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Opinions expressed in signed Notices articles are those of the authors and do not necessarily reflect opinions of the editors or policies of the American Mathematical Society.
My term as associate editor of the Notices comes to an end on December 31, 2024. My next journey is to serve as the 10th president of the National Association of Mathematicians (NAM), Inc. I am honored to have been elected president, and I want to thank all NAM members who voted for me. My term started on February 1, 2024.

NAM is a professional nonprofit organization with a strong executive committee and board of directors. As an organization, NAM plays a critical role in advancing education, service, and research within the mathematics community of the United States. One important and motivating factor for the founding of NAM was to recognize the contributions of African American mathematicians. The mission of NAM is “to promote excellence in the mathematical sciences, and to promote the mathematical development of under-represented American minorities.” Since its founding in 1969, much of NAM’s success has been due to the contributions of its founders and several of its members, supporters, key stakeholders, and partners. For instance, NAM is a longtime partner and member of the Conference Board of the Mathematical Sciences (CBMS). NAM’s signature programs are the NAM Undergraduate MATHFest, the Faculty Conference on Research and Teaching Excellence (FCRTE), and conference partnerships with the Joint Mathematics Meeting (JMM) of the American Mathematical Society (AMS), and with MathFest of the Mathematical Association of America (MAA). It is through these programs that many math students and individuals in the mathematical sciences from institutions throughout the country first learn of NAM and its purpose within the mathematics community. By joining NAM, you can make a difference in this community. NAM membership levels include lifetime, individual, student, and institutional. Serving NAM provides a great opportunity to help others. The NAM board of directors, the president and members further the legacy of NAM by:

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For more information about NAM’s history, programs, membership information and opportunities, academic contributions, how to make donations, and the newsletter, see the link https://www.nam-math.org/ and the references listed below. In closing, it would be remiss not to thank Dr. Johnny Houston, a founding member of NAM, for his many years of dedicated service to NAM and the larger mathematics community.

Asamoah Nkwanta is the president of the National Association of Mathematicians. His email address is president@nam-math.org.

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On the Theory of Anisotropic Minimal Surfaces

Antonio De Rosa

1. Introduction
In the 1870s the Belgian physicist Plateau conjectured that every closed wire bounds at least one soap film. Since soap films tend to minimize the surface tension energy, which is proportional to the surface area, Plateau’s conjecture can be formulated as follows:

Given any closed Jordan curve \( \gamma \) in \( \mathbb{R}^3 \), among all surfaces spanning the boundary \( \gamma \), there exists a surface \( \Sigma \) of least area.

Formalizing and proving Plateau’s conjecture turned out to be extremely challenging for mathematicians. Indeed it led to the development of several new theories in analysis and geometry and paved the way for the rise of geometric measure theory. In 1930, Douglas and Radó solved the Plateau problem for surfaces that arise as the image of a 2-dimensional disk in \( \mathbb{R}^n \). Thereafter, a variety of new theories were developed to describe and solve the Plateau problem in \( \mathbb{R}^n \) for surfaces of general \( d \)-dimensional topology: the theory of finite perimeter sets by De Giorgi, the theory of currents by Federer and Fleming, the theory of currents by Almgren and Allard, the theory of homological spanning conditions by Reifenberg, the theory of sliding minimizers by David, the theory of differential chains and linking number spanning conditions by Harrison and Pugh and several others.

Such a wide variety of tools was crucial because, depending on the natural phenomena to model, one has to use a different notion of surface and boundary. For instance, among boundaries of finite perimeter sets, solutions of the Plateau problem in dimension \( n \leq 7 \) are oriented smooth submanifolds, in contrast to soap films in \( \mathbb{R}^3 \) that can have 1-dimensional singular sets. One possible approach to model soap films and to capture their 1-dimensional singular sets, is the set-theoretic Plateau problem, which consists in minimizing the \( d \)-dimensional Hausdorff measure \( \mathcal{H}^d \) within families of \( d \)-dimensional sets in \( \mathbb{R}^n \) with suitable boundary conditions. The set-theoretic approach was pioneered by Reifenberg, Almgren, and Taylor, and has recently been further developed by David, Harrison and Pugh, De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi, and Fang and Kolasinski.

Since solutions of the Plateau problem are minimizers of the surface area functional, they are in particular critical points of this functional in the sense of the calculus of variations. Critical points of the surface area are referred to as (isotropic) minimal surfaces.

The existence and regularity of a solution \( \Sigma \) of the Plateau problem relies on the study of the so-called tangent cones of \( \Sigma \), obtained through a standard blow-up procedure, described below. We consider the following one-parameter family of minimal surfaces \( \{ \Sigma_{x,r} \}_{r > 0} \), obtained by dilating \( \Sigma \) around a point \( x \in \Sigma \):

\[
\Sigma_{x,r} := \frac{\Sigma - x}{r} \quad \text{with} \quad r > 0.
\]

In the unitary ball \( B(0, 1) \), the surface area of this family \( \Sigma_{x,r} \) enjoys a powerful monotonicity formula, i.e.,

\[
r \mapsto \mathcal{H}^d(\Sigma_{x,r} \cap B(0, 1)) \quad \text{is nondecreasing} \quad (1)
\]

The validity of the monotonicity formula (1) for minimal surfaces implies there is a tangent cone \( T \) such that, as \( r \to 0^+ \), up to subsequences \( \Sigma_{x,r} \to T \) in a measure theoretic sense. One can also prove that \( T \) minimizes the surface area.

However, for several natural phenomena the use of the surface area functional is just a first approximation. In order to capture microstructures, numerous models in applied sciences employ directionally dependent functionals, known as anisotropic energies.

Since anisotropic energies are not invariant under rigid motions, their critical points do not enjoy the same...
conservation laws of isotropic minimal surfaces. For instance the monotonicity formula (1) is not known to hold for minimizers of general anisotropic energies. Consequently, the study of anisotropic minimal surfaces is significantly more challenging than the study of their isotropic counterparts.

1.1. Anisotropic energies. We fix \( d, n \in \mathbb{N} \) with \( 1 \leq d < n \) and we consider a smooth anisotropic integrand

\[
F : \mathbb{R}^n \times G(d, n) \to (0, \infty),
\]

where \( G(d, n) \) denotes the Grassmannian of \( d \)-dimensional linear subspaces of \( \mathbb{R}^n \). \( F \) is usually required to satisfy an ellipticity condition, which can be thought of as a generalization of a uniform convexity property of \( F \) in the Grassmannian variable. We recall that there is a plethora of ellipticity conditions for \( F \), especially in codimension strictly bigger than one, i.e., \( d < n - 1 \). The appropriate ellipticity conditions to assume on \( F \) depend on the types of surfaces and boundaries considered and on the regularity theory one is interested in. To avoid introducing too many conditions, we will sometimes use the generic phrase “\( F \) is elliptic.”

We define the anisotropic energy of a \( d \)-dimensional smooth surface \( \Sigma \) as

\[
F(\Sigma) = \int_{\Sigma} F(x, T_x \Sigma) \, d\mathcal{H}^d(x),
\]

where \( T_x \Sigma \) is the tangent space of \( \Sigma \) at \( x \). We observe that if \( F \equiv 1 \), then \( F(\Sigma) = \mathcal{H}^d(\Sigma) \). Hence anisotropic energies subsume the surface area functional.

The anisotropic energy (2) is well defined also for weaker notions of surfaces: namely, the \( d \)-rectifiable sets. A set is said to be \( d \)-rectifiable if it can be covered, up to an \( \mathcal{H}^d \)-negligible set, by countably many \( C^1 \) manifolds. Since \( d \)-rectifiable sets admit a notion of tangent space at \( \mathcal{H}^d \)-a.e. point, they are well suited to compute anisotropic energies.

The minimal configurations for (2) are clearly not invariant by translation or rotation. This is a common behavior in several problems in materials science. Indeed, anisotropic integrands were introduced by Gibbs to model the surface tension arising at the interface between two different materials. Moreover, Wulff observed that crystal structures are polyhedral, because they are optimal configurations for energies that assign different weights to different tangent planes of the boundary surface. Since the pivotal studies of Almgren, Taylor, and Allard, anisotropic energies have attracted a great interest in the geometric analysis community, leading to important contributions in a variety of applications: crystal structures, capillarity problems, gravitational fields, homogenization problems, and many others.

In solving the Plateau problem with respect to (2) and, more generally, in the study of critical points of (2), which are referred to as anisotropic minimal surfaces, one faces a main obstruction: the lack of the monotonicity formula (1). As observed by Allard, monotonicity (1) is deeply related to the isotropic setting. Nevertheless, solutions to the set-theoretic anisotropic Plateau problem were obtained by several authors, as, for instance, by Harrison and Pugh in [HP17] and by De Philippis, De Rosa, and Ghiraldin in [DPDRG20]. In particular we recall the following theorem:

Theorem 1 ([DPDRG20]). Let \( F \) be elliptic. Let \( \Gamma \subset \mathbb{R}^n \) be a closed set and \( \mathcal{P} \) be a family of relatively closed \( d \)-rectifiable subsets of \( \mathbb{R}^n \setminus \Gamma \), such that \( \mathcal{P} \) is closed by Lipschitz deformations of \( \mathbb{R}^n \setminus \Gamma \). If \( \{\Sigma_j\}_{j \in \mathbb{N}} \subset \mathcal{P} \) is a minimizing sequence for \( F \) in \( \mathcal{P} \), i.e.,

\[
\liminf_{j \to \infty} F(\Sigma_j) = \inf_{L \in \mathcal{P}} \{F(L) : L \in \mathcal{P}\},
\]

then, up to subsequences, the following limit holds in the sense of measures

\[
F(\cdot, T_\Sigma \Sigma_j) \mathcal{H}^d \lfloor \Sigma_j \xrightarrow{*} F(\cdot, T_\Sigma \Sigma) \mathcal{H}^d \lfloor \Sigma \quad \text{in } \mathbb{R}^n \setminus \Gamma,
\]

and consequently

\[
\liminf_{j \to \infty} F(\Sigma_j) \geq F(\Sigma).
\]

Moreover \( \Sigma \) is smooth outside of a relatively closed set \( S \subset \Sigma \) with \( \mathcal{H}^d(S) = 0 \).

In Theorem 1 the set \( \Gamma \) should be interpreted as the assigned boundary of the anisotropic Plateau problem, and \( \mathcal{P} \) is the feasible set of the anisotropic Plateau problem. The assumption on \( \mathcal{P} \) of being closed by Lipschitz deformations can be weakened to include more cases of interest, see [DPDRG20]. A main step in proving Theorem 1 is to show the \( d \)-rectifiability of \( \Sigma \). In the isotropic setting, this can be obtained by the monotonicity formula (1) and by proving a density lower bound, as these two properties allow one to employ a deep rectifiability theorem of Preiss. However, since the monotonicity formula (1) does not hold for anisotropic minimal surfaces, we cannot apply Preiss’s theorem to prove Theorem 1. To overcome this obstacle, in [DPDRG20] it is crucial to utilize the theory of varifolds.

1.2. Varifolds. The \( d \)-varifolds are weak notions of \( d \)-dimensional surfaces. More precisely a \( d \)-varifold is a Radon measure on the Grassmannian bundle \( \mathbb{R}^n \times G(d, n) \). Simple examples are the rectifiable \( d \)-varifolds, i.e., \( d \)-varifolds that can be represented as

\[
V = \partial \mathcal{H}^d \lfloor \Sigma \otimes \delta_{T_x \Sigma},
\]

for some \( d \)-rectifiable set \( \Sigma \) and a Borel function \( \partial : \mathbb{R}^n \to \mathbb{R}^+ \). However there are examples of nonrectifiable varifolds, obtained naturally as limits of surfaces. An example is provided in Fig.1, where the limit varifold is

\[
\mathcal{H}^d \lfloor \Sigma \otimes \frac{1}{2} (\delta_{e_2} + \delta_{e_3}),
\]
with \( \ell_1 \) being the oblique line and \( \ell_2, \ell_3 \) being respectively the horizontal and vertical lines.

The mass \( \|V\| \) of a \( d \)-varifold \( V \) is the Radon measure on \( \mathbb{R}^n \) defined by

\[
\|V\|(A) := V(A \times G(d, n)) \quad \text{for all } A \subset \mathbb{R}^n \text{ Borel.}
\]

For a \( d \)-varifold \( V \) we consider (when it exists) its \( d \)-dimensional density at \( x \in \mathbb{R}^n \):

\[
\Theta(x, V) := \lim_{r \to 0^+} \frac{\|V\|(B_r(x))}{\omega_d r^d},
\]

where \( \omega_d := \mathcal{H}^d(B^d(0,1)) \) is the measure of the \( d \)-dimensional unit ball in \( \mathbb{R}^d \). Note that for a rectifiable \( d \)-varifold \( V \), the \( d \)-dimensional density \( \Theta(x, V) \) is equal to the value \( \delta(x) \) in (3) for \( \|V\| \) a.e. \( x \).

Given the duality between \( d \)-varifolds and \( C^0_\#(\mathbb{R}^n \times G(d, n)) \), the definition of anisotropic energy (2) can be naturally generalized also to \( d \)-varifolds \( V \) as follows:

\[
F(V) := \int_{\mathbb{R}^n \times G(d, n)} F(x, T)dV(x, T).
\]

The anisotropic first variation of \( V \) with respect to \( F \) is defined as

\[
\delta_F V(g) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F((id + \epsilon g)_\#(V)), \quad \forall g \in C^1_\#(\mathbb{R}^n, \mathbb{R}^n),
\]

where for a diffeomorphism \( \psi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \), we define the push-forward of \( V \) with respect to \( \psi \) as the varifold \( \psi_\#V \) satisfying:

\[
\int_{\mathbb{R}^n \times G(d, n)} \Phi(x, T)d(\psi_\#V)(x, T)
= \int_{\mathbb{R}^n \times G(d, n)} \Phi(\psi(x), d_\chi \psi(T))d\psi_\#V(x, T),
\]

for every \( \Phi \in C^0_\#(G(\psi(\Omega))) \). Here \( d_\chi \psi(T) \) is the image of \( T \) under the map \( d_\chi \psi \) and

\[
J\psi(x, T) := \sqrt{\det \left( \left( d_\chi \psi \right)_T \circ d_\chi \psi \right)}
\]

denotes the \( d \)-Jacobian determinant of the differential \( d_\chi \psi \) restricted to the \( d \)-plane \( T \). If \( V = \mathcal{H}^d \mathcal{L} \Sigma \otimes \delta_{T,\Sigma} \) is the \( d \)-varifold canonically associated to a smooth \( d \)-dimensional submanifold \( \Sigma \), the pushforward \( \psi_\#V \) is simply the image of \( \psi(\Sigma) \) of \( \Sigma \) under the map \( \psi \).

A \( d \)-varifold \( V \) is called stationary with respect to \( F \) if

\[
\delta_F V = 0.
\]

We observe that if \( V = \mathcal{H}^d \mathcal{L} \Sigma \otimes \delta_{T,\Sigma} \) for a smooth \( d \)-dimensional surface \( \Sigma \) and \( \delta_F V = 0 \), then \( \Sigma \) is an anisotropic minimal surface.

We say that a \( d \)-varifold \( V \) has \( F \)-mean curvature in \( L^p \) if there exists a map \( H_F \in L^p(\mathbb{R}^n, \mathbb{R}^n; \|V\|) \) such that

\[
\delta_F V(g) = -\int_{\mathbb{R}^n} (H_F(x), g(x))d\|V\|(x), \forall g \in C^1_\#(\mathbb{R}^n, \mathbb{R}^n).
\]

2. Regularity for Anisotropic Minimal Surfaces

In the study of partial differential equations (PDEs), a typical question is about the regularity of weak solutions to certain PDEs. Weak solutions are very useful as it is often easier to prove their existence. However this generates the new task of investigating how close they are to being classical solutions. This is achieved by deducing information about their regularity from the PDE.

This is precisely the case of the stationarity equation (4) for anisotropic minimal surfaces, which can be regarded as a geometric PDE in integral form. Since stationary \( d \)-varifolds are well-defined weak solutions of equation (4), it is crucial to study how close stationary varifolds are to being smooth anisotropic minimal surfaces, by deducing structural and regularity properties from (4). To this aim, our first step in this section is to understand whether \( V \) is a rectifiable \( d \)-varifold, and the second step is to investigate the smoothness of the support of \( V \). As we detail below, this analysis is useful, for instance, to deduce the regularity of the solution \( \Sigma \) of the anisotropic Plateau problem in Theorem 1, which a priori is just the support of the limiting \( d \)-varifold of the minimizing sequence of \( d \)-varifolds

\[
\{F(\cdot, T, \Sigma_j) \mathcal{H}^d \mathcal{L} \Sigma_j \otimes \delta_{T, \Sigma_j} \}_{j \in \mathbb{N}}
\]

Rectifiability. Allard’s rectifiability theorem [All72] asserts that every \( d \)-varifold \( V \) in \( \mathbb{R}^n \) with locally bounded isotropic first variation is rectifiable when restricted to the set of points in \( \mathbb{R}^n \) with positive \( d \)-dimensional density. This result is essential in many applications, such as to prove a compactness theorem for integral varifolds [All72], to solve the Plateau problem, in the min-max theory, and in geometric flows.

De Philippis, De Rosa, and Ghiraldin proved in [DPDRG18] the same rectifiability result for anisotropic energies, provided \( F \) satisfies the atomic condition introduced in [DPDRG18, Definition 1.1]. Since the atomic condition is fairly technical to define, for simplicity of notation we recall below the condition (BC), which was introduced in [DRK20] and proved to be equivalent to the atomic condition.

**Definition 1.** An anisotropic integrand \( F \) satisfies (BC) if and only if for every \( (x, T) \in \mathbb{R}^n \times G(d, n) \), denoting \( F_\mu(y, S) := F(x, S) \) for every \( (y, S) \in \mathbb{R}^n \times G(d, n) \), if \( \mu \) is a probability
measure over $G(d, n)$ such that
\[ \delta_{F_\alpha}[(\mathcal{H}^d \otimes T) \otimes \mu] = 0, \]
then $\mu = \delta_T$.

The rectifiability theorem in [DPDRG18] then reads:

**Theorem 2** ([DPDRG18]). If $F$ satisfies the atomic condition, or equivalently $(BC)$, then for every $d$-varifold $V$ whose anisotropic first variation $\delta_F V$ is a Radon measure, the varifold
\[ V := V \setminus \{ x \in \mathbb{R}^n : \Theta(x, V) > 0 \} \times G(d, n) \quad (5) \]
is rectifiable.

Moreover, if $F$ is autonomous, i.e., $F(x, T) \equiv F(T)$, the reverse implication also holds. That is, if $F$ does not satisfy the atomic condition, then there exists a $d$-varifold $V$ such that $\delta_F V$ is a Radon measure and the associated $V_\epsilon$ as in (5) is not rectifiable.

Some ideas for the proof of Theorem 2 were inspired by the “strong constancy lemma” of Allard [All86, Theorem 4]. Theorem 2 was crucial in the proof of the rectifiability of $\Sigma$ in Theorem 1. Indeed the stationarity of the limit $d$-varifold follows from $\{ \Sigma_j \}_{j \in \mathbb{N}}$ being a minimizing sequence, while the density of the limit $d$-varifold can be proved to be positive via a set-theoretic analog of the Federer-Fleming deformation theorem due to David and Semmes. Thereafter, Theorem 2 has found several other applications, such as in the solution of the anisotropic Plateau problem in classes of integral and rectifiable varifolds [DR18], and in the anisotropic min-max theory [DPDR24] that we will discuss more in detail in Section 3.

**$C^{1,\alpha}$-regularity.** A celebrated theorem of Allard [All72] states that, given a rectifiable $d$-varifold $V$ in $\mathbb{R}^n$ with density greater than or equal to one and isotropic mean curvature $H_{\text{Area}}$ bounded in $L^p$ with $p > d$, for every $x \in \mathbb{R}^n$ such that $\Theta(x, V) = 1$ the support of $V$ is $C^{1,\alpha}$ in a neighborhood of $x$. The proof deeply relies on the monotonicity formula (1). Hence, it is unknown whether this result holds for anisotropic energies, i.e., assuming an $L^p$ bound on the anisotropic mean curvature $H_F$.

In codimension one, i.e., $d = n - 1$, the anisotropic counterpart of this regularity theorem has been obtained by Allard under an additional lower density bound assumption [All86]. However, the proof relies on the maximum principle, hence it cannot be extended to general codimension. In general codimension, De Rosa and Tione [DRT22] proved a partial $C^{1,\alpha}$-regularity theorem in the case the $d$-varifold $V$ is associated to a Lipschitz graph. To this aim, they introduce the uniform scalar atomic condition (USAC), a novel ellipticity condition that can be thought of as a uniform version of the atomic condition or of the equivalent (BC) in Definition 1. USAC allowed them to obtain a Caccioppoli inequality, that was crucial to prove the following theorem:

**Theorem 3.** Let $F \in C^2$ be a functional satisfying USAC, let $p > d$, and consider an open bounded set $\Omega \subset \mathbb{R}^d$. Let $u \in \text{Lip}(\Omega, \mathbb{R}^{n-d})$ be a map whose graph induces a varifold $V$ with $F$-mean curvature $H_F \in L^p(\Omega \times \mathbb{R}^{n-d}, \mathbb{R}^n; \|\cdot\|)$. Then there exists $\alpha \in (0, 1)$ and an open set $\Omega_0 \subset \Omega$ such that $\mathcal{H}^d(\Omega_0) = \mathcal{H}^d(\Omega)$ and $u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^{n-d})$.

It remains a challenging problem to prove the anisotropic counterpart of Allard’s regularity theorem [All72], that is, to extend Theorem 3 to all rectifiable varifolds satisfying a lower density bound.

**3. Existence of Anisotropic Minimal Surfaces: The Min-max Theory**

Since Theorem 2 and Theorem 3 are local results, they can be proved with slightly more work in a Riemannian manifold ambient space, rather than just in the Euclidean ambient space $\mathbb{R}^n$. Working in a closed Riemannian manifold, the problem of finding closed anisotropic minimal hypersurfaces becomes meaningful. This problem was posed by Allard [All83, Page 2]. On the contrary, by the maximum principle, in the ambient space $\mathbb{R}^n$ there are no closed anisotropic minimal hypersurfaces.

For the area functional, the problem of finding closed isotropic minimal hypersurfaces has a long literature. In 1917 Birkhoff introduced a min-max method to prove the existence of a closed geodesic on any 2-dimensional Riemannian sphere. Fet and Lyusternik extended this result, proving that every closed Riemannian manifold admits a closed geodesic. In order to generalize the aforementioned results to higher dimension, Almgren introduced the theory of varifolds and developed a min-max theory to prove the existence of stationary integral $d$-varifolds in any closed $n$-dimensional Riemannian manifold. In codimension one, that is when $d = n - 1$, the regularity of these integral $(n - 1)$-varifold was proved by Pitts [Pit81] for $3 \leq n \leq 6$, and by Schoen and Simon [SS81] for every $n \geq 3$.

**Theorem 4** ([Pit81, SS81]). Let $M$ be an $n$-dimensional smooth closed Riemannian manifold. Then there is a nontrivial embedded isotropic minimal hypersurface $\Sigma \subset M$ without boundary, which is smooth outside of a singular set of Hausdorff dimension at most $n - 8$.

Theorem 4 motivated the development of the min-max theory, which has played a major role in proving a number of conjectures in geometry and topology.

Allard [All83, Page 2] was interested in understanding whether a version of Theorem 4 holds for anisotropic energies. De Philippis and De Rosa [DPDR24] settled this question for $n = 3$ up to at most one singular point, as a corollary of the following more general result about the existence of closed surfaces with prescribed constant anisotropic mean curvature.
Theorem 5 ([DPDR24]). Let $M$ be a 3-dimensional smooth closed Riemannian manifold, $F$ be an elliptic integrand and $c \in \mathbb{R}$. Then there is a nontrivial almost embedded (embedded if $c = 0$) closed surface $\Sigma \subset M$ with constant anisotropic mean curvature $H_F \equiv c$ and which is smooth outside of at most one singular point $p \in M$.

In Theorem 5, the singular point $p$ accounts for the index of $\Sigma$, which in general may be an unstable surface, as it is constructed via min-max methods. However, we expect that this singular point $p$ is removable.

It is interesting to observe that we do not have a direct proof of Theorem 5 for the anisotropic minimal surfaces case, i.e., $c = 0$. Indeed the case $c = 0$ is obtained by first constructing a sequence of surfaces $\Sigma_k$ with constant anisotropic mean curvature $c_k = 1/k$ (each having at most one singular point $p_k$) and then using the local stability of $\Sigma_k$ to deduce the smooth convergence of a subsequence of $\Sigma_k$ away from an accumulation point $p$ of the sequence $p_k$. The limit surface $\Sigma$ has then anisotropic mean curvature

$$H_F \equiv \lim_{k \to \infty} c_k = 0,$$

that is, $\Sigma$ is an anisotropic minimal surface with respect to $F$. Constructing surfaces with positive constant anisotropic mean curvature is easier because their multiplicity can be shown to be either 1 for $\mathcal{H}^2$-a.e. point or 2 for a 1-dimensional touching set. Having multiplicity 1 is critical when performing the blowup analysis to study the tangent cones in the absence of the monotonicity formula (1).

For a closed Riemannian manifold $M$ of general dimension $n$, we expect the validity of a similar result to Theorem 5: the existence of a nontrivial almost embedded closed hypersurface $\Sigma \subset M$, with $H_F \equiv c$ and with a singular set of zero $\mathcal{H}^{n-3}$-measure. The latter regularity of $\Sigma$ agrees with the regularity theory for solutions of the anisotropic Plateau problem in codimension one for finite perimeter sets established by Almgren, Schoen, and Simon [SSA77]. This regularity result is almost sharp, because Morgan [Mor90] constructed an example of a uniformly elliptic anisotropic energy for which the 3-dimensional cone over the Clifford torus is a solution of the codimension one anisotropic Plateau problem in $\mathbb{R}^4$. As this 3-dimensional surface has one singular point, this example confirms that the $\mathcal{H}^{n-4}$ measure of the singular set of an anisotropic minimizer of codimension one in $\mathbb{R}^n$ may be in general positive. However, to date it is not known whether there are 3-dimensional anisotropic minimizers in $\mathbb{R}^4$ with a singular set of fractional dimension within $(0, 1)$.

The dimension of the singular set for solutions of the Plateau problem in codimension one for finite perimeter sets is connected to the Bernstein problem, which asks whether critical points in $\mathbb{R}^n$ which are graphs of scalar functions defined on all of $\mathbb{R}^{n-1}$ are necessarily hyperplanes. In the case of the area functional, Bernstein, Fleming, De Giorgi, Almgren, Simons, and Bombieri, De Giorgi, and Giusti showed that the answer is positive if and only if $n \leq 8$. For elliptic integrands, the answer was shown to be positive for $n = 3$ by Jenkins and for $n = 4$ by Simon, while Mooney and Yang [MY24] recently answered negatively for dimensions $n \geq 5$, concluding the solution of the anisotropic Bernstein problem.

We end this section remarking that, when $F = \mathcal{N}$ is close to the area functional in the $C^\infty$ topology, the anisotropic regularity theory becomes similar to the one of the area functional. More precisely Simon [Sim77] proved that in this case the Bernstein problem has positive answer up to dimension $n = 8$ and Almgren, Schoen, and Simon [SSA77] showed that minimizers of the anisotropic Plateau problem in codimension one for finite perimeter sets are regular up to dimension $n = 7$. Moreover Chodosh and Li [CL23] recently proved that, if $F$ is close to the area functional in the $C^4$ topology, then stable critical points in dimension $n = 4$ are flat. It remains an interesting question whether these results can be achieved assuming closeness to the area functional in a weaker topology, as for instance the $C^2$ topology.

4. Constant Anisotropic Mean Curvature Surfaces and the Anisotropic Isoperimetric Problem

The constant anisotropic mean curvature surfaces constructed in Theorem 5 have an important parallel with anisotropic minimal surfaces. While the latter are critical points of the anisotropic Plateau problem, constant anisotropic mean curvature surfaces are critical points of the anisotropic isoperimetric problem. This problem, also known as the Wulff problem, consists in the minimization of the anisotropic energy of boundaries of finite perimeter sets with a prescribed volume constraint $m > 0$. In short, the Wulff problem reads

$$\inf \{ F(\partial \Omega) : \mathcal{H}^n(\Omega) = m \},$$

where

$$\Omega \text{ is a finite perimeter set in } \mathbb{R}^n. \quad (6)$$

If $F$ is an autonomous integrand in codimension one, i.e.,

$$F : G(n - 1, n) \to (0, \infty)$$

does not depend on the $x$ variable, then the Wulff problem (6) has a (unique up to translation) solution $W$, referred to as Wulff shape, which was constructed by Wulff. Henceforth, for the remainder of this section we will assume $F$ is an autonomous integrand. $W$ plays a central role in crystallography and its minimality has been proved with different techniques by Taylor, Fonseca and Müller, Brothers and Morgan, Gromov, Figalli, Maggi, and Pratelli. For the area functional the Wulff shape coincides with the
Euclidean ball, whose symmetries reflect the rotation invariance of the area functional.

On the other hand, the investigation of critical points of (6) is more subtle. This corresponds to characterizing finite perimeter sets with finite volume whose boundary has constant anisotropic mean curvature in the sense of varifolds. For the area functional, the Euclidean sphere has been long known to be the only smooth closed connected hypersurface with constant isotropic mean curvature. Several proofs have been provided and one of the most geometric is via the Alexandrov moving plane method. This method involves reflecting the hypersurface across a family of parallel hyperplanes and using the maximum principle to deduce the symmetry of the hypersurface with respect to one of these parallel hyperplanes. However the moving plane method is not well suited for anisotropic energies, as the reflection of a constant anisotropic mean curvature surface changes the tangent planes and hence also the geometric PDE solved by the surface. With different techniques, He, Li, Ma, and Ge characterized the Wulff shape boundary as the only closed connected $C^2$ hypersurface with constant anisotropic mean curvature. The $C^2$ assumption was removed for the area functional in an ingenious way by Delgadino and Maggi [DM19], who proved the following theorem.

Theorem 6 ([DM19]). Among sets of finite perimeter and finite volume, finite unions of Euclidean balls with equal radii are the unique critical points of the isotropic isoperimetric problem.

Afterward, De Rosa, Kolasiński, and Santilli [DRKS20] extended Theorem 6 to uniformly elliptic $C^{2,\alpha}$ anisotropic integrands adding an extra assumption, equation (7) below, which corresponds roughly to a density lower bound. More precisely they proved the following theorem.

Theorem 7 ([DRKS20]). Consider an elliptic integrand $F \in C^{2,\alpha}$, with $\alpha \in (0, 1)$. Among sets $\Omega$ of finite perimeter and finite volume satisfying
\[
\mathcal{H}^{n-1}(\partial^+ \Omega \sim \partial^* \Omega) = 0, \tag{7}
\]
finite unions of Wulff shapes with equal radii are the unique critical points of the anisotropic isoperimetric problem (6).

Both Theorem 6 and Theorem 7 are proved by means of an Heintze-Karcher inequality for sets of finite perimeter, obtained by refining an argument due to Montiel and Ros. In particular, the Heintze-Karcher inequality they proved in the anisotropic setting reads as follows:

Theorem 8 ([DRKS20]). Consider an elliptic integrand $F \in C^{2,\alpha}$, with $\alpha \in (0, 1)$, and a set of finite perimeter $\Omega \subset \mathbb{R}^n$ such that (7) holds, the anisotropic first variation $\delta_F(\partial^* \Omega)$ is absolutely continuous with respect to $\mathcal{H}^{n-1}\partial^* \Omega$ and the anisotropic mean curvature $H_F$ is bounded, positive and locally $C^{0,\alpha}$ on the $C^{1,\alpha}$ regular part of $\partial^* \Omega$. Then
\[
\mathcal{H}^n(\Omega) \leq \frac{n-1}{n} \int_{\partial^* \Omega} \frac{F(T_x \partial^* \Omega)}{H_F(x)} \, d\mathcal{H}^{n-1}(x).
\]

Equality holds if and only if $\Omega$ coincides up to a set of zero $\mathcal{H}^n$-measure with a finite union of disjoint open Wulff shapes with radii not smaller than \(\frac{n-1}{\|H_F\|_{L^\infty}}\).

Condition (7) is used to apply Allard $C^{1,\alpha}$-regularity theorem in codimension one [All86]. Condition (7) always holds if $\Omega$ satisfies a density lower bound at every point in $\partial^* \Omega$. In particular it holds for Lipschitz domains, for local minimizers, and for almost minimizers of problem (6). On the other hand, it is not known whether condition (7) holds for every set of finite perimeter $\Omega$ such that $\partial^* \Omega$ has constant anisotropic mean curvature $H_F$. In the isotropic setting, condition (7) follows from the monotonicity formula (1), which as mentioned in Section 1 is not known to hold for general anisotropic integrands. Hence it remains an interesting question whether Theorem 7 holds without condition (7).

Since the Wulff shape is characterized as the unique minimizer for all positive continuous anisotropic integrands, another very interesting question is whether Theorem 7 holds also for anisotropic integrands that are neither $C^{2,\alpha}$ nor elliptic. Particularly relevant for applications in materials science is the case of crystalline integrands, for which the Wulff shapes are obtained by intersecting a finite amount of half-spaces. Since in this case the integrand is convex, but is not $C^1$, the notion of anisotropic first variation and critical points are suitably defined by Maggi using the convexity in time of the functional along any prescribed variational flow. Figalli and Maggi have greatly advanced our understanding of crystalline energies in several works, see for instance [FM11], proving quantitative stability results for crystalline Wulff shapes for almost minimizers under the action of potential energies.

References


Antonio De Rosa

Credits

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Colding-Minicozzi Entropy and Complexity of Submanifolds

Jacob Bernstein and Lu Wang

Introduction

Intuitively, a sea urchin is more complicated than a billiard ball. However, it is not so easy to see how to formalize and quantify this distinction, i.e., what constitute good measures of the complexity of geometric objects. There are many different approaches, but on a high level, satisfactory answers should all involve something with the following properties:

1. Respects the geometric symmetries;
2. Collections of objects, taken together, should be more complex than the constituent elements;
3. The geometrically “simplest” elements in natural subclasses should be extrema of the quantity.

In addition to these general properties, it proves helpful to add a more technical and specific property:

4. Monotonic along a geometric heat flow.

This is justified by the fact that such flows tend to simplify the geometry of the objects being evolved.

In this article we focus on the geometry of submanifolds of Euclidean space and the measure of complexity we introduce will be associated with the mean curvature flow. One aspect we highlight is topological. On the one hand, mean curvature flow, like many geometric flows, should simplify the topology along with the geometry. On the other hand, we observe that, in many cases submanifolds with low complexity in the sense we discuss must be topologically simple.

Mean Curvature Flow

Surface tension is the force that causes the surface of a liquid to behave like a stretched elastic membrane and drives the interface to minimize its surface area. The force induced by surface tension on the interface is mathematically characterized by its mean curvature, which is the sum of the principle curvatures of the geometric surface corresponding to the interface.

This force also gives rise to a potent method for studying geometric and topological properties of submanifolds, especially embedded hypersurfaces. Indeed, by considering the (negative) gradient flow for area one obtains a dynamical process in which submanifolds of Euclidean space, e.g., curves or surfaces, continuously reduce their areas by deforming in the direction of steepest descent. A consequence of the first variation of area formula is that this gradient flow corresponds to the mean curvature flow. This is the flow in which points of the evolving submanifolds move with (normal) velocity determined by the mean curvature of the submanifold they lie in. That is,

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{H}$$

where $\mathbf{H}$ is the mean curvature of the appropriate submanifold in the flow and $\mathbf{x}$ is the position vector. In general, when $e_i$ is an orthonormal frame of the tangent space of a submanifold, its mean curvature satisfies $\mathbf{H} = \sum_i (D_{e_i} e_i)^\perp$ where $\perp$ means the component orthogonal to the submanifold. For instance, the sphere of radius $R$ in $\mathbb{R}^{n+1}$ centered at the origin satisfies $\mathbf{H} = -nR^{-2}\mathbf{x}$.

There are several equivalent characterizations of mean curvature flow. For example, a one-parameter family of $n$-dimensional submanifolds $t \mapsto \Sigma_t \subset \mathbb{R}^N$ is a mean curvature flow if and only if the coordinate functions, $x^i$, of $\mathbb{R}^N$, $i = 1, \ldots, N$, restricted to the submanifolds $\Sigma_t$, satisfy the heat equation

$$\frac{\partial x^i}{\partial t} = \Delta_{\Sigma_t} x^i.$$

Despite appearances, these equations are nonlinear as both the $x^i$ and the Laplacian, $\Delta_{\Sigma_t}$, depend on $\Sigma_t$.

A direct computation shows that the function $Q = |\mathbf{x}|^2 + 2nt$ satisfies the same heat equation on the evolving submanifolds as the $x^i$. If $Q$ is bounded initially, then, by the parabolic maximum principle, it remains bounded with
Colding-Minicozzi Entropy

We now introduce a quantity that quantifies the simplifying process provided by the mean curvature flow of closed convex hypersurfaces. As we will see, this quantity also satisfies the properties laid out in the introduction and so provides a plausible measure of complexity of submanifolds. It was introduced by Colding-Minicozzi in [CM12] where they named it entropy. Its monotonicity relies on a property of mean curvature flow first observed by Huisken [Hui90].

Recall that the heat kernel of $\mathbb{R}^n$ has the profile of a time-varying Gaussian

$$K_n(t,x) = \frac{(4\pi t)^{-\frac{n}{2}}}{2} e^{-\frac{|x|^2}{4t}} \quad \text{for } t > 0.$$ 

Let $t \mapsto \Sigma_t \subset \mathbb{R}^N$ be an $n$-dimensional mean curvature flow with starting time $t = 0$ and extend $K_n$ to $\mathbb{R}^N$ by the same formula. For $x_0 \in \mathbb{R}^N$ and $t_0 > 0$, the time derivative of the integral of $K_n(t_0 - t, \cdot - x_0)$ over $\Sigma_t$ is nonpositive. This is a consequence of the Huisken monotonicity formula [Hui90] whose proof uses the fact that the restriction of the coordinate functions of the ambient Euclidean space satisfy the heat equation on the evolving submanifolds as well as convexity properties of $K_n$.

Translating the flow $t \mapsto \Sigma_t$ by a vector in spacetime, $(s_0, y_0)$, produces a new mean curvature flow $t \mapsto \Sigma_{t-s_0} + y_0$ of the same shape. The same holds for parabolic rescalings of the flow, i.e., the flow $t \mapsto \rho \Sigma_{t-\tau}$ for $\rho > 0$. Integrating the kernels $K_n(t_0 - t, \cdot - x_0)$ over these transformed flows essentially corresponds to changing $t_0$ and $x_0$ in the integrals over the original flow. Here for a submanifold, $\Sigma$, $\rho \Sigma + x_0$ denotes the submanifold obtained by dilating $\Sigma$ by factor $\rho$ followed by translation by vector $x_0$.

One way to unify this family of quantities and also account for the symmetries of the mean curvature flow is the approach taken by Colding-Minicozzi [CM12]. They define the entropy, $\lambda[\Sigma]$, of an $n$-dimensional submanifold $\Sigma \subset \mathbb{R}^N$ by

$$\lambda[\Sigma] = \sup_{x_0, t_0} \int_{\Sigma} K_n(t_0, x - x_0) \, d\text{vol}_\Sigma(x).$$

Thus, $\lambda$ is invariant under rigid motions and dilations of $\Sigma$ and, by the Huisken monotonicity formula, is monotone decreasing along the mean curvature flow.

It is convenient at times to express the entropy in terms of the Gaussian integral centered at the origin with scale 1

$$F[\Sigma] = \int_{\Sigma} K_n(1, x) \, d\text{vol}_\Sigma = \frac{1}{(4\pi)^{n/2}} \int_{\Sigma} e^{-\frac{|x|^2}{4}} \, d\text{vol}_\Sigma.$$ 

Namely, by a change of variables,

$$\lambda[\Sigma] = \sup_{y, \rho} F[\rho \Sigma + y]$$ 

as $(y, \rho)$ ranges over all points of $\mathbb{R}^N \times (0, \infty)$.
The simplest nontrivial example of a self-shrinker is the rescaled mean curvature flow, which satisfies the equation \[ \frac{\partial x}{\partial t} = H \frac{x}{2}. \]

The static solutions of this flow satisfy \( H = \frac{x}{2} = 0 \) and describe the limiting behavior of the flow. A submanifold, \( \Sigma \), satisfying this limiting equation is called a self-shrinker, because \( t \mapsto \sqrt{-t} \Sigma \) is a solution of mean curvature flow. The simplest nontrivial example of a self-shrinker is the round sphere of radius \( \sqrt{2n} \) in \( \mathbb{R}^{n+1} \),

\[ \sqrt{2n}\mathbb{S}^n = \{ x_1^2 + \ldots + x_{n+1}^2 = 2n \}. \]

For the flow of convex hypersurfaces discussed in the previous section, this is exactly the self-shrinker one obtains at the unique terminal singularity.

