prominent Milwaukee doctor who, a few years earlier, had been appointed one of that city's first Black housing inspectors—and who would later rise to become administrator of the state's Equal Rights Division. "Defecting" from Madison to raise their two children, Gilmer returned to the classroom, first in the public schools of exurban Buffalo and then in Milwaukee, where she became that system's first African American math teacher. As she moved through historically Black and predominantly White institutions, Gilmer built a pedagogy grounded in a radical ethic of inclusion and care, one meant to cut against cultural frames that, as Sara Hottinger writes, assume a "normative, white, masculine mathematical subjectivity." One of her Morgan State students later recalled a "dynamically enthusiastic attitude" that was so jarring as to be, at first, "somewhat traumatic," calling Gilmer's classroom manner "demanding, inspirational, and understanding" and stirring him to "never settle for mediocrity in personal achievements."

Returning to tertiary education, in 1965 Gilmer became the first African American mathematics instructor at Milwaukee Area Technical College, taking a second job as the University of Wisconsin-Milwaukee's first Black woman mathematics instructor the following year. In 1973, she began a PhD program in curriculum and instruction at Marquette University. Though feeling unsupported by her advisor and by the institution, the work she did at Marquette allowed her to take a step back, to consider how curricular design and systemic inequities structured classroom instruction and helped produce racial differences in math achievement. Gilmer had long made the National Association of Colored Women's Club's motto, "lifting as we climb," a constant refrain. Now she began to codify her teaching philosophy, thinking more critically about her earliest years in front of the blackboard when she mimicked her own teachers and blamed her students when they failed. "In order to inspire my students," she insisted, "I must like them. In liking them, I shall seek to understand them in the same manner as I feel compelled to understand the subject I teach." It was a humanizing philosophy, one that emphasized personal relationships in the classroom and the impact of teachers who expect their students to succeed, and which urged instructors to empower students to support their own learning. Gilmer ultimately became the first Black woman to graduate from Marquette's School of Education, her commitment to leaving open the doors she had walked through stronger than ever.

By 1979, PhD in hand, the Gilmers were on the move again. Gloria took an associate professorship at Atlanta University and then one at Morehouse before moving to Washington, DC to serve as a research associate for the National Institute of Education. By 1980, she had also become the first Black woman on the Board of Governors of the Mathematical Association of America, and that role brought her into math classrooms all over the country. By 1983, she was in China, touring classrooms with 57 other educators at the behest of the Chinese Mathematical Society, not entirely impressed with what she saw but clearly moved by mathematical stories of the Great Cultural Revolution. Gilmer described math studies that were once "protracted" and "infused with political content," curricula "designed for immediate social needs" taught with intuitive instruction that built on student experiences, and math courses that combined algebra, geometry, and trigonometry. Encountering the foreign through the lens of her own

[4] *The articles were the first non-PhD theses to be published by an African American woman. Luna I. Mishoe and Gloria C. Ford, "On the Limit of the Coefficients of the Eigenfunction Series Associated with a Certain Non-self-adjoint Differential System,"* Proceedings of the American Mathematical Society *7, no. 2 (April 1956), 260–266; Luna I. Mishoe and Gloria C. Ford, "On the Uniform Convergence of a Certain Eigenfunction Series,"* Pacific Journal of Mathematics *6, no. 2 (1956), 271–278.*

experiences, as travelers often do, she began to imagine what useful things she might extract for classrooms back home.

The Gloria Gilmer of the early 1980s, driven by unhesitating ambition and unusual extroversion, had added an array of new skills to her repertoire: in education policy and grantmaking, in committee work and curricular design, and in activism.

Building Power through Connection

In 1968, the James Brown hit "I'm Black and I'm Proud" signaled a time of transition from the Civil Rights Movement to the Black Power era. Gilmer's career, and her activism, bridged both periods. In 1963, she attended the March on Washington. As she moved into college-level instruction in Milwaukee, her teaching and scholarship began to evolve in concert with wider movements dedicated to Black pride and a reclamation of identity, culture, and community. On campus, Gilmer increasingly merged advocacy for her African American students with sharp critiques of institutional racism. In a 1976 speech before the UW-Milwaukee Board of Regents, for instance, she insisted that the university's struggle to retain African American students stemmed from their "academic and social isolation," and called for a suite of changes ranging from multidisciplinary evaluations of minority student achievement to the elimination of dull remedial courses that foreclosed self-directed discovery and stifled innovative thinking. Off-campus, she became involved in a dizzying array of community organizations, her activism often intersecting with her educational concerns.

Gilmer was a dedicated Episcopalian who described her ministry as "anti-racism wherever it leads me." She was a soror, a member of Alpha Kappa Alpha since her time at Morgan State. She was a member of the YWCA, and in 1970 became one of the framers of a revision to its mission statement, an added phrase committing the organization "to eliminate racism wherever it exists and by any means necessary." In the following decades, Gilmer would take on leadership positions with the National Council of Negro Women (as chair of their Commission on Education) and the National Urban Coalition (as member of their Technical Advisory Board). She served as an advisor on the National Urban League's Project PRISM, a national education reform project. She joined the Milwaukee Ethnic Council and the Haiti-based Desgranges Foundation, and attended meetings of the Milwaukee Socialist Party well into her 80s, according to a web profile she drafted in 2000.

Always, Gilmer refused to compartmentalize her faith, commitment to social justice, and focus on solutions for improving educational outcomes. Instead, they combined to bring an intersectional lens to her work no matter where she found herself. When she saw civil rights organizations working to address classroom issues without the aid of professional mathematicians, she offered her aid as an academic. When she saw the academic community working fruitlessly to solve problems for minorities in mathematics, she offered her cultural expertise. By 1986, Gilmer had launched the Family Math Project, an initiative designed to help preschoolers from low-income, primarily African American families develop math skills by enabling their parents to develop their own math literacy. It was a research-driven approach that aimed to help families integrate mathematical concepts into their daily lives, empower parents to teach math to their own children, and foster computer literacy for everyone in the process. It was also a staple of Gilmer's activist philosophy, a nuts-and-bolts solution to systemic issues of racism and poverty, grounded in a faith that both could ultimately be solved. As she had argued at Marquette, children could only "grow and flourish in all the beauties of their inner spirits and outer gifts" when racism had been "eliminated," and she clearly believed the elimination of racism was possible.

Increasingly, as Gilmer moved from institution to institution, found opportunities to observe the teaching of others, and deepened her commitment to social justice, she came to recognize how inequities in math achievement had become a natural byproduct of institutional racism. In the mid-1970s, she wrote that one of the major functions of America's educational system had been to "channel privilege—to determine who will receive wealth, power, and prestige." African American students, she argued, "are painfully aware that the white university was not made to accommodate them." If it had been, she believed, universities would have created learning environments in which Black students could also "achieve academically and develop intellectually and emotionally." It was a problem she believed could be solved.

Remediation efforts were one of her targets. Gilmer argued that remedial mathematics courses limited academic advancement and disempowered African American students by appealing to "the areas of their greatest weaknesses and at the dullest level of the subject." She advocated for strategies that inspired a desire for knowledge and self-directed explorations that allowed students to focus on "areas of personal strength rather than repeating and failing the basics." Students had to be empowered, and connected—first to their peers, then to their teachers. The use of standardized testing was another issue. The tests, she insisted, had often served as gatekeepers, steering African American students into remedial classes, and were inadequate measures of student learning. Recognizing that test designers and administrators may not have intended to discriminate, Gilmer nevertheless argued that "neutral and objective tests which are administered for one purpose but which lead directly to racist policies are racist," and that scores could not be properly interpreted without a recognition of disparate access to resources and differences in curriculum implementation. In a Cold War political environment increasingly concerned over students losing their edge in STEM education, and often turning to standardization for help, Gilmer insistently embraced individuality and cultural distinctiveness as the true routes to effective education.

At times, Gilmer's problem solving even verged into techno-optimism. Already seeing disparities in student access to computers in the 1980s, she argued that, "unless steps are taken now, 'new' inequities will exacerbate current ones." For all children, she saw computers as a tool to strengthen and complement instruction and application, one that might help them apply math to simulations of real life problems. Computers, she believed, could empower students to solve ever more complex problems, to use "inductive approaches that motivate mathematical conjecturing," and to discover graphical interfaces that would help sharpen their intuition and foster a liberating mathematical experimentation. Always attuned to the equitable distribution of educational resources, she envisioned a brighter future where democratized computing equalized access to mathematical skills. But she envisioned a darker future as well, one where an innovative new resource again fell largely into the hands of those that needed it the least.

Gilmer shared these critiques and solutions widely, through publications and workshops, conference papers and conversations. And in 1984, she made another international trip, this time to Adelaide, Australia, for the Fifth International Congress of Mathematical Education (ICME-5). There she presented a paper on the IEA Mathematics Study, an international math assessment test, offering suggestions for future improvements like using the test to inform curriculum planning and professional development. Her attendance at ICME-5 would prove to be fortuitous. Already a professional with a national profile, Gilmer was about to encounter a framework that would help to crystalize her work and activism, and expand her ministry to a global platform.

Ethnomathematics and the Beginning of a Global Movement

As Gilmer was building a pedagogy grounded in a pursuit of equity and the recognition of the classroom as a cultural and political space, Ubiratan D'Ambrosio, a Brazilian mathematics educator and historian, was articulating an approach to make mathematics instruction more accessible to a broader population. D'Ambrosio felt that greater math literacy could serve as a counterweight to inequities among individuals and communities, and contribute to creating a more peaceful society. Inspired by the knowledge and practice of mathematics in Brazilian indigenous cultures and grounded in his own research in the history of mathematics, he found himself at the cutting edge of research on the intersection of mathematics and culture. At ICME-3 in Germany in 1976, D'Ambrosio had organized a panel entitled "Why Teach Mathematics." That panel helped create a new field in mathematics education, ethnomathematics. Eight years later, at ICME-5, D'Ambrosio was delivering the plenary address, entitled "Socio-Cultural Bases for Mathematical Education," in which he argued that mathematics instruction should be grounded in the social and cultural context of students.

D'Ambrosio's remarks must have captured Gilmer's interest right away. He seemed to be giving voice to many of the ideas she had already been formulating. Gilmer later summarized her reaction to the speech:

> "Before that, like Bertrand Russell, I too had experienced the cold and austere beauty of mathematics. Early in my career, as a student and later colleague of Clarence Stephens, I learned to value the rigor, precision and resilience of mathematicians and to appreciate the social and humanitarian values implicit in this scholarly community such as respect, solidarity and cooperation. Then as now, my modes of understanding, learning styles, intuition, emotions, and use of mathematics are all closely bound to my cultural heritage both as an African American Christian and as an active member of the mathematical community. Now, however, I am more aware of the immense potential in the development and acquisition of mathematical knowledge by the inclusion of the concept of ethnomathematics. Ethnomathematics as a field of study connects mathematical concepts, their acquisition and application through cultural origins. In this way, ethnomathematics paves the way for reform in mathematics education and new horizons in mathematical research."

D'Ambrosio framed math as a cultural product, an insight that resonated with Gilmer's own experiences and observations, and with her prior research.

One tenet of ethnomathematics that especially spoke to Gilmer was its assertion that the cultural neutrality of mathematics was a myth, an insight that opened the door to almost limitless explorations of how mathematical ideas might be iterated throughout the world. It also supported her existing commitment to helping students

Figure 2. Gloria Ford Gilmer at her Milwaukee home, c. 1970s.

connect math to their everyday lives. D'Ambrosio's distinction between learned mathematics and ethnomathematics proved useful to her as well. Learned mathematics refers to the formal math most often taught in schools, a body of knowledge that developed slowly over time but largely became disconnected from the social environment of most students. Learned mathematics, D'Ambrosio argued, serves to reinforce privileged pathways into higher education, foreclosing practical applications that could be meaningful to a broader population of students. Ethnomathematics, on the other hand, examines how different cultures conceptualize, communicate, and apply mathematical ideas in their everyday life. The opportunity to introduce students to the social context of mathematics and make lessons more culturally relevant, Gilmer believed, could help build student self-confidence and demonstrate how math could contribute to their success. By 1985 at the National Council Teacher of Mathematics (NCTM) meeting in San Antonio, Gilmer, D'Ambrosio, and Rick Scott of the University of New Mexico decided to form a study group in ethnomathematics. Gilmer would serve as its first chair.

One of Jill Gilmer's earliest memories of her mom was of her sitting on a stool in their kitchen, talking on the phone. "That was her favorite spot," she recalled. "Just sitting on that little stool. She was usually laughing. I mean, she loved people. I call her an extreme extrovert, which is a little unusual for mathematicians. But she absolutely loves people." One of Gilmer's great gifts was as a builder of relationships, and she brought those skills to bear in her longtime role as chair of the International Study Group on Ethnomathematics (ISGEm), a position she held from 1985 to 1996. Gilmer developed an increasingly global footprint, traveling to places such as Haiti, Budapest, Australia, the Soviet Union, Quebec, Bahia, and China as she built and sustained relationships with other organizations in the mathematical community. Her influence allowed for the cross-pollination of ideas and the formation of new partnerships to inform curriculum development, shift assumptions about what underrepresented students could achieve, and share solutions that could broaden participation in mathematics.

Already an experienced hand at committee work, Gilmer began to bring her ethnomathematics commitments to professional mathematics organizations as well. In 1985, she was appointed chair of the AMS-MAA-AAAS joint Committee on Opportunities in Mathematics for Underrepresented Minorities (COMUM) by American Mathematical Society president Julia Robinson. Expanding Gilmer's work as director of the MAA's Blacks and Mathematics program, COMUM's charge was to broaden the membership of the sponsoring organizations and increase participation by underrepresented minorities. "The real issue for people of color," Gilmer argued in 1995, "is the intellectual climate that claims 'this cannot be done by you.'" To change it, she proposed a suite of changes, from publicizing the work of successful achievers, to connecting minority mathematicians to larger professional communities, to developing a directory of minorities in mathematics-related careers and holding information sessions at regional and national meetings. Gilmer also began to coordinate a series of skits, held at Joint Math Meetings as a way to dramatize issues related to gender and racial awareness. Often, she sought out opportunities to interject ethnomathematics research into professional mathematics spaces, and to leverage her inquisitive and inclusive nature to share ideas that could shift the culture of mathematics.

Her work with curriculum development began to intersect with ethnomathematics as well. "Mathematics curricula," she wrote, "have been slow to change, due partly to a failure to separate the universality of truth of mathematical ideas... with the cultural basis of that knowledge." The need had become urgent, she insisted, "to multiculturalize the mathematics curricula." In 1984, Gilmer had founded a small consulting business, Math-Tech, which helped her bring ethnomathematics theories into classroom practice through the development of new programs and evaluations of existing initiatives. Often, Gilmer served as a charismatic interpreter of discoveries made by others. She was fond of sharing the stories she first heard from D'Ambrosio at ICME-5, of the geometrical patterns passed down through generations of indigenous boat and house builders in the Amazon. She spoke about Marcia Ascher's work on a game called Mū tōrere played by the Māori people of New Zealand, and Paulus Gerde's examination of

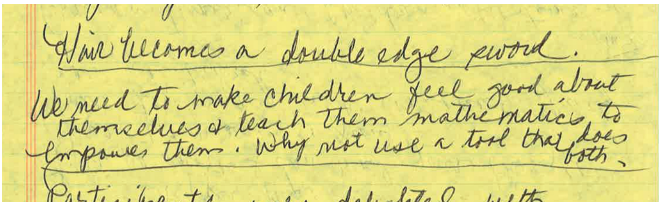

Figure 3. Gilmer's notes following a presentation on mathematics and hair braiding, c.1990s. Gloria Ford Gilmer papers, Manuscript Division, Library of Congress.

diverse arithmetical relationships in Angolan sand drawings. In 1993, she served as project director of a major textbook series published by Addison-Wesley, entitled *Building Bridges to Mathematics: Cultural Connections*, perhaps her most comprehensive curation of ethnomathematical approaches adapted for practical classroom use. And she traveled the country and the world, observing classrooms wherever she could and arguing that mathematics could be found everywhere in everyday life, in functional artifacts ranging from fishing nets and quilts to baskets, chimneys, and calculating devices.

Gilmer developed her own lesson plans as well. One introduced the concepts of mean, variation and standard deviation by examining comparative shopping costs for food at different markets, and further introduced statistical ideas by having students engage in studies of professional journals related to their major. Another showed how the UPC codes found on products could be translated as symbolic representations of a number and product. She designed lessons centered around origami cranes, and the distances, angles, and paths Michael Jordan followed as he traveled down the court to make a basket. Like her ethnomathematics colleagues, she found herself always in search of the perfect metaphor, the everyday activity or object that silently harbored generations of mathematical wisdom.

So in the spring of 1998, Gilmer joined Ron Eglash before an excited, overflow crowd to talk about the geometry of African American hairstyles. It was the National Council of Teachers of Mathematics annual meeting, held that year in Washington, DC, and 40 minutes prior to their start time the large room had nearly filled to capacity. Organizers, with growing concern over the city's fire codes, were ultimately forced to turn some of the curious away. Gilmer was animated. She roamed the crowd and pointed out patterns in the hairstyles of audience members. She showed images of tessellations, spirals, concentric circles, symmetries, and fractal patterns in braided hair. She insisted that students were capable of drawing mathematical concepts from culture and nature, and emphasized the power of math skills to generate community wealth. Eglash traced braiding patterns back to African material culture, and storyboarded a computer math lab students might use to explore them in the future, a descendant of which is still in use by students today.[5]

"The idea," Gilmer later wrote, "was to determine what the hair braiding and hair weaving enterprise can contribute to mathematics teaching and learning what mathematics can contribute to the enterprise." Meticulously researched in hair salons and in beauty magazines, hair braiding became the metaphor Gilmer was most known for, one that linked the beauty of natural hair to the politics of inequitable access to mathematical knowledge, and to the economics of community business. It still resonates.

Conclusion

Mathematicians often use power as a synonym for an exponent's multiplier effect. Gilmer offered her own definition of mathematical power: the ability to "discern and investigate through a variety of mathematical methods the mathematical relationships observed in patterns and structures in one's own surroundings." Other intersections of math and power structured much of her life and career: the inequitable distribution of educational resources by race and class, the underrepresentation of minorities in professional mathematics spaces, the earning power math skills can bring to individuals and communities, the hegemony of a mathematics curriculum that purports to be culture neutral, the geopolitical consequences for a nation falling behind in STEM education. Gilmer also held her own kind of power, an influence born of a radical empathy and charismatic extroversion, and the ability to view deeply ingrained issues like institutional racism as just another problem to solve, if she could only make the right calculations.

What makes Gilmer's work particularly compelling is how relevant it remains. Jill Gilmer has said that she wishes her mother's views had been more widely understood and embraced, particularly her underlying beliefs that the capacity to learn is within everyone and that each person has something to contribute. Like a good teacher, much of Gilmer's legacy can be found in the ways she inspired people to increasingly higher levels of achievement. Yet, of course, inequities still persist in mathematics achievement. "It feels like we're kind of still back where we were," Jill has said, "trying to figure this thing out. So that's a little bit frustrating but, that's the beauty of having her papers at the Library of Congress. Maybe the word will get out and we'll start to realize it's not as hopeless as it looks, there's work in this area that's already been done."

In 2020, the AMS established the Claytor-Gilmer fellowship in order to further excellence in mathematics research and help generate wider and sustained participation by

[5] *See the Cornrow Curves project at:* `https://csdt.org/culture/cornrowcurves/index.html`.

Figure 4. Gilmer in front of the US Supreme Court and Library of Congress during Barack Obama's 2009 presidential inauguration.

Black mathematicians. It is named for William S. Claytor, the first African American man to publish a research article in a peer-reviewed mathematics journal, and Gloria Ford Gilmer.

Tanya Moore

Josh Levy

Credits

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Figure 2 is courtesy of Jill Gilmer.
Photo of Tanya Moore is courtesy of Albina Khazan.
Photo of Josh Levy is courtesy of Tamaki Levy.

An Enchantress of Number? Reassessing the Mathematical Reputation of Ada Lovelace

Adrian Rice

Introduction

Figure 1. Ada Lovelace, pictured around 1843.

If any reader of the *Notices* were asked to name a female mathematician of the 20th or 21st century, they would have no difficulty whatsoever. Giving them the same task but changing the parameter to the 19th century would present more of a challenge, but the names of several notable figures would no doubt still emerge. Mathematicians such as Sophie Germain, Sofia Kovalevskaya, Christine Ladd-Franklin, Charlotte Scott, and Mary Somerville would probably all feature—along perhaps with Emmy Noether, who although born in 1882 did all her significant mathematics in the 20th century. But if this second task were given to the general public, far fewer names would appear, and the occurrence of one would probably outweigh all others by a considerable margin: Ada Lovelace.

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This is not surprising as today Lovelace ranks as one of the most famous female scientists in history. Lauded as "the first computer programmer" and praised for her visionary ideas, she is held up regularly as an inspiration for women in science and mathematics. She has been immortalized in literature, in plays, and on the screen. Exhibits relating to her life and legacy crop up in numerous science museums around the world, while her biography and earnest appraisals of her work adorn countless blogs and websites. Her homeland has been similarly keen to commemorate her as one of its own: between 2015 and 2020 her picture featured inside every newly issued British passport and in 2023 the UK's Royal Mint issued a commemorative £2 coin in her honor. And all of this despite the fact that she made no major mathematical or scientific discovery, proved no theorems, and her entire scholarly output amounts to a single paper published in 1843.

Augusta Ada King, Countess of Lovelace was born in London on December 10, 1815. The only legitimate child of the romantic poet Lord Byron, she lived a privileged but somewhat isolated life. After her parents' separation in early 1816, she was raised largely by her mother, the redoubtable Lady Byron, but never managed to escape the all-embracing effects of her father's immense fame. She was what we would today call a "celebrity," famous in her lifetime simply for being Lord Byron's daughter, with her every move watched and scrutinized by polite society. Even after her death on November 27, 1852, at the early age of 36, her obituaries still focused on her Byronic connection. A fitting modern analogy would be the life of Lisa Marie Presley. Both women were the only children of fathers who died young, having experienced unprecedented levels of fame, enormous wealth, and notoriety. Both women lived lives defined, whether they liked it or not, by their status as the sole offspring of a dead celebrity. And both women themselves sadly succumbed to untimely deaths.

But that is where the analogy ends. Lovelace lived the privileged life of an English Victorian aristocrat, moving in social circles that included royalty and influential nobility, whose servants tended to her every need, and whose leisure activities included riding to hounds, lavish soirées, and nights at the opera. She was unusual for a woman in the 19th century, not only for having interests in science and mathematics, but for having been encouraged to pursue them, first by her mother and then by her husband, the Earl of Lovelace, who also cultivated an amateur interest in scientific matters. Her mathematical proclivity culminated in her publication of the 66-page paper on which her present-day fame rests [Men16]. It contains a theoretical account of a machine called the *analytical engine*, designed by the Victorian mathematician, inventor, and polymath Charles Babbage, which, had it ever been built, would have been the world's first general purpose computer—100 years before the beginning of the modern computer age.

Lovelace's paper contained seven lengthy appendices, or "Notes," with the last one, "Note G" [Men16, pp. 94–105], being her chief claim to fame. In this, in addition to some thoughts on the possibility of artificial intelligence, she outlined an iterative process by which Babbage's machine could compute Bernoulli numbers, coefficients of the $x^n/n!$ terms in the infinite series expansion

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}.$$

Lovelace described how to use this definition to derive a useful identity that would generate each Bernoulli number in turn. Since the above equation is equivalent to

$$\frac{x}{x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{B_{2n-1}}{(2n)!}x^{2n}$$

or

$$\frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots}=1-\frac{x}{2}+B_1\frac{x^2}{2!}+B_3\frac{x^4}{4!}+\cdots,$$

multiplying through by $1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots$ would give

$$1=\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right)\left(1-\frac{x}{2}+B_1\frac{x^2}{2!}+\ldots\right)$$

or

$$\begin{aligned}1=1&+\left(\frac{B_1}{2}+\frac{1}{3!}+\frac{1}{2\cdot 2!}\right)x^2\\&+\left(\frac{B_1}{2}\cdot\frac{1}{2\cdot 3}+\frac{1}{5!}-\frac{1}{2\cdot 4!}+\frac{B_3}{4!}\right)x^4+\cdots.\end{aligned}$$

From this expression, it followed that the general coefficient of the x^{2n} term would be

$$\begin{aligned}&\frac{1}{(2n+1)!}-\frac{1}{2}\cdot\frac{1}{(2n)!}+\frac{B_1}{2!}\cdot\frac{1}{(2n-1)!}\\&\qquad+\frac{B_3}{4!}\cdot\frac{1}{(2n-3)!}+\cdots+\frac{B_{2n-1}}{(2n)!}=0\end{aligned}$$

or, when multiplied by $(2n)!$

$$\begin{aligned}&-\frac{1}{2}\cdot\frac{2n-1}{2n+1}+B_1\cdot\frac{2n}{2}+B_3\cdot\frac{2n\cdot(2n-1)\cdot(2n-2)}{4!}\\&\qquad+\cdots+B_{2n-1}=0.\end{aligned}$$

Lovelace then described how, using a series of punch-cards, the analytical engine could be instructed to successively plug $n = 1, 2, 3, \ldots$ into this formula to produce the famous but nonintuitive sequence of Bernoulli numbers, $B_1 = \frac{1}{6}$, $B_3 = -\frac{1}{30}$, $B_5 = \frac{1}{42}$, $B_7 = -\frac{1}{30}$, etc. (The Bernoulli numbers $B_2, B_4, B_6, \ldots$ are all zero. It should also be observed that Lovelace's notation for the Bernoulli numbers differs somewhat from that commonly used today, in which they would be listed as $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}$, etc.) Although the algorithm she devised was never run and the computer for which it was intended was never built, it is the process based on this algebraic manipulation for which Ada Lovelace is chiefly remembered today. And this is what has gone down in history as the world's "first computer program."

