WHAT IS...?

a Białynicki-Birula Decomposition?

Alberto Franceschini

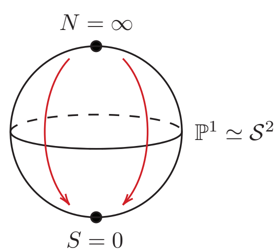

Figure 1. The $\mathbb{C}^*$-action on $\mathbb{P}^1$, pictured as the sphere $\mathcal{S}^2$. The action flows from the north to the south pole.

An introductory example. Consider the complex projective line with nonhomogeneous coordinates, $\mathbb{P}^1 = \mathbb{C} \cup \infty$, and consider the natural action of the multiplicative group $\mathbb{C}^* := \mathbb{C} \setminus 0$ given by

$$\mathbb{C}^* \times \mathbb{P}^1 \ni (t, p) \longmapsto tp \in \mathbb{P}^1.$$

We have that

$$\lim_{t\to 0} tp = \begin{cases} 0 & \text{if } p \neq \infty, \\ \infty & \text{if } p = \infty. \end{cases}$$

We call 0 the *sink* and ∞ the *source* of the action. Such action provides a decomposition of $\mathbb{P}^1$ into the affine spaces $\mathbb{C}$ and ∞. Using the homeomorphism between the projective line $\mathbb{P}^1$ and the sphere $\mathcal{S}^2$, we can draw the action as in Figure 1.

This is an example of *Białynicki-Birula decomposition*. There are some similarities with Morse theory. In that case, one can study the topology of a manifold M via its decomposition provided by the critical points of a function on M. Analogously, we will study a (smooth) algebraic variety via the cell decomposition provided by fixed points of a $\mathbb{C}^*$-action.

The Białynicki-Birula theory. To keep our feet on the ground, we will stick to a very basic set-up (cf. [1]) even though the theory has been developed more generally (cf. for instance [5]). So, X will be a complex nonsingular projective variety endowed with a (nontrivial) $\mathbb{C}^*$-action:

$$\mathbb{C}^* \times X \ni (t, x) \longmapsto t \cdot x \in X.$$

We consider the decomposition of $X^{\mathbb{C}^*}$, the fixed locus of the action, into connected components:

$$X^{\mathbb{C}^*} = \bigsqcup_{Y \in \mathcal{Y}} Y,$$

where $\mathcal{Y}$ denotes the set of connected components. Since X is nonsingular, by a theorem of Iversen (cf. [4]) each connected component Y is also nonsingular, hence irreducible.

One can always extend a $\mathbb{C}^*$-action on a nonsingular projective variety X to an algebraic morphism $\mathbb{P}^1 \times X \to X$ (cf. [8]), which means that for $x \in X$ there exist the limiting points to 0 and ∞ and they are fixed points of X for the $\mathbb{C}^*$-action. Notice that the limiting point to ∞ is just the limiting point to 0 for the opposite action, that is

$$\lim_{t\to\infty} t \cdot x = \lim_{t\to 0} t^{-1} \cdot x.$$

For a given $Y \in \mathcal{Y}$, we define its *Białynicki-Birula cells* (*BB-cells* for short) to be the two subsets

$$X^{\pm}(Y) := \left\{ x \in X \,:\, \lim_{t\to 0} t^{\pm 1} \cdot x \in Y \right\},$$

where $\pm$ will be intended as a shortcut for stating a result both for the $+$ and for the $-$ decomposition. Essentially, the BB-cells of a fixed point component Y consist of all

Alberto Franceschini obtained his PhD at the University of Trento. His email address is alberto.bobech@gmail.com.

Communicated by Notices *Associate Editor Han-Bom Moon.*

DOI: https://doi.org/10.1090/noti2910

the points of X that converge to Y as the parameter t of the action goes to 0 or to ∞.