As entropy is invariant under translations and dilations, it is also monotone decreasing along the rescaled mean curvature flow. Thus, the entropy of the initial submanifold bounds the entropy of the limiting self-shrinkers associated to the singularity.

Starting from a hypersurface that bounds a compact convex region, the flow becomes round as it reaches its extinction time. Monotonicity implies the entropy of the initial hypersurface is larger than or equal to that of a round hypersphere. Hence, round hyperspheres minimize entropy among all closed convex hypersurfaces. In the following sections, we study the minimizers of entropy in more general settings.

**Entropy Minimizers**

We now explore the extrema of entropy in two different classes of submanifolds.

**General submanifolds.** For \( \Sigma \subset \mathbb{R}^N \) an \( n \)-dimensional submanifold, one characterization of the tangent space at \( p \in \Sigma \) is via

\[ \lim_{\rho \to \infty} \rho(\Sigma - p) = T_p \Sigma \]

where the convergence may be interpreted in various senses. Geometrically this means that if we zoom in on smaller and smaller scales about \( p \in \Sigma \), then the zoomed in surface becomes closer and closer to the tangent plane.

An immediate consequence of this is that

\[ \lambda(\Sigma) \geq \limsup_{\rho \to \infty} F(\rho(\Sigma - p)) = F(T_p \Sigma). \]

A straightforward calculus exercise shows us that \( F(T_p \Sigma) = 1 \). Likewise, for any \( n \)-plane, \( P \subset \mathbb{R}^N \),

\[ \lambda(P) = F(P - q) = 1 \]

where \( q \) is a point on \( P \). In other words,

\[ \lambda(\Sigma) \geq \lambda(P) = 1 \]

where \( P \) is an \( n \)-plane. That is, planes are both the geometrically simplest elements and absolute minima of \( \lambda \) within the class of submanifolds. A related observation is that in the larger class of immersed submanifolds, any submanifold that is not embedded has entropy at least 2.

**Closed hypersurfaces.** When restricting to the smaller class of closed hypersurfaces, i.e., those that are compact and without boundary and of codimension one, the geometrically simplest elements become the round hyperspheres. In this case, one may, with more work, verify that

\[ \lambda(\Sigma) = F(2n\mathbb{S}^n). \]

It was computed by Stone that \( \lambda(\mathbb{S}^1) = \sqrt{2\pi} e < 2 \), \( \lambda(\mathbb{S}^2) = \frac{4}{e} \) and, more generally,

\[ \lambda(\mathbb{S}^1) > \lambda(\mathbb{S}^2) > \lambda(\mathbb{S}^3) > \cdots \]

With this in mind, Colding-Ilmanan-Minicozzi-White conjectured in [CIMW13] that round hyperspheres minimize entropy among closed hypersurfaces. Precisely, if \( \Sigma \subset \mathbb{R}^{n+1} \) is a closed hypersurface, then \( \lambda(\Sigma) \geq \lambda(\mathbb{S}^n) \).

Some immediate evidence for the conjecture was provided by earlier work of Gage-Hamilton [GH86] and Grayson [Gra87]. Specifically, they showed that curve shortening flow—i.e., one-dimensional mean curvature flow—evolves a closed simple curve into a round point in finite time. Thus, the reasoning used for convex hypersurfaces also shows the conjecture of Colding et al., [CIMW13], for simple closed curves. If one tries to reproduce this argument in higher dimensions one encounters several difficulties. The first is that there are many more
singularity models. Another, more subtle, issue is that unlike what happens with curve shortening flow, a singularity may form before the flow disappears.

Additional evidence for the general conjecture was provided by work of White [Whi05], which in turn built on earlier work of Brakke and Allard. In particular, one may conclude from these results that any singularity model has entropy a definite amount above 1 and, because the flow of any closed hypersurface forms a singularity in finite time, the same is true of any closed hypersurface. Hence, there is an $\epsilon = \epsilon(n) > 0$, so that when $\Sigma \subset \mathbb{R}^{n+1}$ is closed,

$$\lambda[\Sigma] \geq 1 + \epsilon.$$ 

This shows that a quantitative difference in the properties of entropy appears when restricting to the the class of closed hypersurfaces.

The singularity models of mean curvature flow are the nonflat self-shrinkers. Hence, the optimal inequality is a consequence of the following more ambitious conjecture of Colding et al., [CIMW13]: If $\Sigma \subset \mathbb{R}^{n+1}$ is a nonflat self-shrinker, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$. In the same paper, they prove this is true for closed self-shrinkers. This stronger conjecture was resolved in [BW17] for $n = 2$, but remains open for $n \geq 3$.

At first glance, one may think that the result of Colding et al., [CIMW13], on the entropy of closed self-shrinkers suffices to prove their initial conjecture. However, there is, a priori, no reason for flow of closed hypersurfaces to develop a closed singularity model. The example of a neck-pin—where the first singularity is the noncompact cylinder—illustrates one possible complication. There are even more bizarre possibilities. For instance, the first singularity could be noncompact and asymptotic to a cone. Such exotic singularities can exist—we refer to the paper of Chopp [Cho94] which contains pictures of many numerically computed examples—and there are now several rigorous constructions; for instance, by Kapouleas-Kleene-Møller, X. H. Nguyen, and Buzano-H. Nguyen-Schulz.

Nevertheless, restricting attention to singularities where the flow disappears rules out some of the least well understood behavior. Crucially, the class of terminal singularities are more stable in a dynamical sense than general singularities and stable singularities tend to be more rigid as observed in [CM12]. Terminal singularities may be hidden behind earlier exotic singularities and so one must genuinely work with a weak flow that exists through singularities. This is technically quite challenging and involves deep results in PDEs and geometric measure theory.

With careful work, the authors established in [BW16] the conjecture in low dimensions.

**Theorem.** If $2 \leq n \leq 6$, and $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$.

The bound $n \leq 6$ comes from the use of the regularity theory for stable minimal hypersurfaces. J. Zhu [Zhu20] later relaxed this requirement and extended our argument to prove the result for $n \geq 7$.

**Rigidity of Minimizers**

For a sharp inequality, it is natural to study when it is saturated, i.e., when equality is achieved. This is the question of rigidity of an inequality. Within the two classes considered above there is rigidity for the corresponding optimal inequalities.

For closed hypersurfaces, the proof of the sharp inequality also shows the corresponding rigidity: If $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface with $\lambda[\Sigma] = \lambda[\mathbb{S}^n]$, then $\Sigma$ is a round hypersphere. Somewhat surprisingly, rigidity in the general case proved slightly tricky. Using a rather subtle mean curvature flow construction, L. Chen [Che21] showed that if $\Sigma \subset \mathbb{R}^n$ is a properly embedded $n$-dimensional submanifold with $\lambda[\Sigma] = 1$, then $\Sigma$ is an $n$-dimensional plane. A more elementary argument was later given by the
first-named author when \( n = 2 \). One should also be able to use the rigidity properties of the Gaussian isoperimetric inequality of Sudakov-Tsirel’son and Borell to obtain this result for hypersurfaces.

Stability of Entropy Minimizers

A more robust form of rigidity is the question of stability, that is, whether almost minimizers are close to minimizers. This is particularly relevant, because well-behaved measures of complexity should have the feature that when the measured value of an object is near a minimum, the object is simple in a qualitative or quantitative sense. We discuss two different perspectives on this question for closed hypersurfaces with entropy near that of the round sphere.

Topological stability. One qualitative measurement of the complexity of a submanifold is in terms of its topology. It is now known that, in many dimensions, closed hypersurfaces with entropy below a certain threshold must be, topologically, as simple as possible—i.e., isotopic to the standard sphere. Of course all closed simple curves in the plane are isotopic to the round circle without additional assumptions; this can be proven in many ways, though an appealing approach is to use curve shortening flow.

For surfaces in \( \mathbb{R}^3 \), the authors showed the only singularity models with entropy below that of the round circle is the shrinking round sphere [BW17]—note that an elementary calculation shows that the entropy of the round circle is the same as that of the corresponding cylinder in \( \mathbb{R}^3 \). Combining this with the monotonicity of entropy shows that any closed surface with entropy less than that of a circle evolves smoothly under mean curvature flow until it disappears in a round point—mirroring the behavior of the curve shortening flow. Hence, the flow provides a smooth isotopy between the initial surface and the round two-sphere and so:

**Theorem.** If \( \Sigma \) is a closed surface in \( \mathbb{R}^3 \) with \( \lambda(\Sigma) \leq \lambda(\mathbb{S}^1) = \lambda(\mathbb{S}^1 \times \mathbb{R}) \), then \( \Sigma \) has genus 0.

That is, the entropy provides some measure of the topological complexity of closed surfaces.

In higher dimensions, the classification of low-entropy singularity models is highly incomplete, and a major challenge is to understand cone-like singularities. Nevertheless, the above theorem may be extended up to dimension five.

**Theorem.** For \( 3 \leq n \leq 5 \), if \( \Sigma \subset \mathbb{R}^{n+1} \) is a closed hypersurface with \( \lambda(\Sigma) \leq \lambda(\mathbb{S}^{n-1}) \), then \( \Sigma \) is smoothly isotopic to \( \mathbb{S}^n \).

When \( n = 3 \), the authors utilized the theory of self-expanders to show that all regular time slices of the weak mean curvature flow starting from such an initial hypersurface are in the same isotopy class [BW22]. The \( n = 3 \) case was also independently established by Chodosh-Choi-Mantoulidis-Schulze [CCMS21], who showed that generic one-sided perturbations of the initial hypersurface evolve smoothly under mean curvature flow until disappearing in a round point. Very recently, this second approach was extended by Chodosh-Mantoulidis-Schulze [CMS23] to dimensions \( n = 4, 5 \).

Finally, one may wonder if entropy can detect the presence of more complex topological structure. The expectations in this direction are quite modest. Indeed, consider a collection of distant unit spheres joined by very thin necks. These configurations can have arbitrary positive genus, but still have entropy a little bit above that of the circle. The fact that the increase is small comes from the fact that the Gaussian decays rapidly at large distances.

**Stability as sets.** More quantitatively, one may study the stability in terms of closeness as sets. This is inspired by the classical Bonnesen inequality which relates the isoperimetric defect of a region in the plane to the difference between its inradius and outradius. Here, for a compact set \( \Omega \subset \mathbb{R}^{n+1} \), the inradius, \( R_{\text{in}}(\Omega) \), is the radius of the largest ball contained in \( \Omega \) and outradius, \( R_{\text{out}}(\Omega) \), is the radius of the smallest ball containing \( \Omega \). Clearly, balls are the only sets where these are equal and the difference of \( R_{\text{out}}(\Omega) \) and \( R_{\text{in}}(\Omega) \) is small precisely when \( \Omega \) is close as a set to a ball.

Unlike the Bonnesen inequality, which is a purely two-dimensional phenomenon, there is stability as sets in all dimensions for compact regions whose boundaries nearly minimize entropy:

**Theorem.** For any \( \alpha > 0 \), there is \( \beta = \beta(\alpha) > 0 \) such that, if \( \Omega \subset \mathbb{R}^{n+1} \) is a compact region with \( \Sigma = \partial \Omega \) a closed hypersurface and \( \lambda(\Sigma) < \lambda(\mathbb{S}^n) + \beta \), then

\[
1 \leq \frac{R_{\text{out}}(\Omega)}{R_{\text{in}}(\Omega)} < 1 + \alpha.
\]

Using mean curvature flow, the result was proved by the authors in dimension two in [BW18] and generalized to all dimensions by S. Wang [Wan20].

**Further Developments and Questions**

Finally, we discuss various generalizations of the Colding-Minicozzi entropy and their use as measures of geometric complexity.
Higher codimension. As we previously observed, for the class of $n$-dimensional submanifolds of $\mathbb{R}^N$, the entropy minimizers are the affine $n$-planes. However, the situation for other interesting classes of submanifolds remains largely unexplored.

As in the codimension-one case, it is a consequence of the Brakke regularity theorem that nonflat $n$-dimensional self-shrinkers in $\mathbb{R}^N$ have entropy at least $1 + \varepsilon(n,N)$. Moreover, properties of mean curvature flow of submanifolds imply that the flow of closed submanifolds must form a finite time singularity. Hence, any closed submanifold must have entropy at least $1 + \varepsilon$. It is very plausible that the least entropy is achieved by a round $n$-sphere lying in an affine $(n+1)$-plane. However, there has been very little progress towards proving this. This is due, in part, to our limited understanding of stable minimal submanifolds in higher codimension.

Some compelling evidence that entropy measures complexity in this setting is provided by work of Colding-Minicozzi [CM20]. They show that bounds on a self-shrinker’s entropy provide effective bounds on its codimension. This means there is an estimate for $\varepsilon$ depending only on the dimension $n$ of the submanifold and not on the ambient dimension $N$.

Another interesting direction is to analyze closed entropy minimizers in other classes of higher codimension submanifolds. Of particular interest is the study of the question for closed Lagrangian submanifolds. This is because this condition interacts well with the mean curvature flow and spheres cannot be minimizers in this class.

Other ambient spaces. It is also desirable to generalize Colding-Minicozzi entropy to submanifolds of other ambient Riemannian manifolds.

One extension, to $n$-dimensional submanifolds of the round sphere, $\mathbb{S}^N$, was obtained by J. Zhu in his thesis. Essentially, he showed that the Colding-Minicozzi entropy of a submanifold of $\mathbb{S}^N$, thought of as a submanifold in $\mathbb{R}^{N+1}$ via the usual embedding of $\mathbb{S}^N$ into $\mathbb{R}^{N+1}$ remained monotone under mean curvature flow in $\mathbb{S}^N$. He concluded from this explicit lower bounds on the areas of topologically nontrivial minimal hypersurfaces in $\mathbb{S}^{N+1}$.

In another direction, the first-named author used the heat kernel on hyperbolic space, $\mathbb{H}^N$, to define a notion of entropy for submanifolds of hyperbolic space [Ber21]. This has subsequently been extended to the more general setting of Cartan-Hadamard manifolds—i.e., complete simply connected manifolds of nonpositive sectional curvature. Many of the existing results in Euclidean space discussed above carry over in some fashion to this setting. However, some new phenomena appear. For instance, rigidity of the ambient geometry holds inside certain entropy minimizers. There is also an interesting connection between the entropy of minimal submanifolds of $\mathbb{H}^N$ and a quantity associated to the asymptotic geometry of the submanifold.

In addition to finding further applications, it would be appealing to find a broader framework for these measures of complexity. At present the proposed generalizations are largely ad hoc.

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Uncovering Data Across Continua: An Introduction to Functional Data Analysis

Sophie Dabo-Niang and Camille Frévent

1. Introduction

Nowadays, advancements in data collection technologies like sensors, computer vision, medical imaging, IoT, and wearables have generated vast volumes of high-frequency data across various fields. These data are not just a collection of numbers and tables but a rich, dynamic tapestry of information that captures the essence of change over a continuum. Functional Data Analysis (FDA) [1, 2, 6, 13] efficiently handles large-scale, high-dimensional datasets, extracting valuable insights from data containing structured information.

Unlike traditional statistics dealing with discrete data points, FDA focuses on entire functions, curves, or shapes, providing insights into continuous changes. Whether analyzing time series, spatial data, growth curves, or any structured dataset, FDA excels at capturing ongoing change. FDA’s applications span various fields like medicine, biology, chemistry, economics, and environmental science, offering insights beyond isolated measurements. It aids in patient health tracking, economic trend analysis, and chemical or environmental management by modeling and understanding complex systems. In manufacturing, FDA can be applied to monitor continuous processes, such as chemical reactions, quality control measurements, and equipment performance. It helps detect deviations from the desired process behavior [12]. In computational biology, FDA involves studying complex biomolecular structures and understanding the relationship between organism shapes and functionality. FDA techniques are also applied to analyze longitudinal patient data [18], which are common in clinical trials. This enables the study of disease progression, treatment effectiveness, and personalized medicine. Furthermore, in biology, in the omics data context, gene expression data comprises measurements of gene expression levels across thousands of genes at multiple time points. By considering these data as a function of time, FDA [10] can help researchers better understand the general features and dynamics of gene expression, to identify key genes associated with specific diseases or biological processes, and to identify differences or similarities between genes.

In economics, FDA is employed to analyze longitudinal data, such as stock prices, gross domestic product trends, and inflation rates. It helps identify long-term patterns, cyclic behavior, and structural changes [6]. In environmental science, FDA is used to analyze temporal or space-time environmental data, such as temperature records, precipitation patterns, and ocean currents. It aids in understanding long-term climate trends and variability. FDA can be applied to study spatial data, helping to identify pollution hotspots, or study vegetation growth, and monitor land use changes over time.

In essence, FDA transcends traditional data analysis limitations by leveraging data with functionality, providing valuable statistical tools for researchers and professionals seeking deeper insights and solutions to complex problems. We explore FDA’s significance, mathematical foundations, practical applications, and future prospects to unveil its transformative potential.

2. The Significance of Functional Data Analysis

In various fields, FDA provides a powerful set of methods to model, analyze, and interpret data that exhibit continuous variation, allowing researchers and professionals to gain deeper insights, and make more accurate predictions and informed decisions based on the inherent functional nature of the data. This versatility makes FDA a valuable approach in a wide range of scientific and practical applications [16].

Employing mathematical domains like linear algebra, functional analysis, probability and statistics, FDA
manipulates and analyzes functions by representing data as observations of random variables in a function space. This allows operations like differentiation, integration, and smoothing, facilitating exploration of data structure and variations.

By treating data as functions, FDA helps uncover hidden patterns, relationships, and trends that would be challenging to discern using traditional statistical methods, leading to more informed decision-making and a deeper understanding of complex phenomena.

**FDA versus multivariate statistics.** Is it worthwhile to employ continuous representations, or are we unnecessarily adding complexity to our tasks? Given that discrete data is often needed for computational purposes, what are the benefits of utilizing continuous representations in our analyses?

While discrete data may offer computational convenience, the advantages of working with continuous representations are numerous. By viewing objects as functions, curves, or surfaces, scientists can unlock more powerful analysis techniques, yielding better practical results and more natural solutions. Grenander’s principle of discretizing as late as possible underscores the importance of retaining continuous representations for as long as feasible, highlighting their inherent value in data analysis workflows.

With this in mind, let us consider how continuous representations enhance our understanding and analysis of data.

If data are sampled from an underlying function (e.g., Figure 1 (a)), and time points are synchronized across observations, focusing solely on heights, then analysis can be conducted using the vector \( \mathbf{x} = (x_1, x_2, \ldots, x_L)^T \). If the time points hold significance as well, then it is necessary to retain them alongside the height data: \( \{(t_1, x_1), (t_2, x_2), \ldots, (t_L, x_L)\}^T \). With continuous functions one can interpolate and resample at arbitrary points (e.g., Figure 1 (b)) and easily compare observations with different time points, as elements of a function space.

In traditional data analysis, one might work with data points in a table where each row represents an observation (e.g., \( \mathbf{x} \)) and each column represents a variable. In FDA, the data are treated as functions, where each observation is considered as a function (e.g., in Figure 2 (a)) that maps a continuous variable (often time, frequency, wavelength or a spatial dimension) to a measured value. These functions represent how the data change over the continuum.

Understanding functions necessitates a profound grasp of the structures lying beneath them. Analyzing these structures requires a solid foundation of mathematical representations. The FDA approach empowers researchers to investigate various models extensively, thus expanding the comprehension of data characterized by functional structures across diverse fields of science and engineering. Examples of functional data are illustrated in Figure 2.

### 3. Mathematical Foundations of FDA

FDA involves a variety of specialized statistical techniques for handling functional data [13], including methods for function smoothing, visualization (e.g., plotting entire functions as curves or surfaces), dimension reduction (e.g., functional principal component analysis), functional regression, and clustering. These techniques account for the continuous nature of the data and are designed to capture underlying patterns and structures in functions. As said before, analyzing these functions involves mathematical representations.

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space, \( \mathcal{F} \) a function space (e.g., a separable Banach space or a Hilbert space). A functional random variable is a variable

\[
X = \{X(t), t \in \mathcal{F} \} : \Omega \to \mathcal{F},
\]

taking values in \( \mathcal{F} \) (of eventually infinite dimension). A functional data is then an observation of the functional random variable \( X \).
The latter is viewed as the kernel of the linear Hilbert-Schmidt operator \( \Gamma_X \) on \( \mathcal{F} = L^2(\mathcal{F}, \mathbb{R}) \): \( \Gamma_X : \mathcal{F} \to \mathcal{F} \), \( \Gamma_X f(t) = \int_{\mathcal{T}} C_X(t, s) f(s) \, ds \). Note that \( \Gamma_X \) admits the spectral decomposition \( \Gamma_X = \sum_{j=1}^{\infty} \lambda_j f_j \otimes f_j \), where \( (f \otimes g)(x) = \langle f, x \rangle g, x \in \mathcal{F}, \{f_j\}_j \) is a complete orthonormal system in \( L^2(\mathcal{F}, \mathbb{R}) \) and \( \{\lambda_j\}_j \) is a decreasing sequence of positive real numbers such that \( \sum_{j=1}^{\infty} \lambda_j < \infty \).

Let \( X_1, \ldots, X_n \) be an independent and identically distributed (i.i.d.) sample of \( X \).

The usual estimator of \( \mu_X \) is the method of moments estimator given by \( \hat{\mu}_X(t) = \frac{1}{n} \sum_{i=1}^{n} X_i(t) \).

In this i.i.d. framework, there are several theoretical guarantees regarding the convergence of \( \hat{\mu}_X \) to \( \mu_X \) (such as the law of large numbers, the central limit theorem, and concentration inequalities of the Bernstein type; see Chapter 2 of [1]). For instance, for the Hilbert space \( \mathcal{F} \) equipped with a norm \( \| \cdot \|_{\mathcal{F}} \), if \( E[\|X_i\|_{\mathcal{F}}] < \infty \) then \( \hat{\mu}_X \to \mu_X \) almost surely as \( n \to \infty \). If \( E[\|X_i\|_{\mathcal{F}}^2] < \infty \), \( \hat{\mu}_X \) is asymptotically normally distributed: \( \sqrt{n}(\hat{\mu}_X - \mu_X) \) converges in distribution to \( N(0, C_X) \).

Classic empirical estimators of the covariance operator \( \Gamma_X \) and covariance function \( C_X \) are \( \hat{\Gamma}_X = \frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\mu}_X)^\otimes (X_i - \hat{\mu}_X) \) and \( \hat{C}_X(s, t) = \frac{1}{n} \sum_{i=1}^{n} (X_i(t) - \hat{\mu}_X(t)) (X_i(s) - \hat{\mu}_X(s)) \). Several asymptotic results on \( \hat{\Gamma}_X \) are given in Chapter 4 of [1]. More theoretical details are given in this last reference.

From raw data to functional data: Note that, in practice, we observe raw data (e.g., the average daily temperature in spatial locations described by the first panel of Figure 4: the temperature in each location is measured every day from 1960 to 1994) of the form

\[
x_{i,t_{i,l_i}}, \quad t_{i,l_i} \in \mathcal{F}, \quad i = 1, \ldots, n \quad l_i = 1, \ldots, L_i.
\]

It should be noted that the observation times \( t_{i,l_i} \) can vary in number and value depending on the individual \( i \).

Following Zhang and Wang [20], we should distinguish the case of sparse and dense functional data. Dense functional data are characterized by the fact that all \( L_i \) are larger than some order of \( n \). In this case, it is possible to use a smoothing technique on the raw data \( x_{i,t_{i,l_i}} \) to recover the original curves.

It is common in FDA to assume that the \( L_i \) observations \( \{x_{i,t_{i,l_i}}\}_{i=1}^{n} \) are noisy observations of the smooth latent curve \( X_i(\cdot) \). Namely, we have \( x_{i,t_{i,l_i}} = X_i(t_{i,l_i}) + \varepsilon_{i,t_{i,l_i}} \), where the error terms \( \varepsilon_{i,t_{i,l_i}} \) are zero mean and i.i.d. In the early stages of FDA, this smoothing is typically conducted as an initial step by kernel smoothing, local polynomial smoothing, Fourier, spline, or penalized spline approaches.

The classic smoothing approach is basis expansion by assuming that \( X_i, 1 \leq i \leq n \) can be expressed as a finite combination of the first \( K \) functions of a basis function...
\{\phi_1, \ldots, \phi_K, \ldots\} \text{ of } L^2(\mathcal{T}, \mathbb{R}):

\[ X_i(t) = \sum_{k=1}^{K} a_{i,k} \phi_k(t). \]

This is equivalent to writing \( X_i(t) = \phi(t) a_i \), where \( \phi(t) = (\phi_1(t), \ldots, \phi_K(t)) \) and \( a_i = (a_{i,1}, \ldots, a_{i,K})^T \). Then the estimators of the mean and covariance functions of \( X \) can be defined respectively on \( \mathcal{T} \) by

\[
\hat{\mu}_X(t) = \frac{1}{n} \sum_{i=1}^{n} X_i(t)
\]

and \( \hat{\Sigma}_X(s, t) = \frac{1}{n} \phi(t) A^T A \phi(s)^T \),

where \( A \) is the \( n \times K \) matrix whose \( i \)-th row \( A_i \) is equal to \( (a_i - \frac{1}{n} \sum_{k=1}^{n} a_k)^T \). Depending on the nature of the data, various choices for the \( \phi_k \) are possible. In the case of periodic data, a Fourier basis is appropriate, whereas for nonperiodic data, possible choices are polynomial basis or splines basis. Figure 3 shows a transition from raw data to functional data using a cubic B-splines basis. This process is not applicable for sparse functional data due to the very limited quantity of information available for each curve. Sparse data require more sophisticated approaches not discussed here. For more details concerning basis options, please see [13] and for other smoothing techniques or more details on the sampling design of functional data, see [20].

![Figure 3](image.png)

**Figure 3.** Example of transition from raw data (gray connected points) to functional data (black curve) using a cubic B-splines basis (colored curves) on the gait dataset from the R package fda. The colored points correspond to the B-splines coefficients (with the same color as the splines).

With the growing popularity of functional data analysis, numerous statistical methods have been extended and adapted to this context. In the following discussion, we explore some of the most valuable and widely used ones (principal component analysis, clustering, linear and nonlinear regressions). It should be noted that we only consider the case of univariate functional data, i.e., for all \( t \in \mathcal{T}, X(t) \in \mathbb{R} \). The structure of multivariate functional data \((X(t) \in \mathbb{R}^p, p \geq 2)\) is more complex, please refer to [9] for more details.

**Functional principal component analysis:** Kleffe [8] has expanded upon the conventional and widely employed statistical approach known as principal component analysis (PCA) to accommodate variables with values in a separable Hilbert space. Then PCA has been extended to accommodate specifically univariate functional data (see e.g., [13]). This technique is particularly useful for reducing the dimensions of the data, extracting their main features, providing an indication of their complexity, and gaining insights into the underlying patterns and structures. The subsequent paragraph discusses the functional PCA method within the context of univariate functional data, noting that the case of multivariate functional data has also been investigated in the literature [13].

The above spectral decomposition of \( \Gamma_X \) is linked to the PCA on the \( X_i \). In fact, functional PCA aims to represent the i.i.d. curves \( X_i \) using a few \( (P) \) principal orthogonal eigenfunctions \( f_j \in L^2(\mathcal{T}, \mathbb{R}) \) so that

\[ X_i(t) \approx \mu_X(t) + \sum_{j=1}^{P} c_{i,j} f_j(t). \]

This is an approximation of the Karhunen-Loève (KL) expansion that states that

\[ X_i(t) = \mu_X(t) + \sum_{j \geq 1} c_{i,j} f_j(t), \]

where \( \{f_j\}_{j \geq 1} \) and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \) are the eigenfunctions and eigenvalues of \( \Gamma_X \). The \( c_{i,j} = (X_i - \mu_X, f_j) \) are called the scores, they extract the main features of \( X_i \) and are centered pairwise uncorrelated random variables.

Note that the KL expansion is related to Mercer’s theorem, that states

\[ C_X(s, t) = \sum_{j \geq 1} \lambda_j f_j(t)f_j(s). \]

Since the mean and covariance functions are unknown, in the early stage of FDA, applying a functional PCA is in practice equivalent to find estimated orthogonal eigenfunctions \( \hat{f}_j \) so that

\[ \forall t, \int_{\mathcal{T}} \hat{C}_X(s, t) \hat{f}_j(s) \, ds = \hat{\lambda}_j \hat{f}_j(t). \]

Hence, assuming that \( \hat{f}_j \) can be expressed as \( \hat{f}_j(t) = \phi(t)b_j \), the task is to find \( \hat{\lambda}_j \in \mathbb{R} \) and \( b_j \in \mathbb{R}^K \) so that

\[ \frac{1}{n} A^T A \int_{\mathcal{T}} \phi(s)^T \phi(s) \, ds \cdot b_j = \hat{\lambda}_j b_j. \]
By defining $W = f\phi(s)^T \phi(s) ds$ and $u_j = b_j^TW^{1/2}$, and then multiplying the previous equation on the left by $W^{1/2}$, we arrive at the classic PCA formula (where $Z = AW^{1/2}$):

$$\frac{1}{n} Z^T Z u_j^T = \lambda_j u_j^T.$$ 

Subsequently, it becomes straightforward to ascertain the values of $u_j$ and $\lambda_j$. Additionally, the determination of $b_j$ (and consequently $\hat{f}_j$) can be inferred through the following relationships:

$$b_j = (u_j W^{-1/2})^T, \hat{f}_j(t) = \phi(t) b_j.$$

Ultimately, the estimated scores $\hat{c}_{i,j}$ are given by $\hat{c}_{i,j} = \langle X_i - \hat{\beta}_X, \hat{f}_j \rangle = A_i W b_j$. It should be noted that, although not discussed here, other approaches for functional PCA have been proposed in the literature. A more complete review can be found in [15].

**Functional linear regression:** Numerous studies have explored regression modeling within the context of functional data [7,11]. In the following, we focus on presenting generalized functional linear models, which are designed to model a continuous response variable as a function of functional covariates.

In this framework, we consider a real-valued response variable $Y$ and a functional covariate $\{X(t), t \in \mathcal{T}\} \in \mathcal{L}^2(\mathcal{T}, \mathbb{R})$. Throughout the section, we assume that $X$ has been centered and that we have a sample $(Y_i, \{X_i(t), t \in \mathcal{T}\})_{i=1,...,n}$ of i.i.d. replications of $(Y, \{X(t), t \in \mathcal{T}\})$.

Generalized functional linear regression posits that the relationship between the response variable and the functional covariate is defined as follows:

$$E[Y|X(t), t \in \mathcal{T}] = g^{-1}(\eta),$$

$$\text{Var}[Y|X(t), t \in \mathcal{T}] = V(g^{-1}(\eta)),$$

where $g$ is a monotonic “link function”, $V$ is a positive “variance function” and $\beta$ is a linear predictor defined by $\eta = \alpha + \int_{\mathcal{T}} X(t) \beta(t) dt$.

The model finds practical applications in various scenarios, such as establishing associations between the incidence of respiratory diseases (e.g., asthma, lung cancer) and air pollution levels in the months or years leading up to the study. In such cases, a generalized functional linear Poisson regression model is often employed, where the link function $g$ is the logarithm, and $V$ is the identity function.

The simplest and most popular model is the so-called functional linear model, where $g$ is the identity function, and $V$ is a constant function:

$$Y_i = \alpha + \int_{\mathcal{T}} X_i(t) \beta(t) dt + \varepsilon_i.$$ 

The random variables $\varepsilon_i$ are assumed i.i.d., scalar variables with a mean of zero and a constant variance. Sometimes, an additional assumption of normality is made.

More generally, this linear model may encompass both functional ($X_i$) and nonfunctional ($Z_i \in \mathbb{R}^d, d \geq 1$) covariates, so that:

$$Y_i = \alpha + Z_i^T \theta + \int_{\mathcal{T}} X_i(t) \beta(t) dt + \varepsilon_i,$$

with $\theta \in \mathbb{R}^d$.

The primary challenge in FDA lies in dealing with the infinite dimension of the functional variable. A frequently adopted solution is to approximate $X_i$ as a finite combination of orthogonal basis functions (as mentioned earlier), as well as $\beta$. However, in practice, finding such a basis is not always straightforward. An orthonormal basis can be derived through functional PCA:

$$X_i(t) \approx \sum_{j=1}^P c_{i,j} f_j(t),$$

where the $f_j$ are the orthonormal eigenfunctions. By assuming that $\beta$ can also be written as $\beta(t) \approx \sum_{j=1}^P d_j f_j(t)$, we then obtain the following truncated linear regression model:

$$Y_i = \alpha + Z_i^T \theta + \sum_{j=1}^P c_{i,j} d_j + \varepsilon_i.$$ 

Beyond the linear model, this truncation procedure leads to a classic generalized linear model with covariates $c_{i,j}$ and $Z_i$.

**Functional data clustering:** Clustering is the process of organizing observations into clusters, where observations within each cluster share similar characteristics, while the characteristics of each cluster are distinct from those of others. Clustering methods can be broadly categorized into hierarchical, partitional, and model-based approaches. Researchers have explored adaptations of these categories to the functional data framework. In the case of hierarchical methods, a significant challenge arises in devising an appropriate similarity measure for functional observations. One approach to addressing this challenge was presented by Hitchcock et al. [5]. Among partitional methods, the most well-known technique is the K-means algorithm. It starts by randomly selecting $K$ points as the initial centers of $K$ groups and then assigns each observation to the group with the closest centroid. The centroids of the $K$ groups are then recalculated, and the observations are reassigned to the groups iteratively until convergence.

Then, the K-means algorithm relies on measuring the distance between observations, typically using the Euclidean distance for nonfunctional data. However, when dealing with functional data, this distance metric needs to be adapted. García et al. [4] conducted a study where they compared various approaches for modifying the K-means algorithm in the context of functional data.
Model-based clustering based on mixture of distributions have also been proposed. The interested reader may refer to [19] for a detailed examination of clustering approaches specific to functional data.

For more comprehensive details, methodologies, and applications, please refer to the reviews provided by Horváth and Kokoszka [6], Wang et al. [17], and Koner and Staicu [9].

4. Applications and Future Directions

A multitude of methods for handling functional data have been introduced, with many others yet to be discovered. In this section, we highlight the potential of functional data through an illustration of clustering using the well-known Canadian Weather dataset from the R package fda. We focus on the average daily temperature recorded every day from 1960 to 1994 in 35 spatial locations in Canada.

As said above, in practical applications, functional data are typically observed at discrete points, such as the 365 days of the year. However, it is possible to reconstruct the underlying functions by representing them in a basis of functions. The initial step of this process is depicted in Figure 4, where a B-splines basis has been employed.

To distinguish groups of Canadian cities based on their temperature patterns, we applied the $K$-means algorithm proposed by Sangalli et al. [14] and implemented in the R package fdacluster. The optimal number of groups was determined using the optimal average silhouette index and the resulting clustering results are depicted in Figure 5. We can observe that two distinct groups of Canadian cities emerge from the analysis: the first group (in green) corresponds to cities with consistently lower temperatures throughout the year, while the second group (in red) represents cities with consistently higher temperatures.

In addition to the functional aspect of the data, the spatial dimension is becoming increasingly relevant, particularly in the context of environmental data. Thus, the literature has seen the emergence of numerous methods specifically tailored to the analysis of spatial functional data. Recently, several spatial cluster detection methods have been introduced in this context. These methods can be used, for instance, to identify environmental hotspots characterized by elevated levels of certain pollutants.

In the following example, we will demonstrate a cluster detection approach that incorporates both the spatial and the functional nature of the data. This approach is the distribution-free functional spatial scan statistic (DFFSS) proposed by Frévent et al. [3], which has been implemented in the R package HDSpatialScan. The data used in this example are sourced from the National Air Quality Forecasting Platform (www.prevair.org) and are available within the package. They comprise the daily average concentration of the pollutant NO$_2$ recorded from May 1 to June 25, 2020, in northern France. Figures 6 and 7 present the raw data, their functional reconstruction using a B-splines basis, and their spatial distribution, respectively.

Figure 8 (left panel) displays the statistically significant cluster detected by the DFFSS (highlighted in red) in this air pollution dataset. The right panel compares the NO$_2$ concentration over the time within this cluster (in red)
Figure 6. Observed daily average concentration (left panel) and reconstructed functions (using a B-splines basis) for the daily average concentration (right panel) of NO$_2$ from May 1 to June 25, 2020 in northern France.

Figure 7. Spatial distribution of average NO$_2$ concentration over the period May 1 to June 25, 2020 in northern France.

Figure 8. Visualization of the detected cluster with the DFFSS method as well as the concentration curves of NO$_2$ (in $\mu g/m^3$) inside (in red) and outside (in gray) the cluster.

FDA's significance has grown significantly owing to its relevance across diverse domains and the advancements in data collection technology (please refer to [9] for further insights). The role of FDA in comprehending, analyzing, and harnessing these datasets is poised to expand further. FDA's utilization in burgeoning fields will influence the trajectory of data-driven innovation, decision-making, and issue resolution. Here is a glimpse of FDA's potential past and forthcoming contributions in these areas:

- **Healthcare and Personalized Medicine**: FDA can analyze patient data as continuous functions, allowing for personalized treatment plans based on individual health profiles. Real-time monitoring through wearables and FDA can aid in disease prediction and the optimization of treatment strategies.

- **Artificial Intelligence (AI) and Machine Learning (ML)**: FDA provides a nuanced representation of data, improving AI and ML models' performance in various applications. In speech recognition, it enhances accuracy by capturing the continuous nature of speech signals.

- **Climate Science**: In high-resolution climate data analysis, FDA identifies subtle patterns and trends, aiding in modeling, prediction, and mitigation strategies. Continuous data analysis contributes to precise climate projections and monitoring environmental changes.

- **Digital Marketing and User Behavior**: In the digital realm, FDA uncovers intricate user behavior patterns, optimizing marketing, user experience, and product recommendations. It analyzes continuous data streams from digital platforms for deeper insights.

- **Brain-Computer Interfaces (BCIs)**: FDA enhances BCIs by interpreting continuous brain activity...
data for prosthetics, neurorehabilitation, and cognitive augmentation. It enables precise control of assistive devices and cognitive enhancements.

- **Smart Cities**: In smart cities, FDA optimizes urban planning, transportation systems, and energy consumption by analyzing continuous IoT and sensor data. It helps design sustainable and efficient cities through traffic analysis and energy usage trends.

- **Biotechnology and Synthetic Biology**: In biotechnology, FDA models complex biological systems, facilitating the design of custom organisms and pharmaceuticals. It analyzes longitudinal data to engineer organisms for specific tasks.

**Software**: The Task View Functional Data Analysis on CRAN ([https://cran.r-project.org/web/views/FunctionalData.html](https://cran.r-project.org/web/views/FunctionalData.html)) lists the available R packages in the field of FDA, covering general functional data analysis, unsupervised learning (PCA, clustering, …), supervised learning (regression, classification), visualization and exploratory data analysis, registration, and alignment. Python and MATLAB also offer a few alternatives such as fda or f/da (for Python), and fda or fdaasrf (for MATLAB).

5. Conclusion

Functional Data Analysis (FDA) understands and extracts meaningful insights from data that continuously vary over a continuum. While FDA may be particularly intriguing for those with a mathematical inclination, it invites everyone to explore the process of transforming numbers into valuable insights and offers a statistical approach that allows us to gain a deeper understanding of the world and actively contribute to shaping the future.

Indeed, over time, the methods and frequency of data collection will evolve, and computing and storage capacities will increase. The development of functional analysis methods is therefore essential, and their applications will improve decision-making in a variety of fields, providing biologists, economists, and policymakers with accurate information to make informed choices.