As we will see, in the 180 years since the appearance of Lovelace's paper, opinions on her mathematical proficiency have varied considerably, with some assessments verging on the hagiographic, giving her credit for more than is perhaps due, while other writers have cast considerable doubt on whether she had the ability to understand—let alone write—such a technically proficient account, implying that it was in fact due to Babbage. On the one hand, in order to explain the process of calculating Bernoulli numbers, Lovelace must have had an understanding of the mathematics underlying the procedure, which was by no means trivial. On the other, this would largely have been beyond the capability of anyone who had not had some kind of university-level training in mathematics, which in the mid-19th century was unavailable to women, since they were not allowed to attend college or university. So which view of Lovelace's mathematical ability is correct? Did she have the mathematical competence to write and understand the mathematics contained in her 1843 paper?

Or was she essentially a mathematical ignoramus, as some have argued?

In this article, we will attempt to answer this question. While others have focused their attention on assessing Lovelace's theoretical understanding of the mechanical operations of the analytical engine, we limit our analysis purely to an assessment of her mathematical ability. We begin with a brief survey of the mathematical topics to which she would have been exposed in the ten-year period culminating with the publication of her 1843 paper. We then look at the changing perspectives on Lovelace's intellectual caliber over the years, from the 19th century to the present day, before focusing on the three most fundamental (and compelling) arguments that have influenced almost all negative assessments of her mathematical ability. Finally, we close with a timely postscript on Ada Lovelace and artificial intelligence. Much of the analysis below is informed by research carried out in collaboration with my colleagues Christopher Hollings and Ursula Martin, comprising a detailed study of the mathematical papers in the Lovelace archive, now housed in the Bodleian Library in Oxford. This research resulted in the research papers [HMR17a] and [HMR17b], as well as an illustrated expository book [HMR18], all of which may be consulted for further details.

Lovelace's Mathematical Education

Throughout her childhood, Ada's keen interest in science and mathematics was actively fostered by her mother. By the age of 18, her mathematical education consisted of a thorough grounding in arithmetic and some basic geometry, already surpassing the level attained by most children at the time. Her introduction to London's high society in 1833 led to her meeting two of the foremost British mathematical scientists of the day, Charles Babbage and Mary Somerville, both of whom encouraged the continuation of her mathematical studies.

On the advice of a family friend, Dr. William King, who had studied mathematics at Cambridge in the early 1800s, she began a focused course of reading under his direction. He advised:

> Begin with Euclid, then Plane & Spherical Trigonometry, which is found at the end of Simson's Euclid, then Vince's Plane & Spherical Trigonometry, then Bridge's Algebra [HMR17a, p. 229].

But this guidance, while standard for a British university course circa 1800, was quite out of date by the 1830s. By this time, the old-fashioned synthetic, geometrical methods in vogue in England at the beginning of the century were quickly being replaced by newer and more progressive analytic and algebraic techniques imported from continental Europe, with which Lovelace's friends like Somerville and Babbage were quite familiar. Somerville in particular was able to provide some useful assistance on various mathematical points, later recalling that Ada was "much attached" to her and "always wrote to me for an explanation when she met with any difficulty" [Ste85, p. 50]. Indeed, by 1835, shortly after her marriage, we find the 19-year-old Ada writing to Mrs. Somerville about her current mathematical studies [Too92, p. 83]:

> I now read Mathematics every day & am occupied in Trigonometry & in preliminaries to Cubic & Biquadratic Equations. So you see that matrimony has by no means lessened my taste for those pursuits, nor my determination to carry them on....

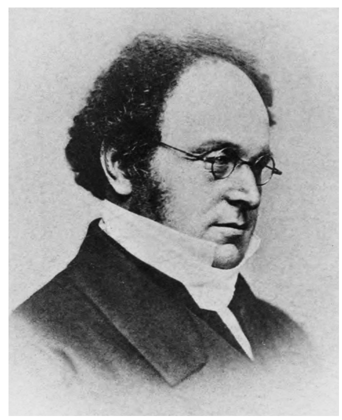


Figure 2. Augustus De Morgan.

Although her marriage caused no immediate interruption of her mathematical education, bearing three children in quick succession between 1836 and 1839 certainly did. But she was eager to resume her studies and, in the summer of 1840, she began a course of study under the tutelage of the mathematician and logician Augustus De Morgan. After about eighteen months, Lovelace had covered much of the material included in a typical undergraduate course of the time: basic algebra, trigonometry, logarithms, complex numbers, functions, limits, infinite series, differentiation, integration, and differential equations. It was during this period that she learned the skills that would be essential for her mathematical commentary on Babbage's analytical engine a couple of years later. In particular, it was via her study with De Morgan that she first came across Bernoulli numbers.

In late 1842, Lovelace was invited by the scientist Charles Wheatstone to translate an account of Babbage's analytical engine, recently published in French by the Italian politician and engineer Luigi Menabrea. On learning that she was working on an English translation, Charles Babbage asked her

> why she had not herself written an original paper on a subject with which she was so intimately acquainted? To this Lady Lovelace replied that the thought had not occurred to her. I then suggested that she should add some notes to Menabrea's memoir; an idea which was immediately adopted [Bab94, p. 136].

This led to the composition of Lovelace's seven lengthy "Notes" which, at 41 pages, when added to her 25-page translation, resulted in a paper more than double the length of the original.

Lovelace spent the first half of 1843 feverishly working on her "Notes," with multiple revisions and corrections flying back and forth between her and Babbage. As she told him, "I am working *very* hard for you; like the Devil in fact" [Too92, p. 198]. It was in July that she came up with the idea for her final note:

> I want to put in something about Bernoulli's Numbers, in one of my Notes, as an example of how an implicit function may be worked out by the engine, without having been worked out by human head & hands first [Too92, p. 198].

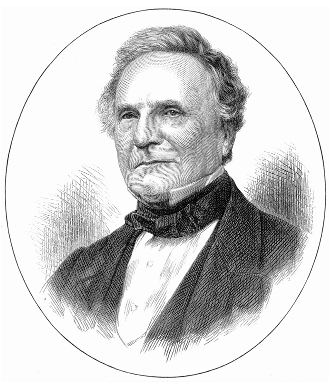


Figure 3. Charles Babbage.

Babbage later recalled that he wrote out the necessary algebra for her, "to save Lady Lovelace the trouble," but that "she sent [it] back to me for an amendment, having detected a grave mistake which I had made in the process" [Bab94, p. 136]. The result was the famous "Note G" containing her well-known "program."

When the paper was published at the end of the summer, Lovelace made no attempt to hide the obvious pride she felt in her work, declaring herself "very much satisfied with this first child of mine" [Too92, p. 211]—although before long this satisfaction had morphed into considerable hubris. Using the word "analyst" as it was often used in those days, to mean "mathematician," she wrote [Too92, p. 215]:

> I do *not* believe that my father was (or ever could have been) such a *Poet* as *I shall* be an *Analyst*.... The more I study, the more irresistable do I feel my genius for it to be.

Yet despite this bold claim (and several others), her paper of 1843 remained her only publication and the only other mathematics she wrote—none of it original—is contained in her surviving letters to Babbage, Somerville, and De Morgan. It is perhaps for this reason that subsequent opinions on her mathematical merits have not always been favorable, as we shall see.

Evolving Perspectives on Lovelace

Given Lovelace's high social rank and the scarcity of women in any branch of science in those days, it is perhaps unsurprising that contemporary evaluations of her mathematical talent were uniformly positive. But, even taking these factors into account, they are still overwhelmingly effusive. Shortly after her paper on his machine had appeared in 1843, Babbage was referring to her as "the Enchantress of Number," going even further in a letter to Michael Faraday, in which he described her as

> that Enchantress who has thrown her magical spell around the most abstract of Sciences and has grasped it with a force which few masculine intellects (in our own country at least) could have exerted over it. [Jam96, p. 164]

The following year, De Morgan, who as her tutor had witnessed her mathematical development firsthand, wrote that her "power of thinking on these matters ... [was] utterly out of the common way for any beginner, man or woman." He went on to say

> Had any young [male] beginner, about to go to Cambridge, shown the same power[s], I should have prophesied ... that they would have certainly made him an original mathematical investigator, perhaps of first-rate eminence [Ste85, p. 82].

After her death, obituarists were at pains to emphasize her mathematical talents. The *Athenæum* reported [Ano52a]: "Like her father's Donna Inez, in *Don Juan*—'Her favourite science was the mathematical,'" while the *Examiner* observed [Ano52b]:

> Her genius, for genius she possessed, was not poetic, but metaphysical and mathematical, her mind having been in the constant practice of investigation, and with rigour and exactness. With an understanding thoroughly masculine in solidity, grasp, and firmness, Lady Lovelace had all the delicacies of the most refined female character.

The years that followed saw similarly fulsome praise come from a variety of sources. In 1860, the English author John Timbs praised "the noble and all-accomplished" translator of Menabrea's memoir, remarking that

> The profound, luminous, and elegant notes forming the larger, and by far the most instructive, part of the work, and signed A. A. L., are all by that lamented lady. [Tim60, p. x]

In his 1864 autobiography, Babbage concurred with this assessment of Lovelace's "Notes," believing that "Their author has entered fully into almost all the very difficult and abstract questions connected with the subject" [Bab94, p. 136].

A generation later, in an effort to collate and preserve documents relating to his father's life's work, Babbage's son Henry published a substantial volume of writings in 1889, including the entirety of Lovelace's 1843 paper. Sixty years later, as the modern computer age dawned, this book was rediscovered by the Cambridge physicist Douglas Hartree, who found himself drawn to Lovelace's paper. Struck by the fact that "some of her comments sound remarkably modern" and impressed by her discussion of the numerical computations and iterative procedures, he observed that "she must have been a mathematician of some ability" [Har49, p. 70].

Hartree's book was read by Alan Turing, who in a famous paper of 1950, challenged Lovelace's views on artificial intelligence [Tur50] (see Postscript). Turing also contributed to a volume of essays on computing edited by the scientist and educationalist B. V. Bowden, who agreed with Hartree's assessment of Lovelace's abilities, describing her as "a mathematician of great competence" [Bow53, p. xi]. But we note here that, with the passage of time and increasing reliance on second- and third-hand information, assessments of Lovelace's intellect, while still positive, had become a little more measured, with words like "some ability" and "competence" replacing the more indulgent uses of "first-rate" and "genius" by Lovelace's contemporaries.

By now, along with Babbage, the name of Ada Lovelace was known and accepted among those in the burgeoning community of computer programmers and engineers as one of the pioneers in the prehistory of their field. It was in this spirit that the US Department of Defense chose *Ada* as the name of its newly developed programming language in 1979.

The 1970s saw the publication of the first full-scale biography of Lovelace, by experienced Byron scholar Doris Langley Moore. This study, like many which were to follow, benefitted substantially from access to the large collection of unpublished Lovelace-Byron manuscript papers, made available to scholars via their recent deposit in the Bodleian Library. Although a fine historian, Langley Moore was less authoritative when it came to mathematics, able to offer little more than a description of Lovelace's "curious letters" to De Morgan as being full of "pages with equations, problems, solutions, algebraic formulae, like a mathematician's cabalistic symbols." [LM77, p. 99]

The first Lovelace biography to focus on the mathematics and science was *Ada: A Life and a Legacy* (1985) by Dorothy Stein. Also the first intellectual study to delve deeply into the archives, Stein's book examined Lovelace's mathematical and scientific papers in painstaking detail, and laid bare lesser-known aspects of her life, including extramarital liaisons, gambling and drug addictions, and physical and mental illnesses. Her key thesis was, essentially, that Lovelace was "a figure whose achievement turns out not to deserve the recognition accorded it" [Ste85, p. xii]. Perhaps inevitably, given Lovelace's own grandiose claims about her mathematical ability, Stein's conclusion is not flattering:

> It is unusual to find an interest in mathematics and a taste for philosophical speculation accompanied by such difficulty in acquiring the basic concepts of science as she clearly displayed. We can only be touched and awed by the questing spirit that induced her to launch so slight a craft upon such deep waters [Ste85, p. 280].

To this day, Stein's text remains a standard work for anyone interested in Lovelace's scientific endeavors, its authority bolstered by the sheer weight of documentary evidence employed to defend its thesis. For this reason, Stein's conclusions have strongly influenced subsequent studies, particularly those with negative viewpoints. For example, in 1990 historian of computing Allan Bromley concluded: "Not only is there no evidence that Ada ever prepared a program for the Analytical Engine, but her correspondence with Babbage shows that she did not have the knowledge to do so" [ABCK+90, p. 89]. More damning still was Bruce Collier who described Lovelace as "mad as a hatter ... with the most amazing delusions about her own talents," calling her "the most overrated figure in the history of computing" [Col90, preface]. Recent studies have been more measured, however, with Thomas Misa's 2016 survey of the debate concluding that the 1843 "Notes" were "a product of an intense intellectual collaboration" between Lovelace and Babbage [Mis16, p. 18].

It was against this background that Hollings, Martin, and I undertook our research project to provide what we hope is a more nuanced and historically accurate assessment of Lovelace's mathematical proficiency. In particular, our work challenged—and, we argue, refuted—Stein's judgement that

> The evidence of the tenuousness with which she grasped the subject of mathematics would be difficult to credit about one who succeeded in gaining a contemporary and posthumous reputation as a mathematical talent, if there were not so much of it [Ste85, p. 90].

Stein's case against Lovelace's mathematical competence boils down to three main arguments, which, at first sight, seem quite compelling. And these arguments have influenced the subsequent judgements of others for nearly four decades. But as we will now show, all three arguments fail when subjected to further contextual analysis.

Argument 1: A Trigonometric Identity

The first argument given by Stein is that "despite hard work, skill and ingenuity in the manipulation of symbols did not come easily to her" [Ste85, p. 56]. And certainly, as we found in our study, many of the letters from Lovelace to her tutor De Morgan, particularly early on in her course of study, contain either algebraic errors or evidence of difficulty in symbolic manipulation. However, the first example given by Stein to illustrate this algebraic incompetence predates these letters by five years. Dating from 1835, it is contained in two letters to Mary Somerville, concerning the algebraic manipulation of trigonometric identities.

At this point, Lovelace was studying trigonometry and had attempted unsucccessfully to derive certain identities from others algebraically. To give a flavor, we will just look at the first (and easiest) problem. Given the addition formulae

$$\sin(a+b) = \frac{\sin a \cos b + \sin b \cos a}{R}$$
$$\cos(a+b) = \frac{\cos a \cos b - \sin a \sin b}{R},$$

the question was to deduce the corresponding difference formulae

$$\sin(a-b) = \frac{\sin a \cos b - \sin b \cos a}{R}$$
$$\cos(a-b) = \frac{\cos a \cos b + \sin a \sin b}{R}.$$

Figure 4. Mary Somerville.

In her letter to Somerville, Lovelace quotes her textbook as saying: "The values for the sine & cosine of the differences $(a-b)$ may easily be *deduced from these two formulas*." But she admits, "Now however easily deduced, I have not succeeded in doing it" [Ste85, p. 55].

On seeing this, the modern reader will undoubtedly have two principal reactions. The first would be to ask "what is R?" And the second would be to entertain serious doubts, along with Stein and others, about Lovelace's mathematical competence if she could not even make simple substitutions such as $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

The convention of teaching trigonometry with reference to a circle of radius 1 is a relatively recent development and had certainly not been uniformly adopted in Lovelace's time. In her day, all trigonometric functions were defined with respect to a circle of arbitrary radius R. Rather than ratios, they were understood as simple chord sections, with respect to an arc, which of course would correspond to an angle, subtended from the center of the circle. So,

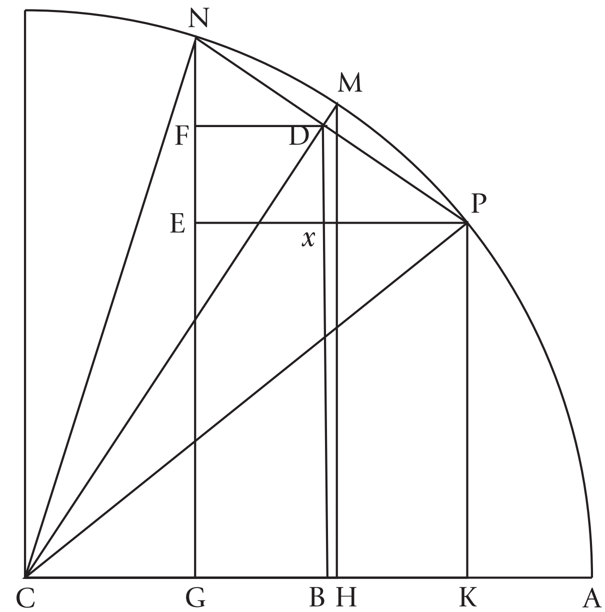


Figure 5.

for example, in Figure 5, the angle ∠ACM would correspond to the arc APM. The sine of that angle was defined as being "a line drawn from one extremity of the arc perpendicular to a diameter drawn through the other extremity" [Vin00, p. 44]. Thus sin ∠ACM is the line segment MH, cos ∠ACM is the line segment CH, and $\sin^2$(∠ACM) + $\cos^2$(∠ACM) = $CM^2 = R^2$.

We referred earlier to Ada's somewhat outmoded course of study in the 1830s, as recommended by her family friend, Dr. King. This included reading on algebra (largely comprising methods for solving polynomial equations), Euclid's *Elements* for basic plane geometry, and *A Treatise on Plane and Spherical Trigonometry* (1800) by the Reverend Samuel Vince. This last work was a highly synthetic, geometrical treatment of the subject, in which there was no mention of negative angles, no use of algebraic substitutions, and in fact, little use of algebra at all.

Here, for example, is the rather verbose—but entirely rigorous—way that Ada would have learnt how to derive the addition formula for the sine, with reference to Figure 5 [Vin00, pp. 55–56]. The reader is invited to follow the reasoning.

> Let ACM, MCN be the two given angles, of which ACM is the greater; make the angle MCP = MCN, and then ∠ACN is the sum of the two given angles, and ∠ACP their difference. Draw the chord PN, intersecting CM in D; then as the angle ∠PCD = ∠NCD, PC = CN, and CD is common to the triangles PCD, NCD, we have (by Euclid, Book I, Prop. 4) PD = DN, and the angle ∠PDC = ∠NDC, therefore each of them is a right angle. Draw NG,

DB, MH, PK, perpendicular to AC, and therefore parallel to each other, and DF, PxE parallel to AC, and therefore parallel to each other (by Euclid, Book I, Prop. 30). Now as ND = DP, we have (by Euclid, Book VI, Prop. 2) NF = FE = Dx, and Px = Ex = DF; therefore BK = GB = DF. Also, the triangle NDF is similar to the triangle CBD, for the angle ∠NDF = ∠CDB, each being the complement of ∠FDC, and the angles at F and B are right angles; but ΔCDB is similar to ΔCMH, the angle at C being common, and the angles at B and H right angles; therefore ΔNDF is also similar to ΔCMH.

The book then uses the similarity of these triangles to deduce that

$$\mathrm{CM} : \mathrm{MH} :: \mathrm{CD} : \mathrm{DB},$$
$$\mathrm{CM} : \mathrm{CH} :: \mathrm{ND} : \mathrm{NF},$$

and therefore

$$\begin{aligned}\frac{\mathrm{MH}\cdot\mathrm{CD}+\mathrm{CH}\cdot\mathrm{ND}}{\mathrm{CM}} &= \mathrm{DB}+\mathrm{NF}\\ &= \mathrm{FG}+\mathrm{NF}=\mathrm{NG}.\end{aligned}$$

Thus, letting $\angle\mathrm{ACM} = a$, $\angle\mathrm{MCN} = b$, and the line length $\mathrm{CM} = R$, it arrives at

$$\sin(a+b) = \frac{\sin a\cos b + \sin b \cos a}{R}.$$

The corresponding difference formula would therefore follow, not via the algebraic substitution of negative-angle identities, but by further geometric reasoning:

$$\begin{aligned}\frac{\mathrm{MH}\cdot\mathrm{CD}-\mathrm{CH}\cdot\mathrm{ND}}{\mathrm{CM}} &= \mathrm{DB} - \mathrm{NF}\\ &= \mathrm{DB} - \mathrm{D}x = \mathrm{B}x = \mathrm{PK},\end{aligned}$$

which gives the desired result.

Given Lovelace's exposure to this style of reasoning in trigonometry, it is not difficult to see why the thought of algebraic substitutions either did not occur to her, or did not come easily at first. Although we might regard such operations as obvious today, if we had not been trained in the algebraic style of doing mathematics, the need for such procedures would be far from evident. Thus, to base an argument of algebraic ineptitude on this example fails to take into account precisely what kind of mathematics and what style of mathematical thinking Lovelace had been exposed to up to this point.

Argument 2: The Case of the Missing Case

Stein's next argument concerned what she believed to be "perhaps the most telling and consequential" [Ste85, p. 90] piece of evidence against Ada's mathematical proficiency. It was contained in the 1843 paper, not in Lovelace's "Notes" at the end of her translation of Menabrea's article, but in the text of the translation itself. In his original article, Menabrea had discussed how the analytical engine might deal with computations that theoretically required an infinite number of steps. The example he gave was the following multiple of the Wallis product

$$2\cdot\prod_{k=1}^{n}\left(\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}\right),$$

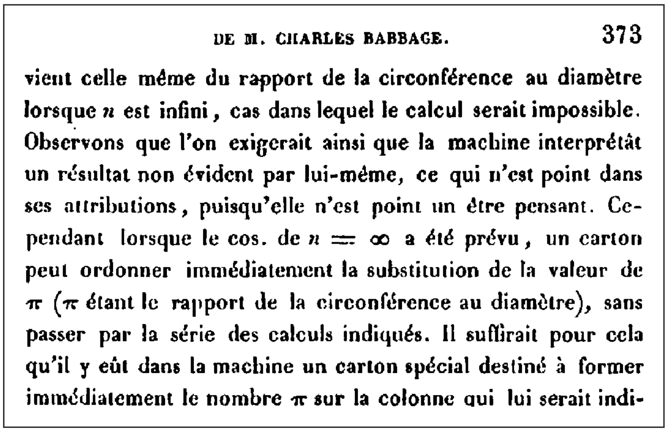
the value of which, as $n\to\infty$, is π.

Since plugging in a value of $n=\infty$ would obviously be impossible, Menabrea suggested that, in this case, a punch-card containing the numerical value of the output could simply be substituted in place of having the machine actually do the calculation. In Lovelace's translation, the full passage reads as follows [Men16, pp. 53–54]:

> Let us now examine the following expression:
>
> $$2\cdot\frac{2^2\cdot4^2\cdot6^2\cdot8^2\cdot10^2\cdots(2n)^2}{1^2\cdot3^2\cdot5^2\cdot7^2\cdot9^2\cdots(2n-1)^2\cdot(2n+1)^2},$$
>
> which we know becomes equal to the ratio of the circumference to the diameter, when n is infinite. We may require the machine not only to perform the calculaiton [*sic*] of this fractional expression, but further to give indication as soon as the value becomes identical with that of the ratio of the circumference to the diameter when n is infinite, a case in which the computation would be impossible. Observe that we should thus require of the machine to interpret a result not of itself evident, and that this is not amongst its attributes, since it is no thinking being. Nevertheless, when the cos of $n=\infty$ has been foreseen, a card may immediately order the substitution of the value of π (π being the ratio of the circumference to the diameter), without going through the series of calculations indicated. This would merely require that the machine contain a special card, whose office it should be to place the number π in a direct and independent manner on the column indicated to it.

Although a little long-winded, the majority of this extract is intelligible enough, but there is one notable exception: the sentence concerning "the cos of $n=\infty$." Not only is this phrase obviously meaningless, but it also raises the question of what on earth the cosine function has to do with this example anyway. Referring back to Menabrea's article in French, we find that sentence originally was rendered as follows:

> Cependant lorsque le cos. de $n=\infty$ a été prévu, un carton peut ordonner immédiatement la substitution de la valeur de π (π étant le rapport de la circonférence au diamètre), sans passer par la série des calculs indiqués.