Theorem (Białynicki-Birula, 1973). *Let X be a complex nonsingular projective variety endowed with a (nontrivial) $\mathbb{C}^*$-action. Consider the induced $+$ and $-$ decompositions. Then the following hold:*

1. $X = \bigsqcup_{Y \in \mathcal{Y}} X^{\pm}(Y)$ *and the BB-cells are locally closed subsets of X for any $Y \in \mathcal{Y}$.*
2. *The natural maps*
$$X^{\pm}(Y) \ni x \longmapsto \lim_{t \to 0} t^{\pm 1} \cdot x \in Y$$
are algebraic: they are locally trivial bundles in the Zariski topology, and the fibers are affine spaces of rank $\nu_{\pm}(Y) \coloneqq \dim X^{\pm}(Y) - \dim Y$.
3. *There are homology decompositions*
$$\mathrm{H}_m(X, \mathbb{Z}) = \bigoplus_{Y \in \mathcal{Y}} \mathrm{H}_{m-2\nu_{\pm}(Y)}(Y, \mathbb{Z}).$$

As a consequence of the theorem, among the fixed point components there exist unique $Y_+, Y_- \in \mathcal{Y}$ such that $X^+(Y_+), X^-(Y_-)$ are dense subsets of X. We call Y_+ and Y_- respectively the *sink* and the *source* of the action, following the notation of the introductory example.

An example of Grassmannian. Let V_+ and V_- be n-dimensional vector spaces. Consider the $\mathbb{C}^*$-action on $V_{\pm}$ given by
$$\mathbb{C}^* \times V_{\pm} \ni (t, v) \longmapsto t^{\pm 1} v \in V_{\pm}.$$
Let $V \coloneqq V_+ \oplus V_-$. Then V has a naturally defined $\mathbb{C}^*$-action such that V_+ and V_- are maximal invariant linear subspaces.

Consider the induced action on $X \coloneqq \mathrm{Gras}(n, V)$, the Grassmannian of n-planes in V. The fixed points of the induced action are the n-planes in V that are $\mathbb{C}^*$-invariant for the action above. Indeed, if $W \subset V$ is a $\mathbb{C}^*$-invariant n-plane, then we can write
$$W = (W \cap V_+) \oplus (W \cap W_-),$$
where $W_{\pm}$ are the eigenspaces on which $\mathbb{C}^*$ acts. One can prove that the sink and the source of the induced action on X are isolated points, representing the subspaces V_+ and V_-, respectively.

For explicit computations, suppose that $n = 2$. Then $X \simeq Q^4$, a smooth quadric hypersurface in the projective space $\mathbb{P}\left(\wedge^2 V\right)$ of dimension 5. One can prove that the action on X is the restriction of the action on $\mathbb{P}\left(\wedge^2 V\right)$ given by
$$t \cdot [x_0 : \ldots : x_5] = \left[t^2 x_0 : x_1 : x_2 : x_3 : x_4 : t^{-2} x_5\right].$$
Then the fixed locus of the action on X is
$$X^{\mathbb{C}^*} = [1 : 0 : \ldots : 0] \sqcup \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \sqcup [0 : \ldots : 0 : 1],$$
and the homology groups of X can be computed using homology of points and of $\mathbb{P}^1 \times \mathbb{P}^1$. The fixed point component $\mathbb{P}^1 \times \mathbb{P}^1$ appear as the intersection of the quadric X with $\mathbb{P}^3 \subset \mathbb{P}^5$ generated by x_1, x_2, x_3, x_4. Notice that the decompositions of the Grassmannian given by BB-cells are particular cases of Schubert decomposition.

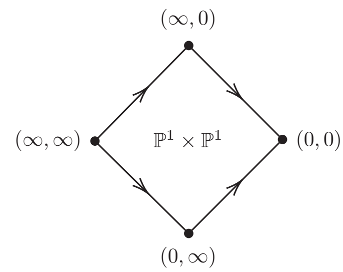

Figure 2. A schematic picture of the $\mathbb{C}^*$-action on $\mathbb{P}^1 \times \mathbb{P}^1$. The action flows from left to right as t moves from ∞ to 0.

Furthermore, consider the action on $\mathbb{P}^1 \times \mathbb{P}^1$ given, in nonhomogeneous coordinates, by
$$\mathbb{C}^* \times \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \ni (t, p, q) \longmapsto (tp, tq) \in \mathbb{P}^1 \times \mathbb{P}^1.$$
There are four fixed points for such action
$$(0,0), \qquad (0,\infty), \qquad (\infty, 0), \qquad (\infty,\infty),$$
as pictured in Figure 2.

