# Notices 

## of the American Mathematical Society

April 2024
Volume 71, Number 4


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ERRATA. An erratum for "Differentiating by Prime
Numbers" [Jack Jeffries, 70 (2023), no. 11, 1772-1779] is now available athttps://www.ams.org/journa1s /notices/202403/cnoti-p1772.pdf.

On p. 301 of "A Word from... Rhonda Hughes" [71 (2024), no. 3], "its founding in 1939" should be "its founding in 1839."

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[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 121892046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society's street address and the other bracketed information is a technical requirement of the US Postal Service.
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## LETTERS TO THE EDITOR



## Letter to the Editor of Notices of the AMS

I was both surprised and disappointed to see the following claim published in the January 2024 issue of Notices of the American Mathematical Society, "Metacognition in the Mathematics Classroom:"

When we taught 82 middle school students in a youcubed summer camp we taught the students these strategies as they worked on open tasks. At the end of the four-week camp the students had increased their achievement on standardized tests by the equivalent of 2.8 years (Boaler et al, 2021, see 3).

Editors should have immediately questioned the validity of such a statement. The magnitude of the gain, representing an effect size of 0.91 standard deviations, defies belief. An analysis by NWEA estimates that US seventh graders will need an additional 5.9 months of learning in math to recover from the pandemic. The claim here is that more than four times that amount can be produced by 18 days of instruction. Matthew Kraft analyzed 747 randomized controlled trials of education interventions, considered a strong design for estimating causal effects, and calculated a mean effect size of 0.16 for both reading and math and 0.11 for math alone. The alleged summer camp gain is several times larger.

The study of the youcubed summer camp did not have a strong design. It didn't identify a control group for estimating learning gains. It featured non-random selection into the treatment, meaning that students were recruited for the summer camps. Those who showed up were in; those who didn't were out. The same four "tasks," created by the Mathematical Assessment Research Service, or MARS, were used for both pre- and post-test assessments. The pre-test was administered on the first day of summer camp and the post-test on the final day. Using the same four problems in such a short interval is a legitimate concern. Moreover, the publishers of MARS describe their assessments and tasks as "prototype materials" that "are still in draft and unpolished form," needing "further trialing before inclusion in a high stakes test."

[^0]For permission to reprint this article, please contact:
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I discuss additional weaknesses of the summer camp studies as part of a critique of the California Math Framework published in Education Next. Two developments subsequent to that publication are important for AMS readers to consider. First, the state board adopted an approved list of assessments that the state's charter schools can use for documenting learning gains. MARS tasks were evaluated but not approved. Second, researchers working for the state of California removed the claim of 2.8 years of summer camp growth from the version of the California State Math Framework ultimately adopted by the state board of education.

It's a pity that AMS Notices allowed this dubious claim to be repeated in its pages.

Tom Loveless

## Sad News: The Passing of Professor Bent Fuglede

Professor Bent Fuglede, University of Copenhagen, passed away on December 7, 2023. Almost to the end of his long life, he was in good shape and he was mathematically very active. In addition to his many important contributions to different parts of mathematics-harmonic analysis (the Fuglede conjecture, spectrum vs. tiling), potential theory/analysis (his long term focus), functional analysis, operator algebras (the Fuglede-Kadison determinant), operator theory (the commutator theorem)-he was also a member of the editorial board of the journal Expositiones Mathematicae from its start.
https://en.wikipedia.org/wiki/Bent_Fuglede
Regards,
Palle Jorgensen, Professor, University of Iowa

# Adverise in the : Notices 

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Levi L. Conant Prize-Winning Articles


[^1]
# A WORD FROM... 

David Manderscheid, Division Director, NSF Division of Mathematical Sciences



The opinions expressed here are not necessarily those of the Notices or the AMS.


I started as Division Director for the Division of Mathematical Sciences (DMS) at the National Science Foundation just over two years ago. I am writing to update you on some of the exciting things going on in the division and more generally at NSF.

DMS has a budget of roughly $\$ 250 \mathrm{M}$ a year, a scientific staff of roughly thirty, and an administrative staff of ten. Based on the reviews provided by panels and ad-hoc reviewers we make decisions to allocate resources in the best interests of the nation and the mathematical sciences. This is embodied in our mission: to support research and discovery in the mathematical sciences; to advance the mathematical sciences impact on society; and to build a vibrant mathematical sciences community. We do this in support of NSF's mission: to promote the progress of science by investing in research to expand knowledge in science, engineering, and education and to invest in actions that increase the capacity of the US to conduct and exploit such research. The overall budget of NSF is close to \$10B a year.

NSF has three priorities in support of its mission: promote discovery in science and engineering, accelerate technology and innovation, and advance diversity in science and engineering. The first has always been a priority. The second is a new thrust for NSF and is more about usedriven science as opposed to the curiosity-driven science

[^2]DOI: https://doi.org/10.1090/noti2911
that one more typically associates with NSF. The third is both about aligning diversity of the scientific community with the diversity of our nation and bringing more people into science. NSF has four cross-cutting themes for these priorities: advance emerging industries for national and economic security, build a resilient planet, create opportunities everywhere, and strengthen research infrastructure. DMS has a significant role to play here. Emerging industries, for example, include AI and Biotechnology where the link is clear. Building a resilient planet includes topics like modeling and uncertainty quantification for extreme events and climate science. Creating opportunities everywhere refers to bringing more people into the sciences, including mathematics and statistics, and then supporting their research. Infrastructure is mostly about large facilities such as telescopes, and there is less of a role here for the mathematical sciences.

DMS pays attention to these priorities and crosscuts to grow potential funding sources for the mathematical sciences while serving the national interest. For example, suppose we have a strong proposal that involves the use of AI to generate new mathematics. As AI is an emerging industry, we would have a good chance to get funds from outside DMS, sometimes up to half of the budget, to support the proposal. Assuming we are successful, that grows the pot of money available to support the mission of DMS. Similarly, if we can partner with another federal agency, such as we do with NIH and DOE, we can grow support for the mathematical sciences. It is important to note that other agencies also might support the mathematical sciences on their own. Indeed, while NSF is still the largest federal supporter of the mathematical sciences, the percentage of funding provided by NSF has shrunk over the years as support from other agencies has grown. Simply put, the goal of DMS is to wisely use taxpayer money to support national priorities and the mathematical sciences and to do so with other government agencies. We also work with private foundations. For example, we recently
partnered with the Biology Directorate here at NSF and the Simons Foundation to fund the National Institute for Theory and Mathematics in Biology at Northwestern with support of $\$ 50 \mathrm{M}$ over five years. The Simons Foundation is a great friend of the mathematical sciences. We partner with them regularly as they can often enhance or complement our funding portfolio.

The vast majority of the DMS budget goes to funding single principal investigators (PIs) or small groups of investigators in our core programs: Algebra and Number Theory; Analysis; Applied Mathematics; Computational Mathematics; Mathematical Biology; Probability, Combinatorics, and Foundations; Statistics; and Topology and Geometric Analysis. While the names of these groups have stayed constant over recent years, their funding portfolios have changed as research interests within the mathematical sciences community have shifted. The next largest portion of DMS funds goes to the Mathematical Sciences Research Institutes (MSRI) Program which is roughly $10 \%$ of our budget with funds going to six institutes: American Institute of Mathematics, Institute for Advanced Study, Institute for Computational and Experimental Research in Mathematics, Institute for Mathematical and Statistical Innovation, Institute for Pure and Applied Mathematics, and Simons Laufer Mathematical Sciences Institute (the institute formerly known as MSRI). We run an institute competition every five years in which the existing institutes must compete with proposals for new institutes if they want further funding. The competition for the most recent round had a closing date of March 14, 2024-yes, Pi Day. The institutes do an excellent job of bringing together mathematical scientists around emerging topics of interest and play a vital role for mathematical scientists who do not otherwise have support from NSF grants through support for visits to the institutes. They also help train the next generations of mathematical scientists through programs for students and visiting appointments for postdoctoral fellows.

As mentioned above, the scientific staff of program directors in DMS is roughly thirty people. Half of the program directors are permanent employees of NSF and half are rotators. Since rotators typically stay at NSF for two or three years before returning to their home institutions this means that in any given year, we hire an average of six rotators. Being a program director is a rewarding job. I encourage you to consider the possibility of becoming one if it is the right time in your career for you to do so. We expect applicants to have several years of successful independent research post PhD as normally would be expected of the academic rank of associate professor or higher. You do not need to be in academia currently, however. You could come from government, industry, or be emeritus, for example. It is very easy to apply, and we would be happy to talk to you about what the job entails. NSF gives rotators both time and resources to maintain their research
programs. When we hire rotators our first consideration is whether their area(s) of mathematical sciences expertise are a good fit for our needs. We also assess their ability to work as part of a team and help build a diverse collection of funded proposals that span the mathematical sciences. We are very proud of the diverse team of program directors that we have built. Just last year we reached gender parity in the group for the first time ever. We did this with my first year of hires, building on work started by my predecessor Dr. Juan Meza.

Diversity is also important in what we fund. We welcome the submission of proposals that include the participation of the full spectrum of diverse talent in STEM, e.g., as PI, co-PI, senior personnel, postdoctoral scholars, graduate or undergraduate students or trainees. This includes historically under-represented or underserved populations. It also includes diverse institutions including Minority-Serving Institutions (MSIs), Primarily Undergraduate Institutions (PUIs), and two-year colleges, as well as major research institutions. Proposals from EPSCoR (Established Program to Stimulate Competitive Research) jurisdictions are especially encouraged.

EPSCoR is an NSF-wide program that provides cofunding to requests from DMS for worthy proposals from PIs in 25 states, Guam, Puerto Rico, and the US Virgin Islands. ${ }^{1}$ These jurisdictions are the jurisdictions that receive the least amount of NSF funds, and the goal of the program is to increase NSF funding to these jurisdictions. The EPSCoR program is part of the opportunities everywhere cross-cut and increased funding to EPSCoR jurisdictions is a priority of Congress.

An opportunity unique to the Mathematical and Physical Sciences (MPS) Directorate, of which DMS is one of five divisions, is the LEAPS (Launching Early-Career Academic Pathways) program. The funds for this program come from MPS. The program is aimed at assistant professors at institutions that traditionally do not receive significant NSF funding, such as minority-serving, predominantly undergraduate or R2 Carnegie Classification institutions. LEAPS proposals require strong proposed research and particularly strong proposed efforts in broadening participation. Likewise, ASCEND (Ascending Postdoctoral Research Fellowship) is an MPS program for postdocs with similarly balanced research/broader impact profiles. New this year in ASCEND is a funding program that ASCEND fellows apply to to help them transition to tenure-track faculty positions. It provides them with resources for research and broadening participation activities that are in addition to initial resources typically provided through institutional start-up packages.
https://new.nsf.gov/funding/initiatives/epscor/epscor -criteria-eligibility

Enhancing diversity also means funding conferences and other activities that place an emphasis on bringing members of groups underrepresented or underserved in the mathematical sciences into the field. Finally, a new program in DMS is PRIMES (Partnerships for Research and Innovation in the Mathematical Sciences). In this program faculty members from MSIs that are not R1 in the Carnegie Classification apply for support for visits to one of the MSRIs and release time to work on research at their home institution. The goal of this program is to raise the level of research at the MSI through leveraging our investment in the MSRIs. The grant is made to the MSI. The response to this solicitation has been strong, and we funded six awards last fiscal year.

We also have several other relatively new solicitations. One is the IHBEM (Incorporating Human Behavior in Epidemiological Models) solicitation. This is a joint venture with the Directorate for Social, Behavioral, and Economic Sciences here at NSF and the National Institute on Drug Abuse at NIH. As the title and collaboration suggest, this program brings together mathematical scientists and social scientists to build and apply epidemiological models that incorporate human behavior, for example, vaccine hesitancy. This program has proven of interest to both the scientific community and policy makers.

In addition, we have been funding workshops to gauge interest in the community in emerging areas of research. In October of 2022 we sponsored the SIAM Convening on Climate Science, Sustainability, and Clean Energy. This addressed the crosscut of building a resilient planet. We partnered with the Department of Defense, Department of Energy, and various parts of the National Science Foundation on the NASEM (National Academies of Science, Engineering, and Medicine) Consensus Study on Foundational Research Gaps and Future Directions for Digital Twins. We also partnered with the Computer and Information Sciences and Engineering (CISE) directorate on workshops on Artificial Intelligence for Mathematical Reasoning. These efforts, and others, help us bring into focus potential future funding efforts for DMS. We are also involved in a series of workshops being run by CISE around accelerating computer enabled discovery.

In all potential future funding initiatives for DMS we consider what is best for the nation and the mathematical sciences, including our core programs. Critical to our consideration is to make sure that we are not viewed as just a service discipline. New initiatives must lead to the advancement of the mathematical sciences and the mathematical sciences workforce consistent with our mission and that of NSF. That means that sometimes we take the lead and other times we are partners. It is often said that mathematics is the language of science. As such we can interact with all parts of the NSF and thus, we strive to be everywhere, just like the mathematical sciences.

If you have any questions or comments, I welcome your email. I can be reached at dmanders@nsf.gov.


International Congress of Mathematics (ICM) July 23- July 30, 2026
Pennsylvania Convention Center Philadelphia, Pennsylvania

Calls for ICM 2026 Satellite Events Proposals
On behalf of the International Mathematical Union, we invite and encourage proposals for "Satellite Conferences" and other possible events to be associated with the upcoming International Congress of Mathematicians (ICM). We hope to encourage a blossoming of activity across the continent to amplify the impact of the ICM.

Such conferences and events should take place within a couple of months of the ICM itself and be held in North America. Please see our webpage at https://www.icm2026.org for submission procedures and criteria for selection. Proposed conferences should involve an international perspective on some aspect of the mathematical sciences. Diversity of participants is strongly encouraged. Organizers should seek sponsorship/funding for their events, as we cannot provide financial support for satellite activities.

ICM 2026 Committee on Satellite Events

# The Quintic, the Icosahedron, and Elliptic Curves 

## Bruce Bartlett

There is a remarkable relationship between the roots of a quintic polynomial, the icosahedron, and elliptic curves. This discovery is principally due to Felix Klein (1878), but Klein's marvellous book [9] misses a trick or two, and doesn't tell the whole story. The purpose of this article is to present this relationship in a fresh, engaging, and concise way. We will see that there is a direct correspondence between:

- "Evenly ordered" roots ( $x_{1}, \ldots, x_{5}$ ) of a Brioschi quintic

$$
\begin{equation*}
X^{5}+10 B X^{3}+45 B^{2} X+B^{2}=0, \tag{1}
\end{equation*}
$$

- Points on the icosahedron, and
- Elliptic curves equipped with a primitive basis for their 5-torsion, up to isomorphism.
Moreover, this correspondence gives us a very efficient direct method to actually calculate the roots of a general quintic! For this, we'll need some tools both new and old, such as Cremona and Thongjunthug's complex arithmetic geometric mean [3], and the Rogers-Ramanujan continued fraction [5, 12]. These tools are not found in Klein's book, as they had not been invented yet!

If you are impatient, skip to the end to see the algorithm.

If not, join me on a mathematical carpet ride through the mathematics of the last four centuries. Along the way we will marvel at Kepler's Platonic model of the solar system from 1597, witness Gauss' excitement in his diary entry from 1799, and experience the atmosphere in Trinity College Hall during the wonderful moment Ramanujan burst onto the scene in 1913.

For the approach I present here, I have learnt the most from Klein's book itself together with the new introduction and commentary by Slodowy [9], as well as [2, 5, 8, 11].

[^3]
## Arnold's Topological Proof of the Unsolvability of the Quintic

We are all familiar with the formula for the roots $x_{1}, x_{2}$ of a quadratic polynomial $X^{2}+a X+b=0$, namely

$$
\begin{equation*}
x=\frac{-a \pm \sqrt{a^{2}-4 b}}{2} . \tag{2}
\end{equation*}
$$

Perhaps we are also familiar with Cardano's formula (1545) for the roots of a cubic polynomial $X^{3}+a X+b=0$,

$$
\begin{equation*}
x=c-\frac{a}{3 c}, \quad c=\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^{2}}{4}+\frac{a^{3}}{27}}} . \tag{3}
\end{equation*}
$$

There is a similar formula for the roots of a quartic polynomial, due to Ferrari (1545). We say formulas like (2) and (3) express the roots of a polynomial in terms of radicals, since the only ingredients necessary are the usual algebraic operations $(+,-, \cdot, /)$ and extraction of $n$th roots.

It is also commonly known that Ruffini (1799) and Abel (1824) showed that there is no such radical formula for the roots $x_{i}$ of a general quintic equation. The standard modern way to understand these results is the algebraic framework of Galois (1832). Namely, we associate to a specific polynomial

$$
\begin{equation*}
P=X^{n}+a_{1} X^{n-1}+a_{2} X^{n-2}+\cdots+a_{n-1} X+a_{n} \tag{4}
\end{equation*}
$$

a finite group $G$, the Galois group of $P$, which is a certain subgroup of the group $S_{n}$ of permutations of the roots $x_{1}, \ldots, x_{n}$ of $P$. Galois showed that there is a radical formula for $x_{i}\left(a_{1}, \ldots, a_{n}\right)$ if and only if $G$ is a solvable group (a tower built from iteratively stacking finite cyclic groups on top of each other, i.e., "built from epicycles" as Ptolemy might have put it). Now, the Galois group of a general quintic is $S_{5}$, which is not solvable. Therefore, there is no radical formula for the roots of a general quintic.

Galois' approach is elegant but requires a semester's worth of abstract algebra to understand. In 1963, the Russian mathematician Vladimir Arnold gave an alternative topological proof of the unsolvability of the quintic in a series of lectures to high school kids in Moscow. In Arnold's approach, instead of focusing on finding an algebraic formula for the roots of a specific polynomial $P$ as in (4), one considers the collection of all polynomials of degree $n$ as
a topological space:

$$
\begin{aligned}
\text { Poly }_{n}=\left\{\text { polynomials } X^{n}\right. & +a_{1} X^{n-1}+\cdots \\
& \left.+a_{n-1} X+a_{n}, a_{1}, \ldots, a_{n} \in \mathbb{C}\right\} .
\end{aligned}
$$

To each polynomial $P \in$ Poly $_{n}$ we may associate the unordered set $\left\{x_{1}, \ldots, x_{n}\right\}$ of its roots, so that we have a covering space

$$
\begin{aligned}
\text { Roots } & \rightarrow \text { Poly }_{n} \\
\left\{x_{1}, \ldots, x_{n}\right\} & \mapsto \prod_{i=1}^{n}\left(X-x_{i}\right)
\end{aligned}
$$

which is branched over the discriminant locus $D \subset$ Poly $_{n}$ of polynomials with multiple roots, and is an $S_{n}$-principal bundle over the complement $\mathcal{P}_{n}=$ Poly $_{n} \backslash D$.

If we start at some fixed basepoint polynomial $P_{0} \in \mathcal{P}_{n}$, and move along a path in $\mathcal{P}_{n}$, the roots of the polynomial move around (see Figure 1). If we loop back to $P_{0}$, then they will have undergone a permutation. We have established a monodromy map

$$
\begin{equation*}
\pi_{1}\left(\mathcal{P}_{n}, P_{0}\right) \rightarrow S_{n} \tag{5}
\end{equation*}
$$

which in fact classifies the covering space $\mathcal{P}_{n}$.
(Note how this approach is more geometric than that of Galois. Instead of caring only about the permutations of the roots, we also care about the journey they undertook to accomplish that permutation.)

Arnold's insight was to show that if there is a radical formula for the roots of a general polynomial, then the "dance of the roots" cannot be overly complex, in the sense that the image of the monodromy map must be a solvable subgroup of $S_{n}$. But, for example, the 1-parameter Brioschi family of quintics

$$
P_{B}(X)=X^{5}+10 B X^{3}+45 B^{2} X+B^{2}, \quad B \in \mathbb{C}
$$

has monodromy group $A_{5}$ (see Figure 1), which is certainly not solvable since it is simple, as we will see by relating it to the icosahedron in the next section. Hence the unsolvability of the quintic.

In fact, after some algebraic manipulations involving at most two square roots [2, Theorem 6.6], finding the roots of the general quintic

$$
\begin{equation*}
X^{5}+a_{1} X^{4}+a_{2} X^{3}+a_{3} X^{2}+a_{4} X+a_{5}=0 \tag{6}
\end{equation*}
$$

reduces to finding the roots of the Brioschi quintic ${ }^{1} P_{B}$ for a certain $B \in \mathbb{C}$.

[^4]

Figure 1. When the Brioschi parameter loops around $B=0$ along the blue loop $\sigma$ as shown on the left, the roots undergo the cyclic permutation (14352) as shown on the right. In a similar way, when $B$ loops around $B=-1 / 1728$ along the red loop $\gamma$, the roots undergo the cyclic permutation (153). These permutations generate $A_{5}$.

## Enter the Icosahedron

Now for a wonderful fact: we will show that there is a natural correspondence between the set of "evenly ordered" d5-tuples ( $x_{1}, \ldots, x_{5}$ ) of roots of a Brioschi quintic $P_{B}$, and the set of points on the icosahedron!

To understand this, recall that the icosahedron

$$
I \subset \mathbb{R}^{3}
$$

that most enigmatic of the Platonic solids, has 30 edges, 20 equilateral faces, and 12 vertices. If we make the identification $\mathbb{R}^{3}=\mathbb{C} \times \mathbb{R}$, we can take the vertices to be situated at:

$$
\begin{equation*}
(0, \pm 1), \quad \pm \frac{1}{\sqrt{5}}\left(-2 \zeta^{i}, 1\right), \quad \zeta=e^{\frac{2 \pi i}{5}} \tag{7}
\end{equation*}
$$

Consider the group $G$ of rotational symmetries of $I$. Each nonidentity $g \in G$ is a rotation about an axis through an antipodal pair of edge midpoints, face midpoints, or vertices of $I$, with order 2,3 , and 5 respectively. In fact, $G$ is naturally isomorphic to $A_{5}$, the group of even permutations of 5 things. What are these 5 things that are being evenly permuted when we rotate the icosahedron? They are the 5 inscribed octahedra which have their vertices on the edge midpoints of $I$ ! See Figure 2.

This natural isomorphism between $G$ and $A_{5}$ gives us a nice way to see that $A_{5}$ is simple (and hence not solvable). This is because $G$ is simple: if a normal subgroup $N$ contains a rotation about an axis through a vertex $v$, then it contains rotations about all axes passing through vertices (since it is closed under conjugation). But a rotation about an edge midpoint equals the product of the rotations about the three vertices in a triangle adjacent to it, while a rotation about a face midpoint equals the product of the rotations about the two vertices in an edge adjacent to it. So if $N$ contains a rotation about $v$, it must be all of $G$, and similarly for edge midpoints and face midpoints. So, $G$ is simple, and therefore $A_{5}$ is simple.


Figure 2. One of the 5 inscribed octahedra in the icosahedron. On the right, Kepler's view of planetary orbits as inscribed Platonic solids from Mysterium Cosmographicum (1596).

## Invariant Polynomials

Let $S^{2}$ denote the 2-dimensional unit sphere. Since $G$ is a group of rotational symmetries, it acts on $S^{2}$. Our goal in this section is to understand the quotient space $S^{2} / G$, which we can think of as the "moduli space" of points on the "round icosahedron" (the soccer ball version of $I$, obtained by inflating $I$ outward onto the sphere $S^{2}$; see Figure 3 ). We are going to need a toolbox of $G$-invariant functions on $S^{2}$.

To write down such functions, we need to keep in mind that $S^{2}$ has the structure of a Riemann surface, since we can identify it with the complex projective plane

$$
\mathbb{C P}^{1}=\left\{1 \text {-dimensional linear subspaces of } \mathbb{C}^{2}\right\}
$$

via stereographic projection from the north pole,

$$
\begin{aligned}
S^{2} & \xlongequal{\rightrightarrows} \mathbb{C} \cup\{\infty\} \\
(a, b, c) & \mapsto \frac{a+b i}{1-c},
\end{aligned}
$$

followed by the identification

$$
\begin{aligned}
\mathbb{C} \cup\{\infty\} & \stackrel{\cong}{\leftrightarrows} \mathbb{C} \mathbb{P}^{1} \\
z & \mapsto \mathbb{C}(z, 1) \\
\infty & \mapsto \mathbb{C}(1,0) .
\end{aligned}
$$

In what follows, I will freely use these identifications; my preference is to use the $S^{2}$ picture because I want the visual image of the icosahedron in $\mathbb{R}^{3}$ to be front and center.

Under this identification, the $\mathrm{SO}(3)$ rotation action on $S^{2}$ translates into an $\mathrm{SO}(3)$ action on $\mathbb{C P}^{1}$, which can be explained as arising from the natural action of its double cover $\operatorname{SU}(2)$ on $\mathbb{C}^{2}$. We define the binary icosahedral group $\hat{G} \subset S U(2)$ as the double cover of the icosahedral group $G \subset S O(3)$. So, we seek $\hat{G}$-invariant homogenous polynomials on $\mathbb{C}^{2}$.

We have the vertex polynomial $V$ (of degree 12, vanishing on the 1 -dimensional subspaces corresponding to icosahedron vertices), the face polynomial $F$ (of degree 20, vanishing on the 1 -dimensional subspaces corresponding to icosahedron face midpoints), and the edge polynomial


Figure 3. The round icosahedron and its stereographic projection. The edge midpoint $\alpha$ and face midpoint $\beta$, together with their stereographic projections $a$ and $b$, have been marked for later use.
$E$ (of degree 30, vanishing on the 1-dimensional subspaces corresponding to icosahedron edge midpoints):

$$
\begin{align*}
V= & u v\left(u^{10}+11 u^{5} v^{5}-v^{10}\right)  \tag{8}\\
F= & -u^{20}+228\left(u^{15} v^{5}-u^{5} v^{15}\right)-494 u^{10} v^{10}-v^{20}  \tag{9}\\
E= & u^{30}+522\left(u^{25} v^{5}-u^{5} v^{25}\right) \\
& -1005\left(u^{20} v^{10}-u^{10} v^{20}\right)+v^{30} \tag{10}
\end{align*}
$$

To see that these polynomials are indeed $\hat{G}$-invariant (instead of picking up phase factors when acting with $g \in \hat{G}$ ), it helps to realize that $G$ is generated by $R_{z}$ and $R_{y}$, the rotations about the $z$ - and $y$-axis by angles $2 \pi / 5$ and $\pi$ respectively. We can take their preimages in $\mathrm{SU}(2)$ to be

$$
\hat{R}_{z}=\left(\begin{array}{cc}
e^{\frac{\pi i}{5}} & 0 \\
0 & e^{-\frac{\pi i}{5}}
\end{array}\right), \quad \hat{R}_{y}=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)
$$

which similarly generate $\hat{G}$, and which act on homogenous polynomials in $u, v$ as:

$$
\begin{equation*}
\hat{R}_{z}: u \mapsto e^{-\frac{\pi i}{5}} u, v \mapsto e^{\frac{\pi i}{5}} v, \quad \hat{R}_{y}: u \mapsto v, v \mapsto-u \tag{11}
\end{equation*}
$$

It is clear that our polynomials (8)-(10) are invariant under these transformations, so they are indeed $\hat{G}$-invariant. In fact, they generate the algebra of $\hat{G}$-invariant homogenous polynomials on $\mathbb{C}^{2}$.

We can play the same game with our 5 inscribed octahedra. Let $O_{i}$ be the vertex polynomial of the $i$ th inscribed octahedron. It has degree 5 and vanishes at the 1 -dimensional subspaces of $\mathbb{C}^{2}$ corresponding to the 8 vertices of $O_{i}$. We compute

$$
O_{1}=\left(u^{2}+v^{2}\right)\left(u^{2}-2 n u v-v^{2}\right)\left(u^{2}-2 m u v-v^{2}\right)
$$

where $m=\zeta+\zeta^{4}=\frac{1}{\phi}$ and $n=\zeta^{2}+\zeta^{3}=-\phi$, with the other $O_{i}, i=2 \ldots 5$ obtained by simply rotating $O_{1}$ around the $z$-axis using the $\hat{R}$-action in (11).

## The Icosahedron and the Quintic

When we rotate the icosahedron, the 5 inscribed octahedra and hence their vertex polynomials $O_{i}$ undergo an (even)
permutation. Consider the quintic

$$
\prod_{i=1}^{5}\left(X-O_{i}\right)
$$

whose coefficients are symmetric polynomials in the octahedral polynomials $O_{i}$ and hence $\hat{G}$-invariant polynomials on $\mathbb{C}^{2}$. If we multiply it out, we obtain a Brioschi-type quintic

$$
X^{5}-10 V X^{3}+45 V^{2} X-E
$$

as the reader will verify! (The coefficients of $X^{4}$ and $X^{2}$ are invariant polynomials of degree 6 and 18 respectively and hence must vanish, while the coefficients of $X^{3}, X$ and 1 are of degree 12, 24, and 30 and hence must be multiples of $V, V^{2}$, and $E$ respectively.)

Let us make this correspondence between points on the icosahedron and ordered roots of a Brioschi quintic clear and precise. Consider the rationalized octahedral functions

$$
x_{i}=-\frac{V^{2}}{E} O_{i}, \quad i=1 \ldots 5
$$

They are of degree zero in $u$ and $v$. Therefore, away from the edge midpoints, they are well-defined complex-valued functions on $S^{2}$. In other words, to each point $z$ on the "round icosahedron" $S^{2}$ (excluding edge midpoints), we can associate an ordered tuple $\left(x_{1}(z), \ldots, x_{5}(z)\right)$ of 5 complex numbers! We map these 5 numbers to the quintic

$$
\begin{equation*}
P_{\left\{x_{i}\right\}}=\prod_{i=1}^{5}\left(X-x_{i}(z)\right) \tag{12}
\end{equation*}
$$

thereby forgetting their ordering. If we multiply out this quintic, we find that it is a Brioschi quintic

$$
P_{B}=X^{5}+10 B X^{3}+45 B^{2} X+B^{2}
$$

with Brioschi parameter $B(z)=-V^{5} / E^{2}$ ! See Figure 4. In summary, we have:

Theorem 1. The map

$$
z \mapsto\left(x_{1}(z), \ldots, x_{5}(z)\right)
$$

is an $A_{5}$-equivariant bijection between points on the round icosahedron (minus edge midpoints) and evenly ordered (defined below) roots of Brioschi quintics $P_{B}$ with Brioschi parameter $B(z)=-V^{5} / E^{2}$. In other words, it is an explicit equivariant isomorphism of covering spaces:



Figure 4. The Brioschi parameter $B(z)$ on $S^{2}$. The absolute value maps to the brightness, while the argument maps to the hue. Note the zeros of $B$ at the vertices of the icosahedron, and the poles at the edge midpoints.

The only subtlety here is the notion of an "evenly" ordered set of roots of a Brioschi quintic $P_{B}$. We need this because there are 60 points on a generic orbit $A_{5} \cdot z$ in the icosahedron, while there are 120 ways to order the 5 roots $x_{1}, \ldots, x_{5}$ of the quintic $P_{B}$. We call an ordering of the roots of a generic Brioschi quintic $P_{B}$ even if, as $B$ approaches 0 through positive real values (which implies that $z$ approaches a vertex of the icosahedron) and track the roots continuously, the roots end up in an even permutation of the numbering shown in Figure 1.

Theorem 1 tells us that in order to find the roots of a Brioschi quintic $P_{B}$ for some Brioschi parameter $B \in \mathbb{C}$, we need to find a point $z$ on the icosahedron such that $-V^{5}(z) / E^{2}(z)=B$. Then our 5 ordered roots will be given by the 5 octahedral numbers $x_{i}(z)$. Actually calculating $z$ in terms of $B$ is hard (we need to solve a polynomial equation of degree 60), but it can be done efficiently using elliptic curves, as we will see shortly!

## A Polynomial Relation

But first, let's return to the subject of invariant polynomials on the icosahedron for a moment, as there is an important relation between $V, E$, and $F$ which we will need later. The point is that these are polynomials on $\mathbb{C}^{2}$, and three polynomials on a 2-dimensional space must satisfy a relation. The first opportunity for this relation to occur is in degree 60, and indeed we have:

$$
\begin{equation*}
E^{2}+F^{3}=1728 V^{5} \tag{13}
\end{equation*}
$$

This reminds us of modular forms, where the Eisenstein series satisfies

$$
E_{6}^{2}-E_{4}^{3}=1728 \Delta
$$

This is the first clue that the icosahedron has something to do with moduli spaces of elliptic curves. But for now, what we need to get out of (13) is that it tells us that

$$
B(z)=-\frac{V^{5}}{E^{2}}=-\frac{V^{5}}{1728 V^{5}-F^{3}}
$$

so that $B$ sends vertices to 0 , edge midpoints to $\infty$, and face midpoints to $-1 / 1728$. This uniquely characterizes it as a holomorphic map from $S^{2}$ to $S^{2}$.

## Enter Elliptic Curves

Expressing a number "in terms of radicals" implies having access to the roots of unity, i.e., the $n$-torsion points (points $p$ such that $n p=0$ ) in the nonzero complex numbers $\mathbb{C}^{*}$, thought of as an additive abelian group. In the nineteenth century, mathematicians discovered that the set of points on an elliptic curve, i.e., the set $E$ of complex solutions to a cubic equation of the form

$$
\begin{equation*}
y^{2}=4 x^{3}-g_{2} x-g_{3} \tag{14}
\end{equation*}
$$

also forms an abelian group (once one works projectively). It was natural to speculate that, while the roots of a quintic could not be expressed in terms of $n$-torsion points on the circle, perhaps they could be expressed in terms of the 5 -torsion points of an elliptic curve somehow associated with the quintic. Remarkably, this is precisely what Hermite (1858) and Kiepert (1878) managed to do! To quote McKean and Moll [10]:

In this way, the solution of the general equation of degree 5 is made to depend upon the equations for the division of periods of the elliptic functions, as they used to say.
Hermite and Kiepert worked with the Bring-Jerrard form of the quintic, and their final expression for the roots of the quintic in terms of the 5 -torsion points of an elliptic curve is, to this humble author, a bit convoluted and indirect. I will present my own streamlined and modernized form of Klein's approach (1878) [9]. We will see that the 60 evenly ordered 5 -tuples of roots ( $x_{1}, \ldots, x_{5}$ ) of a Brioschi quintic directly correspond to the 60 equivalence classes of primitive bases (not points themselves) for the 5-torsion points of an elliptic curve!

## Moduli Spaces of Elliptic Curves

In the previous section we found that the icosahedron is a 60 -sheeted $A_{5}$ equivariant covering space of $S^{2}$ via the Brioschi map $B$. And moreover, we found that away from the edge midpoints, this covering space is explicitly isomorphic to the covering space of evenly ordered roots of Brioschi quintics.

Besides the icosahedron, there is another 60 -sheeted $A_{5}$ equivariant covering space of $S^{2}$ occurring in nature: the moduli space $X(5)$ of elliptic curves equipped with a primitive basis for their 5-torsion!

Recall that an elliptic curve $E \subset \mathbb{C P}^{2}$ given by a cubic equation as in (14) identifies holomorphically with $\mathbb{C} / \Lambda$, the quotient of $\mathbb{C}$ by some rank 2 lattice $\Lambda \subset \mathbb{C}$. So, topologically, $E$ looks like a doughnut. Moreover, under this identification the addition operation on $E$ is just the standard addition in $\mathbb{C} / \Lambda$. Therefore,

## 5-torsion points of $E \cong \mathbb{Z} / 5 \mathbb{Z} \times \mathbb{Z} / 5 \mathbb{Z}$.

Let $p, q$ be 5 -torsion points in $\mathbb{C} / \Lambda$ and let $\omega_{1}, \omega_{2} \in$ $\mathbb{C}, \operatorname{Im}\left(w_{1} / w_{2}\right)>0$ be generators of $\Lambda$. Since the equivalence classes $\left[\omega_{1} / 5\right],\left[\omega_{2} / 5\right]$ generate the 5 -torsion of $\mathbb{C} / \Lambda$, we can write

$$
\begin{aligned}
p & =a_{1}\left[\omega_{1} / 5\right]+a_{2}\left[\omega_{2} / 5\right] \\
q & =b_{1}\left[\omega_{1} / 5\right]+b_{2}\left[\omega_{2} / 5\right]
\end{aligned}
$$

for some matrix

$$
\gamma \in\left(\begin{array}{ll}
a_{1} & a_{2} \\
b_{1} & b_{2}
\end{array}\right) \in \mathrm{GL}_{2}(\mathbb{Z}, \mathbb{Z} / 5 \mathbb{Z}) .
$$

We say that $(p, q)$ is a primitive basis for the 5 -torsion of $E$ if $\operatorname{det} \gamma \equiv 1 \bmod 5$. Write

$$
X(5)=\{(E,(p, q))\} / \sim
$$

for the set of equivalence classes ("moduli space") of pairs $(E,(p, q))$ where $E$ is a complex elliptic curve and $(p, q)$ is a primitive basis for its 5 -torsion (see [4]). Two such pairs $(E,(p, q))$ and $\left(E^{\prime},\left(p^{\prime}, q^{\prime}\right)\right)$ are equivalent if there is an isomorphism $E \rightarrow E^{\prime}$ which carries $(p, q) \mapsto\left(p^{\prime}, q^{\prime}\right)$.

Write $X(1)$ for the "vanilla" moduli space of elliptic curves (no extra torsion information tagged on), and $\mathbb{H}=$ $\{\tau \in \mathbb{C}: \operatorname{Im}(\tau)>0\}$ for the upper half plane. Thinking of an elliptic curve as a quotient $\mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} \tau$ for some $\tau \in \mathbb{H}$, we can identify these moduli spaces as:

$$
X(1) \cong \mathbb{H}^{*} / S L(2, \mathbb{Z}), \quad X(5) \cong \mathbb{H}^{*} / \Gamma(5)
$$

Here, $\mathbb{H}^{*}=\mathbb{H} \cup \mathbb{Q} \cup\{i \infty\}$ is the extended upper half plane (the extra "cusp" points $\mathbb{Q} \cup\{i \infty\}$ are needed to get a compact moduli space; they contribute a single point to the quotient in $X(1)$ and 12 points in $X(5)$ ) and

$$
\begin{aligned}
& \Gamma(5)=\left\{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \operatorname{SL}(2, \mathbb{Z}):\right. \\
& \left.\qquad\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \equiv\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), \bmod 5\right\}
\end{aligned}
$$

is the principal congruence subgroup of level 5. (The congruence relation is done independently entrywise, so the requirement is that $a \equiv 1(\bmod 5), b \equiv 0(\bmod 5), c \equiv 0$ $(\bmod 5)$ and $d \equiv 1(\bmod 5))$.

## Permutation Wizardry

Now, $\Gamma(5)$ is a normal subgroup of $\Gamma:=\operatorname{SL}(2, \mathbb{Z})$, and so we can form the quotient group

$$
\Gamma / \Gamma(5) \cong \operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z}) .
$$

The magic is that $\operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z})$ is isomorphic to $A_{5}$ ! We will need to understand this isomorphism explicitly in terms of the action of $A_{5}$ on the five inscribed octahedra ${ }^{2}$.

[^5]The vertices of an inscribed octahedron $O_{i}$ are located at edge midpoints of the icosahedron $I$. Therefore, $O_{i}$ partitions the vertices of $I$ into pairs, and hence encodes a fixedpoint free involution of the 12 vertices of $I$. This involution commutes with the map $p \rightarrow-p$, and so we conclude that each inscribed octahedron $O_{i}$ encodes (and is in fact encoded by) a fixed-point free involution $o_{i}$ of the 6 vertex axes of $I$.

On the other hand, $\operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z})$ acts naturally on projective space

$$
\mathbb{P}^{1}(\mathbb{Z} / 5 \mathbb{Z})=\{0,1,2,3,4, \infty\},
$$

which also consists of 6 things. So, let us identify the six vertex axes of the icosahedron in $\mathbb{R}^{3}$ with $\mathbb{P}^{1}(\mathbb{Z} / 5 \mathbb{Z})$ in the natural way:

$$
\begin{aligned}
\mathbb{R}\left(-2 \zeta^{k}, 1\right) & \mapsto k, \quad k=0 \ldots 4 \\
\mathbb{R}(0,0,1) & \mapsto \infty .
\end{aligned}
$$

In this way $\operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z}) \subset S_{6}$ acts by conjugation on the 5 octahedral involutions $o_{i} \in S_{6}$, and it turns out that it permutes them evenly, giving our isomorphism $\operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z}) \cong A_{5}$.

Let us pre- and post-compose this isomorphism with the natural isomorphisms $\Gamma / \Gamma(5) \cong \operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z})$ and $A_{5} \cong$ $G$ and record the result explicitly for later use.

Lemma 1. The explicit isomorphism

$$
\Gamma / \Gamma(5) \rightarrow G,
$$

at the level of generators, sends

$$
\begin{aligned}
T & \mapsto R_{z} \\
S & \mapsto R_{\alpha}
\end{aligned}
$$

where $T(\tau)=\tau+1$ and $S(\tau)=-\frac{1}{\tau^{\prime}}$ and $R_{z}(z)=\zeta z$ and $R_{\alpha}(z)=\frac{n z+1}{z-n}$ are rotations by $\frac{2 \pi}{5}$ and $\pi$ counterclockwise about the positive $z$-axis and the axis through the edge midpoint $\alpha$ shown in Figure 3 respectively.

## An Isomorphism of Covering Spaces

The isomorphism $\operatorname{PSL}(2, \mathbb{Z} / 5 \mathbb{Z}) \cong A_{5}$ means that the forgetful map

$$
\pi: X(5) \rightarrow X(1)
$$

is an $A_{5}$-equivariant 60 -sheeted covering space of $X(1)$. Note that we can also make our own direct count of the sheets in the covering. Given an elliptic curve $E \in X(1)$, there are $480=24 \times 20$ pairs ( $p, q$ ) which form a basis for the 5 -torsion, since the only constraint is that $p \neq 0$ and $q \notin\{0, p, 2 p, 3 p, 4 p\}$. Of these 480 pairs, exactly 120 will be a primitive basis. If $E$ is a generic elliptic curve, then we must also account for its solitary nontrivial automorphism $p \mapsto-p$. So there are 60 points in a generic fiber.

Now, the moduli space $X(1)$ of elliptic curves identifies with $S^{2}$,

$$
j: \mathbb{M}^{*} / S L(2, \mathbb{Z}) \xrightarrow{\cong} S^{2},
$$

via the $j$-invariant of an elliptic curve:

$$
\begin{aligned}
j(\tau) & =\frac{1728 g_{2}^{3}}{g_{2}^{3}-27 g_{3}^{2}} \\
& =\frac{1}{q}+744+196884 q^{2}+\cdots
\end{aligned}
$$

Here, $q=e^{2 \pi i \tau}$ and $g_{2}$ and $g_{3}$ are the coefficients appearing in the equation (14) for an elliptic curve $E$. If we identify $E$ with the quotient of $\mathbb{C}$ by a lattice $\Lambda_{\tau}=\mathbb{Z} \oplus \mathbb{Z} \tau$, where $\tau \in \mathbb{H}$, then they are computed in terms of $\tau$ as

$$
g_{2}=60 \sum_{\omega \in \Lambda_{\tau}}^{\prime} \frac{1}{\omega^{4}}, \quad g_{3}=140 \sum_{\omega \in \Lambda_{\tau}}^{\prime} \frac{1}{\omega^{6}},
$$

where the primes on the sums means leaving out $\omega=0$ from the sum.

Ok, so now we know that $X(5)$ is an $A_{5}$-equivariant covering space of $S^{2}$. If $E_{j}$ is an elliptic curve with invariant $j \in S^{2}$, then the number of points in the fiber is given by

$$
\left|\pi^{-1}(j)\right|=\frac{120}{\left|\operatorname{Aut}\left(E_{j}\right)\right|}
$$

as we explained above. A generic elliptic curve has automorphism group $\mathbb{Z} / 2 \mathbb{Z}$, corresponding to the involution $p \mapsto-p$, but precisely two curves have more symmetry, namely those constructed from the square lattice $(j=1728)$ and the hexagonal lattice $(j=0)$ :

$$
\begin{aligned}
\operatorname{Aut}(\mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} i) & \cong(\mathbb{Z} / 2 \mathbb{Z})^{2} \\
\operatorname{Aut}\left(\mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} e^{\pi i / 3}\right) & \cong \mathbb{Z} / 2 \mathbb{Z} \times \mathbb{Z} / 3 \mathbb{Z}
\end{aligned}
$$

We should also figure out how many cusp points there are (i.e., count points in the fiber over $j=\infty$ ). This requires counting the number of orbits of the action of $\operatorname{SL}(2, \mathbb{Z})$ on the extended rationals $\mathbb{Q} \cup\{i \infty\}$. A quick calculation shows there are 12 of these, whose representatives we can take to be:

$$
\begin{equation*}
\text { cusps }=\left\{i \infty, \frac{2}{5}\right\} \cup\left\{\frac{k}{2}: k=0 \ldots 9\right\} \tag{15}
\end{equation*}
$$

So, in summary, $X(5)$ is an $A_{5}$-equivariant branched covering space of $S^{2}$, with three branch points $j=1728, j=0$ and $j=\infty$ having 30,20 , and 12 elements in their fibers respectively. This implies that it is isomorphic, as a branched covering space, to the "round icosahedron" $S^{2}$ ! In particular, it has genus zero.
Tidying things up. Let's tidy up this isomorphism of covering spaces over $S^{2}$ by ensuring that the branch points correspond correctly.

Instead of using $B=-V^{5} / E^{2}$ to identify the icosahedron quotient $S^{2} / A_{5}$ with $S^{2}$ (which was convenient from the viewpoint of identifying points on the icosahedron


Figure 5. The map $j_{5}$ is an $A_{5}$-equivariant map from $\mathbb{H}^{*} / \Gamma$ to $S^{2}$. The standard tesselation of $\mathbb{N}^{*}$ by fundamental domains for $\Gamma$ is shown, together with their image on $S^{2}$. Images taken from [6].
with ordered roots of Brioschi quintics, but not with elliptic curves), we should use Klein's function

$$
\begin{equation*}
J(z)=\frac{F^{3}}{V^{5}}=\frac{1728 V^{5}-E^{2}}{V^{5}}=1728+\frac{1}{B} \tag{16}
\end{equation*}
$$

instead. (Recall the fundamental relation (13)). This $J$ invariant aligns correctly with the projection $\pi: X(5) \rightarrow$ $X(1)$, since it sends the 30 edge midpoints to $J=1728$, the 20 face midpoints to $J=0$, and the 12 vertices to $J=\infty$. Let us record this in a theorem.

Theorem 2. There is an $A_{5}$-equivariant isomorphism $j_{5}$ of covering spaces between the moduli space $X(5)$ of elliptic curves equipped with a choice of primitive basis for their 5-torsion, and the icosahedron, such that the following diagram commutes:


Let us nail down the definition of $j_{5}$. To make (17) commute, we know it must send:

$$
\begin{aligned}
i \infty & \mapsto \text { a vertex } \\
i \mapsto & \mapsto \text { an edge midpoint } \\
e^{\frac{2 \pi i}{3}} & \mapsto \text { a face midpoint. }
\end{aligned}
$$

Since $j_{5}$ is $A_{5}$-equivariant, it must send fixed points of the action of $A_{5}$ on $\Vdash^{*}$ to fixed points of the action of $A_{5}$ on $S^{2}$. This determines $j_{5}$ up to some sign choices, which we now fix.

Refer to Figure 3. A rotation about the $z$-axis in $S^{2}$ by $\pi / 5$ corresponds, by Lemma 1 , to the transformation $\tau \mapsto \tau+1$ of $\mathbb{H}^{*}$, whose only fixed point is $i \infty \in \mathbb{H}^{*}$. So, we must have $j_{5}(i \infty) \in\{0, \infty\} \subset S^{2}$, and we make the natural choice $j_{5}(i \infty)=0$, to line up on-the-nose with $j$.

Similarly, a rotation about the $\alpha$-axis in $S^{2}$ by $\pi$ corresponds to the transformation $\tau \mapsto-1 / \tau$ on $\mathbb{H}^{*}$, whose only fixed point is $\tau=i$. So, $j_{5}(i)$ must equal $\pm a$, and we choose the plus sign. In the same way, since rotation around $b$


Figure 6. Extract from page 9 of Ramanujan's first letter to Hardy on January 16, 1913.
by $2 \pi / 3$ equals the product $R_{a} R_{z}$ and hence corresponds to the transformation $\tau \mapsto-\frac{1}{\tau+1}$, whose only fixed point is $\tau=e^{2 \pi i / 3}$, we must have $j_{5}\left(e^{\frac{2 \pi i}{3}}\right)= \pm b$, and again we choose the plus sign.

In summary, after doing a quick calculation of the stereographic projections $\alpha$ and $\beta$ of $a$ and $b$, we are defining $j_{5}$ as the unique $A_{5}$ equivariant map $\mathbb{H}^{*} / \Gamma(5) \rightarrow S^{2}$ satisfying:

$$
\begin{align*}
j_{5}(i \infty) & =\infty  \tag{18}\\
j_{5}(i) & =\alpha=\sqrt{\frac{5+\sqrt{5}}{2}-\frac{\sqrt{5}+1}{2}}  \tag{19}\\
j_{5}\left(e^{\frac{\pi i}{3}}\right) & =\beta=e^{\frac{-2 \pi i}{10}} \frac{\sqrt{30+6 \sqrt{5}}-3-\sqrt{5}}{4} . \tag{20}
\end{align*}
$$

See Figure 5. This definition is great... but it would be nice to have an explicit formula for $j_{5}(\tau)$, for an arbitrary point $\tau \in \mathbb{H}^{*}!$

## Enter Ramanujan

On Sunday evening of February 2, 1913, Bertrand Russell wrote to his lover Lady Ottoline Morrell from his rooms in Trinity College Cambridge:

> In Hall I found Hardy, and Littlewood in a state of wild excitement, because they believe they have discovered a second Newton, a Hindu clerk in Madras on £20 a year. He wrote to Hardy telling him of some results he has got, which Hardy thinks quite wonderful, especially as the man has had only an ordinary school education. Hardy has written to the Indian Office and hopes to get the man here at once.

Behold the stir which Srinivasa Ramanujan (1887-1920) created when he sent his famous letter to Hardy, England's foremost mathematician at the time, from the Port Trust Office in Madras on January 16, 1913. Figure 6 shows an extract from page 9 of his letter. In formula (4), we see that Ramanujan introduces a continued fraction

$$
\begin{equation*}
r(q)=\frac{q^{\frac{1}{5}}}{1+\frac{q}{1+\frac{q^{2}}{1+\frac{q^{3}}{1+\cdots}}}} \tag{21}
\end{equation*}
$$

and states an identity involving it, while in (5) we find the remarkable evaluation

$$
\begin{equation*}
r\left(e^{-2 \pi}\right)=\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2} \tag{22}
\end{equation*}
$$

Hardy famously wrote [7] that formulas (4) and (5) as well as a similar evaluation of $r\left(e^{-2 \pi \sqrt{5}}\right)$ (from Ramanujan's second letter to Hardy)

> ... defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

The continued fraction $r(q)$ in (21) is called the RogersRamanujan continued fraction, since it had been first written down 20 years earlier by Rogers (1894), who proved some important identities regarding it. Thus Ramanujan had independently rediscovered it, and had proved some remarkable new identities of his own, such as those above.

## Ramanujan and the Icosahedron

Staring at equations (19) and (22), we immediately conjecture the following relationship between the RogersRamanujan continued fraction $r$ and the $j_{5}$ covering map from the world of elliptic curves:

$$
\begin{equation*}
j_{5}(\tau)=r(q), \quad q=e^{2 \pi i \tau} \tag{23}
\end{equation*}
$$

This is indeed the case! From Theorem 2 and the discussion below it, all we need to do is establish equivariance of $r(q)$, thought of as a function of $\tau$ !

Theorem 3 ([5]). The Rogers-Ramanujan continued fraction $r$ is an $A_{5}$-equivariant map

$$
\mathbb{H}^{*} / \Gamma(5) \rightarrow S^{2} .
$$

That is,

$$
\begin{align*}
r(\tau+1) & =R_{z} r(\tau)  \tag{24}\\
r\left(-\frac{1}{\tau}\right) & =R_{\alpha} r(\tau) \tag{25}
\end{align*}=\frac{1-\phi r(\tau)}{\phi+r(\tau)}, ~ l
$$

where $\phi=\frac{1+\sqrt{5}}{2}$.
The idea of the proof is as follows. Equivariance (24) with respect to rotations about the $z$-axis follows immediately from the $q^{\frac{1}{5}}$ factor in the definition (21) for $r$. The transformation formula (25) for $r\left(-\frac{1}{\tau}\right)$ follows from the Rogers-Ramanujan identities which allow us to writer as a ratio of two theta functions, each of whose transformation properties under $\tau \mapsto-\frac{1}{\tau}$ is known.

Once we know $r$ is an $A_{5}$-equivariant map (and hence $r=j_{5}$ ), we immediately have Ramanujan's beautiful formulas (22) and (20) ${ }^{3}$ !

Indeed, the equivariance allows us to calculate $r(\tau)$ at any point $\tau \in \Gamma \cdot\{i, \rho, \infty\}$, since these map to the 62 special points on the icosahedron ( 12 vertices +20 face midpoints +30 edge midpoints). This gives us a bunch of intriguing identities! For instance, what is $r(0)$ ? Well,

$$
\begin{aligned}
r(0) & =r(S(i \infty)) \\
& =R_{\alpha} r(i \infty) \\
& =-\frac{1}{n}=\frac{1+\sqrt{5}}{2}
\end{aligned}
$$

In other words, we have

$$
\frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}=\frac{1+\sqrt{5}}{2}
$$

which is indeed a well-known identity!
Similarly, for which $\tau$ is $r(\tau)$ equal to the edge midpoint $i \in S^{2}$ ? Well, we know that $r(i \in \mathbb{H})=\alpha$, and we know that $R_{z}^{2} R_{\alpha} R_{z}^{2} R_{\alpha} R_{z} \cdot \alpha=i \in S^{2}$, therefore

$$
r\left(T^{2} S T^{2} S T \cdot i\right)=i, \quad \text { i.e., } r\left(\frac{7+i}{5}\right)=i
$$

and hence $r\left(\Gamma(5) \cdot \frac{7+i}{5}\right)=i$.

## Gauss and the Arithmetic-Geometric Mean

We're going to need one final ingredient before we can tie everything together. In our algorithm for finding the roots of a quintic, we will start with a quintic, do some magic, and associate to it an elliptic curve $E$ in Weierstrass form:

$$
\begin{equation*}
E: y^{2}=4 x^{3}-g_{2} x-g_{3} \tag{26}
\end{equation*}
$$

The next thing we will need is to find $\tau \in \mathbb{H}$ such that $E \cong \mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} \tau$. This means we need to find a basis $\omega_{1}, \omega_{2}$ for the period lattice [10] $\Lambda \subset \mathbb{C}$ of $E$,

$$
\Lambda=\left\{\int_{\gamma} \frac{d x}{y}: \gamma \in H_{1}(E, \mathbb{Z})\right\}
$$

and then set $\tau=\frac{\omega_{2}}{\omega_{1}}$. But we don't want to have to calculate integrals in order to solve the quintic! How can we calculate periods efficiently?

This is where the final piece of magic enters the story. In the late 1790 s, Gauss was trying to compute precisely such a period integral, namely

$$
\omega=\int_{-1}^{1} \frac{d x}{\sqrt{1-x^{4}}}
$$

He was excited to discover that such period integrals can be calculated very efficently using an algorithm called the

[^6]

Figure 7. The entry in Gauss' mathematical diary on May 30, 1799, recording his excitement at the discovery that $\omega=\pi / M(1, \sqrt{2})$. Translation from Latin: We have confirmed up to the eleventh figure that the arithmetic-geometric mean of 1 and $\sqrt{2}$ equals $\frac{\pi}{\omega}$. Therefore, once demonstrated, a completely new field in analysis will certainly be opened.
arithmetic-geometric mean (see Figure 7). Given two positive real numbers $a$ and $b$, define sequences $a_{n}$ and $b_{n}$ by starting with $a_{0}=a$ and $b_{0}=b$ and then setting

$$
\begin{align*}
& a_{n+1}=\frac{1}{2}\left(a_{n}+b_{n}\right)  \tag{27}\\
& b_{n+1}=\sqrt{a_{n} b_{n}} \tag{28}
\end{align*}
$$

so that $a_{n+1}$ is the arithmetic mean of $a_{n}$ and $b_{n}$, and $b_{n+1}$ is their geometric mean. These sequences converge very rapidly (the accuracy doubles with each iteration), and the arithmetic-geometric mean of $a$ and $b$ is defined as the common limit

$$
M(a, b)=\lim _{n \rightarrow \infty} a_{n}=\lim _{n \rightarrow \infty} b_{n}
$$

More generally, given an elliptic curve $E$ in Weierstrass form (26) which factorizes as

$$
4 x^{2}-g_{2} x-g_{3}=4\left(x-e_{1}\right)\left(x-e_{2}\right)\left(x_{3}\right)
$$

with real roots $e_{1}<e_{2}<e_{3}$, the period lattice is $\mathbb{Z} \omega_{1} \oplus \mathbb{Z} \omega_{2}$, where:

$$
\begin{align*}
\omega_{1} & =\frac{\pi}{M\left(\sqrt{e_{3}-e 1}, \sqrt{e 3-e 2}\right)}  \tag{29}\\
\omega_{2} & =\frac{\pi i}{M\left(\sqrt{e_{3}-e_{1}}, \sqrt{e_{2}-e_{1}}\right)} \tag{30}
\end{align*}
$$

Finally, in 2010 Cremona and Thongjunthug defined the arithmetic-geometric mean $M(a, b)$ appropriately for complex numbers [3] (one must simply choose the correct square root in (28) at each step) and gave analogous versions of (29) and (30). Armed with these formulas, we can efficiently compute the period of any elliptic curve!

## The Algorithm

Let us now put all the ingredients together.
Algorithm 1. Find the roots of a quintic using elliptic curves and the icosahedron.

Step 1. Start with a general quintic

$$
X^{5}+a_{1} X^{4}+a_{2} X^{3}+a_{3} X^{2}+a_{4} X+a_{5}=0
$$

and transform it to a Brioschi quintic

$$
X^{5}+10 B X^{3}+45 B^{2} X+B^{2}=0
$$

for a certain Brioschi parameter $B \in \mathbb{C}$.
See [2, Theorem 6.6]. This requires extraction of at most two square roots, and works away from a set of measure zero.

Step 2. Determine the associated elliptic curve $E$.
The $j$-invariant of the associated elliptic curve $E$ is $j=$ $1728+\frac{1}{B}$. A Weierstrass equation

$$
\begin{equation*}
y^{2}=4 x^{3}-g_{2} x-g_{3} \tag{31}
\end{equation*}
$$

for $E$ with this $j$-invariant is given by setting

$$
g_{2}=\frac{-3 j}{1728-j}, \quad g_{3}=\frac{j}{1728-j}
$$

since then we have $j=\frac{g_{2}^{3}}{g_{2}^{3}-27 g_{3}^{2}}$ as needed.
Step 3. Find $\tau \in \mathbb{H}$ such that $E \cong \mathbb{C} / \mathbb{Z} \oplus \mathbb{Z} \tau$.
Compute $\omega_{1}, \omega_{2}$ using the complex arithmeticgeometric mean algorithm [3], and then set $\tau=\frac{\omega_{2}}{\omega_{1}}$.

Step 4. Compute the associated point $z$ on the icosahedron as $z=j_{5}(\tau)$.

Use the Rogers-Ramanujan continued fraction $r$, which converges very rapidly.

Step 5. The five roots, delivered to you in "octahedral ordering" free of charge, are $\left(x_{1}(z), x_{2}(z), x_{3}(z), x_{4}(z), x_{5}(z)\right)$ !

The implementation of this algorithm as a Mathematica worksheet, together with all the code for the pictures presented here, can be found at [1].

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# On the Population Size in Stochastic Differential Games Dylan Possamaï and Ludovic Tangpi 

Commuters looking for the shortest path to their destinations, the security of networked computers, hedge funds trading on the same stocks, governments and populations acting to mitigate an epidemic, or employers and employees agreeing on a contact, are all examples of (dynamic) stochastic differential games. In essence, game theory deals with the analysis of strategic interactions among multiple decision-makers. The theory has had enormous impact in a wide variety of fields, but its rigorous mathematical analysis is rather recent. It started with the pioneering work of von Neumann and Morgenstern [vNM44] published in 1944. Since then, game theory has taken center stage in applied mathematics and related areas, especially in economics with several game theorists such as John F. Nash Jr, Robert J. Aumann, and Thomas C. Schelling being awarded the Nobel Memorial Prize in Economic Sciences. Game theory has also played an important role in unsuspected areas: for instance in military applications, when the analysis of guided interceptor missiles in the 1950s motivated the study of games evolving dynamically in time. Such games (when possibly subject to randomness) are called stochastic differential games. Their study started with the work of Issacs [Isa54], who crucially recognized the importance of (stochastic) control theory in the area. Over the past few decades since Isaacs's work, a rich theory of stochastic differential game has emerged and branched into several directions. This paper will review recent advances in the study of solvability of stochastic differential games, with a focus on a purely probabilistic technique to approach the problem. Unsurprisingly, the number of players involved in the game is a major factor of the analysis. We will explain how the size of the population impacts the analyses and solvability of the problem, and

[^7]Communicated by Notices Associate Editor Reza Malek-Madani.
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DOI: https://doi.org/10.1090/noti2908
discuss mean field games as well as the convergence of finite player games to mean field games.

## 1. Two-Player Games

Games involving only two players are arguably the most basic differential games in continuous time. In this section we discuss both zero-sum and nonzero-sum games, where the main difference stems from the existence of some symmetry-or maybe more precisely antisymmetry here-between the players' objectives. Since our goal is to provide intuition, we will use a simple example as our Ariadne's thread throughout the paper, emphasising the differences and new features emerging as we make the modelling more complex.

