The Contributions of Chuu-Lian Terng to Geometry

Karen Uhlenbeck

1. Introduction

Chuu-Lian Terng is a geometer who has made important contributions to both submanifold geometry and integrable systems. My collaboration with her over many years has been rewarding and profitable. This article is an attempt to share my appreciation of her work with a wide audience. I am indebted to her for decades of profitable collaboration, a warm personal relationship and help with writing this article.

The article is incomplete, as space constraints do not allow me to discuss all of her research. I chose to emphasize personal details and her work on integrable systems. Many schools of mathematics contributed to the development of integrable systems and I cannot include them all. The outline given in Section 6 is as I have come to see the subject. This is close, but not identical to Terng's view, and will differ from other views. I hope to give a window opening onto her work, not to provide a definitive treatise. The ideas belong to many people and errors are mine. The reader will surely join me in thanking the referees for helpful comments and important corrections.

2. Background and Education

Chuu-Lian Terng was born in 1949 in Hualian, Taiwan. Her family moved to Taipei when she was three. Her father was in the army under Chiang Kai-shek and had moved from mainland China to Taiwan when the Communists under Mao took over China in 1948. She was the eldest, with three younger brothers, and her family was very poor. She was warned that she had to do well in school, lest she end up working as a maid.

Chuu-Lian was a good student, and graduated the Taipei First Girls' High School. Her comment: "Mathematics was easy. I liked it." When it came time to go to college, she was automatically admitted to the National Taiwan University on the basis of her grades. This was not free, unlike the Normal University, but graduates of the Normal University were expected to teach for ten years in middle and high school after graduation. Her sights were set higher. So she tutored middle school and high school students for ten hours a week to earn tuition, books and money to give her family. And she of course lived at home.

Chuu-Lian's class had 8 women and 24 men. At first it was a big shock for the women to study for the first time with men. In the boys school many students had already studied calculus and read English mathematics books on their own. English was not Chuu-Lian's strong suit. (More about that later!) Her recollection is that the first year there were two math courses: calculus and "What is Mathematics" from Courant and Robbins. However, the students got together, ran a seminar and studied extra material. There was a lot of homework, and the exams were hard. She recalls that in her senior year she just missed passing an algebra course with a 55 when 60 was passing. Only one student had a higher grade. "The atmosphere was good. The students worked together and presented material, teaching themselves. I remember when I once got the highest grades in my freshmen year, and the men called me 'brother Terng.'" She graduated at the top of her class.

At that time there were no PhD programs in Taiwan. The top students routinely went to the United States for graduate study and most of them did get PhD's. They often went to the top graduate schools and did well there. But Chuu-Lian was faced with two problems. The first was her lack of proficiency in English. It was her worst subject and she hated it. All the Taiwanese students had to take the TOEFL exams, and almost all graduate schools in the

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US required at least a score of 550. Chuu-Lian's score was 520. She found three schools that only required 500, applied and was accepted into all three. She elected to go to Brandeis.

The second problem was that her family had no money and she had been giving them most of her tutoring earnings. So she delayed for a year, working as a TA and taking a few more courses. This time she kept her earnings for her plane fare and a nest egg until settled at Brandeis. She again sent money home during her graduate student years.

"I loved my courses at Brandeis. I also found out how good my undergraduate training had been." She still speaks with enthusiasm of Ed Brown teaching topology, Maurice Auslander who gave lot of homework and no lectures and Dick Palais's course on pseudo-differential operators on the circle. And she remembers what many of us discovered: that we loved complex variables, but that the charm does not necessarily carry over to several complex variables. Her subjects were geometry and topology. No PDE and bad memories of a course in harmonic analysis. At the end of the first year she started working with Dick Palais. Her first readings were his book on the Atiyah-Singer index theorem and Spivak's *Differential Topology volume I*. Her thesis was on the classifications of natural vector bundles and natural differential operators.

3. Early Career

Terng's first postdoctoral position was at the University of California at Berkeley, where she went to study differential geometry with S. S. Chern. When she went to ask him for a problem, his reply was "I don't give problems. Go to seminars, library, talk to people and find your own problems." Then Chern became interested in solitons and Bäcklund transformations.

3.1. The sine Gordon equation and Bäcklund transformations. The sine Gordon equation is the equation for $\phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\phi_{uv} = \sin(\phi).$$

We easily see that a trivial solution is given by $\phi = 0$. We use the notation that subscripts denote partial derivatives.

Theorem 3.1.1. *If ϕ is a solution of the sine-Gordon equation and*

$$\begin{cases} \xi_u = \phi_u + 2a \sin \frac{\xi+\phi}{2}, \\ \xi_v = \phi_v + 2a^{-1} \sin \frac{\xi-\phi}{2}, \end{cases}$$

for any constant a, then ξ is also a solution of the sine Gordon equation.

In fact, we see that the two equations for ξ are compatible only when ϕ solves the sine Gordon equation. Just as a test, if $\phi = 0$ is the trivial solution, then, we get

$$\begin{cases} \xi_u = 2a \sin \frac{\xi}{2}, \\ \xi_v = 2a^{-1} \sin \frac{\xi}{2}. \end{cases}$$

Changing variables to $t = a(u + v)$ and $s = 1a^{-1}(u - v)$ gives equations $\xi_s = 0$ and $\xi_t = 2\sin(\xi/2)$, yielding a one-parameter family of solutions. The process can be iterated, giving a chain of solutions known as solitons. After two steps there are relations between solutions known as permutability formulas. We have found a chain of special solutions to the partial differential equation by solving systems of ordinary differential equations. This is the purpose of Bäcklund transformations.