FDA is therefore a powerful field for understanding, analyzing, and using complex datasets. Thus, next time you see a graph, don’t just see points and lines, but look for the continuous story it tells, the hidden patterns it holds, and the insights it offers. Remember what you just read: this is the realm of functional data analysis, where numbers transform into narratives waiting to be discovered.

A longer version of this paper is available on arXiv ([https://arxiv.org/abs/2404.16598](https://arxiv.org/abs/2404.16598)) to provide additional references and details.

**References**


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Uniformization and the Yamabe Problem

Stephen E. McKeown and Cheikh Birahim Ndiaye

Introduction

Conformal geometry is the subfield of differential geometry that studies manifolds on which infinitesimal angles are defined, but not lengths; that is, the crossing angle between any two intersecting curves is well-defined, but their lengths are not.

Given a manifold $M$, recall that a Riemannian metric $g$ is a (smoothly varying) choice $\langle \cdot, \cdot \rangle_g$ of inner product on each tangent space $T_p M$. Now, given any inner product $\langle \cdot, \cdot \rangle$ on a vector space $V$, the angle $\theta$ between two vectors $X, Y \in V$ may be defined by

$$\cos(\theta) = \frac{\langle X, Y \rangle}{\sqrt{\langle X, X \rangle \langle Y, Y \rangle}}.$$ 

The right-hand side remains unaltered if $\langle \cdot, \cdot \rangle$ is replaced by $c\langle \cdot, \cdot \rangle$ ($c > 0$). Thus, angles are well-defined given only a ray of inner products. For this reason, a conformal manifold is usually defined as a manifold $M$ equipped with an equivalence class $[g]$ of metrics, where $\tilde{g} \sim g$ if $\tilde{g} = e^{2\omega}g$, with $\omega \in C^\infty(M)$ a smooth function. (The mode of expression is conventional: the exponential forces the scaling factor to be positive, while the factor $2$ is merely for convenience. Other conventions are also popular.)

One of the numerous reasons for the ubiquity of conformal geometry within differential geometry is that a conformal change—replacing $g$ by $e^{2\omega}g$—is about the simplest nontrivial transformation that can be performed on a metric while preserving some of its geometric content. That the change is scalar also makes it analytically quite practical to study. Just what is preserved under such a change leads to one large set of questions within the field. Thus, for example, the Weyl tensor, which is the totally tracefree part of the Riemann curvature tensor, transforms according to the simple rule $\tilde{W} = e^{2\omega}W$. We will focus on another set of questions: what can be changed in a useful way? Put differently: given a Riemannian metric $g$, is there a conformal metric $e^{2\omega}g$ with especially nice properties? (Can it perhaps be used to study $g$?) This question has been extremely influential in geometric analysis, and we will be concerned with several versions of it.

Beginnings: Two Dimensions

Conformal geometry in two dimensions is a very different subject from that in higher dimensions, because of its intimate relationship with complex analysis. Every nonsingular holomorphic function between regions of the complex plane is angle-preserving (conformal). Thus, if it is also bijective, it is in fact a conformal equivalence—from the point of view of conformal geometry, two regions connected by such a function are equivalent. The Riemann mapping theorem may thus be viewed as saying that every simply connected proper open subset of the plane is conformally equivalent to the unit disk.

A natural question is whether the same might in some sense be true of a compact Riemann surface $(\Sigma^2, g)$. Of course, topological considerations forbid the existence of a homeomorphism between $\Sigma$ and the disk, but might we at least expect the existence of a smooth function $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega}g$ has vanishing Gaussian curvature? (This would imply that $\tilde{g}$ is at least locally isometric to the disk.) In fact, the answer is certainly no: the Gauss-Bonnet theorem asserts that the Gaussian curvature $K_g$ of any metric satisfies

$$2\pi \chi(\Sigma) = \int_{\Sigma} K_g dV_g,$$

where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Thus, unless $\chi(\Sigma) = 0$ (i.e., unless $\Sigma$ is a torus), there can be no such function. However, a significant classical result asserts that we can obtain the best that Gauss-Bonnet allows.

Theorem 1 (Uniformization theorem). Let $(\Sigma, g)$ be a compact Riemann surface. There exists $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega}g$ has constant Gaussian curvature $K_g$. Assuming the normalization $\text{vol}_g(\Sigma) = 1$, we have $K_g = 2\pi \chi(\Sigma)$.

There are many proofs; we mention [3]. Note that, because Gaussian curvature fully determines the curvature in two dimensions, it follows from this theorem that every
surface is conformal to one that is locally isometric to either the plane, the sphere, or hyperbolic space.

One might very well ask what the situation is for a surface with boundary. If $\Sigma$ has a boundary, Gauss-Bonnet reads

$$2\pi \chi(\Sigma) = \int_{\Sigma} K_g dV_g + \oint_{\partial \Sigma} k_g ds,$$

where $k_g$ is the geodesic curvature of the boundary. Thus, in this case, Gauss-Bonnet does not forbid obtaining $K_g \equiv 0$, as long as $k_g$ is nonzero. Sure enough, the uniformization theorem for surfaces with boundary states that a conformal change lets us push the topological data to the boundary, and make it constant there.

**Theorem 2** (Uniformization theorem for surfaces with boundary). Let $(\Sigma, g)$ be a compact Riemann surface with nonempty boundary. There exists $\omega \in C^\infty(\Sigma)$ such that $\tilde{g} = e^{2\omega} g$ has vanishing Gaussian curvature $K_{\tilde{g}}$ and the boundary has constant geodesic curvature $k_{\tilde{g}}$.

See, for example, [1].

**Higher Dimensions: The Yamabe Problem**

In higher dimensions, complex analysis no longer plays an important role in conformal geometry, and the methods of proof are often quite different. Indeed, two dimensions is much more “flexible,” since the Liouville theorem shows that the space of conformal maps of $\mathbb{R}^n$ for $n \geq 3$ is finite dimensional, in contrast to the two-dimensional case. Yet it is natural to ask very similar questions. One of the most celebrated problems of twentieth-century differential geometry is the Yamabe problem, raised by Hidehiko Yamabe in 1960. This asks whether the uniformization theorem holds for compact manifolds $M^n$ when $n \geq 3$. That is, given a Riemannian manifold $(M^n, g)$, does there exist $\omega \in C^\infty(M)$ such that $\tilde{g} = e^{2\omega} g$ has constant scalar curvature $R_{\tilde{g}}$? (To look for constant higher-rank curvature would yield a vastly overdetermined problem, since a conformal change is itself a scalar.) Actually, it is more common in this context to write the conformal change as $\tilde{g} = u^{\frac{4}{n-2}} g$, requiring $u > 0$; this is merely conventional, and is because the problem becomes easier to express and study when written in terms of

$$u = e^{(n-2)\omega/2}. \tag{1}$$

Yamabe believed he had proven that the answer is yes, but seven years after his tragically early death in 1960, Neil Trudinger discovered a significant error in his path-breaking paper. Partial solutions to the Yamabe problem were given over the next ten years by Trudinger and by Thierry Aubin, and in 1984, Richard Schoen finally solved all the remaining cases using the positive mass theorem.

**Theorem 3** (Yamabe, Trudinger, Aubin, Schoen). Suppose $(M^n, g)$ is a compact Riemannian manifold, $n \geq 3$. Then there exists a positive $u \in C^\infty(M)$ such that $\tilde{g} = u^{-\frac{4}{n-2}} g$ has constant scalar curvature $R_{\tilde{g}}$.

Note that in higher dimensions, the Chern-Gauss-Bonnet theorem says nothing about $\int_M R_g dV_g$, so whether the constant given by this theorem is positive, negative, or zero is a property more of the conformal geometry of $[g]$ than of the topology (although the two are not unrelated). The condition $R_{\tilde{g}} = c$ (with $c$ constant) is equivalent to $u$ solving the PDE

$$P_2 u = \frac{u^{n+2}}{n+2}, \tag{2}$$

where $P_2 u = 4\frac{n-1}{n-2} \Delta_g u + R_g u$ is the conformal Laplacian of $u$. (The conformal Laplacian is so named because it satisfies the relatively nice conformal transformation property $\tilde{P}_2(u^{-1}) = u^{-\frac{n+2}{n-2}} P_2(f)$.) One thus wishes to find a strictly positive solution to (2). The original approach to solving the Yamabe problem is variational: (2) is the Euler-Lagrange equation of the functional

$$\mathcal{E}(u) = \frac{\int_M \left(4\frac{n-1}{n-2} |du|^2 + R_g u^2\right) dV_g}{\left(\int_M |u|^{2n} dV_{\tilde{g}}\right)^{\frac{n-2}{n}}} , \quad u \in C^\infty(M), u > 0.$$ 

It follows easily by Hölder’s inequality that $\mathcal{E}$ is bounded below, so one can take a minimizing sequence; the challenge is then to show that it converges to a solution to (2), and that said solution is positive. The analytical and geometric ideas that go toward showing that this is the case are ponderous. First, one replaces the exponent $\frac{n+2}{n-2}$ in (2) by $\frac{n+2}{n-2} - \varepsilon$, as the former is the “critical exponent” for which the relevant Sobolev injection fails to be compact. This process of subcritical regularization yields a compact variational problem. Finding a solution $u_\varepsilon$ via the direct method in the calculus of variations, the question then becomes whether this converges to a solution $u$ as $\varepsilon \to 0$. This is addressed by studying the Euler equation associated to $u_\varepsilon$.

Let $Y([g]) = \inf_u \mathcal{E}(u)$, which one may show is a conformal invariant. It was shown by Trudinger and Aubin that if $u_\varepsilon$ does not converge to a solution $u$, then we must have $Y([g]) \geq Y([S^n, \tilde{g}])$, where the latter is the conformal round sphere; so a solution exists if $Y([g]) < Y([S^n, \tilde{g}])$. The problem is thus reduced to showing that the latter inequality holds. This can be done in many cases by constructing clever test functions $u$ that realize the inequality. In the work of Aubin, the test function is constructed using local geometry. Schoen’s approach in solving the remaining cases used global geometry to construct test functions. Indeed, one of his key ideas in completing the proof was to use the Green’s function of $P_2$ itself as a conformal change.
factor; since it blows up at a point, this transforms the manifold to a noncompact but asymptotically flat space. A careful study of the asymptotics of the Green’s function allows the problem then to be solved by appealing to the positive mass theorem of Schoen-Yau, which arose from General Relativity. The full details, including all the original references, are given in the well-known survey [7]. We will refer to the manner of argument described here as the Aubin-Schoen minimization technique. There is another variational approach to the Yamabe problem by Bahri-Coron and Bahri-Brezis, called the barycenter technique, using algebraic topological tools.

Once again, there are natural related problems to ask in the case of compact manifolds with boundary. The one we will consider here can be considered a natural analog of the Riemann mapping theorem and of Theorem 2. In higher dimensions, the appropriate boundary curvature to consider is the mean curvature $H_g$, rather than the geodesic curvature. The following theorem was first studied by Pascal Cherrier and was established in most cases by José Escobar in 1992; the remaining cases were treated by Sergio Almaraz, Sophie Chen, Fernando Coda Marques, and Martin Mayer and Cheikh Ndiaye.

**Theorem 4.** Let $(M^n, g)$ be a compact manifold with boundary, where $n \geq 3$. There exists $u \in C^\infty(M)$ with $u > 0$ such that $\tilde{g} = u^{-\frac{4}{n-2}}g$ has vanishing scalar curvature $(R_g = 0)$ and constant mean curvature: $H_g = \text{const}.$

(Escobar also treated the reverse problem, where $R_g$ is constant and $H_g = 0$.) The works of Escobar, Almaraz, Chen, and Marques used the Aubin-Schoen minimization approach. Indeed, numerous techniques were developed in the proof of these theorems. Escobar developed a highly refined asymptotic expansion of the Green’s function of the conformal Laplacian near the boundary, as well as a positive mass theorem in the bounded setting. Subsequent developments were often based on ever-finer constructions of local or global test functions. The final step, by Mayer and Ndiaye, used the Bari-Coron barycenter technique. Of the many papers we could cite in this line, we mention [5].

**Fourth Order**

The scalar curvature is not the only scalar function that is important to understanding the curvature of a Riemannian manifold. Since it was introduced by Tom Branson and Bent Ørsted, the so-called Q-curvature has been an object of intense study. This is a local curvature invariant. Define $J = \frac{R}{2(n-1)}$ and the Schouten tensor by $P = \frac{1}{n-2} (\text{Ric} - Jg)$. Then $Q$ is defined by $Q = -\Delta J - 2|P|^2 + \frac{nJ}{2}$; thus, it is fourth-order in derivatives of the metric tensor. We will mention here two of its numerous interesting properties.

First, let the Paneitz operator be the fourth-order operator defined by $P^4 u = \Delta^2 u + \delta((4P - (n-1)J)g)du + \frac{n-3}{2} Qu$, where $\delta$ is the adjoint (with respect to $g$) of the exterior derivative, and the two-tensor in the second term acts as an endomorphism on one-forms via $g^{-1}$. Now, this linear elliptic operator is a pointwise conformal invariant in the sense that, if $\tilde{g} = e^{2\omega}g$, then $\tilde{P}^4 u = e^{(n-2)\omega} P^4 (e^{(n-2)\omega} u)$; this is analogous to the conformal change property of the conformal Laplacian. The first interesting property of the $Q$-curvature, then, is that under a conformal change, when $n = 4$, the $Q$-curvature transforms by $\bar{Q} = e^{-4\omega}(Q + P\omega)$, which is exactly analogous to the relationship between the Gaussian curvature and the Laplacian when $n = 2$; while if $n \geq 5$, it transforms by $\bar{Q} = u^{\frac{n+4}{n-4}} P\omega$, which is analogous to the relationship (for $n > 2$) between the scalar curvature $R$ and $P\omega$. Here, $u = e^{\omega}$, much as in (1). The second interesting property is that, in dimension four, the Chern-Gauss-Bonnet formula can be written in such a way as to include $Q$:

$$4\pi^2 \chi(M^4) = \frac{1}{2} \int_M \left( \frac{1}{4} |W|^2 + Q \right) dV_g,$$

where $W$ is the Weyl tensor of $g$. Now, $|W|^2 dV_g$ is an absolute pointwise conformal invariant, so $\int_M Q dV_g$ is a global conformal invariant. Given these properties, it becomes irresistible to see $Q$ playing a rather exact fourth-order analog to the second-order $R$, at least in conformal geometry.

The question then arises: can one make a conformal transformation $\tilde{g} = e^{2\omega}g$ so that $Q_{\tilde{g}}$ is a constant? In the case $n = 4$, due to the Chern-Gauss-Bonnet formula and the conformal invariance of total $Q$, this has the flavor of the uniformization theorem. In higher dimension, it is the fourth-order analog of the Yamabe problem. The affirmative answer to the fourth-order uniformization question in dimension four was given in two steps in the papers [2, 4] by Alice Chang and Paul Yang and by Zindine Djadli and Andrea Malchiodi, which are landmarks of fourth-order nonlinear geometric analysis.

**Theorem 5.** Suppose $(M^4, g)$ is a Riemannian manifold, and let $k_p = \int_M QdV_g$. Suppose that $\text{ker} P_g \cong \mathbb{R}$ and that $k_p \not\in 16\pi^2 \mathbb{N}$. Then there exists a function $\omega \in C^\infty(M)$ such that $\tilde{g} = e^{2\omega}g$ has constant $Q$-curvature.

The work [2] can be seen as a $Q$-curvature version of the Aubin-Schoen minimization approach. Elaborate additional ideas were needed to approach the problem, using the Adams(-Moser-Trudinger) inequality. The work [4], by contrast, can be seen as a min-max analog of the Bahri-Coron algebraic topological argument, based on an improved version of the Adams inequality.

Just as mean curvature is a natural first-order boundary curvature associated to scalar curvature, in four dimensions there is a natural third-order boundary curvature
associated to $Q$, the so-called $T$-curvature defined by Alice Chang and Jie Qing. This curvature has leading part $T = -\frac{\partial}{\partial n} R_g$, where $\frac{\partial}{\partial n}$ is the inward unit normal field. One can define a linear boundary operator $P_3 : C^\infty(M) \to C^\infty(\partial M)$ with leading part $P_3 u = \frac{1}{2} \partial_n \Delta_g u + \Delta_g \frac{\partial}{\partial n^3} u$ that is conformally invariant in the sense that $P_3 = e^{-2\omega} P_3$, and such that it controls the conformal behavior of $T$ according to the formula $\mathcal{F} = e^{-3\omega}(T + P_3 \omega)$. The curvature $T$ appears in the Chern-Gauss-Bonnet formula with $Q$ according to the equation

$$4\pi^2 \chi(M^4) = \frac{1}{2} \int_M \left( \frac{1}{4} |W|^2 + Q \right) dV_g + \oint_{\partial M} (L + T)dV_\partial,$$

where $L$ is a pointwise conformally invariant curvature quantity. A natural fourth-order generalization of the boundary uniformization problem, in four dimensions, is thus to ask whether we can make a conformal change that will set $Q$ to zero and $T$ to a constant. The answer is generically yes, as shown by Ndiaye [9].

**Theorem 6.** Suppose $(M^4, g)$ is a Riemannian manifold with boundary. Let $H_g$ be the mean curvature, and $k_p = \frac{1}{2} \int_M Q dV_g + \int_{\partial M} T dV_\partial$; and suppose that $H_g = 0$, that $k_p \notin 4\pi^2 \mathbb{N}$, and that only constant solutions exist to the system

$$\begin{align*}
P_3 u &= 0 \quad \text{in } M, \\
P_3 u &= 0 \quad \text{on } \partial M, \\
\frac{\partial}{\partial n} u &= 0 \quad \text{on } \partial M.
\end{align*}$$

Then there exists $\omega$ such that $\bar{g} = e^{2\omega} g$ has $Q_{\bar{g}} = 0$, $T_{\bar{g}} = \text{const}$, and $H_{\bar{g}} = 0$.

The proof uses the min-max method of Djadli-Malchiodi.

Extending the fourth-order problem to higher dimensions than four required significant new techniques. One of the challenges of studying fourth-order elliptic equations, as opposed to second-order ones, is that there is no maximum principle in the case of the former. It was thus a significant achievement when Matthew Gursky and Malchiodi [6] showed that, under certain circumstances, the Paneitz operator on a closed manifold does satisfy a maximum principle:

**Theorem 7.** Suppose that $(M^n, g)$ is a closed Riemannian manifold, with $n \geq 5$, and that

$$\begin{align*}
Q_g &\geq 0 \text{ with } Q_g > 0 \text{ somewhere; and} \\
R_g &\geq 0.
\end{align*}$$

If $u \in C^4$ satisfies $P_3 u \geq 0$, then either $u > 0$ or $u \equiv 0$.

Moreover, under these hypotheses, $P_3$ is a positive operator.

In the same paper, they also proved a fourth-order positive mass theorem for the Green’s function of the Paneitz operator for the global case. Using these remarkable results, they were able to prove a fourth-order Yamabe theorem in higher dimensions.

**Theorem 8.** Suppose that $(M^n, g)$ is a closed Riemannian manifold with $n \geq 5$, and that (3) holds. Then there exists $u \in C^\infty(M), u > 0$, such that $h = u^{4\over n-4} g$ has constant positive $Q$-curvature.

The proof in [6] is by a nonlocal flow. One can also use a variational proof à la Aubin-Schoen using Theorem 7 and the positive mass theorem in [6]; this was pointed out quickly by Emmanuel Hebey and Frédéric Robert. Shortly thereafter, Fengbo Hang and Yang weakened the condition on scalar curvature to nonnegative Yamabe constant. This improvement by Hang-Yang and the observation by Hebey-Robert were made even before final publication of [6]; see the discussion and references in that paper.

Jeffrey Case introduced boundary curvatures analogous to $T$ in dimension higher than 4; the third-order one, he calls $T_3$. There is also an associated conformally invariant boundary operator of third order, which we also call $B_3$. An argument in [10] enabled the proof of the following fourth-order Escobar theorem in higher dimensions.

**Theorem 9.** Suppose that $(M^n, g)$, $n \geq 5$, is a compact Riemannian manifold with boundary. If $H = 0$ and the Green’s function defined by

$$\begin{align*}
P_4 G(x, y) &= 0 \quad \text{in } M \\
B_3 G(x, y) &= \delta_x \quad \text{on } \partial M \\
\frac{\partial}{\partial n} G(x, y) &= 0 \quad \text{on } \partial M
\end{align*}$$

is positive, then there exists a conformal metric $\bar{g} = u^{4\over n-4} g$ satisfying $\bar{Q} = 0$, $\bar{T}_3 = \text{const}$, and $\bar{H} = 0$.

The equation (4) is dual to the problem for the fractional curvature $Q_{3/2}$ in the Poincaré-Einstein setting. The argument in [10] works for the case of (4) as well. The technique of proof is similar to that of Mayer-Ndiaye.

**Higher Order**

The existence of conformally invariant operators and associated curvatures such as $P_5, P_3$ and $R, Q$ is not an isolated phenomenon. In 1992, Robin Graham, Ralph Jenne, Lionel Mason, and George Sparling showed that there exist conformally covariant operators $P_{2k}$ of every order $2k \leq \dim M$, having principal part $\Delta_{g}^{k}$. Branson defined an associated $Q_{2k}$ curvature so that, in dimension $2k$, the $Q_{2k}$-curvature transforms by

$$\bar{Q}_{2k} = e^{-2k\omega}(Q + P_{2k} \omega)$$

(with a somewhat more complicated transformation formula in higher dimensions). Branson’s original construction was by analytic continuation in the dimension. The
GIM5 operators, meanwhile, were defined via the so-called ambient-metric construction of Charles Fefferman and Graham. This allows the study of a conformal manifold by situating it in a Lorentzian manifold of two dimensions higher, much as the sphere could be studied by situating it in the light cone in Minkowski space of two higher dimensions. Relationships between the two were illuminated when Graham and Maciej Zworski showed that each could be defined in terms of the scattering operator on an asymptotically hyperbolic (formally) Einstein space whose boundary was the given manifold.

The higher-order uniformization problem for $Q_{2k}$ (sometimes called the critical case), where $\dim M = 2k$, has been solved by Ndiaye [8] using the min-max argument of [4].

**Theorem 10.** Suppose $(M^{2k}, g)$ is a closed Riemannian manifold, that $\ker P_k = \mathbb{R}$, and that $\int_M Q_{2k}dV_g \notin (2k-1)!\omega_{2k}\mathbb{N}$, where $\omega_{2k}$ is the volume of the $2k$-sphere. Then there exists $\omega \in C^\infty(M)$ such that $e^{2\omega}g$ has constant $Q_{2k}$-curvature.

As for the lower-order curvatures, in the critical dimension there is again a Chern-Gauss-Bonnet formula, due to Spyros Alexakis, connecting the total $Q_{2k}$-curvature and the topology.

The higher-order Yamabe problem (sometimes called the noncritical case), where $\dim M > 2k \geq 6$, was solved quite recently by Saikat Mazumdar and Jérôme Vétois in the setting where the local test function argument works, or if the positive mass theorem holds, using the Aubin-Schoen minimization technique. For example, we mention the following:

**Theorem 11.** Suppose $(M^n, g)$ is a closed Riemannian manifold, that $1 < k < \frac{n}{2}$, and that the Green’s function of $P_k$ is positive. Assume moreover that the $2k$th Yamabe constant $Y_{2k}$ satisfies

$$Y_{2k} := \inf_{g \in [g]} \left( \text{vol}_g(M)^{\frac{n-2k}{n}} \int_M Q_{2k}dV_g \right) > 0.$$ 

If $2k+1 \leq n \leq 2k+3$ or $(M, g)$ is locally conformally flat, then assume also that the mass of the Green’s function of $P_k$ is positive somewhere on $M$. Then there exists $u \in C^\infty(M)$ such that $u^{n-2k}g$ has constant $Q_{2k}$-curvature.

The mass of the Green’s function mentioned here is the constant term in the asymptotic expansion (in especially nice coordinates) of the Green’s function; it is so named because of its relationship, in the $k = 1$ case, with the Arnowitt-Deser-Misner mass of general relativity. One expects that the Bahri-Coron argument should work to yield the remaining cases of this theorem, still assuming the Green’s function is positive.

**Conclusion**

The Yamabe problem and its generalizations have been among the driving forces of conformal geometry in the past sixty years. A natural class of problems, they have required the development of new and powerful insights both analytic and geometric for their solution, as well as novel algebraic-topological arguments. Nor are they of interest merely as curiosities: as the ever-growing number of theorems whose hypotheses include some assumption on the Yamabe constant, once proved they have become one of the most useful tools of the conformal geometer. Leverage in problems can be obtained by making a conformal change so that an appropriate curvature quantity is constant or vanishing.

There remains a large landscape of problems not yet solved, which will require yet new insights; and we have not even discussed all those that have already been attacked. Fully nonlinear Yamabe problems, nonlocal fractional curvatures, higher order boundary curvatures, CR Yamabe questions, extrinsic curvature on corners, and many other problems have seen enormous work and remain at the focus of bustling activity. The source of a whole vast literature, the Yamabe problem stands as an impressive reminder of the power of asking the right question.

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This book contains the proceedings of the International Colloquium on Arithmetic Geometry organized by the Tata Institute of Fundamental Research in January 2020, one of a series of colloquia held every four years since 1956. This volume contains original contributions, as well as expository material, by leading experts on a range of topics in arithmetic geometry, including the arithmetic of local and global fields, Galois representations and their deformations, Shimura varieties, and automorphic forms.

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“What Do You Want?” One Mathematician’s Attempt to Find Balance

Matthew A. Pons

I was chatting with a colleague recently and told her I was writing a piece on mental well-being and work-life balance. I shared some of what I was planning to say, and she asked, “Are you going to tell them it’s a myth?” I was a little astounded by her response and asked her to elaborate. She explained that she had been working as a professor for over 40 years and still was nowhere near a healthy work-life balance. As I pondered her statement, I thought about how much time we spend talking about work-life balance compared to how many of my colleagues feel similarly to my friend, myself included at times. Part of the problem is the high demands of our jobs. Because of the current economic state of our country, budgets are tight and fewer faculty are being hired, which means existing faculty have more demands on them. Another complicating factor is that we are all unique individuals with extremely specific needs. It’s no wonder our employers have a difficult time enacting policies that we all benefit from. By necessity then, we are each responsible for setting our own boundaries so we can achieve a reasonable balance in our lives. And, in my opinion, the populations affected most by this complicated problem are those new to our profession and/or those in positions with variable job security, in other words, those least likely to feel comfortable setting boundaries with their employers. The question then is, what advice can I possibly offer?

As a disclaimer, I am no expert in mental health, though I have been actively sharing my story through the Living Proof Blog (first hosted by the AMS and now by the MAA). I am a human who has collected experiences, both good and bad, and who tries daily to work toward balancing the demands on my time with my own mental and physical well-being. And while I don’t always succeed, I have learned a tremendous amount about myself through my efforts. To be frank, my fear in writing this piece is that I will unintentionally cause harm because so much of this work is deeply personal, and I know that our experiences might not align. Below, I will share what has helped me, and I hope that you can work toward solutions that work for you.

After the first year in my tenure track position, I was fortunate enough to be selected to Project NExT (New Experiences in Teaching), the Mathematical Association of America’s professional development program for new PhDs. In one session, a small group of us were discussing our first year of teaching and I had asked for advice on balancing everything that was expected of me. I was prepared to teach and do research, but the other commitments that came with the position were overwhelming me. Yes, some of them were challenging and enjoyable, and some were not, but sustaining the pace felt impossible. One of the program leaders, T. Christine Stevens, was present and she asked, “What do you want?” I had never been asked that before. Honestly, I didn’t think what I wanted mattered. What mattered was that I did my job well and did everything asked of me. I responded with, “I don’t understand. What do you mean?” Chris has a way of asking a question

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1Project NExT accepts applications every April. The program targets mathematicians in their first few years of postgraduate study and provides workshops and mentoring that touch on all aspects of an academic career.
that gets to the point precisely, while also letting you know that she genuinely wants to know what's on your mind. So, she repeated, "What do you want? What balance would work for you?"

It’s been fifteen years since that conversation, and I still reflect on it regularly. The good news is that I do know what I want. I want to do good work. I want to teach well. I want to be an advocate for students, faculty, and staff who need someone to speak up for them. I want to be recognized for my work. I want to be paid a fair wage for my work. I want to have time to travel and not feel guilty about not working. I want to enjoy my life without always worrying about work. But putting all of that into a formula that will yield a work-life balance that I’m happy with does not always seem possible. However I have developed practices I regularly engage in that help me to be more intentional with my commitments. Below I share what currently works for me. My best advice is to look for what works for you.

1. **Fairness in allocation:** Before each semester, I make a schedule for how long I intend to be on campus working during the week. I usually aim for 40 hours a week, and I do my best to stick to it (that doesn’t mean I’m only working 40 hours a week). I have an hour commute each way and staying at work longer than 8 hours a day eats into time that I need to be spending on other tasks in my life. There might be days I have to stay later, and I try to be as flexible as I can, but I also monitor myself closely. If my mental stamina is low, or I feel like my emotional balance is off, then I borrow time for myself as I need it.

   I think many of us work at odd hours. For me Friday and Saturday nights usually involve what some would call work. I might be doing research, but I don’t consider this work because I think of my research as my own. Of course, it is expected that I do research, but, more importantly, it brings me joy and it stimulates my own development as a mathematician and a human being. And that works for me because I get to decide what goes in the work bucket and what goes in my bucket. Yes, there will be some overlap, but sometimes I think we forget that we can hold things for ourselves.

   The last thing I want to share here is a tough one, especially if you are pretenure or contingent faculty, and is even more complicated for those of us from marginalized demographics. But we all have the right to not allow anyone to impose unfair pressures on us. It’s easier said than done, I know, but please be mindful of yourself. Getting a job, getting research done, and publishing, all while learning how to teach and mentor, and balancing service on top of it all is overwhelming. If the demands of your job do not align with your personal need for balance, then consider a change. That is hard advice. The market is not great and job searches are a lot of work, but there is a job out there that will value what you offer. It might take time to find a fit that works for you.

2. **Mindfulness practice:** I was diagnosed with depression as a graduate student, and that adds a complicating factor to my life. Over the past 20 years, I’ve worked to cultivate a mindfulness practice that helps me manage my depression. Part of this practice is a focus on gratitude. Over the years I have conditioned myself to be aware of things I can appreciate about my day, and this focus provides a way to dispel negative thoughts that creep in. For me, gratitude is like endorphins released after a good workout. Another part of my practice is regular exercise; I lift weights, I do yoga, and I bike. Basically, I try to move my body daily. In working through my depression in therapy, I learned that accomplishing something, anything, or a moment of gratitude can get my mind moving in a functional direction. It’s not about forcing myself to be positive or avoiding the problem. My depression presents as feelings of worthlessness and an inability to function. My therapist helped me develop counter habits that build an outlook on life that helps me see reality from a grateful point of view (even when things are hard). And yes, there are days where it is difficult to function and that is OK. Self-care must include self-compassion.

Mathematics itself helps me in this work. I wrote the following for a blog post about my depression: I’ve also realized that mathematics is actually a great tool for someone like me. Instead of obsessing over proving the theorem, I now end a day of research looking back on what I have learned. It’s about finding the positive. Ok, so I didn’t prove the theorem, but I learned why my method is insufficient. Or, I learned about a new result that may prove useful in my research. Or, I went back to the basics and dug into the foundational aspects of the problem, solidifying my overall understanding. Hell, somedays I just work some fun calculus problems. Learning is one of the strongest tools I have. The fact that I live and work in a discipline that has an endless supply of things to learn is a huge blessing. I know that there is more pressure on graduate students and pretenured faculty to “prove that theorem,” but this perspective can still be useful. You are not defined by your mathematics. It’s a cliché, but I wish that we were more acculturated to focus on the journey rather than the destination.
Once I understood more holistically the value that my mathematical journey adds to my life, it completely changed my relationship with mathematics. It also almost entirely erased impostor syndrome from the equation.

A question I get asked a lot concerns balancing my mindfulness practice with the unexpected. The answer I have come to is that yes, there will always be situations that I cannot control. What I can control are my emotions. We all get to choose how we respond to the things life throws at us. It takes hard work to manage one’s emotions well, and I’m still on my journey, but over the last few years I have learned a lot in this regard, and I recommend Brené Brown’s books and various lectures (available through TED talks, Netflix, and HBO Max) as a resource.

One other habit I have developed here is taking time to sit with my emotions. This is usually Friday and Saturday evening work too, but it gives me a chance to be quiet and look at my reactions/responses to situations that arose during the week. Sometimes I’m proud of how I handled a situation. Sometimes I learn that I over- or under-reacted. Sometimes I’m still feeling anger or frustration and then I have the chance to dig in and figure out what is at the core of those emotions. It’s not easy work but it has helped me learn so much about myself. As a department chair, I have been upfront with my department colleagues that I am engaging in this work, so they know when I come to them with a question or sometimes an apology, that I am dedicated to managing our department well.

3. Develop personal mechanisms for measuring success: For much of my career, my measures of success were distinctly tied to my job performance, and that affected how I valued myself. If a class didn’t go well, I felt terrible. If a paper got rejected, I felt defeated. If I couldn’t solve a problem, I felt incapable as a mathematician. A shift happened a few years ago when I consciously developed new ways to measure my success. I came to realize that my performance in the classroom didn’t always directly correlate to my students’ performance. Yes, I assess my lessons, activities, assessments, and delivery, but I don’t let student outcomes conflict with how I feel about my performance, though I do adapt my approach or clarify expectations as necessary.

In my research, I performed a similar mental shift. This was much more complicated. Here is an excerpt from another blog post I wrote about this process:

After finishing my dissertation, I began a tenure-track position, and my relationship with research continued in the same way for many years. However, around my seventh year I noticed that I didn’t get stuck as often when reading new articles. This revelation helped me immensely. Yes, I surely knew more than I had as a graduate student, but my research capability seemed vastly different, and I couldn’t pin down exactly how or when the change had happened because I still felt inferior. So, I decided to analyze the situation, interrogating my doubts and fears. This led me to reshape my thoughts about mathematics research and exploration. I posed a number of questions to myself.

“Why do you feel like a failure when you can’t prove your latest conjecture?”

“Why do you feel like a failure when you get stuck with an idea?”

“Why do you feel embarrassed to reach out to a colleague to discuss ideas?”

What I realized was that getting stuck is part of the process. It is a frustrating part of the process sometimes, but I can’t control that. The actual “truth” about the process is that we are human, and it is natural to attach our emotions to our work. What we can control are our thoughts and emotional reactions to the process, and I had to develop a way to combat the negative thoughts.

Here’s how I currently handle unkind thoughts I have towards myself regarding mathematics. When I feel like an impostor or inferior because I’m not making progress on my latest project, I read my old work. I remind myself that I am capable and that it isn’t some miraculous feat that I proved results in the past. I take pride in what I’ve accomplished. When I’m feeling isolated, I look for gratitude. Sometimes, this is as simple as being grateful that I had the chance to think about my work. I also take time to be grateful for the colleagues I have who love talking about math. When I feel stuck—really stuck—with a set of ideas, I return to basics. I make sure my foundation is firm or explore a new perspective on ideas I’m familiar with. Basically, I try to learn something. This is the key for me. I can’t prove a new theorem every time I sit down to do research because it takes time for ideas to come together. But I can learn something every time I come to my research, and that is how I measure my progress—by what I learned during the session—not by my productivity. Progress is often slow, and that is OK, but I can learn every day.
This one is tricky because we are still held accountable by the expectations of our positions regardless of our personal measures of success. However, since I have adopted a new attitude toward mathematics and measures of success that are fair to me as a human being, I have become more productive with my output. It has also increased the joy I feel in the doing and teaching of mathematics.

4. Meet people: I have been lucky enough to find delightful colleagues both near and far. But don’t let luck be a factor. Go meet people. Talk to people in your department and across your campus. Talk to others at conferences. Talk to publishers in the exhibit halls. Email an author when you enjoy their work. Find good mentors who can help you grow. Your walk along the mathematical journey will be all the more enriched by your connections. The self-work I’ve done to this point has not been an individual endeavor, and I don’t want to think about what my life would be like without the folks I’ve met along the way.

As mentioned earlier, in order to avoid causing unintentional harm, I have tried to avoid offering blanket advice. But, if you will indulge me, I want to offer a few general recommendations. First, think about what you want frequently and work toward a work-life balance that helps you achieve those goals. Second, be kind to yourself. This will likely translate into kindness for others. Third, don’t compare yourself to others. Finally, invest your time in things which optimize your strengths.

Returning to the original question, do I think work-life balance is a myth? No, I don’t. But, for those of us who want to be excellent at our jobs and live rich lives, I think it requires us to evaluate our commitments frequently and make changes that move us toward balance. The more we talk to each other about our strategies, the less lonely the journey will feel.

Matthew A. Pons

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Photo of Matthew A. Pons is courtesy of North Central College.

Having Children at Critical Career Stages and Flourishing

Theresa Anderson

When I was a beginning graduate student, I thought that to become a successful mathematician I had to postpone having children until I had an established career. Not delaying starting a family is one of the best decisions I have ever made. I describe the challenges and triumphs of having children at critical career stages, with the desire to empower advocacy and to inspire the reader. This article will be almost solely about my personal experience with a (male) partner not in academia, and currently two children, neither with special needs. But whatever your family structure may be, I want to underscore, particularly to women, that with having kids you never “have to settle.” Whether it is subtle coercion to not pass on your last name to your children or being denied a private office to pump milk because office space is tight, speak up and ensure you are listened to and taken seriously. Changes happen when we do these things. This article is my testament to that.

I love analysis and number theory, and despite well-meaning advice to the contrary, I spent my postdoc learning number theory to add to my analysis-based research instead of writing tons of papers. I also wanted to have a child, and instead of timing this after I got a tenure-track job, I decided to not put my life on hold for my career. Before getting an early offer from Purdue, I applied widely and had three interviews. At this point I realized I was pregnant. While the offer was a fantastic opportunity, the early deadline meant not being able to pursue other options. I accepted the offer, realizing that this acceptance did not mean closing other doors forever. Going on the market again may seem like a daunting task, especially with children, but don’t let “daunting” stop you.

In a span of two weeks in 2018, I moved to a different state, bought a house, my husband graduated, I started a tenure-track job, and I had a baby. While I was offered no family leave, I asked anyway and was granted a course reduction that resulted in a semester of no teaching. I happily spent time with my beautiful baby boy, Lucian. My husband also stayed home for significant time periods, which was invaluable. While not every family has this luxury, planning in advance for childcare support jointly with your partner or support system is extremely helpful, especially to ensure that mothers are not overburdened. After a

Theresa Anderson is the Gregg Zeitlin Associate Professor of Mathematical Sciences at Carnegie Mellon University. Her email address is tanders2@andrew.cmu.edu.

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full month completely off, I slowly eased back into math, starting with two hours, two to three days a week, on a focused, independent research problem. With my second child I had a different approach—instead of a focused, individual project I worked on a large-scale, cross-disciplinary, group project. Here I had a very supportive group and a plan to contribute that worked with my schedule. Both times I was amazingly able to get a lot done, even with only short bits of time to focus on work, because I brought a new sense of motivation.

With large caregiving responsibilities, I didn’t dwell on a problem for too long, which actually helped me prioritize. I unapologetically turned down things that I didn’t think would benefit me mathematically. And I acknowledged that the first year after having a child might mean zero research progress. But the flexibility of an academic career in mathematics works well with the flexibility of being a parent. Even now, some time periods contain very little time for research for a wide variety of reasons. My previous experience with schedule flexibility prepared me to be creative with work time and to recognize the flourishing that often happens after taking breaks.

One of the biggest things I learned to do was advocate for myself—no one else can do it for you. I was told by an incredible mathematician that breastfeeding is 70 percent of the effort to care for baby, so definitely get others to help you out. I think she is right (perhaps even an underestimate!). The infrastructure in this country is not as good as it should be to support this, so it is key that you get as much support as you can and advocate for yourself. I did this to get funds for a caregiver to come to conferences with me, something also lacking in infrastructure. I breastfed both my children, and will talk about how I did this during job interviews shortly. To help, my husband would clean my pump parts and set them up for me, change diapers at night and bring baby in to feed, then wake up again (did I mention having kids is tiring!) and bring baby back to the crib. Since I was already up to feed baby, could I have taken the extra minute to bring baby to bed? Yes, but that wouldn’t be very equitable considering I was doing 70 percent of the work! I mention this to underscore how these seemingly small steps can have large impact.