We fix throughout this section a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a one-dimensional Brownian motion $W$, and we let $\mathbb{F}$ be the $\mathbb{P}$-completed natural filtration of $W$. We also fix a time horizon $T>0$, and assume that $\mathcal{F}=\mathcal{F}_{T}$. The players are identified by numbers 1 and 2, and their (uncontrolled) state process is given, for some $\xi$, by

$$
X_{t}:=\xi+W_{t}, t \in[0, T]
$$

The players choose processes $\alpha:=\left(\alpha^{1}, \alpha^{2}\right)$, where both $\alpha^{1}$ and $\alpha^{2}$ are taken in some set of so-called admissible controls, denoted $\mathcal{A}$, which we take here to consist essentially of functions of Brownian paths ${ }^{1}$ and taking values in $\mathbb{R}^{2}$ for simplicity. That is, players make decisions based on the randomness of the problem given by $W$. Such controls are called open-loop. In fact, throughout these notes we consider only open-loop controls. We adopt here the socalled weak formulation, and consider that a choice $\alpha$ from the players generates uncertainty on the distribution of $X$, by considering it under a new probability measure $\mathbb{P}^{\alpha}$ (it is implicitly assumed that definition of $\mathcal{A}$ ensures that this probability measure is well-defined) whose density with respect to $\mathbb{P}$ is given by

$$
\frac{\mathrm{d} \mathbb{P}^{\alpha}}{\mathrm{dP}}:=\exp \left(\int_{0}^{T}\left(\alpha_{s}^{1}+\alpha_{s}^{2}\right) \mathrm{d} W_{s}-\int_{0}^{T} \frac{\left(\alpha_{s}^{1}+\alpha_{s}^{2}\right)^{2}}{2} \mathrm{~d} s\right)
$$

Using standard results of stochastic calculus, ${ }^{2}$ there is then

[^8]an $\left(\mathbb{F}, \mathbb{P}^{\alpha}\right)$-Brownian motion $W^{\alpha}$ such that
$$
X_{t}:=\xi+\int_{0}^{t}\left(\alpha_{s}^{1}+\alpha_{s}^{2}\right) \mathrm{d} s+W_{t}^{\alpha}, t \in[0, T] .
$$

Of course we need to clarify how the players decide which controls they would like to use, and this will be linked to their criterion. These criteria will also be exactly what is going to differentiate zero-sum and nonzero-sum games, which we will exemplify in the next sections.
1.1. Zero-sum games and equilibria. In zero-sum games, the goals pursued by the two players are antagonistic in the following sense: Suppose that player's 1 objective is to minimize to functional criterion

$$
J\left(\alpha^{1}, \alpha^{2}\right):=\mathbb{E}^{\mathbb{P}^{\alpha}}\left[\int_{0}^{T}\left(\frac{\left|\alpha_{s}^{1}\right|^{2}-\left|\alpha_{s}^{2}\right|^{2}}{2}+X_{s}\right) \mathrm{d} s+\left|X_{T}\right|^{2}\right],
$$

with $\left(\alpha^{1}, \alpha^{2}\right) \in \mathcal{A}^{2}$. That is, given some control $\alpha^{2} \in \mathcal{A}$ chosen by player 2 , player 1 aims at solving the minimisation problem ${ }^{3}$

$$
V^{1}\left(\alpha^{2}\right):=\inf _{\alpha \in \mathcal{A}} J\left(\alpha, \alpha^{2}\right) .
$$

Then, given some control $\alpha^{1} \in \mathcal{A}$ chosen by player 1, player 2 aims at solving the minimisation problem

$$
V^{2}\left(\alpha^{1}\right):=\inf _{\alpha \in \mathcal{A}}\left\{-J\left(\alpha^{1}, \alpha^{2}\right)\right\}=-\sup _{\alpha \in \mathcal{A}} J\left(\alpha^{1}, \alpha\right) .
$$

In other words, while player 1 minimizes $J$ player 2 maximizes it. Thus the term "zero-sum," as the sum of the criterion of each players is always 0 . The process $X$ is the (common) path or state of the players 1 and $2, \alpha^{1}$ and $\alpha^{2}$ are their respective actions and $J\left(\alpha^{1}, \alpha^{2}\right)$ the cost (resp. reward) of player 1 (resp. player 2). In this game, we are interested in so-called Nash equilibrium corresponding to a pair $\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right) \in \mathcal{A}^{2}$ such that

$$
V^{1}\left(\hat{\alpha}^{2}\right)=J\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right)=-V^{2}\left(\hat{\alpha}^{1}\right) .
$$

Thus, whenever one player plays the control corresponding to the Nash equilibrium, the other player will never be better off by not also playing according to the equilibrium. Observe that the Nash equilibrium is not necessarily optimal in the sense that it does not yield the highest reward (or lowest cost) to any one player, it is simply a set of strategies that is simultaneously optimal for both players.

The above definition of Nash equilibria anticipates our soon to come discussion of nonzero-sum games in Section 2. But this not the most standard way to attack zero-sum games. Indeed, using the (anti-)symmetry between the players' objectives, one can directly realize that

[^9]if one defines instead a saddle-point as being a process $\hat{\alpha}:=\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right) \in \mathcal{A}^{2}$ such that
$$
J\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right)=\inf _{\alpha^{1} \in \mathcal{A}} \sup _{\alpha^{2} \in \mathcal{A}} J\left(\alpha^{1}, \alpha^{2}\right)=\sup _{\alpha^{2} \in \mathcal{A}} \inf _{\alpha^{1} \in \mathcal{A}} J\left(\alpha^{1}, \alpha^{2}\right),
$$
then it is also a Nash equilibrium. It is important to notice at this stage that while the notion of Nash equilibrium extends to nonzero-sum games, that of saddle-points makes sense only in a zero-sum setting. Moreover, their interpretation is a bit different: a saddle-point corresponds to each player trying to optimize their criterion assuming that the other player has chosen the worst possible control from their point of view. In general, even in the symmetric case we are describing here, not all Nash equilibria are saddle-points, but finding a saddle-point is a standard way to identify a Nash equilibrium.
1.2. Intuition and solution. Let us now use the simple setting we are considering to explain how one can, in general, find a saddle-point or a Nash equilibrium. Exactly as in standard control problems involving only one player, there are two main tools available: analytic one using partial differential equations (PDEs)-the celebrated Hamilton-Jacobi-Bellman (HJB) or Hamilton-Jacobi-Bellman-Isaacs equations-and a probabilistic one using so-called backward stochastic differential equations (BSDEs). This work accents the probabilistic approach (more specifically in the weak formulation), and we refer the interested reader to [FS06] for more details on the alternative one.

Despite these distinctions, the point of both methods is exactly the same: understanding the structure of the socalled best-reaction function of a player, namely what we defined as $V^{1}\left(\alpha^{2}\right)$ and $V^{2}\left(\alpha^{1}\right)$ above. Both these quantities are actually value functions of a control problem faced by each player. As such, it has become part of the folklore in the corresponding literature that under mild assumptions, the following result will be true.

Proposition 1.1 (Best-reaction functions). Under suitable assumptions, for $\left(\alpha^{1}, \alpha^{2}\right) \in \mathcal{A}^{2}$, we have $V^{1}\left(\alpha^{2}\right)=$ $Y_{0}^{1}\left(\alpha^{2}\right)$ and $V^{2}\left(\alpha^{1}\right)=Y_{0}^{2}\left(\alpha^{1}\right)$ where $\left(Y^{1}\left(\alpha^{2}\right), Z^{1}\left(\alpha^{2}\right)\right)$ and $\left(Y^{2}\left(\alpha^{1}\right), Z^{2}\left(\alpha^{1}\right)\right)$ are pairs of processes satisfying appropriate measurability and integrability conditions and solving the following one-dimensional BSDEs, with $t \in[0, T]$

$$
\begin{align*}
Y_{t}^{1}\left(\alpha^{2}\right)= & \left|X_{T}\right|^{2}+\int_{t}^{T}\left(\alpha_{s}^{2} Z_{s}^{1}\left(\alpha_{s}^{2}\right)-\frac{\left|Z_{s}^{1}\left(\alpha_{s}^{2}\right)\right|^{2}+\left|\alpha_{s}^{2}\right|^{2}}{2}\right. \\
& \left.+X_{s}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{1}\left(\alpha_{s}^{2}\right) \mathrm{d} W_{s},  \tag{1.1}\\
Y_{t}^{2}\left(\alpha^{1}\right)= & -\left|X_{T}\right|^{2}+\int_{t}^{T}\left(\alpha_{s}^{1} Z_{s}^{2}\left(\alpha_{s}^{1}\right)-\frac{\left|Z_{s}^{2}\left(\alpha_{s}^{1}\right)\right|^{2}+\left|\alpha_{s}^{1}\right|^{2}}{2}\right. \\
& \left.-X_{s}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{2}\left(\alpha_{s}^{1}\right) \mathrm{d} W_{s} . \tag{1.2}
\end{align*}
$$

Moreover, the corresponding optimal controls are $-Z^{1}\left(\alpha^{2}\right)$ for player 1, and $-Z^{2}\left(\alpha^{1}\right)$ for player 2.

This result relies on appropriately using the so-called $d y$ namic programming principle and the associated martingale optimality principle, as well as results from BSDE theory. We refer for instance to [Zha17] for details. The intuition behind these equations is the following:
(i) $Y_{t}^{1}\left(\alpha^{2}\right)$ (resp. $Y_{t}^{2}\left(\alpha^{1}\right)$ ) represents the value at time $t \in[0, T]$ of the (natural) dynamic version of the value function of player 1 (resp. player 2) whenever player 2 (resp. player 1) has chosen the control $\alpha^{2}$ (resp. $\alpha^{1}$ ). In essence, this corresponds to looking at the game over the time period $[t, T]$ only;
(ii) the functions appearing in the Lebesgue integrals in (1.1) and (1.2) are linked to the Hamiltonians of each player, which in this example are given by maps $H^{1}$ and $H^{2}$ defined on $\mathbb{R}^{3}$ by

$$
\begin{aligned}
& H^{1}(x, z, v):=\inf _{u \in \mathbb{R}} h^{1}(x, z, u, v)=v z-\frac{z^{2}+v^{2}}{2}+x \\
& H^{2}(x, z, u):=\inf _{v \in \mathbb{R}} h^{2}(x, z, u, v)=u z-\frac{z^{2}+u^{2}}{2}-x
\end{aligned}
$$

where, for $(x, z, u, v) \in \mathbb{R}^{4}$,

$$
\begin{aligned}
& h^{1}(x, z, u, v):=(u+v) z+\frac{1}{2}\left(u^{2}-v^{2}\right)+x \\
& h^{2}(x, z, u, v):=(u+a) z+\frac{1}{2}\left(v^{2}-u^{2}\right)-x
\end{aligned}
$$

(iii) the processes $Z^{1}\left(\alpha^{2}\right)$ and $Z^{2}\left(\alpha^{1}\right)$ should be understood at an informal level as "derivatives" ${ }^{4}$ of $Y^{1}\left(\alpha^{2}\right)$ and $Y^{2}\left(\alpha^{1}\right)$, respectively. In practice, they directly allow to compute the optimal control for player 1 (resp. player 2) when player 2 (resp. player 1) plays $\alpha^{2}$ (resp. $\alpha^{1}$ ) in the sense that it corresponds to any maximizer in the definition of $H^{1}\left(X ., Z_{.}^{1}\left(\alpha^{2}\right), \alpha^{2}\right)$ (resp. $\left.H^{2}\left(X ., Z_{.}^{2}\left(\alpha^{1}\right), \alpha^{1}\right)\right)$.
Once we have Proposition 1.1 in hand, it becomes relatively straightforward to realize that to obtain a Nash equilibrium, one should be solving simultaneously (1.1) and (1.2), so that the behaviors of both players are concomitantly optimal. In other words, the following result holds true.

Theorem 1.2. Under suitable assumptions, a pair $\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right) \in$ $\mathcal{A}^{2}$ will be a Nash equilibrium if and only if $\left(\hat{\alpha}^{1}, \hat{\alpha}^{2}\right)=$ $\left(-Z^{1},-Z^{2}\right)$ where the quadruplet $\left(Y^{1}, Y^{2}, Z^{1}, Z^{2}\right)$ of processes satisfies appropriate measurability and integrability conditions

[^10]and solves the two-dimensional BSDE system, with $t \in[0, T]$
\[

$$
\begin{aligned}
Y_{t}^{1}= & \left|X_{T}\right|^{2}+\int_{t}^{T}\left(Z_{s}^{2} Z_{s}^{1}-\frac{\left|Z_{S}^{1}\right|^{2}+\left|Z_{s}^{2}\right|^{2}}{2}+X_{s}\right) \mathrm{d} s \\
& -\int_{t}^{T} Z_{s}^{1} \mathrm{~d} W_{s} \\
Y_{t}^{2}= & -\left|X_{T}\right|^{2}+\int_{t}^{T}\left(Z_{s}^{1} Z_{s}^{2}-\frac{\left|Z_{s}^{2}\right|^{2}+\left|Z_{s}^{1}\right|^{2}}{2}-X_{s}\right) \mathrm{d} s \\
& -\int_{t}^{T} Z_{s}^{2} \mathrm{~d} W_{s}
\end{aligned}
$$
\]

The previous theorem deserves some comments, especially on how one can find Nash equilibria from the Hamiltonians of the player: the point here is that this more or less boils down to finding "fixed points" for the vector-valued function $\left(H^{1}, H^{2}\right)$. For any $\left(x, z^{1}, z^{2}\right) \in$ $\mathbb{R}^{3}$, what we mean by a fixed-point here is a pair $\left(u\left(x, z^{1}, z^{2}\right), v\left(x, z^{1}, z^{2}\right)\right) \in \mathbb{R}^{2}$ such that

$$
\left\{\begin{array}{l}
H^{1}\left(x, z^{1}, v\left(x, z^{1}, z^{2}\right)\right)=h^{1}\left(x, z^{1}, u\left(x, z^{1}, z^{2}\right), v\left(x, z^{1}, z^{2}\right)\right), \\
H^{2}\left(x, z^{2}, u\left(x, z^{1}, z^{2}\right)\right)=h^{2}\left(x, z^{2}, u\left(x, z^{1}, z^{2}\right), v\left(x, z^{1}, z^{2}\right)\right) .
\end{array}\right.
$$

In our simple example, such a fixed-point is trivial to find and is uniquely given by

$$
u\left(x, z^{1}, z^{2}\right)=-z^{1}, v\left(x, z^{1}, z^{2}\right)=-z^{2}
$$

which is how we identified the Nash equilibrium in (1.2).
Moreover, we would like to insist on the fact that finding a Nash equilibrium in a generic two-player game amounts to solving a two-dimensional BSDE system. Intuitively, the dimension of the aforementioned system should increase accordingly with the number of players, and this is exactly what we will make clear in Section 2 below. However, for zero-sum games (where we look for saddlepoints) something interesting happens: It is enough to solve only one equation. In order to understand why, we need to introduce the so-called upper and lower values of the game, respectively denoted by $V^{+}$and $V^{-}$, with

$$
\left\{\begin{array}{l}
V^{+}:=\inf _{\alpha^{1} \in \mathcal{A}} \sup _{\alpha^{2} \in \mathcal{A}} J\left(\alpha^{1}, \alpha^{2}\right) \\
V^{-}:=\sup _{\alpha^{2} \in \mathcal{A}} \inf _{\alpha^{1} \in \mathcal{A}} J\left(\alpha^{1}, \alpha^{2}\right)
\end{array}\right.
$$

as well as the upper and lower Hamiltonians, defined for $(x, z) \in \mathbb{R}^{2}$

$$
\left\{\begin{array}{l}
H^{-}(x, z):=\sup _{v \in \mathbb{R}} H^{1}(x, z, v)=x \\
H^{+}(x, z):=\inf _{u \in \mathbb{R}} \sup _{v \in \mathbb{R}} h^{1}(x, z, u, v)=x
\end{array}\right.
$$

Even if there is one major simplification in our example since $H^{+}$and $H^{-}$only depend on $x$, the main point to notice is rather that in general, it holds $H^{+}=H^{-}$. This is a minimax property which constitutes what is usually refereed to as Isaacs's condition, and is a typical necessary condition for the existence of a saddle-point. Now, in order to characterize these two values, it is useful to rely on the best-reaction functions from Proposition 1.1. More
precisely, one can show that computing $V^{+}$amounts to formally taking an infimum over $\alpha^{1}$ in (the opposite of) (1.2), while computing $V^{-}$amounts to formally taking a supremum over $\alpha^{2}$ in (1.1). This means that we are naturally led to considering the pairs processes $\left(Y^{+}, Z^{+}\right)$and ( $Y^{-}, Z^{-}$) satisfying, for $t \in[0, T]$

$$
\begin{aligned}
Y_{t}^{-} & =\left|X_{T}\right|^{2}+\int_{t}^{T}\left(\frac{\left|Z_{s}^{-}\right|^{2}-\left|Z_{s}^{-}\right|^{2}}{2}+X_{s}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{-} \mathrm{d} W_{s} \\
& =\left|X_{T}\right|^{2}+\int_{t}^{T} H^{-}\left(X_{s}, Z_{s}^{-}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{-} \mathrm{d} W_{s},
\end{aligned}
$$

and

$$
\begin{aligned}
Y_{t}^{+} & =\left|X_{T}\right|^{2}+\int_{t}^{T}\left(\frac{\left|Z_{s}^{+}\right|^{2}-\left|Z_{s}^{+}\right|^{2}}{2}+X_{s}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{+} \mathrm{d} W_{s} \\
& =\left|X_{T}\right|^{2}+\int_{t}^{T} H_{s}^{+}\left(X_{s}, Z_{s}^{+}\right) \mathrm{d} s-\int_{t}^{T} Z_{s}^{+} \mathrm{d} W_{s} .
\end{aligned}
$$

From there, it is immediate (by uniqueness) that $Y^{+}=Y^{-}$, $Z^{+}=Z^{-}$, and this leads to the following result.
Theorem 1.3. Under suitable assumptions, we have $V^{+}=$ $Y_{0}=V^{-}$where $(Y, Z)$ solves the BSDE

$$
Y_{t}=\left|X_{T}\right|^{2}+\int_{t}^{T} X_{s} \mathrm{~d} s-\int_{t}^{T} Z_{s} \mathrm{~d} W_{s}, t \in[0, T]
$$

Moreover, this equation has a unique (and explicit) solution:

$$
Y_{t}=T-t+X_{t}^{2}+(T-t) X_{t}, Z_{t}=2 X_{t}+T-t, t \in[0, T],
$$

corresponding to the saddle-point $(-Z, Z)$ for the zero-sum game.

## 2. Games with an Arbitrary Number of Players

Let us now turn our attention to the analysis of games with $N$ players, for some integer $N \geq 2$. In the two-player games discussed thus far, the main difference that we stressed was the one between zero-sum and nonzero-sum games. With three players or more, things get much harder as the possibility of forming coalitions becomes significant, and this interesting feature has not been studied so far in the stochastic differential game literature due to the apparent difficulty of the question. While one can consider zero-sum and cooperative versions of the $N$-player games, we will focus here on the (fully) noncooperative case where one is interested in Nash equilibria and no coalition is formed. In this case, for tractability of the problem, the symmetry assumption becomes essential! We will come back to this in the final section of the article. By symmetry here, we mean that the game will be set up in a way that players are, roughly speaking, exchangeable. In other words, the game is exactly the same from each player's vantage point. Of course, the reader should remark that symmetric players does not mean independent players, as players will still impact each other's actions and trajectories.

We will again discuss a probabilistic approach to the solvability of $N$-player stochastic differential games through a simple tractable example. Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is now rich enough to carry $N$ independent Brownian motions ( $W^{1}, \ldots, W^{N}$ ). Each player is identified by an index $i \in\{1, \ldots, N\}$ and seeks to solve the stochastic control problem

$$
\begin{aligned}
& V^{i}\left(\alpha^{-i}\right) \\
& \qquad:=\inf _{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{\alpha^{-i}} \otimes_{i} \alpha, N}\left[\int_{0}^{T}\left(\frac{1}{N} \sum_{j=1}^{N} X_{s}^{j}+\frac{1}{2}\left|\alpha_{s}\right|^{2}\right) \mathrm{d} s+\left|X_{T}^{i}\right|^{2}\right],
\end{aligned}
$$

with $X_{t}^{i}=\xi+W_{t}^{i}$, where

$$
\begin{array}{r}
\alpha^{-i}:=\left(\alpha^{1}, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^{N}\right), \\
\alpha^{-i} \otimes_{i} \alpha:=\left(\alpha^{1}, \ldots, \alpha^{i-1}, \alpha, \alpha^{i+1}, \ldots, \alpha^{N}\right),
\end{array}
$$

and for a vector $\alpha$, the probability measure $\mathbb{P}^{\alpha, N}$ is given, for any vector $\alpha:=\left(\alpha^{1}, \ldots, \alpha^{N}\right) \in \mathcal{A}^{N}$, by

$$
\frac{\mathrm{d} \mathbb{P}^{\alpha, N}}{\mathrm{dP}}:=\exp \left(\sum_{j=1}^{N} \int_{0}^{T} \alpha_{s}^{j} \mathrm{~d} W_{s}^{j}-\frac{1}{2} \sum_{j=1}^{N} \int_{0}^{T}\left|\alpha_{s}^{j}\right|^{2} \mathrm{~d} s\right) .
$$

Before going any further, let us describe this problem. Player $i$ has a state process (or trajectory) $X^{i}$ and control process $\alpha^{i}$ which is chosen among admissible controls $\mathcal{A}$, which are $\mathbb{F}^{N}$-predictable, where $\mathbb{F}^{N}$ is the ( $\mathbb{P}$-completed) filtration generated by the Brownian motions ( $W^{1}, \ldots, W^{N}$ ), and satisfy appropriate integrability conditions. This means that information on their controls is entirely encapsulated in the randomness sources ( $W^{1}, \ldots, W^{N}$ ). Player $i^{\prime}$ s goal is thus to minimize the objective function

$$
\begin{aligned}
& J\left(\alpha^{1}, \ldots, \alpha^{N}\right) \\
& \quad:=\mathbb{E}^{\mathbb{P}^{\alpha^{-i}} \otimes_{i} \alpha, N}\left[\int_{0}^{T}\left(\frac{1}{N} \sum_{j=1}^{N} X_{s}^{j}+\frac{1}{2}\left|\alpha_{s}\right|^{2}\right) \mathrm{d} s+\left|X_{T}^{i}\right|^{2}\right],
\end{aligned}
$$

over $\alpha \in \mathcal{A}$, for $\alpha^{-i} \in \mathcal{A}^{N-1}$ fixed. Here, the objective function is essentially the total energy of the player plus the average position of every player in the game. The terminal cost $\left|X_{T}^{i}\right|^{2}$ "forces" player $i$ 's position to be close to zero at the horizon $T$. Observe that since ( $W^{1}, \ldots, W^{N}$ ) are independent, the state processes satisfy

$$
\begin{equation*}
\mathrm{d} X_{t}^{i}=\alpha_{t}^{i} \mathrm{~d} t+\mathrm{d} W_{t}^{\alpha, i}, \mathbb{P}^{\alpha, N}-\mathrm{a} . \mathrm{s} ., X_{0}^{i}=\xi, \tag{2.1}
\end{equation*}
$$

where $W^{\alpha, i}:=W^{i}-\int_{0} \alpha_{s}^{i} \mathrm{~d} s$ is a $\mathbb{P}^{\alpha, N}$-Brownian motion. Thus, the problem we describe aims at choosing the best (in terms of minimising $J$ ) probability space on which the state process satisfies (2.1). Just as in the two player game, we will be interested in Nash equilibria, which, exactly as in the previous section, are defined as admissible strategies
$\left(\hat{\alpha}^{1}, \ldots, \hat{\alpha}^{N}\right) \in \mathcal{A}^{N}$ such that for every $i \in\{1, \ldots, N\}$, we have $J\left(\hat{\alpha}^{1}, \ldots, \hat{\alpha}^{N}\right)$

$$
=\inf _{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{\alpha^{-i}} \otimes_{\otimes_{i}} \alpha, N}\left[\int_{0}^{T} \frac{1}{2}\left|\alpha_{S}\right|^{2}+\frac{1}{N} \sum_{j=1}^{N} X_{s}^{j} \mathrm{~d} s+\left|X_{T}^{i}\right|^{2}\right]
$$

Again, the point here is that a Nash equilibrium corresponds to a situation where, whenever all players but one follow the equilibrium strategy, the one who does not is worse off.

Following the same intuition presented in the twoplayer case, one can rely on a system of backward SDEs for solving and analysing Nash equilibria. In fact, we have to following characterisation result, which directly extends Theorem 1.2.

Theorem 2.1. The $N$-player game admits a Nash equilibrium $\hat{\alpha}^{N}=\left(\hat{\alpha}^{1, N}, \ldots, \hat{\alpha}^{N, N}\right)$ if and only if

$$
\hat{\alpha}_{t}^{i, N}=-Z_{t}^{i, i, N}, \text { and } V^{i}\left(\hat{\alpha}^{-i, N}\right)=Y_{0}^{i, N}, \mathbb{P} \otimes \mathrm{~d} t \text {-a.e., }
$$

for $i \in\{1, \ldots, N\}$, where $\left(Y^{i, N}, Z^{i, j, N}\right)_{(i, j) \in\{1, . . ., N\}^{2}}$ satisfy appropriate integrability conditions and solve the system

$$
\begin{align*}
Y_{t}^{i, N} & =\left|X_{T}^{i}\right|^{2}+\int_{t}^{T}\left(\frac{1}{N} \sum_{j=1}^{N} X_{s}^{j}-\frac{\left|Z_{s}^{i, i, N}\right|^{2}}{2}\right) \mathrm{d} s \\
& -\sum_{j=1}^{N} \int_{t}^{T} Z_{s}^{i, j, N} \mathrm{~d} W_{s}^{\hat{\alpha}^{N}, j}, \mathbb{P}^{\hat{\alpha}^{N}, N_{-a . s . ~}} \tag{2.2}
\end{align*}
$$

The characterisation given by Theorem 2.1 is important mostly because it allows to use the full force of stochastic calculus to analyse the game. As far as solvability is concerned, it does not (necessarily) make the problem any easier because well-posedness of the system (2.2) is no trivial matter in general. The main issues here are the quadratic nonlinearity of the drift and the fact that the system is multidimensional. We will discuss below a very interesting problem for which Theorem 2.1 is particularly relevant. Existence of $N$-Nash equilibria (and thus of the system (2.2)) can also be guaranteed using compactness techniques or analytic methods based on partial differential equations. We refer the reader to [CD18, Vol. I, Chapter 2] and references therein for in-depth discussion on solvability of $N$-player games. Brushing well-posedness issues aside, the numerical simulation of equilibria (which is the end goal in most practical applications) is also a particularly concerning problem, especially so when the number of players $N$ is large. Numerical simulations often reduce to approximating solutions of multidimensional systems such as (2.2) or systems of $N \mathrm{HJB}$ equations. As is well known, most numerical simulation algorithms are subject to the so-called curse of dimensionality. This is the fact that the performance of the algorithm plummets as the dimension increases!

When players are homogeneous (or symmetric) as discussed above and the interaction among the players is sufficiently weak in the sense that the influence of any given player on another one is of order $1 / N$, and if the initial configuration of the particle system is sufficiently chaotic (i.i.d.), then as $N$ increases, interaction plays less and less of a role. One thus speaks of mean field interaction. Systems in mean field interactions have a long history in statistical physics (e.g., thermodynamic models) and mathematical biology (e.g., chemotaxis models). In these areas, a meta theorem (which is by no means obvious) is that a symmetric particle system in weak interaction and with i.i.d. initial positions converges to an independent and identically distributed particle system whose evolution depends on the particle's probabilistic distribution. This phenomenon is known as propagation of chaos as, in the limit, the initial chaotic configuration 'propagates' to the entire path of the particles. This is usually formulated by the fact that, if one focuses on the $k$ joint distribution at a given time $t$ of the first $k$ particles, then for $N$ sufficiently large it should approximately be given by the $k$-fold product distribution of a given particle. Propagation of chaos was first studied by Kac [Kac56] and has had countless applications in pure and applied mathematics, physics, biology and more. Limiting particle systems are usually said to be of McKean-Vlasov type. That is, their dynamics depend on their own law. Inspired by these ideas, Lasry and Lions [LL06] and Huang, Malhamé, and Caines [HMC06] proposed in 2006 a general method allowing to derive approximate Nash equilibria for large population games called mean field games.

## 3. Mean Field Games

The central idea here is to consider a game in which (independent and identical) players interact through the probabilistic distribution of the entire population. Lasry and Lions showed that solutions of these mean field games can be used to construct $N$-player strategies that are "nearly" Nash equilibria for $N$ large enough. This reformulation of the problem provides a decisive advantage for the simulation of equilibria, but the study of mean field games solutions (henceforth MFE for mean field equilibrium) is an intrinsically very interesting mathematical problem. This is particularly due to the fact that, because of the interaction through probabilistic distributions, the study of MFE heavily rests on analysis on the space of probability measures. In fact, studying mean field games often results in the analysis of PDEs as (3.5) below written on the space $\mathcal{P}_{2}(\mathbb{R})$ of Borel measures on $\mathbb{R}$ with finite second moment. We will attempt to present a similar perspective using probabilistic arguments. Once again, we focus on a simple example: the mean field game analogue of the game discussed in the previous section.

Let $\mathcal{C}\left([0, T], \mathcal{P}_{2}(\mathbb{R})\right)$ be the space of continuous functions on $[0, T]$ mapping into the set $\mathcal{P}_{2}(\mathbb{R})$. Let $\mu$ be an element of $\mathcal{C}\left([0, T], \mathcal{P}_{2}(\mathbb{R})\right)$. Intuitively, $\mu_{t}$ should be thought of as the distribution of the (position) of the population at time $t$. Assuming it to be known, a given (representative) player $i$ in the population will consider the stochastic control problem

$$
V^{\mu}:=\inf _{\alpha \in \mathscr{R}} \underbrace{\mathbb{E}^{\mathbb{P}^{\alpha}}\left[\int_{0}^{T}\left(\frac{\left|\alpha_{s}\right|^{2}}{2}+\int_{\mathbb{R}} x \mu_{s}(\mathrm{~d} x)\right) \mathrm{d} s+\left|X_{T}\right|^{2}\right]}_{=: J^{\mu}(\alpha)},
$$

with $\mathrm{d} X_{t}=-k X_{t} \mathrm{~d} t+\mathrm{d} W_{t}^{i}$ and

$$
\frac{\mathrm{d} \mathbb{P}^{\alpha}}{\mathrm{dP}}:=\exp \left(\int_{0}^{T} \alpha_{s} \mathrm{~d} W_{s}^{i}-\frac{1}{2} \int_{0}^{T}\left|\alpha_{s}\right|^{2} \mathrm{~d} s\right)
$$

Because the choice of the representative player $i$ is irrelevant (as the game is the same for all players), henceforth we will simply write $W$ instead of $W^{i}$. A mean field equilibrium is a pair $(\hat{\alpha}, \hat{\mu}) \in \mathfrak{A} \times \mathcal{C}\left([0, T], \mathcal{P}_{2}(\mathbb{R})\right)$ which satisfies

$$
\begin{equation*}
V^{\hat{\mu}}=J^{\hat{\alpha}}(\hat{\alpha}), \text { and } \mathbb{P}^{\hat{\alpha}}\left[X_{t} \in \cdot\right]=\hat{\mu}_{t}(\cdot), \mathbb{P} \otimes \mathrm{d} t \text {-a.e. } \tag{3.1}
\end{equation*}
$$

In this game, $W$ is a standard $\mathbb{P}$-Brownian motion. The set of admissible strategies $\mathfrak{A}$ is the set of processes predictable with respect to the ( $\mathbb{P}$-completion of the) filtration generated by $W$, with appropriate integrability requirements. The measure $\mu_{t}$ is a possible distribution of the path of all players at time $t$. Given such a density, since all players are identical and rational, they will come up with the same optimal strategy $\hat{\alpha}^{\mu}$. Therefore, at equilibrium, we expect (3.1) to be satisfied. This is precisely the same as Nash equilibrium in the sense that if all players use $\hat{\alpha}$, then the law of the trajectories is $\mathbb{P}^{\hat{\alpha}}\left[X_{t} \in \cdot\right]$, so that for a (single) player, not using $\hat{\alpha}$ will be suboptimal. The condition $\mathbb{P}^{\hat{\alpha}}\left[X_{t} \in \cdot\right]=\hat{\mu}_{t}$ is a fixed-point condition sometimes called consistency condition.

Analogous (and actually simpler) computations leading to Theorem 2.1 allow to derive the following probabilistic characterisation of mean field equilibria.

Theorem 3.1. The mean field game admits a mean field equilibrium $(\hat{\alpha}, \hat{\mu})$ if and only if it holds

$$
\begin{equation*}
\hat{\alpha}_{t}=-Z_{t}, \text { and } V^{\hat{\mu}}=Y_{0}, \mathbb{P} \otimes \mathrm{~d} t \text {-a.e. } \tag{3.2}
\end{equation*}
$$

where $\hat{\mu}_{t}(\cdot)=\mathbb{P}^{\hat{\alpha}}\left[X_{t} \in \cdot\right],(Y, Z)$ satisfies appropriate integrability conditions and solves the generalized McKean-Vlasov equation

$$
\begin{align*}
Y_{t}= & \left|X_{T}\right|^{2}+\int_{t}^{T}\left(\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}}\left[X_{s}\right]-\frac{1}{2}\left|Z_{s}\right|^{2}\right) \mathrm{d} s \\
& -\int_{t}^{T} Z_{s} \mathrm{~d} W_{s}^{\hat{\alpha}}, \mathbb{P}^{\hat{\alpha}} \text {-a.s., } \tag{3.3}
\end{align*}
$$

with $W^{\hat{\alpha}}:=W .-\int_{0} \hat{\alpha}_{s} \mathrm{ds}$, and

$$
\frac{\mathrm{d} \mathbb{P}^{\hat{\alpha}}}{\mathrm{dP}}:=\exp \left(\int_{0}^{T} \hat{\alpha}_{s} \mathrm{~d} W_{s}-\frac{1}{2} \int_{0}^{T}\left|\hat{\alpha}_{s}\right|^{2} \mathrm{~d} s\right)
$$

Equation (3.3) is a backward SDE similar to the ones encountered many times in these notes. It is said to be of McKean-Vlasov type because the evolution of $(Y, Z)$ depends on the law, and we call it generalized to stress the fact that the underlying probability measure $\mathbb{P}^{\hat{\alpha}}$ as well as the underlying Brownian motion $W^{\hat{\alpha}}$ should be constructed together with the solution, unlike in standard BSDEs where these are given: this is a consequence of the weak formulation of the game considered here. For games in the strong formulation extensively discussed in [CD18], a stochastic characterisation can be derived on the fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, but the corresponding equation is then a fully-coupled system of forward-backward SDEs of McKean-Vlasov type. The well-posedness of (3.2) is studied in [PT21]. There exist several results on wellposedness of mean field games, see, e.g., [CDL16], [CD18] and the references therein. There is one extra important fact we would like to highlight regarding the above characterisation result:
(i) as mentioned earlier, BSDEs characterizing Nash equilibria in $N$-player games are typically of dimension $N$ themselves;
(ii) there is one notable exception to this rule in the case of two-player zero-sum games, since the symmetry there allows to actually consider a single equation;
(iii) the mean field game framework is similar, at least in spirit: thanks to the overall symmetry of the problem, we can get a characterisation using a one-dimensional equation, despite having infinitely many players.
These observations indicate the reason why both zero-sum games and mean field games are more amenable to numerical simulations than $N$-player games, a fact that is well-recognized even for static and deterministic games. It is also important to point out that so far, the noise in the games we have considered are only idiosyncratic in the sense that all players deal with completely independent sources of randomness. In most applications, a common noise is added to the problem. In such cases, due to issues of measurability, the notion of equilibrium typically needs to be weaken or at least more carefully defined; and for solvability, the set of admissible controls sometimes needs to be enlarged to also consider relaxed (or randomized) controls described in the next subsection. See [CDL16] for details. In addition, uniqueness of the MFE is a delicate issue that often requires more than mere smoothness assumptions. Structural properties and/or monotonicity properties (as the so-called Lasry and Lions monotonicity condition [CDLL19] or the displacement monotonicity condition [JT23]) or sufficient dissipativity of the drift
are usually needed. We will refrain from discussing these-delicate-aspects of the problem here and rather direct our attention to another issue: the link between finite population and mean field games.
3.1. Convergence to the mean field game. To this point, the mean field game has been motivated by heuristic arguments inspired from statistical physics explaining how well-behaved, uncontrolled interacting particle systems converge to independent ones in the macroscopic limit. An interesting question is of course whether (and in which sense) stochastic differential games actually converge to mean field games. This question is interesting in practice because it justifies the mean field game as a proxy of the $N$-player game in a large population, but it is also mathematically interesting in that it would show to which extent propagation of chaos generalizes beyond uncontrolled particles systems to games.

This important problem was considered early on by [LL06] using PDE arguments. The authors considered a case in which, in the $N$-player game, each player has only partial information. More precisely, the strategy of each player $i \in\{1, \ldots, N\}$ depends only on their state process $X^{i}$ or their private Brownian motion $W^{i}$. Under this assumption, the controlled states $X_{t}^{i}=\xi+\int_{0}^{t}-k X_{s}^{i}+\alpha_{s}^{i} \mathrm{~d} s+W_{t}^{i}$, become independent, which is a very useful simplification to prove convergence, but the partial information assumption is very restrictive in practice. Further results have been obtained in the case of finite state space $\Omega$. First general results are due to [Lac16] and [Fis17]. The main idea underlying the results of these authors begins with the embed$\operatorname{ding} a \longrightarrow \delta_{a}$ of the set $A$ in the set $\mathcal{P}(A)$ of probability measures on it. Using this embedding, one can enlarge the set of admissible controls to consider relaxed controls, which are measures $q$ on $[0, T] \times A$ such that $q(\mathrm{~d} t, \mathrm{~d} a)=$ $\mathrm{d} t q_{t}(\mathrm{~d} a)$ for some $q_{t} \in \mathcal{P}(A)$ that is sufficiently integrable. Any (actual or strict) control $\alpha_{t}$ gives rise to a relaxed control $q$ given by $q(\mathrm{~d} t, \mathrm{~d} a):=\mathrm{d} t \delta_{\alpha_{t}}(\mathrm{~d} a)$. Considering relaxed controls, [Lac16, Fis17] essentially show that the sequence of empirical laws of any Nash equilibria is relatively compact and any limiting point is a mean field equilibrium. Further extensions can be obtained when considering closed-loop controls.

These general results suggest several questions; the most interesting of which, in the authors' opinion, being whether one can obtain quantitative convergence results. That is, is it possible to obtain (nonasymptotic) convergence rates informing "how far" the $N$-player game is from the mean field game? The simplest such bound is given by the central limit theorem which claims that, for $N$ i.i.d. square integrable random variables ( $\xi^{i}, \ldots, \xi^{N}$ ), it holds

$$
\mathbb{E}\left[\left|\frac{1}{N} \sum_{i=1}^{N} \xi^{i}-\mathbb{E}\left[\xi^{i}\right]\right|^{2}\right] \leq \frac{\operatorname{Var}[\xi]}{N}, \forall N \in \mathbb{N}^{\star},
$$

where $\operatorname{Var}(\xi)$ is the variance of $\xi^{1}$. Moreover, it seems natural to expect that mean field games would be explained as a form of propagation of chaos just as mean field models are for uncontrolled interacting particle systems.

As it turns out, Theorem 2.1 and Theorem 3.1 are central in answering the above two questions. Just as interacting particle systems model the evolution (or behavior) of stochastically interacting components in time, the N player game can also be described by the interacting particle system given by (2.2). However, in contrast to standard particle systems, (2.2) evolves backward in time in the sense that the terminal configuration of $\left(Y^{i, N}, \ldots, Y^{i, N}\right)$ is known (and i.i.d.), rather than the initial one. Similar to the mean field theory in which the mean field limit is obtained by propagation of chaos, it is now natural to expect that a form of propagation of chaos will allow to show that the (interacting) particle system defining $Y^{i, N}$ converges to $Y^{i}$, the i.i.d. system given by (3.2) and characterizing the mean field game. This is formalized below.

Theorem 3.2. If for every $N \in \mathbb{N}^{\star}$ the $N$-player game admits a Nash equilibrium ( $\hat{\alpha}^{1, N}, \ldots, \hat{\alpha}^{N, N}$ ) and $k$ is large enough, then the value of the $N$-player game converges to that of the mean field game in the following sense

$$
\left|V^{i, N}\left(\hat{\alpha}^{-i, N}\right)-V^{\hat{\mu}}\right|^{2} \leq \frac{C}{N}, \forall N \in \mathbb{N}^{\star},
$$

for a constant $C$ depending only on $T$, and where $\hat{\mu}=\mathbb{P}^{\hat{\alpha}} \circ X^{-1}$. Moreover, for all $N \in \mathbb{N}^{\star}$

$$
\mathbb{E}^{\mathbb{1}}\left[\int_{0}^{T} \mathcal{W}_{2}^{2}\left(\mathbb{P}^{\hat{\alpha}^{N}, N} \circ\left(\hat{\alpha}_{t}^{i, N}\right)^{-1}, \mathbb{P}^{\hat{\alpha}} \circ\left(\hat{\alpha}_{t}^{i}\right)^{-1}\right) \mathrm{d} t\right] \leq \frac{C}{N},
$$

where $\mathcal{W}_{2}$ is the Wasserstein distance.
This result is a particular case of the results derived in [PT21] to which we refer for detailed proofs. Theorem 3.2 quantifies the convergence statement in the sense that it provides a rate for the convergence of the $N$-player game to the MFG. In view of the rate dictated by the central limit theorem, this rate is sharp. Moreover, this result provides convergence of the full sequence of values and controls, not that of a subsequence. This result can also be established under a type of monotonicity condition known as displacement monotonicity, see [JT23]. In this case, we can allow much more general forms of coefficients, including controlled volatility, common noise and infinite horizon.
3.2. Link to partial differential equations. Mostprobably all-of the results discussed in these notes have PDE counterparts that can be derived using purely analytics arguments. Let us briefly elaborate on the link between the newly defined particle systems (2.2)(3.3) and PDE formulations. By the so-called nonlinear Feynman-Kac formula of Peng [Pen91], the solution

$$
\begin{array}{r}
\left(Y^{i, N}, Z^{i, j, N}\right)_{(i, j) \in\{1, \ldots, N\}^{2}} \text { of }(2.2) \text { satisfies } \\
Y_{t}^{i, N}=v^{i, N}\left(t, X_{t}^{1}, \ldots, X_{t}^{N}\right)
\end{array}
$$

where the functions $v^{i, N}:[0, T] \times \mathbb{R}^{N} \longrightarrow \mathbb{R}, i \in$ $\{1, \ldots, N\}$, is a classical solutions of the PDE system, with $\mathbf{x}=\left(x_{1}, \ldots, x_{N}\right)$

$$
\left\{\begin{array}{l}
\partial_{t} v^{i, N}(t, \mathbf{x})+\frac{1}{2} \sum_{j=1}^{N} \partial_{x_{j} x_{j}} v^{i, N}(t, \mathbf{x})+\frac{1}{2}\left(\partial_{x_{i}} v^{i, N}(t, \mathbf{x})\right)^{2}  \tag{3.4}\\
\quad-\sum_{j=1}^{N}\left(\partial_{x_{j}} v^{j, N}(t, \mathbf{x})\right)^{2}+\frac{1}{N} \sum_{j=1}^{N} x_{j}=0 \\
v^{i, N}(t, \mathbf{x})=\left(x^{i}\right)^{2},(t, \mathbf{x}) \in[0, T] \times \mathbb{R}^{N}
\end{array}\right.
$$

This is nothing but the Hamilton-Jacobi-Bellman equation associated with the $N$-player game described earlier, see, e.g., [CDLL19]. On the other hand, consider the socalled master equation given as

$$
\left\{\begin{align*}
& \partial_{t} v(t, x, \mu)+\frac{1}{2} \partial_{x x} v(t, x, \mu)-\left(\partial_{x} v(t, x, \mu)\right)^{2}  \tag{3.5}\\
& \quad+\frac{1}{2}\left(\partial_{x} v(t, x, \mu)\right)^{2}-\int_{\mathbb{R}} \partial_{x} v(t, y, \mu) \partial_{\mu} v(t, x, \mu)(y) \\
& \quad+\frac{1}{2} \partial_{x} \partial_{\mu} v(t, x, \mu)(y) \mu(\mathrm{d} y)+\int_{\mathbb{R}} y \mu(\mathrm{~d} y)=0 \\
& v(T, x, \mu)=x^{2},(t, x, \mu) \in[0, T] \times \mathbb{R} \times \mathcal{P}_{2}(\mathbb{R})
\end{align*}\right.
$$

and characterising the mean field game (see [CDLL19]). Here, $\partial_{\mu} v$ denote the $L$-derivative of the function $v$ which is understood as follows:
(i) lift the function $v(\cdot, \cdot, \mu)$ on $L^{2}\left(\Omega, \mathcal{F}, \mathbb{P}^{\hat{\alpha}}\right)$ by putting $\tilde{v}(\cdot, \cdot, \xi):=v(\cdot, \cdot, \mu)$ whenever $\xi \in L^{2}\left(\Omega, \mathcal{F}, \mathbb{P}^{\hat{\alpha}}\right)$ has law $\mu$;
(ii) denote by $\partial_{\mu} v(\cdot, \cdot, \mu)$ the Fréchet derivative of the lift $\tilde{v}(\cdot, \cdot, \xi)$ on $L^{2}\left(\Omega, \mathcal{F}, \mathbb{P}^{\hat{\alpha}}\right)$.
Equation (3.5) is a one-dimensional equation written on the infinite-dimensional space $[0, T] \times \mathbb{R} \times \mathcal{P}_{2}(\mathbb{R})$ whereas (3.4) is a multidimensional equation written on the finitedimensional space $[0, T] \times \mathbb{R}^{N}$. When (3.5) admits a classical solution, it follows (see, e.g., [CCD22] applied on the probability $\left.\operatorname{space}\left(\Omega, \mathcal{F}, \mathbb{P}^{\hat{\alpha}}\right)\right)$ that

$$
Y_{t}=v\left(t, X_{t}, \mathbb{P}^{\hat{\alpha}} \circ X_{t}^{-1}\right) .
$$

Well-posedness of (3.5) in the classical sense is unfortunately very difficult to obtain in most cases. The work [CDLL19] exhibits cases of games for which the master equation admits a unique solution which in addition has bounded derivatives. In particular, as a consequence of Theorem 3.2 one infers that for $i \in\{1, \ldots, N\}$

$$
\begin{equation*}
\left|v^{i, N}\left(0, X_{0}, \ldots, X_{0}\right)-v\left(0, X_{0}, \delta_{X_{0}}\right)\right|^{2} \leq \frac{C}{N}, \forall N \in \mathbb{N}^{\star} \tag{3.6}
\end{equation*}
$$

[CDLL19] shows that when adequate monotonicity is satisfied, the dissipativity coefficient $k$ is not needed to guarantee (3.6).

## 4. Going Further

Our discussion of stochastic differential games is just a glimpse of a rich and fast-developing theory. These notes focused on how the analysis changes with the number
of players, and we presented only games in the probabilistic weak formulation. This exposition omits important issues such as general conditions guaranteeing wellposedness and properties of equilibria, or the importance of the type of information used by players. For instance, we did not discuss the so-called closed-loop, or Markovian controls, nor the various type of monotonicity conditions on the data often needed to guarantee uniqueness of games such as the Lasry-Lions monotonicity and the displacement monotonicity. We also completely omitted the deep connections and similarities between optimal transport and mean field games: indeed, some particular mean field game problems, such as potential mean field games, can be written as a dynamical optimal transport problem. This opens the way for applying techniques from one field to the other, see for instance [LLO23]. Moreover, we would not want our readers to think that the theory is limited to continuous-time and/or uncountable state spaces: there is plethora of contributions analysing finite state mean field games, and which contribute deeply to the theory, see among many others [GMS13, BCCD21].

Many of the "principles" derived for competitive games in this article extend, albeit with additional technicalities to other types of games which we could not discuss here. This is for instance the case for cooperative games in which a social planner finds the best strategies for individual players to achieve a common objective. This is the kind of problems solved by ride-sharing apps. Games with leaders and followers where some players react to other players actions or decisions (e.g., Stackelberg games) or majorminor players can also be recast in a setting similar to the one discussed here. Such games are central in contract theory applications. Speaking of applications, as explained in the previous section, mean field games arise from finite player games in which players are symmetric and interact with (almost) all other players in the game. This is certainly not true for most applications. It would be interesting to study games in which players (possibly) interact differently with each of their peers; and where the form of interaction can evolve over time. Such games are called games on graphs. Although recent progress have been made, the study of games on graphs is still in its infancy, and it will undoubtedly provide very powerful modelling tools.

The vast majority of works on stochastic differential games, just as the present one, consider Nash equilibria. This has allowed to develop an impressive and successful theory. Notwithstanding this success, it is worth pointing out that the concept of Nash equilibrium itself presents several shortcomings. For instance, Nash equilibria are rarely unique and among the many equilibria, there can be several trivial or nondesirable ones, which poses the questions of selection, approximation and stability of equilibria. In fact, unless players can negotiate to agree on an
equilibrium, they can choose different equilibria, which may in turn result in a nonequilibrium set of strategies. Even when the Nash equilibrium is unique, the rationality assumption might be too restrictive since in most practical cases agents would rather settle for a satisfactory outcome than an optimal one. Analysing stochastic differential games beyond Nash equilibrium seems to be an interesting avenue for future research.

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## EARLY CAREER

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Krystal Taylor and Ben Jaye serve as the editors of this section. Next month's theme will be Research.


## Math Instruction

## Teaching Mathematics Content Courses for Future Elementary Teachers: An Invitation <br> to Innovation

## Gabriela Dumitrascu

Since the development of the academic field of mathematics education, a fundamental tension between subject matter and teaching method has existed and continues to be ongoing. There are conflicting answers to the question

[^11]of whether teaching should rely more on subject matter knowledge or pedagogical methods, which has led to divergent policies in teacher education.

## The Complexity: Teaching Future Elementary School Teachers

Some argue that subject matter knowledge for teaching is defined by the content that elementary school-grade students are expected to learn. Most likely, this perspective is shared by preservice teachers ${ }^{1}$ who are enrolled in the mathematics content courses required by their teacher preparation program. Others argue that teachers should have a broader perspective and deeper content knowledge so that they can understand where their students are heading in their learning journey. This perspective is more likely to be shared by instructors who teach content courses for future teachers.

Add to this tension the facts, that 1) the preservice teacher population is extremely homogeneous (mostly white females) often oriented towards nonmathematical areas [6], and 2) instructors of mathematics content courses for future teachers tend to have doctorates in mathematics with no experience teaching children in grades K-6 [5]. It becomes easy to see why teaching mathematics content courses to future elementary school teachers is one of the most challenging jobs in higher education. The overall homogeneity of elementary math teachers tips the scales in favor of method and pedagogy, but developing the right set of practical subject matter skills is equally important and should be central to any math teaching program.

## Two Frameworks for Curriculum Development: MKT and LT

MKT: Mathematics Knowledge for Teaching. With the technological tools that we now have at hand we can find ideas about how to develop a curriculum specifically for content courses for future teachers. For example, on ChatGPT, you might find the following guidelines: establish a strong foundation (deep and accurate understanding of elementary-level mathematical concepts, number sense, arithmetic operations, geometry, measurement, data analysis, algebraic thinking), model effective teaching strategies, provide opportunities for active learning, encourage

[^12]reflection and self-assessment, incorporate real-life connections, differentiate instruction (adapt instruction for individual learning styles), foster problem-solving skills, stay up to date, supervise practice, collaborate, and mentor. Did I miss anything? Surely, I did! But how do we put those steps into practice in the reality of a classroom setting?

From a theoretical perspective, the "mathematical knowledge for teaching" (MKT) framework has been widely accepted in the past two decades of American mathematics education. The framework "weaves" four main threads of knowledge [1]:

- Common mathematical knowledge (expected to be known by any well-educated adult)
- Specialized mathematical knowledge (strictly mathematical knowledge that is particular to the work of teaching, yet not required, or known, in other mathematically intensive professions: e.g., how to represent the steps and the reasoning behind the division algorithm using base-ten blocks, or illustrate the division of fractions using an aria model)
- Knowledge of mathematics and students (how children learn mathematics; what are common mistakes, misconceptions, or naïve interpretations)
- Knowledge of mathematics and teaching (strategies and teaching practices that are successful in teaching mathematics effectively).

The work done to develop the MKT framework is extensive and impressive, and so far, it has been used primarily to analyze teachers' work or to develop instruments for measuring teachers' knowledge. In my teaching, however, I use MKT as a blueprint for creating a more effective curriculum for instructing future mathematics teachers. This means that I start every lesson planning with a list of learning goals that correspond to each category of knowledge.
LT: Learning Trajectories. Clements and Sarama [4] propose learning trajectories as a framework to guide teachers in helping children develop their math skills effectively. It emphasizes the importance of understanding child development and bridging the gap between research and practice to provide equitable math education for all students. Effective teaching involves meeting students where they are in terms of their mathematical knowledge and helping them build on what they already know. Learning trajectories are proposed as a solution to address these challenges. They consist of three parts:

- Specific mathematical goal (each trajectory has a specific mathematical goal that students are working toward)
- Development path (there is a path along which children develop their mathematical understanding to reach the goal)
- Instructional activities (along this path, there are instructional activities that are fine-tuned for each step to help children progress to the next level).
Learning trajectories are useful resources for exploring the impact of the mathematical content progression on the development of mathematical understanding. In the section Developing Mathematical Knowledge of this article, I describe how I used LTs to teach arithmetic operations in base-r systems.

Note that I used the verb "weave" to describe the MKT framework. The reason is that it requires imagination and creativity to combine all four strains-common, specialized, student and teaching knowledge-and the LTs into a single lesson. In that vein, this essay is an invitation for both new and old generations of instructors to use these frameworks in addition to their own passion and creative powers when they train future mathematics teachers. What follows is an example of how I have incorporated the MKT and LT methods into my teaching, and how doing so has helped to balance theory with practice, which opens creative opportunities that have made my teaching more effective. Others may use or adapt this model to help them bridge the divides of subject matter and pedagogy, to achieve their teaching goals, and maximize their students' math teaching potential. The example addresses three main concerns: developing in-depth mathematical knowledge, reducing math anxiety, and increasing sensitivity in future elementary math teachers.

## The Challenge: Developing Awareness, Trust, and Mathematical Knowledge in Future Teachers

Developing awareness. In my experience, the hardest part of instructing future elementary teachers is figuring out how to help them discover and recognize the need to engage with mathematical content on a deeper level, as doing so will help them to more effectively teach mathematics to young children. As a mathematics educator who also teaches methods courses, I can see (or I've noticed) that recognition for this need develops in future teachers only once they've had direct interaction with children. However, content courses are usually taken before the methods courses, and direct interaction with children is not part of the requirements for a content course. For this reason, I start my content courses with two readings that underscore how mastering impacts both children's mathematical development and teachers' professional development: "Sean's Numbers," which is an extract from the article "Mathematics, mathematicians, and mathematics education" [2] and "Invitation to Learn and Grow," which is a section from the textbook Elementary and Middle School Mathematics: Teaching Developmentally [7].

In our first meeting, students are divided into groups and asked to read "Sean's Numbers," which reveals the types of mathematical knowledge that enable a teacher to navigate complex mathematical interactions skillfully and adaptively in a diverse classroom. The reading describes a classroom episode where students are exploring the concepts of even and odd numbers. One student, Sean, questions whether six can be both an even and an odd number (six can be divided fairly into two groups of three and, also, into three groups of two). The teacher guides the discussion by encouraging the students to articulate their ideas and arguments. Mei, another student, engages in a thoughtful and well-expressed critique of Sean's argument. Mei's astute argument leads to an ironic exchange with Sean which mathematically is a debate about mod 2 arithmetic and $2 \bmod 4$ numbers.

Each group is then asked to answer four questions, listed below.

- What are some characteristics of even numbers? Why, in a sequence of consecutive numbers, do the even and odd numbers alternate? Why do even + even $=$ even; odd + odd $=$ even; even + odd $=$ odd? Do Sean's numbers (odd multiples of two) have similar properties? (Common mathematical knowledge)
- In the article we find three definitions for even numbers: fair share (a number is even if it can be split into two equal groups); pair (a number is even if it is composed of groups of two); and alternating (the even and odd numbers alternate on the number line, with zero being even) (pg. 427). Using visual representations (graphical illustrations), how do you show that the three definitions of even numbers are equivalent? (Specialized mathematical knowledge)
- Follow Sean's mathematical idea about numbers that are odd multiples of two (Sean's numbers) and check if Sean's number + Sean's number $=$ Sean's number (Knowledge of mathematics and students)
- What is the position of Sean's numbers on the number line? How would you use this argument to conclude the discussion? (Knowledge of mathematics and teaching)
The following discussion is focused on the mathematical knowledge that a teacher would need to use Sean's question as an opportunity to help their students grow in their mathematical thinking. In my closing statement of that session, I talk about modular arithmetic as a special type of arithmetic that should be part of the knowledge they draw on to support, and not to impinge on, children's mathematical explorations, regardless of whether they end up teaching it to K-6 students.
Developing trust. In our second meeting, students read "Invitation to Learn and Grow," which describes what skills are required to teach mathematics in the 21st century.

The article lists seven skills, in the form of characteristics, habits, and abilities, in the following order: knowledge of mathematics, persistence, positive disposition, readiness for change, willingness to be a team player, devotion to lifelong learning, focused time to reflect and become selfaware. After we discuss the basic meaning of each skill, I offer my own perspective: that the list represents a learning trajectory for becoming an effective math teacher, but the skills should be developed in reverse order. That is, examining ourselves for areas of improvement and reflecting on our successes are both the markings of a lifelong learner, as well as behaviors that foster intellectual growth and continued development. Additionally, working together as a team and supporting each other will both motivate us and provide us with the support we need to pursue a difficult endeavor.

To learn and to grow is to continually change. But a readiness to change relies on trust-trust in ourselves and our peers-which is why collaboration and support are so important to the learning process. They help build that trust and encourage all of us to take a chance on new ideas, even if those ideas disrupt our equilibrium. Once we become comfortable and willing to learn something new, we will be able to cultivate a positive attitude towards the subject of mathematics. A positive attitude facilitates a person's ability to persist, reflect on, and engage with problem solving and other mathematical thinking. This will lead us to develop a more flexible and adaptive mathematical thinking, which will help expanding our mathematical knowledge. As with any other human skill, these skills can become automatic only through repeated practice and reinforcement. Once they become ingrained in the brain, they can be executed quickly and efficiently without conscious thought. For this reason, focused time to reflect and become self-aware, devotion to lifelong learning, willingness to be a team player, readiness for change, positive disposition, persistence, and knowledge of mathematics define the tone and describe the norms for my classrooms. I have designed a poster listing these norms, which I insert into each course syllabus and website.
Developing mathematical knowledge. The next challenge, for me when instructing future elementary teachers is implementing the MKT framework, which is intended to help future teachers learn to

- Create meaningful learning experiences by navigating complex mathematical interactions skillfully and adaptively.
- Present concepts in multiple ways.
- Make connections between content.
- Think beyond their own perspective and instead focus on:
- Knowledge of mathematics and students
- Knowledge of mathematics and teaching

Since in the K-6 mathematics curriculum the largest section is focused on developing numbers and operations sense, a part of the mathematical content for future teachers is also about understanding number systems and operations. Presenting numbers and arithmetic operations in different number systems provides unique insights that future teachers need to be efficient in developing numbers sense and operations sense in young children. Since using numbers and performing operations in base 10 are already automatic reflexes for an adult, one way to challenge these reflexes is by examining how numbers and operations work in bases other than 10. I have tried several approaches to engage my students, which I failed because some students avoided answering the exam questions related with non-base 10 number systems. Others, who were successfully solving problems with numbers represented in a base other than 10 , were doing so by referring to their base 10 understanding instead of using grouping and place-value.

The bitter taste of failing and the firm conviction that the MKL framework should be the baseline for efficiently preparing mathematics teachers compelled me to look for a better way to introduce numbers and number systems. Consequently, I found an approach that gave me more satisfaction regarding my students' level of engagement.

I start the section on numbers and number systems by presenting the research done on children's mathematics learning trajectories, and I introduce my students to the Learning and Teaching with Learning Trajectories [LT] ${ }^{2}$ website [3] where they watch a couple of videos of children who are at different levels on their learning trajectory for counting. Then, I invite my students to experience for themselves the process of learning to count as the young children do: I tell them they will learn a new way to count, using new names for numbers and new symbols. In the "new" numerical system, which is base 5 , the numbers are alpha for $\operatorname{card}(\{x\})$, beta for $\operatorname{card}(\{x, x\})$, gamma for $\operatorname{card}(\{x, x, x\})$, delta for $\operatorname{card}(\{x, x, x, x\})$, and epsilon for $\operatorname{card}(\})$. I use the Greek alphabet, but you have the freedom to invent your own symbols and words to express the cardinality of sets. As they struggle counting (especially backward and skip counting) they also share their feelings and reflect on what they are experiencing. The goal of the discussion is to understand how children might feel when they are first learning about numbers and operations (MKT framework).

To address knowledge of mathematics and teaching (MKT framework), I divide the class into groups. Each group is given a bag of Q-tips that they are asked to count, then report back to me the number of Q -tips in their bag. Afterward, each group receives an addition and a subtraction problem, each of which they need to solve and represent using Q-tips. Once they are done solving the prob-
lems, I reorganize the groups using the jigsaw strategy (each new group has one member from each of the previous groups). Within their new groups, each student must explain and model how they used the Q-tips to solve the problems.

After these activities, the engagement of future teachers is noticeably higher in exploring how the arithmetic operations work in different numerical systems and they start to recognize the importance of the mathematical concepts of grouping and place-value in the development of number sense.

In conclusion, my process for mathematics teacher preparation blends the following three perspectives:

- Practice to develop skills and habits of mind for effective mathematics teaching.
- Design instruction using the MKT and LT theoretical frameworks.
- Commit to collaboration between research mathematicians and mathematics educators (which should be the topic for another article).

These three perspectives bring the disparate worlds of theory and knowledge and teaching together and open a whole universe where our creativity and imagination can become limitless.

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# Interdisciplinary Teaching of Mathematics with Primary Historical Sources 

## Richard A. Edwards

## A Great Divide

In 1959, the chemist and novelist C. P. Snow (1905-1980) identified what he saw as an increasing and unproductive isolation between scholars of different disciplines. "We have two polar groups: at one pole we have the literary intellectuals, at the other scientists. Between the two there is a gulf of mutual incomprehension." [Sn, p. 4] Snow recalled moments in his career when literary elites would scoff at scientists who were unfamiliar with the sonnets of Shakespeare, while they were themselves ignorant of comparable scientific ideas such as the laws of thermodynamics.

Whether or not Snow accurately described intellectual life in the mid-20th century, and there is an argument that the way he concretized the gulf merely increased academic tribalism, I am fortunate to work at an institution that actively encourages interdisciplinary research and teaching. I have derived great benefit from rubbing shoulders with colleagues in the humanities and social sciences. Their perspectives on education, history, philosophy, and ethics have shaped my views on what effective teaching looks like, and what it means to learn.

What of my students? Do they appreciate interdisciplinary teaching, or wonder about how scientists and novelists can productively collaborate? Many of my students take a consumerist view toward their courses. By this I

[^13]mean that their primary goal is to pass my class and get on with their degree. The actual content of the course is less important than the fact that it moves them one step closer to their career aspirations. If they think about it at all, many tend to think of mathematics primarily as a tool for solving problems in science. They might enjoy my class, but very few of them will think much more about it once the semester has come to a close. Is there a place in my classroom for the humanities? When I began my career this wasn't something I thought about. The only teaching I did that could be considered interdisciplinary amounted to little more than occasionally teaching a history of mathematics course, and infusing my calculus lectures with anecdotes (of questionable veracity) about famous mathematicians. Then a quote from a British educator named Charlotte Mason (1842-1923) captured my imagination:

> There is a region of apparent sterility in our intellectual life. Science says of literature, I'll have none of $i t$, and science is the preoccupation of our age. When we present theorems divested to the bone of all superfluous trappings, we lose the vitality along with what we've stripped away. History expires in the process, poetry cannot come to birth, religion faints; we sit down to the dry bones of science and say, here is knowledge, all the knowledge there is to know. [Ma, p. 317]

Snow challenged his associates in the humanities, while Mason chided the scientists. Can I, as a mathematician, teach in ways that draw from the best of both traditions? I'd like to teach in ways that retain the vitality and effervescence of mathematics. I would like to restore humanity to theorems. I want my students, as Polya described, to experience the tension and enjoy the triumph of mathematical discovery [Po].

I believe I have found a path forward by teaching mathematics via primary historical sources. Moving toward teaching in this way has been one of the most rewarding, yet challenging, efforts of my career. It has also changed how I personally think about learning mathematics.