The three classical "integrable systems" with Bäcklund transformations, scattering and inverse scattering theory, tau functions and Virasoro actions are:

(i) sine Gordon equation (SGE)

$$\phi_{uv} = \sin(\phi),$$

(ii) nonlinear Schrödinger equation (NLS)

$$u_t = 2i(u_{ss} + |u|^2 u),$$

(iii) Korteweg-de Vries equation (KdV)

$$u_t = u_{sss} - (u^3)_s.$$

These three equations have similar properties but very different origins. The sine Gordon equation arose in 1862 in the investigation by Edmund Bour into constant Gaussian curvature -1 surfaces in $\mathbb{R}^3$. The angle between the asymptotic lines satisfies the sine Gordon equation. Bäcklund discovered the transformations which bear his name using line congruences. The Korteweg-de Vries equation dates back to 1895 as a description of shallow water waves. The more recent nonlinear Schrödinger equation describes a plethora of phenomena including the propagation of signals in fiber optic cables. Terng and I wrote an expository article for the *Notices* describing more of the history [TU00]. Chern was, of course, interested in the geometry described by the first of these equations. He wanted to extend Bäcklund transformations to affine minimal surfaces in $\mathbb{R}^3$. Chuu-Lian read Chern's notes on affine geometry, and they obtained results very fast. However, they only obtained a single transformation without a parameter. After they obtained the results, the senior author said "You write it up" [CT80]. It is in stories like this that we get hints of how we might have been better thesis advisors and mentors.

3.2. Higher-dimensional versions. Chern then put Chuu-Lian in touch with Keti Tenenblat, a Turkish-Brazilian mathematician visiting Chern. The two women collaborated on the problem, suggested by Chern, of generalizing the classical Bäcklund transformations to hyperbolic n-dimensional submanifolds in $\mathbb{R}^{2n-1}$. Later

Ablowitz, Beals and Tenenblat found a Lax pair and did the scattering and inverse scattering. Chuu-Lian found a loop algebra given by involutions which have a suitable splitting. There was a lot of interest in these equations at the time. I recall driving Chuu-Lian to the University of Chicago where she had been invited to talk on this work to the theoretical physics group.

3.3. Chinese women mathematians. Chuu-Lian Terng was not an isolated Chinese woman mathematician. As we noted before, students from the National Taiwan University routinely went on to PhD's in the United States. The year before Chuu-Lian, a number of women whose names you may recognize graduated: Alice Chang, Fan Chung, Winnie Li, Gloria Wu. The success of this group of Chinese women mathematicians is not part of a general trend, and cannot be easily explained. Chuu-Lian did know about the other four, and discovered only recently that all of them have similar backgrounds. They all credit their mothers and an excellent education system which was open to women. She took her teachers' questionings "Did your father help you with this?", "Did your boyfriend help you study?" and "You don't like to look up the answers in the back of the book?" as praise. As we attempt to diversify the pool of individuals succeeding in research mathematics, we might pay attention to what worked here. I recommend a forthcoming article by Allyn Jackson about this group of Chinese women mathematicians.

4. Midcareer

After her postdoc at Berkeley, Terng moved on to an assistant professorship at Princeton. Princeton had only started admitting women students in 1969, and admission had become gender blind in 1974. Chuu-Lian came to Princeton in 1978 as the first female assistant professor in the mathematics department. She was one of ten assistant professors who used their time at Princeton to get as much research done and move on to a "real" tenure track or tenured position elsewhere. Inconveniences, such as having an office on the tenth floor when the only women's bathroom was on the third floor, happened all too often in those days. But in general, it was a matter of being ignored rather than treated badly.

When I came as a member to the Institute for Advanced Study for the academic year 1979–1980, we tried to work together for the first time. I recall that we had offices next to each other in Building C, and at least once Professor Langland slammed the door of the office we were working in. Our husbands have also noted that we are not quiet when we work together. Perhaps we were expected "to be seen but not heard"? I had been learning Teichmüller theory and was attending Bill Thurston's course at the university. We learned about hyperbolic three manifolds which fiber over the circle topologically. No natural geometric fibration was known. With my background in minimal surfaces and Chuu-Lian's command of tools in geometry, the problem seemed made for us.

Alas, it was not to be. The problem is still unsolved. When Bill Thurston heard of our project, he suggested that these manifolds might be fibered by minimal surfaces. We thought it more likely that there would be a fibration by constant mean curvature surfaces. In 1979 the problem was esoteric, but it is now known that every compact hyperbolic 3-fold has a cover that topologically fibers over a circle; these are now essential examples in the study of hyperbolic 3-folds.

I return to this problem periodically, but Chuu-Lian's next project explored polar actions and isoparametric submanifolds. Due to the length limit of this article, we will refer the readers to a survey article by G. Thorbergsson [Th] and the book written by Terng and Palais [PT88] on these topics. This research was carried out during Terng's years at Northeastern University. These were not easy years with heavy teaching load, an unpleasant commute, and the kind of mistreatment and outright sexism that was all too common (which none of us enjoy revisiting). On the good side, she had good interactions with Boston area mathematicians and has fond memories of weekly joint seminar with her colleagues in topology, PDE, and geometry. She also benefited from lectures on integrable systems by Mark Adler and Pierre van Moerbeke at Brandeis. She moved from Northeastern University to University of California at Irvine in 2004 and enjoyed very much the differential geometry group, the graduate students, a house on campus, and the wonderful climate.

5. Soliton Equations in Geometry

My road into integrable systems was from the opposite direction from Chuu-Lian's introduction via Chern. Through one of my PhD students Louis Crane, I learned about the interest of the physics community in "loop groups," and read the papers of Louis Dolan on the sigma model. I had kept up with Chuu-Lian and her husband, only partly because I had relatives in the Boston area. After I noted that Dolan's loop group actions were dressing actions and the classical Bäcklund transformations for sine Gordon were given by actions of some rational loops, we started to work together again. This section is intended to connect examples of integrable equations with equations familiar to geometers.

5.1. Finite dimensions. In finite dimensions, there is a definition of completely integrable.

Definition 5.1.1. Let (M, ω) be a symplectic manifold. The Hamiltonian system given by $H : M \to \mathbb{R}$ is

$$\frac{d\gamma}{dt} = X_H(\gamma),$$

where X_H is the dual of dH with respect to ω, i.e.,

$$\omega(X_H, v) = d_\gamma H(v) \quad \text{for all tangent vector } v.$$

Definition 5.1.2. $H_j : M \to R$ is a conservation law for the Hamiltonian system given by H if for every solution γ, $\frac{d}{dt}H_j(\gamma) = 0$, or equivalently,

$$\{H, H_j\} \coloneqq \omega(X_H, X_{H_j}) = dH(X_{H_j}) = 0.$$

A Hamiltonian system given by H on a manifold of dimension $2m$ is *completely integrable* if there are m conservation laws $H = H_1, H_2, \ldots, H_m$, which are in involution with each other, i.e., $\{H_j, H_k\} = 0$.