DE M. CHARLES BABBAGE. 373

vient celle même du rapport de la circonférence au diamètre lorsque n est infini, cas dans lequel le calcul serait impossible. Observons que l'on exigerait ainsi que la machine interprétât un résultat non évident par lui-même, ce qui n'est point dans ses attributions, puisqu'elle n'est point un être pensant. Cependant lorsque le cos. de $n = \infty$ a été prévu, un carton peut ordonner immédiatement la substitution de la valeur de π (π étant le rapport de la circonférence au diamètre), sans passer par la série des calculs indiqués. Il suffirait pour cela qu'il y eût dans la machine un carton spécial destiné à former immédiatement le nombre π sur la colonne qui lui serait indi-

Figure 6. Portion of Menabrea's article containing the original misprint.

This is exactly the same as in Lovelace's translation. Why then is there mention of $\cos\infty$? The answer is simple: Menabrea's paper contained a typo. The relevant phrase should have read: "lorsque le cas de $n = \infty$ a été prévu," or "when the *case* of $n = \infty$ has been foreseen," which obviously makes much more sense. Thus, in Stein's view, "Ada had translated a printer's error" [Ste85, p. 91], thereby revealing her limited understanding of this section of Menabrea's paper.

One can certainly agree that, by giving a literal translation of this passage without noticing the printing error, Lovelace did not show her eye for detail and proofreading skills in the best possible light. But at the same time it's hardly indicative of mathematical incompetence. Several other typographical errors also slipped past her, including the misspelling of the word calculation as *calculaiton* in the same passage and the rendering of her initials A. A. L. as "A. L. L." at the end of the paper. It should also be remembered that the "cos of $n = \infty$" typo did not just slip past Lovelace and Menabrea; it went unnoticed by several scholars who we know read the article in detail, including Babbage and Wheatstone in the 19th century and Turing and Hartree in the 20th.

Stein's case is further weakened by her subsequent frank admission that this mistake "has been reprinted several times" [Ste85, p. 91]. In addition to its first occurrence in Menabrea's paper and subsequent inclusion in Lovelace's translation, it also appeared in Henry Babbage's collection of papers in 1889, where for some unknown reason, he gave the phrase as "the cos of $n = 1/0$," which makes even less sense than before. This rendering was also subsequently reproduced in Philip and Emily Morrison's *Charles Babbage and his Calculating Engines*, published in 1961. In fact, the mistake was only spotted in 1953, when B. V. Bowden reprinted Lovelace's paper in its entirety with the error silently corrected [Bow53, p. 359]. But the fact that it took 110 years before the mistake was noticed gives some indication that it is not quite as "telling and consequential" as Stein would have us believe.

Argument 3: Chronic Algebraic Inability?

The final piece of evidence presented by Stein is, without doubt, potentially the most damaging. Indeed, if correct, this example alone would be enough to cast serious doubt on Lovelace's ability to comprehend anything but the most elementary mathematics. Like the first argument, it implies that she had a chronic inability to manipulate algebraic expressions. That first argument is based on evidence from 1835, when Lovelace was only 19 and had yet to start her intensive study of university-level mathematics with De Morgan. But by contrast, Stein's final argument dates its evidence from the end of 1842—two and a half years after the commencement of those studies and, most worryingly of all, almost exactly contemporaneous with the beginning of her work on her famous paper.

Stein insists that Lovelace continued to display an inability "to assimilate the symbolic processes with which alone highly complex and abstract matters may be rigorously treated," which rendered her incapable of grasping algebraic reasoning and "in the end limited her understanding of science" [Ste85, p. 84]. This insistence is, of course, essential to her overall thesis that by 1843 Lovelace was in no way sufficiently prepared to write on a mathematical subject.

As we have already indicated, there is no doubt that Lovelace found algebra challenging at first, and her letters to De Morgan contain many examples where she has made elementary algebraic errors or displayed a novice's misunderstanding. Several examples are highlighted, for example, in [HMR17b] and [Ric21]. But it is equally clear from their correspondence that not only did her algebra improve over time, but that it was soon applied to far more demanding topics, such as, for example, second-order differential equations, which she was studying by November 1841. Thus, while still prone to the odd algebraic mistake or misunderstanding, these increasingly arose from far more sophisticated mathematics.

To begin our analysis of Stein's third argument, we quote the relevant passage from her book in full, in order to convey the full weight of the alleged evidence [Ste85, p. 90]:

> The last surviving letters in Lovelace's mathematical correspondence with De Morgan are dated 16 and 27 November 1842 (hence shortly before she translated the Menabrea memoir). In them we find her wrestling with an elementary problem in functional equations. (The problem was: Show that $f(x+y) + f(x-y) = 2f(x)f(y)$ is satisfied by

Exercise. Shew that the equation

$$\varphi(x+y)+\varphi(x-y) = 2\varphi x \times \varphi y$$

is satisfied by

$$\varphi x = \frac{1}{2}\left(a^x + a^{-x}\right)$$

for every value of a: and also that

$$\varphi(x+y) = \varphi x + \varphi y$$

can have no other solution than

$$\varphi x = ax$$

Figure 7. Problem from page 206 of De Morgan's *Elements of Algebra*.

$f(x) = (a^x + a^{-x})/2$.) She was still unable to take a mathematical expression and substitute it back into the given equation. It was the same "principle" that had plagued her in her correspondence with Mary Somerville and in earlier letters to De Morgan. On 27 November, after having struggled for at least eleven days, she sighed,

> I do not know when I have been so tantalized by anything, & should be ashamed to say *how* much time I have spent upon it, in vain. These functional Equations are complete Will-o-the-Wisps to me. The moment I fancy I have really at last got hold of something tangible & substantial, it all recedes further & further & vanishes again in thin air.... I *believe* I have left no method untried.

The obvious inference is that if Lovelace was unable to handle algebraic manipulations like these in November 1842, after more than two years of studying mathematics at a much higher level than this, how are we to believe that she was capable of writing such a mathematically fluent account of Babbage's analytic engine in the first half of 1843?

Lovelace's course of study with De Morgan began around July 1840 and, although dominated from fairly early on by calculus-related topics, periodic excursions into more elementary topics were necessary due to gaps in her prior studies. As De Morgan reminded her in September of that year: "You understand of course that your Differential Calculus must be delayed from time to time while you make up those points of Algebra and Trigonometry which you have left behind" [HMR17b, p. 206].

As part of this remedial study, Lovelace was instructed to read De Morgan's 1835 textbook *The Elements of Algebra*, subtitled, appropriately enough, *Preliminary to the Differential Calculus*. By November, she had reached the book's short Chapter 10 entitled "On the notation of functions." There, in just four pages, De Morgan showed how to express functions—writing for instance ϕx where we today would usually write $f(x)$—and introduced the idea of solving functional equations. (A simple example would be something like $\phi(xy) = \phi x + \phi y$, of which one possible solution is $\phi x = \ln x$.) It was this topic that caused Lovelace problems.

In a letter dated November 10, but with no year given, she wrote to De Morgan: "I do not know *why* it is exactly, but I feel I only half understand that little Chapter X, and it has already cost me more trouble with less effect than most things have. I must study it a little more I suppose" [HMR17b, p. 224].

A subsequent letter written on November 16 finds her going into more detail [HMR17b, p. 225]:

> I can explain exactly what my difficulty is in Chapter X.... That I do not comprehend at all the means of deducing from a Functional Equation the form which will satisfy it, is I think clear from my being quite unable to solve the example at the end of the Chapter 'Shew that the equation $\phi(x+y) + \phi(x-y) = 2\phi x \times \phi y$ is satisfied by $\phi x = \frac{1}{2}(a^x + a^{-x})$'. I have tried several times, substituting first 1 for x, then 1 for y but I can make nothing whatever of it, and I think it is evident there is something that has preceded, which I have not understood. The 2nd example given for practice 'Shew that $\phi(x+y) = \phi x + \phi y$ can have no other solution than $\phi x = ax$', I have not attempted.

The problem to which Lovelace is referring appears on page 206, at the end of Chapter 10 of De Morgan's algebra textbook (see Figure 7). Again, she has given the date but no year on this letter. However, immediately below her handwritten date, a 20th-century hand, possibly that of an archivist, has added a further detail: the year "1842" appears quite legibly on the letter in pencil. Lovelace's final contribution to this particular topic is the letter dated November 27 (but again, with no year) quoted by Stein above. It is thus easy to see why Stein believed that this particular episode dates from November 1842, as all the evidence thus far presented points in that direction.

It looks pretty bad for Lovelace doesn't it?

But wait! There is a flaw in Stein's analysis; namely, she does not appear to look at De Morgan's replies to these letters from Lovelace. And it turns out that one letter in particular carries a vital clue.

In a letter seemingly composed in reply to Lovelace's initial letter of November 10, De Morgan wrote back, saying: "The notation of functions is very abstract. Can you put your finger upon the part of Chapt. X at which there is difficulty[?]" [HMR17b, p. 224]. The content of this letter not only makes it obvious that it is an integral part of this specific epistolary conversation, but crucially, reveals it to be the one and only letter from this sequence that actually contains a full date. At the end, just to the left of his signature, De Morgan has written in his distinctive handwriting: "Nov$^{\mathrm{r}}$. 14/40." In other words, this conversation began, not in 1842, but in *1840*. We are now left with the possibility that the first two letters in the discussion (from November 10 and 14) were both written in 1840, with two subsequent letters following two years later, on November 16 and 27, respectively. This looks potentially even worse for Lovelace's mathematical reputation since it would imply that even after two further years of higher mathematical study, she still couldn't solve this relatively simple problem!

But one final possibility also exists, and this only becomes clear when you read *all* of the letters in this particular conversation. There are actually *six* letters in this back-and-forth exchange (three from Lovelace and three from De Morgan) and, when read in the correct order, they shed crucial light onto what was really happening and, most importantly, *when* it was happening.

On November 10, 1840, Lovelace first wrote to complain that she felt "I only half understand that little Chapter X." Four days later, in a letter dated November 14, 1840, De Morgan replied, asking "Can you put your finger upon the part of Chapt. X at which there is difficulty[?]." Lovelace then responded on November 16, "I can explain exactly what my difficulty is in Chapter X," which she then proceeded to do. De Morgan's answer to this, merely dated "Friday," gave her a short hint, to which she then replied on November 27 [HMR17b, p. 225]:

> I have I believe made some little progress towards the comprehension of the Chapter on Notation of Functions, & I enclose you my Demonstration of one of the Exercises at the end of it : "Show that the equation $\phi(x+y) = \phi x + \phi y$ can be satisfied by no other solution than $\phi x = ax$." At the same time I am by no means satisfied that I do understand these Functional Equations perfectly well, because I am completely baffled by the other Exercise: "Shew that the equation $\phi(x+y)+\phi(x-y) = 2\phi x \times \phi y$ is satisfied by $\phi x = \frac{1}{2}(a^x + a^{-x})$ for every value of a."...These Functional Equations are complete Will-o'-the-Wisps to me....

Lastly, in a letter dated "Monday," De Morgan closed the episode with a final reply [HMR17b, p. 225]:

> I can soon put you out of your misery about p. 206. You have shown correctly that $\phi(x+y) = \phi(x) + \phi(y)$ can have no other solution than $\phi x = ax$, but the preceding question is not of the same kind; it is not show that there can be no other solution except $\frac{1}{2}(a^x + a^{-x})$ but show that $\frac{1}{2}(a^x + a^{-x})$ *is* a solution: that is, try this solution. [He then demonstrates the solution.]...I think you have got all you were meant to get from the chapter on functions.

It is therefore clear that this entire discussion, rather than dating from 1842 as Stein claimed, actually took place two years earlier in 1840, close to the very beginning of Lovelace's studies with De Morgan, rather than right at the end of them. And since Stein's argument hinged on these letters dating from 1842, our redating of them leads to the inevitable conclusion that her negative assessment of the state of Lovelace's mathematical abilities by late 1842 was in fact incorrect. (For the first in-depth analysis of the Lovelace–De Morgan correspondence by historians of mathematics, see [HMR17b].)

On the face of it, and taken together, all three of Stein's key arguments seemed at first to present a compelling picture of Lovelace as someone whose mathematical skills were considerably weaker than had previously been supposed. But as this article has shown, further contextual analysis reveals fundamental flaws in each of these arguments, throwing considerable doubt, not just on Stein's negative conclusions, but on those of subsequent authors who accepted her findings. This results in a more nuanced assessment of Lovelace's abilities. And while it does not prove that she was solely responsible for the 1843 "Notes," nor does it in any way suggest that she was a genius, it does provide strong evidence that she did indeed have the mathematical competence to write and understand the mathematics contained in her famous paper of 1843.

Postscript

In recent months, the subject of artificial intelligence (AI) has featured prominently in news stories, due to rapid developments in the abilities of AI software applications such as ChatGPT, Google Bard, and Microsoft Bing. Academics and educators have exhibited a mixture of excitement and terror at the sheer volume of convincingly humanlike responses now capable of emanating from their computers. But of course, speculation about the possibility of AI is nothing new and, as is well known, also appeared in Lovelace's 1843 paper. For this reason, it seems appropriate to conclude this article with a timely postscript on Ada Lovelace and artificial intelligence.

Lovelace's views on the possibility of artificial intelligence can be summed up in her own words on the potential of Babbage's theoretical machine: "The Analytical Engine has no pretensions whatever to *originate* anything. It can do whatever we *know how to order it* to perform"

[Men16, p. 94]. A century later, while agreeing with her that all the thinking had to be done beforehand by the human hardware and software designers, Douglas Hartree countered: "This does not imply that it may not be possible to construct electronic equipment which will 'think for itself,' or in which, in biological terms, one could set up a conditioned reflex, which would serve as a basis for 'learning'" [Har49, p. 70].

This view was famously expanded by Alan Turing in his seminal 1950 paper "Computing machinery and intelligence," in which he posed the classic question: "Can machines think?" [Tur50, p. 433]. Describing himself as being "in thorough agreement with Hartree" on the question of whether machines might one day have the power to think for themselves, he speculated "that the evidence available to Lady Lovelace did not encourage her to believe that they had it" [Tur50, p. 450]. Turing believed that by the end of the 20th century, "one will be able to speak of machines thinking without expecting to be contradicted" [Tur50, p. 442]; he then gave—and attempted to refute—nine counter-arguments to this opinion, the sixth of which was "Lady Lovelace's Objection" [Tur50, p. 450].

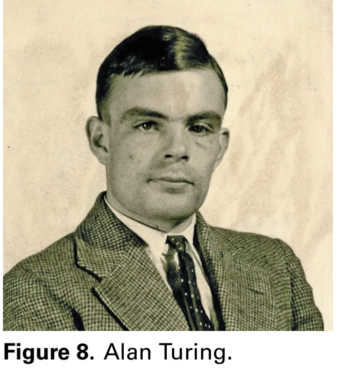


Figure 8. Alan Turing.

Turing chose to intepret Lovelace's view as being that a machine can never "take us by surprise" and tried to counter it with a few rather lame examples of how humans can be surprised by computers, due perhaps to insufficient or erroneous assumptions on the part of the programmer. But he admitted that this wasn't particularly convincing. Of greater importance, of course, was his "imitation game" or Turing test, as it is now known. This was a thought experiment via which, if successful, a computer could produce output that was indistinguishable from that of a human brain. But while most today would agree that the Turing test has been satisfied, Lovelace's objection remains a harder criterion to fulfill.

For this reason, a 21st-century upgrade of the Turing test, known as the Lovelace test, was formulated. Instead of focusing on the mere replication of humanlike intelligence, this new benchmark concentrates on Lovelace's use of the word "originate" to ask whether a machine could ever be able to produce an original, creative piece of work unforeseen by its programmer. Put simply, a computer would pass this test if it created something new and original (such as a poem or a piece of music) in such a way that no programmer could explain the process.

To this end, the author of this article conducted a casual experiment. On one occasion, he instructed ChatGPT to "write a limerick about Ada Lovelace." A limerick was duly composed. He then made up a limerick of his own. Finally, on a separate occasion, he gave the same instruction again to ChatGPT, and the software produced another limerick. The three limericks are presented here, in a different order from their initial composition.

There once was a woman named Ada
Whose writings are said to have played a
Part in the formation
Of machine computation
That's now used to analyze data.

Of computing and numbers a fan,
In mathematics as strong as a man;
With all of her might
She worked day and night
To invent the computer program.

But was she as smart as they say?
Some answer this question, "No way!"
While others maintain
She did all that they claim
And her legend persists to this day.

Three things should now be noted. Firstly, the limericks are all atrocious, for which the author can only apologize. Secondly, in this experiment there is no doubt that ChatGPT passed the Turing test, as the two AI-generated limericks are indistinguishable in quality from the human-composed verse—all three limericks being equally terrible. But thirdly and most significantly, the software failed the Lovelace test because, although it can be seen to have "originated" something, namely two brand-new limericks, it only did exactly what it was instructed to do. And if it had been told to write a *good* limerick, the chances are that it would have failed entirely. It is thus clear from this very unscientific experiment that this particular piece of AI software still has some way to go before it can be said to have satisfied the Lovelace test.

At the beginning of her famous "Note G," while warning of "exaggerated ideas that might arise as to the powers of the Analytical Engine," Lovelace noted

> a tendency, first, to *overrate* what we find to be already interesting or remarkable; and, secondly, by a sort of natural reaction, to *undervalue* the true state of the case, when we do discover that our notions have surpassed those that were really tenable [Men16, p. 94].

It is ironic that she was in fact warning against attributing too much (or too little) to the machine with respect to its abilities, when that is exactly the fate that subsequently befell perceptions of her own mathematical skill. Let us hope that this article goes some way toward setting the record straight.

ACKNOWLEDGMENTS. My thanks are due to Christopher Hollings and Ursula Martin for valuable feedback on an earlier version of this article. Transcriptions of Lovelace's correspondence with De Morgan were made by Christopher Hollings in 2015 and may be viewed online (along with high-quality images of the original letters) at: `https://www.claymath.org/online-resources/ada-lovelaces-mathematical-papers`.

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Adrian Rice

Credits

OPINION

Redefining Math Conferences: A Panel

Robyn Brooks and Padi Fuster Aguilera

Note: The opinions expressed here are not necessarily those of Notices.

Introduction

In January 2023, a panel called "Redefining Math Conferences" took place at the Joint Math Meetings (JMM); this panel grew out of our work as founders of the Math for All conference. The goal of this panel was to create a space where organizers of inclusive conferences could share their experiences, successes, and thoughts about the work involved in organizing these events. In this article, we discuss our intentions for the panel, and its main takeaways.

This article is not meant to be a set of inclusive conference guidelines; for articles of this nature, the interested reader way wish to look at [JSTM+22, CH19] as a starting point. We want to acknowledge that the following comments, except those in quotes, are our own and have been filtered through our own personal biases and experiences.

The Math for All conference was founded with the purpose of fostering inclusivity in mathematics, with an explicit focus on nurturing a sense of belonging amongst attendees. As founders and organizers, we continue to invest time in learning how to provide participants with an open and friendly environment in which to learn and discuss mathematics. A big part of the work that we do relies on having an open mind and the desire to keep learning.

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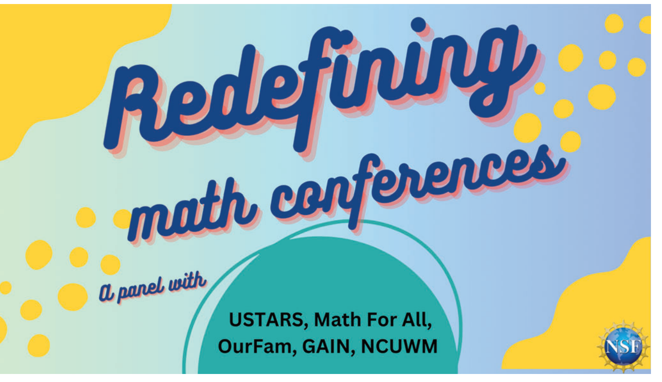

Figure 1. Flyer for the panel.

One way of doing this is to brainstorm and talk to others doing this work. This is why we decided to hold this panel with organizers of several math conferences that are centered around building welcoming math spaces. Moreover, we saw this space as an opportunity to create a networking platform and to foster community for all those engaged and interested in doing this work.

Intention and Panel Organization

The following is a list of the conferences and the panelist representatives. (If you wish to know more about these conferences, we have included their websites as footnotes.)

- **Graduates Achieving Inclusion Now (GAIN).**[1] Panelist: Seppo Niemi-Colvin, postdoctoral fellow at Indiana University at Bloomington

[1] https://sites.google.com/view/gainconference

- **The Math For All in Clemson Conference.**[2] Panelist: Keisha Cook, assistant professor at Clemson University
- **The Nebraska Conference for Undergraduate Women in Mathematics (NCUWM).**[3] Panelist: Christine Kelley, professor at University of Nebraska-Lincoln
- **The Online Undergraduate Resource Fair for the Advancement and Alliance of Marginalized Mathematicians (OURFA^{2}M^2).**[4] Panelist: Zoe Markman, senior math major at Swarthmore College
- **The Underrepresented Students in Topology and Algebra Research Symposium (USTARS).**[5] Panelist: Christopher O'Neill, associate professor at San Diego State University

Figure 2. Conferences represented on the panel.

The purpose of this panel was to both create a networking place for organizers of conferences and use the visibility that the JMM provides to reach out to a wider audience, and create awareness on the criticality of creating inclusive conferences spaces. Our goal for the panel was not to convince others that such spaces are necessary, but instead to work on growing and learning with community on how to effectively create more welcoming spaces for all. We also aimed to disrupt the JMM space by bringing these conversations there, with the hope that the JMM organizers would adopt some of the effective organizational practices discussed by the panelists.

By organizing this panel, we wanted to highlight the importance of creating inclusive conferences because we believe that having a welcoming space is key for participants to really engage and do great mathematics. As graduate students, we had been to many conferences where we had no sense of belonging—even with all the privilege that we carry, being white, cis-women, coming from families with members in academia—and this kept us from enjoying the mathematics. Those of you who have felt this before will immediately understand what we are talking about.

Insecurities and imposter syndrome are invisible barriers which hinder people from being able to enjoy and do mathematics. Welcoming spaces can help remove some of these weights and invite people to bring their full selves and thrive. Mathematics is more fruitful when these invisible barriers are recognized and counteracted. It is naive to believe that mathematicians can focus on just doing mathematics without removing these barriers, because of the mental energy that is required to exist in noninclusive academic spaces. This means that, when organizing a conference, figuring out how to remove barriers and create welcoming spaces must be centered. This focus is as important as thinking about the mathematical content of a conference. Ensuring that conference participants are able to fully contribute should be a central goal, not an afterthought.

We wished for our panel to be a place where we could work towards facilitating genuine community for all, and for it to be a meeting point for those wishing to do this work. By inviting organizers of these exemplary conferences, we ended up with a diverse group of panelists, which suggests that these conferences are already doing work to ensure diverse perspectives within their organizing committee.

The prepared questions for the panel were meant to have the panelists reflect on their intentions and experiences when organizing. We believe that conference organizing requires critical thinking, care, and constant reassessment of the small details. This time and effort is crucial when centering the participants and their experiences. We also wanted to identify and share specific conference practices that have been developed over time, from the creativity of the participants and organizers. We wanted to give the panelists time to share useful tips for other organizers or those who are thinking of doing this work. Finally, acknowledging that doing this work can also cause harm, we wanted to give panelists time to share some of their mistakes, and what they learned.

During the event, panelists introduced themselves, their conferences, and answered the following three questions:

- In the context of creating a conference that has a welcoming environment, what do you think is the most important thing to consider when organizing?
- Are there any good changes/ideas that you implemented in your conference that ended up in somebody being able to enjoy the conference?
- What mistakes have you made while organizing, for example, encountering a tough moment during the conference, and how did you deal with these mistakes?

[2] http://tinyurl.com/mathforallconference
[3] https://math.unl.edu/ncuwm
[4] https://sites.google.com/view/ourfa2m2
[5] https://www.ustars.org

Takeaways

We summarize the main takeaways from this panel as four points: accessibility, intention, diversity of voices, and growth mindset.

Accessibility. Accessibility was one of the main topics around which the panel discussion was centered. One focus of the conversation was that in-person conferences are not accessible to everyone—there are many reasons why in-person conference participation is unfeasible. To name a few, people may be unable to attend due to financial, health, or safety issues, care giving responsibilities, or because conferences are often only designed for able-bodied individuals. Since the pandemic started, in-person conference attendance has been increasingly challenging (or even impossible) for many of our colleagues. Moreover, many country- or state-specific laws can become invisible barriers for participants, such as abortion bans, anti-trans legislation, anti-immigration laws, etc. As a commitment to accessibility, conference organizers can provide a virtual participation option.[6]

During the panel, the OURFA^{2}M^2 panelist spoke about their focus on accessibility on many different levels. The OURFA^{2}M^2 conference is fully virtual. They use *Alt text* in all their advertising and encourage others to do the same. *Alt text* (Alternative text) gives an image description, which can be read aloud by screen-reading tools, among other uses. The OURFA^{2}M^2 panelist also spoke about accessibility, in the context of removing barriers in the form of knowledge of the field. Within their conference and as resources on their website, OURFA^{2}M^2 offers workshops on topics such as using LaTeX, applying for an REU, and crash courses as introductions to math research areas. Finally, OURFA^{2}M^2 organizers are intentional in their work to invite participants and new organizers from community colleges and two-year degree granting institutions. By doing this, students from institutions that typically have less visibility and access to resources have a chance to be exposed to new opportunities to do math research, and venues to participate in the broader math community.