## Primary Source Projects

The benefits and challenges of teaching with primary sources have long been a source of discussion among those interested in the history and pedagogy of mathematics [Ja]. Reading primary source texts allows students to see how individuals first conceptualized an idea, and how mathematical ideas have evolved over time. Many textbooks, almost by their very nature, present mathematical ideas as refined and finished products. In contrast, original sources help to foreground the motivations, cultural contexts, and intellectual atmosphere of their source authors. Primary sources display the patterns of communication that have
characterized the mathematical community, can reveal how those standards have changed over time, and why.

In response to these benefits, over 100 primary source projects (PSPs) have been developed under the NSFfunded TRIUMPHS project (https://digita1commons .ursinus.edu/triumphs/() and its predecessor grants. PSPs are classroom projects designed to replace standard textbook-driven presentations of important mathematical topics. Each PSP features selections from one or more historical sources, supplementary text from the project author that provides both historical and mathematical background, and a series of tasks which help students interact productively with the historical source and learn its mathematical content. Each PSP has been designed to help students reach a level of fluency with a mathematical topic that is at least as strong as if they had learned it via a traditional textbook approach, in roughly the same amount of time. They are replacements, not additions, to my syllabus.

The tasks in a PSP offer students opportunities to engage with mathematics in a variety of powerful ways. These include activities which model how mathematicians actually work; for example, conjecturing, testing, refining, proving, and generalizing relationships between objects. PSPs also include tasks that allow students to interpret results as they were originally presented, and then reformulate these results in modern terms. Such activities encourage robust understanding of mathematics by immersing students in an ongoing conversation which can sometimes span centuries.

For example, I implement "Fermat's Method of Finding Maxima and Minima" [Mo] in order to help students better understand the extreme value theorem, learn methods for finding extrema of functions, and practice their derivative rules. In addition to these object-level themes, the project can help break students out of recipe-thinking with regards to optimization, show how a technique has evolved over time, and generate discussion around the question of what counts as a general method. The source material comes from the writings of Pierre de Fermat (1607-1665), along with some commentary on Fermat's work that Rene Descartes (1596-1650) sent to Marin Mersenne (1588-1648). Students get a glimpse into the personalities of these mathematicians as they struggle to understand and explain a topic (optimization) that firstyear college students struggle to understand and explain today. After presenting his optimization process, Fermat boasted:

We can hardly be provided with a more general method.

However, at this early stage in the project, most of my students appreciate the critique raised by Descartes (in a private letter to Mersenne):

If he [Fermat] speaks of wanting to send you still more papers, I beg of you to ask him to think them out more carefully than those preceding!
I like to use these primary source excerpts to motivate student discussion: Was Fermat's method robust, or did it only work for the specific examples he chose? Was Descartes right to question the generalizability of the method? How is Fermat's method similar to, or different from, the method in our modern textbook? By the end of their correspondence on this subject, Descartes seemed happy with Fermat's method, and wrote,

Seeing the last method that you [Fermat] use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would not have contradicted at all.
Certainly a ringing endorsement for students and faculty alike to communicate our ideas clearly... and show our work.

Primary source projects take students to pivotal moments in the history of mathematics. For example, the PSP "Rigorous Debates Over Debatable Rigor: Monster Functions in Introductory Analysis" [Ba] transports students to the late nineteenth century when Jean Gaston Darboux (1842-1917) and Guillaume-Jules Hoüel (18231886) were beginning to think about properties of functions as something worthy of study in their own right. As with every PSP in the TRIUMPHS collection, the goal is to teach mathematics, not its history. This PSP features core objectlevel themes such as continuity, differentiability, the Intermediate Value Property, Darboux's theorem, and uniform differentiability. ${ }^{1}$ Because the results are presented in their human and historical contexts, instructors can use it to talk about important meta-level themes such as: Why might someone take a critical view of the basic ideas of calculus? Why did mathematicians need to develop new vocabulary, techniques, and theorems in calculus? Other scholars, notably [BCC], have analyzed this particular project in detail with respect to its ability to promote student discussion of metadiscursive rules in Introductory Analysis. Here I restrict myself to sharing some excerpts which illustrate how the project presents the human drama of mathematical correspondence.

The discussion between Darboux and Hoüel began cordially enough... but then descended into a flurry of colorful phrases as the mathematicians become increasingly frustrated with each other.

Darboux began:
Go on then and explain to me a little, I beg you, why it is that when one uses the rule for

[^14]composition of functions, the derivative of $y=$ $x^{2} \sin \frac{1}{x}$ is found to be $-\cos \frac{1}{x}+2 x \sin \frac{1}{x}$, which is indeterminate for $x=0$ even though the true value is $\lim \frac{y}{x}=0$ ?

Darboux to Hoüel, January, 1875
Darboux then critiqued certain proofs which Hoüel had previously provided, using delightful phrases such as "Here is what I reproach in your reasoning. . ." and ". . your proof has nothing but the appearance of rigor." Hoüel's responses hinted at a growing frustration over what he perceived as Darboux throwing up pointless counterexamples, and it seems as if the two correspondents were beginning to talk past each other.

> Yes, I admit as a fact of experience (without looking to prove it in general, which might be difficult) that in the functions that I treat, one can always find $h$ satisfying the inequality $\frac{f(x+h)-f(x)}{h}-$ $f^{\prime}(x)<\epsilon$, no matter what the value of $x$, and I avow to you that I am ignorant of what the word derivative would mean if it is not this. I believe this hypothesis is identical with that of the existence of a derivative.

## Hoüel to Darboux, January 1875

Hoüel's response did not satisfy Darboux, who seemed more concerned with attending to the dependencies between variables in a proof. He tried to get Hoüel to reflect on how variables are introduced, especially those variables that carry universal quantifiers. This is something many students today also struggle with at this point in their mathematical studies, which is one reason why having them read this source material can be powerful.

You have not addressed the nature of my objection...For your methods to be sound, you will need to explain very clearly what part of your reasoning is deficient in this particular case. Without that, your proofs are not proof. As for the question of the derivative, this time you change the question. It is clear that for a value $x_{0}$ of $x$, that saying

$$
\lim \frac{f\left(x_{0}+h\right)-f\left(x_{0}\right)}{h}=f^{\prime}\left(x_{0}\right)
$$

is the same as saying: One can find $h$ such that

$$
\frac{f\left(x_{0}+h\right)-f\left(x_{0}\right)}{h}-f^{\prime}\left(x_{0}\right)<\epsilon
$$

for this value of $h$ and for all values that are smaller. But there is an abyss between this proposition and the following: Being given a function $f(x)$ for which the derivative exists for all values of $x$ between $a$ and $b$, to every quantity $\epsilon$, one can find a
corresponding quantity $h$ such that

$$
\frac{f\left(x_{0}+h\right)-f\left(x_{0}\right)}{h}-f^{\prime}\left(x_{0}\right)<\epsilon
$$

for all values of $x$ between $a$ and $b$.

## Darboux to Hoüel, January, 1875

In addition to exploring issues such as the proper placement of quantifiers in assertions involving multiple variables (e.g., those that define today's properties of pointwise and uniform differentiability), [BCC] note that this PSP gives students opportunities to discuss important questions related to mathematics: What is the purpose of examples? What intuitions are refined by studying them? The Darboux-Hoüel correspondence represents an important turning point in the history of analysis. I love using PSPs such as this one, which take students to the forefront of mathematical developments. I'm convinced that living on this ragged edge is both more exciting for students, and more mathematically satisfying, than some textbookdriven lectures.

Although many PSPs feature work from names my students recognize, such as Euler, Gauss, Cauchy, etc., many of the primary source texts give students exposure to geographically and culturally diverse authors. When I implement A Genetic Context for Understanding the Trigonometric Functions [Ot], my precalculus students get to work through selections from Greek mathematicians Hipparchus and Ptolemy, from Hindu mathematicians such as Varāhamahira, and read selections from The Exhaustive Treatise on Shadows, written in the court of a Turkish sultan in the year 1021. My second-semester calculus students solidify their understanding of series convergence through the PSP Bhāskara's Approximation to and Mādhava's Series for Sine [Mo2]. I'll never forget the day when six of my students simultaneously burst out in delightful surprise at seeing source material written in Sanskrit-a language they had learned to read growing up in India (of course the PSP also provides an English translation). In my multivariable calculus course, we often end the semester with Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem [Ed] in which students read the work of the enigmatic George Green (1793-1841), a working-class miller who didn't begin his college career until middle age, but whose ideas about electromagentism led to the theorem which now bears his name. This year, I look forward to implementing at least one project based on the work of Maria Agnesi [Mo3].

## Challenges, and What it Means to "Learn"

Despite their great potential, teaching with PSPs brings its own set of challenges. Many of my students struggle with reading. Asking them to read (translated) excerpts from long ago is a heavy lift for some of them. It helps to have them work through the projects in groups. It takes me
longer to prepare for class when I'm going to teach with a PSP, although each project features detailed notes for instructors as well as an implementation plan. I don't use a PSP for every topic in a course (although I know instructors who teach courses using only PSPs). In a typical semester, I find time for three or four projects, depending on the course. One method I've recently found success with is putting students into groups and giving each group a different PSP (but all related to a similar topic, such as series convergence) to complete. At the end of the week, we have a "PSP Showcase" in which each group gets to do a mini-presentation of their work to the rest of the class.

Not all of the PSPs are well-aligned with my online homework problem system. However, since each PSP is intended to be a replacement for my standard lesson, I can also use the student's written work to replace the online homework for that lesson. One challenge that I am very aware of is trying to avoid interpreting history through the prejudices of today. This kind of presentism-judging the past by today's standards-can inadvertently give students a sense that all of history was an inevitable sequence of events, of which the 21st century student is its pinnacle.

The greatest-but most rewarding-challenge that I have had to wrestle with in teaching with PSPs has been re-orienting my thoughts about what it looks like to learn mathematics. Of course I want my students to become fluent in the discourse of modern mathematics. Yet being familiar with the modern conception of an idea can sometimes be tantamount to knowing only the last page of a long and richly complex story. Instead, I find it helpful to conceptualize learning as increased participation in the mathematical community [La]. This includes knowing both the current standards of our community, but also how our mathematical ideas have changed over time. PSPs are one means to facilitate that kind of learning.

PSPs give students opportunities to witness mathematicians at work, to "imitate [their] moves while trying to figure out the reasons for the strange things [they are] doing" [Sf, p. 202]. This may be an important step in helping students tell new stories about the world of mathematics, and their own place in that world. I close with a quote attributed to Descartes that frequently comes to mind while I'm teaching:

Scientific truths are battles won.
-Rene Descartes, quoted in [We, p. 162]
His words remind me that much of what we teach has a rich and important history. Perhaps by immersing students in that history, we can help give them a more robust understanding of our subject. PSPs may not bridge Snow's "gulf of mutual incomprehension," but I can testify to their ability to generate enthusiasm and excitement for learning mathematics.

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Richard A. Edwards
Credits
Photo of Richard A. Edwards is courtesy of Lyman Briggs College/Blythe White.

# Getting Your Hands Dirty: Teaching Math Biology with Active Learning Strategies 

## Adrian Lam

Nowadays, mathematical and computational methods are ubiquitous in many areas of biological research, such as genomics, ecology, evolutionary biology, neuroscience, and systems biology, to name a few. It is therefore important to introduce students to the interdisciplinary field of mathematical biology at an early stage, typically during their freshman or sophomore years. It is no surprise that an increasing number of universities are recognizing the importance of mathematical biology and integrating it into their undergraduate curriculum. The integration of mathematics and biology opens a world of possibilities for students to explore various biological phenomena using quantitative techniques. By grounding the mathematical concepts in real-life biological scenarios, students also gain a deeper appreciation for the role of mathematics in shaping their understanding of the living world.

I am an associate professor at the department of mathematics at OSU. My research interest lies in the analysis of partial differential equations. I have worked on systems of reaction-diffusion equations and free-boundary problems which are inspired by applications in biology. I have had the pleasure of teaching and advising students in the math bio track since I joined the faculty at the Ohio State University in 2014. In this article, I aim to share some of my personal experiences with teaching the course Introduction to Mathematical Biology with my colleague Avner Friedman (founding director of Mathematical Biosciences Insti-

[^15]tute) since 2018. While coteaching mathematical biology can be quite different from teaching other more traditional mathematics courses, in terms of the syllabus, audience, and teaching goals, it also offers ample opportunities to apply active learning techniques. Here, I would like to share some of our recent experiences and personal take-aways in interacting with our students.

One of the major differences in teaching mathematical biology compared to traditional mathematics courses lies in the scope and emphasis of the syllabus. In a math biology course, it is crucial to provide students with a thorough understanding of the biological context behind mathematical models. Traditional applied math courses may mention motivation briefly before diving into theorems and proofs. However, in a math biology class, the goal is to establish a strong connection between mathematical methods and their applicability to biological problems. Rigorous proofs are still valuable but take a back seat to explaining the biological rationale behind the models.

Another significant difference is the audience in a math biology course. Many students who enroll in this course do not major in mathematics. While they can be bright individuals (many of whom are premed students), they often find mathematical concepts challenging to grasp or even intimidating. As instructors, building rapport with such students and ensuring effective communication can be a challenging but rewarding task.

The traditional way of teaching mathematics involves presenting the subject logically, defining precise mathematical objects, deriving results, and providing examples of alternative solution methods. After that, students can work on problem sets independently to improve their familiarity with the techniques. For a math biology class, however, there are opportunities to apply active learning methods and dedicate more time to inquiry- and problembased labs.

To foster a more interactive and engaging learning environment, we structured our course into weekly modules. Each week, we introduce a mathematical method alongside one or more biological motivations for its use. For instance, for the module focused on epidemiology in week 6, we introduce the SIR (Susceptible-Infected-Recovered) model, which is a set of ordinary differential equations depicting the transition of the overall infection status among members of a population. Given that most students were affected by the COVID-19 pandemic, lively discussions ensued when exploring how to incorporate real-world details into the model, such as how to incorporate an asymptomatic period in the SIR model before an infected individual becomes symptomatic and can be detected.

In addition to theoretical discussions, we emphasize numerical computation during our teaching. This approach allows students to witness how models work and how to
interpret results in a biological context. For example, while covering birth-death processes, we immediately used laptops to compute the survival probability of right whales in the North Atlantic from given data based on the work of Caswell et al. [Caswell, Fujiwara, and Brault, PNAS, 1999]. This exercise illustrated how small changes in birth rates can significantly impact the outcome.

Numerical computation not only provides students with a practical understanding of mathematical models but also equips them with valuable computational skills that are increasingly essential in modern biology. In today's data-driven scientific landscape, the ability to analyze data and simulate models computationally is critical for making informed decisions and conducting cutting-edge research. To further enhance students' programming skills, our course includes a hands-on computing component primarily using MATLAB. On Fridays, the lecture shifts to a laboratory section with weekly programming assignments. These assignments are individual small projects based on classroom demonstrations. Students are not expected to prove theorems but rather to utilize MATLAB to compute the model with given parameters. Our goal is for them to engage in practical problem-solving using mathematical tools. During the lab, we demonstrate how to translate a biological problem into a mathematical model and guide students on numerically computing and presenting results accurately. The students then work on their assignments on their own computers, and this setting provides ample opportunities for personal interaction. We debug code together, discuss code rationale and purpose, and explore effective presentation and interpretation of results. The numerical assignments take the form of short reports, where students are expected to present and explain the numerical results in terms of how they address the biological question. This setup empowers students to experience real-world problem-solving and leaves them feeling accomplished and knowledgeable.

The course culminates in a final project, spanning four to six weeks. In small groups, students are assigned concrete biological problems along with small data sets. Their first task is to research the biological background to identify where mathematics could be helpful in providing answers. The second step is to select an appropriate mathematical model that represents the relevant biological process. The final step is to use mathematical theories and computational methods in order to fit the model and derive predictions from the model. The choice of model includes those discussed in the course, but students are not limited to them. This part of the course actively involves students in the scientific process of applying mathematical theory to solve biological problems. Regular meetings with the instructor in and outside of class offer opportunities for progress updates and feedback. During the final
two weeks of the semester, the groups take turns presenting their findings to their peers. Here, the various groups take the podium and teach their peers about the biological problem, explain their rationale for mathematical model selection, and showcase their analytical and numerical results. With adequate feedback, most groups successfully complete their projects, providing students with a taste of scientific research in the context of mathematical biology.

As an instructor, teaching this course is a refreshing experience due to the flexibility it offers in content selection. This allows the instructor to include current research topics or address emerging environmental issues like the transmission of COVID-19 or global climate change. Moreover, modeling biological phenomena provides an open-ended inquiry experience that does not carry a single correct answer and demands a different set of skills than those required for solving well-defined mathematical problems. I am delighted by the quality of questions I receive from students, and the course fosters extensive discussions concerning the relation of mathematical models to specific biological situations. The final group project provides an opportunity for me as the instructor to meet with small groups of students outside of regular lectures. While working on their projects, students share with me their reasons for taking the course and their long-term goals. While many took this course to fulfill the math biology track requirements, I am surprised that a large number of them took this course purely out of curiosity. Since a substantial number of them are planning to further their education in medicine and biology, I hope that this experience can train them to think critically as well as quantitatively. Also, this personalized interaction allows me to get to know my students on a deeper level. In fact, I am working with a group of particularly successful students to write up their project findings for submission to a journal dedicated to undergraduate research.

In conclusion, teaching mathematical biology with hands-on, computational component has proven to be effective in enhancing student engagement and understanding. Problem-based labs and projects offer students the skills and experiences needed to use mathematics effectively, which can be challenging to teach using lecturestyle approaches. Additionally, these methods promote communication between math faculty and students. By grounding mathematical concepts in real-life biological scenarios, students gain a deeper appreciation for the significance of mathematics in modern biology. As the field continues to evolve, these courses play a pivotal role in shaping the next generation of biologists equipped with quantitative expertise and problem-solving skills.

ACKNOWLEDGMENTS. This article is dedicated to my late colleague Professor Ching-Shan Chou. ChingShan Chou and Avner Friedman first designed and pioneered this undergraduate mathematical biology course at the Ohio State University.


Adrian Lam
Credits
Photo of Adrian Lam is courtesy of Adrian Lam.

# Extreme Cases: Math Education Within the US Prison System 

Katherine J. Pearce

Pulling into a visitor parking space for the first time, I watch a stray dog hesitantly making her way toward the barbed wire fences. The surroundings are austere and unforgiving. Lockhart Correctional Facility is about an hour's drive from the city of Austin and miles from the nearest town center, situated, intentionally, in the middle of nowhere. I wonder briefly how the dog ended up here and feel a pang of sadness that I have nothing to offer her. It will not be the last time I struggle with these feelings in this parking lot; they only intensify when I think about my students also being left here to fend for themselves.

I pocket my ID and car keys, grab the box of printouts I made for class, and head toward the barbed wire gate. With sunlight still streaming through the lobby's glass doors behind me, the first checkpoint feels similar to airport security. I empty my pockets, place everything on a table to be searched, remove my belt and shoes, and walk slowly through the metal detector. A female correctional officer (CO) outlines my body with her wand before patting me down, paying special attention to the bottoms of my feet, while another CO searches my class materials.

[^16]DOI: https://doi.org/10.1090/noti2917

He warns me that highlighters and papers with "too much ink" are not permitted, but thankfully all my materials are allowed through today. I gather them up and head toward the next security checkpoint, where any glimpse of daylight or sense of familiarity is gone.

The main guard station remotely buzzes me through a door after surveilling me on camera, and I hand over my ID in exchange for a badge. I am buzzed through one last door and suddenly step out into prison, narrowly avoiding the lines of people herded in and out of the cafeteria. I hear my pulse in my ears until I notice that they are all smiling and welcoming me. Someone exclaims, "God, I miss wearing jeans," and we laugh. One of the incarcerated women approaches me to show me to my classroom, and we walk down a hallway under fluorescent lighting and a large hemispherical mirror that shows everything around the corner. "We are so grateful and excited you are here!" she tells me. I feel the same.

When we arrive at the classroom, she explains apologetically that I will have to wait while the COs release my students to come to class. Although class technically begins at 6 p.m., we usually don't start until 6:30, but I enjoy the wait. That half hour has become the most important part of teaching for me; without a book, phone, or electronics to kill time, I end up studying all the art and handouts displayed on the walls. Our classroom is also used by vocational and reentry programs, and the exercises that the students complete are somber reminders of the real stakes here. "My goal is to get sober for my children." "I want to learn how to forgive myself and earn forgiveness from others." "When I get out, I will start a new career with my certifications to support my family." These messages reiterate to me that I am here primarily to build their confidence in themselves and their problem-solving abilities. As a formerly incarcerated friend will later share with me, that confidence gives them a sense of freedom even in incarceration.

The twelve students in my class trickle in, and we introduce ourselves. Even though I learned about average state prison populations during the Texas Prison Education Initiative's (TPEI) orientation, I am still surprised at the large variance in age, from about early 20s to mid 60s. I feel a flutter of anxiety about the pace I've decided on for my lectures. The students need to pass my course as a prerequisite for any credit-bearing TPEI math course offered through UT Austin's extension program, so I want to focus on building their abstract problem-solving abilities. In particular, I have decided to spend a lot of time developing their intuition in subject areas they had likely seen before, like integer addition; I am worried the students will be bored with my decision.
"So, just out of curiosity, what do y'all think about math in general?" I ask, quickly adding, "No judgment here, I
do it for a living and still feel love-hate about it." We laugh, but it's hard to keep smiling as I hear about their previous experiences in math classes. The students are diverse in age, race, ethnicity, and sexual orientation, but they unfortunately had one major thing in common: all of them proclaimed to be "bad at math," and many had dropped or failed out of school specifically because of math. ${ }^{1}$

I tell them my plan for the course: we are going to start over with math and look at it from a new perspective. Anything that they need to know for the course, I will teach them, and it would actually be preferable if they could wipe away any memory of subjects where they had unsuccessful learning outcomes. Even things they have seen before, like integer addition, I want us to consider with fresh eyes. "Our course objective, and the real advantage of math," I tell them, "is using specific instances of a problem to understand how it works in full generality." And that is exactly what we do.

As mathematicians, we often look to "extreme cases" of a problem to gain intuition for a general solution. Mathematics education within the prison system is perhaps the most extreme case, and I believe it sheds light on how we can teach more effectively in a university classroom.

## 1. Satisfying Constraints

In addition to the usual concerns of teaching math, teaching math in prison comes with a unique set of challenges. With respect to the aforementioned issue of bringing class materials inside, there are certain canonical classroom resources that are prohibited or impossible to obtain. There is no computer, internet access, or even a desk for lecture materials, only a small whiteboard at the front of the room which, just like in university classrooms, is unreliably equipped with dry erase markers. There are no office hours, recitations, or means of communicating with students outside of class time. If students miss class for a variety of legitimate reasons, like having to work late at their prison jobs or being confined to their cells during a lockdown, they have no way to access material or get in touch with me until the next class.

During the summer, for example, I taught an elective course called "The Art of Mathematics," in which students investigated several math topics, like algorithms and infinite set cardinalities, that served as inspiration for their own art work. TPEI provided sketchbooks, folders, and colored pencils, but I was not permitted to leave the colored pencils with the students when class was over. The other class materials required certain stickers to denote that TPEI had given them to these students, in hopes they would not accidentally be confiscated.

[^17]I also brought in art books and textbooks that I had at home to pass around the room, which I was told to collect at the end of each class. Though I did not, I was very tempted to ignore this rule on two occasions. Once when an older student was poring through The Math Book by Clifford A. Pickover and asked me excitedly if she could check it out like a library book. Then again when a younger student asked me if she could borrow Baby Rudin ${ }^{2}$ because she was curious about analysis proofs; she'd also asked me earlier that evening which prerequisite textbooks or classes she'd need in order to teach herself analysis some day. Later in the course, I learned the latter student was trying to study astronomy in college before her arrest; unfortunately, she'd been ousted from the STEM major due to her grades in the required math classes.

The culmination of the math art elective course was to be a gallery night, a classroom exhibition of their work on the last day. Together, we would reflect on the various topics we'd covered, discussing the mathematical ideas and artistic choices that spoke to us in each other's work. I'd invited the TPEI program coordinators, Max and Chloe, who were also excited to see what the students had created.

When we arrived at the classroom, we waited for an unusually long time before a couple of students came in and mentioned they'd had a difficult time getting to our room. Around 6:45, when Max went back to the central guard station to ask if all TPEI students had been called for class, we were told that a fight had broken out in one of the dorms, and it was now on lockdown. As a result, despite no involvement in the fight, about half of my students were unable to come and present the art pieces they'd worked on all summer. By the time the ones who could make it finally arrived, we only had about 45 minutes left. Everyone involved was disappointed, but those of us in attendance tried to make the best out of the remainder of the evening. The exhibit was breathtaking, and the influences and creativity in every piece were truly awe-inspiring. Max and Chloe were blown away and agreed that the students' exhibit looked and felt professionally curated.

Since we couldn't bring cameras inside to take pictures, I proposed to the students that, with their permission, I'd borrow their sketchbooks overnight, take photos of their work, then return their sketchbooks to them the next evening at another TPEI instructor's class. That way, students wouldn't have to tear out pages from their sketchbooks and would only be without them for less than a day. However, I was honest about my concern that there could be some unforeseen issue getting the sketchbooks back to them, which was always a risk. I told them I understood completely if they wanted to hold on to their work just in case. I was moved close to tears when every one of them gifted me their art work, tearing pages out of their
${ }^{2}$ Principles of Mathematical Analysis by Walter Rudin (1953)
sketchbooks for me to take home and keep. Those pieces are now framed and displayed in my office, and photos of some of the pieces are shown in Figure 1. In (a), the sketch was created with a straight-edge, colored pencils, and deoderant for shading, inspired by M.C. Escher's work with H.S.M. Coxeter on the "limit of infinite smallness" [Wie10]. Figure 1(b) shows a self-portrait of the aspiring astronomy student contemplating math topics from the course. In (c), the concept of yin and yang is illustrated, playing off of mathematical symmetries. In Figure 1(d), the sketch on the left is inspired by the Poincare disk, created with circular objects of different radii that the student collected from around the prison. On the right in Figure 1(d), the same student interprets Escher's limit of infinite smallness. In (e), we see an artistic rendering of a Fibonacci spiral in nature, and in (f), a self-portrait of the student in front of a tiled background, with a new (imaginary) tattoo showcasing her love of math on her right elbow. The sculpture in (g) is a 3-D quilling of an elephant representing me, named "Lil Kate," made out of small paper strips dyed with food coloring and adhered with a mixture of coffee creamer and water. In (h), the student outlines a proportionally accurate Fibonacci spiral with a meander [CL16].

## 2. Developing Intuition

One of the biggest challenges of teaching in prison is being diligent in my language choices. I actively try to avoid using idioms or explanations that could trigger mental blocks for students, especially about math subjects in which they'd experienced difficulty or unsuccessful learning outcomes. Since I am hoping to build up their intuition on a new mathematical foundation, it's important not to repeat the same explanations that caused them confusion the first time. Moreover, because of the already difficult environment we are in, I don't want to say something that would cause students any more distress.

A small but pervasive example of being more conscious of my language in the classroom was swapping out the term "homework" for "assignment" in my college algebra course. But I also wanted to make sure my writing achieved the same objectives as my classroom language and tone. So every week, I wrote lecture notes for the students to keep that read exactly like I would speak during class, while still including the usual textbook definitions, examples, formulas, etc.-and while trying to avoid anything that would elicit a mental block toward the subject. This style of lecture notes was especially important in situations where students missed class. Since there are no dedicated times like office hours, appointments, or recitations for them to get help, I tried to make my notes reminiscent of our classroom: thorough, but not dry, while still containing the explanations they need to understand the material.


Figure 1. Students' math-inspired art work.

Out loud and in text, before introducing any new topic in the course, I wanted to provide motivation for why we were talking about it, which really addresses why they should care about it. As a stereotypical example, I did not have to ask how my students felt about fractions; they all groaned loudly when I said, "Today, we'll be talking about fractions." I had anticipated this, mostly because of certain family members and friends who have had an identical response to dealing with fractions. I jokingly threw up my hands and acquiesced over the groans: "Okay, fine, we'll talk about one of my favorite things in math first instead ...prime numbers!" We then detoured through prime numbers, prime factorizations, division trees, greatest common divisors, and least common multiples before we finally circled back to fractions. Suddenly, the two biggest obstacles to them mastering fractions previously (adding and simplifying) were reduced to problems they had just tackled in a very different setting. Once we'd gotten around the initial hurdle to approaching a subject they'd already convinced themselves they'd "never understand," my students no longer experienced the same mental block toward it. Instead, I witnessed them appreciate the power of abstraction, to the point that one of them exclaimed, "Holy crap, I can actually help my kids with their homework now!" after adding three fractions with different denominators using their least common multiple.

By the time we got to solving linear equations and word problems, the students had a completely different outlook on the material. After hearing it so many times on the first day, I had banned the phrase "bad at math" from our classroom, but even if I hadn't, that was not how the students felt anymore. We had spent multiple class periods developing intuition for the properties of addition, subtraction, multiplication, and division of real numbers, from an abstract perspective but also with the concrete example of debt to understand computations with negative numbers. What I'd worried would seem boring to the students was actually what initially piqued their curiosity: we took a specific instance of a problem, then abstracted it away with algebraic tools to study the problem in full generality. Once they'd seen a given topic from these two different angles, they also had plenty of "low-stakes" opportunities to practice. I frequently reminded them that it's normal to make mistakes when trying something new. So by the time I said to them, "Today, we are going to solve some word problems," the students were no longer groaning, or feeling anxious about using variables to represent unknowns; their confidence in their abstract problem-solving abilities had been strengthened by the time we spent in the lowstakes material, building a solid foundation from which to work. Upon solving one of the linear systems arising from a word problem, one of my students proudly announced, "I feel like a mathematician!" "You are," I replied.

When I taught the art elective course, I ascribed to a similar ideology. Each mathematical topic (algorithms, proportion, infinity, and abstraction) was paired with complementary art work from different time periods and regions of the world to illustrate the idea. It was surprising how much of the material, which I'd created with the intention of introducing them to advanced undergraduate math they wouldn't have seen before, illuminated other mathematical concepts for them that I hadn't even anticipated. When we talked about proportions, for instance, we discussed how the ancient Greeks thought the ideal relationship between the width $w$ and height $h$ of a building is given by the "golden proportion"

$$
\varphi=\frac{w}{h}=\frac{w+h}{w}
$$

After we talked about its relationship to the Fibonacci sequence, I mentioned offhandedly that we can solve explicitly for the golden ratio $\varphi$ using the quadratic formula; a student immediately raised her hand and asked to see how that would work. After I showed them the trick of setting $h=1$, they were amazed to see an "elementary" formula they'd rotely memorized in school being used to answer such a seemingly unrelated question. Later in the same class, when we were discussing the Archimedean method of approximating $\pi$, I wrote the more familiar equation $C=\pi d$ as $\pi=C / d$ to emphasize the ratio. To my surprise and dismay, the vast majority of them had never seen or considered $\pi$ as a ratio, and it was emotional to see them appreciating the amazing property of circles that every mathematician discovers at some point in their career. I began to think deeply about how we motivate these concepts when students first encounter them.

My students' comments in class cast many mathematical concepts in a new light for me as well. For example, in talking about algorithmic constructions with girih tiles [LS07], we discussed how it is possible with rudimentary tools to fabricate these tiles with precise angular measurements and arrange them into astonishingly intricate designs. We talked about how to decompose the polygonal tiles into triangles, and how an understanding of triangles unlocks a lot of possibilities for the tiles' fabrication and design. One of the students mused casually, "So that's why they teach an entire class about triangles. They're like the atoms of shapes." I've since begun borrowing that phrase to motivate the subject of trigonometry.

During another class period, we talked about the cardinality of the natural numbers. I chuckled to myself as I listened to a familiar debate among the students. "Why would 0 be included? You don't count anything with zero fingers." Fortunately, they were satisfied that it didn't much matter with respect to cardinality after we wrote out the bijection between the natural numbers and the integers. However, that awareness came back to bite me in the form of a deeply
profound and unexpected question that I received while explaining Cantor's diagonalization argument. I had just demonstrated the proof by contradiction: if we try to enumerate all of the elements $s_{1}, s_{2}, s_{3}, \ldots$ of the set $T$ of infinite sequences of 0's and 1's, it is always possible to construct an element $s \in T$ that differs from $s_{k}$ in the $k$ th position, so that element $s$ of $T$ actually wasn't enumerated in our list. "But why can't you just call that element $s_{0}$ ? Then wouldn't you be able to count them all with counting numbers since it doesn't matter if we include 0 ?" The question was so subtle and clever that it caught me off guard. "You're thinking like a mathematician," I replied, before we spent the next few minutes verifying that Cantor had indeed gotten it right, though probably not on his first try.

## 3. Solving Problems

Once I was talking about my experience teaching math in prison with a friend of mine who was formerly incarcerated. He'd served a ten-year sentence over the entirety of his 20s, during which time he had the opportunity to take several math courses for high school degree equivalency. He's now earning his bachelors degree while working fulltime as a water treatment facility operator. While math courses equipped him with necessary skills for his new career, he credits those classes with something even more important. As a creative writer, my friend had always felt more passionate about writing classes in school. He admitted to me that he'd never liked that there was "only one right answer" in his math classes. But once he was incarcerated, solving math problems became a mentally stimulating and comforting activity. With so much time to think and reflect, he realized how many problems in life lack a clear-cut solution, and he began to appreciate the existence and uniqueness of the solutions to the problems in his math assignments.

I told my students about this conversation with my friend, and I asked them if they felt similarly about learning math while incarcerated. Several of them participate in entrepreneurial and vocational training programs, and those students echoed the importance of math for their new career paths. In fact, one student made parole during our spring algebra class, and she still wanted to finish the course remotely postincarceration to help in earning her EMT certifications. (This situation is one of many in which the TPEI program managers' administrative efforts are essential to our success.)

Several other students said that my friend's point about taking comfort in having a right answer really resonated with them, even though they'd had terrible previous experiences with math. One of them was the student who'd had to drop her astronomy major because of her math classes, who'd asked about teaching herself Baby Rudin. Another student had been concurrently earning her high school
diploma during our class ("only 50 years late!" she'd often say), and she explained to me how learning math again had helped her find balance in her life. "Everything in math has an opposite," she said, "I am a 'yin and yang' type of person, and I guess that's why I love math. You can always add back anything you subtract, or multiply anything you divide." Her friend chimed in, "Plus, even just knowing there are infinitely many possible choices to start from gives you options for all of the gray areas in life." Yet another student, who was nervous about her upcoming parole hearing the following week, reiterated that being able to solve a problem and find a correct answer out of infinitely many possible choices was going to be vital to her success postincarceration. Like so many of her classmates, she'd also been interested in STEM growing up; she wanted to be an astronaut when she entered high school, before dropping out because of math. "I'm nervous about finding a job and a permanent place to live, but I have people to help me for now," she said. "I probably can't be an astronaut anymore because of my felony charge, but I still want to enroll in college and earn my BS once I'm back on my feet." In spite of all the problems and uncertainties that she faced postincarceration, she'd already begun identifying solutions.

On the last day of the art elective class, after the exhibition, I announced that I would be teaching a brand new course in the fall that had never been offered by TPEI. It is a credit-bearing math course at UT Austin that is the prerequisite for the calculus sequence. My student-the aspiring astronomer enthralled by Baby Rudin-came up to talk to me after the other students had said their thankyou's and goodbye's. "I really want to take the course, but I've failed precalculus before and I'm worried I will again," she said nervously. "I haven't thought about math in a long time." Remembering all of the insights she'd shared during the course, I told her that I knew she could do it and reassured her that I would be there to help her succeed this time. "Besides," I added, "you've been thinking like a mathematician this whole time."

ACKNOWLEDGMENTS. I am extremely grateful to all of my fellow TPEI volunteers, especially Emily Hsiao, Max Lubell, Chloe Craig, and Sarah Brayne, for their guidance and support. I would also like to thank Ilse Ipsen, Mansoor Haider, Tye Lidman, David White, Justin di Stefano, and especially my students for their invaluable perspectives.

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Katherine J. Pearce
Credits
Figure 1 is courtesy of students of TPEI.
Photo of Katherine J. Pearce is courtesy of Katherine J. Pearce.

## Tips for Saving Time with Grading

## Jennifer Schaefer and Rebecca Swanson

Grading is a way for faculty to both assess student work and provide feedback to students on their level of understanding of a course's material. Whether your grading system is traditional or nontraditional, the feedback-giving process is often one of the most time consuming aspects of teaching. As early career faculty transition from lighter graduate school teaching loads to more sizable teaching loads as new faculty, finding time to grade an increased amount of student work can be challenging. Having a student grader can reduce your grading burden, but not all institutions offer this support. Our goal is to provide you with some tips and tricks to make the grading process more smooth and efficient regardless of institution.

[^18]DOI: https://doi.org/10.1090/noti2918

## Think About Your Policies

A lot of energy can be saved by carefully crafting grading policies when you develop your syllabus. For instance, you could decide to grade student work on proficiency, something both authors do in their classes. What this means is that homework is graded using a scale of proficiency, e.g., mastery, near mastery, tentative, unsure, or didn't attempt. This form of grading saves time because the instructor can focus on feedback instead of agonizing over the number of points to deduct for mistakes. The following are a few other policies you can implement in any course. Regardless of what you decide, we advise that you create grading policies that work for you and your course and stick to them.

Determine whether shorter, more frequent assessments or longer, less frequent assignments work better for your course and your grading style. One of the authors assigns and collects homework assignments each class period. She does this because she finds these shorter assignments less overwhelming, and she can grade the assignments more quickly. However, one of her colleagues prefers to collect a longer homework assignment once per week.

Allow for a small number of low-stakes assignments to be dropped. This removes some of the anxiety when students receive lower grades than they would like and can eliminate the request for make-up assignments.

Think about your late work policy. Trying to balance flexibility with fairness is tough and determining what to do on a case-by-case basis can waste time and emotional energy. If you do decide to allow for make-up assignments or extensions, only give a limited number and keep the extension window short. This reduces the number of submissions you need to keep track of. One of the authors uses a policy that she has found to be equitable and easily enforced. Homework submissions are due on Wednesdays, but solutions are posted on Saturdays. As long as a student asks for an extension before the due date, it is granted, until the solution set is posted, at no penalty.

Incorporate group assignments into your course. This allows you to assess student understanding while grading fewer submissions. At the same time, students develop the skill of working with others, which benefits them outside of class.

Once your policies are in place, how can you make grading and giving feedback easier? We've compiled lists of both tech-free and tech-required tools that can help.

## Tech-Free Grading Tips

Grade a smaller selection of the assigned problems, or grade some assignments and not others. It is OK if you don't grade everything you assign. Let us say that again. It is OK if you don't grade everything you assign. After all, the goal is for the bulk of student learning to happen when students
complete the assignment to begin with. You could do this by grading a subset of the problems you assigned. Or you could consider having some assignments that receive feedback without a grade and other assignments that receive grades without feedback. Either way, you can provide solution sets for students to see where they made mistakes and how they can correct them.

Grade one problem at a time and group problem solutions based on mistakes made. When grading problem sets, your mental load will be higher if you attempt to grade more than one problem at a time. This is because you have to change what you are looking for as you move between problems, and this shift takes time and mental energy. As you move through students' submissions for a given problem, it is helpful to group the assignments by the mistakes the students made. This allows you to quickly provide the same feedback to all students who made similar mistakes and helps ensure fairness with your grading.

Use a rubric. Designing a rubric will take a bit of time up front, but utilizing a rubric while grading can save you a great deal of time overall. This is because a rubric delineates exactly what you are looking for into a limited number of categories and requirements. The fewer buckets you have, the easier and more consistent your grading will be. Moreover, if you pair a rubric with student presentationsgroup work, homework, or projects-you can fill out the rubric while the presentations are happening so that once a student is done, you are done grading as well. This can allow you to assess work in real time without needing an additional block of time just for grading. By giving your rubric to your students ahead of time, you also communicate what your expectations are, which helps them correlate your feedback with the grade they receive.

Have your students self-grade their work. Allowing students to assess their own work can help students develop metacognition skills. It can also help reduce the amount of grading you are required to do. One of our colleagues pairs self-grading with student presentations of homework solutions. While a presentation is occurring in class, students are allowed to use a green pen to mark up their work based on their classmate's solution and any additional feedback the instructor offers. While this does take class time, grading goes more quickly because the students have already marked up their submissions. Another one of our colleagues in the mathematics community chooses to focus on student reflection when grading homework. She posts solutions once students submit work and has them selfgrade and reflect on that work by asking questions about their understanding of what was done wrong, the mistakes they made, and what they still don't understand. This allows her students to think more intentionally about what they are learning. Students who thoughtfully complete this assignment get full credit.

## Tech Tips

There are many ways that technology can help you assess student work too. Some are more conventional and others are relatively new. For instance, requiring students in major courses to complete their homework in LaTeX makes it easier for us to read their solutions. As an added bonus, it also provides them the opportunity to gain expertise in using our discipline's typesetting tool. As a newer option, you could use tools such as Audacity to record your comments over audio in place of or in addition to written feedback. The following are a few other options to consider.

Online homework. Most publishers have online homework systems associated with their textbooks. Such publisher-based systems come with large problem banks that offer students a great deal of practice and immediate feedback. The challenge is that there are often limits to the types of problems that can be posed, and that the focus of these problems is usually on the final answer, not on the work done to get that answer. Additionally, there is a cost to either the students or the institution. Some free alternatives do exist, though, that work with almost any textbook. WeBWorK is a free system supported by the Mathematical Association of America, and MyOpenMath is another free option that works with open educational resources in mathematics. Further, many campuses utilize a learning management system (LMS) in which you can create your own free online homework. Of course, then you are not able to make use of the large question banks available in the other systems. It should be noted that you do not have to go all or nothing with online homework. For instance, one of the authors uses online homework in her calculus courses but supplements it with written work as well.

Tools that help grade written work. Your LMS may also be capable of helping you grade written work. For instance, Canvas has a tool called Speed Grader, Blackboard uses BbAnnotate, and Moodle accepts file submissions through the Assignments feature. Each of these allows students to upload their own work and you to grade their submissions electronically, essentially the equivalent of paper grading, but online. This saves you from writing the same comment by hand repeatedly, entering grades into a gradebook, and spending time returning papers to students in class. An additional benefit is that students can submit work as a group, and your feedback is given to the entire group at once. Finally, you and your students don't have to pay extra to use your institution's LMS!

There are additional resources that can help support faculty in giving feedback that are much more robust than the built-in LMS tools. Some examples include Gradescope, Assign2, Crowdmark, and Pearson's Freehand Grader. (As a disclaimer, both authors have experience working with Gradescope in their classes but have not had the
opportunity to test the others.) What they have in common is that students complete work on paper, a tablet, or a computer, and then upload their work for you to grade electronically. For most, the instructor can also scan and upload student work. These tools all support multiple graders grading different problems from the same assignment at the same time, making them useful in coordinated settings. With any of these resources, you can edit your rubric as you go, and the changes apply to what you have already graded. You can also make use of saved comments as you give your feedback. All of them provide student data that allows the instructor to determine where students are struggling. Syncing with a standard LMS makes grade entry automatic, and as with the LMS tools, you aren't spending class time returning papers. Each of the resources below has one common challenge-they aren't free. Pricing plans vary and are often dependent upon a number of factors. However, you usually have an option to run a free pilot. If you are interested in continuing to use the product, you will need to further discuss those details with representatives at these companies. If you are using a nontraditional grading scheme that isn't points-based, these tools are not directly designed to help you, although each has its own workarounds. Below we highlight some of the differences among these tools:

- Gradescope. Gradescope is a well-developed grading platform by Turnitin. Gradescope provides a "grouping" feature that is currently not available in the other tools on this list. With this feature, the instructor can sort problems into groups based upon mistakes and then apply the same grade and feedback to all problems within a particular group. This can be especially useful when class sizes are large. Gradescope has two plans-a basic plan and an institutional plan. It should be noted that not all of the more desirable features are available through the free basic plan.
- Assign2. Assign2 seems to stand out in the robust statistics it can provide, including data on the amount of time the instructor spends grading a particular problem. For instructors planning to scan and upload student work, the system uses QR code technology to match submissions to students. Additionally, Assign2 is in the process of developing a number of new features, including tools to grade code and a grouping feature similar to that of Gradescope. They seem to have the easiest workaround for non-traditional grading schemes. Finally, they are the most budget-friendly option in this list.
- Crowdmark. Crowdmark, developed by a mathematician and his graduate student, aims to improve dialogue between students and instructors, as well as to scale human-to-human interactions. Crowdmark is very intentional about keeping the ownership of data
in the hands of the students and faculty, making it readily available for distribution and assessment purposes. Crowdmark also utilizes QR-code technology that helps match students to their submissions, which is quite useful for instructors who scan and upload a large amount of student work. The pricing structure depends upon a number of factors. However, unlike with Gradescope, all features are available to all users.
- Pearson's Freehand Grader. For those of you already using Pearson's MyLab and Mastering online homework system, you have access to their Freehand Grader tool. While this tool is less developed than the others in this list, it has the advantage that it is free to those already using MyLab and Mastering. This is the one tool on the list that does not allow the instructor to easily scan and upload student work, as it is aimed more toward homework grading than exam grading, but it does have the other features that help grade studentsubmitted work more quickly, including the ability to build and alter the rubric as you go and to grade problem-by-problem, which is often better for consistency.
We hope you found something new that can save you some time while helping support your students as they learn. By carefully thinking about both your grading policies and which tools you will employ, you can make this aspect of your job easier, more supportive of student learning, and (maybe!) more fun. Happy Grading!

ACKNOWLEDGMENTS. The authors would like to thank the following people for contributing their grading tips: Jill Brautingam Jordan, Jennifer Gorman, Anansa Green, Diana Hubbard, Audrey Malagon, Kate Owens, Amy Parrott, Chris Somari, and Diana Thomas.


Jennifer Schaefer


Rebecca Swanson

Credits
Photo of Jennifer Schaefer is courtesy of Dickinson College. Photo of Rebecca Swanson is courtesy of Rebecca Swanson.

## Dear Early Career

How can I remain positive about my work after the rejection of a grant application?
—Down-in-the-dumps
Dear Down-in-the-dumps,
This is a tough one. After my first NSF rejection I worked very hard to write what I thought was an excellent resubmission which took into account the comments of the reviewers. My subsequent proposal was then promptly rejected with a collection of rather dismissive comments, and it was difficult to think about going through that process again. From a position where now I have had some more failure and a bit of success in the process, my coping strategy is to acknowledge the amount of randomness in the process, and just concentrate on the writing of the proposal itself as an important part of my research process.

To be more detailed, let's look at the process from the perspective of a reviewer. A reviewer on an NSF panel often has around 10-12 proposals to read and write reports on in advance of the panel (in addition to whatever their current workload is). The proposals are discussed in the panel session itself, and this discussion is typically summarized in a review. A panel might consist of around 10 people, and a proposal will typically be read by 3 panelists, but other panelists can make comments during the session. Additionally, external reviewers may be solicited for their opinions of the proposal. However, the views of the 3 panelists will be the predominant factor in the final evaluation of the proposal. I am not suggesting there is a better alternative to this system, but there is a lot of randomness associated to it: Some people are overly confident in their opinions on certain areas of research, and others will be biased by how "well-known" the submitter of the proposal is, whether this is conscious or not. It is much more difficult to read a proposal when you do not have direct research experience with problems closely related to it, and some reviewers misinterpret their own discomfort with a proposal as a weakness of the proposal. It could well be that if a proposal just fell into the inbox of three different panelists, then the outcome would be different. Additionally, it can happen that your proposal is great, but the competition is particularly tough that year, and some reviewers focus on creating reasons why your proposal was not ranked as highly as others, rather than
giving some constructive feedback and encouraging resubmission.

Having said this, I do take feedback seriously and try to incorporate constructive comments in a resubmission. My rule is that if multiple reports all point to the same issue, then this should definitely be addressed, as even if I think their comments are completely off-base and the ravings of a lunatic, my proposal certainly did not express itself as I had intended. When I have received comments that were particularly difficult to process about research, then discussing these with senior colleagues that I trust has been useful.

Finally, remember that the process of writing a grant application is constructive. It can organize your thoughts on your research and force you to do the groundwork on the directions in which you would like to push your research. The work done putting your proposed research into a larger context will help you when you write research papers associated to those problems, and your efforts will help with other documents such as research statements. I look back at my proposals quite regularly when writing papers or thinking about projects to start new collaborations.

Best of luck with your resubmission!
-Early Career editors

Have a question that you think would fit into our Dear Early Career column? Submit it to Tay1or .2952@osu.edu or bjaye3@gatech.edu with the subject Early Career.

DOI: https://doi.org/10.1090/noti2913

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# In Memory of Igor Krichever Alexander Braverman, Pavel Etingof, Andrei Okounkov, Duong Phong, and Paul Wiegmann 

## The Work of Igor Krichever

Professor Igor M. Krichever of the Mathematics Department of Columbia University passed away in New York City on December 1, 2022. He was born on October 8, 1950 in Kuybyshev in the former Soviet Union. He graduated in 1972 from the Department of Mechanics and Mathematics (MechMat) of the Moscow State University, under the direction of Professor Sergei P. Novikov.

Throughout his career, he held research positions at the Krzhizhanovsky Energy Institute, the Institute for Problems in Mechanics, and the Laudau Institute for Theoretical Physics. In 1992, he became a professor at the Independent University of Moscow and later visited Columbia University as the Eilenberg Chair of Mathematics in 1996, becoming a permanent faculty member in 1997. He significantly contributed to the development of the Columbia Mathematics Department and served as its chair from 2008 to 2011. He also taught at the Higher School of Economics in Moscow and served as deputy director of the Institute for Problems of Information Transmission of the Russian Academy of Sciences. In 2016, he founded the

[^19]Center for Advanced Studies at Skoltech, which now bears his name.

The central theme in Igor Krichever's research was the theory of solitons, where he made groundbreaking contributions that shed new light on a wide range of topics in mathematics and physics, notably algebraic geometry, quantum integrable models, statistical physics, condensed matter theory, and string theories. He received numerous awards throughout his career, including the Prize of the Moscow Mathematical Society, and was invited to speak at several International Congresses of Mathematicians, including as a plenary one-hour speaker at the 2022 ICM. Columbia University held week-long conferences in his honor in 2011 and 2022.

Krichever's scientific legacy is profound, extensive, and diverse. It is nearly impossible to encapsulate the full breadth of his contributions within the format of a memorial article. Nonetheless, we can highlight some of his most influential achievements:

- Systematic construction of algebro-geometric solutions of integrable models, notably the KadomtsevPetviashvili (KP) equation, based on the concept of BakerAkhiezer function.
- Development of what is now known as the KricheverNovikov algebra, which extends the Virasoro algebra to a Riemann surface with two marked points (punctures).
- Introduction of a Lax pair for the elliptic CalogeroMoser system, linking the dynamics of its poles with the KdV equation, which later played a crucial role in SeibergWitten theory.
- Development of the Whitham semiclassical approach to nonlinear waves in soliton theory.
- Integrable structure of Laplacian growth.
- Construction of a universal symplectic form for soliton equations based on the Lax pair, leading to a new Hamiltonian theory of solitons applicable to 2D equations.
- Proof of Welters's conjecture of the early 1980s that an indecomposable principally polarized abelian variety is
the Jacobian of a curve if and only if there exists a trisecant of its Kummer variety.
- Characterization of Prym varieties as polarized abelian varieties with Kummer varieties admitting a pair of symmetric quadrisecants.
- Complete solution of Peierls instability leading to the formation of charge density waves, and analysis of finiteband structure in electronic crystals.

The above is only a small sampling of Igor's scientific legacy. More is detailed in the individual recollections below, offering different perspectives on the aforementioned works.

Igor Krichever was truly a great mathematician. Clearly, mathematics in general, and the theory of integrable models with its applications to algebraic geometry and theoretical physics in particular, owe him a lot. However, those of us who had the privilege of knowing him personally or working with him owe even more. He provided us with a model of honesty, kindness, and generosity, and demonstrated equanimity and fortitude in the most trying circumstances. For this, we shall always be most grateful.

## Memories

## Enrico Arbarello

There was something of a Chagall's quality in the watercolor that Tanya painted during her first visit to Rome with her parents, Igor and Natasha, over thirty-five years ago. She was very young, and I remember admiring not just her talent, but the subtle layering of culture, over generations, that had brought her to that point. My mother was still alive then, and this intangible cultural kinship flowed in all of our conversations, across generations and across wildly different experiences. It was as if my mother had always known Igor and Natasha: they shared a deep humanity and sense of humor that knows no limits of age or language. I had a similar experience some twenty years earlier, when, as a young man just landed in New York, I found a safe haven in the home of Mary and Lipa Bers, whose friendship and warmth melted away any feeling of unfamiliarity as soon as I stepped through their door.

In the early 1970s, years before Igor's first visit to Rome, I attended a very crowded lecture by Sergei Novikov at the CNR (Consiglio Nazionale delle Ricerche). It was something completely new to me. He talked about KdV equations and hyperelliptic curves. I did not get much

[^20]out of that lecture, other than the determination to study those beautiful relations from an algebro-geometric point of view. When Igor arrived, he talked indiscriminately with geometers and physicists, like my friend Francesco Calogero, and this presented the opportunity to deepen my understanding of those relations. I recognized his versatility and openness as one of the wonderful traits of the Russian school, which I have observed throughout the years in meeting all the Russian mathematicians passing through Rome.

In retrospect, one of my deepest regrets is that we, in Italy, could not somehow find a way to hold on to all those great mathematicians leaving the Soviet Union. Igor, who was deeply rooted in Moscow, was not among them at that time, but many others were. We could not find a way to attract them to stay in Italy. And it should have been easy! In fact it was clear that the general disorganization of a country like Italy made them feel quite at ease. The cumbersome bureaucracy, the endless forms to fill out, the myriad of obstacles thrown in the way whenever you try to get anything done, as well as the consequent cleverness in finding ways around all these difficulties, beating the system at its own game: this aspect of Italian life was all too familiar to anyone coming from Russia. So I wish that, when Igor decided to get a position abroad, that the very same bureaucracy had not stood in the way when we should have jumped at offering a desirable job to Igor, whose presence would have influenced Italian mathematics in wonderful ways. But Columbia University won, without even a fight.

In the subsequent years, when talking with Igor, there was this constant willingness to understand each other's points of view: the dynamical system point of view, and the algebro-geometrical point of view. From Igor, and in fact from the Russian school, I learned how to keep both points of view in my mind, and it was instructive to see how the miraculous mathematical procedures that Igor often produced had a neat, but also concealed, algebrogeometrical counterpart. A typical one was his use of the Baker-Akhiezer function.

The last time we were in Moscow together was 2018, and it was a truly memorable stay. It was a privilege to enjoy art, theater, music, ballet, and the city in general, with Igor and his friend Irina (who went by Ira). I had the feeling of a very sophisticated city, whose deep culture enriched all aspects of life, and became a wonderful background to our interesting mathematical conversations in the Skoltech Institute, as well as strolling through Moscow's alleys and avenues, to end up at the lively seminars at HSE.

We enjoyed museums with Tanya, and amazing architectural gems with Ira and Igor. Mika, Igor's grandson, was my guide to ungentrified areas of Moscow, to listen to great jazz and enjoy that vibrant unofficial cultural life that is the
hallmark of any great city. That feeling of familiarity and kinship, started so long ago in Rome, has continued, and has been passed on to the next generations. The friendship that tied me to Igor and his family will continue to thrive.

## Serguei Brazovski

I had the good fortune to meet and collaborate with Igor in the 1980s due to the intersection of our interests in the theory of solitonic lattices. For me, this was the problem of spontaneous translational symmetry breaking in a multi-fermion system upon a deformable background (the Peierls-Frohlich model or Gross-Neveu one in field theory), with a special interest in the formation of solitons and their superstructures. Simple cases were found to be solvable by a naive ansatz, and it surprised me that the multi-fermionic self-consistent problem allows for exact solutions. The mathematical theory of S.P. Novikov and coauthors on quasi-periodic solutions of KdV and nonlinear Schrödinger PDEs gave us the opportunity to study physically necessary doubly periodic structures such as overlapping soliton lattices, and envelope and embedded solitons-all this in continuum models. At some point, I realized that the well-known exact solution of the Toda lattice equations gives us for the first time the opportunity to solve multi-fermion problems in a physical discrete system. This perspective resonated with old ideas of I. E. Dzyaloshinskii (my former teacher) about the effect of locking commensurability. I. E., who worked parttime at the MechMat of the Moscow University, shared the idea with Novikov, who suggested Igor Krichever to help us. "Help" turned out to be a vast but lightning work by Igor, which completely subjugated "simple physicists" by introducing incredible elegance in operating with Riemann surfaces, recovering from them all the distributions we needed in real space. For me, a representative of the generic solid state theory, this transcendental view of hyperelliptic functions seemed like some kind of miracle (as well as the whole theory laid down by Novikov). After the first publications treating the most pressing physical questions, I left the team-not wanting to be a coauthor in works whose technique I could not even reproduce. More versatile and mathematically oriented, I. E. continued to work with Igor in the more difficult extensions beyond exact solutions.

This scientific path went through circumstances which were not entirely favorable for Igor Moiseevich Krichever. Such a Jewish patronymic, and especially the infamous

[^21]"line five" in the passport' could not but spoil life in the Soviet Union. Igor was lucky to be young enough to enter the university in the somewhat tolerant 1960s. But the events of 1967 (the Six-Day War) and 1968 (Prague) dramatically changed the climate, putting an end to the remnants of the Khrushchev Thaw. In the early 1970s, despite his already demonstrated talents, Krichever had no chance of staying at Moscow University, or even at a less prestigious university, or at any of the numerous academic institutions of Moscow. Eventually a position for him was found at the industrial Energy Institute, where there was no demand for fundamental science and even less for higher mathematics. According to my recollections, Igor worked there (for 13 years!) practically as a system administrator with a big computer (he wrote computational algorithms and even drivers for printers). It was necessary to have great mental stability, versatility, and speed in order to keep in shape under these conditions, and even more so to keep working at the cutting edge of modern mathematics. This resilience, inner strength, and complete self-control were clearly visible in everything that Igor did. These characteristics were wonderfully combined with seasoned humor, certain skepticism, and sometimes evasiveness, and with the high culture of the Moscow intelligentsia.

Returning to history, Novikov's recommendation of Krichever had a subtext: the prospect of Igor's employment at the Landau Institute (the famous Novikov and Sinai already worked there part-time). Unfortunately, nothing came out of it at that time-we faced a double opposition. Firstly, the director of the institute was under enormous pressure for exceeding the (unofficial, but enforced by authorities) ethnic quota-the number of Jews was already several times higher than any other academic institution could afford. Secondly, among the "old guard" there was an opposition to the deviation from the original purpose of the condensed matter theory, and even more so to the shift towards mathematics.

All the obstacles fell by 1990 thanks to perestroika, the doors to academic institutions were opened for Igor, and he even finally became a member of the Landau Institute for several years. Igor did not hold a grudge for the injustice (e.g., even when he was in the US, he was one of the few who invariably came to the annual meetings of the "Landau days"). Then followed an enchanting series of duties in three different new prestigious institutes in Moscow, when not only his scientific, but also his organizational talents were discovered and in demand. And, even more importantly, the human qualities of Igor Krichever were appreciated.

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## Alexander Braverman

I knew the name Igor Krichever since my early years in graduate school. But then in 1999, I also became his son-in-law. Both fortunately and unfortunately my interaction with him throughout the years was concentrated around family matters much more than around mathematical discussions. So, I will try to gather here some (rather erratic) thoughts about Igor's mathematical life, but I will be writing it more in the capacity of a family member than that of a mathematician.

A long time ago, V. I. Arnold said that mathematicians could be divided into two categories: those who on their arrival to a new untouched land try to climb the highest mountain and those who start by building roads. According to Arnold, the most obvious examples of mathematicians of the two kinds were Kolmogorov and Gelfand (he later wrote that neither of them were happy with this metaphor). In my opinion, Igor Krichever did not fall into either category: he definitely was not just a road constructor (he always cared about the result, not just about the process), but he was not a climber either. In his nonmathematical life he was a semi-professional ping-pong player, and it seems to me that his approach to mathematics was partly governed by that. His philosophy of pingpong was, roughly speaking, as follows: you must train a lot to bring your technical abilities to a very high level, and after that is done you need to invent your own unique shot-this is what will eventually allow you to win important games.

I guess, in mathematics his approach was similar: after getting a very broad education during the "golden years" of the Moscow MechMat, and after acquiring a really remarkable level of technical ability ${ }^{2}$ he did design his own unique shot: it was the bridge between integrable PDE's and geometry of algebraic curves, which started with the famous "Krichever construction" of finite-zone solutions of the KdV hierarchy and then got extended to a huge variety of other situations. As with ping-pong, although the shot itself was public knowledge, nobody could master it at the level of its original designer-mostly because of a lot of small things that had to accompany "the shot" (for example, I still remember the days when, going to an important game, he spent a lot of time gluing the rubber to the racket blade; according to him, the result of the game very much depended for example on the type of glue used).

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Figure 1. Igor with his grandson Mika.

It is interesting to note that although for many years this "shot" was applied in one particular direction (using algebraic geometry to construct solutions to integrable PDE's and their deformations) one of his more recent big cycles of papers did exactly the opposite: Igor was able to prove the so-called Welters's conjecture (which provided a very geometric characterization of Jacobians of curves among all principally polarized abelian varieties) using the theory of integrable equations. ${ }^{3}$ Igor was very proud of this work-he used to say that finally the theory of integrable systems paid back to algebraic geometry for all the good it received from it during the previous 40 years.

Igor was definitely a mathematician and not a physicist, but physics always attracted him; I think he regretted that he did not spend more time working with physicists. For example, he always talked about the time of his collaboration with Dzyaloshinskii and others on the so-called Peierls models (in the beginning of 80's) as one of the happiest periods of his scientific life.

Unlike so many physicists and mathematicians, Igor had absolutely no element of paranoia about having his results stolen from him. He always told everyone everything-including everything unpublished. It seems that he understood well that no one could steal really big results from him anyway-since no one except him would be able to figure out all the details (continuing with the ping-pong analogy-what kind of glue you need to apply on the rubber to fight today's opponent), and "small" things did not bother him so much.

In fact, it is completely unclear when Igor was doing mathematics (e.g. in the last 23 years he was babysitting one of his grandchildren for about $2 / 3$ of the time). He was very fond of his office at Columbia, but he was not an "armchair scientist." He spent time in the office mainly in order to write something down, but he knew how to think "on the go." Even on the day of his death, he recalled how many years ago he was walking along Leninsky Prospekt in Moscow and suddenly he came up with a very interesting idea ("the Lax matrix for the elliptic

[^24]Calogero-Moser system"-nowadays it is a very common thing, but 40 years ago it was new), after which he decided to immediately tell it to his former adviser S. P. Novikov who lived nearby. But it turned out that Novikov had a birthday that day, so instead of talking about the Lax matrix, they got terribly drunk.

His status in the mathematical community was of course very high and it was important to him in a good sense. He was very proud to get an invitation to deliver a plenary talk at ICM-2022. He viewed this invitation as an opportunity to summarize his mathematical legacy and to try to explain it to a wide mathematical audience. At the same time he did remember some of his "failures:" for example it always amused me that even many decades later he was still a little disturbed by the fact that during the International Mathematical Olympiad in 1967 he won only a silver medal, rather than a gold medal.