If two Hamiltonians are in involution with each other, their flows commute. This is the picture in finite dimensions. However, in infinite dimensions, the integrable systems in this article always have an infinite number of conservation laws whose flows commute. Only in very rare cases is there a theory which indicates "completeness." The example I know involves finding action angle coordinates for a special case of KdV. Nevertheless, we shall see that the infinite-dimensional theory is much richer than the finite-dimensional theory.

The symplectic manifolds in this section are the Grassmanians $N = G(k, n)$, which are adjoint orbits in the Lie algebra $u(n)$ of $U(n)$,

$$N = G_{(k,n)} = \left\{g^{-1}Jg \mid J = \frac{i}{2}\operatorname{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})\right\}.$$

Geometric information 5.1.3. For $h \in N$,

(i) $T_hN = \{[a, h] \mid a \in u(n)\}$.
(ii) $\langle a, b\rangle = -\operatorname{tr}(ab)$.
(iii) The orthogonal projection a_h of a on $T_h(N)$ is

$$a_h = -[h, [h, a]] = -(\operatorname{ad} h)^2(a).$$

(iv) The symplectic form ω is

$$\omega_h(a, b) = \operatorname{tr}([h, a]b).$$

(v) The associated complex structure in the tangent space is

$$i(h)a = [h, a].$$

It is important to know at least one example.

Example 5.1.4. Let $a \in u(n)$ and let $H_a(h) = \operatorname{tr}(ah)$. Then the hamiltonian flow for H_a is

$$\frac{dh}{dt} = -[a, h].$$

Note that if $a, b \in u(n)$ commute, then

$$\{H_a, H_b\} = dH_a(X_{H_b}) = \operatorname{tr}(a[h, b]) = \operatorname{tr}(h[b, a]) = 0.$$

However, when $n > 2$, $[a, b_j] = 0$, does not imply that $[b_j, b_k] = 0$. So the integrals for the Hamiltonian flow $\frac{dh}{dt} = -[a, h]$ are not in involution. In this case, the symmetries are Poisson and we can expect to see such symmetries for equations with target $N = G_{(k,n)}$ with $n > 2$.

5.2. Equations of global analysis.

We now switch gears. Geometers are familiar with the nonlinear elliptic equation of harmonic maps between Riemannian manifolds and associated parabolic heat and hyperbolic wave map equations. However, in the case that the image manifold is symplectic, we have a geometric nonlinear Schrödinger (GNLS) equation.

As a general principle, the image negatively curved symmetric spaces correspond to defocusing equations and those with positive curvature correspond to focusing. Formally they appear similar, but technically and geometrically they are very different. Terng and I considered the case of the positively curved symmetric space $N = G_{(k,n)}$ described above.

We briefly describe the GNLS on $\mathbb{R} \times \mathbb{R}^{n-1}$ with image $G_{(k,n)}$. We consider

$$C_J(\mathbb{R}^{n-1}, N) = \{h : \mathbb{R}^{n-1} \to N \mid h_{x_i} \in S(\mathbb{R}^{n-1}, u(n)\},$$

where $S(\mathbb{R}^{n-1}, u(n))$ is the Schwartz space and $\lim_{x\to-\infty} h(x) = J$. Corresponding to the geometry listed in 5.1.3, we have the following geometry in $C_J(R, N)$.

5.2.1. *Geometric information in $C_J(\mathbb{R}^{n-1}, N)$.*

For $h \in C_J(\mathbb{R}^{n-1}, N)$,

(i) $T_hC_J(\mathbb{R}^{n-1}, N) = \{[h, A] \mid A \in S(\mathbb{R}^{n-1}, N)\}$.
(ii) The formal inner product on the tangent space is

$$\langle A, B\rangle = -\int_{\mathbb{R}^{n-1}} \operatorname{tr}(AB)(dx)^{n-1}.$$

(iii) The orthogonal projection A_h of $A \in S(\mathbb{R}^{n-1}, u(n))$ on $T_hC_J(\mathbb{R}^{n-1}, N)$ is

$$A_h = -\frac{1}{4}(ad(h))^2 A.$$

(iv) The symplectic structure on $C_J(\mathbb{R}^{n-1}, N)$ is

$$\Omega(h)(A, B) = -\int \operatorname{tr}([h, A]B)(dx^{n-1}).$$

(v) The associated complex structure in the tangent space to $C_J(\mathbb{R}^{n-1}, N)$ at h is

$$i(h)A = [h, A].$$

Theorem 5.2.2. *The Hamiltonian flow for $H = \frac{1}{2}\langle dh, dh\rangle$ is the GNLS equation*

$$\frac{\partial h}{\partial t} = [h, \triangle h].$$

We will explain in Section 6 why this equation is integrable for $n = 2$.

5.3. Ward harmonic maps and space-time monopoles.

The 2- and $1+1$-dimensional examples of harmonic maps and wave maps into $\mathrm{SU}(n)$ are examples of equations which have many of the usual properties of soliton equations. However, there is a variant of the wave map equation in $1 + 2$ into $\mathrm{SU}(n)$ due to Ward which has solitons and scattering and inverse scattering theories. We refer to

Ward's equation as the modified wave map. The wave map equation for $g : \mathbb{R} \times \mathbb{R}^2 \to \mathrm{SU}(2)$ is

$$\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}g\right)g^{-1}\right) - \frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right) = 0.$$

Ward's equation is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}g\right)g^{-1}\right) - \frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial x}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right)\\ = \left[\left(\frac{\partial}{\partial x}g\right)g^{-1}, \left(\frac{\partial}{\partial t}g\right)g^{-1}\right].\end{aligned}$$

The Lax pair for this equation is more easily seen if we transform using the variables $\xi = \frac{1}{2}(t+x)$ and $\eta = \frac{1}{2}(t-x)$. Then the modified wave map equation simplifies to

$$\frac{\partial}{\partial \xi}\left(\left(\frac{\partial}{\partial_\eta}g\right)g^{-1}\right) - \frac{\partial}{\partial y}\left(\left(\frac{\partial}{\partial y}g\right)g^{-1}\right) = 0.$$

Proposition 5.3.1. *There is a Lax pair for the modified wave map of the form* $[L(\lambda), \tilde{L}(\lambda)] = 0$, *where*

$$\begin{aligned}L(\lambda) &= \lambda\frac{\partial}{\partial \xi} - \left(\frac{\partial}{\partial y} - \left(\frac{\partial}{\partial y}g\right)g^{-1}\right),\\ \tilde{L}(\lambda) &= \lambda\frac{\partial}{\partial y} - \left(\frac{\partial}{\partial \eta} - \left(\frac{\partial}{\partial \eta}g\right)g^{-1}\right).\end{aligned}$$

The frame E_λ is useful for constructing a large number of examples, which are not known for the wave map. The modified wave map equation can be transferred (via a gauge transformation similar to the Hasimoto transformation) to the space-time monopole equation, which is a reduction of anti-self-dual Yang-Mills with signature $(2,2)$ to a monopole equation in $\mathbb{R}^{1,2}$.