Along with these points, the Math For All panelist shared their poster workshops as a way to increase accessibility. These poster workshops are held at local institutions prior to the conference and are usually led by undergrad or grad students. The main focus of these workshops is to demystify the process and to give tools to undergrads for creating a scientific poster from scratch. In particular, in these workshops facilitators introduce students to LaTeX and Overleaf and discuss topics such as how to find a mentor and a poster topic, as well as how to present a poster effectively. Additionally, free poster printing is provided to all participants presenting a poster at the conference.

Another common feature amongst USTARS, NCUWM, and Math For All is that they provide travel funding for participants at different levels. (Note that GAIN and OURFA^{2}M^2 are fully virtual, so the cost of travel is zero.) When possible, conference organizers pay directly for food, travel, and lodging instead of providing reimbursements. This practice can be key in allowing participation of those without the financial resources to pay costs ahead of time.

Intentionality. Intentionality was a recurring topic brought up during the panel in order to set up inclusive and safe spaces. Safe spaces cannot be possible without collective agreement and effort from all participants. In order to create this collective action and mindset, conference organizers must be intentional in setting the stage at the beginning of the conference and throughout the event.

In this spirit, the Math For All panelist talked about their conference opening remarks in which they include snippets of the history of the place where the conference takes place, in order to disrupt the predominantly white cis-male academic space and create space for those who do not identify with this group. Additionally, the conference has an ethical conduct agreement which participants agree to abide by during registration. This agreement is also read out loud at the start of the conference to set the tone and allow participants to reflect on it. The goal of this is to center the importance of the agreement to the conference dynamics, and to remind participants that a safe and welcoming space is not possible without everyone's collaboration.

The NCUWM panelist talked about assigning seating for the opening banquet where each table has a faculty member, undergrads from different institutions, and an invited guest or mentor in order "for participants to meet several others right away and become comfortable talking with others they do not know." The conference intentionally invites speakers and mentors from diverse backgrounds and experiences, such as career trajectory, type of institution, racial, ethnic, and sexual orientation. This is so that "each participant [can] recognize someone who they can identify with who may inspire them (like an "I can do this, too" realization)."

During the NCUWM networking dinner, invited professionals and other mentors are assigned specific tables, and the undergraduate participants choose where to sit. At dessert, seats are rotated, so that participants have the ability to network with a larger group of people. This gives students who may be more shy or less confident a means of meeting role models.

The GAIN conference has the goal of being a space for mathematicians to talk about issues of discrimination and systemic inequity. The GAIN panelist spoke about

[6] *Letter to the AMS: A call to defend bodily autonomy in the mathematics community:* `https://sites.google.com/view/bodily-autonomy-in-math`

how this goal shaped the conference: what kind of topics would be discussed and the structure of the talks and participant interaction. For instance, the conference was spread over several weekends. Each weekend had a different discussion topic including sexism, racism, homophobia, ableism, and mental health. During these weekend events, a mathematician would give an introductory talk about the specific issue, and space was allotted for participants to discuss and debrief on the issue itself, as well as its impact on the mathematical community. The panelists emphasized the importance of bringing these issues, which are often not considered *mathematical topics*, into mathematical spaces. Without talking about and working to address these issues, which affect many mathematicians, inclusivity is not achievable.

Diversity of voices. A recurring theme during the panel was the importance of 1) having a diverse group of organizers in order to better serve the conference participants and 2) listening to the people that you want to serve.

The OURFA^{2}M^2 conference was created by undergraduate students who identified a need to create resources for their peers and themselves. Within the organizational group, there is a diversity of voices to better serve their participants' needs. For example, all the accessibility practices mentioned before came from having organizers with different accessibility needs. The OURFA^{2}M^2 panelist spoke specifically about how the organizers focus on creating an exhaustive list of groups that need to be included and served within the mathematical community.

Similarly, the USTARS conference was originally started by graduate students from underrepresented groups in math. The current set of organizers are past attendees who were positively impacted by the conference, and wanted to continue the work, so that future students would also have a place where they felt that they belonged. Specifically, the USTARS panelist talked about how this practice preserves the founding goals and allows the conference to adapt to the needs of the groups that it works to serve.

As a result of the numerous iterations of USTARS and NCUWM and their continuity in serving students in mathematics, they have built a very strong community both at and across conferences. More generally, all the conferences represented by the panel have made a big impact on the math identity of participants.

Growth mindset. Another common theme was the necessity of having a growth mindset when organizing conferences. This involves not seeing the practices and structure of the conference as fixed, but instead spending time reflecting and updating these practices based on feedback from the participants.

For example, the NCUWM panelist mentioned their implementation of small group discussions during the conference with topics focused on intersectional issues faced by women in math. This conference component was implemented as a suggestion from a past participant, and has become a very valuable space for connection among attendees.

Through explaining challenging moments that the organizers had faced, the Math For All panelist mentioned the importance of having volunteers whom you trust to be able to handle situations that might come up during the conference. However well-intentioned, careful, or trained an organizer is, there may be instances where certain practices or oversight can cause harm. Given this possibility, it is important to have a structure in place to address issues as they arise so that conference organizers can listen and respond to unexpected situations. In addition, making space for reflection and implementing changes based on feedback from participants can help improve future conference iterations.

Apart from these main takeaways, the questions of the audience brought up other important themes. One of the themes was that, while many people attempt to address the presence of sexism, racism, etc in mathematical spaces, often classism and its effects are not mentioned. Classism is present and rooted in academia, not only because of the cost of higher education, but also because of the inherent privilege that comes with being able to access higher education. Additionally, there is a hierarchy within academic spaces, e.g., level of position and salary, that creates classes within the mathematical community. During the panel there was not a concise idea of how to address this issue while organizing conferences, other than creating spaces where academic hierarchy is not a barrier to participant interaction, or reaching out and advertising to colleges with fewer resources. In any case, this remains a subject for us to reflect on.

Another theme that was discussed was how to reach out to those who do not care about the importance of inclusivity and diversity. The general sentiment was that it is not the responsibility of a conference organizer to convince others of the benefits that come from inclusive practices in mathematics. The focus of the work should go towards intentionality and care toward the community. A commitment from each participant to listen, learn, and challenge one's biases is essential to work towards creating safer spaces.

The last theme discussed was how big conferences such as the JMM could implement some of the good practices that had been brought up during the panel. In the later years, big conferences have started adapting ethical conduct agreements and safe practices but given the resources and staff available to the organizations that run these conferences, more can be done. For example, having a conduct agreement but not bringing it up throughout the conference makes it somehow a static document and it can become meaningless. For safe practices to have meaning, they need to be accompanied by actions and commitment on behalf of the organizations responsible for designing the event.

One main practice that was brought up was the fact that JMM has no virtual option for participants and there seems to be no intention of implementing this option in the foreseeable future.[7] In fact, this was one of the downsides when organizing this panel, because we did not have control over having a virtual option for it. This type of accessibility has a huge impact on who is able to attend and benefit from these conferences, and this in turn acts as a gateway, preventing participation and limiting opportunities for many people.

Concluding Remarks

The main message that we would like to convey is that **centering inclusion throughout conferences can only give rise to more prolific mathematics, as everyone is able to engage in mathematical topics and with the conference community. Therefore, if the goal of a conference is truly to maximize the mathematical outcomes, then this side of conference organization cannot be omitted. It is just as important to provide welcoming and safe spaces in which people can collaborate and reimagine how to live in a better, more just world for everybody.**

ACKNOWLEDGMENTS. We would like to thank the panelists Keisha Cook, Christine Kelley, Zoe Markman, Seppo Niemi-Colvin, and Christopher O'Neill for their participation in this event and their feedback on this article. We would also like to thank Selvi Kara, Florencia Orosz Hunziker, and Swati Patel for their thoughtful input and feedback, as well as the editors of the AMS *Notices* for their comments and suggestions. This panel was made possible thanks to the NSF conference grant 2138357, in the form of partial travel funding for the panelists to attend JMM.

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[7] *We reached out to* `meet@ams.org` *to see if they were considering having a hybrid mode for JMM in the future, and they said they were not considering it at this time.*

Robyn Brooks

Padi Fuster Aguilera

Credits

Figure 1 is courtesy of Padi Fuster Aguilera and Robyn Brooks.

Figure 2 is courtesy of Padi Fuster Aguilera, Robyn Brooks, OURFA^{2}M^2, Math for All, and USTARS.

Photo of Robyn Brooks is courtesy of Todd Anderson.

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BOOKSHELF

New and Noteworthy Titles on our Bookshelf
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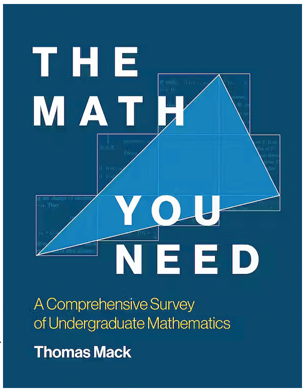

Courtesy of MIT Press.

The Math You Need
A Comprehensive Survey of Undergraduate Mathematics
By Thomas Mack. The MIT Press, 2023, 496 pp., $55.

The Math You Need is a collection of fundamental undergraduate topics in mathematics. I found it to be quite complete; it has chapters for group theory, commutative algebra, linear algebra, topology, real analysis, multivariable analysis, complex analysis, number theory, and probability. The text presents its contents using the traditional "Definition – Theorem – Proof" style of writing to which many mathematicians are accustomed. Since it does not contain excessive prose or many examples, I found the book well-suited to be a reference, likely useful for a graduate student who has already grappled with the ideas and proofs in an introductory course. It could be used as a textbook, however, as it contains exercises at the end of each of its nine chapters. While reading, I was reminded of many of the standard proof techniques of algebra, analysis, and topology that I saw in graduate school. I expect an advanced (and motivated!) undergraduate student could expand upon what they saw in their introductory courses and would appreciate what the book adds to their background.

I believe this book would be an excellent addition to the textbooks and reference books in the libraries of mathematics students, professors, and hobbyists. It might take you back to your graduate school days, as it did for me. I especially think a department could use it to add advanced topics to what is covered in their undergraduate courses or guide graduate students to the background material they will be expected to know as they take graduate courses or study for their preliminary exams.

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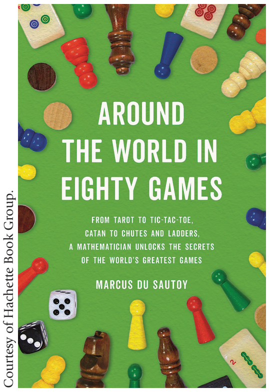

Courtesy of Hachette Book Group.

Around the World in Eighty Games
From Tarot to Tic-Tac-Toe, Catan to Chutes and Ladders, A Mathematician Unlocks the Secrets of the World's Greatest Games
By Marcus du Sautoy. Basic Books, 2023, 384 pp., $30.

As you might have guessed by its title, this book about games is playful and fun. In 80 vignettes, the author explores games from all over the globe. In interludes, he ponders aspects of a "game." What are the qualities that turn an activity into a game? While he and others have tried to define it and we can certainly provide examples, the definition of a "game" is nearly impossible to pin down.

I found joy in reading about games familiar to me: backgammon, chess, Go, mancala, SET, and Nim. There were games that were new to me, such as Carrom from India, Hanafuda from Japan, and Truco from many countries in South America. The author turned the book into a game as well; he encourages a nonlinear reading of the text and suggests that the reader should roll a die to determine the next chapter to explore. He hypothesizes that games are popular in how they build community as well as offer a refuge from the complexities of daily life—they are rule-bound microcosms with a definite ending.

The book includes the general rules of each game, where it originated, and where mathematical concepts appear. The author also writes about parallels between games and mathematics. There are common traits of games across cultures as well as unique aspects provided by each. We often appreciate the aesthetics of a game and how simple rules give rise to complexity. They both offer a chance to overcome barriers to reach an end—a victory in a game or a QED in a proof. "The rules of the game are like the axioms of mathematics. Playing a game is like exploring the consequences of those axioms. Games for me are a way of playing mathematics." This is a fun book about games and how math plays a role in them. Consider using it in a history of math course, as part of a math club, or for personal interest.

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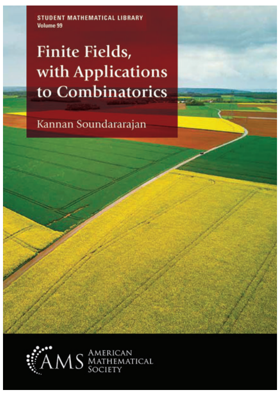


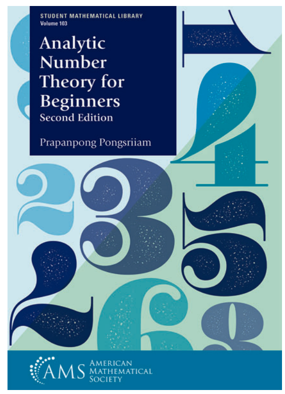
Finite Fields, with Applications to Combinatorics
By Kannan Soundararajan.
STML/99, 2022, 170 pp.

Over the past hundred years, finite fields—nontrivial finite commutative rings where every nonzero element has a multiplicative inverse—have shown themselves to be indispensable throughout number theory, coding theory, and cryptography. The book under review offers a self-contained but expansive development of the theory of finite fields, alongside several applications. It originated as the course text for a class at Stanford targeting highly motivated first-year students.

The first chapter begins with the definitions of "rings" and "groups"; the last (ninth) chapter finishes with the classification theorem for finite fields. The audience is expected to have some calculus under their belt and have had some exposure to "vector spaces." No prior knowledge of abstract algebra is assumed, and the needed theory of groups and rings is developed, cleanly and efficiently, in Chapters 1, 3, 4, and 6. Chapter 2 and Chapter 5 discuss topics in number theory useful for contextualizing results about finite fields.

Chapter 7 showcases a number of delightful applications in combinatorics, specifically to Sidon sets, to perfect difference sets, and to de Bruijn sequences. Chapter 8 is an exposition of the Agrawal–Kayal–Saxena (AKS) test, the first provably fast ("polynomial-time") algorithm for deciding whether a given integer is prime or composite; finite fields play an essential role in the proof of the algorithm's correctness. The material of Chapters 7 and 8 is not contained in standard textbooks, and so the wonderfully lucid treatment here is quite welcome.

This is a splendid introduction to finite fields. The writing is crisp, clean, and down-to-earth, and the mathematics is not only unusually elegant but extraordinarily useful.



Analytic Number Theory for Beginners: Second Edition
By Prapanpong Pongsriiam.
STML/103, 2023, 375 pp.

Students might find it surprising that there is a subject known as "analytic number theory." What does analysis, which investigates the continuous, have to contribute to the study of the discrete set of natural numbers? This book provides an answer to that question, assuming only a modest mathematical background on the part of the reader.

Consistent with the title of the book, the exposition is unusually down-to-earth. Dirichlet's theorem on primes in arithmetic progressions is demonstrated by an elementary method of Gelfond–Linnik and Shapiro, so that this part of the text can be appreciated without a course in complex functions. The prime number theorem, one of the landmark results in the subject, is given its simplest known proof based around ideas of Newman, Zagier, and Korevaar. Complex analysis is needed there, but the relevant facts are given a thorough review in Chapter 8. Throughout, arguments are written in detail and are easy to follow, making the text conducive to self-study.

The author has not been shy about inserting his own tastes and interest. The results on the floor function in Chapter 3, some of which draw on the author's own research, will be new even to experts in number theory. Each chapter features extensive end notes, offering the reader a glimpse of modern results. Exercises are another strong point of the book; there are quite a few, many of which are original and nonroutine. This second edition makes the text available to readers outside Thailand and includes a new chapter on sieve methods and additive number theory.

This AMS Bookshelf is by Paul Pollack, professor of mathematics at the University of Georgia. His email address is `pollack@uga.edu`.

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COMMUNICATION

Shock Waves and PDEs

A Conversation with Barbara Lee Keyfitz

Fariba Fahroo and Reza Malek-Madani

1. Introduction

For the past half century, hyperbolic conservation laws have served as an important subfield of applied mathematics, especially in its relation to physics, chemistry, and biology. The way we model the behavior of elastic materials and fluid flows is inherently through partial differential equations (PDEs). When we explore dynamics (wave propagation), we end up analyzing a special class of PDEs, which are often hyperbolic and nonlinear, and are capable of modeling features such as shocks, dislocations, and fracture, because they are able to support discontinuous functions as their solutions. One of the grand challenges in mathematics in the past fifty years has been to develop a framework to make sense of what it means to have a discontinuous solution of a differential equation.

Barbara Keyfitz has spent her entire career in this field. One of the goals of this article is to highlight her contributions. We have had extensive conversations with Barbara and will describe some of her substantial contributions in her own voice. While we will present some of the details of her technical contributions, our emphasis will be on allowing Barbara to share her mathematical journey over the past several decades, first by describing her triumphs, and some of the challenges she has encountered, but also by sharing some of her unique achievements in providing service to our community, at the national and international level.

Barbara's education started at the University of Toronto in 1962. After earning her bachelors degree in mathematics, she enrolled as a graduate student in the Courant Institute of Mathematical Sciences in 1966 where she studied conservation laws under the direction of Peter D. Lax. Our conversation with Barbara began from this stage of her career, and we proceeded to learn about the path that took her to Columbia University, Princeton, Arizona State University, University of Houston, and Ohio State, where she is now an emeritus professor. In addition, while she has served on numerous national and international committees, we will concentrate on hearing from Barbara about her contributions as the president (2005–2007) of the Association for Women in Mathematics, the president of the International Council for Industrial and Applied Mathematics (2011–2015), and as director of the Fields Institute (2004–2008).

But first a few definitions.

1.1. Hyperbolicity and PDEs. The simplest conservation law is the scalar partial differential equation

$$u_t + (f(u))_x = 0, \tag{1}$$

which is typically supplemented with the initial condition $u(x,0) = u_0(x)$. In (1), t stands for time, x for space, and u for a physical concentration. Standard generalizations of (1) are to one-dimensional systems of equations

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}, \tag{2}$$

where $\mathbf{u} : R \times R_+ \to R^n$, and to multidimensional systems

$$\mathbf{u}_t + \sum_{i=1}^{d} A_i(\mathbf{u})\mathbf{u}_{x_i} = 0, \tag{3}$$

where $\mathbf{u} : R^d \times R_+ \to R^n$. Many fundamental equations of mathematical physics fall in the category of (1), (2), or (3).

Equation (2) is called *hyperbolic* if the $n \times n$ matrix $A(\mathbf{u})$ has n real eigenvalues, which in the case of (1) reduces to $f'(u)$ simply existing. The corresponding definition for (3) is that $\sum A_i(u)\xi_i$ have real eigenvalues.

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Equation (2) shows why conservation laws are special. A standard way of solving the wave equation (2) is to follow the behavior of solutions along *characteristics*, curves defined in the (x, t) plane by the equations

$$\frac{d\hat{x}_i}{dt} = \lambda_i(u(\hat{x}_i, t)), \quad \hat{x}_i(0, t) = \xi_i, \quad i = 1, \dots, n, \tag{4}$$

where λ_i is an eigenvalue of A. In the case of (1), the characteristic curves are defined by the ordinary differential equations

$$\frac{d\hat{x}}{dt} = f'(u(\hat{x}, t)), \quad \hat{x}(0) = \xi. \tag{5}$$

Unlike the linear wave equation, where $f'(u)$ is a constant c and the characteristics are therefore parallel straight lines and never intersect, the characteristics of a nonlinear conservation law depend on the solution u, and could intersect at a finite time $T > 0$. This behavior is the key observation why the analysis of conservation laws is so challenging. To elaborate on this further, consider the case of $f'(u) = u$ in (1), the *inviscid Burgers equation*. Keeping $\xi \in R$ fixed, the characteristic curves $\hat{x}(\xi, t)$ of (5) now take the form

$$\frac{d\hat{x}}{dt} = u(\hat{x}(\xi, t), t), \quad \hat{x}(0) = \xi. \tag{6}$$

A simple computation shows the source of the problem: The quantity $u(\hat{x}(\xi, t), t)$, that is, the solution u confined to a fixed characteristic curve, remains constant in t, as can be seen from

$$\frac{d}{dt}(u(\hat{x}(\xi, t), t)) = u_t + \frac{d\hat{x}}{dt} u_x(\hat{x}(\xi, t)) = u_t + u u_x = 0,$$

and therefore $u(\hat{x}(\xi, t), t)) = u_0(\xi)$. So in fact the equation in (6) reduces to $\frac{d\hat{x}}{dt} = u_0(\xi)$, a constant in t, so the characteristics of Burgers equation are again straight lines, but their slopes now depend on u_0. Consequently, if the initial state u_0 is such that $u_0(\xi_1) > u_0(\xi_2)$ for $\xi_1 < \xi_2$, then the two characteristics that initiate at ξ_1 and ξ_2 will meet in finite time, and the solution u will become multivalued. At this point we say that the solution has developed a discontinuity or a *shock*. Much research in the past several decades has been dedicated to making sense of how a solution of a PDE with smooth initial data can develop a discontinuity in finite time, and once it does, how we are supposed to extend the solution beyond this point. This is the starting point of much of the research in conservation laws, including Barbara's research.

1.2. Student of Peter Lax. One of the first questions we asked Barbara Keyfitz was how she came to study at Courant and how she chose conservation laws as her topic.

"When I think about it, there was a certain amount of serendipity involved. I was an undergraduate at the University of Toronto, where the mathematics honors program was quite advanced relative to US schools. In my final year, one of the courses I took (I am pretty sure we did not have a course in partial differential equations) was an advanced ODE course with F. V. Atkinson. He introduced the method of characteristics for quasilinear scalar equations (see above!), as a way of showing how to use ODEs to solve PDEs. I was entranced. I determined to go to graduate school to study partial differential equations. In 1966, this was not that easy. (Remember, in 1966, PDE was considered 'applied math', and the focus was not on analysis.) At the time, Canadian graduate programs were still quite small, and some were unwelcoming to female graduate students. So I applied to a few programs in the US ... and then another very important person in my life comes into the story. That was Chandler Davis. He said, 'Go to NYU and work with Peter Lax'. I sent a letter to NYU and nothing happened.

"After a while Chandler asked me how the application was going and I said I haven't heard anything. He asked me what I had done. I said I wrote to NYU, and he said 'No, no, no, that's not the way it works, you need to write to the Courant Institute'. So I wrote to the Courant Institute, and they said 'you're late, but since you were recommended by Chandler Davis, we'll consider it'. So I applied and was accepted.

"The Courant institute had just moved into a new building at 251 Mercer Street, and I had a desk in a beautiful office with windows on two sides (there were four desks there) and a telephone on my desk. I picked up the telephone; I phoned Peter Lax and said 'I'd like to take your functional analysis course', and he said, politely, 'well actually this course is meant for people who know something'. I enrolled anyway. (Fortunately, the course was not graded.)

"On a sunny afternoon, beautiful California weather, at an AMS Summer School in Berkeley on Global Analysis in 1968, after rather little further interaction between us, Peter suggested a research topic to me. He described the evolution of a scalar conservation law by the method of characteristics (see above, again!), and how the characteristics collide; and he said, I remember the phrase, 'but the flow must go on'.

"Peter Lax's research has spanned much of modern PDE; he was directing the research of other students, many of whom have gone on to brilliant careers in a number of fields. How did he pick conservation laws for me?

"He talked about weak solutions, and then he left me on my own. When I couldn't think of anything to do, he suggested, 'look at the L^1 difference between two solutions of a scalar equation. As long as the solutions are smooth, it's a simple calculation that the difference remains constant, but once you get weak solutions, I don't know, I think it might go bad.' So that was the challenge."

As the earlier example of the inviscid Burgers equation showed, one expects that solutions to the initial-value problems of conservation laws exist in a classical sense on a maximal time interval. Once a discontinuity is formed, however, we need to consider the solution in a weak sense. For the scalar conservation law $u_t + (f(u))_x = 0$, $u(x,0) = u_0(x)$, the weak formulation of the problem takes the form

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} u(x,t)\phi_t(x,t) + f(u)\phi_x(x,t)\,dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,t)\,dx = 0,$$

where ϕ is any function belonging to $C_0^\infty(R, R_+)$, the space of smooth functions with compact support. While this formulation presents a natural generalization from the classical formulation of the problem, with a natural connection to how conservation laws appear in continuum physics, it does not lead to a well-posed problem: simple examples of initial-value problems exist that exhibit multiple weak solutions. The challenge then became to come up with additional conditions capable of selecting a unique solution. These conditions, often called entropy conditions, have had enormous success for two important classes of conservation laws: scalar conservation laws in R^d (such as the equation in (3) with $n = 1$), and systems of conservation laws in R (such as the one in (2)). In fact, as shown in Chapter 5 of [Daf10], weak solutions of these conservation laws, under appropriate conditions on the nonlinearities and endowed with an entropy condition, are well-posed: they have unique solutions and are stable in L^1.