I found that having a child is joyful in such unexpected ways that it motivated me. I forged ahead with the number theory direction, which was perhaps risky as it was very different from my previous training and I was preparing to go on the job market yet again. In addition, there was now a pandemic and I was pregnant with my second baby. But I loved the math I was doing, so I did not let unpredictability bother me. Putting my children first made decisions easy—even with going on the job market and hoping to maximize my portfolio. In May 2021, my beautiful baby boy Jasper arrived!

Early that fall, I applied to jobs selectively, and got six interviews, some of these virtual due to COVID. Fortunately, I was vaccinated while pregnant, but I still had to factor COVID precautions along with breastfeeding while interviewing. Even for the virtual interviews, I needed breaks to feed Jasper, so when accepting the interview I immediately shared that I needed breaks at certain times. For the in-person interviews, I called the host to explain exactly what I needed (fridge to store milk, timing of breaks), and asked for my schedule in advance so there would be no surprises. Unlike basically every other woman’s experience, my experience in doing this was positive, in part because I was armed with stories of bad experiences and determined to not have these happen to me! My thought was that if a school could not accommodate me during the interview, why would I want to work there? I could tell that at most places I was possibly the first breastfeeding woman to interview there and they didn’t want to “mess up!” I pumped milk at two of the in-person interviews and brought Jasper to the third. I had a soft cooler for storing milk, a hand pump for the airport, and requested 30 minute breaks and a private office with a plug as well as freezer space. And I made sure TSA handled the milk I was traveling with correctly so I could bring it back.

Be prepared to speak up to anyone at your interview. With one dean, I politely walked out during the discussion, after having told him several times that I needed to go feed my baby. This was not the only surprising moment. I once asked someone about campus safety—I was told that because he was a large male that it is no big deal, but he understood why I might be concerned. I responded that safety should be a concern for everyone (and couldn’t help adding that I was one tough woman). I unabashedly asked about parental leave policies and used my mug with my children’s names. Of course you don’t need to do all this, but be yourself. If a school doesn’t like who you are, why would you want to go there? Successfully navigating the pandemic, with two young kids under my care full-time, breastfeeding, and interviewing was stressful but exhilarating. I accepted a great job at Carnegie Mellon and we got ready to move to Pittsburgh!

I then moved with two children—just knowing another young woman with two children who moved across the country was so helpful. Pivotal to my success is my support network; there are so many people that helped me along the way. In particular, the advice and time my PhD advisor Jill Pipher gave was invaluable. My network was not only women—talking to men helped me with both gaining and giving perspective. Before having children, I reached out to someone recommended to me who had been a new mother on the job market (in an interesting turn of events this person interviewed me when I went on the job market again and I could thank her). So please reach out early to...
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build a network of support. During the pandemic a collaborator of mine started a research accountability group that functioned as a huge resource for me. Have people you can talk to in academia and outside of it; plan in advance, but be able to be spontaneous. If work is not going well, you have your kids to light up your life.

With having children, you may need to restructure and prioritize, but don’t feel like you need to give everything up. While pregnant I had months of terrible morning sickness that made work essentially impossible. On the flip side, I was later able to go out running up to the day of Jasper’s birth and envision some of my best mathematical ideas. Everyone’s situation is different, but even if you need to put something on hold for your health, you will find a way to do it again. Children make you flexible, but strong. I sometimes multitask (edit a paper while on the exercise bike), rechannel my interests toward my children (pointing out torii and other mathematical shapes in their lives, or using my interest in art to paint clothes for them), but also make sure to carve out some time, even just a little bit, for yourself. You need to take care of yourself too. Sometimes it is very difficult to balance things, especially if you feel your research is not going anywhere and you have outside pressure, or you hear something disparaging, and you have to tackle both racism and sexism. Remember that research comes in ebbs and flows, especially when you are pursuing something new, adventurous, and difficult, that people are disparaging when they are jealous of you, and that you stand in the face of racism and sexism. Your priorities will naturally work themselves out. Don’t think, “if only I could get into a schedule.” Your schedule always changes, but you are a superhero, so you will do what is truly important.

I am a mother to two beautiful mixed-race boys. Navigating the challenges will always be part of my life. Racism doesn’t wait until your kids are in school. There have been several days when all I can do is reach out to my most intimate support networks to share the ugly scars of racism. But there are things much stronger than all this. This resilience makes me a better mathematician and a better mother. I pass on to my children a mother who rises to meet challenges and celebrates the full richness of our community, and of them.

Advocacy, flexibility, support networks, and the ability to change and be changed in unexpected ways are not simply mechanisms for survival, they are for flourishing. If mathematics cannot work for parents, then mathematics would be limited, and I believe mathematics is limitless. I hope the challenges and triumphs presented herein empower you. Indeed, you, the reader, can take from this what you wish. I hope you take the triumphs. Parenting is a lifelong adventure. If you embark, it will be like nothing else in this world.

Invisible Struggles: When the Mask Stays On at Work

Mikael Vejdemo-Johansson

It has taken me a long, long time to recognize my own story as one of resilience and to recognize my own experiences as a struggle. It is still at times difficult for me to fully embrace it. One core reason for this is that we in academia value work very highly—and I rarely struggled as a student, postdoc, or professor. Instead, the energy I expend masking my issues raises my stress level and brings the pent-up emotional storm to bear at home.

I grew up in Stockholm, Sweden, in the 1980s and 1990s—I’m at the very cusp of the “Millennials.” We had computers at home as far back as I can remember, and my parents both worked with computers already in the late 70s and early 80s. It was a household that encouraged academic endeavor and seeded it with a lot of literature—literature that I devoured as soon as I could read.

I always dreamed of working in academia. We have a family story from when I was four and was asked by a man I was talking to what I wanted to be when I grew up. “Researcher,” I lisped as precociously as I possibly could. Since that day, my plan has not changed noticeably—it has merely become more precise year by year.

I was precocious. I was odd. I was a rampant geek. Kindergarten through 9th grade was a period of constant exclusion and bullying at school. Grades 10 through 12 came with some specialization in the schooling, and with that selectivity came relief.

Credits

Photo of Theresa Anderson is courtesy of Theresa Anderson.

Mikael Vejdemo-Johansson is an associate professor of data science in the Department of Mathematics at the College of Staten Island and deputy executive officer for the Computer Science and Data Science programs at the CUNY Graduate Center. His email address is mvj@math.csi.cuny.edu.

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Selectivity as relief is something I have noticed through my years. Even when bullying was particularly bad, my extracurriculars have always provided me with a haven. Scouting, orchestra, theater, and the Young Scientist’s Association all took their turns as a place of refuge and safety. Making friends through shared interests rather than among peers who just happen to live in the same school district has always led to better outcomes for me.

As far back as my family remembers, I have always had problems with my mood, with my emotions going haywire on me. I have spent decades by now trying to sort through this, trying to treat it, trying to control it. Certainly, there is some sort of PTSD from the years of bullying, but my family has told me stories of mood swings before from Kindergarten. My biggest breakthroughs have come in the past decade or so: I have found that autism and ADHD (abbreviated by some to AuDHD) explain almost everything I am experiencing better than mood disorders do, and where medication and therapies meant for mood disorders have failed, medication and therapies that focus on ADHD and autism have quite noticeable benefits.

I am pretty good at masking, though: keeping stable while out among relative strangers and forcing my mood to stabilize. It is something paid for dearly once I let go of the force. As a result, I tend to have more and more severe mood swings at home than in school or at work. My colleagues have gone for years without knowing that I have any issues whatsoever, because I stay stable and balanced during the day. Instead I will crash at home, reacting with anxiety, tears, and sometimes freezing up completely at the slightest of causes.

There are some particularly noticeable crashes I can remember because they caused me to go and seek help. The first time I got started with psychiatrists and psychologists was thanks to my wife. During her first family dinner with my family, she got to witness my meltdowns firsthand as I crashed out, fled the dinner table, and hid in my room. She came after me and encouraged me to seek help. A few months later I was seeing a psychiatrist, taking Lamictal for mood stability and seeing a psychologist for therapy.

Another time—earlier than this—during a postdoc in Scotland, my wife was enthusiastically looking forward to my cooking a lamb curry for dinner. I did not want to, but I could not bring myself to tell her that I did not want to cook the curry. Instead, I froze up in the middle of the frozen foods aisles and then broke down. That’s the first time I got restarted on mental health care.

During a research semester in Minneapolis, a car turned in front of me at a crosswalk and the world just . . . slowed down. I froze up on the sidewalk and stood frozen still and shaking for several minutes. Once I managed to get myself to move again, I could not push myself beyond an extremely slow pace. Once I got home and inside, I cried so hard the muscles in my face hurt from it. A few days later, I sought out the student health services and saw a psychiatrist for advice on the antidepressants I was taking and a psychologist for a sequence of therapy sessions.

In the summer before I moved to New York to start my current job, I had my regularly occurring anxiety around moving act up: from May, when I vacated the office I was borrowing to August, when I finally moved, I had paralyzing mood swings and anxiety attacks at least three to four times a week.

Some aspects of these examples are widely recognizable. Everybody has at some point been too tired or unmotivated to want to cook. Everybody would rightly have been traumatized if almost hit by a car. But as far as I am able to understand other people’s reactions, my responses are often much more intense than many others would have experienced—to the point where it gets in the way of living my life. Everybody feels stressed over moving, but not everybody is paralyzed from crying almost every second day.

There was a time period when the hacker/security research community faced a sequence of high-profile suicides, almost all of them just one step removed from me: close friends of my nearest friends in that community. Feeling the impact of these deaths, the community stepped up and deliberately built a caring supportive space for its members: with conference panels, support groups, and increased visibility and support for mental health issues. While this was happening, I could see none of this support in my academic circles, leaving each of us feeling alone and isolated in our struggles. I wanted what the hacker community built for my academic world.

One of the visibility steps in the hacker community was people posting texts on blogs and websites, “coming out” with their own struggles to put a spotlight on how widespread issues with depression and anxiety were. Inspired by the posts I read, I wrote up a text about my own mood instability, put it on my website and tweeted about it. A day later I got an email from one of the colleagues I had during my previous postdoc in Scotland—he told me he had depression, and that he had not noticed me struggle at all while we were in adjacent offices and sharing lunch and coffee breaks over a year. I told him about the hacker community and their work—so we founded a group blog for “depressed academics”: https://depressedacademics.org/.

Almost as soon as I started writing openly about my mood swings, I heard from others. Friends with math PhD degrees who talked about fellow students who habitually hid under their desks to cry. Friends who talked about their own struggles. Friends who ended up leaving academia entirely because of their difficulties. And I started talking more and more; together with two friends,
Early Career

we wrote about our own struggles in the August 2019 issue of the AMS Notices; and we organized panel discussions about mental health in the math community at the Joint Mathematics Meetings in 2019 and 2020.

Since starting Depressed Academics, I have been consciously choosing to be open about my mood instability. I write online under my own name about moods, medications, and therapy. Sometimes very personally. I tell people early on about my AuDHD—for instance I often weave a mention into my teaching early in my courses. As a result, people open up to me. Students come to me with their own struggles instead of hiding them.

The autistic reader will have noticed by now that I choose some words more aligned with AuDHD: masking, meltdown. In 2019, they were in my text as bread crumbs, but I was still mostly coming out with mental health struggles. Both autism and ADHD have an impact on mood, through meltdowns, through dysregulation, through hyperfixations and special interests—and the way that my mood swings fits better with these impacts than with my original mood disorder diagnoses. The medications I mention—Lamictal and anti-depressants—might let you know that for a very long time I was treated for bipolar, though with much less success. Now I am on Strattera, and I saw changes within the first week after starting it.

Now I am coming out as autistic, with ADHD. I’m not sure that mood disorder ever was an accurate diagnosis for me, but I know that ADHD medications are effective and autistic coping strategies and explanations work for me.

I found my current diagnoses by noticing how other people’s personal narratives resonated with me. If my narrative and my experiences resonate with you, you may want to look for more support than you are currently getting. I have had good experiences with peer support groups on social media, and some good experiences seeking out professional help. Depending on where you are and what your health care looks like, it might be relatively easy to get professional help, or you might hit long waiting times or limited or absent funding for getting help. And you may have to try more than once before you find support that works for you: I have certainly left several therapists and psychiatrists because their approaches to helping me did not work. Your university might also have support structures in place—it’s most common to support undergraduate students, and many universities are getting better about supporting graduate students. For postdocs and faculty, the universities have less support in place, which is one reason I keep talking and writing about my own experiences: I want our community and our employers to have better support systems for us.

Just because my struggles were not visible in the workplace did not mean I did not struggle. Now, I am making it visible.

Be Open to Unexpected Opportunities: Work-Life Balance Requires Creativity, But it is Worth the Effort

Elizabeth G. Campolongo

From a very early age I had a plan: I was going to get a PhD in math. What I would do after that was always an open question. I have an eclectic set of interests, but the thread of mathematics weaves most of them together. In undergrad, I found algebra to be a natural framework for my approach to math, and its overlap with number theory connected my interests in linguistics and cryptology. My first deviation from the plan was a foray into analysis in my second year of graduate school. This led me, at the beginning of my third year, to knock on the door of the professor who was to be my advisor and ask her to do an independent study with me to explore possible research topics. Enter the study of lattice point counting through the lens of harmonic analysis and fractal geometry. Then came my second deviation from the plan: my son. Four years and a global pandemic later, I successfully defended my dissertation (virtually, while my husband took our son to the park—thank goodness for good weather!).

The key to my success was the support of my family, my department, and—most of all—my advisor. She was determined for me, not just to graduate, but to find a meaningful career afterwards (be that in academia or industry). The funny part being that I’m currently on the “industry

Elizabeth G. Campolongo is the research data manager and technology coordinator for the Imageomics Institute at The Ohio State University. Her email address is e.campolongo479@gmail.com.

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path” but working at an academic institute based at my alma mater.

Let me back up a little in this story…

I read forums suggesting that grad school was the best time to have kids because you will never have as much time in an academic career until it’s far too late. I can’t speak to how well that holds, but I’ll make a supporting argument for the case if you don’t go the academic route, with one caveat: after candidacy.

Flexibility is key. In many ways I was lucky, my son was born at the beginning of a semester and I earned a fellowship so I wouldn’t have to teach. This gave me the flexibility to plan my semester of research in a way that worked for us. I spent much of that semester (once I started itching to get back to math—my cousin was right that staying engaged in research is best) reading papers with my son asleep in my lap, allowing me to feel like both a good mama and a good math researcher. This habit extended far past any maternal leave of which I’m aware (at least in the US), and I even got in the habit of working out loud when he was awake (how many six-month-olds get to learn about Fourier decay?). The flexibility of my schedule even after my fellowship ended allowed me much more quality time with my son in his early years—something for which I am eternally grateful. Having a baby in grad school definitely isn’t easy, and probably leads to much more working at odd hours and the sense of a need to constantly be on, but some of that is just being a parent, and the quality of that time cannot be overstated, though—barring a pandemic—day care is probably a good idea at some point.

Despite the challenges of everything going online in 2020, there were many doors opened; it became feasible to attend seminars around the country and abroad, collaborate with others half a world away, even asynchronously. Unfortunately, this meant often working when my son was asleep, which did not bode well for my sleep.

Believing I could do everything, I rejected the idea of time for myself meaning anything other than work, but I digest…

When my 2020 conference schedule was postponed, I made another pivot. It was time to answer that long-open question of what I would do for a career after getting my degree, and the industry-curious side had the more persuasive argument. I had done an internship in topological data analysis (TDA) in the summer following my candidacy, so I signed up for the (now online) data science bootcamp offered by the Erdős Institute, channeling my COVID anxiety into a machine learning (ML) problem of predicting (or comparing) COVID spread across the country using publicly available data on county case numbers and demographics. I completed the program again the following year and used TDA to analyze football plays, though I knew next-to-nothing about football. Sometimes lack of preconceptions can be an asset.

Don’t be afraid to seek out these opportunities. Grad school is a time to explore opportunities and learn as much as possible—you never know what may inspire your next approach. Certainly, taking a break to chew on another problem can help unstick your brain (many of my best dissertation writing marathons came after a coding session or working through an ML problem). There is a lot of power in positive affirmations, and this is a good one, courtesy of one of my computer science colleagues: “You’re a mathematician, that means you can do anything.”

Mathematics is far more versatile than we are often led to believe—we’ve all been in a class where someone asks “when am I going to use this?” Even the moniker of “Pure Mathematics,” as in, a subject whose only impact is on itself, disguises the breadth and broad applicability of this domain. This common misconception is one of the greatest disservices to our field. I got my current position based on data science programs I pursued outside of my main research area, yet I can still find connections. After telling one of the graduate students I’ve been working with (a biologist) about my dissertation topic, he shared some papers on butterflies’ wing patterns and the mathematics that describes them—yet another connection to lattice points.

Don’t be self-limiting. More and more research involves overlap of various fields, be they within or exterior to mathematics. I currently work in the field of imageomics, which utilizes ML techniques (in particular, computer vision) to answer questions in biology by analyzing images. Though it is not directly mathematical, analyses of embedding spaces to understand the structure and pattern of big data are based in mathematical concepts (from topology to group theory). Even when it is not directly applicable to the problem at hand, there is a method of thinking—the techniques that we learn—which informs our approach to any problem, not just those that are clearly mathematical in nature. Harness that, celebrate it.

During my interview for my current position, I was asked about an experience where I had to balance competing interests and how I handled it. Now, full disclosure, by the time I was asked this I had assumed I was not getting this position. They had asked about experience with a bunch of programs I had not used (which, had they been listed in the job description, would have kept me from applying—there’s a lesson on self-selecting here).1 That assumption released me from the big fear: As a woman, I know mentioning family can be the kiss of death in an industry setting, but—let’s be honest—the greatest competing interest, completely unavoidable, is family. It’s

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1I’ve since learned the ones that have come up (plus others that weren’t mentioned) and I have a team to work with, so it’s not all on my shoulders.
persistent, incessant, and unavoidable, so what better than motherhood to teach one to balance competing interests? Which brings me to my answer:

Spring 2020, the world shut down, and suddenly we were all home, all the time: my husband, my toddler, and myself. I had research, I had students, I had my family, some days I had life flights (emergency helicopters) flying overhead every hour.

I had to balance the competing interests of my child who wanted my attention—constantly—who was no longer really napping, who was supposed to be starting day care. I had to find time for him, for the students, for my research, and I had to coordinate on Zoom with my advisor, holding papers up to my camera where we would otherwise have been writing together on a chalkboard or side-by-side. Balance had to be achieved. I felt responsibility for my students, but I couldn’t put the same care into correcting their proofs writing with a trackpad and keep making progress on my research and be available for my son and sleep. Instead, I compartmentalized: I wrote out detailed solutions for my students, offering explanations to common mistakes at the bottom to reduce the time spent writing out information on their individual assignments.

I set aside designated days for my research, often passing my scrap paper or papers I had read to my son to “make Mommy’s work look pretty” with his crayons.

Don’t underestimate the value of scheduling. Balance doesn’t come easy, and there are definitely times when it’s simpler to work on one project over another, but time is precious, so I binned my tasks. More precisely, I grouped different tasks that were similar to have a selection of tasks to choose from at a particular time that I had reserved for those types of tasks. For instance, writing time vs research time vs grading, etc. It’s amazing how inspiration can strike while typing up a proof with a two-year-old sitting on a stool next to you with his “woo-woo [work] machine” working while Daniel Tiger music is playing in the background.

Having balance between your work-life and home-life doesn’t always take the form of separate time for each. When my son was just a newborn, I found tasks that I could do with him (like reading papers), so that we could be together. Now he’s in school, but my husband and I both work from home, so we can take a break and go for a walk together between meetings—just the two of us, or as a family when our son gets home from school. Squeezing in this time during the day brings us all closer together, and is certainly healthier than sitting at a desk nonstop. Work-life balance is not something of which I’d claim mastery, but I have learned that giving myself that space and time allows me to be more present, both at work and at home.

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2“Woo-woo machine” is what my son called a little folding screen cleaner that he would set up like a little laptop to work with me.

Elizabeth’s son, working with Mama

Credits

Photo of Elizabeth’s son provided by and © Elizabeth G. Campolongo.

Karate and the Art of Mathematical Maintenance

Helene R. Tyler

There has never been a time when testing wasn’t a major part of my life. For the first 30 years, I was usually the one tested, while for the last 30 years, I mostly have been the one who created, administered, and evaluated the tests. Then there have been the periods of time, one of which is ongoing, during which I have been both the tester and the tested. All of it, no matter my role, has informed my assessment of my evolving role in the mathematical community and my contributions to it.

One of my favorite professors in graduate school was the one I had for our program’s notoriously difficult measure theory course. The exams in the course crushed my soul, and the homework sets drove me mad. They made some of my classmates question their fitness for graduate study in mathematics, but the more senior students encouraged me to persevere, to keep getting up each time one of those brutal assessments knocked me down. They promised that it would be worth it, and it was, because making even a little bit of progress on one of those seemingly impossible problems strengthened me for the next challenge. This was the first period during which I was simultaneously the tester and the tested, as it coincided with the beginning of my teaching career. Since then, I have earned a reputation as one of my institution’s tougher professors. Like my measure theory professor, I am known as someone who will make your current course a little bit more challenging, but your next courses a lot easier. What

Helene R. Tyler is a professor of mathematics and chair of the newly merged Department of Mathematics and Physics at Manhattan College. Her email address is helene.tyler@manhattan.edu.

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really made that professor someone who I would want to emulate, though, is how much time he would give his students in preparation for the exams. He would spend hours coaching us through deep and subtle theory and techniques, strongly conveying his confidence in our ability to master them. So in the end, the only students who didn’t pass the course were the ones who stopped getting up.

Twelve years ago, I began training in Kenshikai Karate. I began out of a desire to support my bones with more muscle mass, and I wanted to start before I reached a certain age and it became more difficult to do so. The dojo where I began training and continue to train is only one block from my home, and I’d gone past it hundreds of times before ever stepping inside. I attended a trial class and was instantly hooked. As a mathematician, an algebraist in fact, I find beauty in structure, abstraction, and formalism, so I recognized the importance of the strict etiquette required inside the dojo. I understood that basic techniques form a necessary foundation for more advanced learning, and I felt a familiar collection of emotions when I was put forward for my first belt promotion test.

Over the 30 years that I have been teaching, I’ve learned a lot about the different ways that we can assess our students’ learning. High-stakes exams, like my measure theory exams and the belt promotion exams I take at the dojo, are examples of summative assessments. They evaluate student skill acquisition at the end of a defined period of time by comparing it to some established benchmark. When I was taking that measure theory course, I thought the exams were the sole basis for our grades. I only realized much later that our professor was also continually engaged in formative assessment, which measures student progress while learning is taking place. The many hours of mathematical discussion, both formal and informal, much like my karate instructor’s seemingly spontaneous questions during class, are what kept me engaged and motivated. Many of us use formative assessment without realizing it—by guiding students through in-class activities, encouraging students to ask questions during class, or giving low-stakes quizzes. Such measures are an important part of my educational philosophy because of the way they can improve teaching and learning simultaneously. Responding in real time to students’ questions and answers keeps me flexible, agile, and strong.

Two terms that are often used to describe different approaches to martial arts training and practice are “do” and “jutsu.” The emphasis in a “jutsu” is typically on the practical application of a skill, such as in combat or self-defense. “Do” literally means “way” or “path,” so the dojo where I train is the “place of the way.” In a “do,” there is greater emphasis on personal growth and self-improvement, and the practice includes techniques for cultivating mental and physical discipline. The stated goal of the Kenshikai Karate-Do is “the physical, mental, and spiritual growth of its students.” We are taught that “the competition is with yourself, not with others. Our goal is that each day you train your hardest, slowly pushing your own limitations, so that when you leave you are a little better than when you arrived.” This philosophy precisely describes ipsative assessment, the practice of determining student progress based on earlier work.

Many in the mathematics community are skeptical of ipsative assessment. To be honest, I have a hard time finding a place for such assessment in my courses. Most of the courses I teach are necessary prerequisites for others in a variety of STEM fields. Who would care that my students reached their personal bests if the bridge they designed fell down? However, in the wake of the Covid-19 pandemic, some educators began considering ipsative assessment and other alternative assessment methods as we try to climb out of the hole that Covid dug. So much was lost and our students have fallen so far behind in their skills. Perhaps, at least for a short while, we might focus on student gains, rather than on meeting a particular set of criteria.

But what sort of assessment is appropriate for someone whose personal best is behind them?

The Fields Medal is considered by many to be the most prestigious award in mathematics. It is often described as the “Nobel Prize of Mathematics,” but there is a crucial difference. The Fields Medal has an age limit; a recipient must be under age 40. The intent of the under-40 rule was a desire to recognize the work a recipient had already done, while also encouraging further achievement. The under-40 rule has also created an attitude that mathematics is a young person’s endeavor. Of course, there are many aspects of the culture of mathematics that contribute to this reputation. To wit: I know few people over 40, or even over 30, who would submit to the punishment of a measure theory exam. Certainly I no longer feel equipped to endure one, and I often question whether I could still produce research at the level of my early work.

Karate might also be considered a young person’s endeavor. It is extremely physically demanding, and aging bodies can’t necessarily withstand what they could when they were younger. At the end of 2019, as I marked eight years of dedicated karate practice, I was in the best physical condition of my life. Then I pushed myself a bit too hard and incurred a shoulder injury that took nine months to heal. My healing process was slowed by the Covid-19 pandemic, and I lost even more fitness. Though my shoulder healed and the dojo reopened, I still have not regained my previous strength. It seems that I have reached that certain age. Despite this, in the fall of 2022, I was tested for another belt promotion. I came to that test with my physical personal best behind me. I would not allow that to

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matter, though. I would perform at my personal best of that moment in time. I would perform with the added wisdom and deepened spirit acquired through the challenges and experiences of the previous three years, both in and out of the dojo.

In June 2020, I began a four-year term as chair of the Manhattan College Mathematics Department. I was elected in January of that year, before anyone had any idea what the following years would bring. My new position brought challenges that none of my predecessors had met, nor could any of us have imagined. My colleagues and our students didn’t need a Fields Medalist. They needed someone who was ready to be tested, by enrollment cliffs, budget cuts, AI chatbots, faculty and staff reductions, and diminished student preparedness. They need someone who will continue to get up no matter how the consequences of the pandemic continue to knock us down. They need someone who can lead us through a comprehensive assessment of how well we are recovering.

Our opponents may be formidable, but my technique and my spirit are strong. I’ve got a lot of fight left in me, and I’ve got a lot more to contribute to my community.

Helene R. Tyler

Credits
Photo of Helene R. Tyler is courtesy of Helene R. Tyler.

Dear Early Career

How do you decide where to submit a research paper?

—Publisher

Dear Publisher,

This is a good question! If your work is collaborative, it is essential to ask collaborators how ambitious they want to be in selecting a journal to submit to. That said, I’d encourage you to develop a rather detached view of the role of journals in terms of judging the quality of your own work. When you apply for (postdoc or tenure-track) academic positions, letters of recommendation will likely play a larger role than where your papers are published. I’d also recommend coming up with a backup journal at the time of submission: This can be helpful in case you get a rejection with a referee report that you strongly disagree with—having a backup plan laid out in advance can prevent you (or a collaborator) from making an emotional decision about the future of the paper.

There are journals that will publish articles in all mathematical subfields, while others are more specialized. Consider prioritizing a journal with an editor in your field over the generality of the journal. While several high-profile journals will publish articles across mathematical disciplines, it is not uniformly the case that “general” journals are considered better than subfield specific journals. (Since I’ll be naming some journals, I’ll give the disclaimer that every subfield of math evaluates journals differently, and my biases come from working largely in pure harmonic analysis.) For instance, *Analysis and PDE* is a specialized journal which aims to publish excellent contributions to analysis, and a paper submitted there would have to excite referees more than, say, a submission to the *Pacific Journal of Math*, or the *Michigan Journal of Math*, even though the latter two are well-regarded general journals. Such examples occur in all subfields of mathematics. In either case, I only submit to journals that have an editor in my field (and I often look more specifically for editors who I think might have some appreciation for the paper because of their prior work).

When writing a paper, often a few previous works are significant influences: for instance, we might answer a question (or be attempting to answer a question) raised in a previous paper, or we might be building upon techniques pioneered in a previous paper. I quite like to submit to the journals in which those papers appeared when that feels appropriate. An exception to this is when those papers appeared in “top top journals” such as the *Annals of Math, Acta Math, Invent. Math*, or the *Journal of the AMS*. For a paper to have a good chance of being accepted for publication in a “top top journal,” it should be a really exciting development about which the referees are extremely enthusiastic, so if my paper is an application of ideas developed in a “top top journal” paper to a problem that I find interesting, then I wouldn’t necessarily think it would be appropriate to send to a “top top journal.” If, however, your work is creating excitement in your area, then submitting to one of the top-tier journals’ could be worthwhile.
It takes a bit of practice to gauge your work in this regard, and talking to mentors is often a good idea.

The journey your manuscript goes on after it is submitted is out of your control. Out of the set of all possible reviewers for your paper in the world, most likely some of them will accept the paper, and some won’t. A great mathematician once told me that they knew they were really onto something if a paper got rejected three times, so don’t spend too much time worrying about what might happen and get submitting!

—Early Career editors

Have a question that you think would fit into our Dear Early Career column? Submit it to Taylor .2952@osu.edu or bjaye3@gatech.edu with the subject Early Career.

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“Personally, I include Geometric and Functional Analysis to this list. Some people might also include Duke Math Journal, and most might include Publications Mathématiques de l’IHE’S, as well as other journals.

\(^1\) Journal of the European Mathematical Society, Advances in Math, and Crelle’s Journal would also all be examples of “prestigious” journals in addition to those mentioned above, and (for example) a geometer would likely add the Journal of Differential Geometry, and Geometry and Topology, as well as other journals.
In Memory of Martin Davis

Wesley Calvert, Valentina Harizanov, Eugenio G. Omodeo, Alberto Policriti, and Alexandra Shlapentokh

In 1950, Martin David Davis found the culture in his PhD program at Princeton deeply alienating, and he wanted to be out of there. So he solved an open problem of Kleene (establishing the backbone for a major branch of modern computability theory), wrote down his initial steps toward the eventually successful solution of a Hilbert problem, and graduated.

Davis’s solution to the problem of Kleene became the hyperarithmetical hierarchy, which we will explain in Section 1. His work on Hilbert’s Tenth Problem included what would later be called his “daring hypothesis” [17]: a conjecture, later verified, that the computably enumerable sets were exactly the Diophantine sets, as we will discuss in Section 2.

In spite of the economic restrictions which his family had to face, as immigrants who arrived shortly before the Great Depression, Davis received a high-quality education. He arrived at City College of New York as a freshman in 1944 and soon became interested in the foundations of real analysis and in logic. He approached Post, who introduced him to the writings by Church and Kleene on algorithmic unsolvability and to Hilbert’s Tenth Problem, which would soon become Davis’s “lifelong obsession.” When Davis had to choose where to undertake graduate studies, Post advised him to go to Princeton, where, as Davis later expressed it, the “culture clash” between his Jewish working-class background and the “genteel Princeton atmosphere” made him eager to finish quickly. In fact, he got his PhD in just two years, under the guidance of Church, in 1950.
Davis's first position was at the University of Illinois at Urbana-Champaign, but “the Korean war and the hot breath of the draft” led him to leave that job for the Control Systems Laboratory. He later moved to the Institute for Advanced Study, the University of California at Davis, the Ohio State University, the Rensselaer Polytechnic Institute, Yeshiva University, and New York University. Certain summer projects funded by military and civilian research agencies enabled him to make crucial achievements (“It was in the summer of 1959 that Hilary and I really hit the jackpot,” he says, to describe the original, raw discovery of what would become known as the celebrated Davis–Putnam–Robinson theorem). Over the course of his career, Davis supervised a total of 25 PhD students, including scholars now known for work in mathematics, computer science, and philosophy.

Davis’s expository books have become classics and have been translated into various languages: Computability and Unsolvability; A First Course in Functional Analysis, Applied Nonstandard Analysis; Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science (with Ron Sigal and Elaine J. Weyuker); and The Universal Computer: The Road from Leibniz to Turing. His 1993 Lecture Notes in Logic are a true jewel.

His many honors and awards include the Steele Prize, the Chauvenet Prize (with Reuben Hersch), Fellowship of the AAAS, a Guggenheim Foundation Fellowship, the Herbrand Award of the International Conference on Automated Deduction; and the Pioneering Achievement Award from the ACM SIG on Design Automation.

Martin Davis left two rather comprehensive autobiographic accounts [8, 9] and long interviews [15, 16]. For this reason, the present note will primarily focus on his scientific achievements.

1. Computability

Computably enumerable sets and universal Turing machines. Modern computability theory started in 1936 with Turing’s seminal paper on computable numbers with an application to the Entscheidungsproblem, a decision problem of Hilbert and Ackermann for which Turing provided a negative solution. Turing introduced what today we would call a Turing machine (he called it an a-machine), which is essentially an abstraction of a computer. Turing’s and other formalisms for an intuitive concept of an effectively calculable function, developed by Gödel, Kleene, Church, Post, and others, had profound significance for the emerging science of computing.

One of the main concepts in computability theory (also called recursion theory) is that of a computable function and computable relation. A function \( f : \mathbb{N}^n \to \mathbb{N} \) is computable if there is a Turing machine that on every input \( a_1, \ldots, a_n \) halts and outputs a value \( f(a_1, \ldots, a_n) \). Addition and multiplication on natural numbers are computable functions (operations). A set of natural numbers is computable if its characteristic function is computable. For example, the set of prime numbers is computable. All finite sets are computable. Clearly, the complement of a computable set is computable. Computable \( n \)-ary relations are defined similarly. It can be shown that there is a computable bijection \( h : \mathbb{N}^2 \to \mathbb{N} \), thus allowing algorithmic coding of pairs and, more generally, finite tuples of natural numbers by natural numbers. Decidable problems are encoded by computable relations.

A function \( f : D \to \mathbb{N} \), where \( D \subseteq \mathbb{N}^n \), is partial computable if there is a Turing machine that on every input in the domain \( D \) halts and outputs its value, while on every input in \( \mathbb{N}^n \) that is not in the domain of \( f \) it does not halt, thus computing forever. Clearly, computable functions are partial computable functions that are total. Partial computable functions coincide with partial recursive functions defined by Kleene, starting with some basic functions and applying the operations of composition, primitive recursion, and unbounded search.

Since each Turing machine is a finite list of instructions, Turing machines can be algorithmically enumerated without repetitions as:

\[
M_0, M_1, M_2, \ldots
\]

A Turing machine on a given input may halt and output its value, or it may compute forever. For each Turing machine \( M_e \), we denote the \( n \)-ary partial function it computes by \( \varphi_e^{(n)} \) and its domain by \( W_e^{(n)} \). Hence

\[
\varphi_0^{(n)}, \varphi_1^{(n)}, \varphi_2^{(n)}, \ldots
\]

is a computable enumeration of all \( n \)-ary partial computable functions. For \( n = 1 \), we omit the superscript. Moreover, there is a binary partial computable function \( \psi \) such that \( \psi(e, x) = \varphi_e(x) \).

The above enumeration gives rise to a universal Turing machine, which can simulate any Turing machine on any input and leads to the idea of a stored-program computer. Davis wrote about the universal Turing machine in a number of papers starting in 1956. His lecture in 2012 entitled “Universality is ubiquitous” is available at https://www.youtube.com/watch?v=ZVTgt0DX0Nc.

In a written version, Davis stated: ‘Turing’s concept of ‘universal machine’ will be discussed as an abstraction, as
embodied in physical devices, as present in nature, and in connection with the artificial intelligence project."

In addition to his large body of expository work on universal Turing machines, Davis was also an early technical contributor to the subject. Turing had constructed a universal machine, but had not dealt with universal Turing machines as a class of objects. John McCarthy and Claude Shannon posed the problem of giving a definition of universal Turing machines, which would deal, for instance, with the simplicity of the encoding by which the universal machine simulates arbitrary machines. Davis solved this problem in [3], as we explain next.

Definition 1. Let $S$ be a set. We say that $S$ is computably enumerable (also called recursively enumerable) if and only if $S$ is empty or is the range of a computable function.

It is not hard to see that a set is computably enumerable if and only if it is the domain $W_e$ of some partial computable (equivalently, partial recursive) function $\varphi_e$. If $W_e$ is nonempty, then $W_e$ can be computably enumerated by the procedure that simultaneously runs $M_e(0), M_e(1), \ldots, M_e(k), \ldots$ and enumerates those $k$ for which $M_e(k)$ halts, as soon as the halting occurs. Here, simultaneously means that at each step we add a new input and also run all activated inputs for an additional computational step. The converse, that every computably enumerable set is some $W_e$, is also true.

Since we can computably enumerate the Turing machines, we can also computably enumerate the computably enumerable sets by $W_0, W_1, W_2, \ldots$.

Clearly, every computable set is computably enumerable since a decision algorithm can be transformed into an enumeration algorithm. Computable sets are exactly computably enumerable sets that also have computably enumerable complements.

Definition 2. [3]

1. We say that a set $S$ is c.e. complete if and only if it is computably enumerable and for any computably enumerable set $W$, there is a computable function $\sigma : \mathbb{N} \to \mathbb{N}$ with $x \in W$ if and only if $\sigma(x) \in S$.
2. We define $\delta_M$ to be the set of all initial configurations of $M$ from which $M$ will eventually halt.
3. We say that $M$ is universal if and only if $\delta_M$ is a computably enumerable set that is c.e. complete.

Davis proved that a machine which is universal in this sense does, indeed, simulate all Turing machines. However, having universal machines defined as a class, rather than simply observed as a phenomenon, opened the door to lines of thinking that involve quantification over all universal Turing machines, such as the foundational work of Solomonoff and Kolmogorov on information theory (Kolmogorov complexity), and the theory of algorithmic randomness arising, in part, from it.

The diagonal halting set $H$ consists of all inputs $e$ on which the Turing machine with index $e$ halts. That is,

$$H = \{e : M_e \text{ halts on input } e\}.$$

It can be shown that the set $H$ is computably enumerable. It is not computable since its complement is not computably enumerable. If the complement $\overline{H}$ were computably enumerable, then for some $e_0$, we would have $\overline{H} = W_{e_0}$. Then

$$e_0 \in H \iff e_0 \in W_{e_0} \iff e_0 \in \overline{H},$$

which is a contradiction.

It can also be shown that a set $A$ is computably enumerable if and only if there is a computable binary relation $R$ such that for every $a$,

$$a \in A \iff (\exists x)R(a, x).$$

We can relativize all of these notions using Turing’s notion of an oracle machine. A machine with an oracle for a set $S$ is a Turing machine which carries out its computation with the additional resource of read-only access to the characteristic function of the set $S$. In this way, even if a set $U$ is not computable, it may be computable with an oracle for another set—for instance, its complement. The halting set relative to $S$ (also called the jump of $S$ and denoted $S'$) is defined exactly as before, but replacing the machines with machines with oracle $S$.

The hyperarithmetic hierarchy. In his 1950 PhD thesis at Princeton, Martin addressed a problem posed by Kleene. It was already known that every formula of classical predicate logic is equivalent to a formula consisting of a block of quantifiers (“for all” and “there exists”), followed by a quantifier-free formula. Kleene noted that the optimal form of such an equivalent formula corresponds to the degree of unsolvability of satisfying that formula. For instance, Turing’s halting problem is equivalent to the problem of satisfying a particular sentence with a single existential quantifier, but not to the satisfaction of any quantifier-free formula.

This gives rise to a hierarchy of formulas—and, equivalently, of decision problems, according to the number of quantifiers, and whether those quantifiers are “for all” or “there exists”. Since there is no computational difference between determining the existence of a single element and determining the existence of a finite tuple of elements, we consider only alternations of quantifiers.
Definition 3 (The Arithmetical Hierarchy). Let \( S \subseteq \mathbb{N}^m \).

1. We say that \( S \) is \( \Sigma^0_n \) if and only if there is a computable set \( T \subseteq \mathbb{N}^{m+1} \) such that \( \bar{a} \in S \) if and only if \( \exists x \{ (x, \bar{a}) \in T \} \).

2. We say that \( S \) is \( \Pi^0_n \) if and only if there is a computable set \( T \subseteq \mathbb{N}^{m+1} \) such that \( \bar{a} \in S \) if and only if \( \forall x \{ (x, \bar{a}) \in T \} \).