Proposition 5.3.2 ([DTU06]). *Solutions to the Ward equation can be identified with special solutions to the space-time monopole equation in* $\mathbb{R}^{1,2}$,

$$*D_A\Phi = F_A.$$

Here $D_A = d + A, F_A = \{D_A, D_A\}$ *is the curvature 2-form,* Φ *is the (scalar) Higgs field in* $su(n)$ *and* $*$ *is the map from 1-forms to 2-forms induced by the indefinite metric on* $\mathbb{R}^{1,2}$.

The analytic properties of the wave map problem are hard theorems in geometric analysis and it is difficult to find explicit examples. Description of Euclidean monopoles are only accurate at large spacings. However there is both a scattering theory for the monopole equation and Bäcklund transformations for Ward's equation which are gauge equivalent to the Lax pair for the monopole equation. This is due to the existence of Lax pairs for both variants of the equation which are gauge equivalent. For smooth decaying initial data, the scattering theory shows that solutions exist for all time. Terng's papers [DT07] and [DTU06] contain a wealth of information about these solutions.

6. The Drinfel′d-Sokolov Construction for the MNLS Hierarchy

In this section, we outline a very general procedure developed by many mathematicians of different backgrounds for generating soliton equations and their properties. There is a long list of contributors to this project, starting with Zhakarov and Shabat in 1976. In the last and final section we will come back to Terng's ideas for classifying them. Much of the work of Terng is based on a fundamental paper of Drinfel′d and Sokolov [DdS84]; hence I have called the general scheme a Drinfel′d-Sokolov (DS) construction. The flows are traditionally in the variables $(x = t_1, t = t_2, t_3)$. However, certain pieces of the structure can be applied with only part of the structure available. For example, Bäcklund transformations can be applied when there is only a single flow if there is a Lax pair description. In perhaps most examples, the needed factorizations for scattering theory and Bäcklund transformations are only local. The matrix nonlinear Schrödinger equation (MNLS) is a generalization of the scalar NLS, and our choice is motivated by the fact that the factorizations are well-understood in this case. Also note that there may be multiple descriptions of the same flows. The computations do not need to be done in any fixed order.

6.1. Choice of a triple group. Most flows are based on a triple loop group $(\mathcal{G}, \mathcal{G}^+, \mathcal{G}^-)$. Here $\mathcal{G}^\pm$ are subgroups of $\mathcal{G}$ and elements of $\mathcal{G}$ factor into products in both orders on an open, dense set. The loop parameter plays the role of a spectral parameter λ, and the group factoring is done via Birkhoff factorization. These groups dependent on a spectral or loop parameter are often called "loop groups," especially in the physics literature. There exists an open dense set Ω of $\mathcal{G}$ (*big cell*) such that for $g \in \Omega$, we can factor

$$g = g_+g_- = h_-h_+, \quad g_+, h_+ \in \mathcal{G}_+, g_-, h_- \in \mathcal{G}_-.$$

In our example:

$$\begin{aligned}\mathcal{G} &= \{g(\lambda) \in \mathrm{GL}(n,\mathbb{C}) \,|\, g \text{ defined and holomorphic in}\\ &\qquad \{\lambda : k < |\lambda| < \infty\}, (g(\bar{\lambda}))^\dagger g(\lambda) = \mathrm{I}\},\\ \mathcal{G}^+ &= \{g \in \mathcal{G} \,|\, g \text{ extends holomorphically to } \mathbb{C}\},\\ \mathcal{G}^- &= \{g \in \mathcal{G} \,|\, g(\infty) = \mathrm{I}, g \text{ holomorphic in a}\\ &\qquad \text{neighborhood of } \infty\}.\end{aligned}$$

At the Lie algebra level $(\mathfrak{G}, \mathfrak{G}_+, \mathfrak{G}_-)$ we have $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$. We can decompose a Laurent series converging in $\{\lambda : k < |\lambda| < \infty\}$ into negative powers of λ and nonnegative powers of λ. At the Lie group level, this is called Birkhoff factorization and involves writing a power series as the products of series that converge in the two regions. It fails on the group, but is valid on a dense open subset Ω of what we call we call "the big cell."

6.2. **Choice of a vacuum frame.** A vacuum frame $V_0 \in C(\mathbb{R}^s, \mathcal{G}^+)$ determines the flows. Different flows result from different choices. For our example

$$V_0 = \exp\left(\sum_{j=1}^{n_0} t_j \lambda^j J\right),$$

$J = \frac{1}{2}\,\mathrm{diag}(\pm i)$ with k +'s and $n-k$ minuses. Here the upper limit n_0 is arbitrarily large. We never use infinite sums.

6.3. **The dependent variable in the flow.** Our first step is completely formal. Choose an arbitrary element $f \in \mathcal{G}^-$. Factor

$$V_0 f^{-1} = M^{-1} E$$

for $M^{-1} \in \mathcal{G}^-$, $E \in \mathcal{G}^+$.

Note that the scattering data f does not depend on the flow variables $\vec{t} = (x = t_1, t = t_2, t_3)$. Since V_0 does, M and E do as well. We call E the frame and M the reduced frame (also called the Baker function). The frame E has positive powers of λ and the reduced frame M has negative powers:

$$\partial_x M\, M^{-1} + M \partial_x V_0 M^{-1} = \partial_x E E^{-1},$$
$$\partial_x E\, E^{-1} + [M \partial_x V_0 M^{-1}]_+ = \lambda J + [M_{-1}, J].$$

Then $u = [M_{-1}, J]$ (or technically $\partial_x + u$) will be our dependent variable in the flow, where M_{-1} is the coefficient of λ^{-1} the power series of M. Once the theory is sufficiently developed, ∂_x plays the role of the dual of the element defining a central extension of $\mathcal{G}$. Note that u is in $[J, u(n)]$ and the flows generated by this framework are equations for maps $(x = t_1, t = t_2, \dots, t_\ell)$.