"I made the naive (and very restrictive) assumption that the solution was piecewise smooth, and that the only irregularities were shocks. It then becomes a simple calculus problem to compute the L^1 difference: as long as you have a convex conservation law, like Burgers equation, and the shocks are going in the right direction, which is to say, the left states are higher than the right states, the L^1 norm strictly decreases when shocks are present. If the conservation law isn't convex, then sometimes it decreases and sometimes it doesn't, but I wrote down a condition on discontinuities that would give the desired inequality. I brought this back to Lax and he said, 'that condition looks familiar; I think Oleinik had something to say about that.' So we looked it up and we found that indeed the Oleinik condition was necessary and sufficient for the norm to decrease ([Ole63]). So that became my thesis ... The rule of thumb seems to be that when you prove a result that your adviser didn't believe, you then have a dissertation.

Figure 1. Peter Lax and some of his students, at the conference for the 80th birthdays of Peter Lax and Louis Nirenberg, Toledo, Spain. In the picture, BLK leaning over to talk with Peter Lax (seated). Standing, from left to right: part of David Levermore, Alex Chorin, Stephanos Venakides, and Sebastian Noelle. Visible at lower right corner, Haim Brezis.

"The result itself turned into a paper, [Qui71], published about the same time I got my degree. It attracted some interest: my result showed that the entropy solutions to a scalar conservation law form a contractive semigroup in L^1. That year, 1970, was a prime time for applying semigroup theory to PDEs. I had found a semigroup, but I had no idea of what people were talking about when they spoke about the generator of a semigroup, which I had not looked for or found. (Crandall's paper, a few years later, notes this, and does find the generator [Cra72].) At the same time, the PDE community was also discovering that Soviet mathematicians had been studying conservation laws a little bit earlier. The well-known and much cited paper of Kruzkov [Kru70], translated into English in 1970, had been published in Russian, I think in 1968. There was also work of Vol'pert [Vol86]; he had done genuine analysis. That is, he had looked at weak solutions that didn't have the simplified structure I considered. I was rescued a little bit by Dave Schaeffer who showed that, generically, solutions of a convex conservation law do have a piecewise continuous structure, [Sch73].

"I spent a couple of years trying to figure out more estimates, which turned out not to be very useful."

2. Research in Nonstrictly Hyperbolic Systems of Conservation Laws

Our conversation at this point turned to the realities of life after graduate school; how does one get a job, how to balance personal life with building a career, and what research projects to pursue.

"The real problem was that I didn't know what to do next. For personal reasons I needed to stay in New York. I got a job at Columbia University, in the engineering school. They were willing to hire me because there had been an institute for flight studies, and, okay, conservation laws govern compressible flow. Right after they hired me, that institute was disbanded. And once again, I was rescued by a tremendous piece of good luck.

"Another Courant graduate at Columbia, John (C. K.) Chu, was in the mechanical engineering department. We talked a bit. He tried to get me interested in Tokamak reactors and fusion, but I didn't find anything to work on. But a couple of years later Herb Kranzer, a friend of his and another Courant graduate, had a sabbatical and decided to spend it with John. John introduced me to Herb. One of Herb's students, Dennis Korchinski, in his PhD dissertation [Kor77] had looked at an interesting problem that he had not been able to analyze completely. It involved an extremely simple system of conservation laws, where he had considered the Riemann problem, a simple initial value problem with two constant states. For states that were close to together ('small data'), you could get regular solutions, for other states ('large data') he couldn't analytically find a solution, and when he tried to do it numerically … because what else can you do … solutions seemed to blow up; they became unbounded. Herb described the problem to me, and then he said the magic words, 'Would you like to work on this problem with me?'

"This is now 1974. As we started working on it, we realized that we didn't know very much about systems of conservation laws. It was the early days of the field. There was Glimm's theorem, which had been around by then for almost ten years, which almost nobody understood, ourselves included, and there was not much else. There was a lot of work by Joel Smoller, but Smoller's results didn't help with our problem. Smoller had been a visitor at Courant my last year as a student there; he was extremely kind, and somebody I could talk to about conservation laws, but his parting warning was that life beyond studenthood was brutal.

"Conservation law research on systems, up to that point, had required that characteristics be distinct, and Korchinski's problem didn't have that feature, so we started to think about nonstrictly hyperbolic conservation laws.

In [KK80] Keyfitz and Kranzer introduced the following canonical example of a nonstrictly hyperbolic conservation law:

$$u_t + (\phi u)_x = 0, \quad v_t + (\phi v)_x = 0, \tag{7}$$

where $\phi = \phi(u, v)$. This problem is an example of the system described in (2) where the 2×2 matrix A may not have distinct eigenvalues for a subset Σ of (u, v)-space, and therefore may not be diagonalizable. A motivating example of ϕ is

$$\phi(u, v) = 1 + \delta \frac{(r-1)^2}{r}, \quad \text{where } r^2 = u^2 + v^2$$

which appears in certain stress-strain laws of nonlinear elasticity. The *Riemann problem* for (7) is the solution to this system subject to the initial condition

$$U(x, 0) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0, \end{cases}$$

where $U = (u, v)$ and U_l and U_r are constants.

"When we started looking at nonstrict hyperbolicity, we observed that there were two things that could happen, and you can easily see what they are if you think about a 2×2 matrix with two equal eigenvalues: is it diagonalizable or is it not? We started out by looking at a planar elastic string, modeled by a pair of second order equations, or equivalently, a system of four first-order equations (see [AMM88]) … we knew we couldn't handle the four-dimensional problem, so we created two first order equations, instead of second order equations, and when we did that, we got a matrix that was not diagonalizable. (This was, in fact, the result of a modeling error on our part.)

"Reviewing what was known about large data Riemann problems for strictly hyperbolic systems, we discovered a condition which we called *opposite variation*, somewhat more general than conditions that Smoller and Johnson [SJ69] had advanced. If systems satisfied this condition, we could also solve Riemann problems for any pair of nonstrictly hyperbolic equations where the behavior of eigenvectors near characteristic crossings was of a certain type. The solutions weren't as well-behaved as for strictly hyperbolic Riemann problems, but we could describe the sense in which they satisfied continuous dependence on the data.

"Eventually, we noticed that in fact with the elastic string problem, for the full four-dimensional system, the matrix is diagonalizable where the eigenvalues become equal in pairs. We solved the Riemann problem for that system, as well as our ersatz nondiagonalizable example.

The paper appeared in the Archive for Rational Mechanics and Analysis, [KK80]."

The paper [KK80] has received over four hundred citations. For several years following its publication in 1980, this paper set the stage for much of the research in nonstrictly hyperbolic conservation laws. The paper [SS86] is a notable body of research on these equations where the authors present a classification of the behavior of Riemann problems for 2×2 systems motivated by simulation of oil recovery problems. This particular application, sometimes referred to as the 'reservoir problem', was a focus of research by James Glimm and his associates during the decade of the 80's, and introduced a new set of theoretical and computational challenges, culminating in formulation of the "front tracking" approach for shock waves, first for numerical simulation (see [GGL+98] for details) and eventually, triumphantly, to Alberto Bressan's celebrated results providing the first proofs of the well-posedness of the Cauchy problem for general, strictly hyperbolic systems [Bre00].

"The intersection of our work with Glimm's was the nondiagonalizable problem, which we'd solved by mistake but published anyway along with the result relevant to elastic strings. It was one of Glimm's simple reservoir models. Eli Isaacson and Blake Temple, at that time two of Glimm's postdocs, had solved it, and planned to give a talk about it in an upcoming Joint Math Meeting. I saw their abstract and wrote to them, 'you might want to know that we have a published paper that solves this problem'. They withdrew the paper they had submitted for publication (I still have the preprint), and graciously talked about our result instead at the JMM. We had sent our published paper to Glimm, but of course he hadn't looked at it, and all he said to them was, 'you ought to read the literature'.

"I still feel badly about it. When this happened I had a tenure-track job and they were postdocs; they were junior and it wouldn't have hurt me to be kinder. Why did I have to treat them as though they were stealing my result?

"But then another direction opened up. At the time it seemed totally screwball, but it sustained Herb's and my research for another 15 years or so. We started thinking about what happens if instead of opposite variation, when these characteristics cross, you have the same variation in the two families, and here I give credit to the mathematician Michelle Schatzman, a friend. I was trying to explain this problem to her, that you could have the characteristics crossing in the shape of an X, in opposite orientation, or you could have the characteristics crossing still as an X, but a distorted X, with the lines going from lower left to the upper right, from both sides. Michelle took a marker to my X and simply separated the lines .... So now you have a system that is strictly hyperbolic, as the two characteristics are separated. It didn't take us long to write down an example, to try to solve the Riemann problem, and to find that indeed we couldn't solve it, even for the strictly hyperbolic equation. When we resorted to numerical methods, the solution blew up, just as Dennis Korchinski's example had: solutions became unbounded."

In 2011, Barbara published the paper [Key11] which, as its title indicates, gave a retrospective on her contribution to the concept of *singular shocks* and the inspiration she received from her conversations with Michelle Schatzman.

Singular shocks were first discovered by Keyfitz and Kranzer in their attempt to finding solutions to Riemann problems in certain strictly hyperbolic problems based on a peculiar reformulation of the governing equations for isothermal, isentropic gas dynamics. Despite the fact the system is genuinely nonlinear, they discovered a large region of state space where the Riemann problem cannot be solved using shocks and rarefactions. They produced approximate unbounded solutions which do not satisfy the equation in the classical weak-solution sense. This discovery has led to a rich, new area of conservation laws with deep connections to the Dafermos-DiPerna regularization theory, and to geometric singular perturbation theory (the manifold theory for singular dynamical systems). The paper by Tsikkou [Tsi16] gives an excellent review of this field, in addition to describing how singular shocks appear in chromatography. See also [Sev07] for a systematic exposition of what was known about systems with singular shock solutions fifteen years ago.

Barbara continued to share her recollections of her mathematical journey.

"After my (deservedly) failing to get tenure at two fine institutions—Columbia and Princeton—but by then happily married and a mother, I and my mathematician husband Marty Golubitsky found jobs at Arizona State University. A brief foray on my part into applications of singularity theory (Marty's field) was a springboard to jobs for both of us at the University of Houston, and ultimately at Ohio State.

"In 1992, at Houston, I got a State of Texas grant to hire a postdoc, Sunčica (Sunny) Čanić. A colleague of mine, David Wagner, a former student of Smoller's, had noted an example of a two-dimensional Riemann problem, described in a paper of Brio and Hunter, [BH92]. This was another lucky chance where something came along at exactly the right moment. At the time very little was known about systems of conservation laws in two space dimensions. David suggested that we 'bite off a little bit of multi-dimensional systems'. Sunny and I started looking at these 2D Riemann problems and found that we needed to learn about elliptic equations, something that hyperbolic people usually avoid like the plague. And, serendipitously, Sunny moved to Iowa State University, and met Gary Lieberman, who was the best person in the world to help us, and did, [ČanićKL00]."

Barbara's collaboration with Sunny Čanić was another productive period in her mathematical career. In more than a dozen papers, Barbara and Sunny attacked a number of hard problems in conservation laws: The work on degenerate elliptic PDEs appeared in [ČanićKL00]. The paper proves existence of a solution to a free boundary problem for the transonic small-disturbance equation—the free boundary is the position of a transonic shock dividing two regions of smooth flow; the equation is hyperbolic upstream where the flow is supersonic, and elliptic in the downstream subsonic region. Joined later by another postdoc, Eun Heui Kim, they analyzed a prototype for Mach stems [ČanićKK05]. Gui-Qiang Chen and Misha Feldman and their students and associates have made great strides with this approach, documented in their monograph, [CF18].

One of the remarkable aspects of Barbara's career is her service and mentoring record, the strength of which parallels her research record. In the remainder of this article, rather than bringing readers up to date on the field of conservation laws, now so much more extensive than in 1970, we report on our conversation on how she prepared herself for the high-level community service positions she has held.

Barbara attributed much of her success to her ability to listen and to engage, abilities she acquired and mastered early in her career. The statement 'one thing led to another' was a common refrain as she described how opportunities presented themselves, paralleling the serendipity present in so many of her research collaborations.

3. Community Service and Mentoring

During her long career, Barbara has been on the leadership of many national and international organizations, including serving as the vice president for programs at SIAM and the treasurer of ICIAM; but serving as the president of the Association of Women in Mathematics (see [Key22]), and as the president of ICIAM, particularly stand out. Her tenure as the director of the Fields Institute, and the many initiatives she was able to launch in that position, ended up being the main focus of our conversation.

"In terms of highlights, Fields was the highlight. Being the director of Fields was a job that I think I enjoyed more than anything else in my career. For the first six months I just walked around the campus of the University of Toronto smiling—people would stop me to ask, 'What are you smiling about?'"

Figure 2. Akershus Castle, Abel Prize ceremony 2014 (the winner was Yakov Sinai). This was a banner year for women. From left to right, Maria J. Esteban (member of the Abel Prize Committee), me (president of ICIAM), Ragni Piene (chair of the Abel Committee), Ingrid Daubechies (president of IMU), and Marta Sanz-Sole (president of the European Mathematical Society). [Not at the ceremony, but rounding out the theme, Ruth Williams had just completed a term as president of IMS.]

During the four and half years of her directorship at Fields, Barbara was able to introduce several programs, many of them intended to improve the participation of women in the research programs at the Fields Institute, as well as in programs at other partner institutions in Ontario.

"I wanted to make Fields more accommodating to women. For example, there were distinguished lecture series that had never had women lecturers. There was no kind of affirmative action. I asked our Board of Directors for a mandate, and they said, 'Go ahead'. Yes!

"Another function at Fields was organizing workshops at other universities, as well as thematic programs, which are like the ones in the US institutes. A large fraction of the Fields budget was spent on short-term events, typically one-week workshops that took place at different partner institutions in Canada. We would solicit proposals, and we would have to evaluate them. On one occasion my deputy director wrote to a proposer, 'I see your list of invited speakers doesn't include a single woman. Could you please explain your thinking to me?'"

At the end of our conversation, we asked Barbara about her impressions related to dedicating workshops or awards solely to women.

"I have mixed feelings about that. Some women mathematicians take exception to being recognized in part because they're women. Instead of seeing it as an opportunity, a foot in the door, they don't want to accept an honor that in their interpretation is sullied by this. And they see attending a workshop for women, from which important male scholars are excluded, as an inefficient use of their time. ... On the other hand BIRS, the Banff math institute, now has one workshop a year for women. These appear to have been extremely successful. I think it's because they bring women together physically. There's a story here.

"For a while, BIRS had been rather unfriendly about women's participation; for example, they refused to allow people to bring children—no daycare, no accommodations. If a woman didn't want to come under those conditions, then she didn't need to come. Since many places had started making daycare arrangements, I felt BIRS could have at least thought about it, or at least not complained about my asking. So out of the blue the director said, 'how about this year we give you one week for women, you can do whatever you want with it. It is yours.' It happened it was the week of Rosh Hashanah, which is possibly why nobody wanted it. I was willing to give up Rosh Hashanah for one year and so we ran a workshop. It wasn't primarily a scientific workshop, and had the unimaginative name 'Women in Mathematics'. The idea was to discuss 'what can women do to get ahead.' We had participants from Canada, US and Mexico, because BIRS is run by all three countries. Our workshop report was titled 'Women in the Academic Ranks: A Call to Action'." (It can be found on the BIRS workshop report page at `https://www.birs.ca/workshops/2006/06w5504/report06w5504.pdf`)."

4. Afterword

We started planning this project in June of 2023. Barbara has been enormously generous with her time over the past 4 months: In addition to two sessions on July 24 and Aug 1 where we had the opportunity to meet remotely, we have been in frequent contact with Barbara with our questions. This article is built on extracts from those many, many hours of communications. We hope it gives a glimpse of the impact of Barbara's career.

The article does not do justice to two aspects of our interactions: We haven't been able to adequately depict Barbara's sharp and engaging sense of humor. Our sessions were always enlivened by her infectious laughter, which we felt we couldn't get enough of. Second, while neither of us has been Barbara's student, it was clear during the past few months that she is a master teacher. Our conversation sessions were filled with Barbara describing details of her work, using examples and analogies that we could embrace, which is reminiscent of the important role canonical and reduced models have played throughout her research career ... Barbara's lectures are now available on social media, which we highly recommend.

For the past several decades Barbara Keyfitz has participated in, has led and organized many national meetings and PDE conferences. Her approach to research in applied mathematics has led to posing new questions and pursuing new directions. Her lifelong desire to work to improve the environment of research for women has already had an enduring effect. We have been so fortunate that she has always been approachable and ready to share and to engage.

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Fariba Fahroo

Reza Malek-Madani

Credits

Figure 1 is courtesy of Maria Venikides.

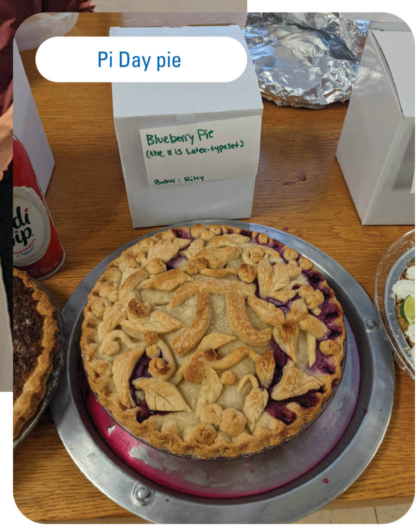
Figure 2 is courtesy of Marty Golubistky.

Photo of Fariba Fahroo is courtesy of Hennie Farrow.

Photo of Reza Malek-Madani is courtesy of Photo Lab: US Naval Academy.

How do AMS Graduate Student Chapters support the mathematical community and beyond?


University of Minnesota chapter members


Pi Day pie

University of Minnesota

Chapter members run a variety of events throughout the academic year, including weekly teas, periodic social events, and a seminar series. Past events have included Pi Day pie and trivia competitions and the Unary Day, featuring Guest Speaker Max Engelstein, Assistant Professor.

Photos courtesy of their respective chapters.

AMS AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

Utah State University

The Utah State graduate students invite guest speakers throughout the academic year. One lecture, by Dr. Charles Torre, Professor and Associate Department Head of the Physics Department at Utah State, focused on general relativity and solutions to Einstein's equations.

Utah State University Chapter


Presentation at the Duke University Chapter

Duke University

This Chapter builds community among students by promoting conversation, collaboration, and friendship across research and institutional milieus. They will host the 9th annual Triangle Area Graduate Mathematics Conference (TAGMaC) in 2023, at which graduate students from Duke, University of North Carolina at Chapel Hill, and North Carolina State University give talks about research-level mathematics.

The Chapter Program makes meaningful contributions to the professional development of graduate students in the mathematical sciences and connects graduate students with AMS offerings.

www.ams.org/studentchapters

FROM THE AMS SECRETARY

AMS Prizes and Awards

NEW! I. Martin Isaacs Prize for Excellence in Mathematical Writing


I. Martin Isaacs

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

About this Prize

The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in *Communications of the AMS*, *Journal of the AMS*, *Mathematics of Computation*, *Memoirs*, *Proceedings of the AMS*, or *Transactions of the AMS*, and is therefore not a prize selection criterion.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups—such as solvable groups—that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/isaacs-prize`.

NEW! Elias M. Stein Prize for Transformative Exposition


Elias M. Stein

The Elias M. Stein Prize for Transformative Exposition is awarded for a written work, such as a book, survey, or exposition, *in any area of mathematics* that transforms the mathematical community's understanding of the subject or reshapes the way it is taught.

About this Prize

This prize was endowed in 2022 by students, colleagues, and friends of Elias M. Stein to honor his remarkable legacy of writing monographs and textbooks, both singly and with collaborators. Stein's research monographs, such as *Singular Integrals and Differentiability Properties of Functions* and *Harmonic Analysis*, became canonical references for generations of researchers, and textbooks such as the Stein and Shakarchi series *Princeton Lectures in Analysis* became instant classics in undergraduate and graduate classrooms. Stein is remembered for his ability to find a perspective to make a method of proof seem so natural as to be inevitable, and for his strategy of revealing the essential difficulties, and their solutions, in the simplest possible form before elaborating on more general settings. This prize seeks to recognize mathematicians at any career stage who, like Stein, have invested in writing a book or manuscript that transforms how their research community, or the next generation, understands the current state of knowledge in their area.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/stein-exposition`.

Mary P. Dolciani Prize for Excellence in Research

The AMS Mary P. Dolciani Prize for Excellence in Research recognizes a mathematician from a department that does not grant a PhD who has an active research program in mathematics and a distinguished record of scholarship. The primary criterion for the prize is an active research program as evidenced by a strong record of peer-reviewed publications.

Additional selection criteria may include the following:

- Evidence of a robust research program involving undergraduate students in mathematics;
- Demonstrated success in mentoring undergraduates whose work leads to peer-reviewed publication, poster presentations, or conference presentations;
- Membership in the AMS at the time of nomination and receipt of the award is preferred but not required.

About this Prize

This prize is funded by a grant from the Mary P. Dolciani Halloran Foundation. Mary P. Dolciani Halloran (1923–1985) was a gifted mathematician, educator, and author. She devoted her life to developing excellence in mathematics education and was a leading author in the field of mathematical textbooks at the college and secondary school levels.

The prize amount is $5000, awarded every other year for five award cycles.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: `https://www.ams.org/dolciani-prize`.

Award for an Exemplary Program or Achievement in a Mathematics Department

This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

About this Award

This award was established in 2004. For the first three awards (2006–2008), the prize amount was US$1,200. The prize was endowed by an anonymous donor in 2008, and starting with the 2009 prize, the amount is US$5,000. This US$5,000 prize is awarded annually. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: A letter of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department in being nominated as well as the achievements which make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). Where possible, the letter and documentation should address how these successes came about by 1) systematic, reproducible changes in programs that might be implemented by others, and/or 2) have value outside the mathematical community. The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Information on how to nominate can be found here: `https://www.ams.org/department-award`.

Award for Impact on the Teaching and Learning of Mathematics

This award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.

Priorities of the award include recognition of:

(a) accomplished mathematicians who have worked directly with precollege teachers to enhance teachers' impact on mathematics achievement for all students, or

(b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

About this Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen.

The US$1,000 award is given annually.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages, and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included. The nonwinning nominations will automatically be reconsidered, without further updating, for the awards to be presented over the next two years.

Information on how to nominate can be found here: `https://www.ams.org/impact`.

Ciprian Foias Prize in Operator Theory

The Ciprian Foias Prize in Operator Theory is awarded for notable work in Operator Theory published during the preceding six years. The work must be published in a recognized, peer-reviewed venue.

About this Prize

This prize was established in 2020 in memory of Ciprian Foias (1933–2020) by colleagues and friends. He was an influential scholar in operator theory and fluid mechanics, a generous mentor, and an enthusiastic advocate of the mathematical community.

The current prize amount is US$5,000, and the prize is awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations require CV of the nominee, a letter of nomination, and a citation.

Information on how to nominate can be found here: `https://www.ams.org/foias-prize`.

David P. Robbins Prize

The Robbins Prize is for a paper with the following characteristics: it shall report on novel research in algebra, combinatorics, or discrete mathematics and shall have a significant experimental component; and it shall be on a topic which is broadly accessible and shall provide a simple statement of the problem and clear exposition of the work. Papers published within the six calendar years preceding the year in which the prize is awarded are eligible for consideration.

About this Prize

This prize was established in 2005 in memory of David P. Robbins by members of his family. Robbins, who died in 2003, received his PhD in 1970 from MIT. He was a long-time member of the Institute for Defense Analysis Center for Communications Research and a prolific mathematician whose work (much of it classified) was in discrete mathematics.

The current prize amount is US$5,000 and the prize is awarded every 3 years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/robbins-prize`.

E. H. Moore Research Article Prize

The Moore Prize is awarded for an outstanding research article to have appeared in one of the AMS primary research journals (namely, the *Journal of the AMS*, *Proceedings of the AMS*, *Transactions of the AMS*, *Memoirs of the AMS*, *Mathematics of Computation*, *Electronic Journal of Conformal Geometry and Dynamics*, and *Electronic Journal of Representation Theory*) during the six calendar years ending a full year before the meeting at which the prize is awarded.

About this Prize

The prize was established in 2002 in honor of E. H. Moore. Among other activities, Moore founded the Chicago branch of the American Mathematical Society, served as the Society's sixth president (1901–1902), delivered the Colloquium Lectures in 1906, and founded and nurtured the *Transactions of the AMS*.

The current prize amount is US$5,000, awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/moore-prize`.

Leroy P. Steele Prize for Lifetime Achievement

The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students.

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$10,000.

Next Prize: January 2025

Nomination Period: February 1 – March 31

Nomination Procedure: Nominations for the Steele Prize for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: `https://www.ams.org/steele-lifetime`.

Leroy P. Steele Prize for Mathematical Exposition

The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper.

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$5,000.

Next Prize: January 2025

Nomination Period: February 1 – March 31

Nomination Procedure: Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: `https://www.ams.org/steele-exposition`.