Igor always had a hyperdeveloped sense of duty, so when it was his turn to be the chair of the Mathematics Department in Columbia, he did not refuse, but initially perceived this position as being sentenced to something between hard labor and scaffold. But in the end he unexpectedly got a taste for administrative work-apparently because he realized that he actually had the ability to do good things in such a post. So in a few more years, when he was invited to create a mathematical center at Skoltech in Moscow, he plunged into this job very deeply (and into Skoltech itself too-for example, for several years he headed the promotion and tenure committee of the entire Skoltech). The center, it seems to me, was a phenomenally successful project, although realizing its original idea (to be a lively and very international mathematical center) is certainly impossible in today's Russia, definitely not after Russia launched the aggression against Ukraine; and it was very sad to see Igor, who was already quite ill, witness the destruction of many of his efforts. After Igor passed away, the center was given his name-it is now called The Igor Krichever Center for Advanced Studies, and I truly hope that some day it will regain the international character it had before the war.

One last thing: Igor, of course, was not in any sense an applied mathematician, but at least in recent years some applied questions attracted him very much. For example, he was fascinated by the area of computational geometry, in particular, questions like how to approximate complicated surfaces by small flat pieces (it turns out that integrable systems also arise there; it is also not a coincidence that he was very fond of the works of Zaha Hadid-she was his favorite architect during the last years of his life). I am sure that he would have managed to do something interesting in this area if his illness hadn't interfered.

## Vladimir Drinfeld

Igor Krichever's work on commutative rings of differential operators was motivated by the theory of integrable systems. Unexpectedly, it greatly influenced the development of the Langlands program for function fields of characteristic $p$.

To explain how this happened, let me first recall two classical algebraic constructions. If $B$ is a commutative ring equipped with a derivation $f \mapsto f^{\prime}$ then one can form the ring of differential operators $B[D]$; this is the associative algebra generated by $B$ and an element $D$ with the defining relations $D f=f D+f^{\prime}$ for $f \in B$. On the other hand, if $p$ is a prime and $C$ is a commutative $\mathbb{F}_{p}$-algebra then one can form the associative algebra $C\{\tau\}$ generated by $C$ and an element $\tau$ with the defining relations $\tau c=c^{p} \tau$ for $c \in C$. The two constructions are somewhat parallel.

In a 1976 article, Krichever described commutative subrings of $B[D]$ in terms of vector bundles on smooth projective curves. Around the same time I was studying so-called elliptic modules, which are commutative subrings of $C\{\tau\}$; I used elliptic modules to prove a particular case of the Langlands conjecture for global fields of characteristic $p$. It turned out that a variant of Krichever's theory about the relation between vector bundles and commutative subrings of $B[D]$ has an analog for commutative subrings of $C\{\tau\}$ (this was realized independently by D. Mumford and me). This led to the notion of a shtuka, which is a generalization of the notion of an elliptic module (more details can be found in an expository article by D. Goss in the Notices of the AMS 2003, vol. 50, no. 1, pp. 36-37). Shtukas were then used by L. Lafforgue and V. Lafforgue to prove the Langlands conjecture for global fields of characteristic $p$.

## Anton Dzhamay

Igor Krichever came to Columbia University in Fall 1997. At the time, I was a graduate student there interested in applications of gauge theory to geometry through the theory of Donaldson and Gromov-Witten invariants. I had already been very intrigued by the appearance of the $\tau$ function of the KdV hierarchy in the statement of Witten's conjecture, so I was very eager to take Igor's course on soliton equations. In that course, Igor explained his unique and beautiful approach to integrable systems through algebraic geometry and the notion of the Baker-Akhiezer function. It took me many years to truly appreciate the depth

[^25]and significance of Igor's ideas. Over the years Igor became my PhD advisor, colleague, and friend, but above all I consider Igor to be my mentor and a true role model.


Figure 2. Igor Krichever with some of his students (from the left: Dmitry Zakharov, Yury Volvovskiy, Fedor Soloviev, Igor Krichever, and Anton Dzhamay).

When I think of Igor, one of the first things that comes to mind is his ability to stay calmly positive even in the most difficult circumstances. I cannot recall Igor ever raising his voice or losing his temper. But with this quiet determination Igor was able to achieve a lot. For three years, Igor was a very effective and respected Chair of the Department of Mathematics at Columbia University. He was also instrumental in the creation of the Center of Advanced Studies at Skoltech and it is very appropriate that this center is now named after Igor. I hope that the Center will survive the present difficult times and Igor's legacy will continue.

In addition to his enormous mathematical legacy, Igor left us many deep and unfinished ideas that, unfortunately, may be difficult to develop completely without his deep and very original insight and technical skills. Still, I am sure that efforts spent in understanding Igor's mathematics will be very fruitful and I hope that the new generation of mathematicians will continue developing these ideas.

When I think about Igor, I always think about Natasha, Tanya, and the rest of his beautiful family. For Igor, the family was very important, he was a very dedicated father and grandfather. It is unbelievable how Igor managed to take care of his grandkids, perform a manifold of administrative duties, and at the same time continue doing new and original mathematics. Igor Krichever was a true pillar of strength to his family, friends, colleagues, and many students. Igor's support made a huge impact on my life during some of its most challenging moments, and for that I am eternally grateful.

Igor Krichever was an outstanding mathematician and a great human being. People like him are very rare and I
have been very fortunate for the opportunity to know him for so many years. His passing leaves a huge void that cannot be filled, and I miss him deeply.

## Pavel Etingof

I met Igor in 1988 when he started working at the Institute for Problems in Mechanics in Moscow. At that time, I was a fifth-year undergraduate working there on my master thesis advised by V. M. Entov. My topic was Laplacian growth with zero surface tension. This problem exhibits an infinite family of integrals of motion which allows one to construct many explicit solutions. Thus my adviser and I wondered how it is related to soliton theory, in which similar phenomena arise. So I was very excited and slightly intimidated to talk to Igor, who was already a world-famous mathematician and a leading expert in the subject. But although I was young, Igor listened very seriously to my story, read my texts and gave me great advice which helped me a lot in future research, in particular when I wrote my first book on this subject with A. Varchenko. He also thought deeply about this topic and later wrote (with P. Wiegmann, A. Zabrodin, and others) a series of papers connecting Laplacian growth to mainstream soliton theory. These works, among others, have made this once fairly specialized subject into a vibrant area deeply connected to integrable systems, random matrices, stochastic processes, quantum field theory, etc.

During this time I recall coming to Igor's apartment, where I met his 13-year-old daughter Tanya. She is now a professor of literature and a dear friend of mine, along with her wonderful family-her husband and my coauthor Sasha Braverman and their children Mika, Asya, and Natasha.

In 1999, Igor called me and asked if I'd like to join the Columbia Mathematics Department. This was my first tenured appointment, and I spent a great year there with Igor and his colleagues, learning a lot from him about mathematics and beyond.

One of my last major interactions with Igor, which I enjoyed a lot, was in the summer of 2019 when he invited me to teach a minicourse at the first International Summer School at the Center for Advanced Studies in Skoltech, which he had founded three years before. Igor had nothing but profound contempt for Putin's regime, as he did for the Soviet regime decades earlier, but he believed that fundamental science and international collaboration should be fostered even in adverse political climates, and he hoped to hold such a school every summer. Sadly, this was not to be-the following year covid arrived, and in two more years Russia unleashed a devastating full-scale war against Ukraine, which in particular wiped out much
of what Igor had built. Yet the seeds he sowed are bearing fruit-the students raised by him at the Center are now doing mathematics in many different parts of the world.

The day before his death, I wrote Igor a short message of gratitude, and he responded "Thank you. You have always been dear to me." These were his last words for me; in a few hours he was no more.

As I remember these words, I think how fortunate I am to have met Igor. Throughout my whole mathematical life, I have been blessed to learn from his wisdom and enjoy a warm professional and personal relationship with him. I will always admire him as one of my teachers and a wonderful human being. I miss him greatly.

## Samuel Grushevsky

I first met Igor Krichever in 2003, more recently than many other contributors to this article. I was then a fresh PhD at the beginning of my life as an independent researcher. Some time later, we were discussing mathematics regularly, and soon thereafter we collaborated on our first joint paper. Our collaboration then continued until Igor's untimely death, with further projects unfinished. My path in mathematics was completely altered by working with Igor. Beyond sharing his mathematical expertise and intuition freely, Igor was a very close friend, and knowing him made me a better human being. His departure from this world is an unhealing wound for me, while his memory will continue to provide mathematical and personal inspiration.

For my PhD dissertation I studied the Schottky problem-the question of characterizing Jacobians of curves among all abelian varieties. This is a field that was transformed by Igor's breakthrough 1970s construction of solutions of integrable hierarchies (Functional Analysis and Its Applications, 1977), using the theta functions of Jacobians. ${ }^{4}$ This further led to Novikov's conjecture proven by Shiota in his 1986 paper, showing that the Schottky problem is solved by the KP equation: if the theta function of a principally polarized abelian variety satisfies the KP equation, then it is a Jacobian of a curve. In 2003, I finished my first post-PhD project arXiv:0310085, and being aware of Igor's stature in the field, I shared the preprint with him, hoping he might find something appealing in our computations with derivatives of theta functions.

In response Igor pointed me to the conjecture on addition theorems for theta functions from his paper with Buchstaber. I was eventually able to resolve this conjecture, in arXiv:0503026. By chance I finished the argument during a visit with Riccardo Salvati Manni in Rome, at the

[^26]same time that Igor was visiting Enrico Arbarello there. Thus I was able to explain the argument to Igor in person, and had my first ever dinner with him (and our Italian colleagues). While I expected that a visiting mathematician would spend all his days at the math department, Igor was instead very diligent in visiting the outstanding art museums (then not as crowded as now), while he was still able to share much mathematical insight over coffees and dinners.

From then on Igor and I became friends and collaborators. One thing that always amazed me in working with Igor was how playful his mathematics was. While sometimes I would come to our next meeting (which occurred regularly for years) to discover pages upon pages covered by computations, most frequently I would observe how Igor would come up with an idea just in passing, while walking, doing the dishes, smoking, or even between the acts of an opera. A lot of hard work and difficult computations went into our eventual first joint paper arXiv:0705.2829 (and Igor's computational prowess was clearly manifest), but many of the basic ideas and concepts arose just casually, and occurred to Igor naturally as he was exploring the circle of ideas around his celebrated proof of Welters's trisecant conjecture, arXiv: 0605625.

That proof of the statement that if the Kummer image of a principally polarized abelian variety has one trisecant line, the abelian variety is Jacobian, is a tour de force that still has not been understood via methods of algebraic geometry. While Igor's methods come from integrable systems, and his related statements for degenerate trisecants (arXiv:0504192) have recently found an algebro-geometric explanation (arXiv:2009.14324), the full statement, and Igor's construction of the family of trisecants and integrable hierarchy, starting with just one trisecant line, remain mysterious.

Igor was interested in curves with a differential for decades, partly motivated by numerous problems from physics, e.g., see his work with Phong arXiv 9604199 . While much of the work in this area had been on applying algebro-geometric methods to solve questions in mathematical physics, in arXiv:0810.2139 we applied realnormalized differentials (meromorphic differentials on Riemann surfaces with all periods real) to reprove a statement in algebraic geometry: Diaz's theorem (1984) that a compact complex subvariety of the moduli space of genus $g$ curves has dimension at most $g-2$. Our proof does not require advanced technology, and the result and the proof can essentially be explained to a good undergraduate student.

This is another amazing property of Igor's oeuvre-the sheer number of different novel ideas that he has come up with, and applied broadly in mathematics. While some of

Igor's papers utilize advanced machinery and show heavy computations, whenever Igor was asked to explain something, the answer was never "you compute for ten pages and then you see." All of Igor's work, even our very technical study with Norton of degenerations of real-normalized differentials arXiv:1703.07806, had a clear underlying idea and philosophy in Igor's mind, which-when followed through to the end-yielded a result, however difficult or technical at a first glance.

Igor's ability to intuitively see through the computation and divine the final result, and then to relentlessly follow such a path to success, was unsurpassed. In everything Igor did, in mathematics and in life broadly, he exuded the impression of unhurried and benevolent strength. All his numerous accomplishments-personal, mathematical, and administrative, did not appear to come to him through blood, sweat, and tears. Igor loved life, loved mathematics, loved people around him, and enjoyed creating new knowledge and sharing his vision. This vision will continue to shape geometry for years to come, and the memory of Igor will continue to push all those who knew him to become better people.

## Nikita Nekrasov

The hot air outside the Columbia housing block on 113th street was thick. The parking next to the building was being renovated, the asphalt drill making a terrible noise. The building elevator broke down. It was August 9, 2022. I climbed the narrow staircase to the fourth floor, and entered the apartment. The door was unlocked. Igor was waiting for me, dressed in a short-sleeved shirt and a pair of slacks. His always trim figure was now too small even for small-sized clothes.

A few days prior, thanks to the eloquent pitch of my girlfriend Nina, Igor agreed to have a recorded conversation. We assembled a film crew, carried a ton of equipment up those stairs, and set up the cameras and sound equipment. Igor's daughter Tanya and her son Mika were helping to get the crew some air, my son Boris helped the sound operator. Family, friends, and mathematics were inextricably mixed up, it was always like that with Igor, both last year, and thirty three years ago...
N. I think I saw you for the first time at your lecture at the Moscow Mathematical Society. Topological gravity and KdV equations were gaining popularity, and you were presenting something on the topic. I remember Sergei Novikov was sitting in the front row and saying something

[^27]like "there is something funny with your limit of small ع..."
I. I think it was my review of the 1990 ICM. Those who attended the Congress were to review the papers at the MMS.
N. But I think you were lecturing about your own work...
I. Perhaps it was when I got interested in Whitham dispersionless hierarchies, partly because of the genus zero topological gravity of Witten, I don't know. .
N. Could you tell us how you came to be a mathematician? When did your interest in mathematics start? Do you recall?
I. I don't remember when I wasn't interested.
$\mathbf{N}$. You don't remember when you weren't interested?
I. I've always been interested in solving problems.
$\mathbf{N}$. And where did you get the problems from?
I. There were problem books that I read. But it probably wouldn't have worked out if I hadn't come across a good math teacher in the city of Taganrog. This wasn't your 57th school, yet she was a great teacher. After only a couple of classes she told me: "Sit quietly in class, do what you want, don't participate, solve whatever problem you want-it's your business," and then set me up with one of her former students, who had gone away to the Kolmogorov Internat (a special science oriented boarding school in Moscow) the prior year.
N. Did your parents just let you go, or what was it like?
I. I still don't understand why my mother let me go. Because I was very. . .
N. Timid child?
I. Totally timid. . .
N. Did your parents have anything to do with science?
I. No, they didn't.
N. They didn't. How did you find out about the Boarding School?
I. It all happened in the city of Taganrog. There were no special schools, nothing there. The first one, the only, sort of, special, English school was opened, when I was in the fourth grade. My parents, of course, as expected, sent me there. At one point I got an $F$ in math for a two-digit number problem. I wrote " $10 a+b$ " and got an $F$ for it. My math teacher had a long argument with me in front of the whole class. She said: "Look, for the number 75 we don't write $10 \cdot 7+5$, so you must write $a b$. You see? And from that we got something completely wrong." When I told my mother about the argument I had with the teacher she pulled me out of that school the same day and transferred lucky me to a nice simple school where Anton Chekhov studied. It was an old grammar school. I can't say that my
teacher was from those Chekhov days, but she was very aware of her limitations. I felt very comfortable with her. She did not make me do " $a b$ " under the guise of double digits. And because one of her students had gone to the Internat the previous year (it was the first or second round), I found out about it.
N. Can you not believe that there is such a thing, a plan, a meaning to the meeting? How can one not be surprised by the weaves of fate? By living through such encounters which define everything. ... How did you meet S. Novikov and what was your style of communication with him?
I. At the end of my second year at MechMat, I had to choose a division and an advisor. I decided to go to Novikov, despite the endless warnings of many of my friends. They said, "Novikov doesn't remember his students, he doesn't know what they look like." It turned out not to be true. In fact, he remembered everything about everyone, and what's more, he had a ledger in his head. When I first came to him, he told me: "if you are hoping to get a problem from me, don't get your hopes up. If I know a good problem..."
N. I will solve it myself!
I. "I will solve it myself."
N. Yes.
I. again, that's...
$\mathbf{N}$. This a perfectly logical thought! That's what I myself say to everyone. Where would I get a good problem? I kind of have my own...
I. You know, Nikita,


Figure 3. Igor in August 2022. it works for some people. It suited me. In all my life I did not get a single problem from Novikov. All our work together occurred when something came up, he never told me "here's the problem, it needs to be solved." I myself tried the same practice on several of my students. It turns out it works for some people and not for others. I can't say anything specific, so to speak, about what Novikov taught me. But he instilled in me a taste for what is good and what is bad. And that is the most valuable thing.
$\mathbf{N}$. Is it ever the case that you understand something, but not because you can build up a logical chain, but because it's somehow there, yet maybe you can't keep track of the precise logic.
I. Nikita, that's quite a complicated question, because I'm deeply a nonreligious person. Yet there is a deeprooted belief inside of me that there is some kind of harmony in the world.

## N. Harmony?

I. Yes, at times I have stubbornly tried to prove some nonsense because I felt that there was one brick missing for beauty and that something must be right because it is beautiful. It's another belief, I don't know what you call it, but in general it's not a chain, it's a feeling that. . .
$\mathbf{N}$. that there is a pattern, an ornament!
I. ...the world is somehow harmonious...you see it, and it must be that way to be good...

A few months later, at our last meeting, I wanted to ask him about his feelings. He knew what was ahead of him. He said: "Nikita, this is not a place for words." And in the same breath: "The Lax flows of the integrable system we are discussing, on the other hand, deserve further discussion."

Today, seven months later, that Columbia apartment is empty. Igor is no longer with us. Yet he is with us, the Lax flows continue, and so does the flow of ideas he weaved so masterfully.

## Sergei Novikov

Igor Krichever started to interact with me in the late 1960s. He was an undergraduate student at that time. In the early 1970s, he became a graduate student and did very good work in topology. He studied actions of compact groups on manifolds, using the whole machinery of algebraic topology-including cobordisms, formal groups, and so on. In 1974, I invited him to work on the theory of solitons and nonlinear waves. In 1975/76 he did very good work constructing algebro-geometric solutions of the KP equation which is a natural 2D extension of the famous KdV equation. Its integrability was established by Driuma-Zakharov-Shabat in 1974 who started to study it. They found the "Lax pair" for it. The method of Krichever was based on the pair of commuting ordinary differential (OD) operators of relatively prime orders.

The role of commuting OD operators in the theory of the KP equation became well-known. In 1977-1978 some British mathematicians found forgotten works done in the 1920s (Burchnall-Chaundy) where commuting operators of relatively prime orders were investigated algebraically. Similarity with some modern studies was impressive. One should say that classical people never considered systems of nonlinear PDE, nor periodic or rapidly decreasing potentials in quantum mechanics. Indeed, while in 1940s reflectionless potentials were classified (Bargmann), the algebraic background remained hidden until the 1970s.

[^28]Famous periodic and quasiperiodic finite-gap potentials were discovered and classified in the 1970s-Dubrovin, Matveev, Its, and myself. The algebraic background became completely clear after Krichever's work.

Then began the study of 2D problems.
I. 2D Schrödinger operators generate $2+1$-dimensional nonlinear systems which we call "Manakov's L-A-B triples" instead of $1+1$-dimensional Lax pairs. They use eigenfunctions of the Schrödinger operator restricted to one energy level only. This was developed by several authors including Dubrovin, Krichever, Veselov, Grinevich, myself, and others.
II. $\theta$-functional solutions of the KP system can be used, according to my conjecture, to recognize $\theta$-functions associated with Riemann surfaces. This is a classical problem known since the nineteenth century. This problem was solved by Dubrovin, Arbarello-De Concini, and completed by Shiota. Krichever improved this approach, replacing it by the operators of Lax pairs.
III. The development as-


Figure 4. Igor Krichever. sociated with commuting operators of nonrelatively prime orders is extremely interesting. The classics (Burchnall and Chaundy) worked with pairs of relatively prime order OD operators. Concerning the nonrelatively prime case they wrote: "This problem (commuting operators of nonrelatively prime orders) is trancendental." Indeed, it is obvious that unlike the relatively prime case, the common eigenfunction cannot be found explicitly in quadratures. But Krichever and I started to work on this in 1978-1979. Drinfeld and Mumford also began working on this problem. It became completely clear that holomorphic vector bundles over Riemann surfaces play a fundamental role. There exist two different methods to study holomorphic vector bundles over algebraic curves-Tyurin's and Mumford's. Drinfeld and Mumford used Mumford's approach and made useful observations. However, their results were noneffective.

Krichever and I, on the other hand, used Tyurin's approach based on "framed" bundles. Choose the Chern class $c_{1}=\ell g$ (where $\ell$ is the rank of the vector bundle, $g$ the genus of the base). Framing means selection of $\ell$ holomorphic sections. Avoiding details, let me say that we developed an effective method how to calculate the coefficients of the OD operators. Even for the pairs of orders
$(4,6)$ and $(6,9)$ the situation is nontrivial. They depend on arbitrary (free) functional parameters of one variable $x$. The whole pair depends on two variables $(x, y)$. Each commuting pair defines a solution of the KP equation. Time dependence of the KP-equation leads to time dependence only of free functions of one variable entering the operators. It leads sometimes to remarkable ( $x, t$ ) systems. The case $(4,6)$ made it possible to correct a mistake in the classification made by the school of Shabat: their list of "integrable" KdV-type systems (i.e. $c_{t}=c_{x x x}+f\left(c, c_{x}, c_{x x}\right)$ ) was not complete-the most complicated system was missing.

In the late 1980s, Krichever did very good work in statistical physics working jointly with some of the best theoretical physicists in the Landau Institute-Dzyaloshinskii and Brazovski. His last work with me was done around the year 2000. It was dedicated to commuting difference operators.

Igor was one of my best students and a remarkable mathematician. He left a lasting legacy in mathematics.

## Andrei Okounkov

Igor Krichever was a bright pulse of light. Just the kind of nonlinear natural phenomenon that can be seen on a poster of a conference on integrable systems, his main field of interest. Except, this pulse was the opposite of solitary. Most of the time, he was this warm smiling globe of light that instantly made everybody feel understood, supported, and loved. And in those, sometimes rare, moments when he was free to do his mathematics, all that light could focus like a laser to dissect any mathematical difficulty and illuminate the core issue from within. About the balance of the two, I don't know if it was a conscious duty or just nature for him, but he always had an unlimited budget of time and energy for helping others. For his family, his many friends, his colleagues, neighbors, et cetera, et cetera, he was just like a bright sun, the source of everyday energy, vitality, and good humor. A source that never ran dry and never failed to be there for them every single day. Until he was suddenly gone.

While always pointing towards big goals and ideas, his compass never pointed away from people. He was very passionate about the future of mathematics and, later in his life, he devoted a very large part of himself to administrative work. Everybody knows it is not easy to provide the institutional foundations on which mathematics can flourish: there will be some difficulties every single day, and it may be hard to resist approaching these difficulties with a formal, corporate logic. Igor was that rare kind of leader who always put people first. Somehow, there never was an obstacle or a disagreement that couldn't be put behind by the combination of his smile and his wisdom. For that, he
was always dearly loved by his colleagues, staff, superiors, and maybe most of all, by the mathematical youth. For the young people, he was both an inspiration as a brilliant researcher and a caring senior colleague, whose rock-solid support they could count on in all possible real-life situations. I hope they will carry that feeling and the memory of Igor with them into the future. And "sorry" is not really the word to describe the fact that most of Igor's dreams were crushed just before the sudden illness overcame him.

But nothing can take away the legacy of Igor the original thinker. It is overwhelming to contemplate the brilliance and the fundamental nature of his contributions to mathematics. Many of them are the central pillars of building bridges between different fields, with a lot of ideas traveling in both directions. For me, he personified the fact that mathematical physics can interact and should interact with every branch of pure mathematics. I always loved discussing mathematics with him. In addition to Igor's universally celebrated landmark papers, I had my own personal favorites among the lesser known ones. While thinking about the monodromy of quantum connections, I spent many months with his "Analytic theory of difference equations ..." by my side. Igor's work on the spectral theory of 2-dimensional periodic difference equations had a big influence on my work with Richard Kenyon on planar dimers and Harnack curves. Igor's elliptic genera and their rigidity were a huge inspiration for elliptic stable envelopes and many rigidity-based computations in enumerative $K$-theory. My understanding of many other areas of mathematics was really deepened and sharpened in discussions with Igor, which continued until the tragic day came.

The closest we came to writing a paper together was when we discussed the limit shapes for planar dimers and their quantum analogs, which I had defined in 2009. Quantum limit shapes $\widehat{Q}$ are defined for domains of finite size and the fact that they converge to the classical limit shape as the size of the domain grows to infinity remained a conjecture at the time. Eric Rains and I constructed Painlevé-like equations that describe how $\widehat{Q}$ changes with the domain. With Igor, we proved an averaging result for these dynamics. We both liked it. It involved realnormalized differentials, 2-dimensional quasiperiodic difference operators, and Igor's other favorite objects. It implied the convergence to the classical limit shape. Still, other projects kept us from completing this one. As I look at the slides of my 2010 lectures about this work with Igor, I feel devastated by the size of the dark void that is left by his departure in mathematics, my life, and many, many other people's lives.

## Duong Phong

Igor Krichever was for me a very dear friend and colleague. I often marvel about how mathematics can bring together people whose paths were most unlikely to meet. The first time I heard about Igor was in 1976, from a beautiful description of his work by David Mumford at a colloquium at the University of Chicago. My admiration for Igor's work only increased in subsequent years, when I participated in the year-long seminar held by Lipman Bers at Columbia on integrable systems and Riemann surfaces. Even so, Igor was a world away, and I did not even dream that we could be colleagues and friends some day. Things changed drastically in the early 1990s. Many outstanding scientists from the former Soviet Union had begun emigrating to the West, and the Columbia mathematics department had lost, through some unfortunate circumstances, many faculty who had to be replaced. But with the economic crisis of 1992, many other Columbia departments were facing difficulties of their own, and the new Columbia administration at that time installed a new policy, where the renewal of faculty slots would not be automatic, but would have to be won by each department on its own merit. This meant that the mathematics department had to propose a truly world-class candidate. So it was with great anticipation that it learnt that Igor could perhaps be interested, and the main task became that of building a strong enough case for the Columbia administration to give priority to his appointment. There I am very happy to report that his letters far exceeded all expectations, and I still recall vividly many enthusiastic comments, including that a particular contribution of his was "an epoch-making work." Thus Igor did join Columbia, where he became a key figure in its renewal, serving as chair in the early 2010s, until his untimely passing away in 2022.

It was a great stroke of luck for me that, at the time of Igor's arrival at Columbia, our scientific interests happened to overlap. Nathan Seiberg and Edward Witten had just made their great breakthrough on supersymmetric gauge theories with spectacular applications to topology, and the central role of symplectic forms and integrable models had begun to emerge with works of Ron Donagi and Witten, and Emil Martinec. At the same time, Igor had just completed his work on the Whitham hierarchy, and had also recognized, in joint work with Alexei Morozov and others, the Seiberg-Witten solution of the SU(2) theory as the spectral curve for the Toda model. Eric D'Hoker and I had been investigating instanton corrections. Motivated by the theory of integrable models, Igor and I decided to focus on the construction of moduli spaces of pairs of differentials and their symplectic forms. We succeeded in this goal, and obtained a unified approach to
all the Seiberg-Witten solutions known at that time in the literature. But rather unexpectedly, this work ultimately led to something that we had not even hoped for, namely a universal symplectic form $\omega$ expressible in terms of Lax pairs $(L, A), \omega=\operatorname{Res}_{\infty}\left\langle\Psi^{*} \delta L \wedge \delta \Psi\right\rangle d k$, and a new approach to hierarchies of 2D integrable models. Here $\Psi$ and $\Psi^{*}$ denote the Bloch and dual Bloch functions for $L$, and $\mathrm{Res}_{\infty}$ denotes taking the residue at $\infty$. In particular, it is very different from the Hamiltonian approach pioneered by Ludwig Faddeev and Leon Takhtajan, and has perhaps the advantage over the approach of Mikio Sato of not involving the infinite number of coefficients of a pseudo-differential operator.

But Igor was not just a colleague with whom I arrived at some of my most cherished works. We became the most mutually trusting of friends, and our families grew to be very close as well. Igor was very generous and thoughtful, and I cannot count all the times when an unexpected gift or kind gesture of his would show how much attention he paid to the smallest wishes of his friends: the book with the painting of "La Princesse Lointaine" on the Hotel Metropole in Moscow, which he brought back for me when he heard how much I liked that piece of theatre; the CD's of Boulat Okoudjava; and the Georgian movies and DVD's of dances from the Caucasus. Equally vivid are the souvenirs of times when I could just drop by his apartment, and be welcomed by his wife Natasha with a warm and aromatic mushroom soup. It is terribly sad to think that there will be no more such occasions, but I can take solace in at least having experienced them with a precious friend such as Igor.

## Leon Takhtajan

Igor Krichever was a great friend and his passing came as a terrible loss. My first encounter with Igor was around 1975; he was giving a talk at the Steklov Institute in Leningrad on his (now classical) results on the periodic problem for the Korteweg-de Vries equation. It was the beginning of the Golden Era of numerous interactions between algebraic geometry and mathematical physics, with many discoveries and triumphs. Igor's paper introducing the Baker-Akhiezer function was one of them. His great gift was a special feeling of perspective in mathematics, distinguishing the foreground and the background, like in art. Using this comparison, Igor belongs to the school of old masters, possibly with some twist of impressionism. Igor realized the great unused potential of the RiemannRoch theorem for curves and masterfully used it in all his

[^29]work, which I always admired (in my dinner speech at his 60th birthday conference at Columbia in 2011 I said that we shared a "secret love" for the Riemann-Roch theorem).

The first Soviet-American Symposium on Solitons in Kiev in the summer of 1979, which Igor and I attended, was a pivotal event for the theory of integrable systems in the Soviet Union and in the USA. I gave a talk on our joint paper with Ludwig Faddeev on the eight-vertex model, solved by Baxter; we successfully applied to this model our recently invented (together with Evgeny Sklyanin) method of the algebraic Bethe Anzatz, which takes full advantage of the Yang-Baxter equation (the term was introduced in our paper). Subsequently, in his 1981 paper, Igor beautifully applied the theory of algebraic correspondences and the Riemann-Roch theorem to the problem of classifying solutions of the Yang-Baxter equation and explained the algebro-geometric meaning of the vacuum vectors in our paper.


Figure 5. Natasha and Igor (and Tanya). Lake Mohonk, New York, 1998.


Figure 6. Igor giving a talk.

In 1996, when Igor joined the Mathematics Department at Columbia University and moved with his wife Natasha to New York City, my wife Tanya and I became very close with them. They often visited us on Long Island and we ventured on a few trips to upstate New York (see Figure 5).

Our scientific interests became closer; Igor and I had many discussions about different definitions of deformation spaces. Igor preferred the algebro-geometric approach based on his favorite differentials of the second kind with real periods, while I was using the Ahlfors-Bers approach, and we were trying to merge them.

Igor and I attended numerous conferences and workshops, including the Summer School on New Principles
in Quantum Field Theory in Cargese in 1991, in Saint Petersburg in 2013, 2014, and 2016, in Ascona in 2015, in Gallipoli in 2017, in Moscow in 2018, and his last birthday conference in New York City in 2022.

Igor was a remarkable husband, father, and grandfather. Caring for his family was always his priority even outpacing mathematics. The passing of his wife Natasha in 2013 was a terrible loss after which Igor was even more involved with his daughter Tanya and her family.

Igor was an outstand-


Figure 7. Igor and L. D. Faddeev. Saint Petersburg, 2016. ing person with deep moral principles he always followed, courageous and brave, and he always kept his word. He had a great organizational talent, whether as chair of the Mathematics Department at Columbia or as director of the Center of Advanced Studies at Skoltech in Moscow. Starting in 2016, he was able to successfully build a new research center on par with the best mathematics and theoretical physics centers worldwide, now called "The Igor Krichever Center for Advanced Studies."

To summarize, I greatly treasure our friendship, conversations about mathematics and life, and the moments we spend together with Igor and Natasha, and later with his friend Irina. He will continue to live through his mathematics, his family and his friends. The last photograph, taken in 2016, shows Igor with my teacher and friend Ludwig Faddeev. They are seen through a mirror, which forever preserves the moment.

## Alexander Varchenko

Igor Krichever was a student at the Moscow boarding high school No. 18 for gifted children organized by Andrei Kolmogorov at Moscow University. The school opened in December 1963. Igor enrolled in the school in 1965, and I enrolled a year earlier. At school, Igor was notable as a winner of mathematical Olympiads. Twenty years later, summarizing the school's work, a meeting was organized by Kolmogorov with the graduates from the school who became doctors of physical and mathematical sciences. In Russia, there are two scientific degrees: a candidate of sciences degree and a much more exceptional doctor of sci-

[^30]ences degree. By that time, there were only eight doctors of sciences, see Figure 8, which was published in one of the central newspapers of the Soviet Union. During that meeting, we had a chance to chat with Kolmogorov about the recently introduced mathematical Olympiads for undergraduates. I recall that Igor expressed his belief that undergraduates would be better served by engaging in real research rather than simply playing Olympiad games. Igor published his first paper when he was 21 .

I played table tennis with


Figure 8. First row from left to right: A. Varchenko, A. Kolmogorov, V. Temlyakov. Second row: Y. Matiyasevich, E. Shchepin, I. Krichever, V. Yanchevsky, S. Voronin, S. Pinchuk. Igor on different occasions in different countries. In his youth, Igor was on the table tennis team of Moscow State University. Once I asked Igor if he played against J.-P. Serre, who was known as a very good table tennis player among mathematicians. Igor replied that he had played against Serre and had actually won.

I had only one joint paper with Igor, which was written in 2019. We constructed a family of commuting flows on the space of solutions of the Bethe ansatz equations in a simplest $\widehat{\mathfrak{l}}_{N}$ XXX model and identified these flows with the flows of coherent rational Ruijesenaars-Schneider systems. The last time I saw Igor was in October 2022. He said that we need to make the next step in our project, but we did not have time.

## Alexander Veselov

Who was Igor Krichever to me? Firstly, the elder scientific brother, the referee of my first paper written during my PhD study under the supervision of Sergei P. Novikov. I remember well my visit to Igor's place at Zhitnaya Ulitsa in Moscow, where he taught me how to write a good paper. I remember also his own PhD defense at Moscow State University, which actually was in the area of algebraic topology. Only recently I had a chance to work with Victor M. Buchstaber on cobordism theory and was able to fully appreciate Igor's contribution to this area, which was overshadowed by his outstanding achievements in

[^31]integrable systems and algebraic geometry. For many years, I was very fortunate to have numerous scientific discussions with Igor, which were always very illuminating and stimulating. Even during our last meeting in New York in October 2022 when I came to say goodbye, Igor used this chance to explain his very revealing understanding of Leon Takhtajan's talk, which concluded the conference celebrating his remarkable scientific career.

Igor was also a close friend with whom I could discuss the most delicate problems of my life. I enjoyed every minute spent with Igor's wonderful family, especially with his wife Natasha and grandson Mika.

I will forever remember Igor Krichever as a very strong and positive person, bringing the sense of optimism to others. He will be sorely missed.

## Paul Wiegmann

With Igor's untimely passing, I lost a dear and trusted friend with whom I shared my values in science, humanity, and morality. He was three years older, and we belonged to the same generation, experiencing life events from a similar perspective. When I was about twenty years old and starting my diploma work at the Landau Institute, I had a memorable conversation with a friend who chose a career in molecular biology. During that discussion, it occurred to me that besides being drawn to theoretical physics by its general prestige in the Soviet Union, what truly attracted me to this field were the wonderful people who were part of it. Since then, I consciously admit that, for me personally, the social comfort and intellectual closeness that creative work brings among the people around me may be just as valuable, if not more, than the new knowledge that this work creates. This adds to the sense of loss I feel, knowing Igor as a friend and having the privilege of working with him.

I knew Igor's name since my early years at Landau as a prominent student of S. P. Novikov. Back then, Igor was working at some obscure Energy Institute, which I felt was unfair, as I believed that some of us, and primarily, myself, with lesser achievements and promise managed to get into premier institutes like Landau. Later on, I heard from Igor that he was quite content there, enjoying a good degree of academic freedom and facing less peer pressure, albeit at the cost of less prestige. Throughout our roughly four decades of friendship, I never heard Igor complain. He was never driven by ambitions for superficial things like titles or public standing. At the same time he found value in seeing his ideas and results being recognized for their merit. Many people who knew him noticed his consistently positive attitude, and as Andrei Okounkov aptly put it, Igor was a bright pulse of light to many of us.

At Landau, S. P. Novikov commanded great respect and admiration. S. P. was known for passionately fostering interaction between physics and pure mathematics (and also for his memorable statements that often carried aphoristic value). When Igor finally joined Landau, the interaction between the two distinct cultures of thinking was a theme of the day, leaving a profound impact on many of us. Igor was a bright pulse not just of light but also of clarity, having the exceptional ability to extract the core essence from abstract complex concepts and deliver it in a straightforward manner. People like myself, who lacked formal mathematical education but regularly bumped into algebraic geometry, owed Igor a great deal for his readiness to remove a mathematical concept from its abstract shell. Igor seemed to derive a sense of pleasure from simplifying an intricate matter to its core. A notable example of such nexus is Igor's early work of 1982 with Serguei Brazovski and Igor Dzyaloshinskii and his 1983 paper with Dzyaloshinskii on the Peierls model. This model is a case of Peierls instability, an important phenomenon of structural distortions in crystalline materials caused by electronic interaction. After that work, it became a major condensed matter application of "Krichever's construction" of finite-zone solutions of integrable hierarchies, such as the Toda chain in this case.


Figure 9. Igor explaining mathematics.

We grew closer in Spring 1989 when we randomly ran into each other on Blvd. Saint-Michel in Paris. We ended up in the first café that came along, in the midst of all the tourists, discussing the unfolding events in Moscow and the upcoming election of Yeltsin the next day. There was a sense of euphoria in the air, and a general feeling that our lives were on the brink of a significant change. I remember that Igor's thoughts were balanced, somewhat subdued. He told me that he had an invitation to stay "longer" in Paris but may travel to Columbia for "reconnaissance." I mentioned that I was considering going to San Diego and then possibly to Princeton. The next day we went to the embassy somewhere on the edge of the Bois de Boulogne to vote for Yeltsin. It was a momentous occasion for both of us as we had never voted before (and I haven't voted since). In the mid-1990s, after settling in the US, we became even closer. I met Natasha and Tanya, and we started exchanging visits and having discussions about physics (or mathematics).

Our first paper arXiv: 9604080 , written with Ovidiu Lipan (then a student) and Anton Zabrodin, explored a mysterious connection between the Bethe Ansatz equations for eigenvalues of quantum integrable systems and a special class of solutions (elliptic solutions) of a classical difference Hirota equation. This problem stemmed from discussions with Igor about the Bethe Ansatz solution of the Hofstadter problem, which Anton and I were working on at that time. Hirota's equation encompasses known integrable hierarchies of classical nonlinear equations when time is treated as a discrete variable. The increment of the discrete time corresponds to the Planck constant of the continuous time quantum equations. In a subsequent paper with Igor and Anton Zabrodin arXiv:9704090, we extended Igor's algebraic-geometric solution to discrete (or difference) integrable equations.

In the late 1990s, Mark Mineev-Weinstein introduced me to the problem of Laplacian growth. This phenomenon, also known as the Hele-Shaw problem in fluid mechanics, has been known to engineers since the mid-19th century. It involves the unstable growth of an interface between viscous and inviscid fluids, resulting in fingering instabilities and the formation of cusplike singularities in finite time. By the mid-twentieth century, the problem had been linked to the deformation of a conformal map of a domain with a variable area and fixed harmonic moments. In its modern context, the focus has shifted to the evolution beyond singularities, leading to the emergence of stable, fractal patterns known as diffusion-limited aggregation. When I discussed this topic with Anton Zabrodin, he noticed a potential connection to Igor's work on the $\tau$-function of the universal Whitham hierarchy (source: arXiv: 9205110). We shared this observation with Igor, and he recognized that the dynamics of the interface is merely a specialization of the deformation of Riemann surfaces with real periods, which he referred to as Boutroux curves. His paper explains that the evolution of such curves is described by the dispersionless limit of the 2D Toda hierarchy (see arXiv:0005259 and arXiv:0311005). Building on this connection, jointly with Razvan Teodorescu and Seung-Yeop Lee, who were students at that time, I conjectured that patterns of diffusion-limited aggregation are real sections of what we now call Krichever-Boutroux curves, experiencing a sequence of changes in genus. An example of this is the pattern of anti-Stokes lines of the asymptote of a genus 1 solution of the Krichever-Novikov string equations. It is quite remarkable that a classical problem in fluid mechanics, dating back 150 years, finds its place in Igor's realm of algebraic geometry, integrable hierarchies, random matrix models, topological field theories, and more. Igor was fascinated by the geometric appearance of Laplacian growth and the countless connections
it had to subjects he had previously worked on. He also enjoyed visualizing the various algebro-geometric objects that arose from the study of fluid dynamics. It was wonderful to observe how seemingly different things fit together so harmoniously.

Igor effortlessly divided his time between New York and Moscow, showing devoted loyalty and an unwaveringly positive attitude towards both places. Unlike many of his peers with similar backgrounds for whom relocation to a new environment was an abrupt change, Igor's life gradually evolved between the two countries. I found myself deeply sympathetic to this approach. Under his influence, I began spending more time in Moscow, and he (and Tanya, Igor's daughter) opened my eyes to the continuous and rapid developments of this metropolis and to the vibrant intellectual and artistic environment a sophisticated city provides to its dwellers. As for other perplexing aspects of life, his attitude toward a lifestyle revolving around the New York-Moscow axis was balanced and generally positive. In the last decade, Igor actively participated in building and revitalizing the environment for fundamental research in mathematics and theoretical physics in Moscow. It was unexpected to see him in this capacity, but he proved to be a creative administrator in science. First at Columbia, and then in Moscow, his actions as a deputy director of the Institute for Problems of Information Transmission, and later as the founder of the Center for Advanced Studies at Skoltech, had a lasting positive effect on the lives of many people dedicated to science.

He will be sincerely missed by many.

## Anton Zabrodin

My older friend and coauthor, Igor Krichever, contributed a lot to different areas in mathematics and mathematical physics. Here I would like to concentrate on the topic related to both soliton equations and integrable many-body systems of classical mechanics. I mean the remarkable connection between singular solutions of soliton equations and many-body systems of Calogero-Moser type.

The study of singular solutions to integrable nonlinear partial differential equations and their pole dynamics was started in 1977 in the seminal work by H. Airault, H. P. McKean and J. Moser. They considered elliptic and rational solutions to the Korteweg-de Vries equation and discovered that poles of such solutions move like particles of the Calogero-Moser many-body system. However, their Calogero-Moser-like dynamics was subject to some essential restrictions in the phase space. Igor showed in his

[^32]1978 paper that in the case of the more general KadomtsevPetviashvili (KP) equation $3 u_{y y}=\left(4 u_{t}-6 u u_{x}-u_{x x x}\right)_{x}$ the correspondence between the pole dynamics and dynamics of Calogero-Moser particles becomes a complete isomorphism. So, the connection between soliton equations and integrable many-body systems becomes most natural: the dynamics of the poles $x_{i}$ of rational (with respect to the variable $x$ ) solutions $u(x, y, t)=-2 \sum_{i=1}^{N}\left(x-x_{i}(y, t)\right)^{-2}$ to the KP equation as functions of the variable $y$ is isomorphic to the Calogero-Moser system of particles with rational interaction potential $\left(x_{i}-x_{j}\right)^{-2}$ without any restrictions in the phase space.

Elliptic (i.e., doubly-periodic in the complex plane of the variable $x$ ) solutions to the KP equation were studied by Igor in his 1980 paper, where it was shown that the (in general, complex) poles $x_{i}$ of elliptic solutions $u(x, y, t)=$ $-2 \sum_{i=1}^{N} \delta_{0}\left(x-x_{i}(y, t)\right)+C$ are subject to the equations of motion $\ddot{x}_{i}=4 \sum_{k \neq i} \wp^{\prime}\left(x_{i}-x_{k}\right)$ of the Calogero-Moser system with elliptic potential of pairwise interaction $\wp\left(x_{i}-x_{j}\right)$ (here $\wp$ is the Weierstrass elliptic function and dot means the derivative in $y$ ). The method first suggested by Igor consists in the substitution of the elliptic solution not in the KP equation itself but in the auxiliary linear equation $\partial_{y} \psi=\left(\partial_{x}^{2}+u\right) \psi$. This allows for separation of variables $y$ and $t$ from the very beginning. In this approach, one should use a special ansatz for the wave function $\psi$ (a linear combination of Lamé functions with some coefficients $c_{i}$ ), depending on a spectral parameter. It is then proved that such a wave function is the Baker-Akhiezer function on the spectral curve. The auxiliary linear problem is then equivalent to an overdetermined system of linear equations for the coefficients $c_{i}$ which follows from cancellation of poles. This method allows one to obtain the equations of motion for $N$ poles in the fundamental domain together with the Lax representation of them: $\dot{L}=[M, L]$, where $N \times N$ matrices $L, M$ depend on $x_{i}, \dot{x}_{i}$, and on the spectral parameter. The Lax equation, on the one hand, is equivalent to compatibility of the linear system mentioned above while, on the other hand, it means that the $y$-evolution of the Lax matrix is an isospectral transformation. The characteristic equation for the spectral parameter dependent Lax matrix $L$ defines the spectral curve which is an integral of motion.

Igor's method turned out to be rather general and productive and later was applied to a number of other problems. For example, it was used in Igor's work arXiv: 9411160 with Babelon, Billey, and Talon for the analysis of singular solutions to the matrix KP equation; they turned out to be connected with a spin generalization of the Calogero-Moser system (in the rational case known earlier as the Gibbons-Hermsen system). The spin de-
grees of freedom are matrix residues at the poles (in the scalar case they are fixed). In 1995, a similar method was used in our first joint work arXiv 9505039 for investigation of pole dynamics of elliptic solutions to the 2D nonabelian Toda lattice. In this case the auxiliary linear problem is a differential-difference first order equation. The Lax equation is equivalent to equations of motion for poles and spin degrees of freedom which define a new integrable many-body system with internal degrees of freedom-the spin generalization of the RuijsenaarsSchneider model with elliptic interaction. Its Hamiltonian formulation is still not known. In the scalar case, one obtains the Ruijsenaars-Schneider system which is a relativistic deformation of the Calogero-Moser system. The equations of motion have the form $\ddot{x}_{i}=\sum_{j \neq i} \dot{x}_{i} \dot{x}_{j} \frac{\wp^{\prime}\left(x_{i}-x_{j}\right)}{\wp_{\rho}(\eta)-\wp_{( }\left(x_{i}-x_{j}\right)}$, where $\eta$ is a parameter playing the role of the inverse velocity of light and having the meaning of the latttice spacing in the differential-difference Toda lattice equations.

Some time ago Igor proposed another method to obtain equations of motion of integrable many-body systems based on the observation that the equations of motion are equivalent, rather mysteriously, to the existence of a meromorphic solution to the linear problem (a linear differential or difference equation). In our last joint paper arXiv:2211.17216, this method was applied to the $B$-version of the Toda lattice introduced previously in our work arXiv:2210.12534. In this way, we introduced a new integrable many-body system which is a deformation of the Ruijsenaars-Schneider system. Our last joint work with Igor appeared on the e-print archive on December 1, 2022, the day of his death. He worked till his last day.

Since 1995, we completed with Igor more than 10 papers on different topics. Besides the pole dynamics, among them are representations of the Sklyanin algebra, functional relations in quantum integrable systems, Whitham equations in free boundary problems and others. It was a blessing to communicate and work with Igor.

## Tatiana Smoliarova

One never knows what random moment of everyday life will stick in one's memory forever. I can clearly see: a Moscow morning in early winter. It's still dark in the street; only the snow on the roofs is glittering. I am eight, my dad is thirty-two. We are doing our regular morning exercises (which, of course, he does for my sake: a serious amateur ping-pong player, he is in excellent shape; I am a plump child). Amid the sit-ups, he says: "By the way, you know,

[^33]yesterday I defended my doctoral thesis." ${ }^{5}$ Understanding that this is probably something nice, but also nothing special, I say, "Ok, that's good?"-and do another sit-up. An eight-year-old, I have no idea what an achievement it is to defend a doctoral thesis, in the grim early 1980s, in the Soviet Union with its state anti-Semitism, for someone who is Jewish, and young to boot. That's how my father always talked about the various pivotal points of his scientific career-casually, timidly, with an embarrassed smile. That's how he worked and lived.

He loved his comfort-


Figure 10. Igor with his granddaughters Asya and Natasha. able and cozy office at Columbia, overlooking Broadway, but his most cherished study was a tiny room in the attic of our dacha (country house outside Moscow), overlooking pine trees, an old well, and a bed of ferns. It was there that he would start each day at 5 a.m., at his old desk, with his fountain pen and a notepad. Otherwise, as was observed by several contributors to this article, nobody knew when he was doing his mathematics. I was always told that my father achieved his most important mathematical results of 1975 while endlessly washing and rewashing my swaddling clothes. When my three kids were born, swaddling clothes were no longer in use, but there was still plenty to do. He was a devoted grandfather, and playing games, reading books, or teaching his grandchildren to ride a bicycle were no less serious creative tasks than teaching students, chairing the department, or writing articles. Yet he was never as super-excited, happy, and proud as when he communicated to me (a nonmathematician) that he had just "come up with a very beautiful thing." As Nikita Nekrasov has said, my father, a profoundly nonreligious person, believed nevertheless in the ultimate harmony of the world. And I have always lived with a deep belief that even if the entire world were falling apart, my father would comewith a beautiful formula or tool, mathematical or not, or simply with the unique calm and reassuring tone of his voice-and fix everything. Sadly, when the world really began to fall apart, he was already too ill to fix it.

[^34]ACKNOWLEDGMENT. We thank Slava Gerovitch for reading the article and making useful comments.


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# Mathematicians Confront Political Tests: The American Mathematical Society and the Red Scare in 1954 Albert C. Lewis and Karen Hunger Parshall 

"Five years after a world war has been won, men's hearts should anticipate a long peace, and men's minds should be free from the heavy weight that comes with war. But this is not such a period-for this is not a period of peace. This is a time of the Cold War. This is a time when all the world is split into two vast, increasingly hostile armed camps-a time of a great armaments race" [McC50]. So Senator Joseph McCarthy (R-Wis.) bellowed from the dais in Wheeling, West Virginia, in a Lincoln's birthday speech in February 1950. But, in McCarthy's view, it was even worse. "I have in my hand," he told his listeners, " 57 cases of individuals who would appear to be either card-carrying members or certainly loyal to the Communist Party, but who nevertheless are still helping to shape our foreign policy" as employees of the US State Department.

Although the process of rooting out communists had already begun prior to McCarthy's speech, the Red-hunting mania that followed it in the 1950s was particularly intense. When four mathematicians-one in New York, two in Michigan, and one in Tennessee-lost their teaching jobs in 1954 for failing to answer questions about their

[^35]political beliefs before different investigative committees, their plight came before an American Mathematical Society unsure of how far it could or should go in asserting itself in the political, as opposed to the mathematical, arena.

## The Attraction and Peril of Communist Party Membership

The Communist Party of the United States of America (CPUSA) (also called the American Communist Party) was founded in 1919 and was affiliated with the Communist International (Comintern) headquartered in Moscow. A legal party, it put forward state and federal candidates for election, including candidates for president between 1924 and 1940. One of the latter, Earl Browder, had three sons all of whom became mathematicians: Felix, long associated with the University of Chicago, William at Princeton, and Andrew at Brown. Though never attaining enough votes to win a seat, the American Communist Party nevertheless appealed to a portion of the populace as a counter to what was deemed the class system that had caused the Great Depression of the 1930s. The fact, however, that the Comintern publicized its purpose to lead a worldwide struggle for overturning non-communist governments, gave ample reason for monitoring the CPUSA. Many state and federal investigative committees-for example, the California Senate Factfinding Subcommittee on Un-American Activities (the so-called Tenney Committee named after the rabidly anti-communist state senator, Jack Tenney) and the Special Committee of the Board of Higher Education of the City of New York-were set up or went
into action as the result of what was perceived as a growing communist threat. Their procedures, for better or for worse, appear largely to have been modelled on those of the House Committee on Un-American Activities (usually styled HUAC), which was established in 1938 and which set a certain precedent for how to go about purging the country of covert communist influence. ${ }^{1}$

Meanwhile, agents of the Soviet Union had infiltrated some of the most security-sensitive areas of the US government and military from at least the 1930s and had gone undetected for years despite the FBI's vigorous efforts. Among the best known, Klaus Fuchs and Ethel and Julius Rosenberg were eventually caught and tried by the early 1950s for passing information on nuclear weapons development to the USSR. Members of Congress seeking to root out subversives of a different sort found it could also pay politically to have high profile influencers of American life testify either as informers, like Walt Disney and Ronald Reagan, or as suspected members or sympathizers of the CPUSA such as the group of film writers and directors known as the Hollywood Ten (Figure 1). Whether or not their suspects subscribed to the cause of overthrowing the US government, the argument went that, since communists were supposed to adhere dogmatically to the party line, they were a risk to the status quo.

This implied that Party members of another very influential-if not so well publicized-group, the teaching profession, also represented a potential threat. Nor did mathematics, seemingly the most objective and apolitical of educational topics, provide immunity for its practitioners. As the New York Times reported on February 26, 1953, a figure no less influential than Dwight Eisenhower had done more than suggest, in his second press conference as president, that even mathematics teaching and textbooks could be used to "put across a doctrine." Yet, when Louis Weisner of Hunter College in New York City, Gerald Harrison at Wayne University in Detroit, Chandler Davis at the University of Michigan in Ann Arbor, and Lee Lorch at Fisk University in Nashville found themselves summoned in 1954, it had essentially nothing to do with the fact that they were mathematicians but everything to do with the fact that they were suspected communists teaching at American institutions of higher education.

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Figure 1. The House Committee on Un-American Activities opened its televised investigation of communist infiltration into the professional, educational, and entertainment fields in Southern California on March 23, 1953. The chair is Representative Harold Velde. From left to right: Gordon Scherer, Kit Clardy, Velde, Morgan Moulder, Clyde Doyle, and James Frazier.

## The American Mathematical Society in the Early Days of the Red Scare

Senior members and officers of the AMS were not unfamiliar with cases similar to those of Weisner, Harrison, Davis, and Lorch, men to whom we will refer collectively as the "group of four." Perhaps the earliest example of the AMS taking a public stand in the anti-communist campaign was in 1948 when the Council joined in the protests of other professional organizations and approved a resolution registering its "grave concern" with the statements, insinuations, and procedures of HUAC in its investigation of Edward Condon, nuclear physicist and Director of the Bureau of Standards [Kli48, p. 629]. The case against Condon was almost comically weak and amounted to nothing in the end, but it served as an early warning to the scientific community that it was being watched. It was a different matter when the committee became better prepared and, for example, called another high-profile physicist, J. Robert Oppenheimer, before it in June 1949. AMS Council members may have had second thoughts about their precedentsetting Condon resolution when scientists and mathematicians avowing communist connections, or who were being informed on, came under the spotlight.

HUAC and its legislative predecessors in the 1930s focused on individuals and organizations. A new front opened in 1947, however, when, faced with growing threats from the Soviet Union, President Truman signed US Executive Order 9835, which put in place the first general loyalty program in the United States. Although it applied only to US government employees, it was soon emulated by some state governments. The California State

Legislature, for example, pressured the Regents of the University of California to require faculty to sign an oath declaring, not just simple allegiance, but also nonmembership and non-belief in organizations that advocated the overthrow of the government [Bla09]. Clearly aimed at communist affiliations, the oath, despite strong faculty opposition, was implemented in 1950. At Berkeley, a number of faculty resignations resulted as did the actual firing of more than thirty others who refused to sign. Among the latter were at least three members of the AMS: topologist John Kelley, analyst Hans Lewy, and differential geometer Pauline Sperry.

In September 1950, the Council once again passed a resolution, this time addressed to the California Regents condemning their actions and calling for retracting the oath requirement. On December 28, 1950, it passed a further resolution declaring that the AMS would hold no meetings at the University of California until the matter was rectified. In what must have been an embarrassment for the Council, however, the general membership present at the Business Meeting the following day disapproved of such a boycott. This dissonance led to efforts over the next several years to clarify procedures, and probably led the Council to be more cautious about generating resolutions on controversial issues. ${ }^{2}$ While the AMS tried to sort itself out, the California affair was largely settled. Fired professors, who sued, won their case and were reinstated; the oath requirement was dropped by 1952.

A new opportunity for Council caution quickly came up, though. In April 1951, Oklahoma instituted much the same oath requirement as had California, and among the non-signers who were fired were five mathematicians at Oklahoma Agricultural and Mechanical College (now Oklahoma State University-Stillwater) and two at the University of Oklahoma. This time in response a committee joint with the Mathematical Association of America was formed to determine the facts. Each organization presented similar resolutions in 1952 to both university and Oklahoma government leaders conveying, without any censure, that their actions were dangerous for individual liberty and freedom of thought. The AMS resolution, in particular, spelled out an obvious but seemingly often unacknowledged argument in connection with the purges: "It is not to be expected that such legislation will be effective in eliminating from the faculties men who are dangerous to the national welfare, so that the injury caused is a useless waste, as at Oklahoma A. \& M. College, where a department of mathe-

[^37]matics which had achieved much recognition for its mathematical work was seriously damaged" [Beg52, pp. 617618]. The largest mathematical contingent targeted at Oklahoma $A \& M$ were members of a research group headed by Ainsley Diamond and Nachman Aronszajn. The chair of the joint committee, William Duren then also chair of the mathematics department at Tulane University in New Orleans, later recalled that

We were not able to do much for Ainsley Diamond, chairman at Oklahoma State University. When I got to Stillwater one of the department members took me aside confidentially and said to me: "Duren, there is something you don't know about this. He is a homosexual. We couldn't say that!" The word gay had not been coined then, so they used communist as a euphemism for homosexual. Diamond was gay but no communist in the real sense. He was an excellent mathematician, a good man, and apparently ran his department well. We said so, but he stayed fired. [Dur96, p. 132]
Diamond and Aronszajn promptly moved with their Office of Naval Research grant to the University of Kansas where they proved a welcome addition [Pri70, pp. 329330].

During this same period in the 1950s, the AMS and MAA, along with many other national organizations, were called upon to boycott cities in which racially integrated gatherings were not possible. Lee Lorch, for one, kept this issue before the two mathematical societies. Since not authorizing meetings in such places would essentially mean no meetings in the South, the AMS Council, in particular, sought compromises to avoid cutting itself off from its members there. ${ }^{3}$ One significant result, however, was finally developing procedures to help ensure that the Council avoided the California embarrassment and to represent more accurately the sense of the total membership. The outcome was the current Article IV, Section 8 of the by-laws which establishes rules under which the Council "shall have the power to speak" for the AMS on "matters affecting the status of mathematics or mathematicians" and which gives examples clearly covering the four cases considered here [Pit88, p. 297].

Although adopted in December 1953, in time to apply to the "group of four," there was never unanimity. Some felt that, while all well and good, this "power to speak" should never actually be used. Two of the most senior members advising the Council on the matter, Theophil

[^38]Hildebrandt, AMS president from 1945 to 1946, and G. Baley Price of the University of Kansas, expressed their misgivings. The former decried a tendency he detected to move away from the founding stricture of the AMS that it concern itself with mathematical scholarship and research. "If we keep on in this way we will be taking over the functions of the American Association of University Professors or become a Labor Union" [Hil]. ${ }^{4}$ For his part, Price concluded a letter of February 23, 1952 to AMS Secretary Ed Begle this way:

I agree with Professor Hildebrandt that the Society should not be concerned with the conditions of employment, salary, promotion, or teaching loads of individuals. I should not like to see the Society develop in the direction of a labor union nor take over activities which belong to the American Association of University Professors. If the conditions of employment of mathematicians throughout the nation, or of a large part of it, were to become such that the nation was weakened mathematically, it might be entirely necessary and proper for the Society to take action. This position appears to me to justify the steps which the Council took concerning the difficulties at Berkeley. [Pri] ${ }^{5}$
For those agreeing with this position, the questions of how many individuals would constitute a "large part" and of how to assess relative mathematical strength might still arise. Nevertheless, four mathematicians at four institutions-hardly a "large part" -would have been disqualified for special attention had these standards been adhered to.

## The "Group of Four" in 1954

Louis Weisner (Figure 2) was the first of the four to find himself facing inquisitors, in his case the Special Committee of the Board of Higher Education of the City of New York. Frank Nelson Cole's last PhD student at Columbia, Weisner wrote his 1923 dissertation on "Groups Whose Maximal Cyclic Subgroups are Independent." After an instructorship at the University of Rochester from 1923 to 1926 and a year at Harvard on a National Research Council fellowship, he landed an instructorship at Hunter College in 1927. By 1936, he had moved up through the ranks to become an associate professor with tenure. He joined the Communist Party two years later.

As early as 1939, the New York State Legislature had enacted a Civil Service Law that stipulated that "no person shall be appointed to or retained in the public service nor

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Figure 2. The Mathematics Department of Hunter College in 1936. Louis Weisner (1899-1989) is the first person on the left in the front row. (Mina Rees (see below) is the second person on the right in the first row.) Photo from The Wistarion (1936), p. 207.
in any public educational institution who becomes a member of any organization which advocates the overthrow of government by force or violence, or by any unlawful means" [Gui]. ${ }^{6}$ Although on the books, it was not actively enforced until after, first, the passage of the infamous Feinberg $^{7}$ Law in March 1949, which required all employees not only to attest that they were not members of the Communist Party but also to inform the president of the State University of New York if they ever had been, and, second, the formation in July 1953 of a Special Committee of the Board of Higher Education. The latter was tasked specifically to "undertake an investigative program designed to obtain all of the facts and available information relating to the membership and activities of any college staff members in or connected with subversive organizations and particularly of the Communist Party, and to take or recommend such actions as the facts warrant." When two of Weisner's colleagues, V. Jerauld McGill in the Department of Psychology and Philosophy and Charles W. Hughes in the Music Department, were summoned before the Special Committee, Weisner voluntarily stepped forward on their behalf. Admitting that he, too, had been a member of the Communist Party, he explained that the meetings that he, McGill, and Hughes had attended merely involved, in Lee Lorch's words, discussions of "general political questions, current events and trade union matters" and "that never had anything illegal ever been done or advocated, nor had force and violence ever been proposed" [Lor88, p. 1118]. Instead of assuaging the committee, however, Weisner's

[^40]admission prompted its members to demand that he provide the names of others who had been present at party meetings so that they, too, could be brought up on charges. Weisner's refusal to name names in his interrogation on March 26, 1954 resulted in the verdict that, in willfully failing to disclose "all of the facts or information" at his disposal, he was guilty of "neglect of duty and conduct unbecoming a member of the staff" [Min54, p. 511 (emphasis in the original)]. Before the Board could formally fire him, though, Weisner took early retirement, enabling him at least to tap into what, after twenty-seven years on the faculty, was a modest pension [Lor88, p. 1118].

The cases of Harrison and Davis were different. Since they were both implicated in HUAC's investigations of higher education in the State of Michigan, they were grilled by a standing committee of the US House of Representatives that not only had its own counsel and investigators but also a covert pipeline to the FBI and its files. Those who failed to answer questions to the satisfaction of this committee ran the risk of being cited for contempt of Congress and so of actual jail time.

Gerald Harrison (Figure


Figure 3. Gerald Harrison (1916-2000) before the House Committee on Un-American Activities in Detroit, May 3, 1954.
and, then, as a researcher at the Harvard Underwater Sound Laboratory in Cambridge, Massachusetts. Immediately postwar, he held positions at MIT's Radiation Laboratory as well as at the Sperry Gyroscope Co. in Lake Success on Long Island. By the fall of 1948, he had become an assistant professor in the mathematics department at Wayne

[^41]University (now Wayne State University) in Detroit, and by May 1954, HUAC had him in its sights.