If $f = I$, u is identically 0. This is the "vacuum" solution.

6.4. **Lax pairs.** Let

$$Q = MJM^{-1} = \sum_{j=0} \lambda^{-j} Q_j = J + \lambda^{-1}[M_{-1}, J] + \sum_{j>1} \lambda^{-j} Q_j$$

$$\left(\frac{\partial}{\partial t_\ell} M\right) M^{-1} = [M \lambda^\ell J M^{-1}]_-.$$

The dependent variable for the flow is $u = Q_1 = [M_{-1}, J]$. To get $\frac{\partial}{\partial t_\ell} u$, we compare the coefficients of λ^{-1} to get $\frac{\partial}{\partial t_\ell} M_{-1} = Q_{\ell+1}$, or

$$\frac{\partial}{\partial t_\ell} u = [J, Q_{\ell+1}] = [\partial_x + u, Q_\ell],$$

the *ℓ-th flow in the MNLS hierarchy*.

It is not necessarily true that the Q_j are determined algebraically, which was first proved by Sattinger for our example by using the following two equations.

$$Q^2 = J^2 = -\frac{1}{4} I,$$
$$\partial_x Q = [(\lambda J + u), Q].$$

Lemma 6.4.1. *The Q_j are polynomials in u and its derivatives in x of order up to j.*

The Lax pair formulation is now

$$L = \partial_x + \lambda J + u,$$
$$\tilde{L}_k = \partial_{t_k} + J\lambda^k + \sum_{j=0}^{k-1} \lambda^j Q_{(k-j)}$$

and $[L, L_k] = 0$. For $\vec{t} = (t_1, \dots, t_{n_0})$, we call $E(\lambda, \vec{t})$ a frame for the MNLS hierarchy if $E(\lambda, \vec{t})$ is holomorphic for $\lambda \in \mathbb{C}$ and $L_k E = 0$ or equivalently

$$E^{-1} E_{t_k} = J\lambda^k + \sum_{j=0}^{k-1} \lambda^j Q_{(k-j)}$$

for all $1 \le k \le n_0$. In particular, for $\lambda = 0$, we get $[\partial_x + u, \partial_{t_k} + Q_k(u)] = 0$, or equivalently, u is a solution of the k-th flow $u_{t_k} = [\partial_x + u, Q_k(u)]$ if and only if the system $g^{-1} g_x = u$, $g^{-1} g_t = Q_k(u)$ is solvable for $g : \mathbb{R}^2 \to U(n)$. It also follows that there exists an open subset Ω_- of the negative group $\mathcal{G}^-$ such that we can factor

$$V_0(\lambda, \vec{t}) f^{-1}(\lambda) = M^{-1}(\lambda, \vec{t}) E(\lambda, \vec{t}),$$
$$M(\cdot, \vec{t}) \in \mathcal{G}^-, E(\cdot, \vec{t}) \in \mathcal{G}^+.$$

Then

$$u_f = [M_{-1}, J]$$

is a solution of the MNLS hierarchy and $E(\lambda, \vec{t})$ is the frame for u_f with $E(\lambda, \vec{0}) = \mathrm{I}$.

We compute a few terms:

$$Q_1(u) = u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad Q_2(u) = \begin{pmatrix} -iq^* q & iq_x \\ iq_x & iq^* q \end{pmatrix},$$
$$Q_3(u) = \begin{pmatrix} -qq_x^* + q_x q^* & -q_{xx} + 2qq^* q \\ q_{xx}^* + 2q^* qq^* & q_x^* q - q^* q_x \end{pmatrix}.$$

6.5. **Bäcklund transformations and dressing actions.** Since factorizations can be carried out on an open dense subset (the big cell) Ω of $\mathcal{G}$, dressing actions are locally defined.

Proposition 6.5.1. *If $E(\lambda, \vec{t})$ is a frame of a solution $u(\vec{t})$ for $\vec{t}$ in a compact domain $\mathcal{O}$, then there exists an open subset Ω_- of $\mathcal{G}^-$ of the identity such that for $g \in \Omega_-$, we can factor $Eg^{-1} = \tilde{M}^{-1} \tilde{E}$ with $\tilde{M} \in \mathcal{G}^-$ and $\tilde{E} \in \mathcal{G}^+$, and*

$$g \sharp E := \tilde{E} = \tilde{M} E g^{-1}$$

is the frame for a new solution in $\mathcal{O}$. In particular, if $u = u_f$ for some $f \in \mathcal{G}^-$, i.e., $E = M V_0 f^{-1} = f \sharp V_0$, then $g \sharp E$ is the frame of u_{gf} and $g \sharp E = \tilde{M} M V_0 (gf)^{-1}$.

There is a fair amount of analysis needed to understand more than this in the case of continuous scattering data. The original results for our example are due to Beals and Coifman, but the proofs and results vary by example.

When $g \in \mathcal{G}^-$ in the above Proposition is meromorphic with a simple pole then the factorization $Eg^{-1} = \tilde{M}^{-1} \tilde{E}$ can be written down explicitly, which gives a Bäcklund transformations. We identify particularly simple meromorphic examples in $\mathcal{G}^-$. For the group we have chosen, with the reality condition $g(\bar{\lambda})^\dagger g(\lambda) = \mathrm{I}$, the examples with simple poles are

$$f_{a,\pi}(\lambda) = \pi + \left(\frac{\lambda - \bar{a}}{\lambda - a}\right)\pi^\perp.$$

Here π is the Hermitian projection on a subspace, $\pi^\perp = \mathrm{I} - \pi$ is the projection on the orthogonal subspace, and a is a complex number whose imaginary part is nonzero. Note that

$$f_{a,\pi}^{-1}(\lambda) = \pi + \frac{\lambda - \alpha}{\lambda - \bar{\alpha}}\pi^\perp.$$

Given a frame $E = E(\lambda, x, t_2)$, we can find a new $f_{a,\pi}\#E$ for a new solution.