Leroy P. Steele Prize for Seminal Contribution to Research

The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Analysis/Probability (2020)
2. Algebra/Number Theory (2021)
3. Applied Mathematics (2022)
4. Geometry/Topology (2023)

5. Discrete Mathematics/Logic (2024)
6. **Open (2025)**

About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US$5,000.

Next Prize: January 2025

Nomination Period: February 1–March 31

Nomination Procedure: Nominations for the Steele Prize for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: `https://www.ams.org/steele-research`.

Levi L. Conant Prize

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

About this Prize

Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon the death of his wife.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006.

The Conant Prize is awarded annually in the amount of US$1,000.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination a complete bibliographic citation for the article being nominated.

Information on how to nominate can be found here: `https://www.ams.org/conant-prize`.

Mathematics Programs that Make a Difference

This Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS's Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are potentially replicable models.

About this Award

This award brings recognition to outstanding programs that have successfully addressed the issues of underrepresented groups in mathematics. Examples of such groups include racial and ethnic minorities, women, low-income students, and first-generation college students.

One program is selected each year by a selection committee appointed by the AMS president and is awarded US$1,000 provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level. Nomination of one's own institution or program is permitted and encouraged.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: The letter of nomination should describe the specific program being nominated and the achievements that make the program an outstanding success. It should include clear and current evidence of that success. A strong nomination typically includes a description of the program's activities and goals, a brief history of the program, evidence of its effectiveness, and statements from participants about its impact. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages. Nomination of the writer's own institution or program is permitted. Nonwinning nominations will automatically be reconsidered for the award for the next two years.

Information on how to nominate can be found here: `https://www.ams.org/make-a-diff-award`.

Oswald Veblen Prize in Geometry

The award is made for a notable research work in geometry or topology that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

About this Prize

This prize was established in 1961 in memory of Professor Oswald Veblen through a fund contributed by former students and colleagues. The fund was later doubled by the widow of Professor Veblen. An anonymous donor generously augmented the fund in 2008. In 2013, in honor of her late father, John L. Synge, who knew and admired Oswald Veblen, Cathleen Synge Morawetz and her husband, Herbert, substantially increased the endowment.

The current prize amount of US$5,000 is awarded every three years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/Veblen-prize`.

Ruth Lyttle Satter Prize in Mathematics

The Satter Prize recognizes an outstanding contribution to mathematics research by a woman in the previous six years.

About this Prize

This prize was established in 1990 using funds donated by Joan S. Birman in memory of her sister, Ruth Lyttle Satter. Professor Birman requested that the prize be established to honor her sister's commitment to research and to encourage women in science. An anonymous benefactor added to the endowment in 2008.

The current prize amount is $5,000 and the prize is awarded every 2 years.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/satter-prize`.

Joint Prizes and Awards

Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student

(AMS-MAA-SIAM)

The Morgan Prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

About this Prize

The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is $1,200, awarded annually.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: To nominate a student, submit a letter of nomination, a brief description of the work that is the basis of the nomination, and complete bibliographic citations (or copies of unpublished work). All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research.

Information on how to nominate can be found here: `https://www.ams.org/morgan-prize`.

JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

About this Award

This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.

Next Prize: January 2025

Nomination Period: open

Nomination Procedure: Nominations should be submitted on mathprograms.org. Note: Nominations collected before September 15th in year N will be considered for an award in year N+2.

Information on how to nominate can be found here: `https://www.ams.org/jpbm-comm-award`.

AMS-SIAM Norbert Wiener Prize in Applied Mathematics

The Wiener Prize is awarded for an outstanding contribution to "applied mathematics in the highest and broadest sense."

About this Prize

This prize was established in 1967 in honor of Professor Norbert Wiener and was endowed by a fund from the Department of Mathematics of the Massachusetts Institute of Technology. The endowment was further supplemented by a generous donor.

Since 2004, the US$5,000 prize has been awarded every three years. The American Mathematical Society and the Society for Industrial and Applied Mathematics award this prize jointly; the recipient must be a member of one of these societies.

Next Prize: January 2025

Nomination Period: February 1–May 31

Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: `https://www.ams.org/wiener-prize`.

AMS Programs and Fellowships

AMS-Simons Travel Grants

The AMS-Simons Travel Grant program acknowledges the importance of research interaction and collaboration in mathematics and aims to facilitate these activities for recent PhD recipients. AMS-Simons Travel Grants are administered by the AMS with support from the Simons Foundation. These grants provide support for committed researchers who have limited opportunities for travel and conferences and for collaborative work. For the 2024–2025 award cycle, each grant will provide an early-career mathematician with $3,000 per year for two years to be used for research-related travel. Annual discretionary funds for the enhancement of a grantee's department will be available to institutions that administer the grant on behalf of the AMS. No additional institutional overhead or indirect costs will be covered with these award funds.

About this Grant

Eligible applicants for the 2024–2025 application cycle are early-career mathematicians who are located in the United States (or are US citizens employed outside the United States) and who have completed the PhD (or its equivalent) within the last four years (between April 1, 2020, and June 30, 2024, inclusive).

The applicant's research must be in a disciplinary research area supported by the Division of Mathematical Sciences at the National Science Foundation. Previous AMS-Simons Travel Grant recipients and early-career mathematicians who already receive substantial external funding for research and travel exceeding $3,000 per year (such as from the National Science Foundation) are not eligible to apply.

Recipients may use grant funds for research-related travel, such as travel to a conference, a university, or an institute, or to visit a collaborator. Funds may also be used for a collaborator to visit the grantee to engage in research activities. Other research-related travel may be supported, subject to the approval of the grantee's mentor. Detailed guidelines will be provided to the grantee. Only eligible travel expenses that have advance approval from the grantee's mentor will be reimbursed.

Application Period: Applications will be collected via MathPrograms.org February 15, 2024–March 31, 2024 (11:59 p.m. EST). Find more application information at `https://www.ams.org/AMS-SimonsTG`. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; `ams-simons@ams.org`.

AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty

With generous funding from the Simons Foundation; the AMS; and Eve, Kirsten, Lenore, and Ada of the Menger family, the AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty program was established in 2023 to foster and support research collaboration by mathematicians employed full-time at colleges and universities that do not award doctoral degrees in mathematics. Each year for three years, grantees will receive $3,000 to support research-related activities. Annual discretionary funds for a grantee's department and administrative funds for a grantee's institution will be available to institutions that administer the grant on behalf of the AMS. No additional institutional overhead or indirect costs will be covered with these award funds.

About this Grant

Mathematicians with an active research program employed full-time in tenured or tenure-track positions at PUIs in the United States are eligible to apply. For the purpose of this program, PUI institutions are those that do not confer doctoral degrees in mathematics. Additionally, to be eligible, applicants must have earned a PhD degree at least five years before the start of the grant. For the 2024 application cycle, applicants must have earned a PhD degree prior to August 1, 2019.

The applicant's research must be in a disciplinary research area supported by the Division of Mathematical Sciences of the National Science Foundation. Faculty with appointments solely in statistics departments are not eligible. The grantees may not concurrently hold external research funding exceeding $3,000 per year and may not be in residence at a National Science Foundation institute.

Activities that will further the grantee's research program are allowed. These expenses include but are not limited to conference participation, institute visits, collaboration travel (grantee or collaborator), computer equipment or software, family-care expenses, hiring a teaching assistant, publication expenses, stationery, supplies, books, and membership fees to professional organizations. During the three-year funding period, the grantee may spend up to $2,500 on electronic devices to support their research activities.

Application Period: Applications will be collected via MathPrograms.org January 9, 2024–March 18, 2024 (11:59 p.m. EST). Find more application information at `https://www.ams.org/AMS-Simons-PUI-Research`. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; `ams-simons-pui@ams.org`.

Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of 12 members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at `www.ams.org/profession/ams-fellows`.

Credits

Photo of I. Martin Isaacs is courtesy of Yvonne Nagel.
Photo of Elias M. Stein is courtesy of William Crow/Princeton University.

AMS Updates

Apply by March 18 for AMS-Simons Research Enhancement Grants for PUI Faculty

Apply for the 2024 AMS-Simons Research Enhancement Grants for Primarily Undergraduate Institution (PUI) Faculty program by 11:59 p.m. ET on Monday, March 18, 2024. Applicants will be notified of their status in June 2024, and grants will begin on July 1, 2024. These research grants provide $3,000 a year for three years.

For in-depth information about the program, download the one-hour webinar for prospective applicants from November 9, 2023 [URL below]. In this webinar, Sarah Bryant, director of AMS Programs, describes the program and its eligibility requirements and shares advice on strengthening your application. Bryant and the 2023 committee cochairs, Francis Su and Deborah Lockhart, also field questions from the real-time webinar attendees.

The presentation is available for download at `https://www.ams.org/ams-simons-pui-research`. This page also contains more information about activities supported by the grant. Questions? Email `ams-simons-pui@ams.org`.

—AMS Programs Department

An Industry Perspective: Alan Lee of Analog Devices, Inc., Briefs Capitol Hill

The American Mathematical Society (AMS) and the Institute for Pure & Applied Mathematics (IPAM) jointly hosted a Congressional briefing titled "Math Changes Everything—The Importance of Mathematics to the US: An Industry Perspective" on December 6, 2023, on Capitol Hill in Washington, DC. The briefing was given by Alan Lee, chief technology officer of Analog Devices, Inc.

The AMS holds annual Congressional briefings as a means to communicate to policymakers the pivotal roles of the mathematical sciences in American life. Speakers bring science directly to Capitol Hill decisionmakers. They explain how federal investment in basic research in math and science pays off for American taxpayers and helps the nation remain a world leader in innovation. Each briefing showcases work connected to one of the National Science Foundation-funded Mathematical Sciences Institutes; this year featured IPAM.

Beginning with the personal story of his start in math, Lee delivered a broad survey of how mathematics is fundamental in a very wide variety of applications. He introduced the audience to the work of mathematicians—Julia Robinson; Ronald Rivest, Adi Shamir, and Leonard Adleman; Terence Tao and Emmanuel Candès; Karen Uhlenbeck—and how their mathematics have advanced technologies and improved our understanding of the universe.

Lee then connected specific mathematics to topics of current concern to Congress, such as artificial intelligence; financial systems; networks, cybersecurity, and the modern defense ecosystem; human health; the health of our planet; and manufacturing, transport, and infrastructure. Noting that other countries are investing more in mathematics education, he gave direction for opportunities for the nation to grow domestic talent and simultaneously attract top talent from abroad.

In a compelling argument for strategic federal investment in mathematics research, Lee highlighted the role of the NSF-funded Mathematical Sciences Research Institutes in advancing this research and supporting the math community at large. It was a beautiful presentation; the audience was very attentive and Congressional staff asked great questions.

Lee's slides, as well as further information about AMS Congressional briefings and previous speakers, will be posted at `https://www.ams.org/government/dc-outreach`.

—Karen Saxe, Associate Executive Director, AMS

DOI: https://doi.org/10.1090/noti2904

Fifty Graduate Programs Participate in 2023 Online Fall Graduate School Fair

Fifty US math graduate programs, as well as Math Alliance, the EDGE Program, and the AMS Programs and Membership departments, exhibited at the annual AMS Online Fall Graduate School Fair, which was hosted on the Gather platform on October 17, 2023.

At the fair, representatives of graduate programs in the mathematical sciences staffed virtual tables and answered questions about their programs. Undergraduates learned about possibilities for graduate study and master's students explored opportunities for PhD study.

Register in late spring 2024 for the fall 2024 online fair. For more information, visit `https://www.ams.org/gradfair` or email the AMS Programs staff at `prof-serv@ams.org`.

—AMS Programs Department

Introduction to Advocacy: AMS Welcomes Virginia High-School Students

Thirty students from John R. Lewis High School in Springfield, Virginia, visited the AMS Office of Government Relations in Washington, DC, in October 2023 as part of the school's Lewis Leadership Program.

After exploring the National Mall using the Mathematical Association of America's *Field Guide to Math*, the students learned about AMS policy and advocacy work; shared their own experiences and perspectives about mathematics education; and explored the role of mathematics as a tool for positive social change.

The group was hosted by Karen Saxe, director of the AMS Office of Government Relations, and Tyler Kloefkorn, associate director. Saxe and Kloefkorn discussed the value of mathematics and its importance to future progress in science, engineering, and technology innovation, including artificial intelligence. They also taught the young leaders how to influence the legislative process through advocacy. As an example, the students were encouraged to use AMS resources to support legislation for a Congressional Gold Medal for Bob Moses, the late founder of the Algebra Project. Moses was a public education advocate, math literacy educator, and like their high school's namesake, an influential civil-rights activist.

In its second year, the Lewis Leadership Program provides Lewis High School's 1,690 students with real-world, hands-on learning about leadership, justice, service, and advocacy inside and outside the classroom. The visit to Washington was part of a broader series of Leadership Program events designed to foster joy and curiosity in mathematics among students who experience systemic barriers to enrichment. Students in grades 9–12 engaged in collaborative learning in Spanish and English with AMS leaders, Fairfax County Public Schools staff from Lewis High School and the Office of Curriculum and Instruction, and community partners from TODOS: Mathematics for ALL.

—AMS Office of Government Relations

Deaths of AMS Members

Lawrence A. Zalcman, of Israel, died on May 31, 2022. Born on June 9, 1943, he was a member of the Society for 55 years.

Asvald Lima, of Norway, died on April 11, 2023. Born on October 1, 1942, he was a member of the Society for 46 years.

Joseph A. Wolf, of Berkeley, California, died on August 14, 2023. Born on October 18, 1936, he was a member of the Society for 66 years.

Eric A. Nordgren, of Durham, New Hampshire, died on August 16, 2023. Born on March 31, 1933, he was a member of the Society for 60 years.

Joseph B. Roberts, of Portland, Oregon, died on August 31, 2023. Born on September 9, 1923, he was a member of the Society for 67 years.

Walker E. Hunt, of San Antonio, Texas, died on September 1, 2023. Born on June 7, 1937, he was a member of the Society for 56 years.

Eugenio Calabi, of Bryn Mawr, Pennsylvania, died on September 5, 2023. Born on May 11, 1923, he was a member of the Society for 73 years.

Margaret W. Taft, of Arlington, Massachusetts, died on October 14, 2023. Born on November 4, 1945, she was a member of the Society for 52 years.

Lawrence Hueston Harper, of Riverside, California, died on October 20, 2023. Born on August 1, 1938, he was a member of the Society for 2 years.

Gary M. Seitz, of Seattle, Washington, died on October 30, 2023. Born on May 10, 1943, he was a member of the Society for 57 years.

Gerald T. Cargo, of Windsor, Colorado, died on November 10, 2023. Born on March 1, 1930, he was a member of the Society for 66 years.

Mathematics People

Hickok, Kooloth Win 2024 AWM Dissertation Prize

Abigail Hickok and Parvathi M. Kooloth have been awarded the 2024 Association for Women in Mathematics (AWM) Dissertation Prize, recognizing exceptional work in a dissertation defended in the last 24 months. The award is intended to be based entirely on the dissertation itself, not on other work of the individual. The prizes were presented at the Joint Mathematics Meetings in San Francisco, CA.

Hickok received her PhD in 2023 at UCLA under the supervision of Mason Porter. She is currently an NSF Postdoctoral Fellow at Columbia University. Hickok's dissertation consists of work from six papers and a book chapter in the area of topological and geometric data analysis.

Kooloth received her PhD in Mathematics in 2022 at the University of Wisconsin-Madison under the direction of Leslie M. Smith. She is currently a postdoctoral research associate at the Pacific Northwest National Laboratory. In her thesis, "Moist potential vorticity and coherent structures in the atmosphere," Kooloth solved a longstanding puzzle in the fluid dynamics of the atmosphere and the ocean.

—Association for Women in Mathematics

Women and Mathematics Program Awarded 2023 AWM Presidential Recognition Award

The Association for Women in Mathematics (AWM) presented its 2023 Presidential Recognition Award to the Women and Mathematics Program (WAM) at the Institute for Advanced Study (IAS) for its 30-year record of celebrating excellent mathematics and for fostering exceptional community citizenship.

Founded by Karen Uhlenbeck and Chuu-Lian Terng in 1993 as part of the Park City Mathematics Institute, WAM was established at IAS in 1994. To date, the program has welcomed more than 1,650 participants, resulting in a powerful network. WAM's core curriculum includes an intensive multiday workshop on the IAS campus that features lectures, seminars, and working sessions on a selected topic.

—Association for Women in Mathematics

Pacific Rim Conference on Mathematics to be Held June 2024

The Ninth Pacific Rim Conference on Mathematics (PRCM) will be held June 17–21, 2024 at the Darwin Convention Centre, in Darwin (Northern Territory, Australia), hosted by the Mathematical Sciences Institute (MSI), Australian National University (ANU).

"The PRCM is a broad mathematical event held every few years that covers a wide range of exciting research in contemporary mathematics," according to its organizers. "Its objectives are to offer a venue for the presentation to and discussion among a wide audience of the latest trends in mathematical research, and to strengthen ties between mathematicians working in the Pacific Rim region."

Scheduled plenary speakers are Binyong Sun (Zhejiang University), Jun-cheng Wei (University of British Columbia), Gabor Lugosi (Pompeu Fabra University Barcelona), Allan Sly (Princeton University), Mohammed Abouzaid (Stanford University), Kay Owens (Charles Sturt University), Minhyong Kim (ICMS Edinburgh and Korea Institute for Advanced Study), Alexandre Ern (University of Paris-Est), and Gerhard Huisken (Oberwolfach Research Institute for Mathematics).

For more information, see `https://maths.anu.edu.au/news-events/events/ninth-pacific-rim-conference-mathematics-darwin`.

—Pacific Rim Conference on Mathematics

DOI: https://doi.org/10.1090/noti2903

JMAA Announces 2022 Ames, Wong Awards

The editorial board of the *Journal of Mathematical Analysis and Applications* (JMAA) has selected and announced the winners of the 2022 Ames Awards and James S. W. Wong Prize.

Awarded annually, the JMAA Ames Awards honor the memory of William F. Ames, former editor-in-chief of the JMAA. The papers, in pure and applied mathematics, were selected by a journal selection committee from all papers published in the journal in the last three years. Each award consists of a certificate of merit and a monetary prize of $2,500.

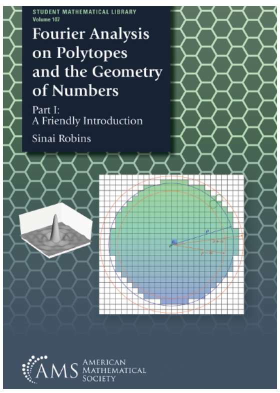
Winners of the 2022 Ames Awards are Paolo Marcellini, for their paper "Growth conditions and regularity for weak solutions to nonlinear elliptic PDEs" *J. Math. Anal. Appl.* 501 (2021), no. 1, 124408, and Davide Barbieri, Eugenio Hernández, and Victoria Paternostro, for their paper "Spaces invariant under unitary representations of discrete groups" *J. Math. Anal. Appl.* 492 (2020), no. 1, 124357.

A biennial award, the Wong Prize recognizes an outstanding paper published in the JMAA in the preceding ten years and consists of a cash award of $10,000. The Wong Prize fund was established in 2012 by the family of Dr. James S. W. Wong and Elsevier in honor of Dr. Wong's contributions to mathematics, career accomplishments, and editorial service to the JMAA. Winners of the 2022 Wong Prize are Angelo Alvino, Friedemann Brock, Francesco Chiacchio, Anna Mercaldo, and Maria Rosaria Posteraro, for their paper "Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^\alpha$" *J. Math. Anal. Appl.* 451 (2017), no. 1, 280–318.

—Journal of Mathematical Analysis and Applications

New Books Offered by the AMS

Analysis



Fourier Analysis on Polytopes and the Geometry of Numbers

Part I: A Friendly Introduction

Sinai Robins, *University of São Paulo, Brazil*

This book offers a gentle introduction to the geometry of numbers from a modern Fourier-analytic point of view. One of the main themes is the transfer of geometric knowledge of a polytope to analytic knowledge of its Fourier transform. The Fourier transform preserves all of the information of a polytope, and turns its geometry into analysis. The approach is unique, and streamlines this emerging field by presenting new simple proofs of some basic results of the field. In addition, each chapter is fitted with many exercises, some of which have solutions and hints in an appendix. Thus, an individual learner will have an easier time absorbing the material on their own, or as part of a class.

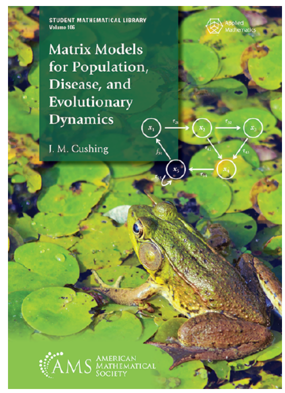
Overall, this book provides an introduction appropriate for an advanced undergraduate, a beginning graduate student, or researcher interested in exploring this important expanding field.

This item will also be of interest to those working in discrete mathematics and combinatorics.

Student Mathematical Library, Volume 107
May 2024, approximately 349 pages, Softcover, ISBN: 978-1-4704-7033-3, 2020 *Mathematics Subject Classification*: 51M20, 11P21, 32A50, 11H06, 11H16, 11H31, 52B20, List US$59, **AMS Institutional member US$47.20**, **MAA members US$47.20**, **All Individuals US$47.20**, Order code STML/107

`bookstore.ams.org/stml-107`

Applications



Matrix Models for Population, Disease, and Evolutionary Dynamics

J. M. Cushing, *University of Arizona, Tucson, AZ*

This book offers an introduction to the use of matrix theory and linear algebra in modeling the dynamics of biological populations. Matrix algebra has been used in population biology since the 1940s and continues to play a major role in theoretical and applied dynamics for populations structured by age, body size or weight, disease states, physiological and behavioral characteristics, life cycle stages, or any of many other possible classification schemes. With a focus on matrix models, the book requires only first courses in multivariable calculus and matrix theory or linear algebra as prerequisites.

The reader will learn the basics of modeling methodology (i.e., how to set up a matrix model from biological underpinnings) and the fundamentals of the analysis of discrete time dynamical systems (equilibria, stability, bifurcations, etc.). A recurrent theme in all chapters concerns the problem of extinction versus survival of a population. In addition to numerous examples that illustrate these fundamentals, several applications appear at the end of each chapter that illustrate the full cycle of model setup, mathematical analysis, and interpretation. The author has used the material over many decades in a variety of teaching and mentoring settings, including special topics courses and seminars in mathematical modeling, mathematical biology, and dynamical systems.

This item will also be of interest to those working in differential equations.

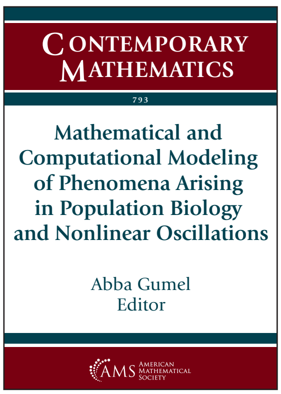
Student Mathematical Library, Volume 106
May 2024, approximately 305 pages, Softcover, ISBN: 978-1-4704-7334-1, LC 2023050951, 2020 *Mathematics Subject*

Classification: 92D25, 92D30, 39A50; 37N25, 15B99, List US$59, **AMS Institutional member US$47.20**, **MAA members US$47.20**, **All Individuals US$47.20**, Order code STML/106

bookstore.ams.org/stml-106

New in Contemporary Mathematics

Applications



Mathematical and Computational Modeling of Phenomena Arising in Population Biology and Nonlinear Oscillations

Abba Gumel, *University of Maryland, College Park, MD*, Editor

This book consists of a series of papers focusing on the mathematical and computational modeling and analysis of some real-life phenomena in the natural and engineering sciences. The book emphasizes three main themes: (i) the design and analysis of robust and dynamically-consistent nonstandard finite-difference methods for discretizing continuous-time dynamical systems arising in the natural and engineering sciences, (ii) the mathematical study of nonlinear oscillations, and (iii) the design and analysis of models for the spread and control of emerging and re-emerging infectious diseases.

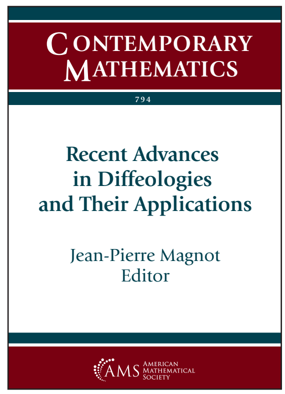
Specifically, some of the topics covered in the book include advances and challenges on the design, analysis and implementation of nonstandard finite-difference methods for approximating the solutions of continuous-time dynamical systems, the design and analysis of models for the spread and control of the COVID-19 pandemic, modeling the effect of prescribed fire and temperature on the dynamics of tick-borne disease, and the design of a novel genetic-epidemiology framework for malaria transmission dynamics and control.

The book also covers the impact of environmental factors on diseases and microbial populations, Monod kinetics in a chemostat setting, structure and evolution of poroacoustic solitary waves, mathematics of special (periodic) functions and the numerical discretization of a phase-lagging equation with heat source.