Harrison had raised red flags for HUAC for several reasons. First and foremost, other testimony the committee had heard alleged that Harrison had been a member of Communist Party sections in both Michigan and Boston. It also suspected him of involvement, after his arrival in Detroit, with the American Federation of Teachers (AFT), an organization thought by HUAC, again based on other testimonies, to be communist-infiltrated in the 1940s. When he came before HUAC in Detroit on May 3, 1954, Harrison was intensely grilled on these questions by Gordon Scherer (R-Ohio) and Kit Clardy (R-Mich.), the latter known as "Michigan's McCarthy."

As a lead-up to hammering the mathematician about his alleged membership in the CPUSA, the HUAC interrogators pressed for details on an array of what they viewed as questionable points in his history. For example, his war work, as well as his work at both the Rad Lab and Sperry, involved government contracts and, at times, work of a classified nature. Indeed, when Sperry discovered that Harrison did not have the appropriate clearance to be working on the projects to which he had been assigned, he was let go [H-GH54, p. 5029]. HUAC suspected that he had actually been fired because he had not honestly filled out a question on his clearance form about his membership in the Communist Party. Then, when asked whether he was an AFT member, Harrison, as he did in response to all such questions of membership, invoked the Constitution. "I feel," he said, "that this is an invasion of my rights under the first amendment which prohibits Congress from legislating and therefore dealing in such questions relating to my right of assembly and free speech" [H-GH54, p. 5022]. Clardy immediately countered, directing Harrison to answer and laying down unequivocally what he and his HUAC colleagues viewed as the ground rules:
[W]e do recognize the right of counsel to advise the witness to invoke the fifth amendment properly so long as it is not done capriciously and, as you know, without any danger of possible recrimination. We do not-and I say this so that everyone may understand it-at any time recognize the right of any witness to refuse to answer on any other ground so far as the Constitution is concerned. ...[S]ince this is the first witness and there are others here, I might as well make it plain that the invocation of those other amendments has been attempted many times, has been rejected, and will not be accepted by the subcommittee as a reason. [H-GH54, pp. 5022-5023 (our emphasis)]

Harrison nevertheless continued to invoke the First Amendment, as had the Hollywood Ten in their trials in 1948 and 1950. Somewhat later in his testimony, however, Harrison also invoked the Fifth and Sixth Amendments [HGH54, p. 5024], as Clardy and Scherer ruthlessly pressed him to answer their question about whether or not he had been a member of the CPUSA. As a result of his testimony, Harrison was first suspended and then dismissed by the administration of Wayne University.

On May 10, 1954, just


Figure 4. Chandler Davis (1926-2022) with his letter of dismissal from University of Michigan President Harlan Hatcher, the lead story in the Michigan Daily, August 3, 1954. a week after Harrison's ordeal before HUAC in Detroit, Chandler Davis (Figure 4)-together with his University of Michigan colleagues, Nathaniel Coburn in mathematics, Mark Nickerson in pharmacology, and Clement Markert in zoology - was called to face Clardy and his colleagues in nearby Lansing.' Coburn, the author of a popular vector and tensor analysis textbook, was excused from appearing due to illness (multiple sclerosis) and no further action was ultimately taken against him [Dav88, p. 420]. Davis, who had joined the Communist Party for the first time when he was in his teens, had quit it prior to enlisting in the Navy during World War II but had rejoined in 1946 when he started graduate school at Harvard. ${ }^{10}$ Four years later, with the PhD in hand on "Lattices and Modal Operators" that he had done under the direction of Garrett Birkhoff, Davis passed up a job at UCLA, owing to the oath controversy then raging in California, and instead accepted an instructorship at Michigan. While in Ann Arbor, he continued his various Communist Party activities and received the first sign of the government's interest in him when officials from the State Department showed up at his apartment in 1952 and demanded that he and his wife relinquish their passports. It came as no real surprise, then, when an actual summons by HUAC followed in 1954 [Dav88, pp. 419-420].

The tenor and content of Davis's testimony was not unlike Harrison's. The committee relentlessly pressed him about his membership and participation in the Commu-

[^42]nist Party, in his case, though, the specific issue was the authorship of a pamphlet, strongly critical of HUAC, that was published and distributed under the auspices of the Council of Arts, Sciences, and Professions, a group in Ann Arbor comprised primarily of faculty and graduate students. Davis, though, steadfastly invoked the First Amendmentwithout subsequently adding other amendments as Harrison had-as he consistently refused to answer each question that he deemed political in nature. He also refused to name names. The committee's frustration is quite apparent in the following exchange:

Dr. Davis. This is a question?
Mr. Clardy. I am telling you the facts, sir. Isn't the reason that you are refusing to answer this question or say anything about it because of its Communist origin, inspiration, and direction?

Dr. Davis. Is this a question also?
Mr. Clardy. Yes, sir. If you don't understand questions, then that line of degrees that you have has misled me terribly. Now, can you answer it?

Dr. Davis. The answer to that question is the same as the answers I have given previously to questions about my political beliefs or affiliations. [H-CD54, pp. 5360-5361]
The exchange concluded:
Mr. Scherer. This witness is clearly in contempt of the Congress of the United States.

Mr. Clardy. There is no doubt about that. He has been in contempt all day here .... [H-CD54, p. 5361]
Indeed, as far as the committee was concerned, it was only the invocation of the Fifth Amendment, against selfincrimination, that would forestall a contempt of Congress citation for failure to answer, as Clardy made resoundingly clear in his response to Harrison and which may have led Harrison eventually to invoke it.

The student-run Michigan Daily reported the next day on petitions being circulated by students and faculty on behalf of Davis, Nickerson, and Markert [Wil]. ${ }^{11}$ The topologist Edwin Moise already had twenty-seven of his fellow mathematics faculty members' signatures in support of Davis by press time. ${ }^{12}$ The university administration also went into action by immediately suspending the three men pending the findings of an internal investigation by

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a Special Advisory Committee to Michigan President, Harlan Hatcher. On May 12, the Michigan Daily's headline read "Clardy Praises Hatcher For Cooperation of 'U'." It also ran a front-page news item on the parallel happenings at Hunter College with Louis Weisner and his colleagues, quoting the Hunter College undergraduate newspaper:

If these men were preaching Communistic ideology to their students, then they are a danger to our community.

But if they remained true to the ethics of their profession, then their suspension, while legal, is contradictory to the basic ethics of democracy.
Hatcher's announcement of the formation of his Special Advisory Committee at a meeting of the University Senate on May 17 not only prompted spirited debate within and outside the university but also served to call the president's judgment into question. As the minutes of that meeting recorded, Hatcher argued that " $[t]$ he recent refusal of three faculty members to answer questions before the House Subcommittee on Un-American Activities ... creates a problem because the questions are ones which need to be answered and because the refusal places on each person the obligation to explain his actions to his colleagues and institution" [Wil]. ${ }^{13}$ It was, however, not just this assumption that drew the fire of members of the faculty and student body as well as of the student paper. Tensions had also been heightened between the Faculty Senate and President Hatcher due to a motion put forward by Raymond Wilder, Michigan topologist and President Elect of the AMS, that called for an investigation of the administration's actions with respect to the implicated professors. The " 'U' family," at least as the associate editorial director of the Michigan Daily saw it on May 21, had become "estranged." Wilder, along with Moise, would soon also become a key player in the mathematical community's response to Red-hunting on a national level (see below). The situation at Michigan gave him first-hand experience of how the search for Reds could disrupt academia [Sch86].

The estrangement only deepened in the months that followed. As far as Davis's case went, an interview before the Special Advisory Committee at the end of May was followed first by a lengthy and detailed statement written by Davis in mid-June and then by Hatcher's ruling in July: Davis would be fired because of his refusal to answer various questions put to him by HUAC as well as by the president and his Special Committee. The Board of Regents officially dismissed both Davis and Nickerson on August 26; it reinstated Markert, who was judged cooperative in answering questions and no longer an adherent of communism. In August, Davis was also indicted for contempt

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Figure 5. Lee Lorch (1915-2014), left, after losing his City College position, being interviewed by James Booker for the New York Amsterdam News, June 24, 1950.
of Congress. As Davis later put it, he next "hung around Ann Arbor, jobless and under indictment, trying to make new plans" [Dav88, p. 424]. The local political fallout for President Hatcher continued to make news well into the fall.

Perhaps the most visible of the four mathematicians who found themselves in seriously hot political water in 1954, however, was Lee Lorch, chair of the Department of Mathematics at Fisk University. Lorch (Figure 5), who had earned his PhD under Otto Szász at the University of Cincinnati in 1941 for a thesis on "Some Problems on the Borel Summability of Fourier Series" before enlisting in the Army during World War II, had been pushing against the system since at least 1949. As an instructor at City College of New York (CCNY), he was not reappointed in his fourth year, a reappointment that would have effectively meant tenure. The New York Herald Tribune on June 9, 1949, described both the appeal Lorch made to the Board of Higher Education and a city councilman's opinion on the reasons behind the decision. According to the latter, Lorch was let go because of "his religion [he was Jewish] and because he had been active as vice-chairman of the Committee to End Discrimination in Stuyvesant Town, where he lives" [Lor]. ${ }^{14}$ The Metropolitan Life Insurance Company, owner and developer of the Lower East Side community of Stuyvesant Town, used the Jim Crow laws of the day to keep Blacks out, and the issue had galvanized the desegregation activities of Lorch and others.

His appeal unsuccessful, Lorch managed to obtain a position at Pennsylvania State University in State College but was let go from that job after just one year. In a widely reported stand, Lorch and his wife, Grace, had invited an African-American family to live rent-free in their Stuyvesant Town apartment while they were in State College, and although the Penn State administration, like that of CCNY, officially justified its action on the basis of

[^45]Lorch's "personal qualifications," an assistant to the Penn State president was reported to have said that Lorch's actions relative to the African-American family in New York City were "illegal, immoral, and damaging to the public relations of the College" [Wil]. ${ }^{15}$ Again, Lorch appealed to no avail, in spite of enlisting the support of no less a figure than Albert Einstein. These difficulties gradually brought Lorch's case within the ambit of the American Mathematical Society.

John Kline, AMS Secretary, topologist at the University of Pennsylvania, and clearinghouse of information mathematical, had first heard of Lorch's troubles at CCNY in 1949 from Samuel Eilenberg and Paul Smith, both at nearby Columbia [Wil]. ${ }^{16}$ In April 1950, Lorch and Kline had met at the AMS meeting held in Washington, D.C., to discuss the situation at Penn State, even though Kline was in no particular position to help. A month later in May, Kline received a letter from the same Raymond Wilder at Michigan who would become involved in Chandler Davis's troubles four years later in 1954, asking Kline what he knew about Lorch's case.

Lorch's next move, in 1950 and with its administration's full knowledge of his prior difficulties, was to Fisk. Popular with both colleagues and students, Lorch settled in well there, becoming active in the local chapter of the National Association for the Advancement of Colored People (NAACP), receiving outside funding for his research, serving as department chair, and working in various ways to put the mathematics department of the historically Black institution on the mathematical map. Even the national scene began to look rosy for those with Lorch's political and social commitments, when on May 17, 1954, the US Supreme Court delivered its unanimous ruling in the case of Brown v. Board of Education of Topeka, Kansas, to the effect that the segregation of public schools was a violation of the Fourteenth Amendment and therefore unconstitutional. The Lorches immediately sought to test this new principle by requesting of the Nashville School Board that their daughter, Alice, be "admit[ted] to the only public school in [their] neighborhood," namely, a segregated school for Blacks only [Wil]. ${ }^{17}$

Almost immediately thereafter, on September 7, Lorch was served with a subpoena by HUAC, commanding that he face that tribunal-comprised in his case of Clardy, Scherer, and Francis Walker (R-Penn.), together with Frank

[^46]Tavenner as HUAC's counsel-just one week later and some 350 miles away in Dayton, Ohio. Due to the extremely short notice, Lorch was forced to present himself without the benefit of counsel [H-LL54, p. 6953]. Clardy, however, made the committee's position clear and set the adversarial tone of the proceedings from the outset. "A week, better than a week's time is ample for anybody, and I should suggest this to you, Mr. Witness, that if the matter should be put over from today you might be put to the trouble and inconvenience at your own expense of coming to Washington. Now that is a matter you should give some consideration to" [H-LL54, p. 6955]. (The rules of the committee allowed witnesses to apply for "travel allowances and attendance fees," but gave no guarantee [USC53, p. 9].)

In the context of its standard questioning regarding educational and employment history, the Committee focused, as it had done in Harrison's case, on Lorch's involvement, while a student at the University of Cincinnati in the early 1940s, with the AFT [H-LL54, p. 6959]. The FBI, usually willing to share its own investigations with HUAC, maintained a file on Lorch that indicates he was a subject of interest from October 1941 to February 1973. ${ }^{18}$ At issue was Lorch's alleged—by husband-and-wife FBI informants in $1950^{19}$ —attendance in July 1941 at the American Youth Congress in Philadelphia as an AFT member. Tavenner asked him point-blank: "Were you a member of the Communist Party at the time you attended the American Youth Congress ... at Philadelphia?" [H-LL54, p. 6960]. When delaying maneuvers worked for only so long, Lorch also opted to invoke the First Amendment "on the grounds that a committee planning, investigating for the purpose of securing legislation cannot take the standpoint that it is [holding] an investigation which can lead to a violation of any of the rights protected by this first amendment" [HLL54, pp. 6962-6963]. ${ }^{20}$ But Lorch did even more. In the context of registering once again his objections to not having counsel present, he stated that "[f]or me to know what to prepare for was actually a very difficult thing. I am [Tennessee]-State vice president of the NAACP. As such, I might be presumed to be interested in bringing before this committee any information which might be valuable concerning efforts being made to subvert the Constitution of the United States in accordance with [the] decision of May 17 as to antisegregation in education" [H-LL54, p. 6963].

[^47]Clardy shot back: "You know you are going deliberately far afield. Come back to the beam."

More questioning, about Lorch's alleged-by the same FBI informants-attendance at a meeting of the CPUSA in Cincinnati in July 1941 resulted in Lorch's invocation yet again of the First Amendment and, this time, in Scherer's unequivocal conclusion: "Witness, I am advising you that you are clearly in contempt, legal contempt of the Congress now under the two answers that you have given. There is no question about it" [H-LL54, p. 6965]. It was only when, in the closing minutes of his testimony, Tavenner asked Lorch if he was "a member of the Communist Party when [he] accepted [his] position at Fisk University in 1950" [H-LL54, p. 6976] that Lorch relented somewhat in his hardline approach to the committee. "Again, for the purposes of safeguarding my institution, against the barrage of newspaper publicity which might accompany this, and which is intended to, by virtue of the public nature of these hearings, I answer that question, too, in the negative, but again with a protest that the committee has no right to pose such a question because of constitutional safeguards and because of its own rules" [H-LL54, p. 6977]. Despite this effort to spare Fisk University, Lorch was called before Fisk's Board of Trustees at the end of October 1954 and told in December that his contract would not be extended beyond the end of June 1955. Also in December, he was indicted for contempt of Congress. In analyzing his situation after the fact, Lorch agreed with the April 16, 1955 assessment of a writer for the Washington, D.C.-based newspaper, Afro-American, namely, that the real reason for Lorch's troubles at Fisk owed to the "bitter-end segregationists in Nashville" and the Dixiecrats who "called upon an agency of the House of Representatives to do their dirty work" [Wil]. ${ }^{21}$

## The AMS Committee on Displaced Persons

By the winter of 1955, the plights of Weisner, Harrison, Davis, and Lorch had come officially to the attention of the American Mathematical Society through the intervention particularly of Lorch's friend and Davis's colleague, Edwin Moise. In his capacity by that time as AMS President, Moise's and Davis's friend and colleague, Raymond Wilder, had received word of his authorization-in a letter from AMS Secretary Edward Begle dated January 12, 1955-"to appoint a committee to consider what could and should be done in order to avoid the termination of the mathematical careers of certain people who have been dismissed for political reasons" [Wil]. ${ }^{22}$ Begle understood this charge for the challenge that it was, however. "It will

[^48]undoubtedly be very difficult to find the proper people for this one," he told Wilder. Still, Wilder had some initial ideas that he jotted down at the bottom of Begle's letter. His (apparently) top choices, as they were bracketed by a large parenthesis and linked by an asterisk to the relevant paragraph, were Moise, Bill Duren, Saunders Mac Lane at Chicago, R(alph) D. James of the University of British Columbia, and Cornell's J. Barkley Rosser.

A first round of letters went out a month later. In reply, Mac Lane, who was then chair of his department, pled press of work in declining to serve, although he felt compelled to "confess in addition that my regard for the wisdom of Mr. Chandler Davis is very low, although this does not disqualify me by itself" [Wil]. ${ }^{23}$ Duren, a Southerner, who, as noted, had served on the AMS-MAA committee investigating the Oklahoma loyalty oath matter, was still department chair and also overworked, but, in reluctantly agreeing to serve, offered that "we can not afford to give up in our efforts for a resolution of the problem of intellectual freedom as it applies in mathematics particularly, and therefore I feel that if I am called on I should serve, even though the Committee would seem to be a rather futile one." A pragmatist, he added that "In my part of the country, we operate as follows in a situation of the sort which you are talking about. We ignore the general principle, and avoid clashes in doctrine; but we say: 'Joe is a friend of mine. I know he is all right. He needs a job.' I think this might work in other parts of the country as well, but the trouble is that most of our dischargees demand satisfaction on the basis of their principles" [Wil]. ${ }^{24}$ James and Rosser also agreed to serve, although the former wondered if the fact that he was Canadian might be a problem for the AMS while the latter feared that his over-stretched commitments in his department would affect his ability to do committee work in a timely way.

Under Moise as chair, this would have constituted a presumably ample committee of four, but Wilder actually had broader considerations in mind. In writing to Claude Shannon at Bell Labs on March 9, he explained that he had "to date, besides the chairman (E. E. Moise) found a suitable man to represent the smaller colleges (Duren), a Canadian member of the Society (R. D. James), and am hoping to find in addition someone from the private eastern universities as well as someone from industry" [Will. ${ }^{25}$ Shannon, although "in sympathy with the idea that these mathematicians should receive a fair break," begged off, but suggested his Bell Labs colleagues, Hendrik Bode and

[^49]Brockway MacMillan [Wil]. ${ }^{26}$ When Wilder extended the invitation to Bode, Director of Mathematical Research, he had no better luck. Bode, however, offered the advice that, if helping to find positions in industry for displaced mathematicians was a priority, then it should be understood that a university mathematician would likely need "to spend at least a few months' extra training in rounding out his physical background or in getting acquainted with some of the obvious applied fields, such as statistics or numerical methods" [Wil]. ${ }^{27}$ He also recommended asking the advice of Hunter College's Mina Rees (Figure 2), who had recently left her position as deputy science director of the Office of Naval Research. In a letter of thanks to Bode, Wilder recounted his meeting with Rees during a visit to New York City and her recommendation of Harold Kuhn, president of the Society for Industrial and Applied Mathematics. Wilder was particularly pleased that he could easily approach Kuhn since the latter, normally at Princeton, was "spending the summer two doors down the hall from me" at the University of Michigan [Wil]. ${ }^{28}$

Kuhn's acceptance over the summer finally completed the committee that Wilder had envisioned, even though Moise, Duren, James, and Rosser had already begun their work in early May. Begle had been right. It had been far from an easy committee to constitute.

That there were concerns in the mid-1950s about the general political climate and its effects on the American mathematical community was evidenced by more than the formation of this committee, however. For example, Wilder received at least two letters from Chicago's Irving Segal urging him as AMS president to create some sort of mechanism that would allow, as he put it in February 1955, the AMS "to receive and disperse voluntary contributions for the financial support of bona fide research mathematicians who are unable to practice their profession for political reasons" [Wil]. ${ }^{29}$ Wilder, probably recognizing that he would not be the only one to see the highly impractical nature of this idea, diplomatically avoided disagreeing and instead made clear that he too was "very much disturbed by the effect of the current hysteria on our profession, particularly in those cases where there is a threat of virtual ostracism. That scientific and mathematical ability is in short supply at present," he continued, "seems to be generally conceded, and the manner in which [we] are rendering some of it unusable, aside from the ethical ques-

[^50]tions involved, is a tragedy in my opinion" [Wil]. ${ }^{30}$ Fortunately, Wilder could pass Segal's idea on to the committee he was forming. In May, Segal reinforced his proposal with the claim that it would be possible to come up with objective criteria for a "bona fide" candidate and for allocating appropriate amounts of money. He also offered a dubious implication of what he cited as a historical precedent, namely, "[t]hat it is legitimate for this Society to act as agent for this purpose is implied by its decision in the California loyalty oath case" [Wil]. ${ }^{31}$ Wilder assured Segal in reply that the matter was now before the committee along with a related proposal from Lorch to create or support some sort of "foundation ... to finance projects for displaced people" [Wil]. ${ }^{32}$ At the same time, he made it clear, contrary to what Segal may have thought, that the AMS "did not collect money for the California professors" and that it would likely "not be feasible" for it "to undertake the handling of funds" in the present situation. As Begle put it, that "would be a great headache for the Society." For his part, Wilder "much prefer[red] to have any such operation carried out by an independent group."

Still, another financial suggestion, from two of Lorch's friends in California, computational number-theorist D. H. "Dick" Lehmer at Berkeley and complex analyst and mathematical physicist Max Schiffer at Stanford, prompted further discussion of such matters. In August 1955, they urged the AMS to "look into the matter of financial aid for legal expenses in order to aid mathematicians who are under political fire, in order that they be not deprived of their legal rights for want of a few thousand dollars" [Wil]. ${ }^{33}$ This became an even more pressing issue for Lorch, in particular, after his December indictment.

That Wilder rendered an opinion to Segal after consulting with Begle but before hearing from the committee may have been because he detected signs of inaction. Unfortunately, under Moise, and despite his initiation of it, the committee proved somewhat rudderless throughout the fall of 1955. Writing to Kuhn on January 24, 1956, Wilder offered that Moise was "apparently a rather poor letter writer" and actually "feels that the crisis no longer exists, since the people involved have generally got other positions" [Wil]. ${ }^{34}$ Nevertheless, Wilder thought "that there are recommendations the committee might make that would be of value should crises again arise," and one of those

[^51]specifically concerned the as-yet unresolved matter of financial support for legal fees.

In March 1956, Moise, spurred into action by a direct appeal from Lorch, and admitting to his "sheer negligence," finally raised the issue explicitly with his committee and asked for their opinions on it at the same time that he shared his own [Wil]. ${ }^{35}$ He was opposed to asking the AMS to collect funds for Lorch, but he was quick to add that he did "not feel happy about this conclusion. The known facts about Lorch and the plausible conjectures which may be based on them do not, in my opinion, justify his being sent to jail; and I think that the hearings on which the contempt citation was based represented very bad public policy. Thus my conclusions on the matter in hand reflect not so much a view of the Lorch case as a view of the Society's proper role in such matters."

The AMS Council apparently agreed, for when Berkeley's David Blackwell, a Council member, proposed that such a committee be set up, his recommendation was not adopted [Wil]. ${ }^{36}$ Instead, the Council suggested that "if a private committee were set up, an announcement of the committee could appear in the News and Notices of the Society," that is, the "News Items and Announcements" section of the Notices of the AMS, a journal that had been created in 1954 in part for the purpose of carrying informal communications of professional interest. Blackwell set to work and soon had such a committee in place. Indicative of the precedent-setting nature of this move and of the AMS's evolving position of maintaining a certain distance from the issue, however, both Wilder as AMS president and Richard Brauer as AMS president-elect declined Blackwell's invitation to serve on his committee. Wilder put it this way: "There's no question how I personally feel about these indictments. ... I am bothered, however, by the fact that I am still president of the Society, and that the Society decided not to take official action. ... I know I could participate as an individual mathematician, but to many it would look otherwise and I believe I'd better say no" [Will]. ${ }^{37}$ He was more candid in a letter to Brauer. "I'll admit," he said, "that I have got personally somewhat tired of writing innumerable letters for people who seem to get into difficulties repeatedly. I have done so in Lorch's case, and am convinced, between you and me, that he isn't happy unless he's under indictment or something comparable" [Wil]. ${ }^{38}$ Wilder's exasperation was understandable given that he had been flooded with documents and re-

[^52]quests from Lorch since early 1955. He did contribute money to Lorch's legal defense fund, however, if not his name to the actual committee, and he continued to respond positively to requests from Lorch in support of various causes for at least the next two decades [Wil]. ${ }^{39}$

When an announcement of Blackwell's committee appeared in the Notices in November 1956, it listed not only the members-Blackwell, Lehmer, Ralph Phillips, John Roberts, Gabor Szegö, and Antoni Zygmund-but also sounded the call for financial donations to Lorch's legal defense fund. Unwilling to establish a central funding agency, the AMS had instead provided a communication channel for members. The decision on legal expenses having been made at the level of the Council, a recommendation from Moise's committee was largely moot, and the AMS went no further. It therefore managed to avoid taking on anything like the trade union functions that Hildebrandt and Begle had warned against.

## And Then...

The year 1954 had unquestionably been a trying one for Weisner, Harrison, Davis, and Lorch. Their professional fates were by no means clear following their respective entanglements with Red-hunting committees. Moreover, their notoriety as "displaced persons" within the American mathematical research community unexpectedly threw them together into an exclusive "club," a band of brothers, to which others hoped never to belong. Davis learned of Lorch's troubles and got in touch with him to offer moral support and advice [Lor]; ${ }^{40}$ Davis, as a fellow Michiganer, was well aware of Harrison's plight; Lorch and Davis came to know of Weisner's situation. Of the four, Weisner and Harrison may not have abandoned their political beliefs but continued to keep lower profiles than the more militant Davis and Lorch who regularly kept in touch with one another after 1954.

After living off his pension for a year-and-a-half, Weisner, thanks to Lorch's active intervention, accepted a full professorship at the University of New Brunswick in Canada that Lorch had turned down in favor of an offer from Philander Smith College, a historically Black college in Little Rock, Arkansas. Weisner happily remained in Fredericton for the rest of his career, continuing his grouptheoretic and combinatorial research and earning a reputation as an outstanding teacher. He was made professor emeritus in 1970.

Harrison's career took a different turn. Unable to find a new job in academe, by 1956, he had taken a position as a systems analyst at the Teleregister Company in Stamford, CT. Teleregister, a pioneer in high-speed data

[^53]transmission and display, engineered such devices as the electronic ticker tape as well as electronic reservations systems for airlines and railroads. From his new industrial post, Harrison published at least one paper on "Stationary Single-Server Queuing Processes with a Finite Number of Sources" in the journal, Operations Research, in 1959.

Both Davis and Lorch, like Weisner, ultimately returned to academe. From 1954 to 1962, Davis managed to cobble together a series of short-term positions: in industry, as a part-time teacher, on a fellowship at the Institute for Advanced Study, in the employ of the AMS as an associate editor of the Mathematical Reviews. During this same period, he unsuccessfully appealed his contempt of Congress charges all the way to the Supreme Court (which refused to hear the case) and spent six months in federal prison in Danbury, CT in 1960 [Dav88, p. 423]. The precedent that had been set in Lorch's case, moreover, namely, of the formation of a private committee of mathematicians for the solicitation of funds to help defray legal costs, also worked in Davis's favor in the capable hands of William Pierce of Syracuse University [Lor]. ${ }^{41}$ By 1962, Davis, owing especially to the efforts of geometer Donald Coxeter, had taken the professorship at the University of Toronto that he would hold until his retirement in 1992. He continued actively to pursue not only his research in operator theory in Hilbert spaces but also his political activism, speaking out about and protesting against the Vietnam War and working for human rights.

As for Lorch, he stayed at Philander Smith College from 1955 to 1958. While there, he remained active in the NAACP and weathered, first, the dropping (by the government) of his contempt of Congress charge (in April 1956), then, the issuing of a second contempt of Congress indictment (in July 1956) on essentially the same grounds as the first, and, finally, a trial (April 1957) and acquittal (November 1957) on that second charge. He, but especially his wife, Grace, also became embroiled in the tensions surrounding the integration of Central High School in Little Rock in the fall of 1957 and found themselves, but again especially Grace, the target of the Dixiecrats [Lor]. ${ }^{42}$ As pressure mounted for the president of Philander Smith to dismiss Lorch and after an attempt was made in February 1958 to dynamite the Lorches' garage, the family made the decision to leave Little Rock. ${ }^{43}$ Although Lorch blanketed North America with applications, he found himself essentially blacklisted and was initially successful in securing only a one-year visiting position at Wesleyan College. In 1959, however, an application he had made the year

[^54]before to the University of Alberta bore fruit. He and his family remained in Edmonton until 1968, when Lorch accepted the professorship at York University in Toronto that he held until his retirement in 1985. Like Davis, Lorch remained active in his mathematical research, in his case in Fourier analysis, as well as in civil rights and other political and social causes.

## Political Lessons

The 1950s presented unforeseen challenges of a political nature for the AMS. Loyalty oaths in the states of California and Oklahoma affected some of its members. The machinations of HUAC and its clones affected others, notably, in our example, the members of the "group of four" who lost their jobs in 1954. Relatively speaking, though, the total number of mathematicians affected was not large, while it has been estimated that from 1947 to 1956 there were 2,700 dismissals and 12,000 resignations as a result of loyalty screening among federal workers alone [Sto13, p. 2]. There appear, however, to be no reliable estimates for the corresponding campaigns conducted by state and local governments and their associated educational institutions, campaigns like those that ensnared Ainsley Diamond in Oklahoma and Louis Weisner in New York. Another number hard to gauge stems from the parallel but less publicized campaigns that took place based on more personal circumstances of risks to security and morality such as alcoholism, homosexuality, and having a communist relative [Sto13, p. 101]. Though documentation is sparse, it is known that in addition to losing jobs, some who had their careers ended took their own lives [Sto13, pp. 193-194].

Given this general atmosphere of suspicion, accusation, and retribution, being called before a body like HUAC was undoubtedly a traumatic experience. Although it was probably of little immediate comfort, contemporaneous countervailing forces in the media and in government nevertheless aimed at putting an end to the anticommunist campaign, especially in its "witch-hunt" mode, on the grounds that it risked being more subversive to American democracy than the CPUSA. Even staunch anticommunists were concerned that pushed too far the campaign could become self-defeating. In 1954, as the bombastic Senator McCarthy, used to attracting media attention, was losing supporters, HUAC and its offspring still carried on, but with less of a public show. Reflective of the cautious optimism that times might be changing, Chandler Davis suggested to Lee Lorch during the fall election season in 1954 that "[t]hough it is unlikely the present Congress will convene again, therefore unlikely you will be cited for contempt, it is possible" [Lor]. ${ }^{44}$ Davis's optimism proved premature, however. Congress did change
${ }^{44}$ Folder 2007-054/026(13): Davis to Lorch, September 24, 1954.
hands from a Republican to a Democratic majority, but the new chair of HUAC, Francis E. Walter (D-Penn.), was at least as avid an anti-communist as his predecessors. Lorch, as noted, was cited; Davis ultimately went to jail.

Anti-communist sentiment was far from dying out, but other motivations for the continuing momentum of HUAC were evidently also at play. The case of Ainsley Diamond exemplified the fact that "communist" was used by some investigators as a cover for "homosexual," thereby enabling two purges to be conducted under one banner. ${ }^{45}$ Racist motives seemed likely to have been behind the persistence of the cases against the Lorches and others in the South. As Grace Lorch described it, committees like HUAC aided the racist cause in at least two ways:

There is direct intervention in which these committees use the power of the federal Congress against individuals and organizations opposed to segregation. Recently the Washington Post observed that "The Senate Internal Security Subcommittee has long made it plain that it regards support of the equal protection clause of the United States Constitution as subversive." (Editorial, October 30 [1957]).

Then there is the indirect help these committees supply by their example, their techniques and the atmosphere they have helped create. [Lor] ${ }^{46}$
HUAC survived long enough to investigate protest movements of the 1960s but began to change its image with a renaming in 1969 to the Internal Security Committee. It was finally terminated in 1975.

Interestingly, the challenges that the political reality of the late 1940 s and 1950 s presented to the mathematical community were reflected in a series of institutional changes within the AMS. It hired its first executive director in November 1949; it moved its headquarters to Providence in 1951; it began to rely less on volunteers [Pit88, pp. 251, 318, and 120-121]. It created its Notices, which quickly became a medium for member news and feedback on a variety of topics. It added Article IV, Section 8 to its by-laws, thereby providing a means for its Council to speak on its behalf on non-scientific matters. The era of the Red Scare dramatically brought out for the AMS the challenge of trying to adhere to its members' core, common interest in mathematics when political and social influences could force certain members out of the profession. The AMS's response to the predicament of the "group of four" in 1954-

[^55]as well as to that of the Californians and Oklahomans earlier in the decade-made plain the differences of opinion that can exist within a professional organization and the difficulties that it can confront in trying to deal with issues not directly related to its charter mission. Still, by putting in place a venue for member discussion and a procedure for voicing the position of the Society, the AMS was arguably better prepared by the end of the 1950s to engage with such challenges in the future.

ACKNOWLEDGMENTS. We gratefully acknowledge the help of: the archival staff at York University; Carol Mead and the staff at the Briscoe Center for American History, University of Texas at Austin; Matty Watson at the Archives of the University of New Brunswick; Sydney Van Nort and Jackie Rees at the Archives of the City College of New York; and the archival staff at Hunter College Library, Hunter College of the City University of New York.

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Albert C. Lewis


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# a Białynicki-Birula Decomposition? 

## Alberto Franceschini



Figure 1. The $\mathbb{C}^{*}$-action on $\mathbb{P}^{1}$, pictured as the sphere $\mathcal{S}^{2}$. The action flows from the north to the south pole.

An introductory example. Consider the complex projective line with nonhomogeneous coordinates, $\mathbb{P}^{1}=\mathbb{C} \cup \infty$, and consider the natural action of the multiplicative group $\mathbb{C}^{*}:=\mathbb{C} \backslash 0$ given by

$$
\mathbb{C}^{*} \times \mathbb{P}^{1} \ni(t, p) \longmapsto t p \in \mathbb{P}^{1}
$$

We have that

$$
\lim _{t \rightarrow 0} t p= \begin{cases}0 & \text { if } p \neq \infty \\ \infty & \text { if } p=\infty\end{cases}
$$

We call 0 the sink and $\infty$ the source of the action. Such action provides a decomposition of $\mathbb{P}^{1}$ into the affine spaces $\mathbb{C}$ and $\infty$. Using the homeomorphism between the projective line $\mathbb{P}^{1}$ and the sphere $\mathcal{S}^{2}$, we can draw the action as in Figure 1.

This is an example of Biatynicki-Birula decomposition. There are some similarities with Morse theory. In that case, one can study the topology of a manifold $M$ via its decomposition provided by the critical points of a function on

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M. Analogously, we will study a (smooth) algebraic variety via the cell decomposition provided by fixed points of a $\mathbb{C}^{*}$-action.
The Biatynicki-Birula theory. To keep our feet on the ground, we will stick to a very basic set-up (cf. [1]) even though the theory has been developed more generally (cf. for instance [5]). So, $X$ will be a complex nonsingular projective variety endowed with a (nontrivial) $\mathbb{C}^{*}$-action:

$$
\mathbb{C}^{*} \times X \ni(t, x) \longmapsto t \cdot x \in X
$$

We consider the decomposition of $X^{\mathbb{C}^{*}}$, the fixed locus of the action, into connected components:

$$
X^{\mathbb{C}^{*}}=\bigsqcup_{Y \in \mathcal{Y}} Y
$$

where $y$ denotes the set of connected components. Since $X$ is nonsingular, by a theorem of Iversen (cf. [4]) each connected component $Y$ is also nonsingular, hence irreducible.

One can always extend a $\mathbb{C}^{*}$-action on a nonsingular projective variety $X$ to an algebraic morphism $\mathbb{P}^{1} \times X \rightarrow X$ (cf. [8]), which means that for $x \in X$ there exist the limiting points to 0 and $\infty$ and they are fixed points of $X$ for the $\mathbb{C}^{*}$-action. Notice that the limiting point to $\infty$ is just the limiting point to 0 for the opposite action, that is

$$
\lim _{t \rightarrow \infty} t \cdot x=\lim _{t \rightarrow 0} t^{-1} \cdot x
$$

For a given $Y \in y$, we define its Biatynicki-Birula cells (BB-cells for short) to be the two subsets

$$
X^{ \pm}(Y):=\left\{x \in X: \lim _{t \rightarrow 0} t^{ \pm 1} \cdot x \in Y\right\}
$$

where $\pm$ will be intended as a shortcut for stating a result both for the + and for the - decomposition. Essentially, the BB-cells of a fixed point component $Y$ consist of all
the points of $X$ that converge to $Y$ as the parameter $t$ of the action goes to 0 or to $\infty$.

Theorem (Białynicki-Birula, 1973). Let $X$ be a complex nonsingular projective variety endowed with a (nontrivial) $\mathbb{C}^{*}$ action. Consider the induced + and - decompositions. Then the following hold:

1. $X=\bigsqcup_{Y \in y} X^{ \pm}(Y)$ and the BB-cells are locally closed subsets of $X$ for any $Y \in Y$.
2. The natural maps

$$
X^{ \pm}(Y) \ni x \longmapsto \lim _{t \rightarrow 0} t^{ \pm 1} \cdot x \in Y
$$

are algebraic: they are locally trivial bundles in the Zariski topology, and the fibers are affine spaces of $\operatorname{rank} \nu_{ \pm}(Y):=$ $\operatorname{dim} X^{ \pm}(Y)-\operatorname{dim} Y$.
3. There are homology decompositions

$$
\mathrm{H}_{m}(X, \mathbb{Z})=\bigoplus_{Y \in \mathcal{Y}} \mathrm{H}_{m-2 \nu_{ \pm}(Y)}(Y, \mathbb{Z}) .
$$

As a consequence of the theorem, among the fixed point components there exist unique $Y_{+}, Y_{-} \in y$ such that $X^{+}\left(Y_{+}\right), X^{-}\left(Y_{-}\right)$are dense subsets of $X$. We call $Y_{+}$and $Y_{-}$ respectively the sink and the source of the action, following the notation of the introductory example.
An example of Grassmannian. Let $V_{+}$and $V_{-}$be $n-$ dimensional vector spaces. Consider the $\mathbb{C}^{*}$-action on $V_{ \pm}$ given by

$$
\mathbb{C}^{*} \times V_{ \pm} \ni(t, v) \longmapsto t^{ \pm 1} v \in V_{ \pm} .
$$

Let $V:=V_{+} \oplus V_{-}$. Then $V$ has a naturally defined $\mathbb{C}^{*}$ action such that $V_{+}$and $V_{-}$are maximal invariant linear subspaces.

Consider the induced action on $X:=\operatorname{Gras}(n, V)$, the Grassmannian of $n$-planes in $V$. The fixed points of the induced action are the $n$-planes in $V$ that are $\mathbb{C}^{*}$-invariant for the action above. Indeed, if $W \subset V$ is a $\mathbb{C}^{*}$-invariant $n$-plane, then we can write

$$
W=\left(W \cap V_{+}\right) \oplus\left(W \cap W_{-}\right),
$$

where $W_{ \pm}$are the eigenspaces on which $\mathbb{C}^{*}$ acts. One can prove that the sink and the source of the induced action on $X$ are isolated points, representing the subspaces $V_{+}$and $V_{-}$, respectively.

For explicit computations, suppose that $n=2$. Then $X \simeq Q^{4}$, a smooth quadric hypersurface in the projective space $\mathbb{P}\left(\wedge^{2} V\right)$ of dimension 5 . One can prove that the action on $X$ is the restriction of the action on $\mathbb{P}\left(\wedge^{2} V\right)$ given by

$$
t \cdot\left[x_{0}: \ldots: x_{5}\right]=\left[t^{2} x_{0}: x_{1}: x_{2}: x_{3}: x_{4}: t^{-2} x_{5}\right] .
$$

Then the fixed locus of the action on $X$ is

$$
X^{\mathbb{C}^{*}}=[1: 0: \ldots: 0] \sqcup\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right) \sqcup[0: \ldots: 0: 1]
$$

and the homology groups of $X$ can be computed using homology of points and of $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The fixed point component $\mathbb{P}^{1} \times \mathbb{P}^{1}$ appear as the intersection of the quadric


Figure 2. A schematic picture of the $\mathbb{C}^{*}$-action on $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The action flows from left to right as $t$ moves from $\infty$ to 0 .
$X$ with $\mathbb{P}^{3} \subset \mathbb{P}^{5}$ generated by $x_{1}, x_{2}, x_{3}, x_{4}$. Notice that the decompositions of the Grassmannian given by BB-cells are particular cases of Schubert decomposition.

Furthermore, consider the action on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ given, in nonhomogeneous coordinates, by

$$
\mathbb{C}^{*} \times\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right) \ni(t, p, q) \longmapsto(t p, t q) \in \mathbb{P}^{1} \times \mathbb{P}^{1}
$$

There are four fixed points for such action

$$
(0,0), \quad(0, \infty), \quad(\infty, 0), \quad(\infty, \infty)
$$

as pictured in Figure 2.
Then, by the homology decomposition of the Białynicki-Birula theorem,
$\mathrm{H}_{0}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}, \mathbb{Z}\right)=\mathrm{H}_{0}((\infty, \infty), \mathbb{Z})=\mathbb{Z}$,
$\mathrm{H}_{2}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}, \mathbb{Z}\right)=\mathrm{H}_{0}((0, \infty), \mathbb{Z}) \oplus \mathrm{H}_{0}((\infty, 0), \mathbb{Z})=\mathbb{Z} \oplus \mathbb{Z}$, $\mathrm{H}_{4}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}, \mathbb{Z}\right)=\mathrm{H}_{0}((0,0), \mathbb{Z})=\mathbb{Z}$.
With this decomposition of the homology groups of $\mathbb{P}^{1} \times$ $\mathbb{P}^{1}$, one can explicitly compute the decomposition of the homology groups of $X$.
The associated birational map via GIT. A birational map $f: Z_{1} \rightarrow Z_{2}$ between algebraic varieties is a rational map such that there are open subsets $U_{i} \subset Z_{i}($ for $i=1,2)$ such that the restriction $\left.f\right|_{U_{1}}: U_{1} \rightarrow U_{2}$ is an isomorphism.

An explicit remark in [7] says that, given a $\mathbb{C}^{*}$-action on a nonsingular projective variety $X$, we can associate a birational map among projective varieties.

Given a finite-dimensional vector space $V$, the projective space $\mathbb{P}(V)$ is the space of lines through the origin of $V$, that is a point of $\mathbb{P}(V)$ is an orbit for the $\mathbb{C}^{*}$-action on $V \backslash 0$ given by

$$
\mathbb{C}^{*} \times V \backslash 0 \ni(t, v) \longmapsto t v \in V
$$

This example leads to the definition of geometric quotient: a space whose points represent $\mathbb{C}^{*}$-orbits of another space, see [6] for a rigorous introduction.

In the context of the Białynicki-Birula decomposition, we consider the subsets $\mathrm{B}_{ \pm}:=X^{ \pm}\left(Y_{ \pm}\right) \backslash Y_{ \pm}$. They are dense because $X^{ \pm}\left(Y_{ \pm}\right)$are dense and $Y_{ \pm}$are closed subsets of $X$. These sets contain all the orbits that flow to $Y_{ \pm}$. The quotients of these sets by the action are denoted

$$
\mathcal{G}_{ \pm}:=\mathrm{B}_{ \pm} / \mathbb{C}^{*} .
$$



Figure 3. A picture of the $\mathbb{C}^{*}$-action on $X$. The action flows from left to right as $t$ goes from $\infty$ to 0 . $C$ We draw an orbit $C$ flowing from the source to the sink, its tangent directions at the source and the sink, and the two geometric quotients $\mathcal{G}_{ \pm}$.

By a theorem of Białynicki-Birula and Święcicka, see [2], the spaces $\mathcal{G}_{ \pm}$are geometric quotients. In particular, they are quasi-projective varieties. Every point of $\mathcal{G}_{+}$(resp. $\mathcal{G}_{-}$) represents an orbit of the action flowing to the sink $Y_{+}$ (resp. from the source $Y_{-}$). Then we can identify such an orbit with its tangent direction at the sink or the source of the orbit.

It remains to describe the birational map between the geometric quotients $\mathcal{G}_{ \pm}$. We consider the intersection $\mathrm{B}_{+} \cap$ $\mathrm{B}_{-} \subset X^{ \pm}\left(Y_{ \pm}\right) \backslash Y_{ \pm}$: It contains all the $\mathbb{C}^{*}$-orbits that flow from the source to the sink of the action. Then we obtain a natural birational map

$$
\psi_{a}: \mathrm{B}_{-} \rightarrow-\mathrm{B}_{+}
$$

which is just the identity on the intersection $B_{+} \cap B_{-}$. The quotient of this map by the $\mathbb{C}^{*}$-action,

$$
\psi: \mathcal{G}_{-} \rightarrow \mathcal{G}_{+}
$$

is the birational map we are looking for. Notice that, since there could be orbits that flow from the source to a fixed point component different from the sink, the map $\psi$ is not an isomorphism in general.

Geometrically, given a $\mathbb{C}^{*}$-orbit $C$ that flows from the source to the sink, $\psi$ associates to its tangent direction at the source and its tangent direction at the sink. Figure 3 gives a schematic picture of the situation.
Torus actions and matrix inversion. In this section, we will go further on the example of $X$ being the Grassmannian $\operatorname{Gras}(n, V)$, where $V=V_{+} \oplus V_{-}$is a $2 n$-dimensional vector space.

The tangent space of the Grassmannian at the point $[W] \in X$ representing $W \subset V$ is

$$
T_{X,[W]} \simeq \operatorname{Hom}(W, V / W)=W^{\vee} \otimes(V / W)
$$

As we stated before, the sink $Y_{+}$and the source $Y_{-}$of the $\mathbb{C}^{*}$-action are isolated points in the Grassmannian. Then
$X^{+}\left(Y_{+}\right)$and $X^{-}\left(Y_{-}\right)$are affine spaces, and there are $\mathbb{C}^{*}$ equivariantly isomorphisms

$$
X^{ \pm}\left(Y_{ \pm}\right) \simeq T_{X, Y_{ \pm}} \simeq V_{ \pm}^{\vee} \otimes V_{\mp} .
$$

with the tangent spaces of $X$ at $Y_{+}$and $Y_{-}$. Then the geometric quotients are isomorphic to projective spaces, $\mathcal{G}_{ \pm} \simeq \mathbb{P}\left(V_{ \pm}^{\vee} \otimes V_{\mp}\right)$, and the induced birational map is

$$
\psi: \mathbb{P}\left(V_{-}^{\vee} \otimes V_{+}\right) \rightarrow \mathbb{P}\left(V_{+}^{\vee} \otimes V_{-}\right)
$$

If we fix bases, then $V_{ \pm}^{\vee} \otimes V_{\mp}=\operatorname{Hom}\left(V_{ \pm}, V_{\mp}\right)$ is the space of $n \times n$ matrices. Then one can show that $\psi$ is the projectivization of the inversion map of $n \times n$ matrices. Moreover, if $n=3$, then $\psi: \mathbb{P}^{8} \rightarrow \mathbb{P}^{8}$ is one of the special quadroquadric Cremona transformations classified in [3].

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The Proof Stage: How Theater Reveals the Human Truth of Mathematics
By Stephen Abbott. Princeton University Press, 2023, 416 pp .
While the connections between mathematics and music have long been explored, there has been less said about math and theater. Until now. If the connections, collaborations, and comparisons between math and theater intrigue you, this book will not disappoint.

Abbott presents details from over 30 plays about mathematicians and mathematical topics. You will read about Hardy and Ramanujan in The Disappearing Number, Alan Turing in Breaking the Code, and a fictional John Nash-like character in Proof. But what I suspect intrigues the author more than using mathematicians as the main characters is the embedding of mathematical ideas into a stage play. For instance, Samuel Beckett utilized Zeno's paradox in Endgame. The playwright Tom Stoppard used the unpredictability of chaos in the plot of Arcadia, and the titular characters from his play Rosencrantz and Guildenstern are Dead humorously discuss the possible explanations of why they flipped heads on a coin 90 times in a row.

Throughout the book, Abbott offers many suggestions for why theater is an ideal tool with which to explore mathematical ideas; truly, a playwright can create the environment of their choosing through the audience's imagination. There are also parallels between the search for truth. Abbott describes mathematics as the search for abstract truth; some might say theater is a presentation of fundamental truths about human nature. This recreational book will expose you to a large variety of playwrights and theatrical pieces, along with short mathematical expositions. After this book reading, I expect that many mathematicians

[^56]will be entertained and intrigued by the numerous connections between math and theater.

## Grading for Growth <br> By David Clark and Robert Talbert. <br> Routledge, 2023, 262 pp.

If a student's grade indicates successful completion of a course, how much has that student learned? Why will some students do almost anything for points in a course except grapple with challenging material? How can we offer meaningful feedback and increase our students' motivation? Clark and Talbert, two mathematics faculty members at Grand Valley State University in Michigan, explore possible answers to these questions in Grading for Growth. By examining alternative grading methods, such as standardsbased grading (where students must obtain proficiency in course standards) or specifications grading (where each assignment has a set of requirements it must meet in order to pass), the authors emphasize that traditional grading systems can be a barrier to student learning in all academic areas, not just mathematics. This book features alternative grading systems for mathematics, engineering, physics, philosophy, and history courses, just to name a few. These systems exist at large schools and small schools, in coordinated sections and lab courses, and as components of traditionally graded courses.

This is an incredibly rich and thought-provoking book; the authors not only tell the reader what alternative grading is, but also how to do it. In a workbook section, they encourage a sincere instructor to set aside 6-8 hours to develop a new grading system. The frequently asked questions section addresses a wide range of issues such as how long it takes to implement these systems and how to handle the increased workload of alternative grading. The authors explain how alternative grading systems often result in a bimodal grade distribution (more $\mathrm{A} / \mathrm{Bs}$ and more Fs ), but besides a student's eventual grade, they reveal that the biggest payout of alternative grading is improved student success with the course content and course outcomes. If you want your students to not only learn the material but also thrive in the learning process, Clark and Talbert make a compelling case that alternative grading systems can help you achieve that goal.

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Topics in Spectral Geometry
By Michael Levitin, Dan
Mangoubi, and Iosif
Polterovich. 2023, 325 pp., https://bookstore.ams .org/gsm-237.

Speaking very generally, spectral geometry investigates relations between the geometry of Riemannian manifolds and spectral properties of geometric differential operators (most commonly, the Laplacian) on them. Origins of spectral geometry can be traced to experiments with vibrating plates by the German physicist and musician Ernst Chiandi in the late 18th century and to the work of the British physicist and mathematician Lord Rayleigh (John William Strutt) about 70 years later. Lord Rayleigh's book The Theory of Sound (1877) is still used by engineers.

This work continued through the 20th century with such mathematicians as Hermann Weyl, Richard Courant, Lars Hörmander, Louis Nirenberg, and many others who asked important questions and proved groundbreaking results. In 1966, the Polish-American mathematician Mark Kac published a particularly noteworthy article entitled "Can one hear the shape of a drum?", where he asked whether there exist isospectral (i.e., with the same spectra of the Laplacian) but noncongruent planar domains.

The book under review invites the reader into the beautiful world of spectral geometry. It starts with the introduction of the Laplace-Beltrami operator on Riemannian manifolds in Chapter 1 presenting, in particular, examples where eigenvalues can be calculated explicitly. Chapter 2 lays out the main analytic tools used in the book, including spectral theory of unbounded self-adjoint operators, Sobolev spaces, and elliptic regularity. In Chapter 3, the variational approach to eigenvalues is described,

[^57]together with basic results about spectra such as domain monotonicity of eigenvalues for the Dirichlet problem, Weyl's law for asymptotic distribution of eigenvalues, and Friedlander-Filonov relations between eigenvalues of Dirichlet and Neumann problems in the same domain.

The nodal set of an eigenfunction $\phi$ of the Laplacian on a Riemannian manifold is the set where $\phi=0$. The connected components of the complement to the nodal set are called nodal domains. Nodal sets and nodal domains are main characters in Chapter 4. It contains, in particular, the proof of the famous theorem of Courant, which says that the eigenfunction corresponding to the $k$ th eigenvalue of the Laplacian has at most $k$ nodal domains. In Chapter 5, various inequalities for eigenvalues in geometrically related domains are presented. One of the first results of this type is the Faber-Krahn inequality that says that the first eigenvalue of the Laplacian in a bounded Euclidian domain $\Omega$ is greater than or equal to the first eigenvalue of the Laplacian in the ball of the same volume.

In Chapter 6 the authors introduce the heat equation associated with the Laplacian and explain the relation of heat kernel asymptotics to the analysis of eigenvalues. This naturally leads to the construction of examples of noncongruent isospectral manifolds and domains. The authors present the 1966 Milnor construction (tori of dimension 16), the 1985 Sunada construction (manifolds of dimension 4), and the 1992 Gordon-Webb-Wolpert construction of planar polygonal isospectral noncongruent domains (thus answering Mark Kac's question).

The final Chapter 7 is devoted to the spectral geometry related to the Steklov eigenvalue problem, where the analogues of eigenvalues appear not in the equation itself but in the boundary conditions. The authors formulate the variational principle for the Steklov problem and prove various inequalities for the Steklov eigenvalues on simply connected planar domains. They also show the relationship between the Steklov problem and the so-called Dirichlet-to-Neumann map, and present some applications of the Steklov problem to the analysis of the flow of ideal fluid in a narrow canal.

With numerous examples and exercises, the book is an excellent text for a graduate class on spectral geometry or for a self-study.

# What Can We Say About "Math/Art"? 

## George Hart



Figure 1. George Hart, Solar Flair. In this 5-foot diameter outdoor sculpture, sixty identical pieces of plasma-cut stainless steel interweave, while meeting at twenty 3 -fold vertices and twelve 5 -fold vertices. Based on the $A_{5}$ symmetry of the icosahedron and grounded in the thirty face planes of the rhombic triacontahedron, the form conveys a sense of intricate geometric regularity.
"Math/art" is becoming a standard term, but what exactly is it or should it be? Mathematical fields tend to mature over time, typically beginning with examples studied in isolation, then connections are observed, generalizations are noted, insightful essences are formalized, and ultimately organizing ideas are articulated that clarify the entire sub-

[^58]

Figure 2. Jasper Johns, Numbers. The composition of this painting features a $2 \times 5$ array of the digits 0 through 9 . The numbers seem to be employed merely as neutral objects for carrying the artist's experiments with color, but they are positioned in a specific sequential structure and the dimensions 2 and 5 derive from factoring 10, raising the question of whether this should be considered "mathematical" art.
ject. In the field of mathematical art there are many fascinating examples about which interesting things can be said, with myriad connections apparent, but the practice of making mathematical artworks has not been organized into any coherent framework. I will not attempt anything so grand here, but I will suggest that mathematicians should be looking harder at mathematical art, both creating more examples and examining them critically. There are many unknowns to clarify and formalize. It is not even clear where to place the field: is it a specific branch of applied mathematics or a separate discipline that emerged from math and art? Studying interesting examples is a key step in this process of maturation.

I consider myself an applied mathematician and sculptor, employing mathematical techniques and ideas in
creating geometric art. Much of my time is spent designing and constructing physical artifacts that I offer to the world as aesthetic objects in the category informally called "math/art." My hope is that viewers, mathematically trained or not, will find these works to be worthy of appreciation. The mathematician in me enjoys discovering, thinking about, and communicating engaging patterns and relationships embodied in sculptural form. The artist in me delights in the self-expression, beauty, and cultural enrichment associated with creative artworks.

From a human perspective I find no contradiction, rather a great resonance, in the blending of mathematics with fine art. It is a central part of my life. Yet when attempting any rational introspection into the nature and power of mathematical art, one is immediately stymied by the fact that the subject seems ill-suited to our usual tools of formal analysis. One can't even define "art" in the rigorous way that elementary mathematical practice would require. And even without a universal definition of "art," if we agree that a particular object is art, people may still disagree on whether it is also "mathematical art."

Without firm foundations, is there any substance here to interest a mathematician? I claim Yes, but one cannot hope to approach the topic as in a textbook with definitions and theorems already laid out. Instead, one must see it more like a challenge at a group problem-solving session, where one ponders examples and counter-examples and enjoys the communal process of beginning to sort through and make sense of an initially confusing cloud of ideas. There are intellectual and aesthetic treasures to be found in the world of math/art, though it may not be easy to express them in rigorous terms. Trained mathematicians are well positioned not only to appreciate the field and move it forward, but also to articulate what it is that makes mathematical art such a worthy human endeavor.

For centuries philosophers have debated varying definitions of art without resolution. Proposed essential characteristics have involved aesthetics (a sufficient degree of beauty), or mimetics (the quality of representation of some subject matter), or technical achievement (original, one-of-a-kind, hand-made mastery), or an affective relationship to the viewer (being thought-provoking or emotionally moving rather than functional), or even the role of culture and institutions (anything in a "museum" by an "artist" is art). For an entry into this formidable literature see [1]. However, such attempts appear to be doomed to failure given the wide range of accepted art objects and categories found throughout the classical, contemporary, and multicultural domains and given the individual variation in how people gauge the artistry of particular examples. Mathematicians, with their professional expertise in making and testing deeply thought-out definitions, are


Figure 3. Sol Lewitt, Incomplete Open Cubes. An example of twentieth-century minimalist and serialist art, this type of work led many contemporary viewers to ask "Is it art?" The artist poses a puzzle by presenting an answer and leaving the viewer to discover the question. A mathematician can phrase it as enumerating all 122 connected subsets of the twelve edges of a cube that span $\mathbb{R}^{3}$ and have cardinality less than twelve, modulo rigid rotations.
no doubt particularly adept at finding faults in any proposed necessary and sufficient conditions claimed to circumscribe the set of objects to be labeled "art."

If we accept that there is no unified essence of "art" to be approached with the mathematician's definition-theoremproof mindset, how then should we proceed? And toward what goal? For years, I have considered writing a "Mathematical Art Manifesto" akin to the twentieth century's celebrated "Cubist Manifesto" and "Surrealist Manifesto" (and dozens of others). Such documents typically spawn from a coherent community trying to prescribe the values and motives of an emerging art movement. For starters, do we have a relevant community?

I am hardly alone as a geometric sculptor; there is a large supportive math/art community. The annual art exhibition at the Joint Math Meetings [2], the Bridges Conference on Mathematics and Art [3], the Journal of Mathematics and the Arts [4], the MAA special interest group in the arts (SIGMAA-ARTS) [5], the semester-long ICERM workshop on Illustrating Mathematics [ 6,7 ], and a great many published books and articles [8] all attest to the vibrant health of the field. A large community is passionately creating and appreciating mathematical art. This consists largely of people who are trained to at least the university level in mathematics, but also includes a wide range of math lovers, educators, and artists who find a natural affinity to patterns and structures expressed in a creative manner. Should some type of Mathematical Art Manifesto be our formal goal? My view is that the congregation has not yet

## COMMUNICATION

## 2023 Joint Mathematics Meetings



Figure 4. Front page of the Joint Math Meetings 2023 Art Gallery web page. An assortment of media and mathematical topics are represented in the curated art exhibition galleries on display at the Joint Math Meetings and the Bridges Conference each year. The various works can be considered to be art, craft, design, models, or visualization.
reached a level of consensus on the core values of mathematical art [9].
(As an aside, I should point out that this essay focuses on math/art creation and ignores other thrusts in this community such as the art historian's perspective on the history of connections between math and art [10], statistical analysis of unsigned art via "stylometry" to determine authorship [11], or studies that interpret what pure mathematicians do to be an artform with the medium being ideas [12].)

As a professional geometric sculptor with a degree in mathematics, deeply involved in the community, I can claim some insight into the field. I have been an associate editor for the Journal of Mathematics and the Arts since its founding. I have been centrally involved with the Bridges Conference since its inception over 25 years ago and have served as president of the Bridges Organization for the past seven years, bringing together hundreds of participants from dozens of countries. I cofounded the Museum of Mathematics in New York City, brought in the artists whose works decorate it, and specified that it must contain an art gallery space for changing exhibits. I know personally a sizable fraction of the community and have no doubt read (and/or reviewed) over a thousand academic papers on the subject.


Figure 5. Albrecht Durer, Melancholia I. This enigmatic engraving includes a $4 \times 4$ magic square, a drawing compass, a sphere, a mysterious polyhedron, and other symbolic references. Various attempts have been made to explain the specific polyhedral shape. Is it a cube elongated into a rhombohedron, then truncated to be inscribable in a sphere? What might it symbolize?

One entry into an understanding of mathematical art is to look at the works that have been accepted into the curated mathematical art exhibitions at the JMM and the Bridges Conference or described in peer-reviewed conference and journal papers. One finds a wide variety of media and subject matter, including computer-generated renderings of fractals, hand-drawn tessellated images inspired by M. C. Escher, or beautiful physical models of mathematical objects such as lattices, knots, topological surfaces, or polyhedra. There are geometric origami, intricately patterned quilts, and amazing handmade examples of beadwork, crochet, knitting, and tatting, each with an underlying mathematical story. In addition to new media such as 3D printing and computer-controlled wood carving, many traditional fine art media, such as oil painting, watercolor, encaustic, ceramics, or block printing are adapted to illustrating mathematical ideas. A knowledgeable viewer will discern inspiration in these works from countless mathematical gems, such as Fibonacci numbers, the Pythagorean
theorem, integer factorizations, the seventeen wallpaper groups, concepts of infinity, and much more.

I am regularly impressed with some of the new works presented in these venues, but we must be clear that these efforts are not of interest to world-class fine art museums or high-ticket auction houses. The community that produces this math/art is its own most enthusiastic audience. One must wonder why.

It is clear that in our culture some artworks are more difficult to appreciate than others. Twentieth-century visual art and music is notoriously opaque for some people, often requiring a more educated eye or ear. Lovers of traditional European academic art or Impressionism may draw the line at cubism, dadaism, expressionism, or more avantgarde conceptual art. Twentieth-century artists such as Max Bill, Sol Lewitt, and M. C. Escher were infusing mathematical ideas into their work at a time when engaging with art began to require more active mental involvement.

Is it possible that math-


Figure 6. Anish Kapoor, Turning the World Upside Down, Jerusalem. This monumental stainless steel sculpture is highly polished to a mirror finish. A mathematician wonders if it is a precise hyperboloid and if so what optical properties such a quartic surface might display. Installed in Jerusalem, it is interesting that the inverting effect of the mirror literally shows heaven on earth (and vice versa). ematical art is simply too hard to understand without the proper education? A quilt containing a visual reference to Cayley tables could be as thematically inscrutable to nonmathematical viewers as a classical painting of a woman holding a swan will be to viewers unfamiliar with Roman mythology. Given the stereotypical "I don't understand math" sentiment that is not uncommon in our culture, it is tempting to propose that this content issue limits the audience for math/art. While it is certainly true that the inspiration for some mathematical art can not be appreciated by everyone, this is not the whole story. Firstly, even many professional mathematicians who are in a position to understand mathematical ideas are not drawn to works of the math/art community. Secondly, a classical painting of Leda and The Swan may be widely appreciated as art for its sensuous surfaces, formal composition, or masterful brushwork, even if the viewer is unfamiliar with the mythological story. And in the same way one expects worthy mathematical art to be recognizable for its form, craftsmanship, aesthetics or other properties, even if an analytical understanding requires explanation. So unfamiliar mathematical references are not the major difficulty here.