3. We say that \( S \) is \( \Sigma^0_{n+1} \) if and only if there is a \( \Pi^0_n \) set \( T \subseteq \mathbb{N}^{m+1} \) such that \( \bar{a} \in S \) if and only if \( \exists x \{ (x, \bar{a}) \in T \} \).

4. We say that \( S \) is \( \Pi^0_{n+1} \) if and only if there is a \( \Sigma^0_n \) set \( T \subseteq \mathbb{N}^{m+1} \) such that \( \bar{a} \in S \) if and only if \( \forall x \{ (x, \bar{a}) \in T \} \).

The \( \Sigma^0_1 \) sets are exactly the computably enumerable sets. The \( \Sigma^0_{n+1} \) sets are exactly those computably enumerable relative to the \( n \)-times iterated jump of the empty set. There are certainly sets of natural numbers that are not \( \Sigma^0_n \) or \( \Pi^0_n \) for any \( n \), and Kleene asked whether the hierarchy could be continued to transfinite levels.

Davis [1] carried out this generalization by iterating the jump starting at an arbitrary set, and also describing a uniform join of infinitely many jumps. In the following definition, \( \omega \) denotes the least transfinite ordinal, the order type of the natural numbers.

Definition 4. Let \( K_0 = \emptyset \).

1. Let \( K_{\alpha+1} = K_\alpha^2 \).

2. Let \( K_{\omega \cdot n} \) be the set defined by \( 2^{x_1} 3^{x_2} \in K_{\omega \cdot n} \) if and only if \( x_1 \in K_\omega^{n-1} + x_2 \).

Davis proved that this hierarchy is proper and that it extends the finite-level Kleene hierarchy to all ordinals less than \( \omega^2 \). After Davis’s work, the major pieces still missing were extension to larger ordinals and the fact that the choice of representative sets at limit levels is not unique. It was five years later that Spector showed that, up to Turing degree, the definition is robust. This transfinite hierarchy is now at the core of modern computable mathematics.

Hypercomputation, neural networks, and unconventional computation. A major effort of the later part of Davis’s career was devoted to defining the bounds of computation. In response to a community of scholars who applied relativistic and quantum theories to propose computing devices more powerful than a Turing machine (an idea they called “hypercomputation”), Davis responded with unbridled skepticism.

Davis’s most significant paper on this subject [7] considered a key proposal for a hypercomputer, a certain kind of neural network. The proposal described a model with parameters. If those parameters range over rational numbers, the machine could determine membership in the computable sets, as expected; however, if the parameters were allowed to range over arbitrary reals, it could determine membership in arbitrary subsets of \( \mathbb{N} \). Davis pointed out that the ability of the model to determine membership in arbitrary sets followed immediately from Turing’s observation that any countable set was computable relative to some oracle, still via a Turing machine. In this way, Davis showed that neural networks reflect computation within the bounds of the Church–Turing Thesis—that is, equivalent to a computation that can be done by a Turing machine.

Many of the hypercomputation models rely on some access to certain full-precision real numbers, to which, Davis pointed out, no scientific observation could give us access.

In a paper entitled, “Why there is no such discipline as hypercomputation” (published as an introduction to a special issue of Applied Mathematics and Computation devoted to papers on exactly this discipline), Davis argued that if there were a hypercomputer, we would be unable to verify its performance, since we could only see finitely many outputs. Moreover, any real computer is subject not merely to the limitation of a Turing machine—that is, the limitation that only a finite time and a finite span of memory can be used—but to a much stricter limitation of a constant bound on these quantities, depending only on the machine, and not on the algorithm or the data. While investigation of algorithms in the Turing machine context has important meaning, both in theory and in practice, they must finally be executed on finite state automata, a much more limiting device.

On a personal note VH.

The first time I met Martin Davis at a conference, I was very impressed by his kindness, modesty, sense of humor, and friendliness. Later, I invited him and Virginia to visit me at George Washington University and give a math colloquium talk. It was in November 2007 and his lecture was entitled “Unsolvability and undecidability in the Diophantine realm,” an ordinary title compared to his 2020 MSRI talk “Here there be monsters.” During the GW talk to a packed room, Davis covered many years of work, progress, and struggles on Hilbert’s Tenth Problem. When a person in the audience asked him how he managed to persist for so many years working on one problem, he replied that his obsession with this problem “was a disease.”

Figure 4. Martin and his wife Virginia as grandparents, holding Katie Rose.
Because of the advances of computer technology, Martin Davis’s 2020 talk, “Here there be monsters,” was possible when MSRI semester program on Decidability, Definability and Computability in Number Theory had to be moved online. His talk was an opening one for the program and is posted on [https://www.smath.org/workshops/25120#videos_workshop](https://www.smath.org/workshops/25120#videos_workshop).

2. Hilbert’s Tenth Problem

A brief history. Martin Davis was one of the four people who collectively solved Hilbert’s Tenth Problem. The other three were Hilary Putnam, Julia Robinson, and Yuri Matiyasevich.

The history of Hilbert’s Tenth Problem starts in 1900 when, during an ICM in Paris, David Hilbert presented a list of 23 problems that had a great influence on mathematics in the twentieth century and continue to influence the subject in the twenty-first. The tenth problem on the list is to devise a process that determines whether any given Diophantine (polynomial) equation with integer coefficients has a solution in the integers.

If we are to rephrase Hilbert’s question in modern terms, we could say that he asked for an algorithm (or a computer program) taking as its input coefficients of a polynomial equation in several variables over \( \mathbb{Z} \) and generating “yes” or “no” to the question of the existence of the roots of this polynomial over \( \mathbb{Z} \).

At the time Hilbert formulated his question a formal notion of an algorithm, let alone a computer program, did not yet exist. He asked for a process terminating in a finite number of steps, and this was later interpreted to mean an algorithm. The theorem proved by Davis, Putnam, Robinson, and Matiyasevich showed that such an algorithm does not exist.

Lagrange’s four-squares theorem from 1770 establishes that every natural number can be expressed as the sum of squares of four integers. Hence, the algorithmic solvability of a Diophantine equation in the integers is equivalent to the algorithmic solvability of a Diophantine equation in the natural numbers.

The first step towards the solution of the problem was made by Davis in 1949 when he showed that any computably enumerable set of natural numbers has the following form:

\[ \{a \in \mathbb{N} \mid \exists y \forall k \leq y \exists x_1, \ldots, x_n : p(a, x_1, \ldots, x_n) = 0 \}, \]

where \( p(...) \) is a polynomial with coefficients in \( \mathbb{Z} \) and all variables range over \( \mathbb{Z} \).

It is not hard to see that a set of natural numbers defined using existential quantifiers and a polynomial equation is computably enumerable. More precisely, let \( p(1, x_1, \ldots, x_n) \) be a polynomial in \( n + 1 \) variables and consider the following set \( S \) of natural numbers:

\[ \{a \in \mathbb{N} \mid \exists x_1 \in \mathbb{N} \ldots \exists x_n \in \mathbb{N} : p(a, x_1, \ldots, x_n) = 0 \}. \]

We can enumerate all \((n + 1)\)-tuples of natural numbers and plug them into the polynomial \( p \). Each time the result is 0, we add the first coordinate of the \((n + 1)\)-tuple to \( S \) eventually listing every element of \( S \). The polynomial \( p \) is called a Diophantine definition of \( S \), and sets defined using existential quantifiers and polynomial equations are called Diophantine sets.

In 1953, Davis [2] established that the collection of Diophantine sets is closed under unions and intersections, but not under complements.

Observe that Davis’s formula defining all computably enumerable sets is very similar to the characterization of Diophantine sets (which clearly defines at least some computably enumerable sets). In 1949, Davis conjectured that every computably enumerable set is definable by an existential polynomial formula. It took 20 years for this conjecture to be proven.

Davis and Putnam proved that, under an additional hypothesis, the bounded universal quantifier can be eliminated, in favor of an “exponential Diophantine equation,” that is, an equation where variables are allowed to appear in the exponents. The additional hypothesis was at the time a conjecture, but is now a theorem, and concerns the lengths of arithmetic progressions of primes. Robinson eliminated the need for the conjecture (Figure 5). Finally in 1970, using Fibonacci numbers, Matiyasevich showed that exponential Diophantine equations can be replaced by polynomial equations.

Figure 5. Julia Robinson announces the possibility of getting rid of the hypothesis made by Davis and Putnam.

It is not hard to see that the theorem proved by Davis, Putnam, Robinson, and Matiyasevich implies a negative
answer to Hilbert’s question. Indeed, assume that an algorithm requested by Hilbert exists. Let the set \( S \) defined above be a computably enumerable set that is not computable. Then we can determine whether an \( a \in \mathbb{N} \) is in \( S \) by determining whether the polynomial \( p(a, x_1, \ldots , x_n) \) has roots in \( \mathbb{N} \). Hence, if the algorithm for solving Diophantine equations exists, then there is an algorithm to determine membership in \( S \). This contradicts our assumption that \( S \) is not computable. Therefore, the algorithm requested by Hilbert does not exist.

In the Preface to the 1982 Dover edition of his book *Computability and Unsolvability*, Davis wrote: “One of the great pleasures of my life came in February 1970, when I learned of the work of Yuri Matiyasevich which completed the proof of the crucial conjecture and thereby showed that Hilbert’s Tenth Problem is recursively unsolvable.”

**Other ramifications.** The atomic diagram of a structure \( \mathcal{A} \) is the set of all atomic and negations of atomic sentences allowing additional constants for elements of the domain, which are true in \( \mathcal{A} \). A structure is computable if the characteristic function of its atomic diagram is computable. The standard model of arithmetic, \( \mathcal{N} = (\mathbb{N}, +, \cdot, S, 0) \), the natural numbers with addition, multiplication, successor function, and zero, is a computable structure. Gödel established that all computable relations are definable in \( \mathcal{N} \).

For any computable relation there are two natural defining formulas: one with a block of existential quantifiers followed by a formula with only bounded quantifiers, \( \forall x < y \) and \( \exists x < y \), and the other one with a block of universal quantifiers followed by a formula with only bounded quantifiers. A block of existential (universal) quantifiers can be replaced by a single existential (universal) quantifier by coding tuples of natural numbers by a single natural number. It follows from the proof of Hilbert’s Tenth Problem that bounded quantifiers can be eliminated from the above formulas, so a computable set is definable in \( \mathcal{N} \) both by an existential and a universal formula.

“Positive aspects of a negative solution”. This was a part of the title of the 1974 paper by Davis, Matiyasevich and Robinson [12]. Perhaps this title was a response to some opinions in the mathematical community that the negative answer to Hilbert’s question about polynomial equations meant that the subject matter was closed. Nothing could have been further from the truth.

Among other things, the authors of the 1974 article explained that in a manner of speaking the negative solution was inevitable in part because a big part of mathematics can be encoded into polynomial equations, e.g., Riemann’s Hypothesis. More precisely, Riemann’s Hypothesis is true if and only if a certain polynomial equation with known coefficients has no integer solutions! Such polynomials exist for many other famous problems: Goldbach’s conjecture, consistency of ZFC, etc. Thus existence of an algorithm to solve polynomial equations would resolve unreasonably many open questions in mathematics.

Perhaps the most positive consequence of the Davis–Putnam–Robinson–Matiyasevich theorem is that it, together with definability results of Robinson, became a foundation of a new field: definability and decidability in number theory. This field seeks to understand what is definable and decidable in the language of rings (i.e., the language of polynomial equations) and its various dialects over rings and fields of interest to number theory. From its inception, this field was situated on the boundary of several areas: number theory, algebraic geometry, model theory, and computability theory; and it has led to some interesting interactions between these fields.

**The question of \( \mathbb{Q} \).** Perhaps the most important question in this new area is the analog of Hilbert’s Tenth Problem for \( \mathbb{Q} \). More precisely, does there exist an algorithm that can determine whether an arbitrary polynomial equation in several variables with integer coefficients has solutions in \( \mathbb{Q} \)? One can show that a positive answer to Hilbert’s question for \( \mathbb{Z} \) implies a positive answer to the question over \( \mathbb{Q} \). However, the reverse implication is not clear.

One way to show that there is no algorithm to determine whether polynomial equations have solutions over \( \mathbb{Q} \) is to construct a Diophantine definition of \( \mathbb{Z} \) over \( \mathbb{Q} \).
However, there are conjectures by Mazur and others implying that such a definition does not exist. The question concerning (non)existence of this Diophantine definition is a major problem in the area.

**On a personal note AS.** I first spoke to Martin (on the phone) in the Spring of 1983 when I was deciding on a graduate school. Martin encouraged me to come to NYU. While there, I was lucky enough to take his class on Hilbert’s Tenth Problem. Martin developed a different method for showing that exponential equations were Diophantine (polynomial) using the Pell equation in place of Fibonacci numbers. His method and its generalizations to norm equations served me well in many a paper.

I continued to be in touch with Martin until his death, seeking his advice on many issues. He was mathematically engaged until the very end. I believe his last talk took place online. It was the opening talk for the MSRI semester on Definability, Decidability, and Computability in Number Theory mentioned above.

**3. Automated Reasoning**

Martin began to program computers in 1951, when he was recruited to a group that developed programs for an ORDVAC machine, in support of the military during the Korean War. He was assigned the task of writing, in absolute binary machine language, the prototype of a system by which ORDVAC was to navigate 100 airplanes in real time. After this tumultuous training on concrete programming that lasted roughly one year, Martin was confident enough of his skills with computers that he managed to receive funding for a project on implementing Presburger’s decision algorithm for integer arithmetic without multiplication. Martin’s implementation of that algorithm took place on a JOHNNIAC computer available at the Institute for Advanced Study in Princeton in the summer of 1954. The prover could only ascertain very simple statements [4, 6], yet its accomplishment marked a milestone in computational logic. The preface of [18] says that it produced “what appears to be the first computer generated mathematical proof,” and this accomplishment qualifies Martin as a trailblazer of the field today known as ‘automated reasoning.’ For some twenty-five years, Martin continued to contribute to that field.

In the late 1950s, the seminal report [13] on computational methods for propositional calculus arose from his collaboration with the distinguished analytic philosopher Hilary Putnam. Martin enjoyed talking with him “all day long about everything under the sun” during the summers of 1957, 1958, and 1959. The Davis-Putnam-Logemann-Loveland procedure [11] (DPLL for short), still fundamental in today’s architectures of fast Boolean satisfiability solvers, was rooted in that collaboration. Between 1958 and 1960, simultaneously with the Davis-Putnam and DPLL projects, three major projects (led by Gilmore, Dunham-Fridshal-Sward, and Wang, respectively) were developing propositional provers. It was Davis and Putnam who set up the overall organization that, after them, would become standard in the automation of quantification theory. They adopted the Conjunctive Normal Form (CNF, a propositional conjunction of disjunctions of affirmed or denied logical variables) in pursuing an unsatisfiability test and embedded their rules for propositional logic into the enhanced proof framework. Martin and Hilary Putnam viewed a tester able to establish whether or not a given CNF formula is truth-functionally satisfiable as a key component in a general-purpose procedure for quantification theory. This general procedure can then be applied to obtain proofs (by contradiction) in virtually any mathematical domain. It is surprising that, to this day, the DPLL-procedure constitutes a kernel of any efficient CNF-tester.

Any propositional formula can be brought to an equisatisfiable CNF formula in linear time, hence

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1. Martin then entered the organization, led by Frederick Seitz, named Control Systems Laboratory. Thanks to this move and, subsequently, enjoying a two-year ONR grant at the IAS in Princeton, he managed to avoid being inducted into the army.

2. Still much later, around 1990, Martin would again look at the problem of automatic proof discovery. In the paper [10] he coauthored, which presents first-order predicate calculus under a very unusual light, he points out: “The above very elementary examples only hint at the kinds of proof procedures which our free variable formulation should make possible. But there is reason to believe that they may turn out to be of interest”.

3. Let us recall that in quantification theory, unlike propositional logic, the problem of validity is semidecidable (that is, computably enumerable) without being algorithmically solvable. That is to say: while a systematic search will sooner or later unearth the proof of a theorem, rejecting an unprovable conjecture may turn out impossible.
CNF-satisfiability would become paradigmatic of the whole collection of NP-hard problems.\(^4\) Martin never took sides on the “\(P \neq NP\)” problem: he argued that we have no reliable intuition of what an algorithm of, say, complexity \(n^2\) can do. He discussed his views on the problem at greater length in [9].

Martin and Hilary Putnam had made it clear that their method would outperform competitors of the time: by exploiting it in a 30-minute hand computation, they in fact validated a claim that a theorem-proving program developed by Paul C. Gilmore had been unable to validate with a 21-minute run on an IBM 704 machine [14]. The improvement was not due to a better handling of quantifiers, but due to an improvement in the propositional part. Later, the promise of these hand computations was realized in computer implementations. Davis and Putnam’s proof procedure for finitely axiomatized theories was implemented by Logemann and Loveland at NYU. They found and removed a bottleneck in the propositional-level component of the procedure. Later an implementation in LISP at Bell Labs gave further incremental improvements.\(^5\)

The Bell Labs implementation of Martin’s method was named Linked Conjunct. The operating principle required that each logical variable in an unsatisfiable CNF formula would be paired with the same variable with the opposite sign in another conjunct. Not long after, there would be a proliferation of new proof search methods arising from John Alan Robinson’s influential resolution principle.\(^6\) It turns out, however, that many of these refinements can be naturally explained from the standpoint of Linked Conjunct.

The original expectations about stand-alone theorem provers have been retrograded over the years, to proof-checking systems that range from highly interactive reasoning assistants to mere proof-script verifiers. Martin also had a role in this change of perspective. One such contribution, jointly authored with his lifelong friend and colleague “Jack” (Jacob T. Schwartz), addressed the issue of metamathematical extensibility in a full-blown program- and proof-verification technology. Which mechanisms can ensure long-term reliable use of a proof checker that undergoes augmentations with new symbols, schemes of notation, and extended rules of inference?

This work stemmed from Martin’s experimentation with Richard Weyhrauch’s FOL proof checker developed at John McCarthy’s Artificial Intelligence Laboratory.\(^7\) Martin recounts in [8]: “I found it neat to be able to sit at a keyboard and actually develop a complete formal proof, but I was irritated by the need to pass through many painstaking tiny steps to justify inferences that were quite obvious”, and then adds: “Using the LISP source code for the linked-conjunct theorem prover..., a Stanford undergraduate successfully implemented an ‘obvious’ facility as an add-on to FOL.”

On a personal note EO. In 1975, Martin offered a summer course on computability in the pretty Italian town of Perugia. The dozen students in his class were initially amazed at the discrepancy between Martin as an unpretentious, easygoing person and his reputation as a distinguished scholar. Admiration quickly prevailed over astonishment when Martin began his lectures: for the entire one-month duration of his course, concepts remained clear, precise, and accessible. Even when he reached his cherished advanced topic: Hilbert’s Tenth Problem.

As a result of having been a student in Martin’s class in Perugia, I was able to do my graduate work at NYU, with Martin as my advisor for a Master’s and then a PhD degree in Computer Science. A stream of Italian students and researchers (three had been my classmates in Perugia; Alberto Policriti and others belong to a successive generation) would, like me, reach Martin overseas in the following decade. This testifies to the influence that Martin’s crystal-clear lectures, and the subtlety with which he addressed issues of philosophical relevance and depth, used to exert on his audience—in Italy much as elsewhere.

On a personal note AP. On the evening of a beautiful day of the fall of 1990, I was invited by Martin to his place in the Upper West Side in New York, to a “party for two Yuri’s.” The two Yuri’s were Yuri Gurevich and Yuri

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\(^{5}\)The mentioned implementation at NYU improved the 30 minute hand computation time to 2 minutes (cf. [11]).


\(^{7}\)Martin would later cooperate, with work conceived in the same stimulating environment at Stanford University, to the launch of non-monotonic logic formalisms.
Matiyasevich, both in town and hosted by Martin. For Yuri Matiyasevich, this was his first visit to the United States. The number and (even more) the names of the people invited to the party were—especially for a young Italian PhD student—rather overwhelming, and I was definitely scared when I was greeted amicably by Virginia upon my arrival. However, every tension promptly dissolved when Martin introduced Yuri Matiyasevich to the audience. He started recalling what remained to be proved after his first reduction of Hilbert’s Tenth Problem and how, at the time, he conjectured that the last pending issue (a number theoretic hypothesis raised by Julia Robinson around 1950) would certainly be solved by a clever young Russian in the near future.

Yuri Matiyasevich was then introduced to everybody as the (constructive and positive) proof of Martin’s conjecture.

4. Conclusions

Recollections of contributions by Martin to computability theory, Hilbert’s Tenth Problem, and automated reasoning have been scattered over the preceding text, and many more could be cited. For example, in [5], Martin stretches the algorithmic unsolvability of Hilbert’s Tenth Problem into this result: For each proper nonempty subset $A$ of $\mathbb{N} \cup \{\aleph_0\}$, no algorithm can establish, given any polynomial $p$ with integer coefficients, whether the number of distinct positive integer solutions to the equation $p = 0$ belongs to $A$.

Beyond the proofs of specific theorems, Martin’s scientific legacy included a broader contribution in the promotion of formal methods and theoretical computer science. He significantly contributed to the recognition of computability theory as an autonomous branch of mathematics. Martin developed a program in logic, and formed a logic group, first at Yeshiva University (being able to involve stars such as Raymond Smullyan) and then at the Bronx campus of the Courant Institute (NYU). In the 1960s, he devoted a good deal of time and energy in preparing an anthology of fundamental articles by Gödel, Turing, Post, Kleene, and Rosser, which he entitled The Undecidable. His involvement with symbolic logic originated in the 1940s from his passionate interest in the foundations of real analysis, which also led him to write a classic book on nonstandard analysis in the 1970s and to serve for decades as the moderator of FOM, an automated e-mail list for discussing foundations of mathematics (see https://cs.nyu.edu/mailman/listinfo/fom).

Over the years, Martin lectured in several countries (to cite a few: Italy, Japan, India, England, Russia, and Mexico), and his lectures have—along with his publications—exerted a wide influence. The centennial of Frege’s Begriffsschrift, Martin reports, “fundamentally changed the direction of my work” [8]. Being invited to place some contemporary trends in a proper historical context, he finds “trying to trace the path from ideas and concepts developed by logicians . . . to their embodiment in software and hardware . . . endlessly fascinating” [8].

BIBLIOGRAPHIC NOTE. The expository nature of this paper has required us to keep bibliographic references to a minimum. A version of this paper with a full bibliography can be found on ArXiv at arXiv:2401.10154.

References


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A Memorial Tribute to David B. Wales

Michael Aschbacher, Arjeh M. Cohen, Richard Foote, Robert Guralnick, Philip Hanlon, and Claire Levaillant

David Bertram Wales was born on July 31, 1939, in Vancouver, British Columbia, and died on July 17, 2023, in London, England. His father was a high school physics and math teacher, and later a school principal. His mother had a degree in library science and spent her time raising him and his two younger brothers.

David began working on the theory of finite groups when he was a PhD student at Harvard. In 1967, he started teaching and doing research at Caltech, where he spent more than 50 years. David contributed significantly to results like the construction of one of the 26 sporadic simple groups and the determination of the finite subgroups of complex Lie groups of types $F_4$ and $E_6$. Besides group theory, David worked in combinatorics and representation theory of algebras.

David loved teaching mathematics and working with students. He enjoyed collaborating with other mathematicians and relaxing with them over a glass of wine. David also enjoyed serving in other roles at Caltech including administrative positions in the math department and the office of student affairs. His interests outside mathematics included hiking and international travel. David was fortunate to be active, healthy, and high on life until his unexpected death due to pneumonia.

The contributions below shed more light on David’s mathematical interests and what it was like to work with him. This article was organized by the second author.

Figure 1. David Wales.

Michael Aschbacher

Canadian by birth, David Wales received his undergraduate and masters degrees from the University of British Columbia. He obtained a PhD from Harvard, working under the direction of Richard Brauer, with a thesis on the representation theory of finite groups. Wales then took a position at Caltech, where he remained until his death. Marshall Hall, one of the most distinguished group theorists of the time, was then at Caltech, making it a good fit for Wales.

In his early years at Caltech, Wales worked on several projects involving the representation theory of finite groups. His expertise in group theory was instrumental in the successful completion of the classification of finite simple groups, a monumental achievement in 20th-century mathematics.

Michael Aschbacher is Emeritus Shaler Arthur Hanisch Professor of Mathematics at Caltech. His email address is asch@caltech.edu.
groups: He determined the finite groups with a faithful representation of small degree. He and his students determined the simple groups of order $2^a 3^b p^c$ for a prime $p$ and with cyclic Sylow $p$-subgroups. Wales and his student Huffman determined the quasiprimitive linear groups of degree $n$ generated by the conjugates of an element with $n − 2$ equal eigenvalues.

During the ten-year period beginning in 1965, finite group theory was rocked by the discovery of 21 of the 26 sporadic groups, that is, those groups that appear in the classification of finite simple groups and are not members of an infinite series. In the sixties, Marshall Hall (and independently Janko) discovered the sporadic Hall–Janko group (HJ). Hall and Wales proved the existence and uniqueness of HJ; Wales proved the uniqueness (subject to suitable constraints) of HJ, using a pretty argument applicable to other sporadics. Later in the seventies, John Conway and Wales proved the existence of the sporadic Rudvalis group.

In later years, Wales turned to questions involving representation theoretic aspects of algebraic combinatorics, particularly Brauer centralizer algebras. He also worked with Arjeh Cohen toward a determination of the finite subgroups of exceptional Lie groups that are maximal among closed subgroups. And he collaborated with numerous people on smaller projects.

Wales had a distinguished record of service at Caltech, particularly with the undergraduate student population. He served terms as master of student houses and dean of undergraduate students, and two terms as executive officer for mathematics. He was faculty secretary for a number of years.

During the seventies Caltech was a center of activity in finite group theory, with many young mathematicians holding positions as research instructors. Examples include Steve Smith, Richard Foote, Joe Carroll, and Bob Guralnick. There were also many visitors, including extended stays by John Conway and John MacKay, and a year-long conference funded by NSF with scores of visitors including Danny Gorenstein and John Thompson. Wales mentored the younger mathematicians and collaborated with older visitors.

There was of course a weekly group theory seminar, but there was also an informal meeting on Friday afternoons in the basement bar of the Athenaeum (the Caltech faculty club) convened over pitchers of margaritas. It was at one of these meetings that David Wales met his wife Kathy.

During David’s tenure as master of student houses, he and Kathy lived in the Master’s House, a large old California house situated on the Caltech campus with various features, such as a large backyard, that made it well suited for entertaining students. The house contained a pipe organ that was two stories high. The Waleses liked cats, and usually owned one or more Abyssinian cats. I recall one week my wife and I were recruited to walk to the Master’s House each day to feed the cats and play with them in the Waleses’ absence. It was during this time that our younger daughter Meredith died in an automobile accident. My family decided to hold a get together for friends of Meredith and the family, to celebrate her life. The Waleses volunteered the yard of the Master’s House as a venue for the event. We’ve always felt a special connection to David and Kathy for this kindness.

David Wales died of a lung infection in London during a trip to England which included the chance to see one of his grandchildren play in an international tournament in ultimate frisbee. He is survived by his wife Kathy, his children Jon and Jennifer, and his grandchildren.

Arjeh M. Cohen

Working with David was a great pleasure and led to more joint papers than I had with anyone else. This is largely due to the many positive aspects of David’s character, which I greatly enjoyed. He was thorough in his treatment of mathematics and at the same time positive in his attitude and joyful to people in general. Where I was the impulsive person with wild ideas, David would provide stability and reassurance that there was something good in what I did. Where I would try to explain an idea in too many words, David would reshape our texts into an understandable format. Where I worked late at night and sent my immature results to him before going to bed, David would check them early in the morning and set the ground for further discussion. When we got stuck, David would present the simplest example that should be examined before trying further bold theories. On a personal

Arjeh M. Cohen is an emeritus professor at Eindhoven University of Technology. His email address is arjehmcohen@gmail.com.
level, we got along very well. I highly enjoyed David and Kathy’s warm welcoming and interested attitude and the many trips we made together to painting exhibitions, nature (mostly beaches), castles, and the beers in Old Town Pasadena to celebrate a new result. David was a great friend and my ultimate math companion.

Let me describe the main parts of our mathematical work. In 1977, David Wales published the article [7] with Cary Huffman on the classification of finite linear groups over the complex numbers containing elements of order 2 with exactly two eigenvalues equal to $-1$. At the time, I worked on the classification of quaternion reflection groups, that is, finite linear groups over the quaternions generated by reflections, that is, elements with exactly one eigenvalue distinct from 1. It struck me that most of my examples should occur in their classification, so I wrote to Wales. David’s kind and enthusiastic response was the start of a collaboration that lasted for the rest of his life.

My thesis advisor, Tonny Springer, suggested that I also classify finite groups generated by homologies (the projective version of reflections) over the octaves $\mathbb{O}$ (also often called the octonions). The first order of business would then be to determine the finite groups of automorphisms of $\mathbb{O}$. Such groups have a faithful linear representation in seven dimensions over the reals, and, since David was an expert on these, he joined in. This led to our first joint publication. Finite groups generated by homologies over the octaves are subgroups of the complex Lie group of type $E_6$, and so we proceeded by trying to classify all finite subgroups of this Lie group and, while we were at it, those of type $E_6$ as well. Early in 1989, we got a chance to work on it during a stay at Caltech that David had organized for me; the result [2] appeared in 1997.

Around that time, our attention shifted to two new developments. First, during David’s sabbatical at Eindhoven University of Technology, we dived into extremal elements in Lie algebras (an element $x$ of a Lie algebra $L$ is called extremal if $[x, [x, L]]$ is contained in the subspace spanned by $x$). The 1-space spanned by such an element in a simple Lie algebra is the Lie algebraic counterpart of a long root subgroup of a group of Lie type. Anja Steinbach and Rosane Ushirobira helped out. The resulting paper [3] was followed by a lot of interesting algebra and geometry. Second, a former PhD student of mine, Daan Krammer, proved that the braid groups are linear. David and I soon realized that his proof could be extended to Artin groups of finite type. David suggested we study analogs of the Birman-Murakami-Wenzl algebra related to these Artin groups, observing that the Artin groups mentioned should embed in their groups of units. This started a challenging project which we worked on with students like Bart Frenk, Dié Gijsbers, and Dan Roozemond until about 2013. David and I coauthored at least seven papers on this topic.

From 2015 on, I started writing interactive mathematics courses and tests for a private enterprise, which was set up by my son and a friend of his. David got involved as my conscience since he checked the English language version and the soundness of proofs. As was the case with our joint writing of papers, when he would report that he did not understand a proof, I could expect there to be an error in it. The construction of the courses was concluded shortly before David died.

### Richard Foote

I first met David in fall, 1976, when I began my two-year tenure as a Bateman Research Instructor at Caltech, although I was familiar with his work on groups of order $p^3q^2$ before then. David soon became a mentor, colleague, and friend. Caltech was one of the epicenters of the Classification of Finite Simple Groups (CFSG) program, with David, Michael Aschbacher, Marshall Hall, myself, and, in 1977, Robert Guralnick on the faculty. As an expert in representation theory, David was an invaluable resource to me, a newly minted PhD; and since he was associate dean of students, I could always count on him to support my undergraduate students with expert advice and compassion.

David was vital to the weekly Caltech group theory seminars. To typify his participation, I remember that we once worked through some notes on the recently published Lusztig-Deligne theory of characters of the groups of Lie type. Not satisfied with the theory alone, David insisted that we work through an example. So we spent a couple of hours after the seminar computing the complete character table of a substantially large linear group using the Lusztig-Deligne formulas, and then checked all the orthogonality relations by hand. That was the kind of mathematician David was: expert and knowledgeable in the theory, but down-to-earth and practical in its workings.

My sole coauthored paper with David, [8], was written in the spring of 1988, when I returned to Caltech for a few

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**Richard Foote** is a professor emeritus at the University of Vermont. His email address is rfoote@uvm.edu.
was also an avid adventurer, hiker, and camper. One of my most enduring happy memories of David was camping with him by the shadow of Half Dome in Yosemite National Park. His mathematics and his life as an educator and a friend are personified to me by that pinnacle.

Robert Guralnick

I first met David in September 1977 when I started my post-doctoral position at Caltech. I had just received my degree the previous June. It was a quite exciting year. Caltech was having a special year in group theory and had quite a number of visitors such as Danny Gorenstein, Graham Higman, and many others. Caltech also had a very strong group theory presence with Michael Aschbacher, Marshall Hall, and David. Richard Foote was another group theory postdoc who had started a year earlier.

The abstract algebra class at Caltech was a sophomore-level class and there were three sections taught in the fall and winter terms. They were taught by Richard, David, and myself. Of course, David gave us a lot of advice (although Richard and I were quite enthusiastic and so I think we covered more material than David; I had Peter Shor in my class). Richard’s notes were the birth of the Dummit-Foote book on abstract algebra.

I got to know David well during my two years at Caltech and he became a lifelong friend in addition to a mathematical colleague. We had many lunches at the Athenaeum. During my first year, the membership dues was only $1 for me but were considerably higher for Michael and David and so I always signed for the lunches.

I continued to come over to Caltech for the group theory seminar and other events for many years after leaving Caltech for USC and my friendship with David grew. During the pandemic, I did not go to Caltech but fortunately in February and March of 2022, Tim Burness was visiting Caltech and he and I had a shared office. I was coming to Caltech at least twice a week for that period and saw David quite a lot. We had lunch several times as well as dinner and many conversations. Currently, David’s last PhD student, Claire Levaillant, works at USC under the joint supervision of Aaron Lauda and myself.

David had a very strong mathematical career. He was involved in the construction of one of the sporadic simple groups (a group discovered independently by Hall and by Janko of order 604,800) and much of his early work was on classifying simple groups under various assumptions and studying low-dimensional representations.

Over his long career, he wrote papers on many different topics (including many joint works with Arjeh Cohen and Phil Hanlon). The ones that were most relevant to my work were his papers mostly with Cohen about subgroups of exceptional groups. In [1], Wales and Cohen proved that the group $GL(4, k)$ acting on an irreducible 16-dimensional module where $k$ is a field of $F$. 

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Robert Guralnick is a professor of mathematics at University of Southern California. His email address is guralnic@usc.edu.
characteristic 3, has only finitely many orbits (by working in the Lie algebra). This was an interesting fact that came up in my work with Liebeck, Macpherson, and Seitz [4] classifying irreducible linear subgroups with only finitely many orbits on Grassmannians. In another article, [3], Wales and Cohen considered generation of Lie algebras by “extremal” elements (which are usually long root elements). This was generalized in work of mine with Garibaldi that we used for classifying generic stabilizers in irreducible linear representations of simple algebraic groups.

In three papers [6, 7, 10], two with Huffman, finite primitive subgroups of $GL(n, \mathbb{C})$ containing semisimple elements with an eigenspace of codimension 2 were classified. The case of pseudoreflections is a classical result of Shephard and Todd and has connections with invariant theory. This used some deep group-theoretic results (but not the classification of simple groups) and also has connections with when the invariant ring is a complete intersection. Jan Saxl and I classified all finite groups in linear groups over finite fields generated by elements which had a fixed space of codimension two. Our proof did require the classification. His joint paper with Zassenhaus [11] on $L$-groups (a concept introduced by Olga Taussky) also influenced a later paper of mine on the topic.

Although David and I often discussed mathematics, we had only one joint paper, [5]. We were considering pairs of nonconjugate subgroups $A$ and $B$ of a finite group $G$ such that their induced modules are isomorphic. If the characteristic is zero, this is saying that the intersections of $A$ and $B$ with each conjugacy class of $G$ have the same cardinality. This is related to prime splitting properties of algebraic number fields and also to the problem of whether you can hear the shape of a drum. We considered the situation for other fields where more subtle issues of modular representations come in. We also classified all primitive permutation groups of degree $pq$ with $p$ and $q$ primes. In particular, we classified the pairs of nonconjugate subgroups of index $pq$ with the property described above. In particular, we showed that if $G$ is solvable, there are no such pairs.

**Philip Hanlon**

I feel very fortunate to have had David Wales as a mentor and collaborator. He was an outstanding algebraist and I learned much from our mathematical work together. Perhaps more importantly, he was someone whose emotional intelligence rivaled his mathematical intelligence. He never lost sight of the fact that mathematics is ultimately a human endeavor, and one which is challenging and often solitary. And so he generously supported those around him with kindness, humor, and empathy. This was an important part of what made him such a special colleague and friend.

I was a PhD student at Caltech from 1977 to 1981. At the time, Caltech’s graduate program in Math was small and cohesive. We bonded over coursework, teaching experiences, and Friday-evening libations in the basement of the Athenaum. By then a full professor, David was always there for us. Encouraging us, sharing our frustrations, joining in our evening gatherings at the Athenaum. For us, as PhD students, David represented the faculty with kindness and support. And his care for others extended well beyond our community of graduate students. For many years, he served as dean of students assisting countless Caltech undergraduates navigate challenges big and small. And he was first in line to recognize the contributions of the Math Department’s hard-working staff.

David had passions outside of mathematics. He loved his family—Kathy, Jonathan, and Jennifer—and often spoke with great pride about their accomplishments. He relished his Canadian heritage. And he had a playful side and a sense of humor that helped put all things into perspective. For many years, he and I bet $1 each year on the outcome of the Harvard-Dartmouth football game. Sadly, I found myself on the losing end of this bet most years. And then one year, as an apparent act of consolation, David offered me “a deal”—a chance to get back the dollar I had just lost. He would bet me $1 that there was at least one rouge during the Grey Cup—the Canadian Football League championship game. I asked “what in the world is a rouge”? He went on to explain it in terms that made it seem that a rouge was very unlikely to happen. And so, I enthusiastically agreed to this offer only to lose the bet and find out later that rouges are a common occurrence in Canadian Football games.

David was also a first-rate mathematician with an avid curiosity. After earning my PhD in 1981, and two years at MIT, I returned to Caltech as a Bantrell Fellow in 1983. At the time, I was interested in the Brauer centralizer algebras, a family of algebras that can be defined via a combinatorial construct indexed by two parameters—a positive integer $n$ and a multiplication factor $x$. My immediate interests were to identify those values of $n$ and $x$ for which the Brauer algebra is semisimple and the structure of the radical for those values of $x$ where semisimplicity fails. David was excited by these questions and immediately jumped in to help.

In some sense, this was an ideal collaboration. David brought deep expertise in algebra generally, and semisimplicity specifically, to the table. And I brought...
combinatorial methods as well as advanced computational tools to bear—simplifying the discriminant calculations by applying the representation theory of the symmetric groups—to make vast calculations just within the reach of what were at the time cutting-edge computational tools: the CRAY 1 and CRAY 2 machines at the Minnesota Supercomputer Center. Our collaboration resulted in a series of papers that included conjectures (most notably that the Brauer algebras were semisimple for noninteger values of $x$) later proved by Hans Wenzl using Jones’ tower construction. What I remember most about that work was what it was like to have David as a collaborator. He had a passion for mathematical discovery that was infectious. David loved doing concrete examples. But in our work on the Brauer algebras, examples quickly became so large and complex that hand calculations were impossible. And so David felt a true excitement at what could be gleaned by combining high end computational methods with the tools of algebra and combinatorics.

I’ve recently been working on a family of algebras (the Okada algebras) that share many of the properties of the Brauer algebras. This has caused me to look back at those papers that David and I wrote together all those many years ago. And I can’t help but imagine how much David would have enjoyed being part of this new project. I miss having him as a collaborator and as a friend.

Claire Levaillant

David and I met for the first time in 2001 when I was 21 years old and a summer intern in neuroscience at Caltech. At that time I was a math major at the Ecole Normale Supérieure de Cachan in France, thus I quite naturally went to visit the math department. There I was asked in which field of mathematics I was interested and my answer was “algebra.” This led me to David’s office. I found him standing, looking towards the ground and in very deep mathematical thoughts. He simply said to me “Come back at 11.” It turned out simplicity, kindness and professionalism described him well. David started advising my PhD work two years later.

During my graduate studies, David’s door was always open to me, except when his close collaborator Arjeh Cohen was visiting. At these times, both of them were so focused on mathematics that an earthquake could very well occur without them noticing. During and after my studies with him, David always enjoyed wishing me a happy Bastille Day on July 14, which I found nice. Conversely, if I missed wishing him a happy Canada Day on July 1, I fortu-

![Figure 4. David (left) with Arjeh Cohen in David’s office.](image)

nately had a backup three days later. As an advisor, David was serious and demanding. Studying the bibliography before my candidacy exam was taking more than a week and he was upset as I came into his office for my weekly meeting showing him only a half page. A week later, I came back with 7 pages and as he saw the first page numbered as 1 slash 7, he exclaimed to me: “You have written 117 pages, you have made good progress.”