Contemporary Mathematics, Volume 793
April 2024, approximately 356 pages, Softcover, ISBN: 978-1-4704-7104-0, LC 2023045277, 2020 *Mathematics Subject Classification*: 34C15, 34D05, 34D20, 34D23, 37M20, 39A28, 39A30, 92B05, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code CONM/793

bookstore.ams.org/conm-793

Geometry and Topology



Recent Advances in Diffeologies and Their Applications

Jean-Pierre Magnot, *Université d'Angers, France, and Lycée Jeanne d'Arc, Clermont-Ferrand, France*, Editor

This volume contains the proceedings of the AMS-EMS-SMF Special Session on Recent Advances on Diffeologies and Their Applications, held from July 18–20, 2022, at the Université de Grenoble-Alpes, Grenoble, France.

The articles present some developments of the theory of diffeologies applied in a broad range of topics, ranging from algebraic topology and higher homotopy theory to integrable systems and optimization in PDE.

The geometric framework proposed by diffeologies is known to be one of the most general approaches to problems arising in several areas of mathematics. It can adapt to many contexts without major technical difficulties and produce examples inaccessible by other means, in particular when studying singularities or geometry in infinite dimension. Thanks to this adaptability, diffeologies appear to have become an interesting and useful language for a growing number of mathematicians working in many different fields. Some articles in the volume also illustrate some recent developments of the theory, which makes it even more deep and useful.

Contemporary Mathematics, Volume 794
April 2024, approximately 264 pages, Softcover, ISBN: 978-1-4704-7254-2, LC 2023049999, 2020 *Mathematics Subject Classification*: 14D23, 14M25, 37K10, 46T05, 49Q10, 53D20, 57R30, 57R55, 58B10, 81R50, List US$135, **AMS members US$108**, **MAA members US$121.50**, Order code CONM/794

bookstore.ams.org/conm-794

New in Memoirs of the AMS

Algebra and Algebraic Geometry

The Generation Problem in Thompson Group F

Gili Golan Polak, *Ben Gurion University of the Negev, Be'er Sheva, Israel*

Memoirs of the American Mathematical Society, Volume 292, Number 1451
January 2024, 94 pages, Softcover, ISBN: 978-1-4704-6723-4, 2020 *Mathematics Subject Classification*: 20F10, 20F65; 20E28, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1451

bookstore.ams.org/memo-292-1451

Finite Groups Which Are Almost Groups of Lie Type in Characteristic p

Chris Parker, *University of Birmingham, United Kingdom*, **Gerald Pientka**, *Halle, Germany*, **Andreas Seidel**, *Magdeburg, Germany*, and **Gernot Stroth**, *Universität Halle–Wittenberg, Germany*

Memoirs of the American Mathematical Society, Volume 292, Number 1452
January 2024, 182 pages, Softcover, ISBN: 978-1-4704-6729-6, 2020 *Mathematics Subject Classification*: 20D05; 20D06, 20D08, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1452

bookstore.ams.org/memo-292-1452

Hyperbolic Actions and 2nd Bounded Cohomology of Subgroups of $\mathrm{Out}(F_n)$

Michael Handel, *CUNY Lehman College, New York, NY*, and **Lee Mosher**, *Rutgers University–Newark, NJ*

Memoirs of the American Mathematical Society, Volume 292, Number 1454
January 2024, 170 pages, Softcover, ISBN: 978-1-4704-6698-5, 2020 *Mathematics Subject Classification*: 20F65; 20E05, 20E36, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1454

bookstore.ams.org/memo-292-1454

Analysis

Global Regularity for Gravity Unstable Muskat Bubbles

Francisco Gancedo, *Universidad de Sevilla, Spain*, **Eduardo García-Juárez**, *Universidad de Sevilla, Spain*, **Neel Patel**, *University of Maine, Orono, ME*, and **Robert M. Strain**, *University of Pennsylvania, Philadelphia, PA*

Memoirs of the American Mathematical Society, Volume 292, Number 1455
January 2024, 87 pages, Softcover, ISBN: 978-1-4704-6764-7, 2020 *Mathematics Subject Classification*: 35A01, 35D30, 35D35, 35Q35, 35Q86, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1455

bookstore.ams.org/memo-292-1455

Geometry and Topology

SYZ Geometry for Calabi-Yau 3-folds: Taub-NUT and Ooguri-Vafa Type Metrics

Yang Li, *Massachusetts Institute of Technology, Cambridge, MA*

This item will also be of interest to those working in analysis.

Memoirs of the American Mathematical Society, Volume 292, Number 1453
January 2024, 126 pages, Softcover, ISBN: 978-1-4704-6782-1, 2020 *Mathematics Subject Classification*: 53C25, 53C21, 32Q20, 32Q25; 14J30, 14J32, 14J33, 97I80, 53C38, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1453

bookstore.ams.org/memo-292-1453

Eulerian Spaces

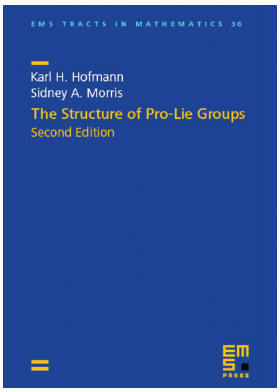
Paul Gartside, *University of Pittsburgh, PA*, and **Max Pitz**, *Universität Hamburg, Germany*

Memoirs of the American Mathematical Society, Volume 292, Number 1456
January 2024, 86 pages, Softcover, ISBN: 978-1-4704-6784-5, 2020 *Mathematics Subject Classification*: 54F15, 54C10; 05C45, 05C63, 57M15, 54F50, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code MEMO/292/1456

bookstore.ams.org/memo-292-1456

New AMS-Distributed Publications

Algebra and Algebraic Geometry



The Structure of Pro-Lie Groups

Karl H. Hofmann, *Technische Universität, Darmstadt, Germany, and Tulane University, New Orleans, LA*, and **Sidney A. Morris**, *La Trobe University, Bundoora, Australia, and Federation University Australia, Ballarat, Australia*

Lie groups were introduced in 1870 by the Norwegian mathematician Sophus Lie. A century later Jean Dieudonn quipped that Lie groups had moved to the center of mathematics and that one cannot undertake anything without them.

A *pro-Lie* group is a complete topological group G in which every identity neighborhood U of G contains a normal subgroup N such that the quotient G/N is a Lie group. Every locally compact connected topological group and every compact group is a pro-Lie group. While the class of locally compact groups is not closed under the formation of arbitrary products, the class of pro-Lie groups is.

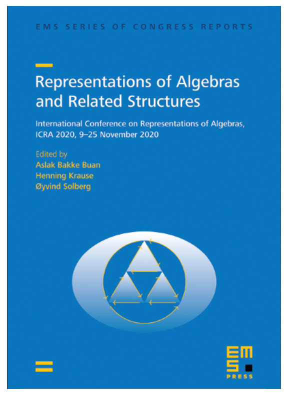
For half a century, locally compact pro-Lie groups have drifted through the literature; yet this is the first book which systematically treats the Lie theory and the structure theory of pro-Lie groups irrespective of local compactness. So it fits very well into that current trend which addresses infinite dimensional Lie groups. The results of this text are based on a theory of pro-Lie algebras which parallels the structure theory of finite dimensional real Lie algebras to an astonishing degree even though it has to overcome technical obstacles.

A topological group is said to be almost connected if the quotient group of its connected components is compact. This book exposes a Lie theory of almost connected pro-Lie groups (and hence of almost connected locally compact groups) and illuminates the variety of ways in which their structure theory reduces to that of compact groups on the one hand and of finite dimensional Lie groups on the other. It is, therefore, a continuation of the authors' monograph on the structure of compact groups (1998, 2006, 2014, 2020, 2023) and is an invaluable tool for researchers in topological groups, Lie theory, harmonic analysis and representation theory. It is written to be accessible to advanced graduate students wishing to study this fascinating and important area of research, which has so many fruitful interactions with other fields of mathematics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Tracts in Mathematics, Volume 36
November 2023, 840 pages, Hardcover, ISBN: 978-3-98547-048-8, 2020 *Mathematics Subject Classification*: 22–02; 22A05, 22D05, 22E20, 22E65, List US$129, **AMS members US$103.20**, Order code EMSTM/36

`bookstore.ams.org/emstm-36`



Representations of Algebras and Related Structures

International Conference on Representations of Algebras, ICRA 2020, November 9–25, 2020

Aslak Bakke Buan, *Norwegian University of Science and Technology, Trondheim, Norway*, **Henning Krause**, *Universität Bielefeld, Bielefeld, Germany*, and **Øyvind Solberg**, *Norwegian University of Science and Technology, Trondheim, Norway*, Editors

This volume contains a collection of articles devoted to representations of algebras and related topics. Distinguished experts in this field presented their work at the International Conference on Representations of Algebras in 2020.

The book reflects recent trends in the representation theory of algebras and its interactions with other central branches of mathematics, including combinatorics, commutative algebra, algebraic geometry, topology, data analysis, Lie algebras, quantum groups, homological algebra, and theoretical physics.

There are thirteen independent articles, written by leading experts in the field. Most are expository survey papers, but some are also original research contributions. This collection is addressed to researchers and graduate students in algebra as well as to a broader mathematical audience. It contains open problems and new perspectives for research in the field.

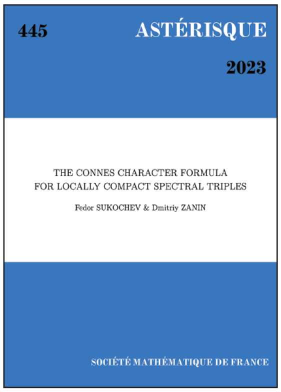
A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Series of Congress Reports, Volume 19
December 2023, 428 pages, Hardcover, ISBN: 978-3-98547-054-9, 2020 *Mathematics Subject Classification*: 16–06;

18–06, 13–06, 14–06, List US$95, **AMS members US$76**, Order code EMSSCR/19

bookstore.ams.org/emsscr-19

Differential Equations



The Connes Character Formula for Locally Compact Spectral Triples

Fedor Sukochev, *University of New South Wales, Kensington, Australia*, and **Dmitriy Zanin**, *University of New South Wales, Kensington, Australia*

A fundamental tool in noncommutative geometry is Connes's character formula. This formula is used in an essential way in the applications of noncommutative geometry to index theory and to the spectral characterization of manifolds.

A non-compact space is modeled in noncommutative geometry by a non-unital spectral triple. The authors' aim is to establish Connes's character formula for non-unital spectral triples. This is significantly more difficult than in the unital case, and they achieve it with the use of recently developed double operator integration techniques. Previously, only partial extensions of Connes's character formula to the non-unital case were known.

In the course of the proof, the authors establish two more results of importance in noncommutative geometry: an asymptotic for the heat semigroup of a non-unital spectral triple and the analyticity of the associated ζ-function.

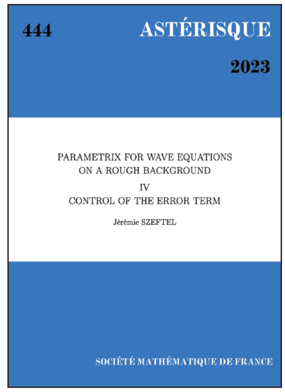
The authors require certain assumptions on the underlying spectral triple and verify these assumptions in the case of spectral triples associated to arbitrary complete Riemannian manifolds and also in the case of Moyal planes.

This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 445
December 2023, 150 pages, Softcover, ISBN: 978-2-85629-982-1, 2020 *Mathematics Subject Classification*: 83C05, 35Q75, 58J45, 35S30, 58J40, List US$65, **AMS members US$52**, Order code AST/445

bookstore.ams.org/ast-445



Parametrix for Wave Equations on a Rough Background IV: Control of the Error Term

Jérémie Szeftel, *CNRS & Laboratoire Jacques-Louis Lions, Sorbonne Université, France*

This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi=0$, where $\mathbf{g}$ is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$ is a major step of the proof of the bounded L^2 curvature conjecture, solved jointly with S. Klainerman and I. Rodnianski.

On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

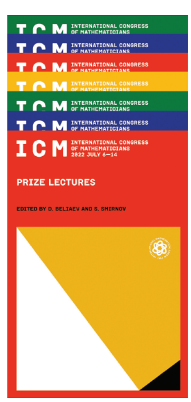
This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 444
November 2023, 314 pages, Softcover, ISBN: 978-2-85629-978-4, 2020 *Mathematics Subject Classification*: 83C05, 35Q75, 58J45, 35S30, 58J40, List US$90, **AMS members US$72**, Order code AST/444

bookstore.ams.org/ast-444

General Interest



International Congress of Mathematicians: July 6–14, 2022

Volumes 1–7 (Set)

Dmitry Beliaev, *University of Oxford, UK*, and **Stanislav Smirnov**, *Université de Genève, Switzerland*, Editors

Following the long and illustrious tradition of the International Congress of Mathematicians, these proceedings include contributions based on the invited talks that were presented at the Congress in 2022.

Published with the support of the International Mathematical Union and edited by Dmitry Beliaev and Stanislav Smirnov, these seven volumes present the most important developments in all fields of mathematics and its applications in the past four years. In particular, they include laudations and presentations of the 2022 Fields Medal winners and of the other prestigious prizes awarded at the Congress.

The proceedings of the International Congress of Mathematicians provide an authoritative documentation of contemporary research in all branches of mathematics and are an indispensable part of every mathematical library.

December 2023, 5900 pages, Hardcover, ISBN: 978-3-98547-058-7, 2020 *Mathematics Subject Classification*: 00Bxx, List US$999, **AMS Individual member US$899.10**, **AMS Institutional member US$799.20**, Order code EMSICM/2022

bookstore.ams.org/emsicm-2022

Meetings & Conferences of the AMS
March Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: `stovall@math.wisc.edu`; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: `mamanes@hawaii.edu`; telephone: (808) 956-4679.

Meetings in this Issue

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at `https://www.ams.org/welcoming-environment-policy`.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Tallahassee, Florida

Florida State University

March 23–24, 2024

Saturday – Sunday

Meeting #1193

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: To be announced
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Wenjing Liao, Georgia Institute of Technology, *Exploiting low-dimensional data structures in deep learning*.
Olivia Prosper, University of Tennessee, Knoxville, *Modeling Malaria at Multiple Scales*.
Jared Speck, Vanderbilt University, *Singularity Formation for the Equations of Einstein and Euler*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Numerical Methods for Partial Differential Equations and Their Applications, **Seonghee Jeong**, Louisiana State University, **Sanghyun Lee**, Florida State University, and **Seulip Lee**, University of Georgia.

Advances in Financial Mathematics, **Qi Feng**, University of Southern California, and **Alec N Kercheval** and **Lingjiong Zhu**, Florida State University.

Advances in Shape and Topological Data Analysis, **Emmanuel L Hartman**, **Eric Klassen**, and **Ethan Semrad**, Florida State University.

Algebraic Groups and Local-Global Principles, **Suresh Venapally**, Emory University, and **Daniel Reuben Krashen**, University of Pennsylvania.

Bases and Frames in Hilbert Spaces, **Laura De Carli**, Florida International University, and **Azita Mayeli**, City University of New York.

Combinatorics in Geometry of Polynomials, **Papri Dey**, Georgia Institute of Technology.

Control, Inverse Problems and Long Time Dynamics of Evolution Systems, **Shitao Liu**, Clemson University, and **Louis Roder Tcheugoue Tebou**, Florida International University.

Data Integration and Identifiability in Ecological and Epidemiological Models, **Omar Saucedo**, Virginia Tech, and **Olivia Prosper**, University of Tennessee/Knoxville.

Diversity in Mathematical Biology, **Daniel Alejandro Cruz** and **Skylar Grey**, University of Florida.

Fluids: Analysis, Applications, and Beyond, **Aseel Farhat**, Florida State University, and **Anuj Kumar**, Indiana University Bloomington.

Geometric Measure Theory and Partial Differential Equations, **Alexander B. Reznikov**, **John Hoffman**, and **Richard Oberlin**, Florida State University.

Geometry and Symmetry in Data Science, **Dustin G. Mixon**, The Ohio State University, and **Thomas Needham**, Florida State University.

Homotopy Theory and Category Theory in Interaction, **Ettore Aldrovandi** and **Brandon Doherty**, Florida State University, and **Philip John Hackney**, University of Louisiana at Lafayette.

Mathematical Advances in Scientific Machine Learning, **Wenjing Liao**, Georgia Institute of Technology, and **Feng Bao** and **Zecheng Zhang**, Florida State University.

Mathematical Modeling and Simulation in Fluid Dynamics, **Pejman Sanaei**, Georgia State University.

Mathematical Models for Population and Methods for Parameter Estimation in Epidemiology, **Yang LI**, Georgia State University, and **Guihong Fan**, Columbus State University.

Moduli Spaces in Algebraic Geometry, **Jeremy Usatine**, Florida State University, **Hulya Arguz** and **Pierrick Bousseau**, University of Georgia, and **Matthew Satriano**, University of Waterloo.

Nonlinear Evolution Partial Differential Equations in Physics and Geometry, **Jared Speck** and **Leonardo Abbrescia**, Vanderbilt University.

Numerical Methods and Deep Learning for PDEs, **Chunmei Wang**, University of Florida, and **Haizhao Yang**, University of Maryland College Park.

PDEs in Incompressible Fluid Mechanics, **Wojciech S. Ozanski**, Florida State University, **Stanley Palasek**, UCLA, and **Alexis F Vasseur**, The University of Texas At Austin.

Recent Advances in Geometry and Topology, **Thang Nguyen**, **Samuel Aaron Ballas**, **Philip L. Bowers**, and **Sergio Fenley**, Florida State University.

Recent Advances in Inverse Problems for Partial Differential Equations and Their Applications, **Anh-Khoa Vo** and **Thuy T. Le**, UNC Charlotte.

Recent Development in Deterministic and Stochastic PDEs, **Quyuan Lin**, Clemson University, and **Xin Liu**, Texas A&M University.

Recent Developments in Numerical Methods for Evolution Partial Differential Equations, **Thi-Thao-Phuong Hoang**, **Yanzhao Cao**, and **Hans-Werner Van Wyk**, Auburn University.

Regularity Theory and Free Boundary Problems, **Lei Zhang**, University of Florida, and **Eduardo V. Teixeira**, University of Central Florida.

Stochastic Analysis and Applications, **Hakima Bessaih**, Florida International University, and **Oussama Landoulsi**, FLORIDA INTERNATIONAL UNIVERSITY.

Stochastic Differential Equations: Modeling, Estimation, and Applications, **Sher B Chhetri**, University of South Carolina Sumter, **Hongwei Long**, Florida Atlantic University, and **Olusegun M. Otunuga**, Augusta University.

Theory of Nonlinear Waves, **Nicholas James Ossi** and **Ziad H Musslimani**, Florida State University.

Topics in Graph Theory, **Songling Shan**, Auburn University, and **Guantao Chen**, Georgia State University.

Topics in Stochastic Analysis/Rough Paths/SPDE and Applications in Machine Learning, **Cheng Ouyang**, University of Illinois At Chicago, **Fabrice Baudoin**, University of Connecticut, and **Qi Feng**, University of Southern California.

Topological Algorithms for Complex Data and Biology, **Henry Hugh Adams**, **Johnathan Bush**, and **Hubert Wagner**, University of Florida.

Topological Interactions of Contact and Symplectic Manifolds, **Angela Wu**, University College of London and Louisiana State University, and **Austin Christian**, Georgia Institute of Technology.

Contributed Paper Sessions

AMS Contributed Paper Session, **Brian D. Boe**, University of Georgia.

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday

Meeting #1194

Eastern Section
Associate Secretary: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 13, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Ryan Charles Hynd, University of Pennsylvania, *Extremals of Morrey's Inequality*.

Jinyoung Park, Institute for Advanced Study, *Threshold Phenomena for Random Discrete Structures*.

Jian Song, Rutgers, State University of New Jersey, *Geometric Analysis on Singular Complex Spaces*.

Talitha M Washington, Clark Atlanta University & Atlanta University Center, *The Data Revolution* (Einstein Public Lecture in Mathematics).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Mathematical Methods in Naval Engineering Research (Code: SS 1A), **Michael Traweek**, Office of Naval Research, and **Anthony Ruffa**, Emeritus Naval Undersea Warfare Center.

Algebraic and Enumerative Combinatorics (Code: SS 2A), **Samuel Francis Hopkins**, Howard University, **Joel Brewster Lewis**, George Washington University, and **Peter R. W McNamara**, Bucknell University.

Analysis of PDE in Inverse Problems and Control Theory (Code: SS 3A), **Matthias Eller**, Georgetown University, and **Justin Thomas Webster**, University of Maryland, Baltimore County.

Artificial Intelligence Emergent From Mathematics and Physics (Code: SS 4A), **Bourama Toni**, Howard University, and **Artan Sheshmani**, MIT IAiFi.

Automorphic Forms and Langlands Program (Code: SS 5A), **Baiying Liu** and **Freydoon Shahidi**, Purdue University.

Automorphic Forms and Trace Formulae (Code: SS 6A), **Yiannis Sakellaridis**, Johns Hopkins University, **Bao Chau Ngo**, University of Chicago, and **Spencer Leslie**, Boston College.

Coding Theory & Applications (Code: SS 7A), **Emily McMillon**, **Eduardo Camps**, and **Hiram H. Lopez**, Virginia Tech.

Commutative Algebra and its Applications (Code: SS 8A), **Hugh Geller**, West Virginia University, and **Rebecca R.G.**, George Mason University.

Complex Systems in the Life Sciences (Code: SS 9A), **Zhisheng Shuai**, University of Central Florida, **Junping Shi**, College of William & Mary, and **Seoyun Choe**, University of Central Florida.

Computability, Complexity, and Algebraic Structure (Code: SS 10A), **Valentina S Harizanov**, George Washington University, **Keshav Srinivasan**, The George Washington University, and **Philip White** and **Henry Klatt**, George Washington University.

Computational and Machine Learning Methods for Modeling Biological Systems (Code: SS 11A), **Christopher Kim**, **Vipul Periwal**, **Manu Aggarwal**, and **Xiaoyu Duan**, National Institutes of Health.

Control of Partial Differential Equations (Code: SS 12A), **Gisele Adelie Mophou**, Universite des Antilles en Guadeloupe, and **Mahamadi Warma**, George Mason University.

Culturally Responsive Mathematical Education in Minority Serving Institutions (Code: SS 13A), **Lucretia Glover**, **Lifoma Salaam**, and **Julie Lang**, Howard University.

Elementary Number Theory and Elliptic Curves (Code: SS 14A), **Sankar Sitaraman** and **Francois Ramaroson**, Howard University.

Fresh Researchers in Algebra, Combinatorics, and Topology (FRACTals) (Code: SS 15A), **Dwight Anderson Williams II**, Morgan State University, and **Saber Ahmed**, Hamilton College.

GranvilleFest 100: A Celebration of the Legacy of Evelyn Boyd Granville (Code: SS 16A), **Edray Herber Goins**, Pomona College, **Torina D. Lewis**, National Association of Mathematicians, and **Talitha M Washington**, Clark Atlanta University & Atlanta University Center.

Interactions Between Analysis, Geometric Measure Theory, and Probability in Non-Smooth Spaces (Code: SS 17A), **Luca Capogna**, Smith College, **Jeremy Tyson**, University of Illinois at Urbana-Champaign, and **Nageswari Shanmugalingam**, University of Cincinnati.

Mathematical Modeling, Computation, and Data Analysis in Biological and Biomedical Applications (Code: SS 18A), **Maria G Emelianenko** and **Daniel M Anderson**, George Mason University.

Mathematical Modeling of Type 2 Diabetes and Its Clinical Studies (Code: SS 20A), **Joon Ha**, Howard University.

Mathematics of Infectious Diseases: A Session in Memory of Dr. Abdul-Aziz Yakubu (Code: SS 21A), **Abba Gumel**, University of Maryland, **Daniel Brendan Cooney**, University of Pennsylvania, and **Chadi M Saad-Roy**, University of California, Berkeley.

Modeling and Numerical Methods for Complex Dynamical Systems in Biology (Code: SS 22A), **Hye Won Kang** and **Bradford E. Peercy**, University of Maryland, Baltimore County.

Moduli Spaces in Geometry and Physics (Code: SS 23A), **Artan Sheshmani**, MIT IAiFi.

New Trends in Mathematical Physics (Code: SS 24A), **W. A. Zuniga-Galindo**, University of Texas Rio Grande Valley, and **Tristan Hubsch**, Howard University.

Nonlinear Hamiltonian PDEs (Code: SS 25A), **Benjamin Harrop-Griffiths**, Georgetown University, and **Maria Ntekoume**, Concordia University.

Optimization, Machine Learning, and Digital Twins (Code: SS 26A), **Harbir Antil**, **Rohit Khandelwal**, and **Sean Carney**, George Mason University.

Permutation Patterns (Code: SS 27A), **Juan B Gil**, Penn State Altoona, and **Alexander I. Burstein**, Howard University.

Post-Quantum Cryptography (Code: SS 28A), **Jason LeGrow**, Virginia Tech, **Veronika Kuchta**, Florida Atlantic University, **Travis Morrison**, Virginia Tech, and **Edoardo Persichetti**, Florida Atlantic University.