Proposition 6.5.2. *Given a simple element $f_{a,\pi}$ and the frame E of a solution u of the flow, then $\tilde{u} = u + (a - \bar{a})[J, \tilde{\pi}]$ is a new solution with frame*

$$f_{a,\pi}\#E = f_{a,\tilde{\pi}(\vec{t})} E f_{a,\pi}^{-1},$$

where $\tilde{\pi}(x,t)$ is the Hermitian projection onto $E(a,\vec{t})^{-1}(\Im\pi)$.

Proof. Note that the residues of the right-hand side of $f_{\alpha,\pi}\#E$ are zero at $\lambda = \alpha$ and $\lambda = \bar{\alpha}$ if and only if

$$\tilde{\pi}(\vec{t}\,)E(a,\vec{t}\,)\pi^\perp = 0,$$
$$\tilde{\pi}(\vec{t}\,)^\perp E(\bar{a},\vec{t}\,)\pi = 0.$$

These are compatible. To see this, apply the complex transpose to the second equation to see that $\pi E(a,\vec{t})^{-1}\tilde{\pi}^\perp = 0$. But both equations are equivalent to $\pi(x,t)$ being the projection on the subspace orthogonal to $E(a,\vec{t})^{-1}(\Im\pi)$. □

The permutability formulas follow easily from the fact that a product of two such f's with poles at a and b can be factored with either pole first. This yields a rather complicated identity between applications of two Bäcklund transformations which is known as the permutability formula. When Bäcklund transformations are generated by an alternative argument, these permutability formulae are mysterious. Solitons are generated by applying Bäcklund transformations to the vacuum frame V_0 of the trivial zero solution.

6.6. **Symplectic structure and Hamiltonians.** We have generated flows using Lax pairs without mentioning either a symplectic structure or Hamiltonians. These are, however, already embedded in our computations. For $u_1, u_2 \in S(\mathbb{R}, \mathcal{P})$, let

$$\langle u_1, u_2\rangle = -\int_{-\infty}^{\infty} \mathrm{tr}(u_1 u_2)dx.$$

We start with a basic symplectic structure (more about this later)

$$\theta(u_1, u_2) = \langle [J, u_1], u_2\rangle = -\langle [J, u_2], u_1\rangle.$$

If we compute the gradient ∇H of a Hamiltonian H as

$$dH(\delta u) = \langle \nabla H(u), \delta u\rangle,$$

then the Hamiltonian flow for H is

$$u_t = [J, \nabla H].$$

Proposition 6.6.1 ([Ter97])**.** *The j-th flow is the Hamiltonian flow for*

$$H_j(u) = -\frac{1}{j+1}\int_{-\infty}^{\infty} \mathrm{tr}(Q_{j+2}(u)J)dx.$$

6.7. **Sequences of symplectic structures.** Terng tells me she first noticed this bi-Hamiltonian structure in a very general context in Drinfel'd and Sokolov [DdS84]: not only can we formulate the flows as Hamiltonian flows of a sequence of Hamiltonians, we can fix the Hamiltonian and vary the symplectic structure ([Ter97]).

Recall that our soliton equations have a phase space of rapidly decaying smooth maps $u \in \mathcal{S}(R, [J, u(n)])$, which can be identified as a coadjoint orbit. Coadjoint orbits come equipped with an orbit symplectic form, and the symplectic form we used in Section 5 was exactly this form. New symplectic structures are found by finding embeddings of the phase space in another coadjoint orbit (cf. [Ter97]).

6.8. **Relation between MNLS and GNLS.** It was proved in [TU06] that given $\gamma : \mathbb{R} \to N = G_{(k,n)}$ there exists $g : \mathbb{R} \to U(n)$ such that

$$\gamma = gJg^{-1}, \quad u := g^{-1}g_x \in [J, u(n)].$$

We call such g an *adjoint frame* along γ, and u the *adjoint curvature* defined by g. Moreover, if g_1 is also an adjoint frame along γ, then there exists a constant $c \in K = U(k)\times U(n-k)$ such that $g_1(x) = g(x)c$. Hence the adjoint curvature u_1 defined by g_1 is $u_1 = c^{-1}uc$.

Given $\gamma : \mathbb{R} \to N = G_{(k,n)}$ and $\xi : \mathbb{R} \to u(n)$ such that $\xi(x)$ is tangent to N at $\gamma(x)$, we define

$$\begin{aligned}\nabla\xi = &\ \text{the orthogonal projection of } \xi_x \text{ onto } T_\gamma N\\ = &-\mathrm{ad}(\gamma)^2(\xi_x).\end{aligned}$$

A direct computation implies that $\gamma_x = g[u, a]g^{-1}$ and $\nabla^{(j)}\gamma_x = g[u_x^{(k)}, a]g^{-1}$. Hence we have

$$gu_x^{(j)}g^{-1} = \mathrm{ad}(\gamma)(\nabla^j\gamma_x).$$

We have seen that the adjoint curvatures u_1, u defined by adjoint frames g and g_1 along γ are related by $u_1 = c^{-1}uc$ for some constant $c \in K$. Hence $g[Q_j(u), J]g^{-1} = g_1[Q_j(u_1), J]g_1^{-1}$. This implies that

$$\gamma_{t_j} = g[Q_j(u), J]g^{-1} \qquad ((*)_j)$$

defines a flow on $C^\infty(\mathbb{R}, N)$ (independent to the choice of adjoint frame) and is the j-th flow in the GNLS hierarchy.

Use the formulas for $Q_j(u)$ given in Section 6 to see that the first three flows are

$$\gamma_{t_1} = \gamma_x,$$
$$\gamma_{t_2} = [\gamma, \gamma_{xx}],$$
$$\gamma_{t_3} = -\nabla^2\gamma_x - (\gamma_x)^3.$$

Notice that the second flow is the GNLS.