Then, by the homology decomposition of the Białynicki-Birula theorem,
$$\mathrm{H}_0\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}\right) = \mathrm{H}_0\left((\infty,\infty), \mathbb{Z}\right) = \mathbb{Z},$$
$$\mathrm{H}_2\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}\right) = \mathrm{H}_0\left((0,\infty), \mathrm{Z}\right) \oplus \mathrm{H}_0\left((\infty,0), \mathbb{Z}\right) = \mathbb{Z} \oplus \mathbb{Z},$$
$$\mathrm{H}_4\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}\right) = \mathrm{H}_0\left((0,0), \mathbb{Z}\right) = \mathbb{Z}.$$
With this decomposition of the homology groups of $\mathbb{P}^1 \times \mathbb{P}^1$, one can explicitly compute the decomposition of the homology groups of X.

The associated birational map via GIT. A birational map $f : Z_1 \dashrightarrow Z_2$ between algebraic varieties is a rational map such that there are open subsets $U_i \subset Z_i$ (for $i = 1, 2$) such that the restriction $f|_{U_1} : U_1 \to U_2$ is an isomorphism.

An explicit remark in [7] says that, given a $\mathbb{C}^*$-action on a nonsingular projective variety X, we can associate a birational map among projective varieties.

Given a finite-dimensional vector space V, the projective space $\mathbb{P}(V)$ is the space of lines through the origin of V, that is a point of $\mathbb{P}(V)$ is an orbit for the $\mathbb{C}^*$-action on $V \setminus 0$ given by
$$\mathbb{C}^* \times V \setminus 0 \ni (t, v) \longmapsto tv \in V.$$
This example leads to the definition of *geometric quotient*: a space whose points represent $\mathbb{C}^*$-orbits of another space, see [6] for a rigorous introduction.

In the context of the Białynicki-Birula decomposition, we consider the subsets $\mathrm{B}_{\pm} \coloneqq X^{\pm}(Y_{\pm}) \setminus Y_{\pm}$. They are dense because $X^{\pm}(Y_{\pm})$ are dense and $Y_{\pm}$ are closed subsets of X. These sets contain all the orbits that flow to $Y_{\pm}$. The quotients of these sets by the action are denoted
$$\mathcal{G}_{\pm} \coloneqq \mathrm{B}_{\pm}/\mathbb{C}^*.$$

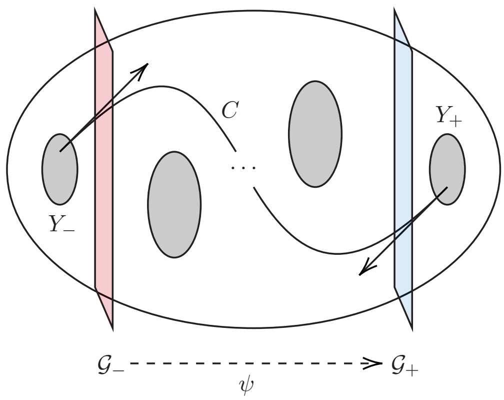

Figure 3. A picture of the $\mathbb{C}^*$-action on X. The action flows from left to right as t goes from ∞ to 0. C We draw an orbit C flowing from the source to the sink, its tangent directions at the source and the sink, and the two geometric quotients $\mathcal{G}_\pm$.

By a theorem of Białynicki-Birula and Święcicka, see [2], the spaces $\mathcal{G}_\pm$ are geometric quotients. In particular, they are quasi-projective varieties. Every point of $\mathcal{G}_+$ (resp. $\mathcal{G}_-$) represents an orbit of the action flowing to the sink Y_+ (resp. from the source Y_-). Then we can identify such an orbit with its tangent direction at the sink or the source of the orbit.