The bigger issue in the world of math/art, I feel, is that much of what is presented in our art exhibitions and publications is not truly "fine art." Of course, without a formal definition of art I cannot rigorously support this claim. But it is evidently justified by observing the accepted authorities concerning art: the fine art institutions such as museums, galleries, auction houses, and university art departments. The sad truth is that no experts from these organizations are rushing over to our mathematical art exhibitions and being impressed by what they find. We must admit that in terms of their culturally accepted notions of art, something is lacking.


Figure 7. Kenneth Snelson, Needle Tower. Snelson conquers infinity with a series of modules that are proportionally smaller along the height of the 26-meter sculpture. The viewer misinterprets them as identical and so imagines the structure as extending to the vanishing point. In this "tensegrity"
design, compressive elements never touch, so that the weight of the upper beams is supported along a chain of alternating compressive and tensional elements. ing or less valuable within our community, but they may distance the object from accepted notions of fine art.

To reconcile these issues, perhaps what we call an "Art Exhibition" should be rebranded as something like "Exhibition of Mathematical Art, Craft, Design, Models, and Visualization." This conveniently covers the entire collection without having to be definitionally specific about
individual items. Beyond any benefits of self-honesty, this labelling might aid us in appearing more modest to any fine art communities that consider our math/art to be below their artistic standards. A venue that is unwilling to present our works under the rubric "Art" might be happy to display it if labelled less grandiosely.

Artists generally aim to


Figure 8. Max Bill, Fifteen Variations on a Single Theme, (Variation 1). In a series of lithographs, Max Bill explores constructions based on a nested sequence of regular polygons, decreasing from eight to three sides while sharing a common edge length. A mathematically inclined viewer will want to work out the rule that determines on which side of each polygon the next one is attached. communicate something to their viewers. In contemporary fine art, the message is often a social or political viewpoint, with the artist daring to push boundaries and speak truths not otherwise heard. Math/art is characteristically tamer. Our artist statements typically indicate an interest in communicating mathematical truths with formal or aesthetic appeal. Our exhibition and publication venues promote the sharing of creative works that inform and inspire a specific well-educated community. If this content isn't being more widely received, it might be argued that we need to find a way to strengthen and clarify our messages, for certainly mathematician artists have a wealth of fascinating raw material available as intellectual inspiration. Whether or not any particular work meets anyone's particular notion of "fine art" is irrelevant to the community's larger purpose in aspiring to create art while communicating something of the wonders of mathematics. I am truly delighted to be part of a community that works in a heartfelt manner to create new forms of art and supports each other in this endeavour.

All cultures have found ways to combine mathematics and art, whether it is a hand-pinched frieze pattern along the rim of an ancient clay vessel or an algorithmically generated aperiodic monotile tessellation laser-cut from plastic polymer. Math and art are essential parts of being human. They are powerful languages and tools for communicating and making sense of the world. So it is perfectly natural that our community of math lovers and professional mathematicians is developing art that reflects our own interests and culture. The themes found in mathematical art often embody the kinds of patterns and structures
that have brought joy to mathematical thinkers through the ages.

Mathematics appeals because of the delights to be found in rigorous reasoning and understanding with clarity. I believe mathematical art succeeds in our community because it alludes in various ways to these same pleasures. When art mimetically suggests objects and relationships in some Platonic world, those delights may be evoked for viewers who have spent enough time in that realm to understand the reference. A painting that simply portrays a triangle or, say, the digit zero might seem to have a "mathematical" subject to a nonmathematician, but an isolated reference of that sort is unlikely to reach deeply enough into the graph of logical relationships to evoke any joy in a mathematically educated viewer. The art that is mathematical art must bring to mind a landscape of mathematical pleasure.


Figure 9. Ceramic tiling from the Alhambra palace, in Granada, Spain. Tessellations and frieze patterns are a staple of geometric art across many cultures. This fourteenth- or fifteenth-century example illustrates what is called the 333 orbifold type (in the Thurston-Conway notation) because there are three different types of centers of 3 -fold rotation.

The works produced in our community that I characterize as craft, design, models, and visualization also evoke these mathematical landscapes. This is part of our motivation for creating them and helps explains why we are our own best audience. These objects often are created with another purpose as well: education. Many of us are teachers accustomed to taking every opportunity to share our knowledge. Math/art is often used as a hook to engage students in new topics. After they are invested emotionally, mathematical conversations are a natural way for a teacher to gently introduce the technicalities. If one's definition of "fine art" requires that it be nonfunctional, this utilitarian genesis seems to preclude such objects from being art, but no matter, as this pedagogical math/art may still be inspiring, engaging, and thought provoking.

So math/art might not be a formalizable branch of mathematics and it might not actually be art, in which case it may be a separate discipline emerging between math and art. The "bridge" that the Bridges Conference makes between math and art might best be visualized as stopping over at a volcanic island rising between the two lands. The inhabitants of this ground are developing a culture, tradition, and corpus of work with its own internal logic, and
growing a garden of hybrid varieties yet to be fully sorted out. We should all revel in this flowering, even if only a fraction of it reaches a level of "art."

I encourage everyone to aim for making art, especially mathematicians. Feeling the joy of creation is enormously rewarding. It is an authentic form of self-expression that lets you discover things within yourself and that gives you a powerful channel for communicating to a new audience. Mathematicians in particular are attuned to fascinating ideas that might be shown in original artistic forms. This is easier said than done, of course, as there are many conceptual, material, and expressive challenges, but mathematicians are adept at solving difficult problems. I expect that anyone who enjoys creative mathematics will also enjoy exploring some domain of the arts in which they can be equally creative. Along the way, you might ask yourself "is this really art?" and try to articulate an answer. Work out the details of your own mathematical art manifesto as you push yourself to raise the bar of your artistry.

For me, creating sculpture is a compulsion. I might try to rationalize it as educational, I might be gratified to receive kind words from an appreciative audience, and I might enjoy the profit of making a sale, but I do it as a compulsive behavior. It is a therapy that soothes some inner need. I work in a haze of healthy confusion about the exact nature of mathematical art, but I don't need a formal framework to know that envisioning new designs and bringing them to reality is somehow saying something beyond words. Finishing a new sculpture is as satisfying as solving a difficult problem, when one's thoughts crystallize and the fog becomes clarity.

For me, mathematics is a source of great intellectual joy, to be shared as widely as possible, whereas art is the highest purpose of being human. Art can uplift the spirit or invite introspection as it communicates deeper meanings. A world rich in art is full of ideas and inspiration at every turn. I trust that as society evolves, more and more people will be freed to create art. And as a fundamental humanist expression, the scope of art needs to be enriched by the viewpoint of mathematicians. Those who have journeyed through mathematical lands have unique stories to tell of what they found and how they now see the world. I look forward to sampling these tales as more mathematicians become engaged in creating art. And as examples accrue, these discussions will continue, various organizing principles will start to emerge, and our understanding of mathematical art can only mature.

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George Hart

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## AMS Prizes and Awards

## new! I. Martin Isaacs Prize for Excellence in Mathematical Writing


I. Martin Isaacs

The I. Martin Isaacs Prize is awarded for excellence in writing of a research article published in a primary journal of the AMS in the past two years.

## About this Prize

The prize focuses on the attributes of excellent writing, including clarity, grace, and accessibility; the quality of the research is implied by the article's publication in Communications of the AMS, Journal of the AMS, Mathematics of Computation, Memoirs, Proceedings of the AMS, or Transactions of the AMS, and is therefore not a prize selection criterion.

Professor Isaacs is the author of several graduate-level textbooks and of about 200 research papers on finite groups and their characters, with special emphasis on groups-such as solvable groups-that have an abundance of normal subgroups. He is a Fellow of the American Mathematical Society, and received teaching awards from the University of Wisconsin and from the School of Engineering at the University of Wisconsin. He is especially proud of his 29 successful PhD students.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/isaacs-prize.

# new! Elias M. Stein Prize for Transformative Exposition 



Elias M. Stein

The Elias M. Stein Prize for Transformative Exposition is awarded for a written work, such as a book, survey, or exposition, in any area of mathematics that transforms the mathematical community's understanding of the subject or reshapes the way it is taught.

## About this Prize

This prize was endowed in 2022 by students, colleagues, and friends of Elias M. Stein to honor his remarkable legacy of writing monographs and textbooks, both singly and with collaborators. Stein's research monographs, such as Singular Integrals and Differentiability Properties of Functions and Harmonic Analysis, became canonical references for generations of researchers, and textbooks such as the Stein and Shakarchi series Princeton Lectures in Analysis became instant classics in undergraduate and graduate classrooms. Stein is remembered for his ability to find a perspective to make a method of proof seem so natural as to be inevitable, and for his strategy of revealing the essential difficulties, and their solutions, in the simplest possible form before elaborating on more general settings. This prize seeks to recognize mathematicians at any career stage who, like Stein, have invested in writing a book or manuscript that transforms how their research community, or the next generation, understands the current state of knowledge in their area.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/stein-exposition.

## Ivo and Renata Babuška Thesis Prize

The Ivo and Renata Babuška Thesis Prize is awarded annually to the author of an outstanding PhD thesis in mathematics, interdisciplinary in nature, possibly with applications to other fields. The current prize amount is US\$3,000.

## About this Prize

Ivo Babuška (1926-2023) was a Czech-American mathematician whose honors include five doctorates honoris causa, the Czechoslovak State Prize for Mathematics, the Leroy P. Steele Prize, the Birkhoff Prize, the Humboldt Award of Federal Republic of Germany, the John von Neumann Medal, the Neuron Prize Czech Republic, the ICAM Congress Medal (Newton Gauss), the Bolzano Medal, and the Honorary Medal De Scientia Et Humanitate Optime Meritis. Asteroid 36060 Babuška was named in his honor by the International Astronomical Union.

Renata Babuška (nee Mikulášek) was Ivo's wife and partner for 63 years. Renata grew up in Prague, Czechoslovakia and graduated from Charles University in 1953 with a degree in mathematical statistical engineering. Upon graduation, she was assigned to the Education Department as an administrator evaluating universities and technical schools. Two years later she became an assistant professor of mathematics at the Czech Technical University. After moving to the US, Renata worked as a data and computing management consultant for different government agencies in Washington, DC. She liked to point out that behind every successful man is a strong woman and he often said that without Renata, he would not have accomplished all that he did.

Babuška was a Distinguished Professor at the University of Maryland at College Park and then the Robert B. Trull Chair in Engineering, TICAM senior research scientist, professor of aerospace engineering and engineering mechanics, and professor of mathematics at the University of Texas, Austin. He was a Fellow of SIAM, ACM, and ICAM; a member of the US National Academy of Engineering, the Academy of Medicine, Engineering, and Sciences of Texas, and the European Academy of Sciences; and an honorary foreign member of the Czech Learned Society.

Babuška's work spanned the fields of theoretical and applied mathematics with emphasis on numerical methods, finite element methods, and computational mechanics. His interest in fostering collaboration among mathematicians, engineers, and physicists led him to establish this prize to encourage and recognize interdisciplinary work with practical applications.

The Ivo and Renata Babuška Thesis Prize is awarded in line with other AMS prizes and awards, according to governance rules and practice in effect at that time.
Next Prize: January 2025
Nomination Period: February 1-June 30

## Nomination Procedure:

1. The prize will recognize a thesis for a PhD granted between July 1 of year -1 and June 30 of year 0 (the year of nomination and selection) and will be presented at the Joint Mathematics Meetings in January of year +1 wherever it appears.
2. The nominating institution will be a PhD-granting institution that is either (a) located in the United States of America (USA), or (b) located outside the USA and an institutional AMS member at the time of the nomination.
3. One PhD thesis may be nominated by a nominating institution.
4. The nominating institution will submit a copy of the thesis along with a letter in support of the nomination, and both will be written in English.
5. A selection committee will be appointed by the AMS president.
https://www.ams.org/babuska

## Mary P. Dolciani Prize for Excellence in Research

The AMS Mary P. Dolciani Prize for Excellence in Research recognizes a mathematician from a department that does not grant a PhD who has an active research program in mathematics and a distinguished record of scholarship. The primary criterion for the prize is an active research program as evidenced by a strong record of peer-reviewed publications.

Additional selection criteria may include the following:

- Evidence of a robust research program involving undergraduate students in mathematics;
- Demonstrated success in mentoring undergraduates whose work leads to peer-reviewed publication, poster presentations, or conference presentations;
- Membership in the AMS at the time of nomination and receipt of the award is preferred but not required.


## About this Prize

This prize is funded by a grant from the Mary P. Dolciani Halloran Foundation. Mary P. Dolciani Halloran (1923-1985) was a gifted mathematician, educator, and author. She devoted her life to developing excellence in
mathematics education and was a leading author in the field of mathematical textbooks at the college and secondary school levels.

The prize amount is $\$ 5000$, awarded every other year for five award cycles.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Nominations should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: https://www.ams.org/dolciani-prize.

## Award for an Exemplary Program or Achievement in a Mathematics Department

This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

## About this Award

This award was established in 2004. For the first three awards (2006-2008), the prize amount was US $\$ 1,200$. The prize was endowed by an anonymous donor in 2008, and starting with the 2009 prize, the amount is US $\$ 5,000$. This US $\$ 5,000$ prize is awarded annually. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: A letter of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department in being nominated as well as the achievements which make the program(s) an outstanding success, and may include any ancillary documents which support the success of the
program(s). Where possible, the letter and documentation should address how these successes came about by 1 ) systematic, reproducible changes in programs that might be implemented by others, and/or 2) have value outside the mathematical community. The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Information on how to nominate can be found here: https://www.ams.org/department-award.

## Award for Impact on the Teaching and Learning of Mathematics

This award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.

Priorities of the award include recognition of:
(a) accomplished mathematicians who have worked directly with precollege teachers to enhance teachers' impact on mathematics achievement for all students, or
(b) sustainable and replicable contributions by mathematicians to improving the mathematics education of students in the first two years of college.

## About this Award

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen.

The US $\$ 1,000$ award is given annually.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Letters of nomination may be submitted by one or more individuals. The letter of nomination should describe the significant contributions made by the nominee(s) and provide evidence of the impact these contributions have made on the teaching and learning of mathematics. The letter of nomination should not exceed two pages, and may include supporting documentation not to exceed three additional pages. A brief curriculum vitae for each nominee should also be included. The nonwinning nominations will automatically be reconsidered, without further updating, for the awards to be presented over the next two years.

Information on how to nominate can be found here: https://www.ams.org/impact.

## Ciprian Foias Prize in Operator Theory

The Ciprian Foias Prize in Operator Theory is awarded for notable work in Operator Theory published during the preceding six years. The work must be published in a recognized, peer-reviewed venue.

## About this Prize

This prize was established in 2020 in memory of Ciprian Foias (1933-2020) by colleagues and friends. He was an influential scholar in operator theory and fluid mechanics, a generous mentor, and an enthusiastic advocate of the mathematical community.

The current prize amount is US\$5,000, and the prize is awarded every three years.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Nominations require CV of the nominee, a letter of nomination, and a citation.

Information on how to nominate can be found here: https://www.ams.org/foias-prize.

## David P. Robbins Prize

The Robbins Prize is for a paper with the following characteristics: it shall report on novel research in algebra, combinatorics, or discrete mathematics and shall have a significant experimental component; and it shall be on a topic which is broadly accessible and shall provide a simple statement of the problem and clear exposition of the work. Papers published within the six calendar years preceding the year in which the prize is awarded are eligible for consideration.

## About this Prize

This prize was established in 2005 in memory of David P. Robbins by members of his family. Robbins, who died in 2003, received his PhD in 1970 from MIT. He was a longtime member of the Institute for Defense Analysis Center for Communications Research and a prolific mathematician whose work (much of it classified) was in discrete mathematics.

The current prize amount is US $\$ 5,000$ and the prize is awarded every 3 years.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nom-
inated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/robbins-prize.

## E. H. Moore Research Article Prize

The Moore Prize is awarded for an outstanding research article to have appeared in one of the AMS primary research journals (namely, the Journal of the AMS, Proceedings of the AMS, Transactions of the AMS, Memoirs of the AMS, Mathematics of Computation, Electronic Journal of Conformal Geometry and Dynamics, and Electronic Journal of Representation Theory) during the six calendar years ending a full year before the meeting at which the prize is awarded.

## About this Prize

The prize was established in 2002 in honor of E. H. Moore. Among other activities, Moore founded the Chicago branch of the American Mathematical Society, served as the Society's sixth president (1901-1902), delivered the Colloquium Lectures in 1906, and founded and nurtured the Transactions of the AMS.

The current prize amount is US $\$ 5,000$, awarded every three years.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/moore-prize.

## Leroy P. Steele Prize for Lifetime Achievement

The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students.

## About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published
mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US $\$ 10,000$.
Next Prize: January 2025
Nomination Period: February 1 - March 31
Nomination Procedure: Nominations for the Steele Prize for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: https://www.ams.org/steele-1ifetime.

## Leroy P. Steele Prize for Mathematical Exposition

The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper.

## About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US $\$ 5,000$.
Next Prize: January 2025
Nomination Period: February 1 - March 31
Nomination Procedure: Nominations for the Steele Prizes for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations
will remain active and receive consideration for three consecutive years.

Information on how to nominate can be found here: https://www.ams.org/stee1e-exposition.

## Leroy P. Steele Prize for Seminal Contribution to Research

The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Analysis/Probability (2020)
2. Algebra/Number Theory (2021)
3. Applied Mathematics (2022)
4. Geometry/Topology (2023)
5. Discrete Mathematics/Logic (2024)
6. Open (2025)

## About this Prize

These prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein, and are endowed under the terms of a bequest from Leroy P. Steele. From 1970 to 1976 one or more prizes were awarded each year for outstanding published mathematical research; most favorable consideration was given to papers distinguished for their exposition and covering broad areas of mathematics. In 1977 the Council of the AMS modified the terms under which the prizes are awarded. In 1993, the Council formalized the three categories of the prize by naming each of them: (1) The Leroy P. Steele Prize for Lifetime Achievement; (2) The Leroy P. Steele Prize for Mathematical Exposition; and (3) The Leroy P. Steele Prize for Seminal Contribution to Research.

The amount of this prize is US\$5,000.

## Next Prize: January 2025

Nomination Period: February 1-March 31
Nomination Procedure: Nominations for the Steele Prize for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

Information on how to nominate can be found here: https://www.ams.org/steele-research.

## Levi L. Conant Prize

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the Notices of the AMS or the Bulletin of the AMS in the preceding five years.

## About this Prize

Levi L. Conant was a mathematician and educator who spent most of his career as a faculty member at Worcester Polytechnic Institute. He was head of the mathematics department from 1908 until his death and served as interim president of WPI from 1911 to 1913. Conant was noted as an outstanding teacher and an active scholar. He published a number of articles in scientific journals and wrote four textbooks. His will provided for funds to be donated to the AMS upon the death of his wife.

Prize winners are invited to present a public lecture at Worcester Polytechnic Institute as part of their Levi L. Conant Lecture Series, which was established in 2006.

The Conant Prize is awarded annually in the amount of US $\$ 1,000$.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Nominations with supporting information should be submitted online. Nominations should include a letter of nomination, a short description of the work that is the basis of the nomination a complete bibliographic citation for the article being nominated.

Information on how to nominate can be found here: https://www.ams.org/conant-prize.

## Mathematics Programs that Make a Difference

This Award for Mathematics Programs that Make a Difference was established in 2005 by the AMS's Committee on the Profession to compile and publish a series of profiles of programs that:

1. aim to bring more persons from underrepresented backgrounds into some portion of the pipeline beginning at the undergraduate level and leading to advanced degrees in mathematics and professional success, or retain them once in the pipeline;
2. have achieved documentable success in doing so; and
3. are potentially replicable models.

## About this Award

This award brings recognition to outstanding programs that have successfully addressed the issues of underrepresented groups in mathematics. Examples of such groups include
racial and ethnic minorities, women, low-income students, and first-generation college students.

One program is selected each year by a selection committee appointed by the AMS president and is awarded US $\$ 1,000$ provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity.

Preference is given to programs with significant participation by underrepresented minorities. Note that programs aimed at pre-college students are eligible only if there is a significant component of the program benefiting individuals from underrepresented groups at or beyond the undergraduate level. Nomination of one's own institution or program is permitted and encouraged.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: The letter of nomination should describe the specific program being nominated and the achievements that make the program an outstanding success. It should include clear and current evidence of that success. A strong nomination typically includes a description of the program's activities and goals, a brief history of the program, evidence of its effectiveness, and statements from participants about its impact. The letter of nomination should not exceed two pages, with supporting documentation not to exceed three more pages. Up to three supporting letters may be included in addition to these five pages. Nomination of the writer's own institution or program is permitted. Nonwinning nominations will automatically be reconsidered for the award for the next two years.

Information on how to nominate can be found here: https://www.ams.org/make-a-diff-award.

## Oswald Veblen Prize in Geometry

The award is made for a notable research work in geometry or topology that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

## About this Prize

This prize was established in 1961 in memory of Professor Oswald Veblen through a fund contributed by former students and colleagues. The fund was later doubled by the widow of Professor Veblen. An anonymous donor generously augmented the fund in 2008. In 2013, in honor of her late father, John L. Synge, who knew and admired Oswald Veblen, Cathleen Synge Morawetz and her husband, Herbert, substantially increased the endowment.

The current prize amount of US $\$ 5,000$ is awarded every three years.

Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/Veblen-prize.

## Ruth Lyttle Satter Prize in Mathematics

The Satter Prize recognizes an outstanding contribution to mathematics research by a woman in the previous six years.

## About this Prize

This prize was established in 1990 using funds donated by Joan S. Birman in memory of her sister, Ruth Lyttle Satter. Professor Birman requested that the prize be established to honor her sister's commitment to research and to encourage women in science. An anonymous benefactor added to the endowment in 2008.

The current prize amount is $\$ 5,000$ and the prize is awarded every 2 years.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/satter-prize.

## Joint Prizes and Awards

## Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student

(AMS-MAA-SIAM)
The Morgan Prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who was enrolled as an undergraduate in December at a college or university in
the United States or its possessions, Canada, or Mexico is eligible for the prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

## About this Prize

The prize was established in 1995. It is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. It is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

The current prize amount is $\$ 1,200$, awarded annually.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: To nominate a student, submit a letter of nomination, a brief description of the work that is the basis of the nomination, and complete bibliographic citations (or copies of unpublished work). All submissions for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research.

Information on how to nominate can be found here: https://www.ams.org/morgan-prize.

## JPBM Communications Award

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to non-mathematical audiences.

## About this Award

This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US\$2,000 are made annually. Both mathematicians and non-mathematicians are eligible.
Next Prize: January 2025
Nomination Period: open
Nomination Procedure: Nominations should be submitted on mathprograms.org. Note: Nominations collected before September 15th in year N will be considered for an award in year $\mathrm{N}+2$.

Information on how to nominate can be found here: https://www.ams.org/jpbm-comm-award.

## AMS-SIAM Norbert Wiener Prize in Applied Mathematics

The Wiener Prize is awarded for an outstanding contribution to "applied mathematics in the highest and broadest sense."

## About this Prize

This prize was established in 1967 in honor of Professor Norbert Wiener and was endowed by a fund from the Department of Mathematics of the Massachusetts Institute of Technology. The endowment was further supplemented by a generous donor.

Since 2004, the US $\$ 5,000$ prize has been awarded every three years. The American Mathematical Society and the Society for Industrial and Applied Mathematics award this prize jointly; the recipient must be a member of one of these societies.
Next Prize: January 2025
Nomination Period: February 1-May 31
Nomination Procedure: Submit a letter of nomination describing the candidate's accomplishments including complete bibliographic citations for the work being nominated, a CV for the nominee, and a brief citation that explains why the work is important.

Information on how to nominate can be found here: https://www.ams.org/wiener-prize.

## AMS Programs and Fellowships

## AMS-Simons Travel Grants

The AMS-Simons Travel Grant program acknowledges the importance of research interaction and collaboration in mathematics and aims to facilitate these activities for recent PhD recipients. AMS-Simons Travel Grants are administered by the AMS with support from the Simons Foundation. These grants provide support for committed researchers who have limited opportunities for travel and conferences and for collaborative work. For the 2024-2025 award cycle, each grant will provide an early-career mathematician with $\$ 3,000$ per year for two years to be used for research-related travel. Annual discretionary funds for the enhancement of a grantee's department will be available to institutions that administer the grant on behalf of the AMS. No additional institutional overhead or indirect costs will be covered with these award funds.

## About this Grant

Eligible applicants for the 2024-2025 application cycle are early-career mathematicians who are located in the United States (or are US citizens employed outside the United States) and who have completed the PhD (or its equivalent) within the last four years (between April 1, 2020, and June 30, 2024, inclusive).

The applicant's research must be in a disciplinary research area supported by the Division of Mathematical Sciences at the National Science Foundation. Previous AMS-Simons Travel Grant recipients and early-career mathematicians who already receive substantial external funding for research and travel exceeding $\$ 3,000$ per year (such as from the National Science Foundation) are not eligible to apply.

Recipients may use grant funds for research-related travel, such as travel to a conference, a university, or an institute, or to visit a collaborator. Funds may also be used for a collaborator to visit the grantee to engage in research activities. Other research-related travel may be supported, subject to the approval of the grantee's mentor. Detailed guidelines will be provided to the grantee. Only eligible travel expenses that have advance approval from the grantee's mentor will be reimbursed.

Application Period: Applications will be collected via MathPrograms.org February 15, 2024-March 31, 2024 (11:59 p.m. EST). Find more application information at https://www.ams.org/AMS-SimonsTG. For questions, contact the Programs Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294; ams-simons@ams.org.

## Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of 12 members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at www.ams.org/profession/ams-fellows.

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Abigail Hickok

## The Ivo and Renata Babuška Thesis Prize

Established in 2022 by Ivo Babuška, the Ivo and Renata Babuška Thesis Prize is awarded annually to the author of an outstanding PhD thesis in mathematics, interdisciplinary in nature, possibly with applications to other fields.

Ivo Babuška was a Czech-American mathematician whose honors include five doctorates honoris causa, the Czechoslovak State Prize for Mathematics, the Leroy P. Steele Prize, the Birkhoff Prize, the Humboldt Award of Federal Republic of Germany, the John von Neumann Medal, the Neuron Prize Czech Republic, the ICAM Congress Medal (Newton Gauss), the Bolzano Medal, and the Honorary Medal De Scientia Et Humanitate Optime Meritis. Asteroid 36060 Babuška was named in his honor by the International Astronomical Union.

Renata Babuška (nee Mikulášek) was Ivo's wife and partner for 63 years. Renata grew up in Prague, Czechoslovakia and graduated from Charles University in 1953 with a degree in Mathematical Statistical Engineering. After a career in Czechoslovakia as an educational administrator, Renata worked for different government agencies in Washington, D.C. as a data and computing management consultant. She liked to point out that behind every successful man is a strong woman and he often said that without Renata, he would not have accomplished all that he did.

Babuška's work spanned the fields of theoretical and applied mathematics with emphasis on numerical methods,
finite element methods, and computational mechanics. His interest in fostering collaboration among mathematicians, engineers, and physicists led him to establish this prize to encourage and recognize interdisciplinary work with practical applications.
Citation: Abigail Hickok. The 2024 Ivo and Renata Babuška Thesis Prize is awarded to Abigail Hickok of UCLA in recognition of the outstanding contributions in her PhD thesis "Topics in Geometric and Topological Data Analysis."

The Babuška Thesis Prize is a new AMS prize, to be awarded annually. In view of Ivo Babuška's broad interests across applied and theoretical areas of mathematics, it is awarded to the author of "an outstanding PhD thesis in mathematics, interdisciplinary in nature, possibly with applications in other fields". One candidate thesis can be nominated each year by a university in the USA or an AMS institutional member university in other countries.

Dr. Hickok, whose PhD was granted in May 2023, works in the very active areas of Topological Data Analysis (TDA) and Geometric Data Analysis (GDA). These rapidly growing fields use ideas from algebraic topology, differential geometry, computational geometry, and statistics to analyze data, often in high dimensions. For example, one of the well-known ideas of TDA is persistent homology, which measures the connected components, holes, and higher-dimensional voids of a data set and tracks how these voids emerge and disappear at different scales. GDA adds the goal of extracting geometric information beyond topological invariants, such as curvature.

This thesis spans the theoretical and the applied. It begins with a beautifully written chapter introducing necessary concepts of topology and TDA and then introduces Hickok's new notion of Persistence-Diagram Bundles, which provide a new TDA approach to datasets that depend on more than one parameter. The next chapter then introduces an algorithm for computing PD Bundles.

This is followed by two chapters on applications to geospatial data, the first related to COVID-19 and the second to the distribution of resources such as polling stations. The final chapter addresses the GDA topic of computing curvature in data sets when all the information known is pairwise distances, not an embedding in Euclidean space.

It is a magnificent thesis, whose contributions will have fruitful consequences. As an appendix it also contains a further contribution, published in 2022 in the SIAM Journal on Applied Dynamical Systems with Hickok as lead author, on modelling opinion dynamics on hypergraphs.

Dr. Hickok was a mathematics undergraduate with pure inclinations in the Class of 2018 at Princeton, and as a graduate student at UCLA, she turned to more applied areas. This breadth of background has contributed to a thesis that is at once strongly founded theoretically, deeply involved in applications, and beautifully written. In Fall 2023 she took up an NSF Postdoctoral Fellowship with Andrew Blumberg at Columbia University.
Biographical Note. Abigail Hickok completed her PhD at UCLA (2023) under the supervision of Mason Porter, after receiving her undergraduate degree from Princeton University (2018). She is currently an NSF postdoctoral fellow at Columbia University, where she works with Andrew Blumberg.
Response from Abigail Hickok. I am very honored to receive the Ivo and Renata Babuška Thesis Prize. I would like to thank Ivo and Renata Babuška for their generosity in establishing this prize, as well as my graduate institution, UCLA, for nominating me. I am deeply grateful for the mentorship of my PhD advisor Mason Porter, who shaped my interest in using mathematics to study complex social systems and other interdisciplinary subjects. I'm also very thankful for the guidance of my postdoctoral mentor, Andrew Blumberg, for introducing me to geometric data analysis and its role in biology research. Additionally, I'd like to acknowledge my other wonderful collaboratorsBenjamin Jarman, Michael Johnson, Jiajie Luo, Deanna Needell-whose contributions formed part of my dissertation. Finally, I wish to express my appreciation for the constant support of my parents, siblings, and partner.


## Chevalley Prize in Lie Theory

The Chevalley Prize was established in 2014 by George Lusztig to honor Claude Chevalley (1909-1984). It is awarded for notable work in Lie Theory published during the preceding six years; a recipient should be at most twenty-five years past the PhD. The prize is awarded in evennumbered years, without restriction on society membership, citizenship, or venue of publication.
Citation: Victor Ostrik. The 2024 Chevalley Prize in Lie Theory is awarded to Victor Ostrik for his fundamental contributions to the theory of tensor categories, which have already found deep applications in modular representation theory and Lie theory.

This award is based on three papers of Ostrik: "On symmetric fusion categories in positive characteristic," published in Selecta Mathematica, "Frobenius exact symmetric tensor categories" (joint with Kevin Coulembier and Pavel Etingof), published in Annals of Mathematics, and "New incompressible symmetric tensor categories in positive characteristic" (joint with Dave Benson and Pavel Etingof), published in Duke Mathematical Journal.

Fundamental to the representation theory of groups and Lie algebras is the ability to take tensor products of representations. For example, all simple representations occur inside tensor powers of any faithful representation. Abstracting the tensor product structure omnipresent in representation theory leads to the notion of a (symmetric) tensor category. For example, the category of finitedimensional representations of a group over a field forms a symmetric tensor category. Another example is that of super-vector spaces, which formalize the sign rules that emerge when using differential forms. Super vector spaces lead naturally to superalgebras and supergeometry.

A remarkable theorem of Deligne from 2002 shows that when the coefficients underlying the tensor category are of characteristic zero, all symmetric tensor categories (of "moderate growth") arise as representations of groups or supergroups. In other words, any such category admits a tensor functor to super vector spaces.

In Lie theory, for every complex simple Lie algebra $\mathfrak{g}$ there is a symmetric tensor category of representations. According to Weyl, the simple objects in the category (irreducible representations) are indexed by a free abelian semigroup (the "dominant weights"). A variant of this theory
emerged from physics. This is the Verlinde category $V(\mathfrak{g}, k)$ where the "level" $k$ is a non-negative integer. As in the Weyl theory, these categories have simple objects indexed by a set of dominant weights, but now a finite set of weights (depending on $k$ ). The simple objects are not representations of $\mathfrak{g}$, but they have representation-theoretic interpretations, in the context of affine Lie algebras or quantum groups.

The Verlinde categories are remarkable for making connections between different representation theories. They are not symmetric but braided, a weaker condition, so they do not appear in Deligne's theorem. Apart from their importance in physics, the braiding of the Verlinde category was used by Witten and Reshetikhin-Turaev to define invariants of knots and 3-manifolds.

It was observed by S. Gelfand and Kazhdan, and by Georgiev and Mathieu that these Verlinde categories have analogs in characteristic $p$, and if the level $k$ is chosen carefully, the characteristic $p$ Verlinde category is symmetric (not 1 just braided!). In 2015, Ostrik made the bold proposal that one particular such symmetric Verlinde category, where $\mathfrak{g}=\mathfrak{g l}_{2}$ and $k=p-2$ can serve as a "universal" Verlinde category needed to complete Deligne's theorem in characteristic $p$. This category is denoted $V e r_{p}$. As a proof of concept, he was able to prove this conjecture for symmetric fusion categories, i.e., semi-simple symmetric tensor categories admitting finitely many simple objects. His proof introduces a beautiful idea: he shows that functors resembling the Frobenius twist are internal to any tensor category in characteristic $p$. This observation proved crucial to further developments in the theory. These results were published in "On symmetric fusion categories in positive characteristic".

Ostrik's work, together with works of Etingof-Ostrik and Coulembier highlighted the importance of "Frobenius exact" tensor categories. Ostrik conjectured that such categories of moderate growth admit a tensor functor to the Verlinde category. This conjecture was proved in the paper "Frobenius exact symmetric tensor categories". An important example of such Frobenius exact categories are given by the semi-simplifications of representations of finite groups in characteristic $p$. When applied to this example, their theorem gives surprising applications to modular representation theory, namely precise information about the growth exponent of the number of indecomposable summands of dimension coprime to $p$ in the $n$-th tensor power of a modular representation of a finite group (an area where any general results are very scarce).

What of Deligne's theorem in general in characteristic $p$ ? In "New incompressible symmetric tensor categories in positive characteristic," Benson, Etingof and Ostrik define "higher Verlinde categories" Ver $_{p^{n}}$ and these objects are
connected to yet another representation theory, namely the modular representations of Chevalley groups. This paper conjectures that every symmetric tensor category of moderate growth in characteristic $p$ admits a fibre functor to a nested union of such categories. This conjecture, if true, would provide a complete analog of Deligne's theorem in characteristic $p$.

Ostrik's work breathed new life into the theory of tensor categories. He pursued these ideas for many years before making the breakthroughs sketched above. His earlier work includes a text (joint with P. Etingof, S. Gelaki and D. Nikshych) that has become indispensable for researchers in the field.
Biographical Note. Victor Ostrik was born in Mariupol, Ukraine, in 1973. He received an undergraduate degree from Moscow State University in 1995. In 1999 he received his PhD from Moscow State University, under the supervision of Alexei Ivanovich Kostrikin and Michael Finkelberg (from Independent University of Moscow). He was a postdoc at MIT before becoming a faculty member at the University of Oregon in 2003. He works in representation theory and in the theory of tensor categories. He was an invited speaker at the 2014 ICM.
Response from Victor Ostrik. It is a great honor to receive an award linked to the names of Sophus Lie, Claude Chevalley, and George Lusztig. I am very grateful to my advisors, Alexei Ivanovich Kostrikin and Michael Finkelberg, who introduced me to Lie theory and Ernest Borisovich Vinberg, whose lectures further deepened my fascination with it. I became interested in the theory of tensor categories when I studied George Lusztig's work on the asymptotic Hecke algebra and tried proving some of his conjectures. I am thankful to Roman Bezrukavnikov who showed me how the theory of tensor categories can help in such problems. The work that earned this prize was made possible due to the remarkable results of Pierre Deligne, Sergio Doplicher, and John E. Roberts. I also want to express my appreciation to my collaborators, David Benson, Kevin Coulembier, and Pavel Etingof. Their insight helped to overcome seemingly insurmountable obstacles and make our results more complete.


## Frank Nelson Cole Prize in Algebra

This prize (and the Frank Nelson Cole Prize in Number Theory) was founded in honor of Professor Frank Nelson Cole on the occasion of his retirement as Secretary of the American Mathematical Society after twenty-five years of service and as Editor-in-Chief of the Bulletin for twenty-one years. The original endowment was established by the Cole family and Society members, was augmented in 2018 by an anonymous donor, and continues to receive support from the family. The prize is for a notable paper in algebra published during the preceding six years. The work must be published in a recognized, peer-reviewed venue.
Citation: Jessica Fintzen. The 2024 Frank Nelson Cole Prize in Algebra is awarded to Jessica Fintzen for her work transforming our understanding of representations of $p$ adic groups.

The prize is awarded in particular for the article, Jessica Fintzen, "Types for tame p-adic groups." Ann. of Math. (2) 193 (2021), no.1, 303-346.

It has long been understood that many questions about arithmetic can be studied by embedding the rational numbers in real or $p$-adic numbers, where methods of analysis are available. Because of this idea, the representation theory of groups defined over archimedean and $p$-adic fields has come to be a central tool in the study of automorphic forms.

Harish-Chandra in the 1960 s used deep ideas about differential equations to describe in detail the representations of groups over archimedean fields. Within a few years, Robert Langlands found a formulation of HarishChandra's results that made sense also for $p$-adic fields, and conjectured that this gave a detailed description of the representations of groups over $p$-adic fields.

Proving such a description has been a central goal of $p$ adic representation theory for more than fifty years. The case of $G L(n)$ was completed, following work of Roger Howe and Allen Moy, by Colin Bushnell and Philip Kutzko in 1993, using the notion of a "type," which is a very particular kind of representation of a compact open subgroup.

Because classical reductive groups are centralizers of involutive automorphisms of $G L(n)$, Shaun Stevens in 2012 was able to use Bushnell and Kutzko's idea to construct some types for classical groups. These methods have a lot to say about what ought to be true for general reductive groups, but offer little in the way of proofs.

In 2001, Jiu-Kang Yu found a construction of representations that works when all of the relevant $p$-adic field extensions are tamely ramified. Yu's construction can be understood as a construction of Bushnell-Kutzko types for general reductive groups in this tamely ramified setting. But he did not extend the Bushnell-Kutzko exhaustion theorem: that every representation of $G L(n)$ contains a type.

Julee Kim in 2007 proved such an exhaustion theorem if the field has characteristic zero and the residual characteristic is (extremely) large.

What Fintzen accomplishes in "Types for tame p-adic groups" is to prove an exhaustion theorem for Yu's construction in all characteristics, and under the weakest possible hypotheses on the residual characteristic $p$ : just those needed for Yu's construction to make sense. For example, in the case of the exceptional group $E_{8}$, this is all residual characteristics except for $2,3,5$, and 7 . Fintzen does this with a fundamental reworking of Yu's ideas, making them into the powerful tools that they have long promised to be.

Earlier work of Fintzen shed new light on $p$-adic representation theory at very small residual characteristic. Fintzen is leading the field toward a much deeper and sharper understanding of $p$-adic group representations.
Biographical Note. Jessica Fintzen received a bachelor's degree in mathematics and one in physics from the international Jacobs University Bremen before completing her PhD at Harvard University. After holding postdoctoral positions at the University of Michigan, the Institute for Advanced Study in Princeton and Trinity College in Cambridge, she became a lecturer (equivalent of assistant professor) and Royal Society University Research Fellow at the University of Cambridge and an assistant professor (later full professor) at Duke University. In 2022 she took up a professorship at the University of Bonn.
Response from Jessica Fintzen. Receiving the Frank Nelson Cole Prize in Algebra is a great honour and at the same time a big encouragement for me. I would like to thank those who nominated me for the prize and those who decided to award it to me. I would also like to thank those who supported me at various stages of my career, those who believed in my potential and offered me positions or opportunities, those who showed interest in the math I am doing and discussed mathematics with me, those who supported me when I faced obstacles, those with whom I could share my experiences and those who shared their experiences with me, and those who show by example how to be a responsible member of our math community. I am privileged that the list of people above that I am grateful for is longer than I can list here, but I would like to mention at least a few of them by name: I am particularly grateful to

Ana Caraiani, Samit Dasgupta, Stephen DeBacker, Tasho Kaletha, Lillian Pierce and Sug Woo Shin for their support in very different ways as well as their friendship.

## Levi L. Conant Prize



Jennifer Hom

This prize was established in 2000 in honor of Levi L. Conant to recognize the best expository paper published in either the Notices of the AMS or the Bulletin of the AMS in the preceding five years. Levi L. Conant (1857-1916) was a mathematician who taught at Dakota School of Mines for three years and at Worcester Polytechnic Institute for twenty-five years. His will included a bequest to the AMS effective upon his wife's death, which occurred sixty years after his own demise.
Citation: Jennifer Hom. The 2024 Levi L. Conant Prize is awarded to Jennifer Hom for her article "Getting a handle on the Conway knot," which was published in the Bulletin of the American Mathematical Society, 59 (2021), 19-29. This article is a wonderful resource for the community on timely and important material.

The topic of Hom's article is a 2020 proof by Lisa Piccirillo that the Conway knot is not slice. When we view this knot K as sitting in the 3 -sphere $\mathrm{S}^{3}$, the boundary of the 4-ball $B^{4}$ saying that $K$ is not slice means it cannot be the boundary of a smoothly embedded disk in $\mathrm{B}^{4}$. (For comparison, every knot is the boundary of a topologically embedded disk.) The Conway knot was the simplest knot for which this question remained unresolved: the problem had been open for fifty years, and it resisted all known invariants and approaches. Piccirillo's solution attracted attention across the mathematical community, and many curious mathematicians wished for an accessible introduction.

Hom's paper gives a beautiful account of Piccirillo's work and its broader context. She starts at the beginning, with basic terminology and background, and then masterfully introduces increasing levels of detail and complexity as she tells the story. Her writing is vivid and engaging, always getting straight to the point with the immediacy of a spoken lecture, and it is full of illuminating diagrams, as well as motivation and commentary. Readers are left with new understanding and a sense of excitement for the future of this field.

Biographical Note. Jennifer Hom grew up in Massachusetts watching Square One TV. She earned a BS in Applied Physics from Columbia University, but decided to pursue graduate studies in mathematics after taking abstract algebra her junior year. She earned a PhD in mathematics from the University of Pennsylvania, under the supervision of Paul Melvin at Bryn Mawr College. She was a postdoc at Columbia and member of the IAS before joining the faculty of Georgia Tech, where she is currently a professor. Her research centers on low-dimensional topology, which she usually studies using Heegaard Floer homology. She has held a Sloan Fellowship, an NSF-CAREER award, and a Simons Fellowship. She is a Fellow of the AMS and spoke in the topology section of the 2022 ICM. She enjoys running and board games.
Response from Jennifer Hom. I am honored to receive the 2024 Levi L. Conant Prize. An extremely important, but often under-valued part of our job as mathematicians is communication, and I'm grateful to the AMS for valuing high-quality exposition in their publications.

I'd like to thank the organizers of the Current Events Bulletin session at the JMM for giving me the opportunity to speak and write about Lisa Piccirillo's beautiful result. I at times wondered whether any article about her work could live up to the remarkable clarity of her original paper; I'm grateful to Lisa for setting such a high standard, and for continually doing such interesting mathematics. I'd also like to thank my PhD advisor Paul Melvin for instilling in me the importance of clear exposition. Lastly, I am grateful to my friends, colleagues, and mentors in the lowdimensional topology community for the support and encouragement, and for helping to keep this whole mathematics thing a lot of fun.


Angel Pineda

## Award for Distinguished Public Service

The Award for Distinguished Public Service was established by the AMS Council in response to a recommendation from their Committee on Science Policy. The award is presented every two years to a research mathematician who has made recent or sustained contributions through public service.
Citation: Angel Pineda. The 2024 AMS Award for Distinguished Public Service is presented to Angel Pineda, Professor of Mathematics at

Manhattan College, in recognition of his tireless work at the grassroots level supporting mathematicians living in challenged, resource-poor environments around the world and of the impact his example has had on national and international scientific organizations.

As a young researcher at Cal State Fullerton, he was one of the first mathematicians to answer the call issued by the AMS through its 2008 summer chairs letter for help in rebuilding the mathematics community in Cambodia, which had been destroyed in the late 1970 s by the Khmer Rouge. He went to Phnom Penh in 2009, and again a year later, to teach an intensive one-month course in numerical analysis which led, eventually, to the creation of a Master's degree program at the Royal University of Phnom Penh, a major step in rebuilding the mathematics community in Cambodia.

With eventual financial support from the US National Committee on Mathematics and the AMS, a Volunteer Lecturer Program (VLP) was established that allowed others to follow Pineda's example. Subsequently, the VLP was incorporated into the portfolio of the Commission for Developing Countries (CDC) as established by the International Mathematical Union (IMU).

Professor Pineda has continued his participation in the VLP and has been involved with the work of the CDC ever since including contributing to a report on mathematics in Latin America for the IMU. Currently, he coordinates the IMU's program called Graduate Research Assistantships in Developing Countries. It grants financial support to deserving graduate students in mathematics in the developing world where graduate assistantships, teaching or otherwise, are essentially unheard of. In addition to the impact of his work as an individual, the model set by Prof. Pineda's leadership, steadfastness, and modesty has attracted others to the work, to the benefit of untold numbers of students of mathematics around the world.

Biographical Note. Dr. Angel R. Pineda is a professor of mathematics at Manhattan College. He was born in Honduras where his parents' work as medical doctors in a public hospital inspired his commitment to service and his research area. He received his BS in chemical engineering from Lafayette College, his PhD in applied mathematics from the University of Arizona and his postdoctoral fellowship from the Radiology Department at Stanford University. Before teaching at Manhattan College, he taught at California State University, Fullerton. His research studies human performance in detection tasks using MRI reconstructions generated by machine learning. He is currently the principal investigator (PI) of a research grant from NIH and was previously the PI of a mentoring grant for underrepresented students from NSF. In 2009 and 2010, he was a volunteer lecturer in Cambodia. He served in the Com-
mission for Developing Countries (CDC) and currently serves on the Committee on Graduate Assistantships in Developing Countries (GRAID) of the IMU. He is a member of "Run For GRAID", a group of mathematicians who fundraise to support mathematics students in developing countries through running races.
Response from Angel Pineda. I feel deeply honored to receive the 2024 AMS Award for Distinguished Service. I accept the award on behalf of the many mathematicians who give their time and money to support mathematics in developing countries in general, and programs of the CDC in particular. In every project I have been involved in, I was just one member of a team who did the work. It is all of our work which is being recognized by this award.

I thank the selection committee for bringing attention to mathematics in developing countries. The ability to develop our mathematical talent depends heavily on the circumstances and countries in which we are born. Working to provide opportunities to those whose circumstances prevent them from developing their talent, both in the US and abroad, is rewarding and impactful. When I think of the Cambodian students with their deep appreciation of their teachers and hunger for knowledge or the African graduate students who can solely focus on their research instead of also having to work full time, I am reminded of the small and large ways in which we make a difference.

Finally, I take this opportunity to thank the members of the AMS who support mathematicians in developing countries through their donations to the IMU in the membership renewal form. Those donations add up to a consistent and meaningful source of funding for our programs. Thank you.

## Award for an Exemplary Program or Achievement in a Mathematics Department



BYU ACME Program

The annual AMS Award for an Exemplary Program or Achievement in a Mathematics Department was established in 2004, first awarded in 2006, and fully funded by a gift to the AMS's permanent endowment by an anonymous donor in 2008. This award recognizes a department which has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of society. Departments of mathematical sciences in North America that offer at least a bachelor's degree in mathematical sciences are eligible.
Citation: BYU ACME Program. The 2024 Award for an Exemplary Program or Achievement in a Mathematics Department is awarded to the Applied and Computational Mathematics Emphasis (ACME) program in the Mathematics Department at Brigham Young University. The ACME program has been highly successful in providing students with a rigorous foundation in mathematics as well as a broad interdisciplinary experience in applied mathematics.

During the first two years of the program, students take the traditional courses in mathematics. In their junior and senior year students join a tight-knit cohort in which traditional mathematics courses are supplemented with computing labs where students learn to convert sophisticated mathematical ideas into efficient working code. On top of this, they take courses in algorithms, optimization, dynamics, modeling with uncertainty, and other applied mathematics courses. Students also choose a concentration in a subject where they can apply the mathematics they have learned. The areas of concentration include biology, engineering, chemistry, data science, machine learning, economics, and many other important areas of science. This allows students to learn how to think about real-world problems, communicate across disciplines, and work on collaborative projects.

The evidence of success of ACME is apparent in many ways. The number of mathematics majors has increased from 276 in the Fall of 2013 to 415 in the Fall of 2021. The first graduating class of ACME had 15 students, and by 2021 the number rose to 52. In 2021 there were 250 declared ACME students. Students have had great success in getting internships and positions with major companies and also being accepted into top PhD programs in pure mathematics, applied mathematics, statistics, and other disciplines.

It should also be noted that the Mathematics Department has made efforts to broaden participation in the program and recruit more women and underrepresented minorities. These efforts include support groups to help students achieve success, recruiting students through summer programs and student clubs, and having current ACME
women and underrepresented students reach out to firstand second-year students and encourage them to apply to the ACME program. The cohorts in the junior and senior year provide substantial social and academic support to students in the program and are especially valuable to students from underrepresented groups.

The ACME program is a valuable resource to the mathematics community. The mathematics faculty, in consultation with an advisory board whose members are from industry, has created curriculum materials that are available to other programs that wish to adopt similar applied mathematics courses.
Biographical Note. BYU ACME Program was the idea of Jeff Humpherys, who recognized many students loved math but were leaving the major because they didn't see rewarding jobs for math majors, and many math alumni had rewarding jobs using math, but weren't using the math taught in the traditional major. He proposed a new undergraduate program in applied and computational mathematics, modernizing the math major and better integrating it in the broader STEM community, with a curriculum written by Jeff Humpherys, Tyler Jarvis, Emily Evans, and Jared Whitehead, focused on mathematical analysis, algorithm design, mathematical modeling, and interdisciplinary study. The first cohort of 15 students began the program in 2013. Since then, the program has attracted many new students into mathematics, and now graduates about 60 students per year, who go on to rewarding jobs and graduate study in both pure and applied mathematics, as well as other STEM fields.
Response from BYU ACME Program. We are honored and delighted to receive the AMS Exemplary Program Prize for the BYU Applied and Computational Mathematics (ACME) Program. ACME has had a significant positive impact on students and our department. Students are attracted by the chance to use math to solve problems they care about, by rewarding career opportunities, by strong prospects for advanced study, and by the strong social support network of the ACME cohorts. This has led to a significant increase in the number of math majors in our department and a corresponding influx of resources.

ACME provides a rigorous education in the theory and practice of applied and computational mathematics. The main things that have made the ACME program successful are the following:

1. A challenging curriculum of rigorous mathematics integrated with applications, and a special focus on mathematical analysis, algorithms, and modeling. The challenging curriculum attracts strong students to the program, motivates students to collaborate, helps students become strong problem solvers, and opens rewarding career paths.
2. Required computing labs connected to every advanced mathematics course in ACME. These motivate theory with applications, improve students' mathematical understanding, teach students to convert sophisticated mathematical ideas into efficient working code, and enhance students' employability.
3. Interdisciplinary concentration in a student-chosen area of application. Students learn to communicate across disciplines and see how math is used. Many students are attracted to ACME because the concentration allows them to study both math and another subject they love instead of choosing between them.
4. Lockstep cohorts for the junior and senior years. After students complete the foundations of programming, calculus, linear algebra, and first-semester analysis, they join a lockstep cohort, taking advanced math and computing courses with the same classmates for two hours every day, five days a week. These cohorts provice significant social, emotional, and academic support for students, encourage teamwork, and build loyal alumni.

One alumna sums up her experience: "ACME is a great major. Its strongest suit is, of course, the combination of math, stats, and coding, but also the friendships and support one gets from other students and professors" -Erika Ibarra Campos '22.

We hope other programs will consider adopting some of the things that make ACME successful. We also hope the curriculum materials we have developed with support of the NSF will be useful to others, including textbooks published by SIAM and open-source lab manuals.


Tilmann Gneiting

## Ulf Grenander Prize in Stochastic Theory and Modeling

The Grenander prize, established in 2017 by colleagues in honor of Ulf Grenander (1923-2016), recognizes exceptional theoretical and applied contributions in stochastic theory and modeling. It is awarded for seminal work, theoretical or applied, in the areas of probabilistic modeling, statistical inference, or related computational algorithms, especially for the analysis of complex or highdimensional systems. Grenander was an influential scholar in stochastic processes, abstract inference, and pattern theory. He published landmark works throughout his career, notably his 1950 dissertation, Stochastic Processes and Statistical Interference at Stockholm University, Abstract

Inference, his seminal Pattern Theory: From representation to inference and General Pattern Theory. A long-time faculty member of Brown University's Division of Applied Mathematics, Grenander was a fellow of the American Academy of Arts and Sciences, the National Academy of Sciences and was a member of the Royal Swedish Academy of Sciences.
Citation: Tilmann Gneiting. The Ulf Grenander Prize in Stochastic Theory and Modeling is awarded to Tilmann Gneiting for seminal work in environmental and stochastic modeling, with applications to computational weather forecasting, and for research in probability theory and mathematical statistics.

Gneiting is most widely known for foundational work on probabilistic forecasting. A simple example concerns the familiar weather forecast format " $40 \%$ chance of rain tomorrow": it would be more informative to state a probability distribution over the amount of rain. To assess the relative accuracy of different such forecasts one needs a scoring rule. Gneiting authored two seminal and very highly cited 2007 papers, "Strictly proper scoring rules, prediction, and estimation" (with Adrian Raftery) and "Probabilistic forecasts, calibration and sharpness" (with Adrian Raftery and Fadoua Balabdaoui). These laid out the fundamental theory. As the latter paper argued, as well as calibration (that events forecast to have 80\% probability should occur about $80 \%$ of the time) one seeks sharpness (individual forecasts should be as concentrated as possible). These theoretical developments have attracted immense attention in the real-world weather forecasting community. Gneiting has conducted extensive research in collaboration with the European Centre for Medium-Range Weather Forecasts (ECMWF), which is both a research institute and a real-time operational service. His foundational work on statistical post-processing for numerical weather forecasts provides the basis for current practice worldwide.

He had previously worked on spatial statistics and related covariance models. A major obstacle in the field was that researchers often were unable to determine whether many proposed models satisfied the hypotheses of a genuine covariance model. Gneiting turned to this area with zeal and published a lengthy series of papers (such as "Nonseparable, stationary covariance functions for spacetime data," 2002) in which he provided necessary and sufficient conditions for a proposed model to be a genuine covariance model. Covariance models which had been used in the statistical analysis of spatio-temporal environmental data for wind-borne pollution were proved by Gneiting to be invalid. This work led to substantial literature amongst environmental researchers revising previous work modeling such data.

This early work relied on his deep understanding of the classical analysis concerning characteristic functions and
positive definite functions. A sequence of papers, culminating in "Convolution roots of radial positive definite functions with compact support," 2004 (with Werner Ehm and Donald Richards), concerned topics such as positive definite functions with symmetry properties, convolution root properties, Pólya criteria and uncertainty relations for characteristic functions.
Biographical Note. Tilmann Gneiting is Scientific Director of the Heidelberg Institute for Theoretical Studies (HITS) and Professor of Computational Statistics at Karlsruhe Institute of Technology (KIT). Previously, he held faculty positions at the University of Washington in Seattle and at Heidelberg University. He received his PhD in mathematics in 1997 from Bayreuth University under the supervision of Peter Huber. His research uses probability and statistics across a range of applications: spatial and spatio-temporal models, theory and practice of forecasting in contexts of Atmospheric, Environmental and Earth Sciences, Epidemiology, Economics and Finance. He is a Fellow of the Institute of Mathematical Statistics (IMS) and a Fellow of the American Statistical Association (ASA) and received the (highest award) Distinguished Achievement Medal from the ASA section on Statistics and the Environment. He served as Editor-in-Chief of the Annals of Applied Statistics.
Response from Tilmann Gneiting. It is a great honor to receive the 2024 Ulf Grenander Prize in Stochastic Theory and Modeling, and I am deeply grateful to my coauthors, students, and colleagues in Heidelberg, Karlsruhe, Seattle, and elsewhere. Their support and their contributions to our joint work are immense and cannot be overstated.

A common thread of my research is a thorough theoretical treatment that is deeply rooted in analysis, probability theory, and mathematical statistics, yet driven by applications, particularly in the atmospheric, environmental, and earth sciences. While mathematical and statistical techniques are instrumental in solving a wealth of real world problems, inspiration goes both ways, and intense interaction with applied problems continues to prompt advances in our fields. In my case, insightful enquiries from meteorologists have prompted and facilitated theoretical and methodological advances in the generation and evaluation of probabilistic forecasts; more recently, collaborators from epidemiology and seismology have contributed fruitful challenges.

Undoubtedly, application oriented mathematical and statistical research will continue to thrive when theoretical foundations meet interdisciplinary fertilization.

## AMS-MAA-SIAM Frank and Brennie Morgan Prize

The AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student is awarded annually to an undergraduate student (or students for joint work) for outstanding research in mathematics.

The prize recipient's research needn't be confined to a single paper. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate. Publication of research is not required.

The prize was established in 1995 and is entirely endowed by a gift from Mrs. Frank (Brennie) Morgan. The prize is made jointly by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

Citation: Faye Jackson. The


Faye Jackson recipient of the 2024 AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student is Faye Jackson of the University of Michigan at Ann Arbor. Jackson worked on a wide range of topics in combinatorics and number theory. In particular, she discovered and theoretically explained several new and unexpected phenomena in analytic number theory. She has co-authored eight research papers, four of which have already been published or accepted, including in journals such as Journal of Number Theory, Discrete and Computational Geometry, and The Fibonacci Quarterly.

Jackson worked extensively on biases of distributions of parts in partitions. A partition ( $m_{1}, \ldots, m_{l}$ ) of a positive integer is a nonincreasing sequence of positive integers $m_{j}$ whose sum is $n$. The individual integers $m_{j}$ are called the parts of the partition. Recent work focused on understanding the distribution of parts modulo a given integer $t$. Beckwith and Mertens showed that the parts of arbitrary partitions are not equidistributed $\bmod t$ and also provided asymptotics for large $n$. Craig generalized these results to the class of partitions with distinct parts. With fellow undergraduate Misheel Otgonbayar, Faye Jackson proved that $k$-regular partitions, where each part occurs less than $k$ times, also exhibit biases in the distributions of their parts and provided detailed asymptotics for large $n$ with improved error estimates.

Jackson then considered the case of partitions with parts that are not multiples of $k$. Given $k$ and $n$, the number of
such partitions coincides with the number of $k$-regular partitions, and Jackson was curious whether these two classes share the same distributions of their parts mod $t$. Unexpectedly, the answer is no: Jackson and Otgonbayar not only worked out heuristics for what these distributions should be for partitions with parts that are not multiples of $k$ but also proved a beautiful general theorem that explains what these distributions are and how they converge to the distributions for $k$-regular partitions as $k$ becomes large.

Jackson used an impressive range of techniques from analytic number theory, including modular forms, EulerMaclaurin summation, $L$-functions, and the circle method, to establish these unexpected results.

At the University of Michigan, Jackson founded the Mathematics Undergraduate Student Advisory Council, served as the President of the Society of Undergraduate Mathematics Students, and was a member of the Mathematics Climate Committee. She also supported many outreach activities of the Ypsilanti Math Corps at Michigan as a mentor and instructor.

She received a Goldwater Scholarship in 2022 and the Alice T. Schafer Prize from the AWM in 2023. Jackson will continue her studies as a PhD student in the Department of Mathematics at the University of Chicago.
Biographical Note. Faye Jackson is a math PhD at the University of Chicago and a former undergraduate at the University of Michigan. She strives to become an educator for equity and to discover beautiful phenomena in mathematics. In Summer 2021 she participated in the SMALL REU at Williams College and played a major role in four different research projects. This work led to three published papers, two submitted preprints and two papers in preparation. Her mentor praises her creativity, generosity and the clarity of her exposition. In Summer 2022 she participated in the REU at the University of Virginia and coauthored one published paper and two submitted papers. Her mentor praised the beauty of her work and her impressive contributions to the life of the community.

Faye's instructors are similarly enthusiastic about her abilities and enthusiasm, and they describe her as a delight to have in class who helps spark important discussions. They are particularly excited about her contributions to outreach, and they describe her as a talented teacher for the Math Mondays in Ypsi, Super Saturday and Math Corps programs.
Response from Faye Jackson. Firstly I would like to thank the AMS for selecting me to win the Frank and Bennie Morgan Prize. I would also like to thank my incredible mentors such as Stephen DeBacker, Sarah Koch, Steven J. Miller, Ken Ono, and Jenny Wilson. They have intro-
duced me to amazing mathematics, people, and opportunities. Sarah Koch and Stephen DeBacker especially have served as role models in outreach, which is such an important part of being a good mathematician. I also want to thank the mathematical community at the University of Michigan broadly. My classmates have provided me with friendships, shoulders to lean on, mathematical insights, and more laughter than I could have ever imagined. As I move to my PhD studies I will deeply miss all of them. I am also extremely thankful for my collaborators at both SMALL and the UVA REUs. Finally I would like to thank my mother and my partner for their continual support and encouragement. My mom is my personal superhero, and my partner has been my confidant and my rock for the past four years. Moving forward, I would like to continue to be a part of and to build communities that encourage kindness and greatness.


Rupert Li

Citation: Rupert Li. Rupert Li is recognized with an Honorable Mention for the 2024 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student. Li is an undergraduate student of mathematics at the Massachusetts Institute of Technology. Li's work has focused on problems in combinatorics and has resulted in ten co-authored mathematical research papers.
Li's exceptional talent and dedication to research are evident through his three significant contributions to combinatorics, probability, dynamical systems, linear programming, and sphere packing. Impressively, his collaboration with Colin Defant, James Propp, and Benjamin Young on "Tilings of Benzels via the Abacus Bijection" settled two open problems regarding tilings. At the same time, his work on "Dual Linear Programming Bounds for Sphere Packing via Discrete Reductions" showcases his versatility in tackling mathematical challenges. His collaboration with James Propp on "A Greedy Chip-Firing Game" also introduces the "hunger game," enriching the fields of dynamical systems and probability. His impressive research achievements have earned recognition from reputable journals and experts.
Biographical Note. Rupert Li hails from Portland, Oregon. He was first introduced to math research in high school through the MIT PRIMES-USA program, and has loved researching ever since. His research interests lie in discrete geometry, probability, and combinatorics. Li is a
senior double majoring at MIT, his primary major being mathematics and his secondary major being computer science, economics, and data science. In his free time, he enjoys hiking, watching movies, playing puzzle games, and alpine sliding.
Response from Rupert Li. I am incredibly honored to receive an Honorable Mention for the 2024 Morgan Prize. I wish to thank the Morgan family and the AMS, MAA, and SIAM for establishing this award.