Qualitative Dynamics in Finite and Infinite Dynamical Systems (Code: SS 29A), **Roberto De Leo**, Howard University, and **Jim A Yorke**, University of Maryland.

Quantum Mathematics: Foundational Mathematics for Quantum Information Theory, Science and Communication (Code: SS 30A), **Tepper L. Gill**, Howard University.

Recent Advances in Harmonic Analysis and Their Applications to Partial Differential Equations (Code: SS 31A), **Guher Camliyurt** and **Jose Ramon Madrid Padilla**, Virginia Polytechnic Institute and State University.

Recent Advances in Optimal Transport and Applications (Code: SS 32A), **Henok Mawi**, Howard University (Washington, DC, US), and **Farhan Abedin**, Lafayette College.

Recent Advances on Machine Learning Methods for Forward and Inverse Problems (Code: SS 33A), **Haizhao Yang**, University of Maryland College Park, and **Ke Chen**, University of Maryland, College Park.

Recent Developments in Geometric Analysis (Code: SS 34A), **Yueh-Ju Lin**, Wichita State University, **Samuel Perez-Ayala**, Princeton University, and **Ayush Khaitan**, Rutgers University.

Recent Developments in Noncommutative Algebra and Tensor Categories (Code: SS 35A), **Kent B. Vashaw**, Massachusetts Institute of Technology, **Van C. Nguyen**, U.S. Naval Academy, **Xingting Wang**, Louisiana State University, and **Robert Won**, George Washington University.

Recent Developments in Nonlinear and Computational Dynamics (Code: SS 36A), **Emmanuel Fleurantin** and **Christopher K. R. T. Jones**, University of North Carolina.

Recent Developments in the Study of Free Boundary Problems in Fluid Mechanics (Code: SS 45A), **Huy Q. Nguyen**, University of Maryland, and **Ian Tice**, Carnegie Mellon University.

Recent Progress on Model-Based and Data-Driven Methods in Inverse Problems and Imaging (Code: SS 37A), **Yimin Zhong**, Auburn University, **Yang Yang**, Michigan State University, and **Junshan Lin**, Auburn University.

Recent Trends in Graph Theory (Code: SS 38A), **Katherine Perry**, Soka University of America, and **Adam Blumenthal**, Westminster College.

Riordan Arrays (Code: SS 39A), **Dennis Davenport** and **Lou Shapiro**, Howard University, and **Leon Woodson**, SPIRAL REU At Georgetown.

Skein Modules in Low Dimensional Topology (Code: SS 40A), **Jozef Henryk Przytycki**, George Washington University.

Spectral Theory and Quantum Systems (Code: SS 41A), **Laura Shou**, University of Maryland, and **Shiwen Zhang**, U Mass Lowell.

Stochastic Methods in Fluid Mechanics (Code: SS 42A), **Hussain Ibdah**, Univeristy of Maryland, **Theodore D. Drivas**, S, and **Kyle Liss**, Duke University.

Tensor Algebra & Networks (Code: SS 43A), **Giuseppe Cotardo**, **Gretchen Matthews**, and **Pedro Soto**, Virginia Tech.

Variational Problems with Lack of Compactness (Code: SS 44A), **Cheikh Birahim Ndiaye**, Howard University, and **Ali Maalaoui**, Clark University.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), **Steven H Weintraub**, Lehigh University.

Milwaukee, Wisconsin

University of Wisconsin-Milwaukee

April 20–21, 2024

Saturday – Sunday

Meeting #1195

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 20, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Mihaela Ifrim, University of Wisconsin-Madison, *The small data global well-posedness conjecture for 1D defocusing dispersive flows*.

Lin Lin, University of California, Berkeley, *Title To Be Announced*.

Kevin Schreve, LSU, *Homological growth of groups and aspherical manifolds*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic methods in graph theory and applications I (Code: SS 1A), **Tung T. Nguyen**, University of Chicago/ Western University, **Sunil K. Chebolu**, Illinois State University, and **Jan Minac**, Western University.

Algorithms, Number Theory, and Cryptography I (Code: SS 3A), **Jonathan P Sorenson**, Butler University, **Eric Bach**, University of Wisconsin at Madison, and **Jonathan Webster**, Butler University.

Applications of Algebra and Geometry I (Code: SS 8A), **Thomas Yahl**, University of Wisconsin - Madison, and **Jose Israel Rodriguez**, University of Wisconsin Madison.

Applications of Numerical Algebraic Geometry I (Code: SS 14A), **Emma R Cobian**, University of Notre Dame.

Artificial Intelligence in Mathematics I (Code: SS 9A), **Tony Shaska**, Oakland University, **Alessandro Arsie**, The University of Toledo, **Elira Curri**, Oakland University, Rochester Hills, MI, 48126, and **Mee Seong Im**, United States Naval Academy.

Automorphisms of Riemann Surfaces and Related Topics I (Code: SS 4A), **Aaron D. Wootton**, University of Portland, **Jennifer Paulhus**, Grinnell College, **Sean Allen Broughton**, Rose-Hulman Institute of Technology (emeritus), and **Tony Shaska**, Research Institute of Science and Technology.

Cluster algebras, Hall algebras and representation theory I (Code: SS 5A), **Xueqing Chen**, University of Wisconcin, Whitewater, and **Yiqiang Li**, SUNY At Buffalo.

Combinatorial and geometric themes in representation theory I (Code: SS 23A), **Jeb F. Willenbring**, UW-Milwaukee, and **Pamela E. Harris**, University of Wisconsin, Milwaukee.

Complex Dynamics and Related Areas I (Code: SS 16A), **James Waterman**, Stony Brook University, and **Alastair N Fletcher**, Northern Illinois University.

Computability Theory I (Code: SS 25A), **Matthew Harrison-Trainor**, University of Illinois Chicago, and **Steffen Lempp**, University of Wisconsin-Madison.

Connections between Commutative Algebra and Algebraic Combinatorics I (Code: SS 10A), **Alessandra Costantini**, Oklahoma State University, **Matthew James Weaver**, University of Notre Dame, and **Alexander T Yong**, University of Illinois at Urbana-Champaign.

Developments in hyperbolic-like geometry and dynamics I (Code: SS 11A), **Jonah Gaster**, University of Wisconsin-Milwaukee, **Andrew Zimmer**, University of Wisconsin-Madison, and **Chenxi Wu**, University of Wisconsin At Madison.

Geometric group theory I (Code: SS 28A), **G Christopher Hruska**, University of Wisconsin-Milwaukee, and **Emily Stark**, Wesleyan University.

Geometric Methods in Representation Theory I (Code: SS 2A), **Daniele Rosso**, Indiana University Northwest, and **Joshua Mundinger**, University of Wisconsin - Madison.

Harmonic Analysis and Incidence Geometry I (Code: SS 17A), **Sarah E Tammen** and **Terence L. J Harris**, UW Madison, and **Shengwen Gan**, Massachusetts Institute of Technology.

Mathematical aspects of cryptography and cybersecurity I (Code: SS 24A), **Lubjana Beshaj**, Army Cyber Institute.

Model Theory I (Code: SS 15A), **Uri Andrews**, University of Wisconsin-Madison, and **James Freitag**, University of Illinois Chicago.

New research and open problems in combinatorics I (Code: SS 12A), **Pamela Estephania Harris**, University of Wisconsin, Milwaukee, **Erik Insko**, Central College, and **Mohamed Omar**, York University.

Nonlinear waves I (Code: SS 22A), **Mihaela Ifrim**, University of Wisconsin-Madison, and **Daniel I Tataru**, UC Berkeley.

Nonstandard and Multigraded Commutative Algebra I (Code: SS 13A), **Mahrud Sayrafi**, University of Minnesota, Twin Cities, and **Maya Banks** and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Panorama of Holomorphic Dynamics I (Code: SS 21A), **Suzanne Lynch Boyd**, University of Wisconsin Milwaukee, and **Rodrigo Perez** and **Roland Roeder**, Indiana University - Purdue University Indianapolis.

Posets in algebraic and geometric combinatorics I (Code: SS 26A), **Martha Yip**, University of Kentucky, and **Rafael S. González D'León**, Loyola University Chicago.

Ramification in Algebraic and Arithmetic Geometry I (Code: SS 29A), **Charlotte Ure**, Illinois State University, and **Nick Rekuski**, Wayne State University.

Recent Advances in Nonlinear PDEs and Their Applications I (Code: SS 27A), **Xiang Wan**, Loyola University Chicago, **Rasika Mahawattege**, University of Maryland, Baltimore County, and **Madhumita Roy**, Graduate Student, University of Memphis.

Recent Advances in Numerical PDE Solvers by Deep Learning I (Code: SS 7A), **Dexuan Xie**, University of Wisconsin-Milwaukee, and **Zhen Chao**, University of Michigan-Ann Arbor.

Recent Developments in Harmonic Analysis I (Code: SS 6A), **Naga Manasa Vempati**, Louisiana State University, **Nathan A. Wagner**, Brown University, and **Bingyang Hu**, Auburn University.

Recent trends in nonlinear PDE I (Code: SS 19A), **Fernando Charro** and **Catherine Lebiedzik**, Wayne State University, and **Md Nurul Raihen**, Fontbonne University.

Stochastic Control and Related Fields: A Special Session in Honor of Professor Stockbridge's 70th Birthday I (Code: SS 18A), **Chao Zhu**, University of Wisconsin-Milwaukee, and **MoonJung Cho**, U.S. Bureau of Labor Statistics.

The Algebras and Special Functions around Association Schemes I (Code: SS 20A), **Paul M Terwilliger**, U. Wisconsin-Madison, **Sarah R Bockting-Conrad**, DePaul University, and **Jae-Ho Lee**, University of North Florida.

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday

Meeting #1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: Expired
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Julia Yael Plavnik, Indiana University, *Title to be announced*.

Mandi A. Schaeffer Fry, University of Denver, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Commutative and Noncommutative Algebra, Together at Last (Code: SS 1A), **Pablo S. Ocal**, University of California, Los Angeles, **Benjamin Briggs**, University of Copenhagen, and **Janina C Letz**, Bielefeld University.

Diagrammatic Algebras in Representation Theory and Beyond (Code: SS 2A), **Mee Seong Im**, United States Naval Academy, **Liron Speyer**, Okinawa Institute of Science and Technology, **Arik Wilbert**, University of Georgia, and **Jieru Zhu**, University of Queensland.

Extremal Combinatorics and Connections (Code: SS 3A), **Sam Spiro**, Rutgers University, and **Van Magnan**, University of Montana.

Geometry and Topology of Quantum Phases of Matter (Code: SS 4A), **Ralph Martin Kaufmann**, Purdue University, and **Markus J Pflaum**, University of Colorado.

Geometry, Integrability, Symmetry and Physics (Code: SS 5A), **Birgit Kaufmann** and **Sasha Tsymbaliuk**, Purdue University.

Groups and Representations (associated with Invited Address by Mandi Schaeffer Fry) (Code: SS 6A), **Nathaniel Thiem**, University of Colorado, **Mandi A. Schaeffer Fry**, University of Denver, and **Klaus Lux**, University of Arizona.

Homological Methods in Commutative Algebra & Algebraic Geometry (Code: SS 7A), **Ritvik Ramkumar**, Cornell University, **Michael Perlman**, University of Minnesota, and **Aleksandra C Sobieska**, University of Wisconsin - Madison.

Inverse Problems (Code: SS 8A), **Hanna E. Makaruk**, Los Alamos National Laboratory, Los Alamos, NM, and **Robert M. Owczarek**, University of New Mexico.

Mathematical Fluid Dynamics (Code: SS 9A), **Igor Kukavica** and **Juhi Jang**, University of Southern California, and **Wojciech S. Ozanski**, Florida State University.

Mathematical Modeling of Complex Ecological and Social Systems (Code: SS 10A), **Daniel Brendan Cooney**, University of Illinois at Urbana-Champaign, **Mari Kawakatsu**, University of Pennsylvania, and **Chadi M Saad-Roy**, University of California, Berkeley.

Partial Differential Equations and Convexity (Code: SS 11A), **Ben Weinkove**, Northwestern University, **Stefan Steinerberger**, University of Washington, Seattle, and **Albert Chau**, University of British Columbia.

Partial Differential Equations of Quantum Physics (Code: SS 12A), **Israel Michael Sigal**, University of Toronto, and **Stephen Gustafson**, University of British Columbia.

Probability Theory and Related Fields (Code: SS 13A), **Terry Soo** and **Codina Cotar**, University College London.

Random Structures, Computation, and Statistical Inference (Code: SS 14A), **Lutz Warnke**, University of California, San Diego, and **Ilias Zadik**, Yale University.

Recent Advances in Differential Geometry (Code: SS 15A), **Lihan Wang**, California State University, Long Beach, **Zhiqin Lu**, UC Irvine, and **Shoo Seto** and **Bogdan D. Suceavă**, California State University, Fullerton.

Recent Developments in Commutative Algebra (Code: SS 16A), **Arvind Kumar**, **Louiza Fouli**, and **Michael DiPasquale**, New Mexico State University.

Representations of Lie Algebras and Lie Superalgebras (Code: SS 17A), **Dimitar Grantcharov**, University of Texas At Arlington, **Daniel Nakano**, University of Georgia, and **Vera Serganova**, UC Berkeley.

Research in Combinatorics by Early Career Mathematicians (Code: SS 18A), **Nicholas Mayers**, North Carolina State University, and **Laura Colmenarejo**, NCSU.

Special Session in Celebration of Bruce Reznick's Retirement (Code: SS 19A), **Katie Anders**, University of Texas at Tyler, **Simone Sisneros-Thiry**, California State University- East Bay, and **Dana Neidmann**, Centre College.

Tensor Categories and Noncommutative Algebras, I (associated with invited address by Julia Plavnik) (Code: SS 20A), **Ellen E Kirkman**, Wake Forest University, and **Julia Yael Plavnik**, Indiana University, Bloomington.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), **Michelle Ann Manes**, University of Hawaii.

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: Expired

For abstracts: To be announced

San Antonio, Texas

University of Texas, San Antonio

September 14–15, 2024

Saturday – Sunday

Meeting #1198

Central Section

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 3

Deadlines

For organizers: February 13, 2024

For abstracts: July 23, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

James Alvarez, University of Texas, Arlington, *To Be Announced*.

Jason Schweinsberg, University of California, San Diego, *To Be Announced*.

Anne J. Shiu, Texas A & M University, *To Be Announced*.

Savannah, Georgia

Georgia Southern University

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: Not applicable

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 5, 2024

For abstracts: August 13, 2024

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Peter Bubenik, University of Florida, *To Be Announced*.

Akos Magyar, University of Georgia, *To Be Announced*.

Sarah Peluse, Princeton/IAS, *To Be Announced*.

Albany, New York

University at Albany

October 19–20, 2024

Saturday – Sunday

Meeting #1200

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub, Lehigh University

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 19, 2024
For abstracts: August 27, 2024

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Meeting #1201

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: March 26, 2024
For abstracts: September 3, 2024

Auckland, New Zealand

December 9–13, 2024

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 24, 2024
For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 8–11, 2025

Wednesday – Saturday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 16, 2024
For abstracts: September 10, 2024

Submit Your Proposals for AMS Special Sessions at the 2025 Joint Mathematics Meetings

All members of the mathematics community are invited to submit proposals for American Mathematical Society (AMS) Special Sessions at the 2025 Joint Mathematics Meetings (JMM). If you have a topic that you would like to explore in a special session, now is the time to put your great idea into motion.

The 2025 JMM will be held January 8–11, 2025 in Seattle, WA. On behalf of the American Mathematical Society, Prof. Brian D. Boe (`brian@math.uga.edu`), the AMS Associate Secretary responsible for the AMS program at this meeting, solicits proposals for AMS Special Sessions for this meeting. Proposals that reflect the full spectrum of interests of the mathematical community are welcome.

A special session is a collection of talks devoted to a single area of mathematics or a single topic. Special sessions can be proposed by teams of organizers.

Please go to the submission form at `https://meetings.ams.org/math/jmm2025/cfs.cgi` and provide the following information:

1. the title of the session;
2. the name, affiliation, and email address of one organizer designated as the contact person for all communication about the session (any additional organizers should be listed as "co-organizer");
3. a brief public description of the topic of the proposed special session (character limit 500 with spacing and punctuation);
4. a *sample* list of speakers whom the organizers plan to invite (it is not necessary to have received confirmed commitments from these potential speakers, but the sample list should demonstrate sufficient potential interest in such a session);
5. description of proposed special session, limited to 3000 characters (including spaces and punctuation). This description will be used for the purposes of review only, it will not be published;
6. either the primary two-digit MSC (Mathematics Subject Classification) number that most closely matches the topic (see `http://www.ams.org/mathscinet/msc/msc2020.html`) or one of the following new code numbers adopted for topics (see `http://www.ams.org/journals/notices/202010/rnoti-p1602.pdf`):
   - 101: Teaching and learning
   - 102: Recreational mathematics
   - 103: Professional development and professional concerns
   - 104: Wider issues

The deadline for submission of proposals is **April 16, 2024**. Late proposals will not be considered. No decisions will be made on proposals until June 2024. For questions about using the submission form, contact `meet@ams.org`.

Organizers are encouraged to read the *AMS Manual for Special Session Organizers* at `https://www.ams.org/meetings/meet-specialsessionmanual` in its entirety.

Some key information: Special sessions will in general be allotted between 4 and 8 hours in which to schedule speakers. To enable maximum movement of participants between sessions, organizers must schedule each session speaker for either (a) a 20-minute talk with 5-minute discussion and 5-minute break or (b) a 45-minute talk with 5-minute discussion and 10-minute break. A special session may include any combination of 20-minute and 45-minute talks that fits within the time allotted to the session, but all talks must begin and end at the scheduled time.

The number of special sessions in the AMS program at the JMM is limited, and because of the large number of high-quality proposals, not all can be accepted. Please be sure to submit as detailed a proposal as possible for review by the Committee on Special Sessions and Contributed Paper Sessions, and address the diversity aspects of your list of proposed speakers. Decisions will be made on acceptance and scheduling of sessions by June 2024. At that time, contact organizers will be notified whether their proposal has been accepted. If so, they will be informed of their session's schedule and will be sent additional information about organizational details.

We look forward to reviewing your proposals.

Call for Proposals for Professional Enhancement Programs (PEPs) at the Joint Mathematics Meetings in Seattle, WA, January 8–11, 2025

Send by Tuesday, April 16 to `https://meetings.ams.org/math/jmm2025/cfs.cgi`.

The AMS solicits proposals for Professional Enhancement Programs (PEPs) at the 2025 Joint Mathematics Meetings (JMM) to be held January 8–11, 2025 in Seattle, Washington. Proposers of a PEP should keep in mind the aim of PEPs is to improve mathematical education at all levels, contribute to building a more mathematically literate society, make mathematics and statistics more attractive as a discipline and career pathway, and enhance the careers of mathematical scientists.

Information: Mathematical and statistical instruction has grown in importance due to the expanding connections between mathematical/statistical ideas and computing, interdisciplinary foundational connections, as well as the ever-growing impact of technology on the social, economic, and cultural fabric of our society. In educating a skilled and diverse workforce at both the undergraduate and graduate levels, departments must be able to provide a broad overview as well as learning opportunities for advanced topics in mathematics and statistics. In doing so, departments face a wide range of recent and historical issues and challenges, including the interface between online teaching, active learning, and inclusive pedagogy; the development of effective foundational courses that address the importance of data analysis and the need for modeling; effective outreach to regional and national communities; collaboration with broader scholarly communities; and increasing diversity and reducing implicit bias in recruitment, hiring, and promotion practices. A major part of this challenge is to broaden access to mathematical and statistical education at all levels, which has important implications for addressing society's most pressing and complex problems and for erasing long-standing socioeconomic inequities in our society.

JMM brings together the entire mathematical sciences community. With the aim of PEPs in mind, members of the community are invited to propose a PEP that will provide professional development opportunities for participants and serve to strengthen departments' efforts to meet their responsibilities and challenges.

- PEPs are expected to be hands-on and participants should leave with a tangible product.
- PEPs are scheduled for two, 2-hour sessions.
- AMS will provide a total of $1000 to the designated organizer of approved PEPs.
- Participants will pay a program fee.

Suggested topics:

- Effective pedagogical and assessment practices for online instruction in the mathematical and statistical sciences (especially given lessons learned from the pandemic)
- Inclusive pedagogy in the mathematical and statistical sciences and curricular practices that promote diversity, equity, and inclusiveness in undergraduate and graduate programs
- Pedagogical practices that promote active learning in specific mathematical and/or statistical courses
- Novel methods and applications for incorporating data science into established undergraduate and/or graduate courses and programs or into established faculty research agendas
- Innovative implementation of software
- Interdisciplinary research collaborations within the mathematical community or new connections between mathematics and/or statistics and other disciplines
- Any mathematical or statistical topic that enhances the professional development for faculty and is suited for the PEP format

Proposals should include:

1. Title of the event
2. Session format (select "PEP Program")
3. Name, affiliation, and email address of each organizer, with one organizer designated as the contact person for all communication about the event, and their website(s), if available
4. A brief public description of the topic of the proposed PEP (character limit: 500 with spacing and punctuation)
5. A detailed (character limit: 3000) description of the proposed event to include participant takeaways
6. Some sessions associate their event with an organization; list the organization(s) agreeing to be associated with your event
7. Time slots to avoid (please indicate in the Notes section where possible)
8. Other JMM events to avoid (please indicate in the Notes section where possible)
9. A list of other events proposed by the organizer (please indicate in the Notes section where possible)
10. If applicable, a two-digit MSC (Mathematics Subject Classification) number (`https://mathscinet.ams.org/mathscinet/msc/msc2020.html`) matching the topic of the event or a three-digit code from the New Expanded Classification System (`https://www.ams.org/journals/notices/202010/rnoti-p1602.pdf`) adopted for topics included previously in an MAA or other JMM session

Proposals for PEPs should be submitted to the JMM Program Committee (`https://meetings.ams.org/math/jmm2025/cfs.cgi`) by Tuesday, April 16, 2024.

We look forward to receiving your submissions.

Call for Proposals for Panels, Workshops, and Other Events at the Joint Mathematics Meetings in Seattle, WA, January 8–11, 2025

Send by April 16, 2024 to `https://meetings.ams.org/math/jmm2025/cfs.cgi`.

The AMS solicits proposals for panels, workshops, and other events at the 2025 Joint Mathematics Meetings (JMM) to be held January 8–11, 2025 in Seattle, Washington.

We invite proposals that reflect the full spectrum of interests of the mathematical sciences community.

The topics of proposals could be in areas in the mathematical sciences, but there are others, such as: creating an inclusive atmosphere in the mathematical sciences community; teaching and learning; professional development and professional concerns; recreational mathematics; and wider issues. Events that connect communities are welcome, such as interdisciplinary workshops or collaborations between mathematical sciences researchers and mathematics education researchers.

Proposals should include:

1. Title of the event
2. Session format (panel, workshop, other [specify])

3. Name, affiliation, and email address of each organizer, with one organizer designated as the contact person for all communication about the event, and their website(s), if available
4. A brief public description of the topic of the proposed panel, workshop, or other event (character limit: 500 with spacing and punctuation)
5. Some sessions associate their event with an organization; list the organization(s) agreeing to be associated with your event
6. A detailed (character limit: 3000) description of the proposed event, including the intended audience
7. A proposed list of panelists/moderators/speakers and their institutional affiliation whom the organizers plan to invite. Have they agreed to participate?
8. Duration of time requested for the event (please indicate in the Notes section: 1 hour, 1.5 hours, 2 hours, 4 hours)
9. Time slots to avoid (please indicate in the Notes section where possible)
10. Other JMM events to avoid (please indicate in the Notes section where possible)
11. If applicable, a two-digit MSC (Mathematics Subject Classification) number (`https://mathscinet.ams.org/mathscinet/msc/msc2020.html`) that matches the topic of the event or a code from the New Expanded Classification System (`https://www.ams.org/journals/notices/202010/rnoti-p1602.pdf`) adopted for topics included previously in an MAA or other JMM session

Proposals for panels, workshops, and other events should be submitted to the JMM Program Committee (`https://meetings.ams.org/math/jmm2025/cfs.cgi`) by Tuesday, April 16, 2024.

Further information: Panels, workshops, and other events approved by the Program Committee will normally be allotted a 60–90-minute slot, with the exception of a Professional Enhancement Program (PEP) that could run for 4 hours. Decisions will be made on proposals after the submission deadline has passed.

The number of panels, workshops, and other events in the AMS program at the Joint Mathematics Meetings is limited, and because of the large number of high-quality proposals anticipated, not all can be accepted, nor can all scheduling requests be honored.

Please submit as detailed a proposal for your event as soon as possible for review by the JMM 2025 Program Committee. Organizers of proposals will be notified whether their proposal has been accepted by **end of May**. Additional instructions and the event schedule will be sent to the contact organizer of each accepted proposal shortly after that deadline.

We look forward to receiving your submissions.

Hartford, Connecticut

Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

April 5–6, 2025

Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

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