It remains to describe the birational map between the geometric quotients $\mathcal{G}_\pm$. We consider the intersection $\mathrm{B}_+ \cap \mathrm{B}_- \subset X^\pm(Y_\pm) \setminus Y_\pm$: It contains all the $\mathbb{C}^*$-orbits that flow from the source to the sink of the action. Then we obtain a natural birational map

$$\psi_a : \mathrm{B}_- \dashrightarrow \mathrm{B}_+,$$

which is just the identity on the intersection $\mathrm{B}_+ \cap \mathrm{B}_-$. The quotient of this map by the $\mathbb{C}^*$-action,

$$\psi : \mathcal{G}_- \dashrightarrow \mathcal{G}_+,$$

is the birational map we are looking for. Notice that, since there could be orbits that flow from the source to a fixed point component different from the sink, the map ψ is not an isomorphism in general.

Geometrically, given a $\mathbb{C}^*$-orbit C that flows from the source to the sink, ψ associates to its tangent direction at the source and its tangent direction at the sink. Figure 3 gives a schematic picture of the situation.

Torus actions and matrix inversion. In this section, we will go further on the example of X being the Grassmannian $\mathrm{Gras}(n, V)$, where $V = V_+ \oplus V_-$ is a $2n$-dimensional vector space.

The tangent space of the Grassmannian at the point $[W] \in X$ representing $W \subset V$ is

$$T_{X,[W]} \simeq \mathrm{Hom}\,(W, V/W) = W^\vee \otimes (V/W).$$

As we stated before, the sink Y_+ and the source Y_- of the $\mathbb{C}^*$-action are isolated points in the Grassmannian. Then $X^+(Y_+)$ and $X^-(Y_-)$ are affine spaces, and there are $\mathbb{C}^*$-equivariantly isomorphisms

$$X^\pm(Y_\pm) \simeq T_{X,Y_\pm} \simeq V_\pm^\vee \otimes V_\mp.$$

with the tangent spaces of X at Y_+ and Y_-. Then the geometric quotients are isomorphic to projective spaces, $\mathcal{G}_\pm \simeq \mathbb{P}(V_\pm^\vee \otimes V_\mp)$, and the induced birational map is

$$\psi : \mathbb{P}(V_-^\vee \otimes V_+) \dashrightarrow \mathbb{P}(V_+^\vee \otimes V_-).$$

If we fix bases, then $V_\pm^\vee \otimes V_\mp = \mathrm{Hom}\left(V_\pm, V_\mp\right)$ is the space of $n \times n$ matrices. Then one can show that ψ is the projectivization of the inversion map of $n \times n$ matrices. Moreover, if $n = 3$, then $\psi : \mathbb{P}^8 \dashrightarrow \mathbb{P}^8$ is one of the special quadro-quadric Cremona transformations classified in [3].

References

[1] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497, DOI 10.2307/1970915. MR366940

[2] Andrzej Białynicki-Birula and Joanna Święcicka, *Complete quotients by algebraic torus actions*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 10–22, DOI 10.1007/BFb0101505. MR704983

[3] Lawrence Ein and Nicholas Shepherd-Barron, *Some special Cremona transformations*, Amer. J. Math. **111** (1989), no. 5, 783–800, DOI 10.2307/2374881. MR1020829

[4] Birger Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236, DOI 10.1007/BF01425495. MR299608

[5] Joachim Jelisiejew and Łukasz Sienkiewicz, *Białynicki-Birula decomposition for reductive groups* (English, with English and French summaries), J. Math. Pures Appl. (9) **131** (2019), 290–325, DOI 10.1016/j.matpur.2019.04.006. MR4021177

[6] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906

[7] Gianluca Occhetta, Eleonora A. Romano, Luis E. Solá Conde, and Jarosław A. Wiśniewski, *Small bandwidth $\mathbb{C}^*$-actions and birational geometry*, J. Algebraic Geom. **32** (2023), no. 1, 1–57, DOI 10.1090/jag/808. MR4548671

[8] Andrew John Sommese, *Extension theorems for reductive group actions on compact Kaehler manifolds*, Math. Ann. **218** (1975), no. 2, 107–116, DOI 10.1007/BF01370814. MR393561

Alberto Franceschini

Credits

Figures 1–3 and author photo are courtesy of Alberto Franceschini.