Theorem 6.8.1 ([TU06]). *If $\gamma(x,t)$ is a solution of $((*)_j)$, then there exists $g : \mathbb{R}^2 \to U(n)$ such that $g(\cdot,t)$ is an adjoint frame along $\gamma(\cdot,t)$ such that $u := g^{-1}g_x$ is a solution of the j-th flow $u_t = [Q_{j+1}(u), J]$ in the MNLS hierarchy. Conversely, given a solution $u(x,t)$ of the j-th flow, let $g : \mathbb{R}^2 \to U(n)$ be a solution of $g^{-1}g_x = u$ and $g^{-1}g_t = Q_j(u)$. Then $\gamma(x,t) = g(x,t)Jg(x,t)^{-1}$ is a solution of $((*)_j)$. Hence the Hamiltonian theory of the MNLS can be translated to that of the GNLS.*

6.9. **A sketch of the construction of τ functions.** Integrable systems (of the type of KdV) appear in conformal quantum field theory, and τ functions are the partition functions. A τ function is a function associated to a solution $u(\vec{t}\,)$. Terng found a definition due to Wilson which applies to all the systems under discussion. The second derivatives of u can be derived from τ; however, she also discovered that in the general case, the map from $u \to \tau$ is not injective. The definition of τ using second derivatives of u appears in the physics literature [AvdL03].

The general definition of τ for a system depends on a choice of central extension of the Lie group $\mathcal{G}$, which in turn depends on a skew symmetric bilinear form w on its Lie algebra $\mathfrak{G}$ which is compatible with the splitting $\mathcal{G} = \mathcal{G}^+\mathcal{G}^-$. The choice of the bilinear form for our example is

$$w(A,B) = \left(\frac{\partial}{\partial\lambda}A, B\right) = \sum_j j\,\mathrm{tr}(A_j, B_{-j}).$$

Here both $\mathfrak{G}_\pm$ are isotropic subspaces for w.

The central extension $\hat{\mathcal{G}}$ is the principal $\mathbb{C}\setminus 0$ bundle over $\mathcal{L}$ with first Chern class w. Because $\mathcal{G}^\pm$ are isotropic subspaces for w, the central extension is canonically trivial over $\mathcal{G}^\pm$, so there are canonical liftings of $\mathcal{G}^\pm$ to $\hat{\mathcal{G}}$. We use this to define Wilson's μ functional over the big cell Ω in which factorizations occur: For $c \in \Omega$, factor $c = f_+f_- = g_-g_+$ with $f_\pm, g_\pm \in \mathcal{G}^\pm$. Then $\tilde{f}_+\tilde{f}_-$ and $\tilde{g}_-$ and $\tilde{g}_+$ lies in the same fiber, where $\tilde{f}_\pm$ and $\tilde{g}_\pm$ are natural lifts of $f_\pm, g_\pm$. Hence they differ by a nonzero complex number.

Definition 6.9.1. For $c \in C$, define $\mu(c)$ as the difference between the canonical lifts given by two factorizations.

Definition 6.9.2 (Wilson). Let $E = MV_0f^{-1}$ represent a (partial) solution to an integrable system. Then the tau function defined by $f \in \mathcal{G}^-$ is

$$\tau_f(\vec{t}\,) = \mu(V_0f^{-1}).$$

The collective papers of Terng and myself work out many of the details. For example, we prove in [TU16] (i) the solution u_f is determined by the second derivatives of $\ln(\tau_f)$ for the the Gelfand-Dickey n-KdV hierarchy, and (ii) the second derivatives of $\ln\tau_f$ determines the solution u_f up to a conjugation by a constant in $U(k)\times U(n-k)$.

6.10. **The Virasoro action.** We found in the previous section that the τ function was in fact a function on the scattering data. The same is true for the Virasoro action. Again, we only give the prototype for the example we are using.

Definition 6.10.1. The Virasoro algebra is the real Lie algebra $\mathcal{V}$ with generators ξ_j and relations

$$[\xi_j, \xi_k] = (k-j)\xi_{k+j}.$$

The positive algebra $\mathcal{V}_+$ is the subalgebra generated by $j \geqq -1$.

It is an important fact that the Lie algebra of the conformal group of S^2 is generated by $\{\xi_{-1}, \xi_0, \xi_1\}$. It is best to think of $\mathcal{V}_+$ as generated by the operators

$$\xi_j = \lambda^{j+1}\frac{\partial}{\partial\lambda}, \quad j \geq -1.$$

Again, we only give the action on our particular choice of splitting.

Lemma 6.10.2. *The formulae for a representation of $\mathcal{V}_+$ on f is*

$$\delta_j f(\lambda)f^{-1}(\lambda) = \textit{negative powers of } (\lambda^j f'(\lambda)f^{-1}(\lambda)).$$

Note that we will be able to compute the induced representation on the flow variables from

$$E = MV_0f^{-1},$$
$$\delta_j E = \delta_j MM^{-1}E - E\delta_j ff^{-1}.$$

Theorem 6.10.3 ([TU16]). *The action of the positive half of the Virasoro algebra on the scattering data f induces an action on the flows. In many cases this can be shown to be via partial differential operators.*

7. Terng's Contributions

The previous section was an attempt to convey the ideas and techniques used in investigating integrable systems. Explaining in detail which piece is due to which mathematician would not make for interesting or enticing reading, although I have attempted to cite the major references. I have also cited a number of Terng's publications. Many of those are joint with me, but anybody familiar with me will reinforce my assertion that the bulk of the geometry is due to Chuu-Lian. We cite here some of her most important results.

Chuu-Lian served as AWM president [Ter22] from 1995–1997 and a co-organizer of the Women and Math Program at the Institute for Advanced Study in Princeton. Her influence on mathematics is evident in many ways.

The observation that the gauge transformation between the matrix nonlinear Schrödinger equation and the geometric nonlinear Schrödinger equation used by Fadeev-Tahkajen and Hasimoto is an application of moving frames and the details surrounding this is due to her, as is the worked out relationship between the modified wave map and the space-time monopole equation. In order to get geometric realization of Drinfel′d-Sokolov KdV-type systems associated to a noncompact simple Lie group G, she constructed curve flows on flat space with the symmetry group G. For example, affine curve flows on $\mathbb{R}^n$ with symmetry $\mathrm{SL}(n,\mathbb{R})$ are related to the n-KdV system ([TZ19]).

Terng made many contributions to the scheme of the DS flow described in the previous section. The details of applications to the various symmetric spaces are all hers. But the novel contribution is the extension of the 2×2 KdV flow to $n\times n$ KdV flows using the structure of the Lie group $\mathrm{SL}(n,\mathbb{C})$. We describe briefly the 2×2 KdV flow.