For my PhD, David had me work on representations of diagrammatic algebras named after Joan Birman, Jun Murakami, and Hans Wenzl. These algebras contain the braid group. This was shortly after the linearity of the braid group had been shown by Daan Krammer algebraically and by Stephen Bigelow topologically, using a representation of Ruth Lawrence. The latter representation became known as the Lawrence-Krammer representation of the braid group and to this date, it is the only known faithful representation of the braid group. As part of my work, David had me build my own representation of the braid group, namely one equivalent to the Lawrence-Krammer representation but of simpler expression. Later on, I worked on representations of Artin groups of types $D_n$ and $E_6$ based on the beautiful constructions for tangles of type $D_n$ by Cohen, Gijsbers, and Wales.

David was an open-minded person with a good sense of humor. He was a happy person and had a balanced life. His happiness was highly communicative. His personal honesty was particularly well conveyed through mathematics. His mathematical knowledge was immensely broad but there were always more things that we did not know than things that we knew. It always made the interactions true and pleasant.

David was always very supportive and encouraging to the students. He was specifically involved in the undergraduate students’ well-being. He was also generous to the graduate students who were not his own students. David’s overall generosity to the students was an important part of his life. The postdocs also liked him. I recall talking with a postdoc who said to me about David closing both eyes “he is extremely nice.”

Claire Levaillant is an assistant professor at the University of Southern California. Her email address is levail1a@usc.edu.
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References


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2023 TexPREP-Lubbock students pose proudly next to the gumball machine they designed during this hands-on math, science, and engineering summer program.

Photo courtesy of TexPREP Staff
Our colleague Miguel Abánades asked Spanish first-year university students to calculate \( \frac{d}{dx} \log(x^2) \). Of 40 students, 39 solved the task using the chain rule:

\[
\frac{d}{dx} \log(x^2) = \frac{1}{x^2} \frac{d}{dx} x^2 = \frac{1}{x^2} \cdot 2x = \frac{2}{x}.
\]

Only one student started by manipulating the logarithm:

\[
\frac{d}{dx} \log(x^2) = \frac{d}{dx} (2 \log x) = \frac{2}{x}.
\]

Both approaches, or solution strategies, are correct, but many mathematicians might consider the latter more “clever.” The ability and propensity to use solutions tailored to features of a problem is what we call strategy flexibility.

This phenomenon has been observed over a long time (e.g., Wertheimer, 1945; see Section 2 below). In the past year, two research literature reviews of the field have appeared ([5] and [15]) each with over 70 references. We refer the reader to these surveys for a comprehensive overview of the research and comparisons of different conceptualizations of the phenomenon. As is usual, most of the research has been conducted at the elementary and middle school levels, even though the impact of strategy flexibility is perhaps even greater at higher levels.

In this expository article, we make the case that strategy flexibility is relevant for university mathematics and its teaching. We include concrete mathematical examples throughout the text which hopefully will allow the readers to connect the presented ideas with their own experience and perhaps draw some inspiration for their own teaching development journey.

We first present additional manifestations of flexibility, or lack thereof, as well as most research from the university level (Section 1). We also extrapolate from flexibility research at other levels and draw on our personal experience. Specifically, we describe mathematics education research related to flexibility (Section 2) and provide some ideas for practical steps that can be taken in university classes (Section 3). We argue that university mathematics teachers are well-positioned to cultivate the skills and attitudes necessary for strategy flexibility. Finally, challenges and caveats to teaching flexibility are presented.

We drop the word “strategy” from “strategy flexibility,” since this is the only type of flexibility that we consider; see [5, 15] for other types flexibility in representations. We also note that some authors use the term “adaptive” instead of “flexible.”

1. Examples of Flexibility

Those who teach mathematics have undoubtedly observed the flip side of the coin: weaker students are prone to select unexpectedly inefficient solution strategies. As an example from this year’s Finnish high-school leaving exam, far too many students used the quotient rule as follows

\[
\frac{d}{dx} \frac{x^3 - 15x^2 + 50x}{1000} = \frac{(3x^2 - 30x + 50) \cdot 1000 - (x^3 - 15x^2 + 50x) \cdot 0}{1000^2}.
\]

Unsurprisingly, this approach exposed the students to a host of slip-up opportunities that are absent from a more efficient approach.

The opportunity to be clever—to choose one’s approach wisely—occurs not only at university but at every level of mathematics. In arithmetic, one can calculate 5002 – 3997
with a standard “borrowing” approach or by noticing that adding 5 to 3997 gives 4002, after which another 1000 is needed to reach 5002. Already, back in 1945, Wertheimer lamented that students calculated
\[
\frac{274 + 274 + 274 + 274 + 274}{5}
\]
by first adding the five numbers in the numerator and then performing a division. In elementary algebra, the equation \(3(x - 4) + 4(x - 4) = 14\) can be solved by the standard algorithm which involves distributing both parentheses or by a situationally appropriate strategy which starts with combining like terms:
\[
3(x - 4) + 4(x - 4) = 14
\]
\[
7(x - 4) = 14
\]
\[
x - 4 = 2
\]
\[
x = 6.
\]
University mathematics contains both similar opportunities for flexibility as well as more profound ones. In university calculus courses, we can use tasks like
\[
\frac{d}{dx} \log(x^2) \quad \text{and} \quad \frac{d}{dz}(z(z^3 + z)^{-1}),
\]
from the first paragraph (above) and Maciejewski–Star’s study [7] in the US, respectively. In both cases, a simplification before differentiation leads to a more efficient solution. This is analogous to the linear equation mentioned above. Schoenfeld (1985) makes the opposite observations about his students’ approaches to integration tasks such as \(\int \frac{x}{x^2 - 9} \, dx\): an initial algebraic manipulation (partial fractions) leads to much more work. Beyond calculus, the inefficient solution strategy may be so technically demanding that students are not able to complete it at all: a quantitative difference in performance becomes a qualitative one.

Broley and Hardy [2] studied tasks such as showing that the set \(\{ \frac{n^p}{n+1} : n \in \mathbb{N} \}\) is unbounded when \(p > 1\) in an analysis course in Canada. They found that students mostly fell back on calculus routines and had difficulty connecting solutions to concepts from analysis. However, they suggested that flexibility in calculus could improve students’ ability to make connections between calculus and analysis.

In a linear algebra course at a New Zealand university, Kontorovich [6] similarly connected flexibility to the use of general properties over matrix calculation-based approaches. Students were for instance asked to find a basis of the space of vectors \(b \in \mathbb{R}^3\) for which the equation \(Ax = b\) has a solution, where
\[
A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}.
\]
In this case the most efficient approach was based on observing that the column vectors of \(A\) are independent since their coordinates are not proportional (this was a lemma proved in the course). A computational approach was to use Gaussian elimination; this involved many more steps, which were completed in more or less efficient ways by different students.

2. The Nature of Flexibility

To recap, flexibility refers to having access to a variety of solution strategies and the ability and inclination to choose a situationally appropriate one that is well-suited to the task at hand.

Traditionally, mathematical proficiency has been framed with the constructs of procedural and conceptual knowledge. Procedural knowledge is often seen merely as the ability to execute memorized algorithms (reflecting superficial understanding) whereas conceptual knowledge is seen as indicating deeper connections between mathematical objects. Star [11] argued that the procedural–conceptual dimension should be distinguished from the superficial–deep dimension. He proposed that flexibility is a form of deep procedural knowledge where one needs to understand the limitations and strengths of procedures in relation to specific tasks.

In investigations of elementary and secondary students’ mathematics learning, there has been long-standing interest in this aspect of mathematical proficiency under a variety of labels, including productive thinking (Wertheimer), relational understanding (Skemp), adaptive expertise (Hatano), and strategic competence (Kilpatrick). Conceptualizing this competency as flexibility has gained traction in the field in the past 20 years (see [5, 15] for recent reviews), perhaps due to its more precise definition, the development and use of validated measures of flexibility among researchers, and the ease of applicability of this construct to secondary as well as tertiary mathematics.

One such instrument is the triphase test developed by Xu, Star, and colleagues. We used it in a fairly large study of middle- and high-school students’ flexibility and accuracy in linear equation solving in three countries [14]. Unsurprisingly, we found students with low accuracy and low flexibility and others with high accuracy and high flexibility. Furthermore, many had high accuracy and low flexibility, but virtually no one had high flexibility and low accuracy. This results in the lower-triangular pattern in Figure 1. The fact that there were highly accurate students with all levels of flexibility indicates that flexibility is not merely a matter of having greater procedural skills or fluency; rather, flexibility involves a different kind of (deeper) procedural knowledge.
Deep knowledge is needed since deciding on the situationally appropriate strategy requires paying attention to the special structure of a problem and assessing the importance of these features in relation to the problem-solving objective [11]. While one might think that most problems have generic rather than special structure, experience suggests that problems with special structures are far more common in mathematical research and applications than their "statistical probability" would suggest (e.g., if you generate polynomials with random coefficients, then clever shortcuts will almost never apply). Indeed, things lining up, canceling, and simplifying often indicates to a mathematician that they are on the right track to solving the problem.

The ability to view objects and expressions as the combination of parts has been called structure sense by researchers. This concept also appears in the Common Core High School objective "Seeing Structure in Expressions." To summarize the previous paragraph: flexibility requires structure sense and flexibility also allows students to capitalize on their structure sense by choosing the most suitable strategy to the structure at hand. Most research on flexibility has dealt with primary and secondary school. Yet structure sense often comes into play far beyond arithmetic and linear equations. Indeed, much of calculus involves pattern-matching expressions with formulas and carrying out correct algebraic manipulations. However, structure sense carries even further. For instance, in real analysis Hölder’s inequality

$$\int f(x)g(x) \, dx \leq \|f\|_p \|g\|_p', \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is simple enough for students to understand, but applying it in the proof of Minkowski’s inequality requires interpreting the integrand of $\int |u|^p \, dx$ as $|u| \cdot |u|^{p-1}$ and identifying the first term with $f$ and the second term with $g$ in Hölder’s inequality. This kind of application is much more challenging, especially for students who are less adept at spotting patterns in algebraic expressions.

Flexibility involves not only skills and knowledge; it is also a matter of disposition. Core to flexibility is striving for the most appropriate or best solution in a given situation. Some researchers have sought to quantify "best" in terms of the speed and accuracy with which the strategy can be applied by a given individual to a given problem. A good summary of this point-of-view is given by Hickendorff [4]. While a situationally appropriate strategy often involves fewer steps and can be executed quicker, we consider this a side-effect rather than the goal of flexibility. It is more important to contemplate a variety of approaches and weigh their pros and cons, rather than going with the first idea that springs to mind.

Let us speculate on more wide-ranging consequences of such a mindset of optimization beyond tasks in a mathematics lesson. Expanding on the point that it impacts all levels of mathematics, even reading research papers one sometimes wonders if this is just the first proof that the authors managed to produce and if additional effort could have led to more streamlined and elegant proofs. In programming, there is a world of difference between a brute-force solution and an elegant one. Choosing data-structures unwisely will stifle opportunities to evolve one’s program with changing demands. More generally still, one can assess and improve most processes ranging from engineering to administration in terms of efficiency if one spends time and effort on optimization.

The critical reader may wonder whether all this “flexibility” is actually a manifestation of the same trait. We see a conceptual relationship between these different examples.
of flexibility and at least personally utilize the same flexible thinking also outside procedural mathematics. However, research suggests that transfer of skills between contexts is usually difficult and the jury is still out on the level of transfer of (acquired) flexibility to other mathematical areas and other subjects. What the research does demonstrate is that it is possible to impact students’ flexibility positively by suitable adjustments to teaching practices.

3. Teaching for Flexibility

Much has been written about mathematical problem solving (in the sense of Pólya), including at the university level (e.g. Schoenfeld, 1985); teaching problem solving skills is likely to also support flexibility. However, despite decades of effort it is fair to say that teaching problem solving has not had a large-scale impact, due in part to disagreement on what “problem solving” is. Compared to the daunting task of improving mathematical problem solving, aiming for flexibility offers several advantages. Both a traditional routine-oriented approach and flexibility involve a focus on procedural knowledge. However, adding an element of flexible procedure use changes students’ perceptions of the mathematical task, from merely parroting back what a teacher has demonstrated to a wider, yet still limited, set of creative options. It is the limited range of creativity which sets flexibility apart from problem solving. Thus the reorientation required of the teacher and the students is smaller in flexibility than in mathematical problem solving.

Star and Newton [12] studied mathematics experts’ flexibility. Participants recounted that looking for a situationally appropriate strategy in a mathematical task is intellectually challenging and interesting. Furthermore, they reported not having been taught to search for optimal strategies, but rather considered doing so a personal desire that they brought to problem solving. While these mathematics experts had been able to figure out flexible strategy use on their own, earlier mentioned studies have shown that this is not the case for most students.

In reference to Figure 1, we mentioned that there were virtually no students with high flexibility and low accuracy in our study [14]. The figure indicates that the Spanish schools in the sample placed more emphasis on accuracy than on flexibility compared to the Nordic countries. This is evidence that it is possible to impact flexibility by teaching and curriculum choices. We are not aware of similar cross-national comparisons of flexibility at the university level. However, Ernvall-Hytönen, Hästö, Krzywacki and Parikka [3] found that Finnish university students showed fair levels of flexibility whereas Shaw, Pogossian, and Ramirez [10] found that students in a US university showed less flexibility than they expected. It should be mentioned that these studies measured flexibility with very different instruments. Nevertheless, variability across contexts suggests that flexibility is not (only) a consequence of being mathematically gifted, but rather depends on the education system and can be cultivated through teaching. It is important not to view flexibility as the ability to think of unusual, clever solutions out of nowhere (“fixed mindset”), since such a view can be dispiriting and counterproductive. Rather, flexibility is a combination of skills and attitudes that can be achieved through systematic work and supported through instruction (“growth mindset”).

Star and Seifert [13] carried out an intervention which illustrates a fairly straightforward yet generalizable teaching approach. Students were introduced to equation solving by teaching them what operations are permitted on equations (add a constant to both sides, combine like terms, etc.) but were not told to follow a specific sequence of steps or rules (such as the standard algorithm). Furthermore, students were instructed to solve each problem with at least two different strategies. This focused the attention on the solution process instead of the answer and showed the students that there are multiple viable solution strategies. As a result, the intervention participants exhibited more varied solution strategies while also achieving the same proficiency on the standard algorithm when compared to the control group. In a calculus context, a corresponding shift could be to tackle multiple differentiation rules simultaneously with tasks that match several rules at the same time. An additional layer is that calculus also involves algebraic manipulations that may invite flexibility; the examples in the introduction utilize this. A one-session intervention along these lines at a Canadian university was studied by Maciejewski and Star [7].

The students in the interventions [7, 13] had more freedom to work things out than in traditional teaching, but the number of options offered to them was not overwhelming. Furthermore, instead of instilling students with an “always distribute first” mantra, this approach to equation solving encourages them to pause and think for a while before plunging into solving a problem. Thinking before acting should help reduce the frequency of strangely inefficient solutions that surface far too often, such as solving \((x + 1)(2x - 3) = 0\) by distributing and using the quadratic formula on \(2x^2 - x - 3 = 0\). Scaling back on the “always follow this set of steps” approach also enables different task types where students reverse the steps, start in the middle, analyze given solutions, etc. For instance, students could start with an answer and then manipulate it to construct an equation for the rest of the class to solve; the process could look like

\[
x = 6 \quad \rightarrow \quad 3x = 18 \quad \rightarrow \\
3x - 4 = 14 \quad \rightarrow \quad \frac{1}{2}(3x - 4) = 7.
\]
This task highlights the invertibility of the steps in equation solving. An analogous task in calculus could eluci-
date the relationship between differentiating and integrat-
ing. To avoid overly trivial tasks, students should be en-
couraged to aim for “sneaky” problems that will give their
classmates a run for their money.

Although flexibility is sometimes framed as using the
optimal choice among a set of known strategies, the inter-
vention [13] also shows that it is possible for students to
construct their own strategies. As long as the work is done
in a suitably restricted domain, the complexity and cogni-
tive load can be kept at reasonable levels. The fewer the
restrictions, the closer the situation is to problem solving.

We emphasize that flexibility is more than just gener-
ating multiple strategies: the strategies should also
be compared and evaluated using mathematical crite-
ria. The teacher is instrumental in negotiating the socio-
mathematical norms with the class regarding the appropri-
ate criteria to use in the comparison as well as making sure
that every student’s contribution is appreciated while still
discussing different proposed strategies critically. Compa-
ration between solutions can be carried out with any task
for student-generated solutions. Furthermore, selecting
the structure of the formulas and numbers appropriately
can encourage applying an opportune, nongeneric stra-
 tegy. For instance, the integral
\[
\int_1^2 x^2 - 2x + 1 \, dx
\]

invites the change of variables \(z := x - 1\), even though a
direct calculation is also quite feasible.

On the other hand, it is also possible to take an ex-
 plicit flexibility focus in a task. For example, students can
be asked to provide two (or more) different solutions to
a given problem. The comparison of the solutions can
be part of the task or organized by the teacher as a class-
room discussion. Supporting flexibility by digital tools is
another interesting, yet underdeveloped area [1].

A task format specifically designed to support flexibility
is the worked example pair (WEP). A WEP presents two hy-
pothetical students’ solutions to a problem (see Figure 2).
In the simplest case one solution represents the generic
approach and the other is situationally appropriate. The
WEP also contains some prompts such as:

- Can you solve the problem in yet another way?
- Which solution is correct?

As the last question hints at, one subtype of WEPs involves
an error in one or both of the solutions. Such intentional
errors may help shift students’ attention from the answer
to the argumentation and lessen their anxiety about pre-
senting in class a solution with an error [9]. Furthermore,
debugging a solution is an important learning opportunity
in itself.

Students’ attitudes are a potential challenge to a focus
on flexibility. Prior experience from the education system
may have led to the belief that the most, or only, important
thing in mathematics is to obtain the correct answer, and
so discussions about multiple solutions are a waste of time.
Students may believe that it is the teacher’s job to explain
the correct strategy for a problem-collection and their job
is to apply it as instructed. Such beliefs hinder teaching
higher levels of thinking, including flexibility.

In the short term, an answer-centered orientation may
lead to seemingly good results. When students are drilled
on a set of tasks for a few weeks, most will display ade-
quate performance on a test with similar tasks at the end
of the training period. However, transfer of the knowledge
to different situations, application and retention to a later
time are likely to be quite limited (Boaler, 1993). In a
middle-school intervention for flexible equation solving
in Finland many teachers suggested that the initial slow
progress was more than compensated for by not needing to
start reviewing equations from scratch in subsequent years.
Good short-term test results arise from students being re-
lieved of many processes involved in solving the problem,
such as identifying the problem type and relevant solution
methods or combining techniques and concepts from sev-
eral areas. But these are important habits of mind that the
students should be learning. In the long term, this leads
to worse structure sense and weaker meta-cognitive skills
such as the ability to assess the viability of a strategy in a
situation.

One source of such unhelpful beliefs is assessment. In
some education systems and institutions, teachers assess
students’ work by only scoring the correctness of the final
answer. Scoring only the answer implicitly communicates
to students that the argument is not important. In sec-
ondary and tertiary education, points are often allocated
also to the argument. However, in mathematics usually
any correct argument gives full points, irrespective of its
other qualities such as elegance, parsimony, or readabil-
ity. In particular, the immediate pay-off from flexibility
may be quite low, resulting only from occasionally saving
a little time on calculations. Thus students may need addi-
tional incentives to strive for flexibility and the concomi-
tant small advantages that accumulate over time. For this
We have argued that procedural knowledge can have more depth than it seems at first glance, specifically when it comes to choosing or constructing a solution strategy tailored to a specific problem. The ability and inclination to use situational and appropriate strategies is called strategy flexibility, and it has been an active area of mathematics education research in the past two decades with roots stretching much further back.

Research has shown that students across skill levels can learn flexibility with appropriate support and encouragement. Flexibility is furthered by shifting the focus from answers to solutions strategies and the ideas behind them. Most tasks can be used to support flexibility; however, there are also tasks specifically designed to promote flexibility, such as worked example pairs. Most research has focused on flexibility pre-university, but many of the tools and task types can be applied in a university context. University mathematics teachers are well positioned to appreciate the value of flexibility and convey it to their students, which may be the most important factor for adopting a flexible mindset.

Flexibility provides a framework for teaching mathematics with a balance of conceptual and procedural focus. It allows teachers and students to expand their working habits gradually toward a more problem-solving oriented style. Teaching for strategy flexibility is a small step for the teacher, but it may just be a small leap for students who realize that mathematics involves decisions and multiple correct solution strategies with different strengths.

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References


Figure 2. A worked example pair.


Credits
Figure 1 is courtesy of Peter Hästö and Jon R. Star.
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Mathematics Departments that Offer a PhD in Mathematics Education

What Courses in Mathematics Education Are Required?

Robert Reys

In the early part of the 20th century the first doctoral programs in mathematics education were initiated in mathematics departments at two institutions: Columbia University (Teachers College) and the University of Chicago [1]. At both of these institutions the doctorate in mathematics education program required mathematics content courses like those taken for a PhD in mathematics, but the dissertation research focused on teaching and learning of mathematics rather than pure mathematics.

For over 100 years doctoral programs in mathematics education have evolved, both in terms of where they are programmatically housed and in terms of course requirements. Today some doctoral programs in mathematics education are hosted entirely within mathematics departments (e.g., Illinois State University and University of Northern Colorado); others entirely within colleges/schools of education (e.g., Auburn University and University of Missouri); and there are a few institutions where a doctorate in mathematics education can be obtained from either the mathematics department or the college/school of education (e.g., University of Arizona and Oklahoma State University).

Doctorates in mathematics education from mathematics departments include a substantial body of work in mathematics, typically the equivalent of a master’s degree in mathematics or more. In addition, doctoral students take courses in mathematics education designed to increase their expertise in learning and teaching mathematics. This foundational knowledge of mathematics plus their interest in teaching makes graduates with a doctorate in mathematics education from a mathematics department highly desired candidates for teaching collegiate level mathematics in regional institutions [2].

Survey. A national conference on doctoral programs in mathematics education was held in October 2022 (NSF Grant No. 1932697). About 100 participants from more than 60 institutions with such programs attended the Conference. Thirteen of the institutions at the conference offered their doctorate in the mathematics department. A survey of these participants was done to gather information about the mathematics education courses required to complete a doctorate.

While required courses in a doctoral program are not indicative of the institution’s total course offerings available in mathematics education, they are one indicator of foundational or core knowledge in mathematics education that the department values. This article reports the required courses in mathematics education to complete a doctorate.
in mathematics education in mathematics departments of the institutions that participated in the conference.

Thirteen institutions participated in the survey. One or more faculty members from each of these 13 institutions provided information about required courses in mathematics education. More specifically they were asked to provide a course title, course description, and semester credit hours for each required course. Due to length considerations, the course descriptions are not included in this article, but could be obtained from the institutional contact.

**Survey results.** Here are the titles of the required courses for each of the 13 institutions, along with the faculty contact person(s), and the credit hours in parentheses:

- **Arizona State University** (Naneh Apkarian)
  - MTE 501: Research in Undergraduate Mathematics Education I (3)
  - MTE 502: Research in Undergraduate Mathematics Education II (3)
  - MTE 503: Research in Undergraduate Mathematics Education III (3)
  - MTE 504: Research in Undergraduate Mathematics Education IV (3)

- **Illinois State University** (Craig Cullen and Jennifer Tobias)
  - MATH 401: Current Research in School Mathematics (3)
  - MATH 403: Theories of Mathematics Learning (3)
  - MATH 598: Mathematical Thinking and Learning (3)
  - MATH 581: Seminar in Research and Development in Mathematics Education (3)
  - MATH 582: History of Mathematics Education Curriculum (3)
  - MATH 585: Topics in Mathematics Education Seminar (3)
  - MATH 586: Mathematics Teaching and Teacher Education (3)
  - An additional 9 graduate semester hours of electives in mathematics education are required for a total of 30 hours of graduate courses in mathematics education.

- **Middle Tennessee State University** (Sarah Bleiler-Baxter)
  - MATH 6900: Research in Mathematics Education (3)
  - MATH 7320: Mathematical Problem Solving (3)
  - MATH 7330: Ethics in Mathematics Education (3)
  - MATH 7340: History, Curriculum, and Policy in Mathematics Education (3)
  - MATH 7900: Teaching and Learning Mathematics (3)
  - MATH 7310: Theoretical Frameworks in Mathematics and Science Education (3)
  - MATH 7900: The Nature of Mathematics, Science, and STEM (3)
  - SPSE 7270: Learning Theories in Mathematics and Science Education (3)

- **Montana State University** (Mary Alice Carlson)
  - M 528: Curriculum Design (3)
  - M 529: Assessment Models and Issues (3)
  - M 534: Research in Mathematics Education (3)

- **Montclair State University** (Steven Greenstein and Nicole Panorkou)
  - MATH 745: The Use of Teacher Knowledge in Mathematics Teaching (3)
  - MATH 811: Mathematics Education Leadership (3)
  - MATH 813: Geometric and Spatial Thinking and Learning (3)
  - MATH 814: Algebraic and Analytic Thinking and Learning (3)
  - MATH 815: Theories of Learning Mathematics (3)
  - MATH 816: Mathematics Curriculum (3)
  - MATH 821: Mathematics Education in Higher Education (3)
  - MATH 822: Mathematics Education in Higher Education Practicum (1)

- **Portland State University** (Iva Thanhleier)
  - MTH 692: Research Methodology and Design (3)
  - MTH 693: Research on the Learning of Mathematics (3)
  - MTH 694: Research on the Teaching of Mathematics (3)
  - MTH 695: Topics in Research in Mathematics Education (3)
  - MTH 696: Mathematical Knowledge for Teaching and Educational Research, Grades K–12 (3)
  - MTH 697: Advanced Mathematics for Teaching and Educational Research (3)

- **San Diego State University/University of California-San Diego** (Susan Nickerson and Jeff Rabin)
  - SDSU is on the semester system—3 hours. UCSD is on the quarter system—all courses are 4 hours.

- **University of Arkansas** (Shannon Dingman)
  - MATH 610V: Directed Readings—Teaching and Teacher Education (3)
  - MATH 610V: Directed Readings—Technology and Curriculum (3)
  - MATH 610V: Directed Readings—Instructional Strategies and Assessments (3)
  - MATH 610V: Directed Readings—Curriculum Ideologies and Learning Theories (3)

- **University of New Hampshire** (Karen Graham)
  - MATH 958: Foundations in Mathematics Education (1)

- **University of Northern Colorado** (Golden Karakok)
  - MED 731: Learning Theories in Mathematics Education (3)
  - MED 732: Mathematics Curriculum Design (3)
  - MED 733: Models of Teaching in Mathematics (3)
  - MED 740: Equity in Mathematics Education (3)
  - MED 752: Research in Mathematics Education Mentorship (3)

- **Virginia Tech** (Estrelia Johnson)
  - MATH 5624: Research on Mathematical Knowing and Learning (3)

- **Western Michigan University** (Laura Van Zoon and Ok-Kyong Kim)
  - MATH 6570: Issues and Trends in Mathematics Education (3)

- **Western Washington University** (Jennifer Czocher and Sharon Strickland)
  - MATH 6540: Secondary School Mathematics Curriculum Studies (3)
  - MED 754: Critical Analysis of Mathematics Education Research (3)

- **York College** (Esther Graff)
  - MATH 690: Research in Undergraduate Mathematics Education (3)
  - MATH 690: Research in Undergraduate Mathematics Education (3)

An examination of the course information shows a range of three to over twelve mathematics education courses were required with a mean of about six, and the course titles varied greatly. However, some common themes across programs include research, curriculum, learning theories, teaching, and equity.

One institution provides an in-depth focus on Research in Undergraduate Mathematics Education (RIUPE), while others include a single course on RIUPE. Some courses are targeted for doctoral students focusing on teacher preparation at the elementary/middle school or secondary school, with the latter often including a specific focus on...
mathematical content, such as the learning/teaching of algebra or geometry.

The word “research” appeared in over 25 of the required courses, but reflected many different courses, such as Current Research in School Mathematics, Introduction to Research in Mathematics Education, Research in Mathematics Education, Research on the Learning of Mathematics, Research on the Teaching of Mathematics, Research on Undergraduate Mathematics Education, Critical Analysis of Mathematics Education Research, or Topics in Research in Mathematics Education. An examination of the course descriptions revealed a wide range of topics and issues being addressed as well as the resources used.

Each of these institutions offered additional optional doctoral courses in mathematics education that were not required for completion of the doctorate in mathematics education. A few of these mathematics departments offer both a PhD in mathematics and a PhD in mathematics education. In these institutions, it was reported that some of their PhD students in mathematics can earn a specialization in collegiate teaching by taking some of their required courses in mathematics education thereby enhancing their employment opportunities.

**Using the survey findings.** Some institutions have a philosophy that the fewer required courses the better as it provides flexibility and allows a PhD program to be individually tailored to address different career goals. Therefore, the number of required courses in a program are not necessarily representative of the overall learning goals of the program. In fact, some institutions rely heavily on “out of course” experiences such as internships and independent readings. To have a clear understanding of the course offerings and/or learning goals of a doctoral program, it is necessary to carefully examine an institutional website or contact faculty members individually.

When this required course information was shared at the conference it prompted many questions, exchanges, and dialogues among participants. Participants wanted to learn more about the various course offerings at other institutions. Some participants requested syllabi and sought to learn about the textbooks/resources that were used in these courses. This course information also encouraged mathematics faculty members to do some self-examination on their course offerings and think about if and how some of these courses could be included in their own program.

Issues and cultures change, so self-reflection and review of doctoral requirements should be done regularly to strengthen and update programs [3]. The goal of this article is to inform faculty members in mathematics departments, stimulate discussion, and encourage self-reflection on required course offerings in mathematics education for their doctoral students. If you have a doctoral program in mathematics education in your department, how do your required courses compare in number and scope? Are you interested in learning more about some of these courses? Are there courses you think would serve your students well? If so, the name(s) of the faculty member(s) providing their course information are provided for those interested in following up to learn more.

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**References**


**Credits**

Photo of Robert Reys is courtesy of Robert Reys.
**BOOKSHELF**

New and Noteworthy Titles on our Bookshelf

August 2024

**What If? 2**
By Randall Munroe

To many, Randall Munroe needs no introduction. His popular webcomic [https://xkcd.com](https://xkcd.com), self-described as "a webcomic of romance, sarcasm, math, and language," has delighted members of our community since 2005. I have personally followed the webcomic throughout my undergraduate, graduate, and professional career. I have shared xkcd links with office mates, classmates, friends, and students. Some of my favorites are posted to my office door. If you haven’t perused the website before (or want to return to it after a hiatus!), I suggest checking out “2658: Coffee Cup Holes,” “2492: Commonly Mispronounced Equations,” and “2595: Advanced Techniques.”

In 2014, Munroe started writing books. The *What If?* series seeks to answer bizarre hypothetical questions, submitted by readers, in a serious way. This book, *What If? 2*, is a sequel to his initial foray into answering absurd questions. How absurd, you ask? Have you ever considered what would happen if you fill the solar system with soup? Perhaps you’ve wondered if you could heat your house using only toaster ovens or what would happen if you were transported to the surface of the sun for one nanosecond. If you are intrigued by these ridiculous questions and desire a serious answer, you are sure to enjoy this book. Many of the calculations Munroe does require some knowledge of physics, but I could follow along despite not having taken physics in decades. Unlike the first book, Munroe includes a list of “Weird and Worrying” questions submitted by readers, each of which only received a classic xkcd-style drawing in response.

This book is not a math textbook, nor is it written by a mathematician (Munroe is a physicist and worked at NASA for a short period of time). However, the webcomic xkcd has been adjacent to math culture for so long, it’s hard to imagine a mathematician who wouldn’t enjoy this book. I certainly did.

**M is for Math**
619 Wreath, 2023, 32 pp.
By Krystina K. Leganza

There can be so much joy and satisfaction in discussing mathematical concepts with children, especially if the math is beyond what they would typically see in school or until they are much older. You might consider the challenge of talking with an elementary-school-age student about an abacus or a Möbius band. Igniting their curiosity is an effective and enjoyable way to influence the next generation of mathematicians.

The new book *M is for Math* provides at least one mathematical item for every letter of the alphabet. I especially appreciate the exposure to advanced concepts, since many children’s books focus on only the very basic structures of mathematics. You can tell this book was written with care by a mathematician. Alongside colorful illustrations and clever alliteration, the reader will discover the hypotenuse of a right triangle, infinity, the Julia set, and Klein bottles. One of my favorite lines from the book is, “The quail quivered under a quilt of quadrilaterals.” An appendix at the end defines the terms found throughout the text, which is a wonderful way to bring in readers of all mathematical levels. I would expect this book would be enjoyed most by infants, toddlers, and preschoolers. A kindergartener might enjoy reading parts of it to you! This children’s book hits the mark for my family, and I hope it will for the children in your life, too.

This Bookshelf was prepared by Notices Associate Editor Emily J. Olson.

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1. Introduction

What is predictive policing? The RAND Corporation, for example, defines predictive policing as “the application of analytical techniques—particularly quantitative techniques—to identify likely targets for police intervention and prevent crime or solve past crimes by making statistical predictions.” This brief survey serves as a mathematical supplement to recent discussions of predictive policing such in [20] and various investigative reports, and perhaps most importantly the boycott by several thousand mathematicians of collaboration with the police, appearing in the Notices of the AMS [3].¹ The foundations of predictive policing are highly mathematical, while modern usages also heavily incorporate machine learning to optimize models.

There are many surveys and explanations of predictive policing from various points of view, such as surveillance technology, algorithmic fairness, and criminal justice, but few specifically address its mathematical aspects. The purpose of this article is to present a selective overview of the mathematics underlying the model used by the company PredPol, followed by critiques and further developments of the model. We focus on PredPol as it is a well-known method of predictive policing and the methodology is arguably less oblique than other predictive policing algorithms that rely more heavily on black-boxed machine learning algorithms.

PredPol is often singled out in discussions surrounding predictive policing, in part because it is arguably the progenitor of the field, although it is far from the only player in the game. According to investigative reports, PredPol was the most widely used predictive policing company/software in the US as of 2019, having contracts in states including Utah, Washington, and California (including the University of California, Berkeley). The company was born out of a collaboration with the LAPD, the FBI, and UCLA in the early 2010s. In 2023, PredPol, which was rebranded as Geolitica, was effectively bought by SoundThinking, previously known as ShotSpotter. The same company also acquired HunchLab in 2018, another predictive policing software company. This growing consolidation shows that predictive policing remains a relevant method in modern policing, and deserves the attention of the mathematical community. Indeed, a recent Wired report called the acquisition of Geolitica its “latest step in becoming the Google of crime fighting.”

2. Summary of the Basic PredPol Model

We begin by discussing two early models: the epidemic type aftershock (ETAS) model that the PredPol patent is built on (US Patent No. 8,949,164), and the reaction-diffusion model that ETAS in turn draws upon. Reaction-diffusion modeling is modeled using partial differential equations, with original applications to the physical sciences. The
ETAS model, on the other hand, is a popular statistical model for earthquake occurrence. PredPol is thus based on equations that are used to model earthquake occurrences and chemical reactions which produce certain “hotspots,” and where adding “hotspot policing” can be viewed as an inhibitor to the process. From a statistical point of view, the introduction of police to given hotspots is then viewed as applying “treatments” or interventions which are meant to reduce crime. Such models are also referred to as spatiotemporal models.

2.1. The reaction-diffusion model. We review the mathematics underlying [21,22]. Reaction-diffusion models are typically used to describe chemical reactions, in which activators and inhibitors move, mix, and interact. In [22], the model describes houses and burglars, while [21] describes “motivated offenders,” targets or victims as activators, and law enforcement as inhibitors. Their reaction-diffusion system involves “mobile criminal offenders” within a square environment with periodic boundary conditions. (See Section 4.1 for a discussion on the biases implicit in the language of crime.)

2.1.1. Discrete model. In the discrete model, houses are placed on a lattice in the plane with constant spacing $\ell$. Within the plane $s = (x, y)$, the score in question $A_s(t)$ is interpreted either as the attractiveness of a house to a burglar, or the risk of victimization, “representing general environmental cues about the feasibility of committing a successful crime and/or specific knowledge offenders possess about target or victim vulnerability in the area.” The risk is given by

$$A_s(t) = A_0^s + B_s(t),$$

where $A_0^s$ is a fixed value and $B_s(t)$ is a dynamic value that models the idea that if a site has been attacked, it has a higher risk of being revictimized shortly after the first incident. The first approximation is

$$B_s(t + \delta t) = B_s(t)(1 - \omega \delta t) + \theta E_s(t),$$

where $\omega$ sets a time scale over which repeat victimizations are most likely to occur, and $\theta$ is a multiplier of $E_s(t)$, the number of burglary events that occurred at site $s$ since time $t$. The authors then modify this model to account for “near-repeat victimization,” and the broken windows effect,” by allowing the quantity $B_s(t)$ to spread spatially to its neighbors. Equation (1) is replaced with

$$B_s(t + \delta t) = \left[ B_s(t) + \frac{\eta \ell^2}{z} \Delta B_s(t) \right] (1 - \omega \delta t) + \theta E_s(t),$$

where $\Delta$ is the discrete Laplacian, whereby

$$\Delta B_s(t) = \frac{1}{\ell^2} \left( \sum_{s' \sim s} B_{s'}(t) - z B_s(t) \right).$$

Here $z$ is the number of sites $s'$ neighboring $s$, and $0 \leq \eta \leq 1$ measures the significance of neighborhood effects. Computer simulations are then run to show that the model produces certain dynamic and stationary “hotspots.”

The “criminal agent” is modeled as either committing a crime at site $s$ or moving to a neighboring location based on a biased random walk so that site $s'$ is visited with probability

$$q_{ss'}(t) = \frac{A_{ss'}(t)}{\sum_{s'' \sim s} A_{ss''}(t)}.$$

The probability of occurrence for each burglar located at site $s$ between times $t$ and $t + \delta t$ given by $p_s(t) = 1 - e^{-A_s(t)\delta t}$ in accordance with a standard Poisson process in which the expected number of events during the time interval of length $\delta t$ is $A_s(t)\delta t$. In the discrete model, burglars are removed after committing a crime, and regenerated at each lattice site at a constant rate $B$. We write $E_s(t) = n_s(t)p_s(t)$, where $n_s(t)$ is the number of criminals at the site $s$ at time $t$.

2.1.2. Continuous model. From the discrete model we form the difference quotient,

$$\frac{B_s(t + \delta t) - B(t)}{\delta t}$$

and take the limit as $\ell$ and $\delta t$ approach 0 to arrive at the differential equation

$$\frac{\partial B}{\partial t} = \frac{\eta D^2}{z} \nabla^2 B - \omega B + \epsilon \Delta \rho A.$$

Here we have denoted

$$D = \frac{\ell^2}{\delta t}, \quad \epsilon = \theta \delta t, \quad \rho(s,t) = \frac{n_s(t)}{\ell^2},$$

where $\rho$ is the density of criminal agents, and

$$\frac{\partial \rho}{\partial t} = \frac{D}{z} \nabla \cdot \left( \nabla \rho - \frac{2 \rho}{A} \nabla A \right) - \rho A + \gamma,$$

where offenders exit the system at the rate $\gamma B$ and are reintroduced at the constant rate $\Gamma/\ell^2$. The PDE for $\rho$ is obtained by a difference quotient for $n_s(t)$, using the equation

$$n_s(t + \delta t) = A_s \sum_{s' \sim s} \frac{n_{s'}(t)(1 - p_{s'}(t))}{T_{s'}(t)} + \Gamma \delta t,$$

where

$$T_{s'}(t) = \sum_{s'' \sim s'} A_{s''}(t),$$

which simply means that any agents that are present at $s$ after one time step must have either arrived from a neighboring site after having not committed a crime there, or have been generated at $s$ at rate $\Gamma$. The coupled differential equations (2) and (3) thus describe the continuous model.
In [21] the authors study these coupled PDEs to show that crime risk will form dense, well-spaced hotspots when the diffusion of risk by individual crimes is spatially broad enough. Police suppression is modeled by instantaneously setting the crime rate \( \rho \) at the locations of current crime hotspots and maintaining this suppression for a fixed time period. The authors then claim that subcritical crime hotspots may be permanently eradicated with police suppression.