I have been most fortunate to have amazing mentors supporting me on my mathematical journey. I extend my deepest gratitude towards Professor Joseph Gallian for his unwavering support. His unflagging effort and care for the Duluth REU and its students fosters a wonderful environment, both mathematically and socially. I am immeasurably grateful for Professor Henry Cohn and all his help and guidance. He has opened my eyes to incredible areas of math, and I have immensely enjoyed working with him ever since my first year at MIT. I extend my heartfelt thanks to Professor James Propp, who I have had the great pleasure of working with and learning from multiple times, enjoying every single project. I deeply thank Professor Nike Sun for her gracious mentorship and time, introducing me to new areas of math and fascinating problems. I am grateful to Dr. Colin Defant for being a wonderful mentor and collaborator. I also wish to thank my advisor, Professor Julee Kim, for her guidance and help throughout the years.

Citation: Daniel Zhu. Re-


Daniel Zhu ceiving an Honorable Mention for the 2024 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student is Daniel Zhu. Daniel was a recent undergraduate student of mathematics at the Massachusetts Institute of Technology and will now pursue a PhD in Mathematics at Princeton University. Introducing completely novel ideas and unexpected connections, Zhu and Moshkovitz addressed the conjecture that the notions of analytic rank and partition rank for higher-order tensors are equivalent up to a constant factor, proving nearly linear dependence (off by only polylog factors), a huge improvement over previous knowledge. In an accepted paper to Combinatorial Theory, he focused on estimating the number of numerical semigroups of a given genus. In a paper published in Annals of Combinatorics, Zhu made progress on the problem of list coloring bipartite graphs as well. Daniel's undergraduate work has re-
sulted in two solo and two co-authored mathematical research papers, each of which represents a meaningful contribution to different areas of combinatorics. Zhu's letter writers have described him as "exceptional, careful and precise" and possessing "extraordinary drive," whose work would be considered strong "even for a professor."
Biographical Note. Daniel Zhu is a first-year graduate student at Princeton University studying combinatorics. A native of Rockville, Maryland, Daniel became interested in mathematics at an early age, and frequently participated in academic competitions throughout middle and high school, winning a gold medal at both the International Physics Olympiad in 2018 and International Math Olympiad in 2019. At MIT, Daniel received degrees in both math and physics and conducted research in several different areas within combinatorics at both the Duluth and Baruch College REUs. Outside of research, you can often find Daniel following national and local politics and going on walks around the Princeton area.
Response from Daniel Zhu. I am honored to receive an honorable mention for the 2024 Morgan Prize, and am grateful to the AMS, MAA, and SIAM for their continued recognition of undergraduate research. I would like to thank Guy Moshkovitz for being an outstanding collaborator and mentor, Alejandro Morales for introducing me to the research process and for his patience and encouragement, Joe Gallian and Adam Sheffer for organizing vibrant REU programs, and Yufei Zhao for providing invaluable advice and support throughout my time at MIT. More broadly, I would like to thank everyone who has supported my mathematical endeavors over the years, especially my parents, who have been there from the very beginning.


Natalie Dean

## JPBM Communications Award

The Joint Policy Board for Mathematics (JPBM) Communication Award was established by the JPBM in 1988 and is given annually to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to nonmathematical audiences. The JPBM is a collaborative effort of the American Mathematical Society, American Statistical Association, Mathematical Association of America, and Society for Industrial and Applied Mathematics.

Citation: Natalie Dean. The 2024 JPBM Communications Award is presented to Natalie Dean for a remarkable record of public engagement providing clear meaning and context to COVID models and predictions through traditional and social media.
Biographical Note. Natalie Dean is an Assistant Professor in the Department of Biostatistics and Bioinformatics at Emory's Rollins School of Public Health. She received her PhD in Biostatistics from Harvard University. She previously worked as a World Health Organization consultant and as faculty at University of Florida. Her primary research area is in methods for infectious disease epidemiology and vaccine study design, and she co-directs the Emory Alliance for Vaccine Epidemiology. She became active in communications at the start of the COVID-19 pandemic, via Twitter and engaging with the press. She authored op-eds in the New York Times, Washington Post, Stat News, and Slate. She appeared on TV and radio, including CNN, MSNBC, Good Morning America, and NPR's All Things Considered. She has over 300 press quotes across national and international outlets. She was previously honored as a member of the Committee of Presidents of Statistical Societies' Leadership Academy.
Response from Natalie Dean. What an absolute honor it is to receive the JPBM Communications Award. This recognition by the JPBM, including my home society the American Statistical Association, means the world to me. The magnitude of the COVID-19 pandemic has necessitated an enormous response that includes keeping policymakers and the public up-to-date. It has been my privilege to be able to contribute to this effort along with so many others.

I would like to thank my community of infectious disease researchers and biostatisticians-an incredibly hardworking community who leapt headfirst into the pandemic response, many in less visible but absolutely critical roles. Thank you to my family, friends, and colleagues who encouraged me with kind words of support, with a special shout-out to my pal Caitlin Rivers. Thank you to the incredible science reporters I have worked with and learned from along the way, and to the communications professionals at the University of Florida and Emory. I'd like to thank my sweet babies, Owen and Noelle, who keep me present and balanced. Most of all, I'd like to thank my husband Ethan, my biggest and best supporter.


Ronald Coifman

AMS-SIAM George David Birkhoff Prize in Applied Mathematics

The Birkhoff Prize is awarded for an outstanding contribution to applied mathematics in the highest and broadest sense. The prize was established in 1967 in honor of Professor George David Birkhoff, with an initial endowment contributed by the Birkhoff family and subsequent additions by others. The American Mathematical Society (AMS) and the Society for Industrial and Applied Mathematics (SIAM) award the Birkhoff Prize jointly. The prize is awarded every three years to a member of AMS or SIAM.
Citation: Ronald Coifman. The 2024 AMS-SIAM George David Birkhoff Prize in Applied Mathematics is awarded to Ronald Coifman for his profound impact on pure and applied harmonic analysis, and for the introduction of tools developed from these areas to address modern challenges of data science.

Coifman is one of the most influential mathematicians of our time. Coifman's research is foundational and has impacted many branches of modern analysis and applied mathematics. His work transformed the theory of Hardy spaces, singular integrals, the theory of homogeneous spaces, factorization theorems in complex analysis, the BMO theory, and the Coifman-Meyer theory of paraproducts. As Terence Tao put it, the theory of paraproducts "is a cornerstone of the para-differential calculus that has turned out to be an indispensable tool in the modern theory of nonlinear PDE".

Coifman is one of the pioneers in the realm of wavelets (a type of wavelet with vanishing moments is now known as Coiflet), and this marks the beginning of his profound impact on applied mathematics. More recently, he has developed powerful methods for dimensionality reduction of high-dimensional point sets, and in particular the Coifman-Lafon Diffusion Mapshave became a powerful standard tool in data science. Coifman has established one of the first theoretical results about what types of functions can be represented via neural networks used in deep learning. Coifman's influence on the scientific prominence of next generations of mathematicians is attested by the long list of his trainees who are today's leaders in their own right.

Coifman's distinguished research career has been recognized by a number of honors and awards, including being
elected to the American Academy of Arts and Sciences in 1994 and to the National Academy of Sciences in 1998. He is a recipient of the 1996 DARPA Sustained Excellence Award, the 1996 Connecticut Science Medal, the 1999 Pioneer Award of the International Congresses on Industrial and Applied Mathematics. In 1999 Coifman was awarded the National Medal of Science, and in 2018 the Rolf Schock Prize for Mathematics.
Biographical Note. Ronald Coifman is Sterling professor of mathematics and professor of Electrical Engineering at Yale University. He obtained his PhD in Geneva in 1965 under the direction of J. Karamata, and was simultaneously mentored by Guido Weiss, and later by A. Calderon and A. Zygmund, while an instructor in Chicago. He joined Guido Weiss at Washington University in St Louis in 1968, until 1980 when he moved to Yale. Through the eighties he pursued an intensive collaboration with Yves Meyer and his team in Paris, later, followed by broad collaborations in Israel with Amir Averbuch and his group. Our current network joint with Y. Keverkidis is developing "mathematical empirical languages" to enable modeling empirical observations.
Response from Ronald Coifman. I would like to thank the AMS and the Society for Industrial and Applied Mathematics (SIAM) for the honor of being named the recipient of the 2024 George Birkhoff Prize in Applied Mathematics. And joining the previous recipients of the award who provided my mathematical inspiration. My view has always been that the boundary between pure and applied mathematics has never existed. Many people guided me throughout an exciting mathematical journey, starting with Zygmund who with Calderon and their students propagated the vision that a principal goal of a Harmonic analyst is to develop "methods" of analysis through a deep "understanding", replacing the miracles of complex analysis with "hard real variable methods or geometric book-keeping". I was mentored by Guido Weiss leading to a long and productive collaboration. Our program was to translate the ideas and tools of classical analysis to a general setting replacing the Fourier transform by adapted transformations that enabled us to go beyond linear convolutions and simple Euclidean structures. Throughout this journey collaborations with Yves Meyer opened the door to applications to nonlinear Fourier analysis and in signal processing. This is currently part of "computational Harmonic Analysis", and is a fundamental step for providing natural latent variables as "languages" essential for scientific models. This extended collaboration with Yannis. Kevrekidis, Yuval Kluger, Amir Averbuch, Vladimir Rokhlin, Jacques Peyriere and many, many, others has opened a world of merged visions.


Sybilla Beckmann

Award for Impact on the Teaching and Learning of Mathematics

The Award for Impact on the Teaching and Learning of Mathematics was established by the AMS Committee on Education (COE) in 2013. The endowment fund that supports the award was established in 2012 by a contribution from Kenneth I. and Mary Lou Gross in honor of their daughters Laura and Karen. The award is given annually to a mathematician (or group of mathematicians) who has made significant contributions of lasting value to mathematics education.
Citation: Sybilla Beckmann. Sybilla Beckmann, Josiah Meigs Distinguished Professor of Mathematics, Emeritus, at the University of Georgia, is nationally recognized for her seminal contributions to mathematics teacher education, combining experiences in mathematics, public school classrooms, textbook writing, national service, and research. Through her work in mathematics education policy, she has advocated for high-quality teacher education and rigorous mathematical curriculum, impacting students and teachers across the country.

Dr. Beckmann started her career in arithmetic geometry working on problems related to the Inverse Galois Problem, an open problem that asks whether every finite group occurs as a Galois group of a Galois extension of the rational numbers. She received her PhD from the University of Pennsylvania and taught at Yale University as a J. W. Gibbs Instructor of Mathematics before moving on to the University of Georgia.

When her children were in school, Dr. Beckmann became increasingly interested in $\mathrm{K}-12$ mathematics education. To gain a deeper understanding of $\mathrm{K}-12$ mathematics teaching, she taught one period of 6th grade mathematics in Clarke Middle School, a public school near the University of Georgia, for the entire 2004-2005 school year. Her direct experience in the classroom laid the foundation for her approach to teaching mathematics to future elementary and middle school teachers.

In particular, Dr. Beckmann's textbook, Mathematics for Elementary Teachers with Activities, first published in 2002 and now in its 6th edition, was groundbreaking in the space of mathematics teacher training. Dr. Beckmann's book focuses on arithmetic operations, giving a coherent understanding of K-12 mathematics, and it does so through activities, allowing students to experience the
interactive, engaged teaching supported by research. This enables future teachers to develop deep insights into algebraic structures-making connections and drawing comparisons between different number systems. In addition, Dr. Beckmann incorporated best practices from mathematics curricula from Singapore and Japan in her textbook. Specifically, strip diagrams, which help students connect topics in arithmetic and algebra, feature prominently. Not only has Dr. Beckman influenced many future teachers who learned from her book, she truly changed the education of mathematics teachers with her innovative approach and perspective.

Dr. Beckmann has also been involved in writing many policy documents with the goal of improving the quality of mathematics education. She was on the Work Group for the Common Core State Standards for Mathematics, the closest thing the United States has to a rigorous set of standards shared across states. Dr. Beckmann was also one of the lead writers for The Mathematical Education of Teachers II, a document published jointly by the Mathematical Association of America and the American Mathematical Society that gives recommendations for high-quality mathematics teacher education. In addition, she helped to develop two Institute of Education Sciences Practice Guides, including Improving Mathematical Problem Solving in Grades 4 through 8: A Practice Guide. These nationally-recognized initiatives have had a broad impact on mathematics teachers and students.

In recognition of her commitment to excellence in math education and lasting impact on mathematics teacher education, the AMS awards Sybilla Beckmann the 2024 Award for Impact on the Teaching and Learning of Mathematics.
Biographical Note. Sybilla Beckmann is Josiah Meigs Distinguished Professor of Mathematics, Emeritus, at the University of Georgia. She earned a PhD in mathematics from the University of Pennsylvania and taught at Yale University as a J. W. Gibbs Instructor of Mathematics before teaching at the University of Georgia for 32 years.

Beckmann began her career doing research in Arithmetic Geometry, but she became interested in mathematics education as her children entered school. She developed courses for prospective elementary and middleschool teachers that were designed to go deeply into the ideas of elementary and middle-school mathematics. Her textbook for such courses is now in a sixth edition. Beckmann was a member of a number of national committees and writing teams to develop recommendations, guidelines, and standards for the mathematical education of students and teachers. She continues to do research in mathematics education.

Response from Sybilla Beckmann. Thank you so much to the AMS and the selection committee for this wonderful honor. When I began my career, I never imagined the path it would take. I followed my interests and took opportunities as they arose, and I developed a passion for mathematics education that sent my career in a non-standard direction. I am deeply grateful to the mathematics community as a whole, and to my colleagues and the heads of my department who valued and encouraged my work. I am so grateful to have been part of a culture in which the work I chose to do could thrive and flourish. It has been a joy and a privilege to think deeply about mathematics at many levels, from elementary school to the forefront of research, to work with so many dedicated and enthusiastic scholars, and to teach so many wonderful students. Huge thanks to my friends and colleagues in mathematics education who worked patiently with me and taught me so much, especially Andrew Izsák. And finally, thank you to my family, Will, Joey, and Arianna, for your love and inspiration, which made everything possible.

## Mathematics Programs that Make a Difference Award



Three generations of MPM Organizers: (top row, left to right): Kim Klinger-Logan, McCleary Philbin, Esther Banaian, Sarah Brauner. (Bottom row, left to right): Patricia Commins, Marcella Manivel, Elise Catania, E Koenig. Not pictured: Alice Nadeau and Harini Chandramouli

In 2005, the American Mathematical Society, acting upon the recommendation of its Committee on the Profession, established the Mathematics Programs that Make a Difference Award. The award, provided by the Mark Green and Kathryn Kert Green Fund for Inclusion and Diversity, highlights programs that are succeeding and could serve as a
model for others in addressing the issues of underrepresented groups in mathematics.
Citation: Mathematics Project at Minnesota. The AMS is proud to recognize the Mathematics Project at Minnesota (MPM) at the University of Minnesota (UMN) with the 2024 Mathematics Programs that Make a Difference Award.

Founded in 2018, the MPM is a week-long workshop that brings together undergraduate and graduate students and faculty at all levels to engage in activities and discussions aimed to build community, provide academic and professional development, and foster an environment committed to inclusivity and equity within the department. Through the efforts of graduate student leaders, the MPM has demonstrated success within the undergraduate program in recruiting, training, and retaining students who identify as underrepresented, women and gender minorities. With this recognition, the AMS envisions that the MPM at UMN can serve as a model and inspiration for similar workshops at peer institutions.

In the words of one of the graduate student organizers, "The workshop activities aim to deconstruct prior notions of what it means to be 'good' at math or to be a 'math person.' Often, students have internalized a rigid interpretation of who can be a mathematician and what types of skills this requires. Many of these assumptions are gendered and racialized. MPM engages students by (1) Asking them to critically think about the validity and origins of these assumptions; (2) Explaining that enjoying math is a sufficient reason to study it, as opposed to assessments of their own abilities (which they tend to underestimate); and (3) Fostering positive and collaborative mathematical experiences in a non-competitive environment. . . Community building underlies the entirety of the MPM workshop; in addition, several hours of each day of MPM are devoted to games, group meals and socializing. Throughout the workshop, small groups of participants are paired with advanced undergraduate students or graduate students who guide them through the various activities and sessions. Professors and post-docs who can serve as good mentors are invited to lead or attend sessions. The academic development sessions include group problem sessions and short individual presentations on a mathematics topic (with extensive help from graduate students along the way). Professional development workshops discuss career opportunities for a mathematics major, including a panel and dinner with local professionals. The equity and inclusivity discussions focus on imposter syndrome, growth mindset, implicit biases and privilege."

Each year, approximately ten graduate student volunteers organize and develop all workshop programming, recruit students and workshop volunteers, and obtain fund-
ing for the workshop (most recently, from the MAA Tensor Women and Mathematics Grant). To ensure that the program is sustainable, the MPM has thoughtfully created a tiered leadership structure, with advanced graduate students working together with early graduate students to convey experience and knowledge of running the workshop. Thus, graduate student volunteers gain experience in organizing workshops, building community, and seeking external funding. As an example of one of the many very positive statements from volunteers, "MPM has been a safe place for me and so many people, in which we have been able to explore and claim the identity of loving mathematics. MPM has given me a language to talk about struggles and growth in and around math and math communities, and tools to help me and others grow... I have made connections through MPM that have lasted years, and seen many people seek out opportunities they never would have known about."

The MPM builds community across academic levels: students early in their undergraduate careers are paired with their more advanced peers and graduate students. The workshop also builds connections among students, post-docs, faculty and local professionals in the mathematical sciences. The MPM mentoring relationships between paired undergraduate and graduate students continue long after the workshop. As noted in a voluntary comment from an undergraduate participant, "If I had not participated in MPM, I may not have stayed in the math department. The program gave me a community and the confidence to stay." And from another, "MPM helped me feel less alone on campus. As the only girl in my class, I often felt really lonely and scared in my pursuit in math and often questioned why I chose it as my major. MPM reminded me that I enjoy learning about math and there are many new careers I can do!"

In summary, the MPM is an invaluable and highly effective graduate student-led initiative developed with the goal of exposing and removing known barriers in retaining and advancing the careers of women, gender minorities and underrepresented groups in the mathematical sciences at the University of Minnesota. The workshop creates lasting mentoring relationships and provides academic and professional enrichment for participants at all levels through thoughtful activities, discussions, and local networking. This replicable program has the potential to have a profound and positive impact on students and faculty at similar graduate programs, as well as the power to make a significant difference to our greater mathematical community.
Biographical Note. Mathematics Project at Minnesota is a graduate student initiative that was founded in the fall of 2017 by Harini Chandramouli, Kim Klinger-Logan, and

Alice Nadeau. It was then organized by Esther Banaian, Sarah Brauner, and McCleary Philbin. The current organizers are Elise Catania, Patricia Commins, E Koenig, and Marcella Manivel. In addition to organizers, each year 510 graduate students help implement sessions and mentor participants.

The workshop is planned during the fall semester, and takes place the week before the University of Minnesota (UMN) spring semester starts. The program has grown substantially since 2017; the number of participants has gone from roughly 15 to 30 , and the number of volunteers (MPM alumni, postdocs and faculty) has also nearly doubled. The workshop aims to recruit UMN undergraduates who come from underrepresented groups in mathematics and are early in their studies. Students are identified by department lists and instructor/teaching-assistant recommendations.
Response from Mathematics Project at Minnesota. The organizers of the Mathematics Project at Minnesota (MPM) are very honored to receive this award.

The MPM is a workshop held annually at the University of Minnesota (UMN). The goal of the program is to increase participation of undergraduate students from underrepresented groups in mathematics, and to encourage their success. The MPM is organized by graduate students at UMN, and typically has around 30 undergraduate participants. The week-long workshop contains approximately 20 sessions on various topics in mathematics, equity and diversity issues, professional development panels and information sessions, and social events.

The workshop has a strong focus on community building. Participants are paired with graduate student mentors who provide individualized advice and mathematical support throughout (and beyond the end of) the program. Postdocs and faculty members participate in various events, and several external speakers are invited to speak on professional development panels.

This sense of community is fundamental to the program; it encourages engagement, and reinforces strong positive messages that these students are welcome in the mathematical community. Participants are invited to return to the program in future years, and many are excited to return to take on supporting and mentoring roles. In anonymous surveys conducted after the program, an overwhelming majority of undergraduate participants indicated that they would recommend the program to other students and that they were more likely to participate in math research or pursue upper-level mathematics courses as a result of their experience at the MPM. Many workshop alumni have said that the workshop helped persuade them to stay in the mathematics major or go to graduate school.

We are indebted to the many graduate student volunteers who have worked tirelessly to help make the MPM a success and the mathematics major at UMN more inclusive. We also thank the MAA Tensor Grant for Women in Mathematics for providing funding for the workshop for the past three years, and to Paul Carter and Max Engelstein for serving as faculty liaisons.

We hope that this award helps MPM become a permanent, internally funded fixture of the mathematics department at UMN. We would also like to take this opportunity to invite members of the mathematics community to start similar programs at their institutions. We are happy to advise about logistics and to provide samples of all workshop materials we have used. Please visit our website for more information: https://sites.google.com /view/mpm-umn.


Susan Landau

## Bertrand Russell Prize of the AMS

The Bertrand Russell Prize of the AMS was established in 2016 by Thomas Hales. The prize looks beyond the confines of the profession to research or service contributions of mathematicians or related professionals to promoting good in the world. It recognizes the various ways that mathematics furthers fundamental human values. Mathematical contributions that further world health, our understanding of climate change, digital privacy, or education in developing countries, are some examples of the type of work that might be considered for the prize. The prize is awarded every three years.
Citation: Susan Landau. The 2024 Bertrand Russell Prize of the AMS is awarded to Susan Landau. Landau is Bridge Professor in Cyber Security and Policy at The Fletcher School and the School of Engineering, Tufts University. She is a leading scholar in encryption policy and digital privacy, an area of great importance currently. Writing technical research papers and opeds, publishing public-facing work, briefing policymakers, and participating in national studies, Landau has effectively coupled the mathematics of digital privacy and encryption to policy-making. Her strengths and energy in communications, in testimony but especially in books, have helped to illuminate essential properties of the digital world that limit the range of policy and the degree of protection that digital methods can offer.

Biographical Note. Susan Landau is Bridge Professor in Cyber Security and Policy at The Fletcher School and the School of Engineering, Department of Computer Science, Tufts University. She works at the intersection of privacy, surveillance, national security law, and cybersecurity. Landau is the author of four books: People Count: ContactTracing Apps and Public Health, Listening In: Cybersecurity in an Insecure Age; Surveillance or Security? Risks Posed by New Communications Technologies; and co-author, with Whitfield Diffie, Privacy on the Line: The Politics of Wiretapping and Encryption. Landau has testified before Congress and briefed US and European policymakers on encryption, surveillance, and cybersecurity issues. She has served on various advisory boards, including the National Academies Computer Science and Telecommunications Board, NSF Computer and Information Science Advisory Board, and NIST's Information Security and Privacy Advisory Board. Landau has received multiple awards, including a Lifetime Achievement Award from USENIX in 2023. She received a BA from Princeton, an MS from Cornell, and a PhD from MIT.
Response from Susan Landau. I am deeply honored to receive the Bertrand Russell Award, which is quite meaningful to me in three ways.

The first is because of the momentous change in the mathematics community since I entered it in the early 1970s. At the time, reaching out to the wider world was deemed an unnecessary distraction from proving deep theorems. So the view embodied in the Bertrand Russell Award makes it particularly meaningful to me.

The second stems from the winds of change that blew across the AMS and math community in the late 1970s. In the early 1980s, the AMS Notices began publishing expository work. My first works on cryptography policy were for the Notices. Thus, those winds of change had a direct effect on my career.

The third reason the Bertrand Russell Award is so personally meaningful is that Russell and Joseph Rotblat founded the Pugwash Conferences on Science and World Affairs, an international organization of scientists working to eliminate weapons of mass destruction. Pugwash efforts lie behind the 1963 nuclear test ban treaty and multiple other international arms treaties. In 1981 I attended a Student Pugwash Conference; the meeting's indelible impression has guided my thinking and actions ever since. I feel greatly privileged to receive this award and thank the AMS and the Bertrand Russell Award Committee for this honor.

## Elias M. Stein Prize for New Perspectives in Analysis



This prize was endowed in 2022 by students, colleagues, and friends of Elias M. Stein to honor his remarkable legacy in the area of mathematical analysis. Stein is remembered for identifying many deep principles and methods which transcend their original context, and for opening entirely new areas of research which captivated the attention and imagination of generations of analysts. This prize seeks to recognize mathematicians at any career stage who, like Stein, have found exciting new avenues for mathematical exploration in subjects old or new or made deep insights which demonstrate promise to reshape thinking across areas.
Citation: Marcel Filoche and Svitlana Mayboroda. The 2024 Elias M. Stein Prize for New Perspectives in Analysis is awarded jointly to Marcel Filoche and Svitlana Mayboroda for their original, powerful, elegant and impactful theory of the "localization landscape," initially developed in Proc. Natl. Acad. Sci. USA, 109 (2012), no. 37 and Contemp. Math., 601 (2013). The theory evolved in scope and impact through multiple subsequent collaborative works, including in Adv. Math. 390 (2021), Paper No. 107946, 34. In this theory, the localization of eigenfunctions to a Schrödinger type operator is controlled in various senses by a single, easily computed "landscape function." This discovery is supported by theoretical results, striking numerics, and physical experiment, and it provides a novel way to look at eigenfunctions that goes beyond existing methods such as semiclassical analysis or probabilistic approaches, greatly clarifying the phenomenon of wave localization.

Biographical Note. Marcel Filoche graduated from Ecole Polytechnique in 1985 and received his PhD from Université d'Orsay in 1991. He is currently CNRS Research Director at the Langevin Institute of the Ecole Supérieure de Physique et Chimie Industrielle (ESPCI), Paris.

Marcel Filoche is interested in transport and propagation phenomena in systems with complex geometries, both classical and quantum. Over the past ten years, he has developed together with Svitlana Mayboroda the mathematical theory of the localization landscape, unveiling the properties of eigenfunctions of wave operators in random potentials. Since 2018, he has been one of the leaders of the international Simons collaboration project on wave localization.
Biographical Note. Svitlana Mayboroda was born in Kharkiv, Ukraine. She received her PhD at the University of Missouri in 2005, and after that held postdoctoral positions at the Ohio State University, Australian National University and Brown University. She worked at Purdue University from 2008 to 2011 and moved to the University of Minnesota in 2011. Professor Mayboroda has been the McKnight Presidential Professor of Mathematics at the University of Minnesota since 2020. In 2023 she joined ETH Zurich.

Svitlana Mayboroda's awards include, in particular, the US Blavatnik National Award in 2023, the AWM Sadosky Prize in Analysis in 2014, the Alfred P. Sloan Research Fellowship in 2010. She has enjoyed continuous NSF support since 2008 and has been the Director of the Simons Collaboration on the Localization of Waves since 2018. She was an invited speaker at the ICM in 2018.
Response from Marcel Filoche. I am very honored and thrilled to receive the inaugural Elias Stein Prize. I would like to thank the American Mathematical Society for this prestigious award, and Svitlana Mayboroda who was my partner all along during the development of the localization landscape theory. It is truly a privilege to work with her. I would like to especially thank my professor who introduced me to the beauty of harmonic analysis, Yves Meyer, for his constant enthusiasm and care. I am also deeply grateful to Guy David, David Jerison, and Douglas Arnold for years of joyful and intense collaboration. I learned immensely working with them, and it is always a privilege. Finally, I am very thankful to my family for all the love and support.
Response from Svitlana Mayboroda. It is an immense honor to receive the inaugural Elias M. Stein Prize for New Perspectives in Analysis. Stein's legacy as a mathematician and educator has shaped my field, and I am incredibly grateful to my peers and to the selection committee for this remarkable recognition. This award has a special meaning to me. It not only endorses my individual contributions, but also pays homage to many years of the exciting collaboration that had such a deep impact on my mathematics, my life, and my career. I am deeply grateful to Marcel Filoche for challenging and inspiring me on this incredible
journey, for sharing his vision and pushing us to fearlessly cross boundaries between mathematics and physics, and to Jill Pipher, Doug Arnold, and Guy David whose unwavering support has made this project possible.


Leo Corry

## Albert Leon Whiteman Memorial Prize

This prize was established in 1998 using funds donated by Mrs. Sally Whiteman in memory of her husband, the late Albert Leon Whiteman. Mrs. Whiteman requested that the prize be established for notable exposition on the history of mathematics. Ideas expressed and new understandings embodied in the exposition awarded the Whiteman Prize will be expected to reflect exceptional mathematical scholarship. The prize is awarded every three years at the Joint Mathematics Meetings.
Citation: Leo Corry. The 2024 Albert Leon Whiteman Prize of the American Mathematical Society is awarded to Leo Corry of Tel Aviv University (currently President, the Open University of Israel) for his exceptional scholarship and exposition elucidating the roles of axioms and structures in the practice of modern mathematics and physics, as well as for his many contributions to the field of history of mathematics as an editor, mentor, and communicator.

Across an impressive array of publications, Leo Corry has insightfully examined pivotal developments in modern mathematical sciences with technical nuance, philosophical sophistication, and narrative flair. His first two books, Modern Algebra and the Rise of Mathematical Structures (Springer-Birkhäuser Verlag, 1996; 2nd ed. 2004) and David Hilbert and the Axiomatization of Physics (18981918): From Grundlagen der Geometrie to Grundlagen der Physik (Springer, 2004), treat what, at first glance, may seem like two largely distinct, if not actually contradictory, trends in twentieth-century mathematics: the development of modern algebra and the mathematical reorientation of physical theory. Corry's work in these books and numerous associated writings identifies the common background that informed the trends, clarifying their apparent tensions and demonstrating how structuralism and axiomatic foundationalism functioned at the dynamic interface between the philosophical characterization, and the actual practice, of mathematics. His analysis has illuminated the significance of the sometimes-paradoxical diversity of conceptions of structures, abstraction, and
universalism, opening up a critical perspective on a rich and challenging era of major mathematical change that has inspired considerable further research by historians and others.

More recently, Corry has explored the interaction between theory building and intensive calculation in pure mathematics, especially number theory, both before and after the advent of the electronic computer. His work in this area has led to the widely-read and translated popular book, A Brief History of Numbers (Oxford, 2015), two books in collaboration with Raya Leviathan, WEIZAC: An Israeli Pioneering Adventure in Electronic Computing (19451963) and Chaim L. Pekeris and the Art of Applying Mathematics with WEIZAC (1955-1963) (Springer Verlag, 2019 and 2023, respectively), and a series of innovative and illuminating articles, notably "A Clash of the Mathematical Titans in Austin: Robert Lee Moore and Harry Schultz Vandiver (1924-1974)" (The Mathematical Intelligencer, 2007). Concurrently, Corry has studied the interrelation between arithmetic and geometry in the Euclidean tradition, focusing on the consolidation of algebraic methods in the early modern period and the rise of analytic geometry, on the one hand, and on the historical question of the interpretation of "geometrical algebra" in classical Greek geometry, on the other. In this vein, he has produced two, one-hundred-page monographs, Distributivity-like Results in the Medieval Tradition of Euclid's Elements: Between Geometry and Arithmetic and British Versions of Book II of Euclid's Elements: Geometry, Arithmetic, Algebra (1551-1750) (Springer Verlag, 2021 and 2022, respectively).

As a generous editor (notably of the leading journal, Science in Context, for most of the period of 1999-2013) and mentor, Corry has shaped the field of history of mathematics and its connections to allied fields in Israel and internationally. He has lectured around the world and shared his insights with many and varied audiences, most notably as an invited session speaker at the International Congress of Mathematicians in Madrid in 2006 and as a keynote lecturer in the Turing Centennial Conference of the Royal Flemish Academy of Belgium for Science and the Arts in 2012.

Taken collectively, Leo Corry's body of research has led to a new understanding of the very notion of "modern mathematics" as well as to insights in earlier traditions.
Biographical Note. Leo Corry is Professor Emeritus of History and Philosophy of Science at Tel Aviv University, currently serving as President of the Open University of Israel. He graduated in mathematics at Universidad Simón Bolívar (1977), in Caracas, and continued his studies at TAU, earning an MSc in mathematics (1982), and a PhD in history and philosophy of science (1990).

At TAU, Corry has been Director of the Institute for History and Philosophy of Science, Director of the Graduate School of Historical Studies, and Dean of Humanities. He has been visiting professor at the Max Planck Institute in Berlin, ETH Zurich and MIT.

Corry is co-author of twenty US patents in the field of electronic data storage. He has published scholarly work on Latin American literature, and has translated into Hebrew such writers as Borges, Vargas Llosa and Carpentier. He is an enthusiastic connoisseur of salsa and Venezuelan music, and a skilled maraquero.
Response from Leo Corry. I am thrilled and honored by being selected to receive this award. My work would not have been possible without the prolific community of historians of mathematics of my generation, whose work over the last few decades turned our discipline into a vibrant field of research. Because of its high quality and the range of topics and periods that it addresses, the ever-growing body of knowledge thus produced has received increased attention and recognition in the mathematical world at large. My sincere thanks go to all of my colleagues with whom I have had the privilege to interact professionally, and to learn from their work.

The remarkable mathematical education I received at Universidad Simón Bolívar, and was later complemented at TAU, provided the rock-solid basis of whatever I have done ever since. The all-encompassing intellectual ecosystem of the Cohn Institute of History and Philosophy of Science at TAU was for more than four decades a world-class academic home that shaped my career.

## Leroy P. Steele Prize for Mathematical Exposition



Benson Farb


Dan Margalit

The Leroy P. Steele Prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein and are endowed under the terms of a bequest from Leroy P. Steele. Prizes are awarded in up to three categories.

The AMS Leroy P. Steele Prize for Mathematical Exposition is awarded annually for a book or substantial survey or expository research paper.
Citation: Benson Farb and Dan Margalit. The 2024 Steele Prize for Mathematical Exposition is awarded to Benson Farb and Dan Margalit for their Princeton Mathematical Series book A Primer on Mapping Class Groups. The authors are leading researchers in group theory as well as allied areas of topology and geometry. Their expertise shines through with masterful and clear expositions of the combinatorial, algebraic, geometric and analytic viewpoints that mapping class groups enjoy. Many of the classical theorems, for example the work of Dehn, are presented from a modern perspective and in particular through the work of Thurston, which was introduced and developed decisively in the short time since the primer was published.

The book has proved to be a valuable resource not only for the graduate students to whom it is addressed, but also for experts. It is already a classic and it sets the standard for accessible, clear and inviting writing. It stands as the very model of scholarship.
Biographical Note. Benson Farb is Professor of Mathematics at the University of Chicago. He was born and raised in Norristown, Pennsylvania, a suburb of Philadelphia. Farb graduated from Cornell University in 1989, obtained a PhD at Princeton University in 1994 under the direction of Bill Thurston, and then went to the University of Chicago as a postdoc and never left. Farb was a Sloan fellow, an NSF Career Award recipient, an inaugural Fellow of the AMS (2012), and an invited speaker at the 2014 ICM (Topology section). He was elected to the American Academy of Arts and Sciences in 2021. Farb has written papers on geometric group theory, low-dimensional topology, dynamical systems, differential geometry, Teichmuller theory, cohomology of arithmetic groups, representation stability, Hilbert's 13th problem, algebraic geometry, 4-manifold theory and the connections among all of these topics. He has supervised 52 PhD students and has been senior scientist for 15 NSF postdocs.
Biographical Note. Dan Margalit grew up in Flanders, NJ, the son of two Israeli immigrants. He received his ScB in Mathematics from Brown University in 1998 and his PhD in Mathematics from the University of Chicago in 2003 under the direction of Benson Farb. He was on the faculty at the University of Utah, Tufts University, and Georgia Institute of Technology before becoming Stevenson Chair and Chair of Mathematics at Vanderbilt University in 2023.

Margalit received a Sloan Research Fellowship in 2009 and an NSF CAREER Award in 2010. He received the Levi L. Conant Prize from the AMS in 2021. He was the Maryam Mirzakhani Lecturer at the 2022 JMM. Margalit was elected
as Fellow of the AMS in 2019 "for contributions to lowdimensional topology and geometric group theory, exposition, and mentoring."

Margalit enjoys music, hiking, and juggling. He is married to Kathleen Margalit. They have two children, Lily and Simon.

Response from Benson Farb. I am grateful to the AMS for this honor. I am lucky to have learned so much about this topic from my advisor Bill Thurston (from whom I also learned how to encounter mathematics), from Curt McMullen (whose course in 1993 served as an inspiration for this book), and from Lee Mosher and Howard Masur. Thanks to Joan Birman for her support throughout the years. Joan was a pioneer in this area, and has served as a role model for so many of us. Finally, thanks to my family: Amie, Bea and Felix, for their love and support.

This project began with me teaching Dan Margalit this subject, and it ended with Dan teaching me much more. I am grateful to him for this, and for catching and explaining the many (alas) subtle points I'd missed.
Response from Dan Margalit. I am honored and grateful to be a co-recipient of the 2024 Leroy P. Steele Prize for Mathematical Exposition.

In Winter 2001, I was a third year graduate student. A struggling third year graduate student. Benson Farb, my advisor (and co-recipient), took a chance and asked me if I would take notes on his course, with the goal of writing a book. I thought this was a one-year project. Our book was published a decade later.

I am incredibly grateful to Benson for bringing me into this project. It gave me an avenue for deepening my feel for mathematical argumentation, for nurturing my intuition for groups and topology, and for developing my skills as a writer. Most of all, I benefitted from Benson's broad vision, impeccable taste, and joy for beautiful mathematics. We had many disagreements over the usage of commas, the ordering of sections, and the proofs of theorems (full disclosure: most of the time he was right). We often marvel (jokingly) at how we are still on speaking terms.

In working on the book, I relied heavily on conversations with Bob Bell, Mladen Bestvina, Joan Birman, Tara Brendle, Ken Bromberg, Chris Leininger, Andy Putman, Steven Spallone, and Kevin Wortman. I am grateful for their intellectual and emotional generosity. I would also like to thank Thomas Banchoff for drawing me into mathematics.

I am grateful to my wife, Kathleen, for giving me inspiration for this project and all my other endeavors. I am further grateful to her and our two children, Lily and Simon, for supporting my long hours of writing during weekends and winter vacations. My siblings, Ron and Thalia, are
constant sources of love. Finally, I would like to thank my parents, Batya and Zamir, who sacrificed endlessly so their children could be successful and realize their dreams.

## Leroy P. Steele Prize for Seminal Contribution to Research



József Balogh


Wojciech Samotij


## Andrew Thomason

The Leroy P. Steele Prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein and are endowed under the
terms of a bequest from Leroy P. Steele. Prizes are awarded in up to three categories.

The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research. The prize is awarded according to the following six-year rotation of subject areas: Open, Analysis/Probability, Algebra/Number Theory, Applied Mathematics, Geometry/Topology, and Discrete Mathematics/Logic.
Citation: József Balogh, Rob Morris, and Wojciech Samotij; David Saxton and Andrew Thomason. The 2024 Steele Prize for Seminal Contribution to Research is awarded jointly to József Balogh, Rob Morris, and Wojciech Samotij for their paper "Independent sets in hypergraphs," published in the Journal of the American Mathematical Society, 28 (2015), 669-709, and to David Saxton and Andrew Thomason for their paper "Hypergraph containers," published in Inventiones Mathematicae, 201 (2015), 925-992.

An independent set in a hypergraph is a subset of vertices containing no hyperedge. It is well understood that several important theorems and conjectures in combinatorics, such as Szemerédi's theorem on arithmetic progressions and the Erdös-Stone Theorem in extremal graph theory, can be cast as questions about families of independent sets in certain uniform hypergraphs. The above two papers, written independently of each other, distilled the property that many of these hypergraphs exhibit: a certain clustering phenomenon that explains why certain intriguing results hold.

A hypergraph container theorem claims the existence of a collection of subsets of vertices (containers) such that (a) every independent set is a subset of one of the container sets, (b) the number of containers is not large, and (c) every container set spans only few hyperedges. Each of the above papers proved such a container theorem, and gave numerous applications of it. Specifically: i) new proofs of random analogues of Turán and Szemerédi Theorems, ii) tight estimates for the number of H -free graphs and hypergraphs (for a fixed graph H), and a counting version of Szemerédi's theorem, and iii) a proof of the celebrated Kohayakawa-Łuczak-Rödl conjecture. Although these papers were published only eight years ago, their importance has been amply demonstrated by numerous deep results proved subsequently, in which a key step has been an application of the hypergraph container theorem or a variant of it.
Biographical Note. József Balogh grew up in a small thermal spa town, Mórahalom, in South Hungary. He attended the top mathematics secondary school at the time:

Ságvári, in Szeged. While in high school, Balogh won two silver medals at the International Mathematical Olympiad. He completed his undergraduate and master's studies at Szeged University and earned his PhD at the University of Memphis under the supervision of Béla Bollobás. Balogh held postdoctoral positions at AT\&T Research, the Institute for Advanced Study at Princeton, and The Ohio State University; visiting positions at Szeged University, IPAM at UCLA, and the University of Cambridge; and a tenured position at UCSD. Currently, he is a professor at the University of Illinois in Urbana-Champaign, where he has advised 16 doctorate students. Balogh was a recipient of the George Pólya Prize in Combinatorics from SIAM (2016), an ICM speaker (2018), and a Simons Fellow $(2013,2020)$. Outside of mathematics, he enjoys racquet sports, chess, and soccer-both as a player and as a former coach.
Biographical Note. Rob Morris grew up in the north of England, but received his PhD from the University of Memphis, where he was a student of the famous Hungarian mathematician, Béla Bollobás. He fell in love with Rio de Janeiro during a visit to IMPA in 2004, and spent a year there as a postdoc in 2006-2007. After spending time in Cambridge, Tel Aviv and Tokyo, he returned to Rio (and to IMPA) in 2010, where he has been ever since. He has been awarded numerous prizes, including the MCA Prize, the Prêmio Reconocimiento de UMALCA, the Prêmio SBM, the Prêmio Elon Lages Lima, the Fulkerson Prize, the George Pólya Prize in Combinatorics, and the European Prize in Combinatorics. He was an invited speaker at the 2018 ICM, and in 2022 he was elected to the Brazilian Academy of Sciences. He lives in Rio de Janeiro, a few minutes walk from IMPA, with his wife and two daughters.
Biographical Note. Wojciech Samotij was born in Wrocław, Poland in 1983. After receiving MSc degrees in mathematics and in computer science from the University of Wrocław, he moved to the University of Illinois at Urbana- Champaign, where in 2010 he obtained his doctorate, advised by József Balogh. Samotij spent his postdoctoral years between Trinity College in Cambridge and Tel Aviv University, where he was appointed as a faculty member in 2014. He has worked at the School of Mathematical Sciences of Tel Aviv University ever since.
Biographical Note. David Saxton was born in Hampshire, England, and studied mathematics at Cambridge, where he did his PhD (2008-2012) in combinatorics under the supervision of Andrew Thomason. He continued combinatorics research in a postdoctoral position (20122014) at IMPA, Brazil, and is a recipient of the 2016 Pólya Prize. Since 2015 he has worked at DeepMind as a machine learning researcher, where his research interests and projects have included improving the reasoning abilities
of neural networks, and applying generative modelling to protein design. Outside of professional life, David enjoys writing, meditation, and various sports, including climbing and cycling.
Biographical Note. Andrew Thomason was an undergraduate at Peterhouse, Cambridge, and received his PhD from Cambridge under the supervision of Béla Bollobás. Following a research fellowship at St John's College Cambridge, and tenured positions at Louisiana State University and at the University of Exeter, he returned to Cambridge as a faculty member and also a Fellow of Clare College. His research has been largely in the area of graph theory. He is now retired.
Response from József Balogh, Rob Morris, and Wojciech Samotij; David Saxton and Andrew Thomason. We are deeply honoured to receive the Leroy P. Steele Prize for Seminal Contribution to Research. We are extremely grateful to the colleagues who nominated us and to the wider combinatorial community for their unfaltering support over the years; in particular, we thank Béla Bollobás, the research supervisor of three of us.

We would like to stress that our work on hypergraph container theorems was only made possible by earlier works of many other mathematicians. The early foundations for the graph container method were laid already in the 1980s and 1990s by Daniel Kleitman and Kenneth Winston and, independently, by Alexander Sapozhenko, who was the first to use the term "container" in this context. Curiously, our journeys to a useful notion of hypergraph containers, and effective ways to implement it, were different. Two of us were pointed to the start of the trail by Sapozhenko, whilst the other three, following the spirit of Kleitman and Winston, were heavily influenced by the seminal work of Penny Haxell, Yoshi Kohayakawa, Tomek Łuczak, and Vojta Rödl in the 1990s, and by the wonderful papers of David Conlon and Tim Gowers and of Mathias Schacht, which independently developed two versions of the so-called "transference principle" in the context of extremal properties of random structures.

Further, a large portion of the credit that the five of us have received for the development of the "container method" should in fact extend to a much larger group of mathematicians who have found a great many, often very surprising, applications of our hypergraph container theorems. The container method is an achievement of the entire combinatorics community, and we would like to dedicate this prize to all of the mathematicians who contributed to its development over the years.
Response from József Balogh. I would like to express my deepest gratitude to my parents. My father, although equally talented in math during middle school, faced the
unfortunate circumstance of not being able to attend high school due to financial constraints. The same was true for my mother. Despite their own limitations, they provided me with unwavering support for my education. I am profoundly grateful for their sacrifices and dedication to my future.

I extend my heartfelt appreciation to all the exceptional teachers who have played pivotal roles in my mathematical journey. In high school, Tamás Tarcsay, József Csúry, and Lajos Pintér ignited the flames of my passion for advanced mathematics. Later, as an undergraduate student, the influence of Péter Hajnal and András Pluhár sparked my interest with combinatorics, eventually leading me to join the research group of Béla Bollobás.

As a first-generation high schooler, the academic path was, and continues to be, far from smooth. Along this journey, solving mathematical problems often proved to be the easiest part of the journey.

One of the most valuable lessons I have learned from my teachers is that mathematics is fun; it's an exhilarating adventure. We should always focus on interesting problems and always enjoy the journey, not just the destination. I am dedicated to passing on this philosophy to the next generation of mathematicians.

On a more lighthearted note, my journey into the world of mathematics began when I participated in a Hungarian TV show-a math competition designed for 6 th graders. You can catch a glimpse of my early mathematical enthusiasm in this video: see https://www.youtube.com /watch?v=0E7GeTCTBtg.


Haïm Brezis

## Leroy P. Steele Prize for Lifetime Achievement

The Leroy P. Steele Prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein and are endowed under the terms of a bequest from Leroy P. Steele. Prizes are awarded in up to three categories.

Presented annually, the AMS Leroy P. Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students.
Citation: Haïm Brezis. The 2024 Steele Prize for Lifetime Achievement is awarded to Haïm Brezis for his outstand-
ing and seminal contributions in several fields of Nonlinear Functional Analysis and Partial Differential Equations, and for his remarkable influence in mathematics, in particular through his exceptional training of PhD students.

Brezis has greatly contributed to leading and shaping the fields of Nonlinear Analysis and Partial Differential Equations and how the main questions are posed. He has started and animated several different areas of analysis, for example maximal monotone operators, gradient flows and weak notions of degree. His papers contain gems with beautiful unexpected statements. His philosophy of action, which always starts with simple and easily understandable questions, has been adopted by many of his numerous students. Although a pure mathematician at heart, his mathematics has often been motivated by, or found its way back to, applications, for example to liquid crystals and to Ginzburg-Landau vortices in the theory of superconductivity.

Brezis is a fine lecturer and expositor. His beautiful book on Functional Analysis, Sobolev Spaces and Partial Differential Equations, first published in French in 1983 and then reprinted and expanded along the years and translated into eight different languages, has been used for forty years as a classical textbook in many universities worldwide.

The legacy of Haïm Brezis is measured not only by his work but also by that of his students and associates, many of whom have had, and continue to have, outstanding careers. He has supervised 58 PhD theses. In addition to his role as a teacher, leader and researcher, he has contributed greatly to the community through his many editorial roles and through influential posts such as Vice-President of the American Mathematical Society.
Biographical Note. Haïm Brezis was born in 1944 in Riom-es-Montagnes, a hamlet in the mountainous Auvergne region of France. His parents were Jewish refugees hiding under precarious conditions in the woods surrounding this hamlet. After WWII they settled in Paris, where Haïm received his entire education in various institutions of the celebrated Latin Quarter. He earned a Doctorate in 1971 from the Universite de Paris, under the supervision of G. Choquet and J.L. Lions.

In 1972 he was appointed at the Université Paris VI (Associate Professor 1972-1976, Full Professor 1976-2007, Emeritus since 2008).

In 1987 he accepted an offer from Rutgers as Distinguished Visiting Professor for several months every year; he held it until 2022 when he became Emeritus. He was also a regular visitor at the Technion (2008-2022).

Brezis is a member of Académie des Sciences, Paris. He is a foreign member of the American Academy of Arts and Sciences, the National Academy of Sciences, USA, and
several European national academies (Belgium, Italy, Romania, Spain).

He received Honorary degrees from various universities in Belgium, Greece, Israel, Italy, Netherlands, Romania, and Spain. He holds a Honorary Professorship from the Institute of Mathematics, Academia Sinica, Beijing, from Fudan University, and from Beijing Normal University.
Response from Haïm Brezis. I am delighted to have been awarded the 2024 Steele Prize for lifetime achievement and honored by the generous citation.

My encounter with Partial Differential Equations (PDEs) was accidental. During the 1960s French academia (perhaps still under the influence of Bourbaki) largely overlooked PDEs, with the notable exception of J.-L. Lions. Given my interest in Nonlinear Functional Analysis, my PhD advisor, Choquet, gave me papers by F. Browder to read. Some of them contained applications to PDEs that I did not yet understand, and so I taught myself basic PDEs. With Lions' support, I later deepened my understanding of the field under three leading experts who became my mentors and collaborators: Browder (Chicago), Nirenberg (NYU), and Stampacchia (Pisa).

Later, in the early 1970s, I witnessed in France a revolution: students were encouraged to learn PDEs because of their potential applications to many real-life problems. I received a position at the University of Paris where I taught PDEs to large groups of outstanding students (including from Ecole Normale Supérieure and Polytechnique). I had to generate open problems for my PhD students. Many of them and their descendants have become leaders in PDEs and adjacent fields. I was fortunate to work with brilliant collaborators to whom I am immensely grateful. Their list is much too long to be inserted in the limited space I have here.

Today, PDEs are thriving in France and worldwide; many new results and research directions have emerged, and some challenging open problems remain. Looking back, fifty years later, I am proud to have been part of this success story.

## Credits

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## AMS Updates

## On-Demand Webinar Features AMS MathViewer Accessibility

AMS journal readers may be familiar with MathViewer, the dual-panel reading platform that offers an interactive alternative to print or PDF. The platform's responsive design perfectly renders formulas, theorems, and references on all devices, while its layout makes navigating articles simple. MathViewer leverages these features to make mathematical content easier to use-especially for users of assistive technologies such as screen readers.

MathViewer manager and developer Peter Krautzberger recently created a webinar demonstrating various MathViewer capabilities, including assistive tools such as changing display colors, enabling speech subtitles, and activating speech synthesis. You can find the video on the AMS YouTube channel under the title "How MathViewer's dual-panel platform improves accessibility for all readers" or at the URLhttps://www. youtube.com/watch?v=P2y -4bGBwec.
-Tyler Kane, AMS

## 2024 JMM Hosts Graduate School Fair

More than seventy US graduate programs and the National Science Foundation exhibited at the 2024 JMM Graduate School Fair on January 5, 2024, at the Moscone Center in San Francisco. The fair was a great opportunity for undergraduates to learn about possibilities for graduate study and for master's students to learn more about opportunities for PhD study. Representatives of graduate programs in the mathematical sciences staffed tables and answered students' questions about their programs.

Registration for the 2024 Online Fall Graduate School Fair will open in late spring 2024. For more information,
visit https://www.ams.org/gradfair or email the AMS Programs staff at prof-serv@ams.org.
-AMS Programs Department

## Panova Awarded AMS Birman Fellowship

Greta Panova, Gabilan Distinguished Professor of Science and Engineering and professor of mathematics at the University of Southern California, has been awarded the 20242025 AMS Joan and Joseph Birman Fellowship for Women Scholars.

Panova's research is in algebraic combinatorics with connections to representation theory, computational complexity theory within theoretical computer science, and probability and statistical mechanics. Additionally, she works with a team of molecular biologists on modeling DNA repair dynamics.

The Joan and Joseph Birman Fellowship for Women Scholars is a midcareer research fellowship made possible by a generous gift from Joan and Joseph Birman. The fellowship seeks to address the paucity of women at the highest levels of research in mathematics by giving exceptionally talented women extra research support during their midcareer years. The award for the 2024-2025 academic year is $\$ 50,000$.
-AMS Communications

## Castella Earns AMS Centennial Fellowship

Francesc Castella, an associate professor of mathematics at the University of California, Santa Barbara, has been awarded the 2024-2025 AMS Centennial Research Fellowship for the 2024-2025 academic year. The primary selection criterion is the excellence of the candidate's research.

Castella's research interests are number theory and arithmetic geometry: more specifically, $p$-adic $L$-functions, Euler systems, and Iwasawa theory.

In 1973, the AMS established a Research Fellowship Fund that was renamed in 1988 to honor the AMS Centennial. Applicants for the fellowship must have held a doctoral degree for between three and twelve years and must currently serve in a tenured, tenure-track, postdoctoral, or comparable position (at the discretion of the selection committee) at a North American institution. The amount of the fellowship varies each year. For 2024-2025, the fellowship amount is $\$ 50,000$.
-AMS Communications

## Meier Named AMS Executive Director



The American Mathematical Society Board of Trustees is pleased to announce the appointment of John Meier, the provost and David M. and Linda Roth Professor of Mathematics of Lafayette College, as the new executive director of the AMS. He will begin a five-year term on July 1, 2024.

Meier will replace Catherine Roberts as executive director. Lucy R. Maddock, who has served as interim executive director since July 2023, will return to her duties as the AMS chief financial officer.
"We are delighted to welcome John Meier as the new AMS Executive Director," said AMS President Bryna Kra. "John has extensive experience as a high-level academic leader and as an engaged member of the mathematics community. His successes across these forums will be invaluable to the AMS's future efforts."

Meier, who received his PhD in mathematics from Cornell University in 1992, brings considerable experience in both mathematics and administration to the AMS. He joined Lafayette's faculty as an assistant professor in 1992, was promoted to associate professor in 1999, to professor in 2004, and was named Roth Professor in 2017. During sabbaticals, he held visiting appointments at the University of Pennsylvania; the Mathematical Sciences Research Institute (now Simons Laufer Mathematical Sciences Institute, or SLMath); University of California, Santa Barbara; Barnard College/Columbia University; The Ohio State University; Binghamton University; and Cornell.

Meier's service as provost began in 2019, a term which spanned the COVID-19 pandemic. Prior to this role he also held several other administrative positions at Lafayette.

A recipient of the AMS Centennial Research Fellowship (2003-2004), Meier has served the AMS on the Centennial Fellowship Committee and the Eastern Section Program Committee, of which he was chair in 2011. He also has served on several committees of the Mathematical Association of America.
"We are confident that John's strategic mindset, passion for mathematics, and track record of bringing people together will further build upon the strong foundation of the AMS and help us navigate future challenges," AMS Board of Trustees Chair Joseph Silverman said.

Meier's research is in geometric group theory, an area that sits in the liminal space between algebra, geometry, and topology. He has authored or coauthored three books and more than 50 articles, and has presented his work at sixteen special sessions of the AMS.
"I am excited to be moving from being a longstanding, active member of the AMS into the executive director role," Meier said. "I look forward to advancing the mission of the AMS: to support research in the mathematical sciences; to provide critical support to the mathematics community; and to increase the visibility and the understanding of the importance and beauty of mathematics."

## Deaths of AMS Members

Louis Brickman, of Albany, New York, died on December 2, 2023. Born on December 7, 1930, he was a member of the Society for 65 years.

Sigurdur Helgason, of Belmont, Massachusetts, died on December 3, 2023. Born on September 30, 1927, he was a member of the Society for 70 years.

Bent Fuglede, of Denmark, died on December 7, 2023. Born on October 8, 1925, he was a member of the Society for 69 years.
D. L. Johnson, of the United Kingdom, died on December 14, 2023. Born on June 10, 1943, he was a member of the Society for 52 years.

Lambertus A. Peletier, of the Netherlands, died on December 16, 2023. Born on March 29, 1937, he was a member of the Society for 52 years.

## Credits

Photo of John Meier is courtesy of Adam Atkinson, Lafayette College.

# Mathematics People 

## Kiltz, Viehmann Awarded Leibniz Prizes

Eike Kiltz, University Bochum, and Eva Viehmann, University of Münster, are among the ten winners of the Gottfried Wilhelm Leibniz Prizes. The prizes were to have been presented by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) on March 13, 2024, at the Berlin-Brandenburg Academy of Sciences and Humanities in Berlin.

Kiltz received the Leibniz Prize 2024 for his fundamental and pioneering work in the field of public key cryptography, which has had a lasting impact on theory and practice. He obtained his doctorate in mathematics from the Ruhr University Bochum in 2004, after which he spent a year as a postdoctoral researcher at the University of California, San Diego. He then moved to the Centrum Wiskunde \& Informatica in Amsterdam as a research assistant, before returning to the University Bochum in 2010. He now holds the Chair of Cryptography there and is one of the spokespersons for the Cluster of Excellence "Cyber Security in the Age of Large-Scale Adversaries (CASA)." Sources of funding for his research include an ERC Consolidator Grant (2013) and an ERC Advanced Grant (2021).

Viehmann received the Leibniz Prize for her influential work on arithmetic algebraic geometry in connection with the Langlands program. One of her strengths is the elaboration of group-theoretic formulations behind various structures, phenomena, and constructions. After obtaining her doctorate at the University of Bonn in 2005, Viehmann completed her postdoctoral lecturing qualification in Bonn (2010), following research stays in Chicago and Orsay, near Paris. Following a fellowship under the DFG's Heisenberg Programme, she took up a professorship at the Technical University of Munich in 2012. Since 2022, she has held a chair in arithmetic geometry and representation theory at the University of Münster, where she conducts research in the Cluster of Excellence "Mathematics Münster: Dynamics - Geometry - Structure." In 2012 she received the DFG's von Kaven Award. She has been
awarded an ERC Starting Grant (2011) and an ERC Consolidator Grant (2018).

The Gottfried Wilhelm Leibniz Prize has been awarded annually by the DFG since 1986. Up to ten prizes can be awarded per year, each endowed with prize money of $€ 2.5$ million. Including the ten prizes in 2024, a total of 418 Leibniz Prizes have been awarded to date. The winners each receive $€ 2.5$ million in prize money. They are entitled to use these funds for their research work in any way they wish, without bureaucratic obstacles, for up to seven years. The award ceremony for the Leibniz Prizes will be held in Berlin on March 13, 2024.
-Deutsche Forschungsgemeinschaft

## Tsimerman Receives 2023 <br> Ostrowski Prize

The Ostrowski Prize for 2023 is awarded to Jacob Tsimerman of the University of Toronto in recognition of his work at the interface of transcendence theory, analytic number theory, and arithmetic geometry, including recent breakthroughs on the André-Oort and Griffiths conjectures.

Tsimerman is a Canadian mathematician who received his doctorate from Princeton University in 2011 under the supervision of Peter Sarnak. He held a postdoctoral position at Harvard University as a Junior Fellow of the Harvard Society of Fellows. In July 2014 he was awarded a Sloan Fellowship and he started his term as assistant professor at the University of Toronto, where he is now a full professor.

The Ostrowski Foundation was created by Alexander M. Ostrowski, who was for many years a professor at the University of Basel. He left his entire estate to the foundation and stipulated that the income should provide a prize for outstanding achievements in mathematics. The prize is awarded every other year and is currently 100,000 Swiss
francs.
-Ostrowski Prize Citation

## 2023 CRM-Fields-PIMS Prize Awarded to Genest

The Centre de recherches mathématiques (CRM), the Fields Institute, and the Pacific Institute for the Mathematical Sciences (PIMS) have awarded the 2023 CRM-FieldsPIMS Prize to Christian Genest of McGill University.

Genest is one of the leading statisticians in Canada, whose work has had dual impact on both theory and realworld applications. He is best known for his contributions to multivariate analysis and was a pioneer in the expansive use of copula models in science. Together with a few close collaborators, he combined nonparametric methods and the asymptotic theory of empirical processes to design a broad array of rank-based inference tools for building, selecting, fitting, and validating stochastic models within this class. Additionally, Genest has also contributed to group decision-making, prioritization techniques, multivariate extreme-value theory and, most recently, to spacetime modeling of rare events in environmental science.

The CRM-Fields-PIMS Prize is the premier Canadian award for research achievements in the mathematical sciences. It is awarded jointly by the three largest Canadian mathematics institutes: the CRM in Montréal, the Fields Institute in Toronto, and the PIMS in Vancouver. This annual prize comes with a monetary award.
-Fields Institute

## CRM-ISM-AMQ Prize Awarded for 2023

The 2023 CRM-ISM-AMQ Prize is awarded to Ashay Burungale (University of Texas at Austin), Francesc Castella (University of California, Santa Barbara), Christopher Skinner (Princeton University) and Ye Tian (University of Chinese Academy of Sciences) for their article " $p^{\infty}$-Selmer groups and rational points on CM elliptic curve," published in the special issue of Annales Mathématiques du Québec (AMQ) in honor of Bernadette Perrin-Riou.

In recent years, an important breakthrough in the study of the Birch and Swinnerton-Dyer conjecture is the socalled $p$-converse theorem pioneered by Skinner. The BSD conjecture predicts that the algebraic and analytic ranks of an elliptic curve are equal. The $p$-converse theorem states that if the algebraic rank given by the Selmer group of an elliptic curve is 1 , then the analytic rank is 1 , which proves the BSD conjecture for a large family of elliptic curves.

The article studies a new method that generalizes previous results to a number of new settings. It is also the building block of further generalizations to totally real fields announced by the authors. This article can be an important stepping-stone for many further results in the study of the BSD conjecture.

The CRM-ISM-AMQ Prize is awarded annually for an outstanding publication in the AMQ . The prize was created in collaboration between the Centre de recherches mathématiques (CRM), the Institut des sciences mathématiques (ISM), and the AMQ.
-Centre des Recherches Mathématiques

## Nazaryan Awarded 2024 Emil Artin Junior Prize in Mathematics

Aram Nazaryan of Yerevan State University has been awarded the 2024 Emil Artin Junior Prize in Mathematics for his paper "Equilateral triangles have minimal area and perimeter among all triangles containing a given circle in Hilbert planes," Journal of Geometry 114 (2023), no. 3, Paper No. 25.

Established in 2001, the Emil Artin Junior Prize in Mathematics is awarded under the auspices of the Armenian Mathematical Union and carries a cash prize of US $\$ 1,400$. It is presented usually every year to a student or former student of an Armenian educational institution who is under the age of thirty-five, for outstanding contributions to algebra, geometry, topology, and number theory: the fields in which Artin made major contributions.
-AMS Communications

# Thomas Yizhao Hou won William Benter Prize in Applied Mathematics 2024 



Professor Thomas Yizhao Hou
Professor Thomas Yizhao Hou, Charles Lee Powell Professor of Applied and Computational Mathematics, California Institute of Technology, US, won the William Benter Prize in Applied Mathematics 2024.

Professor Hou, an outstanding applied mathematician with exceptional strengths in both numerics and analysis, has made pioneering and ground-breaking contributions in several areas of applied mathematics. For fluid interface problems, Professor Hou and collaborators developed the Small Scale Decomposition method which has many applications ranging from fluid dynamics to materials science and biology. The first level set method to study incompressible multiphase flows was developed by Chang, Hou, Merriman and Osher in 1996. The work has generated a significant impact in the computational fluid dynamics community.