Here the major change is in the first step, the choice of the Lie groups. Instead of restricting the flows to the subgroups in which $g(\bar{\lambda})^\dagger g(\lambda) = I$, we introduce a reality condition on g:

$$G(g,\lambda) = \Phi(\lambda)g(\lambda)\Phi(\lambda)^{-1}G(g,\lambda) = G(g,-\lambda),$$

where $\Phi = \mathrm{diag}(1,-1)\lambda + e_{2,1}$.

Now the flows are generated by the vacuum sequence

$$V_0 = \exp\Big(\sum_{j\geq 0} J^{2j+1}t_j\Big),$$

where $J = \mathrm{diag}(1,-1)\lambda + e_{1,2}$.

The extension to $\mathrm{sl}(n,\mathbb{C})$ turns out to generate the Gelfand-Dickey or n-KdV flows. If we let $\alpha = e^{(2\pi i/n)}$, we similarly define a $G(g,\lambda)$ with a different choice of Φ, and require

$$G(g,\lambda) = G(g,\alpha\lambda).$$

Now the vacuum sequence will be

$$V_0 = \exp\Big(\sum_{j\neq 0 \bmod n} J^j t_j\Big),$$

with $J = \mathrm{diag}(1,\alpha,\dots,\alpha^{n-1})\lambda + \sum_{k=1}^{n-1} e_{(k,k+1)}$.

The definition of Φ is very complicated involving the roots of $\mathrm{sl}(n,\mathbb{C})$. See Theorem 2.1 of [TU11]. However, one can see from that description that it can be extended to loop groups constructed from other simple Lie algebras.

Theorem 7.0.1. *With the appropriate choice of Φ, the DS flows correspond to the Gelfand Dickey n-KdV integrable systems. In particular, the τ function and Virasoro actions correspond to the constructions of Segal-Wilson* [SW85].

This latter result is most satisfactory. The Segal-Wilson construction uses pseudo-differential operators on the line, and does not involve Birkhoff factorization. Note in these examples involving complicated splittings, the map from flows to τ functions is injective, meaning the τ functions determine the flow. As just commented, with some work, this construction may generalize to other Lie groups and root systems.

8. Interesting Questions and Open Problems

The approaches of Terng and her collaborators to integrable systems, when they are rigorous, involve rapidly decaying fields on the line (or $\mathbb{R}^2$). Any question involving periodic boundary conditions is intimately tied up with algebraic geometry. The formal construction of the flows is the same. However, the factorization

$$V_0 f^{-1} = M^{-1}E$$

cannot be made global so easily, even in specially chosen examples. This is not an open problem, but a problem which is addressed by a separate literature.

As we comment at the end of the previous section, the Drinfel′d-Sokolov scheme of Section 6 appears to have a good many abstract applications in geometry and an internal consistent structure derived from loop groups and Birkhoff factorization. The problems we discuss now are less abstract and have more to do with physical applications. The approach of integrable systems, with rapidly decaying initial data, has very little overlap with questions of analytic well-posedness. The goal of well-posedness is to prove existence and smooth dependence on initial data of the flows for small times t_j, $j > 1$ in Sobolev spaces of minimal regularity and decay, and then to separately investigate finite time blow-up and the behavior at infinity. To study well-posedness, it is necessary to choose a suitable gauge. An integrable system such as the space-time monopoles equation can look very different in different gauges [Czu10], [HY18]. There is also extensive work by authors such as Deift [Dei19], [DZ02] on numerics for Riemann Hilbert problems which is less familiar to me. It should be possible to combine these approaches into an organized scheme which takes advantages of what I might call the geometric, the analytic and the algebraic approaches to integrable systems.

It should not be impossible to approximate other nonlinear equations (i.e., harmonic maps in $1+2$) by integrable ones (i.e., Ward wave maps) and derive new approximation schemes for certain nonlinear equations using integrable equations. Much as we study nonlinear equations using their linearizations. However, I am not myself aware of any references.

My own favorite problem at the moment is the question of integrability for the Gross-Pitaevskii hierarchy, an infinite hierarchy of coupled linear inhomogeneous PDE appearing in the derivation of nonlinear Schrödinger from quantum many-particle systems. In fact, the entire hierarchy of flows appears in describing special solutions. Recently Mendelson, Nahmod, Pavlovic and Staffilini [MNPS19] have shown the existence of an infinite number of conservation laws, but as mentioned in the very beginning of Section 5, this is not equivalent to showing integrability. And in fact, it is not clear what integrability is. The key tool in their analysis is the existence of an invariant measure (due to Finetti). From the point of view of the integrable systems in this article, the measure itself would be the key ingredient to describing the geometric structure of the GP hierarchy. While the problem has interested many analysts, it cries out for the geometric interpretation present in Terng's work.

We return to a point addressed in Section 5. What is an integrable system? Certainly any system of the Drinfel'd-Sokolov type addressed in Section 6 is integrable. The modified wave map, monopoles and self-dual Yang-Mills are all equations with Lax pairs and posses at least some of the same properties, as do equations in higher dimensions which fail to be "algebraic" in the sense of algebraic geometry, but which also have special rigidity properties. Perhaps we should think of integrable as referring to the circle of ideas presented in this article, rather than having a precise definition (we mathematicians do love precision).

The final point is a philosophic one. Integrable systems are very specialized algebraically rigid equations. Why do they appear in so many subjects of mathematics? The systems considered by Terng and collaborators can most easily be described by the geometry connected with finite- or infinite-dimensional group theory. However, KdV arises in the study of shallow water waves, nonlinear Schrödinger arises in the description of waves in optical fibers and the equations of 2-dimensional gravity are integrable. Even more confusing, space-time monopoles and Ward wave maps are gauge equivalent. How should we regard the subject?

If we were describing fundamental particles in physics, we wouldn't be surprised if they corresponded to equations determined by the geometry of loop groups. And certainly there was a short period of time in which this was a hope. So I am not surprised that integrable systems are important in conformal field theory. But these are somewhat esoteric equations appearing in completely different areas of mathematics.

There are two wildly different guesses as to why this is. Maybe, in some Platonic sense, these equations are all that is available to our human consciousness. So in modeling phenomena, we mathematicians can only use what is there. The opposite view would be that these equations represent fundamental phenomena in the world we are trying to describe. Hence we will use them in many different applications to describe this same behavior.

The subject of integrable systems and Terng's work in it can be appreciated without knowing the answer.

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