2.2. Epidemic-type aftershock (ETAS). This section covers the mathematics underlying [14], which is the basis for the PredPol patent (US Patent No. 8,949,164). The authors treat the dynamic occurrence of crime as a continuous time, discrete space epidemic-type aftershock sequence point process. In seismology, point processes are used by considering a “parent earthquake” and subsequent background events or aftershocks. The ETAS model estimates long-term and short-term hotspots and systematically estimates the relative contribution to risk of each via an expectation-maximization (EM) algorithm.

The ETAS model can be intuitively understood as a branching process: first-generation events occur according to a Poisson process with constant rate \( \mu \), then events (from all generations) each give birth to \( N \) direct offspring events, where \( N \) is a Poisson random variable with parameter \( \theta \). As events occur, the rate of crime increases locally in space, leading to a contagious sequence of “aftershock” crimes that eventually dies out on its own or is interrupted by police intervention.

In this model, policing areas are discretized into square boxes. The probabilistic rate of events in box \( n \) at time \( t \) is defined to be

\[
\lambda_n(t) = \mu_n + \sum_{t_n < t} \theta \omega e^{-\omega (t - t_n)},
\]  

where \( t_n \) are the times of events in box \( n \) in the history of the process. The background rate \( \mu \) is a (nonparametric histogram) estimate of a stationary Poisson process.

The expectation, or E-step in the EM algorithm, sets

\[
\omega = \frac{\sum_n \sum_{i < j} p_{ij} n}{\sum_n \sum_{i < j} p_{ij} n (t_n - t_i)}, \quad \theta = \frac{\sum_n \sum_{i < j} p_{ij} n}{\sum_n \sum_{i < j} 1}, \quad \mu = \frac{\sum_n \sum_j p_{ij} n}{T},
\]

where \( T \) is the length of the time window of observation.

2.3. Field trials and case studies. The authors of the model underlying the PredPol patent conducted a randomized controlled field trial to test the effectiveness of their model [14]. In their experiment, hotspots were generated daily by the ETAS model and a crime analyst. These hotspots were randomly assigned to foot patrols who then decided independently how to patrol the area as long as they remained within the prescribed area. The study was split across three LAPD divisions where each division studied effects for approximately six to eight months.

The authors compared how well their model predicted crime events against the predictions of crime analysts using standard methodologies of generating hotspot maps. Their ETAS model generated hotspots that successfully predicted crimes at a higher rate—ranging from 1.4 to 2.2 times better—than the hotspots generated by a crime analyst. They estimated a 7.4% decrease in crime (4.3 fewer crimes reported per week) at mean patrol levels when hotspots from their ETAS model were used compared to no patrol at all. When police used hotspots created by the crime analysts there was only a 3.5% decrease (2.0 fewer crimes per week) compared to no patrol at all. This corresponds to a decrease in crime that is 2.1 times larger in magnitude when comparing standard practice to the ETAS model, in line with the 1.4–2.2 times improvement in the prediction rate when comparing the analyst and the ETAS model.

Given this information, researchers asked whether this algorithm was susceptible to bias. Brantingham and Mohler, in collaboration with Valakis, expanded upon their previous work and analyzed the results of their field trial to see if the use of predictive policing results in arrest bias [4]. They compared the arrest data of police on patrol in algorithm-based hotspots to the arrest data of police on patrol in hotspots allocated by the crime analysts. After performing Cochran-Mantel-Haenszel and Woolf tests on their data, the authors found that the difference in arrest percentage with respect to ethnic group (i.e., Black, Latino, White) between the crime analyst hotspots and the ETAS generated hotspots was not statistically significant.

Although differences were not deemed statistically significant, they are worthy of further examination. There was a multiplicative increase of 1.8 in arrests in Foothill LAPD division on days where algorithms were used, but a 4.5
increase in arrests for Blacks compared to a 1.6 and 1.9 multiplicative increase in arrests for Latinos and Whites respectively. There was a 2.0 multiplicative increase in arrests on algorithm days in the Southwest division, but no increase in arrests for Whites compared to a 2.2 and a 1.8 increase in arrests for Blacks and Latinos respectively. The population distributions in the regions studied could have led to the statistical tests considering the differences observed in arrest rates as insignificant. For example, there were few arrests of Blacks in Foothill and few arrests of Whites in Southwest.

Furthermore, Brantingham, Mohler, and Valasik do not check if the arrest data is already biased. This would lead to their predictive policing model replicating the existing bias in policing and not removing the bias.

There have been many instances where the police were found to implement practices that would bias arrest data. For example, former New York City detective Stephen Anderson testified that he along with other police officers planted drugs on innocent civilians in order to boost their arrest numbers [13]. Sometimes even basic recording errors lead to a significant change in crime statistics, for example, a Los Angeles Times investigation found that from 2005 to 2012 the LAPD incorrectly classified 14,000 serious assaults as minor offenses [17]. It has been demonstrated that several predictive policing instruments (including PredPol) have been implemented in several jurisdictions where and when the police in that jurisdiction were either under government investigation into illegal policing practices or agreed to a federal settlement in response to illegal policing practices [19]. For example, police in Maricopa County have been found to conduct racially biased stops, searches, and arrests from 2007 to 2011 and from 2014 to 2017.

Of course, alterations to the algorithm have been theorized. Mohler included a fairness condition in order to control for bias due to group affiliation. An example of the impact of such a condition is given by Akpinar, De-Arteaga, and Chouldechova [1]. In their work they attempt to quantify and test the influence of crime reporting rates on predictive policing. A natural bias of PredPol is that it can only create hotspots based on crimes that are reported. Crime reporting rates are known to depend on factors such as age, gender, race, and socioeconomic class. Therefore the authors attempted to see how crime reporting rates would affect PredPol hotspots in Bogotá, where district-wide crime reporting rates are given.

By running various hotspot-creating models, the authors find that districts that report crime at higher rates will receive a number of hotspots disproportionately high compared to the amount of crime in the district, and districts that report crime at lower rates receive a disproportionately low number of hotspots.

Moreover, considering Bogotá records crime reporting rates on a district-wide level, normalizing by crime reporting rates creates skewed data within districts, as all cells in a given district are normalized by the same factor rather than on a cell-wide basis. Therefore, they hypothesize that the only way to remove such bias is if we had cell-specific crime-reporting data.

3. Mathematical Critiques of Predictive Policing

There have been critiques of the use of predictive policing both from a social and a mathematical perspective. We focus on the mathematical critiques first, then turn to the broader social critiques, as academics frequently miss the social and political consequences of the topics studied.

3.1. Statistical bias using a synthetic population

The most notable quantitative study of the PredPol system is the study of Lum and Isaac [12], which simulates a synthetic population in Oakland, CA, based upon census data and applies a model based on data from the 2011 National Survey on Drug Use and Health (NSDUH) in order to predict an individual’s probability of drug use within the past month based on their demographic characteristics. The resulting data set acts as a replacement for the “ground truth” of drug crime use data, giving estimates of illicit drug use from a noncriminal justice, population-based data source. Compared with police records, the authors find that drug crimes known to police are not a representative sample of all drug crimes.

Applying their reconstruction of the PredPol algorithm as outlined above, the authors conclude that rather than correcting for the apparent biases in the police data, the model reinforces these biases, suggesting that predictive policing of drug crimes results in increasingly disproportionate policing of historically overpoliced communities. In particular, this is despite PredPol’s claim that they use “only three data points in making predictions: past type of crime, place of crime and time of crime. It uses no personal information about individuals or groups of individuals, eliminating any personal liberties and profiling concerns.”

3.2. Runaway feedback loops via a generalized Pólya urn model

These concerns can be given a mathematical framework. Ensign et al. [6] used a generalized Pólya urn model to model a predictive policing algorithm. In their model, they assumed that police patrolled two areas A and B at a rate based on their prior beliefs—this is represented by initial ratio of balls in the urn. When the crime rates in the two areas were equal, the rate at which police were sent to a specific area does not converge to the actual rate of crime, rather the probability that police were sent to an area was a beta distribution dependent on the initial beliefs of the police. This means the police do not actually
learn the crime rates are the same in each area. When the crime rates were not equal, the probability that the police visited the area with the higher crime rate asymptotically approached one (hence, the police will eventually ignore the area with a lower crime rate).

This is a general problem of what is called traditional batch-mode machine learning, and the theory of urns is a common framework in reinforcement learning. In the generalized Pólya urn model, an urn contains balls of two colors, say red and black. At each time step, a ball is drawn, and based on its color a number of balls are replaced. If red, we add a red and b black balls; and if black we add c red and d black balls. This is represented by the replacement matrix \((a \quad b)\) where the standard case is when \(a = d = 1\) and \(c = b = 0\). As a toy model for predictive policing, A and B are two policing blocks, and the goal is to distribute police officers according to the proportion of crime in each area. Let \(d_A\) be the rate at which police in A discover crimes, \(r_A\) the rate at which crimes are reported in A, and \(w_d, w_r\) the respective weights such that \(w_d + w_r = 1\) and \(w_A(d_A + w_r)\) represents the total rate of incident data from A.

For the rest of the section we make the following assumptions:

1. (Predictive model) The officer decides where to go next with probability based on current statistics. This means that the model uses some form of statistical information to make predictions on crime.

2. (Context) The only information retained about a crime is a count.

3. (Truth in crime data) If an officer goes to a location A with an underlying ground truth crime rate of \(\lambda_A\), the officer discovers crime at a rate of \(d_A = \lambda_A\). Reported incidents are also reported at a rate that matches the underlying ground truth crime rate, i.e., \(r_A = \lambda_A\).

4. (Discovery only) Incident data is only collected by an officer’s presence in a neighborhood, i.e., \(w_d = 1\) and \(w_r = 0\).

3.2.1 Uniform crime. First assume that crime rate is uniform, so \(\lambda = \lambda_A = \lambda_B\). In this case, sending police to area A or B is given by drawing a red or black ball. We then sample a Bernoulli distribution. If 1, we simulate one step of the standard Pólya urn, and if 0, we simply replace the ball that was drawn. In this case, the probability of drawing a red ball has a limiting distribution equal to the beta distribution with parameters \((\alpha, \beta) = (n_A, n_B)\) where \((n_A, n_B)\) are the number of red balls and black balls initially in the urn (c.f. [18]). Recall that the beta distribution is the distribution on \([0, 1]\) given by probability distribution function

\[
\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} x^{\alpha-1}(1-x)^{\beta-1},
\]

where \(\Gamma(x)\) is the usual gamma function. This means that as the number of draws increases, the probability of visiting A or B does not necessarily converge to 1/2. Rather, this probability remains dependent on the initial data.

3.2.2. Nonuniform crime. Now we drop the uniformity assumption. The urn is now modelled by the stochastic addition matrix \(\begin{pmatrix} X_A & 0 \\ 0 & X_B \end{pmatrix}\), where \(X_A, X_B\) are Bernoulli variables with parameter \(\lambda_A, \lambda_B\), respectively. Let \(n_A(t), n_B(t)\) be the number of red and black balls respectively at time \(t\). The probability of adding any ball to the urn is given by

\[
P(\text{adding a ball}) = \frac{n_A(t) \lambda_A + n_B(t) \lambda_B}{n_A(t) + n_B(t)},
\]

while the probability of adding a red ball conditioned on adding any ball, is given by

\[
P(\text{adding a red ball}) = \frac{n_A(t) \lambda_A}{n_A(t) \lambda_A + n_B(t) \lambda_B},
\]

and similarly for a black ball. In particular, this is the same as the deterministic Pólya urn in which an \(i\)-colored ball is sampled, replaced, and then \(i\) more balls of the same color are added. Thus the stochastic matrix reduces to \(\begin{pmatrix} \lambda_A & 0 \\ 0 & \lambda_B \end{pmatrix}\). The authors then deduce a runaway feedback loop for this toy model.

**Proposition 3.1 ([6, Lemma 4]).** The asymptotic probability of sampling a red ball is 1 if \(\lambda_A > \lambda_B\) and 0 if \(\lambda_A < \lambda_B\).

This tells us that as long as A has a ground truth crime rate that is even slightly higher than that of B, the update process will lead to police being eventually completely sent to A.

3.2.3. Incorporting feedback. In order to learn the crime rate, the Pólya urn should contain balls in proportion to the relative probability of crime occurrence. The following update rule guarantees that the urn proportion will converge to the ratio of replacement (i.e., crime) rates. Consider the probabilities \(\lambda_A\) and \(\lambda_B\) now conditioned on a ball of the respective color having been sampled. This makes the probability of adding a red ball equal to

\[
\frac{n_A(t) \lambda_A}{n_A(t) \lambda_A + n_B(t) \lambda_B},
\]

rather than \(\lambda\), and the expected fraction of red balls being added to the urn after one step of the process equal to (7) instead of \(\lambda_A/(\lambda_A + \lambda_B)\).

We introduce the following change: instead of always adding the new balls, we first sample another ball from the urn, and only add the new balls if the colors are different.
This makes the probability of adding a red or black ball
\[ \frac{n_A^{(t)} \lambda_A}{n_A^{(t)} + n_B^{(t)}} \quad \text{or} \quad \frac{n_B^{(t)} \lambda_B}{n_A^{(t)} + n_B^{(t)}} \]
respectively, where we see that the probabilities are proportional to \( \lambda_A, \lambda_B \) up to the common factor \( n_A^{(t)}, n_B^{(t)}/(n_A^{(t)} + n_B^{(t)})^2 \). This is an example of rejection sampling, where sampled values are dropped according to some probability scheme to affect the statistic collected.

Another related scheme is importance sampling, where
\[
\min\ 
\frac{L(\alpha, \omega, \theta)}{\sum_{m \in m^*} p_m - p_{m'}}^2, 
\]
so that when \( F = 0 \), each group \( m \) receives the same number of patrols per individual.

We then add \( F \) to the log-likelihood (8) as a fairness penalty and maximize with respect to \( \alpha, \omega, \theta \) the loss function \( L(\alpha, \omega, \theta) - \chi F \), where \( \chi \in \mathbb{R} \) is a penalty parameter that controls the balance between accuracy and fairness in the point process model. Note that this loss function is not everywhere differentiable due to potential changes in the \( k \) hotspots \( K_t \).

### 3.4. Impact of machine learning

One point not yet addressed is the influence of machine learning in modern predictive policing algorithms. For example, Fitzpatrick, Gorr, and Williamson compare the ability to predict crime in Pittsburgh of a perceptron-based model versus a nonperceptron predictive policing model, and show that perceptron-based models lead to a greater entropy in hotspot locations [8]. Nevertheless, these models still have 43.6% of all hotspots remain hotspots for at least 75% of the study period, and these seem to be less predictive than other, nonperceptron models.

In this study, researchers found a significant decrease in serious violent crime at both temporary and chronic hotspots with predictive policing. Moreover, rather than crime displacement, they found a weak spillover effect of crime prevention in adjacent areas. However, this study predicated on the adoption of nonaggressive tactics by police, and arrests for commonly overpoliced crimes did not increase. Moreover, these conclusions are based on a
relatively small scale, with a reduction of 24 crimes over a 12 month period. It should also be noted that any results will be specific to the social context of the city and police department implementing policy.

The problem of guaranteeing fairness of algorithms is a highly active research area. See [2] for a survey of the implications of fairness in machine learning on predictive policing.

To give an idea of the fundamental algorithmic issue of fairness, we consider the work of Kleinberg, Mullainathan, and Raghavan [10]. Within a population, each individual is assigned a feature vector $\sigma$. Our goal is to classify individuals into positive and negative classes; therefore define $p_\sigma$ to be the fraction of people with classification $\sigma$ that are positive. To each population with a given set of characters $\sigma$, we assign a distribution of scores $X_\sigma$, so we say $X_{\sigma b}$ is the fraction of people in $p_\sigma$ assigned to bin $b$, with associated score $v_b$.

The authors propose three goals of a fair algorithm between two groups, group 1 and group 2.

1. For each group and bin $b$, the fraction of people assigned to the positive class should be $v_b$. Namely, within each bin, the group should not affect classification.
2. The average score $v_b$ of people in the positive class should not depend on group.
3. The average score of people in positive class should not change per group.

The authors show that any classification that satisfies all three of these conditions is in some way trivial.

**Theorem 3.3.** Any assignment satisfying conditions $(1), (2), (3)$, has either

1. Perfect prediction: for each feature vector $\sigma$, $p_\sigma \in \{0, 1\}$.
2. Equal base rates: The average does not depend on the group.

Therefore for any nontrivial scenario, the classification could be classified as biased. Moreover, this theorem is robust in the sense that if approximate versions of the three fairness criteria are satisfied, then the selection must be approximately close to perfect prediction or equal base rates. Therefore, “any assignment of risk scores can in principle be subject to natural criticism on the grounds of bias.” To give an idea of the proof, we follow the authors’ proof sketch.

**Sketch of proof.** By assumption $(1)$, for any fixed bin $b$, the total score given to those in group $t$ in $b$ is equal to the number of people in group $t$ in $b$ in the positive class. Summing over all bins gives that the sum of scores of people in group $t$ is equal to the number of people in the positive class in group $t$. Call this number $\mu_t$.

Define $x$ to be the average score of someone in the negative class, and $y$ the average score of someone in the positive class. By assumption $(2)$ and $(3)$, $x$ and $y$ do not depend on the group. Because the total score is $\mu_t$, if there are $N$ total individuals, then we must have

$$(N - \mu_t)x + \mu_ty = \mu_t.$$ 

This is a linear equation in $x, y$ that always has a solution at $(0, 1)$. This corresponds to perfect prediction. If there is another solution, then $\mu_t$ cannot depend on $t$, and we must have equal base rates, and $x, y$ can be arbitrary. □

This gives a strong suggestion that any nontrivial assignment of people to individual scores must be, in some sense, unfair. Liu et al. gave another example of this, considering a theoretical model based on banks assigning credit to potential loan applicants (such a model also is relevant to, for example, college admissions) [11]. They assign average score $\mu_j$ to group $j$ and check the change in the average score $\Delta(\mu_j)$ under three different scenarios, and check whether group scores increase or decrease.

1. Maximum Utility: The bank makes no attempt at fairness and instead only tries to maximize the utility of giving out loans to people who would give them back and limiting giving loans that will not be repaid.
2. Democratic Parity: The bank attempts to optimize utility under the condition that the same fraction of loans are accepted from group $A$ and group $B$.
3. Equal Opportunity: The bank optimizes utility under the condition that across groups, the probability of success if selected does not depend on the group.

The authors show that in their model, there is no guarantee that adding a fairness condition will help a disadvantaged group. A summary of their various theorems is as follows.

**Theorem 3.4.** There are population proportions for which for an underrepresented group $j$, $\Delta(\mu_j^{\text{DemParity}}) > \Delta(\mu_j^{\text{MaxUtil}})$ and $\Delta(\mu_j^{\text{EqOp}}) > \Delta(\mu_j^{\text{MaxUtil}})$.

Therefore these fairness criteria promote the typical score associated with group $j$ at a faster rate than an algorithm that is not conditioned for fairness. However, the opposite is also possible.

**Theorem 3.5.** There are population proportions for which for an underrepresented group $j$, $\Delta(\mu_j^{\text{DemParity}}) < \Delta(\mu_j^{\text{MaxUtil}})$ and $\Delta(\mu_j^{\text{EqOp}}) < \Delta(\mu_j^{\text{MaxUtil}})$.

Namely, the condition of fairness unintentionally harms the underrepresented group $j$. This is analyzed by creating a linear program with the given conditions, then optimizing utility by taking the derivative across parameters.
These results suggest that although the promise of better and stronger algorithms is appealing, there are general issues with any algorithm that assigns individuals scores based on available data. These potential pitfalls in creating a fair algorithm have been considered real world reviews of machine learning algorithms, such as concerning racial bias in medical diagnoses [16].

4. Social Critiques of Predictive Policing

In this section, we bring forth questions from a social perspective about the theoretical understanding of crime that predictive policing embodies and the use of police to deal with crime within the predicted hotspots. There is an extensive literature, both academic and nonacademic, regarding the social critiques and impacts of predictive policing that only focusing on mathematical models do not capture. We mention only several aspects below, deferring to social scientists and theorists for these critiques, and encourage the reader to explore the question more deeply and broadly on their own.

4.1. On the notion of crime. Social scientists and community advocates have long argued that the notion of “crime” is itself a political matter (e.g., [5]). To label a specific act as a crime assumes a particular understanding of the social contract, whereby certain crimes are policed and others are not. Following this line of critique, The New Inquiry published a tool called White Collar Crime Risk Zones predicting where financial crimes will happen in the US, trained on incidents of financial malfeasance. As the authors note, “unlike typical predictive policing apps which criminalize poverty, White Collar Crime Risk Zones criminalizes wealth.” As argued by Proposition 3.1, due to the focus of predictive policing on crime data that is skewed towards communities of color in poor neighborhoods, the biases that we have discussed above in prediction will lead to overpolicing in those areas.

4.2. Policing as a response to prediction. Researchers argue that predicting crimes—or rather, particular undesirable incidents or activities—is not problematic in and of itself, but rather it is the policing response, known as hotspot policing, or informally as cops on dots, that is problematic and what actually leads to overpolicing (see, for example references cited in Section 4.1 of [2]).

4.3. The broader surveillance complex. Perhaps most importantly, predictive policing is only a single aspect of a much larger phenomenon at the intersection of policing and surveillance. Predictive policing is only a small part of a large network of surveillance systems such as gunshot detectors, automated license plate readers, facial recognition systems, geofencing, and CCTV cameras, as well as predictive systems such as risk terrain modeling, recidivism risk scores, and pretrial detention risk assessments [7]. As such, in order to holistically assess the relevance and impact of predictive policing, it is necessary to consider it in the context of this broader system, which will take us beyond simple mathematical models. Indeed, thinking at this systemic level will require collaboration with social scientists, lawyers, and activists who are familiar with crime and the effects of policing on communities to inform our modeling decision.

5. Conclusion

Predictive policing comes from an interesting mathematical background. However, analyzing the mathematical framework shows fundamental theoretical and empirical issues that have yet to be properly addressed (principally expressed in Section 3). Moreover, there are mathematical results that caution against any algorithmic model that inputs a set of features and outputs a risk score (see Theorem 3.3). Furthermore, possibility/impossibility theorems such as the latter suggest that broad theoretical work on algorithmic fairness can have major implications for the application of such prediction systems, which have immense impact in society.

Persistent questions surround the practice of predictive policing: If it replicates the same outcomes as conventional police methods, should it also replicate the biases inherent in those methods? Why do marked disparities persist in patrolled areas between predictive policing and traditional approaches? Should our response to predicted crime prioritize alternatives like health responders or financial relief over police intervention?

While the mathematics and scientific communities aspire to maintain neutrality, the reality is that we introduce our own biases through the fundamental assumptions of the models we create (Proposition 3.1). We must be acutely aware of the potential applications of our mathematical innovations. It is not surprising when after mathematical research is handed over to law enforcement, it is then used to further oppress those already victimized by police violence.

We believe that our awareness of the broader societal context should inform the ethical standards guiding our decisions of which projects we choose to undertake. We hope that this examination of predictive policing prompts a reflection on the projects that the mathematical and scientific broadly opt to pursue.

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A Field Guide to Ethics in Mathematics

Maurice Chiodo and Dennis Müller

Introduction

Mathematics has become inescapable in modern, digitized societies: there is hardly any area of life left that isn’t affected by it, and we as mathematicians play a central role in this. Our actions affect what others, in particular our students, decide to do with mathematics, and how mathematics affects the world [Por23, p. 2], for better or worse. In return, the study of ethics in mathematics (EiM) has become increasingly important, even though it is still unknown to many. This exposition tries to change that, by motivating ethics in mathematics as an interesting, tractable, non-trivial, well-defined and good research area for mathematicians to consider.

What is ethics in mathematics? Historically, many of those working on ethics in mathematics have focused on questions related to specific mathematical subdisciplines (e.g. finance, cryptography, or statistics), but over time the field has begun to ask questions that are fundamental to any area of mathematics [MCF22]. Today, this emerging area of research studies the moral principles of all mathematical practice, addressing the ethical questions related to its applications, teaching, and research, by focusing on the responsibilities of those engaged in mathematics. Ethics in mathematics usually builds on the assumption that the majority of “mathematicians want to be ethical even when [some of them] don’t think ethics apply to them” [BPT22, p. 4].

This work often transcends the boundaries of mathematics, and involves consulting not just the obviously relevant disciplines such as philosophy and law, but also the political and social sciences and psychology [CM23a]. Consider, for example, recent advances in automated theorem proving by artificial intelligence. “What constitutes a proof?” and “Who counts as an author?” aren’t merely practical questions; they’re also ethical concerns with deep psychological and social components: Are we prepared to hand over authorship to an AI and trust it? Similarly, in applied mathematics, the use of approximative formulas also leads to complicated ethical issues. For example, the use of Gaussian copulas to model dependencies between random variables played a fundamental role in the global financial crisis of 2008, leading to some authors calling it the “formula that killed Wall Street” [MS14]. Studying the ethics of using approximate formulas to model complex situations requires us to look beyond mathematics, and use methodologies from many other disciplines.

Mathematics affects our modern societies and our planet with an ever-increasing impact. Today, we cannot hide from mathematics. Someone will have done mathematics that affected us in one way or another, and our mathematics will have affected someone else—positively or negatively. Ethics in mathematics studies this, and guides mathematicians through this jungle of problems. It is, ultimately, an area of philosophy of mathematics, that deeply values the “problems emerging from the actual mathematical practice” and its people [Sko23, p. 115].

Ethics in mathematics and other efforts. The difference between ethics in mathematics and other similar endeavors appears to be mostly a question of perspective and focus. For example, many researchers working on “mathematics for social justice” are predominantly focused on (solving) problems surrounding power and identity, opportunities and injustices, including economic, judicial, racial and other privileges, in relation to mathematics and...
its teaching. They might study the role that mathematics plays in (countering) gerrymandering, or how we can create better classroom environments for teaching. On the other hand, while ethics in mathematics is certainly concerned with these questions, it also studies ethical questions not restricted to balance, equity, and fairness. Over time, the different foci have led to complementary methodologies in research and teaching [Mü12], with many fruitful interactions between them. Questions of social justice are often related to other ethical questions, and almost all ethical questions have a social justice component. All of these can be understood within the context of a sociopolitical turn in mathematics and its education [Gut13], claiming that mathematics is not a neutral and pure player in (modern) societies [Ern20b].

The structure of ethics in mathematics. Embedded throughout this paper are examples from three (partially overlapping) areas of concern:

- the ethics of ensuring that mathematics remains a stable and continuing body of knowledge,
- the ethical and social questions surrounding our mathematical, and other, communities, and
- the impact of mathematics and its practise, on wider society and our planet.

Most questions in ethics in mathematics fall into one or more of these areas, which we can use to start thinking about the ethics of our, and others’, mathematical work and research. Does our work make mathematics a better area of research? Is the way we do and teach mathematics good for the mathematical community? Does our work have a positive impact on wider society? For applications of mathematics, the "Manifesto for responsible development" [CM23a] acts like a handbook to provide a structured list guiding the reader through the most common ethical questions arising from the use of mathematics.

But, as shown later in this article, these areas of concern are not restricted to applied mathematics. Pure mathematics can give rise to ethical concerns, which include: Is the motivation that goes into a piece of mathematics good? Are we writing down our work in a way that other researchers can understand, or are we trying to fudge the boundaries because we want to hide problems? Is the way we do and teach mathematics inclusive to others, enhancing their understanding? Are we working on problems that matter (to us or someone else)? How can our (pure) mathematics be used? And are there areas of dual-use we should be concerned about? Regardless of the type of mathematics we do, be it pure or applied, it’s often necessary to look beyond our area of concern to fully understand the ethics of our work.

How to use this article. This article is intended to give mathematicians a brief introduction to ethics in mathematics. It’s not a philosophical treatise of the subject, but rather an explanation of why doing ethics in mathematics is not so different from doing mathematics, and how we can contribute to the effort or even just use existing resources in our work and teaching. This article is a research exposition of a slightly different kind, explaining why ethics in mathematics satisfies the characteristics of a problem worthy of a mathematician’s attention: it is interesting, tractable, nontrivial, well-defined, and (morally) good. Each of these aspects is presented in one page with the hope of making this article usable in many settings—they are written at a level and length to allow mathematicians with various backgrounds and experiences to have access to these conversations as part of a course or professional development. And finally, the references have been chosen to serve as an introductory reading list on the topic. Many of those papers contain examples of where mathematics has gone wrong, thus motivating the study of ethics in mathematics.

Why is EiM Interesting?

Many of us do mathematics because we find it interesting, and often we choose problems to work on that generate interesting mathematics. Andrew Wiles already argued that “it’s fine to work on any problem, so long as it generates interesting mathematics along the way” [NOV00]. At face value, ethics in mathematics might not seem like something that would pique our interest. After all, how could studying the ethics of mathematical work generate new problems of mathematical interest? But actually, it does.

As mathematicians, we know that theorems often create more open problems; the more we know, the more we want to ask about. The same is true for ethics in mathematics; mathematics creates ethical questions, and in return ethical questions generate interesting mathematics. Consider, for example, the vast technical research on algorithmic fairness. As algorithms now increasingly influence many facets of our lives, new ethical questions have emerged. To answer these, we need new mathematics. “What is fairness?” is a deep philosophical question, but also an inherently mathematical one, and it led mathematicians and computer scientists to develop new metrics to measure individual and group fairness, connecting their mathematics with ethics, law, and philosophy.

During this process, mathematics often helps to determine the limits of these ethical questions. When generative AI (e.g. ChatGPT) “hallucinates” and tells us something untrue, we may ask if it’s always possible to identify these untrue statements, or even better, create stable
neural networks that accurately output true statements. Mathematics shows this is sometimes provably impossible [CAH22]. Creative, deep, and new mathematics goes hand in hand with difficult and urgent ethical questions.

But ethics in mathematics is more than the technical work of individuals. Due to its inherent interdisciplinary and often transdisciplinary nature, this young research area is currently made up of several small, widely dispersed communities, straddling many areas beyond the boundary of mathematics. Training people for this work is an interesting and necessary challenge in itself. To properly address many ethical questions raised by mathematical work, we will need to come together as a community. Some questions (e.g. ethical questions about proof standards) necessitate community consensus, but all questions benefit from assembling our unique knowledge, experience, and perspective. Bringing these separate communities together is the first important step towards establishing more systematic thinking on ethics in mathematics, as explained further in [MCF22].

Ethics in mathematics gives us a way to use our mathematics, and our mathematical insight, to make a tangible, positive impact on society, often by bringing our skills and insights to other areas, people, and disciplines. Without a “translator” to explain what is happening in the mathematical community, the ability of external experts (e.g. lawmakers regulating algorithms) is severely curtailed. Similarly, communicating external contributions to mathematicians is equally important. Mathematicians working on ethics in mathematics can act as “bridges” between those doing mathematical work, and those considering its effects on society. This work is very interesting in itself, bringing meaning and worth to our mathematical understanding, in addition to being an area of active recruitment in industry and beyond.

Finally, there’s the interesting challenge of teaching ethics in mathematics. Consider the cryptography problem posed in [CM23b]: Alice and Bob use RSA private keys \((N, e_1)\) and \((N, e_2)\) with the same modulus \(N\), and their colleagues send messages to each of them encrypted to their respective keys. An eavesdropper Eve monitors their communications. Using number theory we can show that Eve can always decrypt the messages. So who should we tell, and what should we tell them? What if Alice and Bob have an arrest warrant out for them, and Eve is a police analyst; does that change our answer? Now what if the warrant is from a country with history of human rights violations; any change? Alice, Bob, and Eve may not know that decryption can occur, but as mathematicians we do. So how might we teach the consequences of our unique mathematical insight? These questions don’t stop with applied number theory. One needs only to look at how Galois theory—a poster child for pure mathematics—is purported to have been used to compromise the Diffie-Hellman key exchange by exploiting the Logjam vulnerability.

**Why is EiM Tractable?**

Progress in mathematics is more than the number of theorems and published papers we produce; it is, as Thurston describes, the “advance of human understanding of mathematics” [Thu94]. Making progress on ethics in mathematics is also possible, and even goes beyond Thurston’s criteria of “advancing knowledge”—it can be interpreted much more broadly by appreciating the theoretical and practical nature of the field. As a (relatively) nascent area of concern, ethics in mathematics has received comparatively little academic attention so far, thus there remains much to be done: from the foundational level, where we need to get better at establishing and promoting the notion that there are indeed ethical issues that arise from mathematical work; to the practical applications of turning our theoretical knowledge into being morally good mathematicians and combating morally bad uses of mathematics. Many important, critical problems need to be addressed, and therefore much work can, and should, be done.

We can’t expect to “solve” ethics in mathematics with one publication, in the same way we can’t solve topology with one publication. Our contributions are incremental, and, just like in mathematics, we make progress by standing on the shoulders of giants. There are many examples and case studies in this paper of mathematics that raises ethical questions, many of which would benefit from further analysis, and there are even more that didn’t fit into this paper or are yet to be found. There are rudimentary ethics guidelines and frameworks from various mathematical institutions and societies which are constantly being refined and updated (cf. [MCF22, BPT22]). And there are introductory teaching modules on ethics in mathematics, and associated resources for pure and applied courses (e.g. [CM23b]), all of which could be substantially improved with additional time, effort, and insight. Work has already begun, but as mentioned in the previous section, it needs a larger community effort to make a more substantial impact.

Some of the problems associated with ethics in mathematics may seem daunting. The proliferation of AI through all aspects of society, and its many drawbacks and risks, is vast. The ever-increasing production of complicated financial instruments, and the threat they may pose to the global economy and to wealth inequality, might seem overwhelming. As mathematicians, we can’t solve any of these problems by ourselves; they’re simply too big. But by working on ethics in mathematics, we can find ways
to make a positive contribution, and thus make the world a little bit better. And in particular, we can contribute in a unique way and address issues only mathematicians are properly placed to address, or even identify! Ours might only be a small contribution, but it is critical, and one that only we can make.

In doing so, we can often draw ideas and inspiration from adjacent fields, including “leveraging guidelines for ethical practice of [...] related professions” [BPT22, p. 5]. Engineering and computer science already have ways to recognise and address many of their ethical issues. Medicine has a long history of addressing ethics, and well-established ways to research and teach such problems. And associated mathematical disciplines such as statistics have been addressing ethics for over a century. There are certainly lessons that, with work, can be carried over to mathematics more generally, as has begun in [MCF22],[BPT22]. As mathematics is deeply connected to the rest of the world, there are numerous open (mathematical) problems with connections to existing and well-established ethics research areas.

Ethics in mathematics has become an academic discipline, including peer-reviewed articles, books, conferences, and dedicated courses [Em20a]. Some large-scale funders of PhD studentships (e.g. UK government Centres for Doctoral Training) now require dedicated ethical training to be incorporated in those centres, with other higher education agencies advocating for the inclusion of sustainability requirements into degrees due to the subject's increasing impact on society [Age23]. Ethics in mathematics transcends academia, and is critical to modern industries: the EU AI Act proposes new regulations for AI, and trustworthiness has become central in many areas, including finance, insurance, cryptography, and social media, all of which employ many mathematicians.

Overall, ethics in mathematics is something we can do in a variety of ways, each giving clear avenues for us to make progress.

Why is Ethical Mathematics Nontrivial?

As mathematicians we may be tempted to devalue ethics and think that, as a nontechnical area, it is “easy” and “obvious.” But neither is the case. Ethics, and in particular ethics in mathematics, is surprisingly challenging to work on. Often, it is only after seeing the ethical questions in one part of mathematics (e.g. in our own area of expertise, or something receiving media coverage) that we start to grasp how ethics weaves itself through all mathematics.

When dealing with ethics in mathematics, it’s not only that the answers aren’t obvious; even the very questions may not be evident. When setting up an optimisation problem, is it obvious that we should consider an objective function other than time or money? And what about the constraints; is it obvious that we should ask about aspects we were not initially briefed on? We know this from our mathematical research: being able to ask the right questions is often more than half the game. This is why mathematicians, and mathematics, are needed in the study of its ethics; in most cases, it can’t be done without mathematical insight. We’re often the only ones who can understand and ask about the true significance of adding more constraints. And as for choosing the “right” optimisation objective, that’s often a highly contentious, and thus non-trivial, question in itself.

Operations research (OR) has begun developing “soft OR techniques” to help establish the ethics and responsibilities implicit in the OR process, with some even calling for a Hippocratic oath due to its potential for harm. But ethics in mathematics goes further, and at some point involves reverse engineering the process of carrying out mathematics to find out how to do it responsibly [CM23a]. In doing so, the answers we find can take us completely by surprise. For example: a Hippocratic oath for mathematicians may be necessary, but it would likely not be sufficient to address the shortcomings in the ethical awareness of ethically untrained mathematicians [MCF22].

But it’s not just the ethical problems themselves that pose a challenge; bringing together the right people to address these can also be challenging. Ethics in mathematics can’t be done without help, support, and contribution from others in the mathematical community [BPT22] and from funding bodies. This requires mathematicians and institutions to come together in the right way. But growing such a community is hard, as not everyone will agree on what should be done, let alone how. Assembling people without due care can lead to conflict. But only assembling people who always agree with each other is no solution, either, as we may end up lacking diversity and thus hinder ourselves from asking the right questions. Community building—especially across different (research) cultures—is one of the open challenges of doing ethics in mathematics effectively. There is extensive literature and many experts, to read, learn from, and work with [Tow20]. But there’s no easy answer for building such a (professional) community.

Insight and communication are thus key challenges of ethics in mathematics. Even if we identify a problem in our, or someone else’s, mathematical practice, we must do more than just scream “everything is on fire”; we must offer pathways towards a solution, or we risk being ignored completely. Offering insight into what is, or has, gone wrong in the mathematical process (e.g. inaccurate model assumptions) is a good start, but suggesting ways to resolve them is often more appreciated (e.g. suggesting to
consult domain experts to discuss how well our assumptions match reality). This requires astuteness to detect the problem, insight to realise what went wrong, foresight to see what needs to be done, and perspective to see how the proposal will be received by others, some of whom may not necessarily understand all the mathematics.

That ethics in mathematics is hard shouldn’t be surprising. Its problems require a good grasp of many hard areas, including mathematics, ethics, philosophy, and sociology of science, as well as people and context. All of these are far from trivial. We quickly realize that ethics in mathematics is neither one question nor one answer, but rather is woven through the entire mathematical process, so we cannot just “do the ethics” in one afternoon at the start of a project.

Working on ethics in mathematics helps us appreciate that mathematics is everywhere and connects to all parts of life, though not necessarily always for good. This is a challenge for us mathematicians in itself, as we may desire to see mathematics as morally pure, or neutral at worst [Ern20b].

**Why is EiM Well-defined?**

As early as primary school, we’re taught that mathematics is certain and that its truth status cannot be influenced by external powers and interests [BS97]. In contrast, we encounter many debates and arguments about ethics which just appear to be exchanges about matters of opinion. However, a closer look at ethics and mathematics quickly reveals that the situation is more complex [Nic22]. Reasoning in ethics can be just as rational as reasoning in mathematics. It’s just that its starting points are often less clear, and rarely communicated well in practice.

Just like mathematics, much of ethical reasoning begins with finding good starting points or axioms, from which we build further. These can be very different, depending on how we view and value different aspects of life. For some, this may mean being a utilitarian and calculating the good of an action by measuring its outcomes. Some may take a deontological position, applying universal rules to everyday ethics. Others may be more concerned with virtue, a duty of care for people, animals, and the environment, or religious-centred ethics. The list of ethical positions is long and positions may potentially overlap but they may also regularly disagree about what it means to be, and do, good (cf. [AE21]). However, what they all have in common is that they produce well-thought-out answers subject to an initial set of assumptions; something we’re all very familiar with as mathematicians. We also find these philosophical positions in ethics in mathematics. And so, ethics in mathematics is not a nebulous, fuzzy concept. Rather, there is logic and reasoning to it, and its problems are identifiable—that is what we mean by “well-defined.”

But this means that to do ethics in mathematics effectively, we need to engage with others, understand their points of view, and obtain their input on the problem. It’s often necessary to work towards consensus on a particular issue or challenge. We might seek consensus from colleagues on what assumptions to make, what to optimise over, how to interpret and convey our results to others, or even whether we should be carrying out the work or project in the first place. We seek consensus for decisions and choices which no one person can, or should, make on their own. Consensus is again a well-defined notion, which as mathematicians we use all the time in scenarios such as multirefereed papers, PhD exam boards, and hiring panels. And while these processes are not without their shortcomings and can certainly be improved, consensus is the primary tool used.