The Multiscale Finite Element Method (MsFEM) developed by Hou and Wu in 1997 has generated a considerable impact in both the applied math and engineering communities. Some major oil companies have adopted a version of MsFEM in their next generation flow simulators. The Generalized Multiscale Finite Element Method (GMsFEM) developed by Efendiev, Galvis and Hou is another remarkable contribution. The GMsFEM has been used to derive macroscopic equations for a variety of applications and has found many applications in geoscience and materials science. It also provides a rigorous justification for the widely used multicontinuum theories in the engineering community.

Whether the 3D incompressible Euler equations can
develop a finite time singularity from smooth initial data is considered as one of the most challenging problems. Professor Hou and collaborators established a localized non-blowup criterion for 3D Euler equations, discovered and analyzed the surprising stabilizing effect of advection, and proved the existence of globally smooth solutions for the 3D Navier-Stokes equations with large smooth initial data of finite energy. In 2014, Lou and Hou discovered a new blowup scenario for the 3D axisymmetric Euler equations with boundary. They designed an extremely effective adaptive mesh strategy to achieve a remarkable level of resolution, and obtained strong numerical evidence of finite time singularity. Recently, Professor Hou and his former PhD student, Jiajie Chen, made a major breakthrough by providing a rigorous computer-assisted proof of the Hou-Luo blowup scenario. Their method is very powerful and it can be potentially used to study selfsimilar blowup of other nonlinear PDEs. Very recently, Professor Hou made another important breakthrough by discovering a new class of potentially singular solutions of the axisymmetric Navier-Stokes equations.

For his outstanding contributions in applied mathematics, Professor Hou has received many honors and awards. He was an ICM invited speaker in 1998 and a plenary speaker of ICIAM in 2003. He was elected to Fellow of American Academy of Arts and Sciences in 2011, an inaugural SIAM and AMS Fellow. He also co-founded the highly influential SIAM interdisciplinary Journal on Multiscale Modelling and Simulation Journal in 2002.

The William Benter Prize will be presented during the opening ceremony for the International Conference on Applied Mathematics (ICAM 2024), which is coorganized by the Liu Bie Ju Centre for Mathematical Sciences (LBJ) and the Department of Mathematics of City University of Hong Kong.

The William Benter Prize in Applied Mathematics was set up by LBJ in honour of Mr William Benter for his dedication and generous support to the enhancement of the University's strengths in mathematics. The prize recognizes outstanding mathematical contributions that have had a direct and fundamental impact on scientific, business, finance and engineering applications. The cash prize of US $\$ 100,000$ is given once every two years.

- City University of Hong Kong


# Classified Advertising Employment Opportunities 

## NORTH CAROLINA

## Sterling Montessori Lead Teachers

We are actively recruiting Lead Teachers and Teacher Assistants for the 2024-2025 academic year. Explore our school website for the latest job at
https://sterlingmontessori.org/index.php/page
-types/employment
and apply today.

[^59]DOI: https://doi.org/10.1090/noti2926

# New Books Offered by the AMS 

# Algebra and Algebraic Geometry 



## Applications of Group Theory in Cryptography <br> Post-quantum Group-based Cryptography

Delaram Kahrobaei, Queens College, The City University of New York, Flushing, NY, and University of York, United Kingdom, Ramón Flores, University of Seville, Spain, Marialaura Noce, University of Salerno, Italy, Maggie E. Habeeb, Pennsylvania Western University, California, PA, and Christopher
Battarbee, University of York, United Kingdom
This book is intended as a comprehensive treatment of group-based cryptography accessible to both mathematicians and computer scientists, with emphasis on the most recent developments in the area. To make it accessible to a broad range of readers, the authors started with a treatment of elementary topics in group theory, combinatorics, and complexity theory, as well as providing an overview of classical public-key cryptography. Then some algorithmic problems arising in group theory are presented, and cryptosystems based on these problems and their respective cryptanalyses are described. The book also provides an introduction to ideas in quantum cryptanalysis, especially with respect to the goal of post-quantum group-based cryptography as a candidate for quantum-resistant cryptography.

The final part of the book provides a description of various classes of groups and their suitability as platforms for group-based cryptography.

The book is a monograph addressed to graduate students and researchers in both mathematics and computer science.

[^60]Mathematical Surveys and Monographs, Volume 278
May 2024, approximately 142 pages, Softcover, ISBN: 978-1-4704-7469-0, LC 2023044735, 2020 Mathematics Subject Classification: 94A60, 81P94, 20F05, 20F99, 20F10, List US\$135, AMS members US\$108, MAA members US\$121.50, Order code SURV/278
bookstore.ams.org/surv-278

## Discrete Mathematics and Combinatorics



## Introduction to Quantum Algorithms

Johannes A. Buchmann, Technical University of Darmstadt, Germany

Quantum algorithms are among the most important, interesting, and promising innovations in information and communication technology. They pose a major threat to today's cybersecurity and at the same time promise great benefits by potentially solving previously intractable computational problems with reasonable effort. The theory of quantum algorithms is based on advanced concepts from computer science, mathematics, and physics.

Introduction to Quantum Algorithms offers a mathematically precise exploration of these concepts, accessible to those with a basic mathematical university education, while also catering to more experienced readers. This comprehensive book is suitable for self-study or as a textbook for one- or two-semester introductory courses on quantum computing algorithms. Instructors can tailor their approach to emphasize theoretical understanding and proofs or practical applications of quantum algorithms, depending on the course's goals and timeframe.

This item will also be of interest to those working in mathematical physics.

Pure and Applied Undergraduate Texts, Volume 64
April 2024, 371 pages, Softcover, ISBN: 978-1-4704-73983, 2020 Mathematics Subject Classification: 11-XX, 68-XX, 81-XX, 94-XX, List US\$89, AMS members US\$71.20, MAA members US\$80.10, Order code AMSTEXT/64
bookstore.ams.org/amstext-64

## Geometry and Topology



## Alexandrov Geometry Foundations

Stephanie Alexander, University of Illinois, Urbana-Champaign, IL, Vitali Kapovitch, University of Toronto, ON, Canada, and Anton Petrunin, Pennsylvania State University, College State, PA

Alexandrov spaces are defined via axioms similar to those of the Euclid axioms but where certain equalities are replaced with inequalities. Depending on the signs of the inequalities, we obtain Alexandrov spaces with curvature bounded above (CBA) and curvature bounded below (CBB). Even though the definitions of the two classes of spaces are similar, their properties and known applications are quite different.

The goal of this book is to give a comprehensive exposition of the structure theory of Alexandrov spaces with curvature bounded above and below. It includes all the basic material as well as selected topics inspired by considering Alexandrov spaces with CBA and with CBB simultaneously. The book also includes an extensive problem list with solutions indicated for every problem.

Graduate Studies in Mathematics, Volume 236
April 2024, approximately 277 pages, Hardcover, ISBN: 978-1-4704-7302-0, LC 2023038358, 2020 Mathematics Subject Classification: 53C23, 53C45, 30L99, List US\$135, AMS members US\$108, MAA members US\$121.50, Order code GSM/236
bookstore.ams.org/gsm-236

April 2024, Softcover, ISBN: 978-1-4704-7536-9, LC 2023038358, 2020 Mathematics Subject Classification: 53C23, 53C45, 30L99, List US\$89, AMS members US\$71.20, MAA members US\$80.10, Order code GSM/236.S

[^61]
# New in Contemporary Mathematics 

Algebra and<br>Algebraic Geometry



## LuCaNT: LMFDB, Computation, and Number Theory

John Cremona, University of Warwick, Coventry, UK, John Jones, Arizona State University, Tempe, AZ, Jennifer Paulhus, Grinnell College, IA, Andrew V. Sutherland, Massachusetts Institute of Technology, Cambridge, MA, and John Voight, Dartmouth College, Hanover, NH, Editors

This volume contains the proceedings of the LuCaNT (LMFDB, Computation, and Number Theory) conference held from July $10-14,2023$, at the Institute for Computational and Experimental Research in Mathematics (ICERM), Providence, Rhode Island and affiliated with Brown University.

This conference provided an opportunity for researchers, scholars, and practitioners to exchange ideas, share advances, and collaborate in the fields of computation, mathematical databases, number theory, and arithmetic geometry. The papers that appear in this volume record recent advances in these areas, with special focus on the LMFDB (the L-Functions and Modular Forms Database, http://1mfdb.org), an online resource for mathematical objects arising in the Langlands program and the connections between them.

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This item will also be of interest to those working in applications.

Contemporary Mathematics, Volume 796
April 2024, 373 pages, Softcover, ISBN: 978-1-4704-72603, LC 2023045755, 2020 Mathematics Subject Classification: 11-06, 14-06, 11-04, 11-11, List US\$80, AMS members US\$63.96, MAA members US\$71.96, Order code CONM/796

## Applications



## Mathematical Analyses of Decisions, Voting and Games

Michael A. Jones, Mathematical Reviews, American Mathematical Society, Ann Arbor, MI, David McCune, William-Jewell College, Liberty, MO, and Jennifer M. Wilson, Eugene Lange College, The New School, New York, NY, Editors

This volume contains the proceedings of the virtual AMS Special Session on Mathematics of Decisions, Elections and Games, held on April 8, 2022.

Decision theory, voting theory, and game theory are three related areas of mathematics that involve making optimal decisions in different contexts. While these three areas are distinct, much of the recent research in these fields borrows techniques from other branches of mathematics such as algebra, combinatorics, convex geometry, logic, representation theory, etc. The papers in this volume demonstrate how the mathematics of decisions, elections, and games can be used to analyze problems from the social sciences.

Contemporary Mathematics, Volume 795
April 2024, 198 pages, Softcover, ISBN: 978-1-4704-69788, LC 2023042077, 2020 Mathematics Subject Classification: $91-06,90 \mathrm{~B} 06,91 \mathrm{~A} 05,91 \mathrm{~A} 18,91 \mathrm{~A} 46,91 \mathrm{~B} 06,91 \mathrm{~B} 12,91 \mathrm{~B} 14$, 91B32, List US\$135, AMS members US\$108, MAA members US\$121.50, Order code CONM/795
bookstore.ams.org/conm-795

## New in Memoirs of the AMS

## Algebra and Algebraic Geometry

## Semi-Infinite HighestWeight Categories

Jonathan Brundan, University of Oregon, Eugene, OR, and Catharina Stroppel, University of Bonn, Germany

Memoirs of the American Mathematical Society, Volume 293, Number 1459

February 2024, 152 pages, Softcover, ISBN: 978-1-4704-6783-8, 2020 Mathematics Subject Classification: 16G10, 16-02, 17B10, 08C20, 20G05, 20G42, 17B67, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1459
bookstore.ams.org/memo-293-1459

## p-DG Cyclotomic nilHecke Algebras

Mikhail Khovanov, Columbia University, New York, NY, You Qi, University of Virginia, Charlottesville, VA, and Joshua Sussan, The City University of New York (CUNY) Medgar Evers, Brooklyn, NY

Memoirs of the American Mathematical Society, Volume 293, Number 1462
February 2024, 102 pages, Softcover, ISBN: 978-1-4704-6871-2, 2020 Mathematics Subject Classification: 16E35; 16E45, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1462
bookstore.ams.org/memo-293-1462

## p-DG Cyclotomic nilHecke Algebras II

You Qi, University of Virginia, Charlottesville, VA, and Joshua Sussan, The City University of New York (CUNY) Medgar Evers, Brooklyn, NY

Memoirs of the American Mathematical Society, Volume 293, Number 1463
February 2024, 106 pages, Softcover, ISBN: 978-1-4704-6872-9, 2020 Mathematics Subject Classification: 16E35; 16E45, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1463
bookstore.ams.org/memo-293-1463

## Analysis

## Classification of $\mathcal{O}_{\infty}$-Stable $C^{*}$-Algebras <br> James Gabe, University of Southern Denmark, Odense, Denmark

Memoirs of the American Mathematical Society, Volume 293, Number 1461
February 2024, 115 pages, Softcover, ISBN: 978-1-4704-6793-7, 2020 Mathematics Subject Classification: 46L35, 46L80, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1461
bookstore.ams.org/memo-293-1461

# Discrete Mathematics and Combinatorics 

Milliken'sTree Theorem and Its Applications:<br>A Computability-Theoretic Perspective

Paul-Elliot Anglès D'Auriac, Université Claude Bernard Lyon 1, France, Peter A. Cholak, University of Notre Dame, IN, Damir D. Dzhafarov, University of Connecticut, Storrs, CT, Benoît Monin, Laboratoire d'Algorithmique, Complexité et Logique (LACL), Paris, France, and Ludovic Patey, Université Claude Bernard Lyon 1, France

Memoirs of the American Mathematical Society, Volume 293, Number 1457
February 2024, 118 pages, Softcover, ISBN: 978-1-4704-6731-9, 2020 Mathematics Subject Classification: 05D10, 03D80, 03E05; 05C55, 05D05, 03E75, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1457
bookstore.ams.org/memo-293-1457
Mathematical Physics
Angled Crested Like Water Waves with SurfaceTension II: Zero Surface Tension Limit
Siddhant Agrawal, Instituto de Ciencias Matemáticas (ICMAT), Madrid, Spain

This item will also be of interest to those working in differential equations.

Memoirs of the American Mathematical Society, Volume 293, Number 1458
February 2024, 124 pages, Softcover, ISBN: 978-1-4704-6738-8, 2020 Mathematics Subject Classification: 35Q35, 76B15, 76B45, List US $\$ 85$, AMS members US $\$ 68$, MAA members US\$76.50, Order code MEMO/293/1458
bookstore.ams.org/memo-293-1458

## Probability and Statistics

Empirical Measures, Geodesic Lengths, and a Variational Formula in First-Passage Percolation Erik Bates, North Carolina State University, Raleigh, NC

Memoirs of the American Mathematical Society, Volume 293, Number 1460

February 2024, 92 pages, Softcover, ISBN: 978-1-4704-6791-3, 2020 Mathematics Subject Classification: 60K35, 60K37, 60J80, 82B43, List US\$85, AMS members US\$68, MAA members US\$76.50, Order code MEMO/293/1460
bookstore.ams.org/memo-293-1460

## New AMS-Distributed Publications

## Algebra and

 Algebraic Geometry

## Curves Over Finite Fields: Past, Present and Future

Edited by: Alp Bassa, Boğaziçi University, Istanbul, Turkey, Elisa Lorenzo Garcia, Université de Neuchâtel, Suisse, Neuchâtel, Switzerland, and Christophe Ritzenthaler, Université de Rennes, Rennes, France

These proceedings contain the plenary lectures that were presented during the online conference Curves Over Finite Fields: Past, Present and Future in May 2021. Each lecturer was asked to do a survey on a particular aspect of this research area and to point out some compelling open questions.

The six lectures gathered here show the richness and diversity of the results in the domain: Isogeny classes of abelian varieties, large automorphism groups, recursive towers, er-ror-correcting codes, moduli spaces, and arithmetic statistics are gates to this vivid area of research.

This item will also be of interest to those working in number theory.
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30\% discount from list.

Panoramas et Synthèses, Number 60 January 2024, 171 pages, Softcover, ISBN: 978-2-85629-9807, 2020 Mathematics Subject Classification: 15B52, 28A80, 60K35, 81T40, 82B20, 82B27, 82B40, 82B44, 82C05, 82D60, List US\$65, AMS members US\$52, Order code PASY/60
bookstore.ams.org/pasy-60

## Math Education



Twenty-one Articles
for Mathematics Competitions and Enrichment
Titu Andreescu, University of Texas at Dallas, Richardson, TX

This is a collection of articles written by the coordinating author and many of his dear friends. These articles were arranged in this book so that math lovers and anyone interested in expanding their mathematical horizons can benefit from it. Articles range from intermediate to advanced. They delve into several branches of mathematics, such as algebra, geometry, combinatorics, and number theory-precisely those realms of the queen of sciences that are subject to the International Mathematical Olympiad and also mathematical analysis.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

## XYZ Series, Volume 51

December 2023, 184 pages, Hardcover, ISBN: 979-8-9890528-1-3, 2020 Mathematics Subject Classification: 00A05, 00A07, 97U40, 97D50, List US\$59.95, AMS members US\$47.96, Order code XYZ/51
bookstore.ams.org/xyz-51
 feature problems involving angles, some specifically asking students the measures of angles in triangles or quadrilaterals. Book 1 responds to the need of those who would like to improve their knowledge and skills in geometric problems in which angles play an important role.

Book 1 is divided into three chapters that are designed to engage readers in the study of angles. The first chapter
begins by presenting essential concepts concerning angles in the plane and provides a multitude of applications. The second chapter shifts focus to other classical results in Euclidean geometry. It delves into the tools that use angles as their primary elements, unveiling the power of angles in problems related to triangle similarity, quadrilaterals, and circles. This awesome journey culminates with the third chapter where a collection of thought-provoking problems are being offered. These problems draw upon the insights and skills acquired in previous chapters, inviting the reader to explore angles in novel ways. All solutions to the proposed problems in the book are included at the end.

Readers will encounter relevant examples that illustrate key concepts and principles, which will help them grasp the material and solve the proposed problems around the book. Numerous significant examples and problems have been carefully selected from recent mathematical Olympiads, providing the readers with the opportunity to put their knowledge to test.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

XYZ Series, Volume 50
December 2023, 231 pages, Softcover, ISBN: 979-8-9890528-2-0, 2020 Mathematics Subject Classification: 00A05, 00A07, 97U40, 97D50, List US\$59.95, AMS members US\$47.96, Order code XYZ/50
bookstore.ams.org/xyz-50


Introduction to
Number Theory in Mathematics Contests Book 1
Titu Andreescu, University of Texas at Dallas, Richardson, TX, and Marian Tetiva, Gheorghe Rosca Codreanu National College, Barlad, Romania

Introduction to Number Theory in Mathematics Contests is a project divided into three volumes: Books 1, 2, and 3. This is Book 1 of the three-part series and contains an introductory part and basic concepts of the division theorem, divisibility, and congruences. Readers are introduced to a short exposition of the fundamentals on numbers. This book assumes that the audience is already familiar with some fundamental concepts such as sets, functions in general, and particular functions such as exponential and logarithmic, polynomials, or algebraic equations.

The authors' main goal is to establish a strong connection with the readers in the hopes of increasing their understanding of number theory and inspiring them to discover the beauty of number theory.

Several topics are discussed, such as the division theorem, divisibility, congruences, prime numbers and the unique factorization theorem (the fundamental theorem of arithmetic), which are the most important tools for understanding more advanced subjects in number theory. Different concepts are gradually explained, in their natural order, making the exposition self contained. Topics presented have several examples, theorems, and lots of problems to complete each chapter. Most of the problems have full solutions (in many cases more than one), but the authors strongly advise the reader to try solving each problem independently before reading the solution provided.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

XYZ Series, Volume 49
October 2023, 271 pages, Softcover, ISBN: 979-8-9890528-0-6, 2020 Mathematics Subject Classification: 00A05, 00A07, 97U40, 97D50, List US\$59.95, AMS members US\$47.96, Order code XYZ/49
bookstore.ams.org/xyz-49

NEW FROM THE

## EII EUROPEAN MATHEMATICAL SOCIETY



## An Introduction to the Mathematical Fluid Dynamics of Oceanic and Atmospheric Flows

Robin S. Johnson, Newcastle University, UK

The study of the movement of the atmosphere and the oceans is intriguing, challenging, and important, particularly in the context of current concerns about the climate. The familiar and tested approach to these problems is based on the construction of model equations tailored to address specific flow scenarios. In this book, the author presents a single, overarching approach which uses the thin-shell approximation -and nothing more -applied to the general equations of fluid dynamics.
ESI Lectures in Mathematics and Physics, Volume 11; 2023; 176 pages; Softcover; ISBN: 978-3-98547-029-7; List US\$55; Individual member US\$44; Order code EMSESILEC/11

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the American Mathematical Society.

# Meetings \& Conferences of the AMS April Table of Contents 

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www. ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https:// www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams .org/cgi-bin/abstracts/abstract.p7. Questions about abstracts may be sent to abs-info@ams. org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

## Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of WisconsinMadison, 480 Lincoln Drive, Madison, WI 53706; email: stova11@math.wisc.edu; telephone: (608) 262-2933.
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Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: mamanes@hawai i . edu; telephone: (808) 956-4679.

## Meetings in this Issue

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- 2026

January 4-7 Washington, DC
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The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment athttps://www.ams org/welcoming-environment-policy.

## Meetings \& Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www. ams .org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.
New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

## Tallahassee, Florida

## Florida State University

March 23-24, 2024
Saturday - Sunday

## Meeting \#1193

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced Issue of Abstracts: Volume 45, Issue 2

## Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm7.

## Invited Addresses

Wenjing Liao, Georgia Institute of Technology, Exploiting low-dimensional data structures in deep learning.
Olivia Prosper, University of Tennessee, Knoxville, Modeling Malaria at Multiple Scales.
Jared Speck, Vanderbilt University, Singularity Formation for the Equations of Einstein and Euler.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.p1.

Advanced Numerical Methods for Partial Differential Equations and Their Applications, Seonghee Jeong, Louisiana State University, Sanghyun Lee, Florida State University, and Seulip Lee, University of Georgia.

Advances in Financial Mathematics, Qi Feng, Alec N Kercheval, and Lingjiong Zhu, Florida State University.
Advances in Shape and Topological Data Analysis, Emmanuel L Hartman, Eric Klassen, and Ethan Semrad, Florida State University.

## MEETINGS \& CONFERENCES

Algebraic Groups and Local-Global Principles, Suresh Venapally, Emory University, and Daniel Reuben Krashen, University of Pennsylvania.

Bases and Frames in Hilbert Spaces, Laura De Carli, Florida International University, and Azita Mayeli, City University of New York.

Combinatorics in Geometry of Polynomials, Papri Dey, Georgia Institute of Technology.
Control, Inverse Problems and Long Time Dynamics of Evolution Systems, Shitao Liu, Clemson University, and Louis Roder Tcheugoue Tebou, Florida International University.

Data Integration and Identifiability in Ecological and Epidemiological Models, Omar Saucedo, Virginia Tech, and Olivia Prosper, University of Tennessee/Knoxville.

Diversity in Mathematical Biology, Daniel Alejandro Cruz and Skylar Grey, University of Florida.
Fluids: Analysis, Applications, and Beyond, Aseel Farhat and Anuj Kumar, Florida State University.
Geometric Measure Theory and Partial Differential Equations, Alexander B. Reznikov, John Locke Hoffman, and Richard Oberlin, Florida State University.

Geometry and Symmetry in Data Science, Dustin G. Mixon, The Ohio State University, and Thomas Needham, Florida State University.

Homotopy Theory and Category Theory in Interaction, Ettore Aldrovandi and Brandon Carl Leigh Doherty, Florida State University, and Philip John Hackney, University of Louisiana at Lafayette.

Mathematical Advances in Scientific Machine Learning, Wenjing Liao, Georgia Institute of Technology, and Feng Bao and Zecheng Zhang, Florida State University.

Mathematical Modeling and Simulation in Fluid Dynamics, Pejman Sanaei, Georgia State University.
Mathematical Models for Population and Methods for Parameter Estimation in Epidemiology, Yang LI, Georgia State University, and Guihong Fan, Columbus State University.

Moduli Spaces in Algebraic Geometry, Jeremy Usatine, Florida State University, Hulya Arguz and Pierrick Bousseau, University of Georgia, and Matthew Satriano, University of Waterloo.

Nonlinear Evolution Partial Differential Equations in Physics and Geometry, Jared Speck and Leonardo Enrique Abbrescia, Vanderbilt University.

Numerical Methods and Deep Learning for PDEs, Chunmei Wang, University of Florida, and Haizhao Yang, University of Maryland College Park.

PDEs in Incompressible Fluid Mechanics, Wojciech S. Ozanski, Florida State University, Stanley Palasek, UCLA, and Alexis F Vasseur, The University of Texas At Austin.

Recent Advances in Geometry and Topology, Thang Nguyen, Samuel Aaron Ballas, Philip L. Bowers, and Sergio Fenley, Florida State University.

Recent Advances in Inverse Problems for Partial Differential Equations and Their Applications, Anh-Khoa Vo, Florida A\&M University, and Thuy T. Le, UNC Charlotte.

Recent Development in Deterministic and Stochastic PDEs, Quyuan Lin, Clemson University, and Xin Liu, Texas A\&M University.

Recent Developments in Numerical Methods for Evolution Partial Differential Equations, Thi-Thao-Phuong Hoang, Yanzhao Cao, and Hans-Werner Van Wyk, Auburn University.

Regularity Theory and Free Boundary Problems, Lei Zhang, University of Florida, and Eduardo V. Teixeira, University of Central Florida.

Stochastic Analysis and Applications, Hakima Bessaih, Florida International University, and Oussama Landoulsi, University of Massachusetts, Amherst.

Stochastic Differential Equations: Modeling, Estimation, and Applications, Sher B Chhetri, University of South Carolina Sumter, Hongwei Long, Florida Atlantic University, and Olusegun M. Otunuga, Augusta University.

Theory of Nonlinear Waves, Nicholas James Ossi and Ziad H Musslimani, Florida State University.
Topics in Graph Theory, Songling Shan, Auburn University, and Guantao Chen, Georgia State University.
Topics in Stochastic Analysis/Rough Paths/SPDE and Applications in Machine Learning, Cheng Ouyang, University of Illinois At Chicago, Fabrice Baudoin, University of Connecticut, and Qi Feng, Florida State University.

Topological Algorithms for Complex Data and Biology, Henry Hugh Adams, Johnathan Bush, and Hubert Wagner, University of Florida.

Topological Interactions of Contact and Symplectic Manifolds, Angela Wu, University College of London and Louisiana State University, and Austin Christian, Georgia Institute of Technology.

## Contributed Paper Sessions

AMS Contributed Paper Session, Brian D. Boe, University of Georgia.

## Washington, District of Columbia

## Howard University

April 6-7, 2024
Saturday - Sunday

## Meeting \#1194

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced Issue of Abstracts: Volume 45, Issue 2

## Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Ryan Charles Hynd, University of Pennsylvania, Extremals of Morrey's Inequality.
Jinyoung Park, Courant Institute of Mathematical Sciences, NYU, Threshold phenomena for random discrete structures.
Jian Song, Rutgers, State University of New Jersey, Geometric Analysis on Singular Complex Spaces.
Talitha M Washington, Clark Atlanta University \& Atlanta University Center, The Data Revolution (Einstein Public Lecture in Mathematics).

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.p1.

Advanced Mathematical Methods in Naval Engineering Research, Michael Traweek, Office of Naval Research, and Anthony Ruffa, Emeritus Naval Undersea Warfare Center.

Algebraic and Enumerative Combinatorics, Samuel Francis Hopkins, Howard University, Joel Brewster Lewis, George Washington University, and Peter R. W McNamara, Bucknell University.

Analysis of PDE in Inverse Problems and Control Theory, Matthias Eller, Georgetown University, and Justin Thomas Webster, University of Maryland, Baltimore County.

Artificial Intelligence Emergent From Mathematics and Physics, Bourama Toni, Howard University, and Artan Sheshmani, MIT IAiFi.

Automorphic Forms and Langlands Program, Baiying Liu and Freydoon Shahidi, Purdue University.
Automorphic Forms and Trace Formulae, Yiannis Sakellaridis, Johns Hopkins University, Bao Chau Ngo, University of Chicago, and Spencer Leslie, Boston College.

Coding Theory \& Applications, Emily McMillon, Eduardo Camps, and Hiram H. Lopez, Virginia Tech.
Commutative Algebra and its Applications, Hugh Geller, West Virginia University, and Rebecca R.G., George Mason University.

Complex Systems in the Life Sciences, Zhisheng Shuai, University of Central Florida, Junping Shi, College of William \& Mary, and Seoyun Choe, University of Central Florida.

Computability, Complexity, and Algebraic Structure, Valentina S Harizanov, George Washington University, Keshav Srinivasan, The George Washington University, and Philip White and Henry Klatt, George Washington University.

Computational and Machine Learning Methods for Modeling Biological Systems, Christopher Kim, Vipul Periwal, Manu Aggarwal, and Xiaoyu Duan, National Institutes of Health.

Control of Partial Differential Equations, Gisele Adelie Mophou, Universite des Antilles en Guadeloupe, and Mahamadi Warma, George Mason University.

Culturally Responsive Mathematical Education in Minority Serving Institutions, Lucretia Glover, Lifoma Salaam, and Julie Lang, Howard University.

Elementary Number Theory and Elliptic Curves, Sankar Sitaraman and Francois Ramaroson, Howard University.

## MEETINGS \& CONFERENCES

Fresh Researchers in Algebra, Combinatorics, and Topology (FRACTals), Dwight Anderson Williams II, Morgan State University, and Saber Ahmed, Hamilton College.

GranvilleFest 100: A Celebration of the Legacy of Evelyn Boyd Granville, Edray Herber Goins, Pomona College, Torina D. Lewis, National Association of Mathematicians, and Talitha M Washington, Clark Atlanta University \& Atlanta University Center.

Interactions Between Analysis, Geometric Measure Theory, and Probability in Non-Smooth Spaces, Luca Capogna, Smith College, Jeremy Tyson, University of Illinois at Urbana-Champaign, and Nageswari Shanmugalingam, University of Cincinnati.

Mathematical Modeling, Computation, and Data Analysis in Biological and Biomedical Applications, Maria G Emelianenko and Daniel M Anderson, George Mason University.

Mathematics of Infectious Diseases: A Session in Memory of Dr. Abdul-Aziz Yakubu, Abba Gumel, University of Maryland, Daniel Brendan Cooney, University of Illinois Urbana-Champai, and Chadi M Saad-Roy, University of California, Berkeley.

Modeling and Numerical Methods for Complex Dynamical Systems in Biology, Hye-Won Kang, University of Maryland, Baltimore County, and Bradford E. Peercy, Department of Mathematics and Statistics, University of Maryland, Baltimore County.

New Trends in Mathematical Physics, W. A. Zuniga-Galindo, University of Texas Rio Grande Valley, and Tristan Hubsch, Howard University.

Nonlinear Hamiltonian PDEs, Benjamin Harrop-Griffiths, Georgetown University, and Maria Ntekoume, Concordia University.

Optimization, Machine Learning, and Digital Twins, Harbir Antil, Rohit Khandelwal, and Sean Patrick Carney, George Mason University.

Permutation Patterns, Juan B Gil, Penn State Altoona, and Alexander I. Burstein, Howard University.
Post-Quantum Cryptography, Jason LeGrow, Virginia Tech, Veronika Kuchta, Florida Atlantic University, Travis Morrison, Virginia Tech, and Edoardo Persichetti, Florida Atlantic University.

Qualitative Dynamics in Finite and Infinite Dynamical Systems, Roberto De Leo, Howard University, and Jim A Yorke, University of Maryland.

Recent Advances in Harmonic Analysis and Their Applications to Partial Differential Equations, Guher Camliyurt and Jose Ramon Madrid Padilla, Virginia Polytechnic Institute and State University.

Recent Advances in Optimal Transport and Applications, Henok Mawi, Howard University (Washington, DC, US), and Farhan Abedin, Lafayette College.

Recent Advances on Machine Learning Methods for Forward and Inverse Problems, Haizhao Yang, University of Maryland College Park, and Ke Chen, University of Maryland, College Park.

Recent Developments in Geometric Analysis, Yueh-Ju Lin, Wichita State University, Samuel Perez-Ayala, Princeton University, and Ayush Khaitan, Rutgers University.

Recent Developments in Noncommutative Algebra and Tensor Categories, Kent B. Vashaw, Massachusetts Institute of Technology, Van C. Nguyen, U.S. Naval Academy, Xingting Wang, Louisiana State University, and Robert Won, George Washington University.

Recent Developments in Nonlinear and Computational Dynamics, Emmanuel Fleurantin and Christopher K. R. T. Jones, University of North Carolina.

Recent Developments in the Study of Free Boundary Problems in Fluid Mechanics, Huy Q. Nguyen, University of Maryland, and Ian Tice, Carnegie Mellon University.

Recent Progress on Model-Based and Data-Driven Methods in Inverse Problems and Imaging, Yimin Zhong, Auburn University, Yang Yang, Michigan State University, and Junshan Lin, Auburn University.

Recent Trends in Graph Theory, Katherine Perry, Soka University of America, and Adam Blumenthal, Westminster College.
Riordan Arrays, Dennis Davenport and Lou Shapiro, Howard University, and Leon Woodson, SPIRAL REU At Georgetown.

Skein Modules in Low Dimensional Topology, Jozef Henryk Przytycki, George Washington University.
Spectral Theory and Quantum Systems, Laura Shou, University of Maryland, and Shiwen Zhang, U Mass Lowell.
Stochastic Methods in Fluid Mechanics, Hussain Ibdah, Univeristy of Maryland, Theodore D. Drivas, S, and Kyle Liss, Duke University.

Tensor Algebra \& Networks, Giuseppe Cotardo, Gretchen Matthews, and Pedro Soto, Virginia Tech.
Variational Problems with Lack of Compactness, Cheikh Birahim Ndiaye, Howard University, and Ali Maalaoui, Clark University.

## Contributed Paper Sessions

AMS Contributed Paper Session, Steven H Weintraub, Lehigh University.

## Milwaukee, Wisconsin

## University of Wisconsin-Milwaukee

## April 20-21,2024

Saturday - Sunday
Meeting \#1195
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced Issue of Abstracts: Volume 45, Issue 2

## Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Mihaela Ifrim, University of Wisconsin-Madison, The small data global well-posedness conjecture for 1D defocusing dispersive flows.

Lin Lin, University of California, Berkeley, Linear combination of Hamiltonian simulation.
Kevin Schreve, Louisiana State University, Homological growth of groups and aspherical manifolds.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.p1.

Algebraic methods in graph theory and applications I, Tung T. Nguyen, Western University, Sunil K. Chebolu, Illinois State University, and Jan Minac, Western University.

Algorithms, Number Theory, and Cryptography I, Jonathan P Sorenson, Butler University, Eric Bach, University of Wisconsin at Madison, and Jonathan Webster, Butler University.

Applications of Algebra and Geometry I, Thomas Yahl, University of Wisconsin - Madison, and Jose Israel Rodriguez, University of Wisconsin Madison.

Applications of Numerical Algebraic Geometry I, Emma R Cobian, University of Notre Dame.
Artificial Intelligence in Mathematics I, Tony Shaska, University of Michigan, Alessandro Arsie, The University of Toledo, Elira Curri, Oakland University, Rochester Hills, MI, 48126, and Mee Seong Im, United States Naval Academy.

Automorphisms of Riemann Surfaces and Related Topics I, Aaron D. Wootton, University of Portland, Jennifer Paulhus, Grinnell College, Sean Allen Broughton, Rose-Hulman Institute of Technology (emeritus), and Tony Shaska, University of Michigan.

Cluster algebras, Hall algebras and representation theory I, Xueqing Chen, University of Wisconsin-Whitewater, and Yiqiang Li, SUNY At Buffalo.

Combinatorial and geometric themes in representation theory I, Jeb F. Willenbring, UW-Milwaukee, and Pamela E. Harris, University of Wisconsin, Milwaukee.

Complex Dynamics and Related Areas I, James Waterman, Stony Brook University, and Alastair N Fletcher, Northern Illinois University.

Computability Theory I, Matthew Harrison-Trainor, University of Illinois Chicago, and Steffen Lempp, University of Wisconsin-Madison.

Connections between Commutative Algebra and Algebraic Combinatorics I, Alessandra Costantini, Oklahoma State University, Matthew James Weaver, University of Notre Dame, and Alexander T Yong, University of Illinois at Urbana-Champaign.

Developments in hyperbolic-like geometry and dynamics I, Jonah Gaster, University of Wisconsin-Milwaukee, Andrew Zimmer, University of Wisconsin-Madison, and Chenxi Wu, University of Wisconsin At Madison.

Geometric group theory I, G Christopher Hruska, University of Wisconsin-Milwaukee, and Emily Stark, Wesleyan University.

## MEETINGS \& CONFERENCES

Geometric Methods in Representation Theory I, Daniele Rosso, Indiana University Northwest, and Joshua Mundinger, University of Wisconsin - Madison.

Harmonic Analysis and Incidence Geometry I, Sarah E Tammen and Terence L. J Harris, UW Madison, and Shengwen Gan, University of Wisconsin - Madison.

Mathematical aspects of cryptography and cybersecurity I, Lubjana Beshaj, Army Cyber Institute.
Model Theory I, Uri Andrews, University of Wisconsin-Madison, and James Freitag, University of Illinois Chicago.
New research and open problems in combinatorics I, Pamela Estephania Harris, University of Wisconsin, Milwaukee, Erik Insko, Central College, and Mohamed Omar, York University.

Nonlinear waves I, Mihaela Ifrim, University of Wisconsin-Madison, and Daniel I Tataru, UC Berkeley.
Panorama of Holomorphic Dynamics I, Suzanne Lynch Boyd, University of Wisconsin Milwaukee, and Rodrigo A. Perez and Roland Roeder, Indiana University - Purdue University Indianapolis.

Posets in algebraic and geometric combinatorics I, Martha Yip, University of Kentucky, and Rafael S. González D'León, Loyola University Chicago.

Ramification in Algebraic and Arithmetic Geometry I, Charlotte Ure, Illinois State University, and Nick Rekuski, Wayne State University.

Recent Advances in Nonlinear PDEs and Their Applications I, Xiang Wan, Loyola University Chicago, Rasika Mahawattege, University of Maryland, Baltimore County, and Madhumita Roy, Graduate Student, University of Memphis.

Recent Advances in Numerical PDE Solvers by Deep Learning I, Dexuan Xie, University of Wisconsin-Milwaukee, and Zhen Chao, University of Michigan-Ann Arbor.

Recent Developments in Harmonic Analysis I, Naga Manasa Vempati, Louisiana State University, Nathan A. Wagner, Brown University, and Bingyang Hu, Auburn University.

Recent trends in nonlinear PDE I, Fernando Charro and Catherine Lebiedzik, Wayne State University, and Md Nurul Raihen, Fontbonne University.

Stochastic Control and Related Fields: A Special Session in Honor of Professor Stockbridge's 70th Birthday I, Chao Zhu, University of Wisconsin-Milwaukee, and MoonJung Cho, U.S. Bureau of Labor Statistics.

The Algebras and Special Functions around Association Schemes I, Paul M Terwilliger, U. Wisconsin-Madison, Sarah R Bockting-Conrad, DePaul University, and Jae-Ho Lee, University of North Florida.

## San Francisco, California

## San Francisco State University

May 4-5, 2024
Saturday - Sunday

## Meeting \#1196

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced Issue of Abstracts: Volume 45, Issue 3

Deadlines
For organizers: Expired
For abstracts: March 12, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm7.

## Invited Addresses

Julia Yael Plavnik, Indiana University, Title to be announced.
Mandi A. Schaeffer Fry, University of Denver, Counting with blocks and hide-and-seek with character tables: Brauer's problems and beyond.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Commutative and Noncommutative Algebra, Together at Last, Pablo S. Ocal, University of California, Los Angeles, Benjamin Briggs, University of Copenhagen, and Janina C Letz, Bielefeld University.

Diagrammatic Algebras in Representation Theory and Beyond, Mee Seong Im, United States Naval Academy, Liron Speyer, Okinawa Institute of Science and Technology, Arik Wilbert, University of Georgia, and Jieru Zhu, University of Queensland. Extremal Combinatorics and Connections, Sam Spiro, Rutgers University, and Van Magnan, University of Montana.
Geometry and Topology of Quantum Phases of Matter, Ralph Martin Kaufmann, Purdue University, and Markus J Pflaum, University of Colorado.

Geometry, Integrability, Symmetry and Physics, Birgit Kaufmann and Sasha Tsymbaliuk, Purdue University.
Groups and Representations (associated with Invited Address by Mandi Schaeffer Fry), Nathaniel Thiem, University of Colorado, Mandi A. Schaeffer Fry, University of Denver, and Klaus Lux, University of Arizona.

Homological Methods in Commutative Algebra \& Algebraic Geometry, Ritvik Ramkumar, Cornell University, Michael Perlman, University of Minnesota, and Aleksandra C Sobieska, University of Wisconsin - Madison.

Inverse Problems, Hanna E. Makaruk, Los Alamos National Laboratory, Los Alamos, NM, and Robert M. Owczarek, University of New Mexico.

Mathematical Fluid Dynamics, Igor Kukavica and Juhi Jang, University of Southern California, and Wojciech S. Ozanski, Florida State University.

Mathematical Modeling of Complex Ecological and Social Systems, Daniel Brendan Cooney, University of Illinois Urba-na-Champaign, Mari Kawakatsu, University of Pennsylvania, and Chadi M Saad-Roy, University of California, Berkeley.

Partial Differential Equations and Convexity, Ben Weinkove, Northwestern University, Stefan Steinerberger, University of Washington, Seattle, and Albert Chau, University of British Columbia.

Partial Differential Equations of Quantum Physics, Israel Michael Sigal, University of Toronto, and Stephen Gustafson, University of British Columbia.

Probability Theory and Related Fields, Terry Soo and Codina Cotar, University College London.
Random Structures, Computation, and Statistical Inference, Lutz Warnke, University of California, San Diego, and Ilias Zadik, Yale University.

Recent Advances in Differential Geometry, Lihan Wang, California State University, Long Beach, Zhiqin Lu, UC Irvine, and Shoo Seto and Bogdan D. Suceavǎ, California State University, Fullerton.

Recent Developments in Commutative Algebra, Arvind Kumar, Louiza Fouli, and Michael Robert DiPasquale, New Mexico State University.

Representations of Lie Algebras and Lie Superalgebras, Dimitar Grantcharov, University of Texas At Arlington, Daniel Nakano, University of Georgia, and Vera Serganova, UC Berkeley.

Research in Combinatorics by Early Career Mathematicians, Nicholas Mayers, North Carolina State University, and Laura Colmenarejo, NCSU.

Special Session in Celebration of Bruce Reznick's Retirement, Katie Anders, University of Texas at Tyler, Simone Sisner-os-Thiry, California State University- East Bay, and Dana Neidmann, Centre College.

Tensor Categories and Noncommutative Algebras, I (associated with invited address by Julia Plavnik), Ellen E Kirkman, Wake Forest University, and Julia Yael Plavnik, Indiana University, Bloomington.

## Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), Michelle Ann Manes, University of Hawaii.

## Palermo, Italy

July 23-26, 2024
Tuesday - Friday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## San Antonio, Texas

## University of Texas, San Antonio

September 14-15,2024
Saturday - Sunday
Meeting \#1198
Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: Not applicable Issue of Abstracts: Volume 45, Issue 3

## Deadlines

For organizers: Expired
For abstracts: July 23, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

James A M Alvarez, The University of Texas at Arlington, Title To Be Announced.
Jason R Schweinsberg, University of California San Diego, Title To Be Announced.
Anne Shiu, Texas A\&M University, Title To Be Announced.

## Savannah, Georgia

## Georgia Southern University

October 5-6,2024
Saturday - Sunday
Meeting \#1199
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: Not applicable Issue of Abstracts: Volume 45, Issue 4

## Deadlines

For organizers: March 5, 2024
For abstracts: August 13, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Peter Bubenik, University of Florida, To Be Announced.
Akos Magyar, University of Georgia, To Be Announced.
Sarah Peluse, Princeton/IAS, To Be Announced.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.p1.

Advanced Topics in Graph Theory and Combinatorics., Songling Shan, Auburn University, and Zi-Xia Song, University of Central Florida.

Commutative Algebra, Saeed Nasseh, Tricia Muldoon Brown, and Alina C. Iacob, Georgia Southern University.
Deterministic and Stochastic PDEs: Theoretical and Numerical Analyses, Pelin Guven Geredeli, Clemson University, and Xiang Wan, Loyola University Chicago.

Extremal and structural graph theory., Ruth Luo, University of South Carolina, and Zhiyu Wang, Georgia Institute of Technology.

Fluids, Waves, and Free Boundaries., David M. Ambrose, Drexel University.
Geometric Maximal Operators and Related Topics., Paul Hagelstein, Baylor University, and Alex Stokolos, Georgia Southern University.

Modules over Commutative Rings, Laura Ghezzi, New York City College of Technology and The Graduate Center-Cuny, and Joseph P Brennan, University of Central Florida.

Recent Advances of PDEs in Modern Mathematical Physics: Theory and Applications, Yuanzhen Shao, The University of Alabama, and Yi Hu and Shijun Zheng, Georgia Southern University.

Recent Progress in Numerical Methods for PDEs, Xuejian Li and Leo Rebholz, Clemson University.
Topics in commutative algebra and algebraic geometry, Prashanth Sridhar, Charles University, Prague, and Michael Brown, Auburn University.

Topological Data Analysis, Theory and Applications, Peter Bubenik and Kevin P. Knudson, University of Florida.

## Albany, New York

State University of New York at Albany

October 19-20, 2024
Saturday - Sunday

## Meeting \#1200

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: Not applicable Issue of Abstracts: Volume 45, Issue 4

## Deadlines

For organizers: March 19, 2024
For abstracts: September 1, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs /sectional.htm7.

## Invited Addresses

Jennifer Balakrishnan, Boston University, Title to be announced.
Jose Perea, Northeastern University, Title to be announced.
Richard Rimanyi, UNC, Title to be Announced.

## Riverside, California

University of California, Riverside

October 26-27, 2024
Saturday - Sunday

## Meeting \#1201

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable Issue of Abstracts: Volume 45, Issue 4

## Deadlines

For organizers: March 26, 2024
For abstracts: September 3, 2024

## Auckland, New Zealand

December 9-13,2024
Monday - Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## MEETINGS \& CONFERENCES

## Seattle, Washington

## Washington State Convention Center and the Sheraton Seattle Hotel

January 8-11,2025 Issue of Abstracts: To be announced
Wednesday - Saturday
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: To be announced

## Deadlines

For organizers: April 16, 2024
For abstracts: September 10, 2024

## Lawrence, Kansas

## University of Kansas

March 29-30,2025 Issue of Abstracts: To be announced
Saturday - Sunday
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison
Program first available on AMS website: To be announced

## Deadlines

For organizers: To be announced
For abstracts: To be announced

## Hartford, Connecticut

Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

April 5-6,2025
Saturday - Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## St. Louis, Missouri

## St. Louis University

October 18-19,2025 Issue of Abstracts: To be announced
Saturday - Sunday
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison
Program first available on AMS website: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Washington, District of Columbia

# Walter E. Washington Convention Center and Marriott Marquis Washington DC 

January 4-7, 2026
Sunday - Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Hep Create Opportunity SUPPORT AMS UNDERGRADUATE OPPORTUNITY AWARDS

The AMS Undergraduate Opportunity Awards are one-time grants awarded to undergraduate mathematics majors, helping to ensure that financial hardship does not stand in the way of completing their degree program. Over 200 promising math students have received Waldemar J. Trjitzinsky Memorial Awards (established by the Trjitzinsky family in 1991) and Edmund Landau Awards (established by an anonymous donor in 2020).


Kaia De Vries

Mahdi Rahman, a 2022 Trjitzinsky Award winner, is a junior at the University of BuffaloSUNY, majoring in mathematics. He never considered mathematics as an option until he was watching YouTube videos to prepare for the math section of the SAT. As more math videos were recommended to him, he realized that he actually enjoyed mathematics and decided to choose mathematics as his major. Over the years, he has become even more interested in mathematics, particularly in algebra and analysis.
2022 Landau Award winner Kaia De Vries, a sophomore at the University of Maine, Orono, is pursuing a mathematics major and a computer science minor. De Vries first became interested in math as a child and decided she wanted to pursue a degree in mathematics after her experience of being on the Fryeburg Academy (Maine) math team. Encouraged by support of the math department faculty, she is excited to continue studying math at the University of Maine and hopes to become more involved in mathematics research.



Mahdi Rahman
Thank you to AMS donors who have given to these awards in support of young mathematicians.

## Thank You

To learn more and make a gift, visit www.ams.org/opportunityawards

# New AMS Textbook 

## Explore important, interesting, and promising innovations in information and communication technology

Introduction
to Quantum Algorithms

Johannes A. Buchmann

## Introduction to Quantum Algorithms

Johannes A. Buchmann, Technical University of Darmstadt, Germany
Quantum algorithms are among the most important, interesting, and promising innovations in information and communication technology. They pose a major threat to today's cybersecurity and at the same time promise great benefits by potentially solving previously intractable computational problems with reasonable effort. The theory of quantum algorithms is based on advanced concepts from computer science, mathematics, and physics.

Introduction to Quantum Algorithms offers a mathematically precise exploration of these concepts, accessible to those with a basic mathematical university education, while also catering to more experienced readers. This comprehensive book is suitable for self-study or as a textbook for one- or two-semester introductory courses on quantum computing algorithms. Instructors can tailor their approach to emphasize theoretical understanding and proofs or practical applications of quantum algorithms, depending on the course's goals and timeframe.

Pure and Applied Undergraduate Texts, Volume 64; 2024; 371 pages; Softcover; ISBN: 978-1-4704-7398-3; List US\$89; AMS members US\$71.20; MAA members US\$80.10; Order code AMSTEXT/64
bookstore.ams.org/amstext-64


[^0]:    Submit letters to the editor at notices-1etters@1ists.ams.org.

[^1]:    2023 prize-winning article: Greene J., Heegaard Floer Homology. Notices of the AMS, 68 (1):19-33, 2021.
    2021 prize-winning article: Margalit D., The Mathematics of Joan Birman. Notices of the AMS, 66 (3):341-353, 2019.
    2018 prize-winning article: Cohn H., A Conceptual Breakthrough in Sphere Packing. Notices of the AMS, 64 (2):102-115, 2017.
    2017 prize-winning article: Bailey D., Borwein J., Mattingly A., and Wightwick G., The Computations of Previously Inaccessible Digits of $\pi^{2}$ and Catalan's Constant. Notices of the AMS, 60 (7):844-854, 2013.

[^2]:    David Manderscheid is the division director of the Division of Mathematical Sciences at the NSF, and a professor of mathematics at the University of Tennessee, Knoxville. His email address is dmanders@nsf.gov.
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[^3]:    Bruce Bartlett is an associate professor in mathematics at Stellenbosch University and an associate member of the National Institute for Theoretical and Computational Sciences (NITHECS). His email address is bbartlett@sun . ac.za. He dedicates this article to John Baez.
    Communicated by Notices Associate Editor William McCallum.
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    DOI: https://doi.org/10.1090/noti2923

[^4]:    ${ }^{1}$ Using the Brioschi quintic as normal form is one trick which Klein missed, as he instead used a different form - the Bring-Jerrard quintic - where the maximal number of coefficients $a_{i}$ in (6) are zero.

[^5]:    ${ }^{2}$ I learned this argument from David Speyer and Beren Gunsolus [13].

[^6]:    ${ }^{3}$ This formula appears in Ramanujan's "lost notebook."

[^7]:    Dylan Possamaï is a full professor of mathematics at ETH Zürich, Switzerland. His email address is dylan.possamai@math.ethz.ch.
    Ludovic Tangpi is an assistant professor of operations research and financial engineering at Princeton University. His email address is ludovic.tangpi @princeton.edu.

[^8]:    ${ }^{1}$ More precisely we consider $\mathfrak{F}$-predictable processes.
    ${ }^{2}$ Notably Girsanov's theorem.

[^9]:    ${ }^{3}$ We will not discuss practical applications in these notes. To simplify the exposition, we rather present generic examples which extend to more general situations we will consider later. For practical examples, we refer the readers for instance to [Car16].

[^10]:    ${ }^{4}$ This statement can be made rigorous using for instance Malliavin calculus.

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    Gabriela Dumitrascu is an associate professor of mathematics and statistics at Eastern Michigan University. Her email address is gdumitra@emich.edu. DOI: https://doi.org/10.1090/noti2914

[^12]:    ${ }^{1}$ Preservice teachers, teacher candidates, or future teachers are college students who are currently enrolled in teacher education programs or courses and are in the process of preparing to become licensed or certified teachers.

[^13]:    Richard A. Edwards is a teaching professor at Lyman Briggs College, Michigan State University. His email address is aedwards@msu. edu.
    DOI: https://doi.org/10.1090/noti2915

[^14]:    ${ }^{1}$ In many modern texts the notion of uniform differentiability does not appear explicitly. A function is continuously differentiable if and only if its derivative is uniformly continuous.

[^15]:    Adrian Lam is an associate professor of mathematics at The Ohio State University. His email address is 1am.184@osu.edu.
    DOI: https://doi.org/10.1090/noti2916

[^16]:    Katherine J. Pearce is a Peter O'Donnell Jr. Postdoctoral Fellow in the Oden Institute at the University of Texas at Austin and volunteers with the Texas Prison Education Initiative. Her email address is katherine. pearce @austin.utexas.edu.

[^17]:    ${ }^{1}$ According to the Bureau of Justice Statistics, fewer than 4 out of 10 incarcerated people have completed high school, versus 9 out of 10 in the general population. The average incarcerated person in state prison is 39 years old with a 10th grade education. [WSHW22]

[^18]:    Jennifer Schaefer is a professor of mathematics at Dickinson College. Her email address is schaefje@dickinson.edu.
    Rebecca Swanson is a teaching professor in applied mathematics and statistics at Colorado School of Mines. Her email address is swanson@mines. edu.

[^19]:    Alexander Braverman is a professor of mathematics at the University of Toronto and the Perimeter Institute. His email address is sashabraverman@gmai1 . com.
    Pavel Etingof is a professor of mathematics at the Massachusetts Institute of Technology. His email address is petingof@gmai1.com.
    Andrei Okounkov is a professor of mathematics at Columbia University. His email address is okounkov@math.columbia.edu.
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    Communicated by Notices Associate Editor Daniela De Silva.
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    DOI: https://doi.org/10.1090/noti2909

[^20]:    Enrico Arbarello is a professor of mathematics at the Sapienza University of Rome. His email address is enrico. arbare11o@gmail.com.

[^21]:    Serguei Brazovski is a research director emeritus at the Laboratoire de Physique Théorique et des Modèles Statistiques - LPTMS, CNRS, Orsay, France. His email address is brazovsk@gmai1.com.

[^22]:    ${ }^{1}$ In the Soviet Union, line five in the internal passport was ethnicity.

[^23]:    ${ }^{2}$ For example, his computational skills never ceased to amaze me-essentially until his last day; a couple of weeks before his death he complained that a certain mathematical problem would probably remain unsolved forever: "Nobody except me can get through this calculation, and I can't do it because I can't use a pen anymore."

[^24]:    ${ }^{3}$ It is worth emphasizing that unlike, say, the famous Novikov conjecture proved by Shiota, where the sought-for characterization of Jacobians is built into the formulation, Welters's conjecture is formulated purely in the language of elementary algebraic geometry-thus it is highly nonobvious that integrable equations have anything to do with it.

[^25]:    Vladimir Drinfeld is a professor of mathematics at the University of Chicago. His email address is drinfeld@math.uchicago.edu.
    Anton Dzhamay is a professor of mathematics at the University of Northern Colorado. His email address is Anton.Dzhamay@unco.edu.

[^26]:    Samuel Grushevsky is a professor of mathematics at the University of New York at Stony Brook. His email address is sgrushevsky@scgp.stonybrook.edu. ${ }^{4}$ see also his 1977 paper in the Russian Mathematical Surveys.

[^27]:    Nikita Nekrasov is a professor of physics at the Simons Center of Geometry and Physics, Stony Brook, New York. His email address is nikitastring@gmai 1 . com.

[^28]:    Sergei Novikov is a Distinguished Professor Emeritus at the University of Maryland College Park. His email address is sergeynvkv040@gmai1.com.

[^29]:    Leon Takhtajan is a professor of mathematics at the University of New York at Stony Brook. His email address is leontak@math.stonybrook.edu.

[^30]:    Alexander Varchenko is a professor of mathematics at the University of North Carolina at Chapel Hill. His email address is anv@emai1. unc.edu.

[^31]:    Alexander Veselov is a professor of mathematics at the Loughborough University. His email address is A.P.Veselov@1boro.ac.uk.

[^32]:    Anton Zabrodin is a professor of mathematical physics at the Higher School of Economics (Moscow). His email address is zabrodin@itep.ru.

[^33]:    Tatiana Smoliarova is an associate professor of Russian literature at the University of Toronto. Her email address is tsmoliarova@gmai1.com.

[^34]:    ${ }^{5}$ As Alexander Varchenko mentioned in his memoir in the present article, in Russia, there are two advanced degrees: Candidate of Sciences (equivalent to a PhD) and the much more exceptional Doctor of Sciences (equivalent to the French or German habilitation).

[^35]:    Albert C. Lewis is an independent scholar. His email address is albert @neomon.us.
    Karen Hunger Parshall is professor emerita in the Departments of History and Mathematics at the University of Virginia. Her email address is khp3k @virginia.edu.

    Communicated by Notices Associate Editor Laura Turner.
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[^36]:    ${ }^{1}$ There was an equally drastic reaction to the communist threat in the US for a period after the Russian Revolution, 1917-1920. A general account of the "second Red Scare" of concern here is included in [Sto13], which acknowledges the different assessments historians have made of the overall harm done by the anticommunist investigations of the time. Relative to the impact of the second Red Scare on academia per se, see [Sch86]. Indeed, the literature on the period, on the roles of the FBI and HUAC in it, and on many other specific aspects of it, is vast. The Notice's restrictions on the maximal number of references permitted per article has meant that we have been unable to direct our readers more fully to specific secondary sources.

[^37]:    ${ }^{2}$ The AMS actually formed an ad hoc committee in 1952 on "Controversial Questions" specifically to consider what, as a professional society, its boundaries were or should be in cases like the California oath controversy. For the events leading up to this committee's formation, see [Pit88, pp. 297-299] and [Bar20].

[^38]:    ${ }^{3}$ Determining suitable meeting places is a perennial issue facing national organizations. For an account of the development of the AMS's procedures to deal with it in the 1950s, see [Pit88, pp. 297-300].

[^39]:    ${ }^{4}$ See Box 5, Folder: Organizations, American Mathematical Society, Committees: Loyalty Oath 1950-1951, Minority Report, p. 1.
    ${ }^{5}$ See Folder 93-372/5: Committee on Controversial Questions, AMS (1952).

[^40]:    ${ }^{6}$ The quotation that follows may also be found on this site.
    ${ }^{7}$ Benjamin F. Feinberg was a lawyer and member of the New York State Senate from 1933 to 1949.

[^41]:    ${ }^{8}$ The biographical information about Harrison that follows is from his testimony before HUAC [H-GH54, pp. 5013-5018].

[^42]:    ${ }^{9}$ The recent work [Bat23] provides much detail on Davis's encounter with HUAC, his ensuing troubles at Michigan, and his later legal battles.
    ${ }^{10}$ Davis had been an undergraduate at Harvard from 1942 to 1945 and had earned his BS there prior to his naval tour of duty.

[^43]:    ${ }^{11}$ Folder 86-036/41: Faculty Dismissals, Chan Davis, etc. 1954. The news clippings from the Daily Michigan referred to here and below may also be found in this folder.
    ${ }^{12}$ We have not found a copy of Moise's petition, but it would be interesting to know if Hildebrandt, the department chair, signed, given his opinion, quoted above, of how the AMS should respond in such a situation.

[^44]:    ${ }^{13}$ Folder 86-036/41: Faculty Dismissals, Chan Davis, etc. 1954.

[^45]:    ${ }^{14}$ Folder 2007-054/024(8).

[^46]:    ${ }^{15}$ Folder 86-036/31: Lorch case, 1950-1957, Julian Blau et al. to the Faculty of Pennsylvania State College, undated (but spring of 1950).
    ${ }^{16}$ Folder 86-036/31: Lorch case 1950-1957, Kline to Wilder, May 12, 1950.
    ${ }^{17}$ Folder 86-036/31: Lorch case 1950-1957, Lorch to the Fisk Board of Trustees, October 28, 1954. Lorch's Mathematics Department colleague, Robert Rempfer, and his wife, Gertrude, a physics professor at Fisk, also tried (unsuccessfully) to have their children enrolled at the same Black school.

[^47]:    ${ }^{18}$ Letter from the National Archives and Records Administration to Albert Lewis, July 21, 2023.
    ${ }^{19}$ Lorch only saw the record of the 1950 hearing in which he was named for the first time on October 9, 1954, three weeks after he had testified before HUAC.
    ${ }^{20}$ A lightly annotated copy of Lorch's testimony before HUAC may be found in [Lor, 207-054/025(27)]. The words in square brackets here and below are his penciled emendations of the transcript.

[^48]:    ${ }^{21}$ Folder 86-036/31: Lorch case 1950-1957.
    ${ }^{22}$ Folder 86-036/31: Lorch case 1950-1957 (the next quotation is also from this letter).

[^49]:    ${ }^{23}$ Folder 86-036/31: Lorch case 1950-1957, Mac Lane to Wilder, February 22, 1955.
    ${ }^{24}$ Folder 86-036/31: Lorch case 1950-1957, Duren to Wilder, February 25, 1955.
    ${ }^{25}$ Folder 86-036/31: Lorch case 1950-1957.

[^50]:    ${ }^{26}$ Folder 86-036/31: Lorch case 1950-1957, Shannon to Wilder, April 6, 1955.
    ${ }^{27}$ Folder 86-036/31: Lorch case, 1950-1957, Bode to Wilder, June 8, 1955.
    ${ }^{28}$ Folder 86-036/31: Lorch case, 1950-1957, Wilder to Bode, June 21, 1955.
    ${ }^{29}$ Folder 86-036/31: Lorch case 1950-1957, Segal to Wilder, February 27, 1955.

[^51]:    ${ }^{30}$ Folder 86-036/31: Lorch case 1950-1957, Wilder to Segal, March 3, 1955.
    ${ }^{31}$ Folder 86-036/31: Lorch case, 1950-1957, Segal to Wilder, May 4, 1955.
    ${ }^{32}$ Folder 86-036/31: Lorch case, 1950-1957, Wilder to Segal, May 26, 1955. The quotations that follow in this paragraph are also from this letter.
    ${ }^{33}$ Folder 86-036/31: Lorch case 1950-1957, Lehmer and Schiffer to Wilder, August 22, 1955.
    ${ }^{34}$ Folder 86-036/7: Ki-Kw (the next quotation is also from this letter).

[^52]:    ${ }^{35}$ Folder 86-036/31: Lorch case 1950-1957, Moise to Duren, James, Kuhn, and Rosser, March 22, 1956 (the quotation that follows is also from this letter).
    ${ }^{36}$ Folder 86-036/7: Le-Ly, Blackwell to Wilder, September 4, 1956 (the quotation that follows is from this letter).
    ${ }^{37}$ Folder 86-036/7: Le-Ly, Wilder to Blackwell, October 5, 1956.
    ${ }^{38}$ Folder 86-036/7: Le-Ly, Wilder to Brauer, October 5, 1956.

[^53]:    ${ }^{39}$ Folder 86-056/7: Le-Ly. Lorch to Wilder, October 25, 1976.
    ${ }^{40}$ Folder 2007-054/026(13): Davis to Lorch, September 24, 1954.

[^54]:    ${ }^{41}$ Folder 2007-054/016(02): Pierce to "Fellow Mathematicians," February 27, 1960.
    ${ }^{42}$ See folders 2007-054/034(13), (17), (18), (19), (22), and (24).
    ${ }^{43}$ For a historical analysis of the Lorches' time in Little Rock, see [New17].

[^55]:    ${ }^{45}$ The first substantial historical account of the government investigations of homosexuals is provided in [Joh04].
    ${ }^{46}$ Folder 2007-054/021(08): Remarks by Mrs. Grace Lorch, Guest of Honor, Bill of Rights Dinner of the Emergency Civil Liberties Committee, Hotel New Yorker, New York, December 17, 1957, p. 4. The Senate Internal Security Subcommittee was the Senate version of HUAC.

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