As for identifying whose responsibility it is to do ethics in mathematics, it quickly becomes evident that few mathematicians are actively engaging with it. Most mathematics departments have no ethics committees to guide mathematicians with the many ethical issues specific to our subject and its theoretical nature. Often the only solution is to do it ourselves, but even then we don’t have to reinvent everything. By now there are well-defined research areas and standard methods within ethics in mathematics. And alongside adapting existing guidelines and frameworks from other areas, there is plenty to choose from within mathematics too, e.g. building on the vast literature on mathematics education and social justice to produce resources, exercises, and assessments to be incorporated into higher-level mathematics courses. And there are also many well-known problems: AI safety, cryptography, finance, mathematics in warfare, and more.

A feasible workflow of ethics in mathematics is often quite similar to that of mathematical research:

1. Find an area of concern and interest, and begin to understand your ethical position. Where are you right now? What are your starting points?
2. Review the existing literature, including literature not published by mathematicians. A lot of ethics in mathematics is published by other people, including philosophers, science and technology scholars, and domain experts. This is particularly true for mathematics education, where historically the community of research mathematicians can be quite distinct from those specialising in education.
3. Engage with your peers and begin tackling the problem: ethics in mathematics is often a collaborative effort and you’ll quickly find that those from outside mathematics mentioned in (2) are often very happy to
work with mathematicians on an ethical issue, problem, or paper.

Why is EiM Good?

Is it not simply a tautology to say ethics in mathematics is “good”? Well, yes, but we can go further than just saying that it is doing good for good’s sake, and articulate exactly what this good is, and how it comes about. We all have different perspectives on what is “good,” and doing ethics in mathematics helps us explore these and improve on them. Mathematics is not a tool for unmitigated good; it can cause harm, examples of which can be found in [MCF22].

By doing ethics in mathematics, we do direct good for mathematics as a body of knowledge and its research practices. We can combat plagiarism and publication irregularities, thus preserving our corpus of knowledge. We can ensure that jobs are appointed fairly to those best placed to preserve and further mathematics. We can prevent mathematics from being used incorrectly, or misused, and thus enhance public trust in our work and mathematics more generally. We can give ourselves a better perspective on the problems we are solving, providing us with greater challenges and pushing our mathematical output to be even better. And we can ensure that mathematical work produces the best possible outcomes in the places and situations where it is used.

By doing ethics in mathematics, we do direct good for the mathematical community and other communities. We can help turn mathematics into a field that’s open and welcoming to all, regardless of background or circumstance. We can help ensure that mathematics is inclusive and diverse, thereby encouraging as many as possible to join the discipline. We can teach in ways that allow all students to learn and appreciate as much mathematics as possible. We can train upcoming mathematicians to be aware of the role, importance, and responsibility they have to other communities when carrying out their mathematical work. We can work towards ensuring that mathematics, as a tool and an output, doesn’t discriminate or harm those individuals or groups who are most vulnerable in society. We can create mathematical tools that actively combat existing discrimination and oppression in the world. And we can make progress to ensure that mathematics works for the benefit of all. Ethics in mathematics allows us to identify the most urgent problems and enables us to properly work towards a solution using our unique skillset.

By doing ethics in mathematics, we do direct good for the world. We can check what impact mathematics is having, and intervene in a preventative way before harm is caused. We can provide frameworks for mathematicians to follow, to help them avoid producing mathematics detrimental to society. We can create tools to measure the performance and impact of mathematical output, to see where it may be causing harm in ways otherwise invisible to its creator. We can guide mathematicians to work on the right problems, and in the right ways. We can encourage mathematicians to factor in the sociopolitical context of their work, to facilitate it being well-received rather than rejected. We can address the long-term effects of mathematics on society, the environment, and the planet. And we can promote humility in mathematicians, so that they better understand when they need to seek help and guidance from others, potentially outside mathematics. Ethics in mathematics allows us to be more effective and efficient in our attempts to solve society’s most pressing and difficult problems.

And by doing ethics in mathematics, we do direct good for ourselves as individual mathematicians. We can empower ourselves, allowing us to better navigate the world of mathematics and its culture. We expand our knowledge and expertise so that we can do more interesting and advanced work with better impact and minimal harm. We can extol the virtues of our work, making it more attractive for employers and funders. We can develop greater kindness and humility, towards those near to us, as well as those impacted far away whom we might never see. We can become part of new and different communities, interacting and working with people from other backgrounds and disciplines. We can find new opportunities for work and employment, bringing our now-broadened understanding of the role and place of mathematics into other domains and jobs. And we can be happy knowing that our work and input has made a positive contribution to the field, to those around us, to those in need, and to wider society.

Doing ethics in mathematics helps us to become better mathematicians: better people doing better mathematics for a better world.

Going Forward with EiM

A useful observation, and subsequent resource, to have come out of the Ethics in Mathematics Project at Cambridge University is the following decomposition of ethical problems in mathematics, split up into what has been named the “10 pillars for responsible development” [CM23a]:

1. Deciding whether to begin
2. Diversity and perspectives
3. Handling data and information
4. Data manipulation and inference
5. The mathematization of the problem
6. Communicating and documenting your work

1Ethics in Mathematics Project: https://www.ethics.maths.cam.ac.uk
7. Falsifiability and feedback loops
8. Explainable and safe mathematics
9. Mathematical artefacts have politics
10. Emergency response strategies

Building on practical works, such as the ethics framework from Digital Catapult [Cat21], these pillars are further broken up into questions and actionable steps that provide a good starting point for studying the ethics of one’s mathematics (see figure 1). While these questions and steps do not form a complete cover of every ethical issue, the authors found them particularly helpful to guide those with little or no previous exposure to ethics in mathematics. But ultimately, to find satisfactory answers, it'll often be necessary to consult other existing literature including works on general ethical reasoning (such as the ethics framework from the Markkula Center for Applied Ethics [fAE21]), engage in self-reflection, and talk with other people (see also [Tow20]).

The closer we look at an ethical aspect in mathematics, the more it breaks apart into several subsaspects to consider. This can be seen as an opportunity for those of us wishing to work on ethics in mathematics, as we can pick our desired level of specialisation and comfort. And just like ethics in mathematics education [Dub20], the ethics of mathematical practice is done most effectively if it isn’t done with a narrow definition of ethics in mind. Instead of being stumped by the question “whose ethics am I supposed to consider?”, it is often better to acknowledge that the answer may be found after working on an ethical problem for a while. This is not surprising to us as mathematicians: only after studying an object and how it connects to other objects, do we establish how to properly define it and lay out its axioms. Writing down the hypotheses of a theorem is never the first step of its discovery, and neither is having an answer to the question “whose ethics? a necessary requirement to start working on ethics in mathematics. Indeed, it’s always part of exploring the issue at hand, and history has shown us that there can be more than one answer to the relationship between mathematics and ethics [Nic22].

Ethics in mathematics—in research and practice—is a spectrum of actions and roles where everyone can find something important to contribute to. As with any area, we can look at the full breadth of a problem, we can look at a particular (technical) aspect of a problem in depth, and we can do many things in between. Some may work on overwhelmingly large problems, while others may want to work on something smaller and more manageable. This can be done in a complementary, nonconflicting way that respects others. And often, there are hidden connections between the small problems and the biggest challenges. It’s important not to excoriate someone’s work on a problem just because we perceive a different problem as more important—we may not properly understand why it was chosen, or be unaware of its depth and connections. Ethics in mathematics is often a lesson in kindness and humility. It requires from its participants an understanding that mathematics and its practice can change over time and across cultures and groups [D’A85], and that its ethics may follow similar contingent circumstances.

This work employs a vast array of methods and is inherently interdisciplinary and multidisciplinary; we need many different types of understanding to be able to answer all the different types of questions. And different personal goals, and personal work preferences, can fit into ethics in mathematics. We can do deep mathematics that aims to make the world better (e.g. algorithmic fairness). We can do ethics that makes our mathematics better (e.g. ensuring sufficient perspective in a team). We can apply elements from education, psychology, sociology, and other social sciences or humanities to gain insight into the ethical issues of mathematical work (e.g. to understand how existing ways of teaching affect our students’ perception and use of mathematics). We can choose to treat ethics in mathematics as an oil-in-water suspension, where we add some mathematical insight into a predominantly ethical problem. Or we can treat it as water-in-oil, where we add ethical insight into a predominantly mathematical problem. Both of these are useful and productive contributions. Such work might not be 100% mathematics, and may lack the “exactness” of abstract mathematics, but that shouldn’t be a reason to disregard it. We often need to deal with nonmathematical factors, but these enhance our mathematics by providing motivation for problems to work on, adding challenging constraints, and giving our output purpose and meaning.

Starting this does not require any formal education in ethics, but simply for us to approach the subject and our work with an open mind and a willingness to do better. This is where looking at neighbouring subjects and debates can be very helpful. Many of us have some question or problem in mind, but simply don’t know how to start working on it. This can be in our teaching, in our department, in our research, or somewhere else entirely. Everyone starts doing ethics in mathematics with very little knowledge, but if we are commencing it today we have a clear advantage over those who worked on it 30, 10, or even 5 years ago. While Hersh deemed “research work [to be] almost devoid of ethical content” in his initial contributions to ethics in mathematics in the 1990s [Her90, p. 23], times have changed, and advances in knowledge have shown that there are ethical questions in all areas of mathematics, both applied and pure. Even a humble number theorist might one day produce a fast...
factorisation algorithm—an issue of the deepest ethical concern. As a community, we are only just beginning to explore and understand these, but nobody has to start from zero anymore.

We may feel uncomfortable and tempted into inaction by thinking “But I’m not trained to do this; I’m not an expert.” Ethics in mathematics is a very new area, and so there are few experts in the same way that one might be an expert in PDEs. This should not put us off, as we can still make a meaningful contribution despite being relatively inexperienced in the subject matter. Anyone spending a decent amount of time, probably just a few months, looking into ethics in mathematics would develop the necessary knowledge to make a novel and valuable contribution. We do not need to be the best in the world, nor have 20 years of experience, to work on ethics in mathematics. And that goes for teaching ethics in mathematics as well. We often teach undergraduate courses in areas where we are not internationally recognised experts, but nonetheless know enough about, and sufficiently more than the students, to teach them. Ethics in mathematics need not be any different.

Many, though certainly not all, of these ethical concerns in mathematics apply just as easily to other fields, such as computer science and engineering, or even apply more generally to all vocations and lines of work. But that is not a reason to dismiss them. If such issues apply to other mathematical disciplines, or even to everyone, then they must surely apply to us as well. And as outlined earlier, this means we can draw insight from other fields and carry those over in a more mathematics-centric way, to be maximally relevant for mathematicians.

Whatever our concerns are about mathematics, be they about continuing the body of knowledge, about caring for our mathematical and other communities, about the impact of mathematics on wider society, or indeed anything else, ethics in mathematics presents itself as a useful and good mechanism to highlight and address them. To us, the authors of this article, the study and practice of ethics in mathematics will only become more important in light of the ever-increasing use of mathematics. Only if we, as a community, reflect on and engage with the ethical questions that our field raises, can we ensure that its impact remains positive in the future.

References


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Human Rights of Mathematicians

Mohammad K. Azarian and Karen Saxe

Introduction
The topic of human rights has been an important focus of the American Mathematical Society (AMS) for decades. Most current cases that the AMS Committee on the Human Rights of Mathematicians (CHRM) has been following for the last several years are based in Asia, Africa, and Eastern Europe. Even though imprisonment and allegations of torture of mathematicians have not been found domestically, individuals who have voiced their opinions, which are traditionally protected as a first amendment right, have been fired from their teaching positions here in the United States. The AMS was forward thinking in 1977 to create a committee to focus on human rights of mathematicians. Expanding the effort to consider domestic cases as well as international ones was a move made in 2019. Arguably, the main question today should not be whether to maintain these efforts, but how to increase awareness of these efforts, as well as how to increase the committee's overall effectiveness.

History, Goals, and Mission of the AMS CHRM
The CHRM held its first meeting on January 28, 1977, in St. Louis. At that meeting it was decided that it would be appropriate for the Committee to advocate on behalf of foreign mathematicians whose human rights had been violated. The Committee was established to assist the AMS Council, which had intervened on behalf of individuals or groups of mathematicians in other countries who were being mistreated by their government. From the original charge, we read:

By violations of human rights, the Committee is to understand violations of freedoms enumerated in the Universal Declaration of Human Rights and in the Affirmation adopted by the National Academy of Sciences, USA: in particular, torture, imprisonment for political reasons, dismissal from a job which deprives a mathematician of the opportunity to function professionally.

We further read that:
It would be impractical to consider less drastic offenses like denial of promotion or of professional recognition.

At the beginning and indeed for many years, the authority of the Committee was limited to “foreign” mathematicians:

not because human rights of American mathematicians are less important, or completely safe. The AMS should be, of course, particularly concerned with the rights of American mathematicians. However, matters concerning Americans should not be referred to this Committee.

The earliest cases in which the AMS CHRM took some action—actions including letters sent and Notices articles written—involved mathematicians in Russia (former Soviet Union), Mali, Argentina, Germany, Chile, and Uruguay. Members of the first AMS CHRM committee were Nathan Jacobson (Chair), Lipman Bers, Charles Herbert Clemens, Chandler Davis, Morris Hirsch, and John

Disclaimer: The opinions expressed below are those of the authors.

Mohammad K. Azarian is a professor at the University of Evansville, and past chair of the AMS Committee on the Human Rights of Mathematicians. His email address is azarian@evansville.edu.
Karen Saxe is senior vice president, Government Relations, at the AMS. Her email address is kxs@ams.org.
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Nohel. Around the same time that the AMS launched the CHRM, the broader scientific community mobilized to act in the human rights field; notably, the National Academy of Sciences formed its Human Rights Committee.

Over the years, the AMS committee focus has not changed much, though some shifting in the countries of concern has taken place. Cases considered over the past five years are from Russia, Turkey, and Egypt. Cases from Palestine and China have also been considered over the past decades.

The Committee charge has been updated on several occasions with, arguably, the most significant change coming in 2019. By the mid-1980s AMS records show that a discussion of expanding the charge to include cases involving American mathematicians took place. In 2019, this change was made, and the charge no longer solely addresses foreign mathematicians. It now reads:

The AMS is committed to speaking whenever mathematicians are deprived of the opportunity to practice their profession due to violations of the freedoms enumerated in the Universal Declaration of Human Rights [http://www.un.org/en/udhr/] and the Affirmation adopted by the US National Academy of Sciences.

The AMS does not work alone on these issues. In 2013, for example, AMS became a full member of the American Association for the Advancement of Science (AAAS) Human Rights Coalition.

Challenges and Opportunities

Information source challenges. While the original CHRM charter states that to make a “recommendation for action,” “any source of information available” may be used. The state of information sharing in 2024 differs greatly from what it was even a decade ago. We no longer rely only on a few traditional sources of obtaining news, sources which admittedly could be biased with their own agenda. Now, we have access to information from a multitude of sources on an almost minute-by-minute basis. More is not necessarily better, however, and the integrity of those sources can be especially challenging to verify. While misinformation may be shared due to sheer negligence, disinformation can be, and is, used purposely to mislead. Artificial intelligence is one tool that can be used to mislead the reader and manipulate viewpoints. The existence of deepfakes is particularly disturbing. Authenticating claims of abuse must be done consistently and objectively. One challenge for the committee is to remain aware of claims made about human rights violations in a timely manner. Another challenge is to verify the authenticity of those claims.

Actionable steps. Yet another challenge of continuing human rights work in 2024 is how to expand meaningful involvement within the mathematics community. That is, interest in sustaining the work of the committee must be maintained by recruiting new members with fresh perspectives. A committee is only as strong as its members and leadership, and each has a role to play in continuing the committee objectives as set forth in its charter. Namely, the CHRM is charged with several possible courses of action when confronted with a case. These steps could include (i) recommending action based on advice obtained from agencies which deal with human rights (such as Amnesty International or Scholars at Risk); (ii) requesting a letter of support written by the AMS president on behalf of the individual in question; (iii) informing the AMS membership about cases; and (iv) posting news items to the AMS website. Our partnerships with established human rights organizations are especially useful. By collaborating with experts in the field of human rights work, a practical course of action can be considered and undertaken by those who know best how to approach a claim. A checklist might be devised so that all cases can be afforded the same degree of attention and review. It may not be sufficient only to bring awareness to a situation: further purposeful action may be warranted. Standing by silently should not be our default status.

Build greater awareness within the AMS. The key to a strong human rights component within the AMS is to build awareness of the Committee’s work, and share the work being done to address human rights violations against mathematicians around the world. Mathematics is the basis of every other science and, therefore, has a wide reach. Encouraging other mathematicians to join these human rights efforts will bring fresh energy and interest so the work continues and remains relevant and effective. As educators, it is our job to educate with factual information, without distortion or bias. The human rights cases that we are aware of are most likely but a small number of actual cases in the world. Given the increasing conflict and atrocities being committed in the world right now, the moral obligation of those of us who are outside of the troubled areas must be to express our humanity by being the voice of those who cannot speak for themselves.

Closing

Since its inception, the AMS Committee for the Human Rights of Mathematicians has been dedicated to raising awareness about injustices targeting mathematicians around the globe. It is painful to observe that situations which may have seemed remote 50 years ago have remained and, in many cases, grown even more dire in the intervening years. International events over just the last
two years remind us that we must not ignore these injustices wherever they occur. Indeed, we must look both outwardly and inwardly through a social justice lens. We must remain vigilant in our goal to shed light on these victimized individuals and their circumstances. To be most effective, we must connect with partner agencies that perform the actual work of advocating for and assisting in the resolution of these cases. We must translate awareness and compassion into significant action.

You can find the current CHRM charge, list of members, coverage of individual cases, and ways to engage, at the CHRM web page: [https://www.ams.org/about-us/governance/committees/humanrights](https://www.ams.org/about-us/governance/committees/humanrights).

Credits

Photo of Mohammad K. Azarian is courtesy of Mohammad K. Azarian.

Photo of Karen Saxe is courtesy of Macalester College/David Turner.
The Defense Science Study Group: Another Way to Help the Nation

Skip Garibaldi and Mark Taylor

Readers of the Notices are likely familiar with the AMS-CBMS-AAAS Congressional Fellowship program, which places mathematics faculty to work in the legislative branch for a year. But that is just one way that US mathematicians can give back to their country. We’re here to tell you about another one, the Defense Science Study Group (DSSG). One of us, Mark, is in charge of the program and the other, Skip, participated in it during his earlier career as a math professor. The goal of DSSG is to introduce outstanding science and engineering faculty to US national security challenges and to encourage them to apply their talents to these issues. It is divided up into cohorts, which meet for about 20 days per year over two years, broken up into short trips, and scheduled to be approximately compatible with a regular academic calendar. A trip typically involves members visiting military bases, national laboratories, Congress, intelligence agencies, or other parts of the US government. During these visits, members meet with a wide range of people, focusing on top-level officials while also including talking with less-senior folks like privates.

On all of these visits, the DSSG members are joined by a group of mentors, who are mostly retired from distinguished careers in the military or government. They are a rich source of knowledge and wisdom and are regularly cited by former members as a treasured part of the program.

The first few cohorts of DSSG, starting in the mid-1980s, included mathematicians. But over the last two decades, there have been few mathematicians in the pool of nominees. One of our aims with this article is to encourage more mathematicians to apply to be in the program.

**Why you might want to participate.** There is so much to get out of DSSG. Most members talk about their time with the mentors and with their fellow members, who are other outstanding faculty at a similar career stage. And most people come away with an appreciation for, if not awe of, the breadth and depth of Department of Defense operations and the dedication of the Department’s personnel. (One member commented: “I was uniformly impressed with the military personnel I met. There was a sense of competence, dedication, and professionalism conveyed by almost everyone we met” [Spafford].)
For some members, DSSG has a strong effect on their careers. It is exciting to be presented with interesting scientific problems you have never seen before that come with a built-in audience of people who are very interested in their solution. Some members change the direction of their research to focus more on these problems. Members are better able to advise their students on finding jobs in national security. Some members (including one of us) choose to leave academia for jobs in government or national security as a result of their experience with DSSG.

You might be wondering how you could contribute to solving national security problems. In our experience, there is a lot you can do, even if your current research focus may not be directly applicable. First, as a mathematician you bring analytic skills and can break down a problem, even when you are not an expert on the narrow subject. Second, you can read dense technical papers to get up to speed on a topic. Amazingly, “there are lots of places where one person, with the right ideas, can make a huge impact” [Spafford].

It’s good for the nation. DSSG is sponsored by the Defense Advanced Research Projects Agency (DARPA), which is part of the Department of Defense (DoD). This sponsorship exists because technological advantage is fundamental to our nation’s security, and maintaining this advantage requires strong links between emerging scientific leaders and the national security community. This is not an isolated view. In our experience, the DoD hungers for learned and impartial scientific advice and insight. For example, when the Navy plans to renovate a port, they have strong incentives to get accurate forecasts for sea levels decades from now.

One concrete way DSSG contributes is that the members, either individually or in small groups, write “think pieces” on national security issues of their choice. These allow members to focus on a particular area of importance to the DoD, to bring their knowledge from an academic environment to bear on issues of concern, and to interact with individuals in the DoD with related interests. Examples of topics have included protecting soldiers from traumatic brain injury, oxygen recycling processes for submarines, new chemical synthesis techniques for energetic materials, and a process to treat conventional wet suits that improves survival times of Navy divers in frigid waters by a factor of three.

Another way DSSG contributes is that after completing the program, members are much more familiar with the DoD thanks to the visits, interactions with the mentors, and their work on their think pieces. Consequently, they are better prepared to serve on other advisory boards. DSSG graduates have served in over 300 different government-related science advisory roles or leadership positions.

A few nuts and bolts. DSSG is administered by the Institute for Defense Analyses (IDA), a nonprofit corporation headquartered in Alexandria, Virginia. Participating in DSSG requires a security clearance, and all members must be US citizens. More information about the program can be found at https://dssg.ida.org.

How to participate. Every two years, IDA solicits nominations from senior leaders at universities, often a president or a provost. Contacting your provost might be a good way to start. Selection for DSSG is based on academic excellence, breadth of interests, references, consideration of discipline, and geographic distribution. For information on the next nomination cycle, contact Mark Taylor at mtaylor@ida.org.

Further Reading

Credits
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Sooner or later every professional mathematician has the experience of attending a party and getting in an awkward conversation with a non-mathematician about their research. Eventually, they always ask, What is the point? But no one ever asks where is the point. Sometimes this question is harder to answer than you might think.

If you are a mathematician and not a social psychologist, group dynamics refers to the study of group actions on spaces: we have a group $G$, a topological space $X$, and an action of $G$ on $X$, i.e. a homomorphism $G \to \text{Aut}(X)$ where $\text{Aut}(X)$ denotes the group of self-homeomorphisms of $X$ preserving some structure or other. We gain a lot of insight into the action by considering the orbits—i.e. how $G$ and its elements act on particular points: fixed points, periodic points, recurrent points, and so on. But occasionally it happens that the points per se are ...well, they're beside the point. We give three examples.

1. Mapping Class Groups

Sometimes we don’t really have a group action at all, but only a homomorphism $G \to \Gamma$, where $\Gamma$ is a quotient group of $\text{Aut}(X)$; thus, we could think of elements of $G$ as equivalence classes of automorphisms of $X$, or (equivalently) that $G$ only acts on $X$ up to equivalence. One of the most familiar examples is that of mapping class groups. If $X$ is a manifold, and we give $\text{Aut}(X) = \text{Homeo}(X)$ the compact-open topology, path components in $\text{Homeo}(X)$ correspond to homeomorphisms that differ by isotopy (i.e. that can be connected to each other by a continuous one-parameter family of homeomorphisms). The path component of the identity is a normal subgroup, and the quotient is the group $\pi_0(\text{Homeo}(X))$, usually denoted $\text{Mod}(X)$ or sometimes $\Gamma_X$.

The best studied case is when $X$ is a surface $S$. If $S$ is oriented, we typically restrict our attention to the group of orientation-preserving homeomorphisms of $S$ (denoted $\text{Homeo}^+(S)$) and, by an abuse of notation, let $\text{Mod}(S)$ denote the group of orientation-preserving mapping classes. Given a subgroup $G$ of $\text{Mod}(S)$, it is natural to wonder whether it comes from an “honest” action of $G$ on $S$; i.e. whether there is a section from $G$ to $\text{Homeo}^+(S)$. The hardest instance of this question is the universal one: is there a section $\text{Mod}(S) \to \text{Homeo}^+(S)$? In other words, is there some function $\sigma : \text{Mod}(S) \to \text{Homeo}^+(S)$ taking mapping classes to representative homeomorphisms and satisfying $\sigma(g)\sigma(h) = \sigma(gh)$ for all $g, h \in \text{Mod}(S)$?

If $S$ is the 2-sphere, then $\text{Mod}(S)$ is trivial, so this is easy. If $S$ is the torus, $\text{Mod}(S)$ is $\text{SL}(2, \mathbb{Z})$, and the linear representatives of each mapping class group constitute a section (after one fixes a basepoint and a flat structure on the torus). But in every other case the answer turns out to be no: Vladimir Markovic and Dragomir Šarić showed [3] if $S$ is a surface of genus at least 2, that there is no such section.

This fact is not as disappointing as it seems: it is possible to turn $\text{Mod}(S)$ into an honest group of transformations in a few different ways. If we pick a (any) Riemannian metric on $S$ and denote the bundle of unit tangent vectors to $S$ by $UTS$, then (rather surprisingly?!) there is always a
section \(\text{Mod}(S) \to \text{Homeo}^+(UTS)\) coming from the action on the circle at infinity of the universal cover of \(S\). And there is a natural action of \(\text{Mod}(S)\) on spaces that themselves parameterize homotopy classes of structures on \(S\), such as the Teichmüller space of \(S\), or the complex of curves.

2. The Cremona Group

Sometimes we have a group action, but the transformations associated to group elements are not defined everywhere, and it might easily happen that the common domain of definition of the entire group is empty.

A birational transformation between two algebraic varieties is an isomorphism between Zariski open subsets of the domain and range. An example is the map \(f : (x, y) \to (1 - y)/x\) between the open subset of the circle \(x^2 + y^2 = 1\), where \(x \neq 0\) and the (affine) line. A birational transformation which is defined on the entire domain is a birational morphism. Birational transformations between smooth curves extend to honest birational isomorphisms between their projective completions. But in higher dimensions, the situation is more complicated. Birational transformations may be composed (after further restricting the domain), and the collection of all birational transformations from a fixed variety \(V\) to itself is a group.

Let \(K\) be a field, and let \(x, y, z\) be homogeneous coordinates on \(P^2_K\). The group \(\text{PGL}(3, K)\) acts on \(P^2_K\) by automorphisms in an obvious way. The involution
\[
\iota : [x : y : z] \to [yz : xz : xy]
\]
is an isomorphism on the open subset where no coordinate is zero. Near the point \([1 : 0 : 0]\), the map behaves rather badly: the line \([1 : t : 0]\) (for \(t \neq 0\)) is collapsed to the point \([0 : 0 : 1]\), whereas the line \([1 : 0 : t]\) (for \(t \neq 0\)) is collapsed to \([0 : 1 : 0]\). The point \([1 : 0 : 0]\) wants to be spread out over the entire line \([0 : t : s]\) joining these two points, with the result that there is no way to continuously extend the domain of the function to all of \(P^2\). The same situation occurs at \([0 : 1 : 0]\) and at \([0 : 0 : 1]\).

The group of all birational transformations of \(P^2_K\) is known as the Cremona group and is denoted \(\text{Cr}_2(K)\). When \(K = \mathbb{C}\) the Cremona group is generated by \(\text{PGL}(3, \mathbb{C})\) and \(\iota\), but this is not true for general \(K\) (for example, for \(K = \mathbb{R}\)).

Every element of \(\text{Cr}_2(K)\) may be represented by an honest birational morphism if we are prepared to first blow up the domain a finite number of times. The transformation \(\iota\) can be promoted to an honest involution on \(P^2_K\) blown up at three points in general position (Figure 1 shows the action of \(\iota \in \text{Cr}_2(\mathbb{R})\) acting on \(P^2_{\mathbb{R}}\) blown up at three points), but for most elements \(f \in \text{Cr}_2(K)\) there is no finite blowup of \(P^2_K\) on which \(f\) acts as an honest automorphism.

The two-dimensional cohomology of \(P^2_{\mathbb{C}}\) is \(\mathbb{Z}\), generated by the class of \(P^1_{\mathbb{C}}\), topologically a 2-sphere with self-intersection number 1. If we blow up at a point \(p\), the preimage of \(p\) is an exceptional divisor \(E_p\)—topologically an embedded 2-sphere with self-intersection number \(-1\).

Yuri Manin [2] defines a bubble space \(B\) whose points are the equivalence classes generated by the relation \((V, p) \sim (V', p')\) if there is a birational morphism between two (iterated) blowups \(V \to V'\) sending \(p \in V\) to \(p' \in V'\) and which is an isomorphism in a neighborhood of \(p\). This space is a little bit like the Bruhat-Tits tree of \(\text{SL}(2, \mathbb{C}(z))\), which one can think of as a kind of infinite fractal cactus, obtained from \(\mathbb{CP}^1\) by repeatedly attaching new copies of \(\mathbb{CP}^1\) to every point.

Cohomology pulls back under blowup, and the direct limit of \(H^2\) under all possible sequences of finite blowups is a free abelian group \(\mathbb{Z}\) with one copy of \(\mathbb{Z}\) for the generator of \(H^2(P^2_{\mathbb{C}})\), and one copy of \(\mathbb{Z}\) for every point in \(B\). The intersection form on \(\mathbb{Z}\) has signature \((1, \infty)\), and the Cremona group acts by isometries of an infinite dimensional hyperbolic space (which lives inside a suitable \(L^2\) completion of \(\mathbb{Z}\)). Serge Cantat and Stéphane Lamy [1] famously used this action to construct an enormous number of proper quotients of \(\text{Cr}_2(\mathbb{C})\), thus demonstrating that this group is very far from simple—a question that had been open since the 19th century!

3. Symmetries of Pseudoline Arrangements

A pseudoline arrangement is a collection of finitely many embedded oriented circles in \(P^2_{\mathbb{R}}\) so that each pair intersects transversely in exactly one point; it is simple if there are no triple points. We consider such arrangements up to the equivalence relation of isotopy. An arrangement of straight lines in general position is a pseudoline arrangement, but not every pseudoline arrangement is isotopic to a line arrangement (those that are are said to be stretchable). One example is illustrated in Figure 2, which would violate Pappus’s theorem if it were stretchable. A simple arrangement with nine lines is due to Ringel.
Peter Shor [4] gave an elegant example of a symmetrical pseudoline arrangement (it is invariant under an involution) that is stretchable, but not symmetrically stretchable. It is nevertheless true that there is a degenerate arrangement of lines (i.e. one with more coincidences) with the same symmetry. In fact, any finite group of symmetries of a pseudoline arrangement is isomorphic to a finite subgroup of PGL(3, ℜ).

It is slightly subtle even to define the symmetry group of an infinite arrangement; one way is as follows. A marked pseudoline arrangement of n circles is a choice of bijection of the components with {1, ⋯ , n}. Let ℳn denote the set of marked pseudoline arrangements; note that this is a finite set for any n.

An n-cochain on a group G with values in ℳn is a function c from distinct ordered n-tuples of group elements (g1, ⋯ , gn) to ℳn which is invariant under the (left) diagonal action of G on n-tuples. Such a c is a n-cocycle if there is some (n + 1)-cochain c′ that restricts to c (in the obvious sense) under all order-preserving inclusions from {1, ⋯ , n} to {1, ⋯ , n + 1}. Note that if n = 4, then the cocycle property for a 4-cochain c actually implies the existence of a (unique) m-cochain cm for all m ≥ 5 that restricts to c (in the obvious sense); i.e. a 4-cochain which is compatible on 5-tuples is compatible on m-tuples for all m ≥ 5. Thus if c is a 4-cocycle on G, then for any finite subset S of G there is an arrangement of |S| pseudolines which is compatible with restriction and the natural (left) action of G on itself. So, for example, if G is a finite group, a 4-cocycle c on G gives rise to a pseudoline arrangement of |G| pseudolines for which there is a natural permutation action of G extending to all of P2 ℜ.

If G acts on P2 ℜ by homeomorphisms, freely permuting the elements of some (possibly infinite) pseudoline arrangement, we obtain a 4-cocycle. Thus the 4-cocycle in a sense encodes the symmetries of some arrangement.

Conjecture. For any 4-cocycle c on any countable group G with values in ℳ4, there is an arrangement of straight lines in P2 ℜ, and an action of G on P2 ℜ by homeomorphisms preserving the arrangement and freely permuting the lines.

One cannot in general expect a projective action of G, i.e. a faithful representation G → PGL(3, ℜ). For instance, every group of (orientation-preserving) homeomorphisms of the circle acts by homeomorphisms of P2 ℜ with a common fixed point, permuting the set of straight lines through that point. At the other extreme, any homeomorphism of P2 ℜ permuting all the straight lines is actually in PGL(3, ℜ).

Question. If G is a countable group of homeomorphisms of P2 ℜ freely permuting some arrangement of straight lines and no point in P2 ℜ has a finite G-orbit, is G abstractly isomorphic to a subgroup of PGL(3, ℜ)?

AUTHOR’S NOTE. The author would like to thank David Eppstein, Benson Farb, Seraphina Lee, Rich Schwartz, and the anonymous referees for their help.

References


Danny Calegari

Credits

Figure 1, Figure 2, and photo of Danny Calegari are courtesy of Danny Calegari.
Joint Prizes

**JPBM Communications Award**

This award is given each year to reward and encourage communicators who, on a sustained basis, bring mathematical ideas and information to nonmathematical audiences.

**About this award.** This award was established by the Joint Policy Board for Mathematics (JPBM) in 1988. JPBM is a collaborative effort of the American Mathematical Society, the Mathematical Association of America, the Society for Industrial and Applied Mathematics, and the American Statistical Association.

Up to two awards of US$2,000 are made annually. Both mathematicians and nonmathematicians are eligible.

**Next prize.** January 2025

**Nomination period.** Open

**Nomination procedure.** Nominations should be submitted on MathPrograms.org. Note: Nominations collected before September 15 in year N will be considered for an award in year N+2.

Information on how to nominate can be found here: https://www.ams.org/jpbm-comm-award.

Fellowships and Programs

**Joan and Joseph Birman Fellowship for Women Scholars**

The Joan and Joseph Birman Fellowship for Women Scholars is a midcareer research fellowship specially designed to fit the unique needs of women. This program is made possible by a generous gift from Joan and Joseph Birman. One award will be made for the 2024–2025 academic year in the amount of US$50,000. AMS membership will also be offered to the recipient for the duration of the fellowship.

**About this fellowship.** The fellowship seeks to address the paucity of women at the highest levels of research in mathematics by giving exceptionally talented women extra research support during their midcareer years. The most likely awardee will be a midcareer woman whose achievements demonstrate significant potential for further contributions to mathematics. Applications will be accepted from mathematicians currently holding a tenured, tenure-track, postdoctoral, or comparable (at the discretion of the selection committee) position at a US institution.

The fellowship will be directed toward those for whom the award will make a real difference in the development of their research career. Candidates must have a statement regarding the applicant’s overall program of research, past and planned, that is meaningful to mathematicians who are not specialists. The statement should be no more than three pages, including bibliographical references. Special circumstances (such as time taken off for care of children or other family members) may be taken into consideration in making the award. Awardees may use the fellowship in any way that most effectively enables their research—for instance, for release time, participation in special research programs, travel support, childcare, etc. The award is issued through the recipient’s institution, and no part of it may be utilized for indirect costs.

**Application period.** Applications will be collected via MathPrograms.org July 15, 2024–September 30, 2024 (11:59 p.m. ET). Find more information at https://www.ams.org/birman-fellow. For questions, contact the Programs Department at fellowships@ams.org.

**Centennial Research Fellowship**

The AMS Centennial Fellowship Program makes an award annually to an outstanding mathematician to help further their career in research. One award will be made for the 2024–2025 academic year in the amount of US$50,000. Acceptance of the fellowship cannot be postponed. AMS membership will also be offered to the recipient for the duration of the fellowship.
About this fellowship. Eligibility: The eligibility rules are as follows:

The primary selection criterion for the Centennial Fellowship is the excellence of the candidate’s research.

- Preference will be given to candidates who have not had extensive fellowship support in the past.
- Recipients may not hold the Centennial Fellowship concurrently with another research fellowship such as a Sloan, NSF Postdoctoral fellowship, or CAREER award.
- Under normal circumstances, the fellowship cannot be deferred.
- A recipient of the fellowship shall have held his or her doctoral degree for at least three years and not more than twelve years at the inception of the award (that is, received between September 1, 2013, and September 1, 2022).
- Applications will be accepted from mathematicians currently holding a tenured, tenure-track, postdoctoral, or comparable (at the discretion of the selection committee) position at a US institution.

Applications should include a detailed research plan for the fellowship period that is contextualized by the research statement. The plan should include a description of how the fellowship will support the applicant’s success. It should be no more than one page. The selection committee will consider the plan in addition to the quality of the candidate’s research and will try to award the fellowship to those for whom the award would make a real difference in the development of their research careers. Work in all areas of mathematics, including interdisciplinary work, is eligible.

Application period. Applications will be collected via MathPrograms.org July 15, 2024–September 30, 2024 (11:59 p.m. ET). Find more information at https://www.ams.org/centfellow. For questions, contact the Programs Department at fellowships@ams.org.

Claytor-Gilmer Fellowship

The AMS established the Claytor-Gilmer Fellowship to further excellence in mathematics research and to help generate wider and sustained participation by Black mathematicians. One award will be made for the 2024–2025 academic year in the amount of US$50,000. AMS membership will also be offered to the recipient for the duration of the fellowship.

About this fellowship. Awardees may use the fellowship in any way that most effectively enables their research—for instance, for release time, participation in special research programs, travel support, childcare, etc. The award is issued through the recipient’s institution, and no part of it may be utilized for indirect costs. Given the aims of the fellowship, the most likely awardee will be a midcareer Black mathematician whose achievements demonstrate significant potential for further contributions to mathematics. Applications will be accepted from mathematicians currently holding a tenured, tenure-track, postdoctoral, or comparable (at the discretion of the selection committee) position at a US institution.

Application period. Applications will be collected via MathPrograms.org July 15, 2024–September 30, 2024 (11:59 p.m. ET). Find more information at https://www.ams.org/claytor-gilmer. For questions, contact the Programs Department at fellowships@ams.org.

Stefan Bergman Fellowship

The Stefan Bergman Fellowship was established in 2023 with the proceeds of the Stefan Bergman Trust to support the advancement of the research portfolio of a mathematician who specializes in the areas of real analysis, complex analysis, or partial differential equations. One award will be made for the 2024–2025 academic year in the amount of US$25,000. AMS membership will also be offered to the recipient for the duration of the fellowship.

About this fellowship. Applications will be accepted from mathematicians at a US institution who have not received tenure or comparable (at the discretion of the selection committee) and have not held significant fellowship support.

Awardees may use the fellowship in any way that most effectively enables their research—for instance, for release time, participation in special research programs, travel support, childcare, etc. The award is issued through the recipient’s institution, and no part of it may be utilized for indirect costs.

Application period. Applications will be collected via MathPrograms.org July 15, 2024–September 30, 2024 (11:59 p.m. ET). Find more information at https://www.ams.org/bergman-fellow. For questions, contact the Programs Department at fellowships@ams.org.
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—Notices of the American Mathematical Society

In February 2022, Notices of the AMS ran an article inviting applicants to the AMS’s first Business, Industry, and Government (BIG) Mathematics Research Community (MRC), to be held that summer.

The article was written by six of the seven organizers of this inaugural BIG MRC, who hailed from Los Alamos, Oak Ridge, and Pacific Northwest national laboratories. Unlike the summer research communities where academic researchers work with early-career mathematicians, these organizers explained, this MRC was designed to mimic the configuration of research teams found in business, industry, and government (or BIG) organizations.

“The most crucial characteristic of the applicants is the desire to build a community that is willing to teach and learn about other disciplines and to form true interdisciplinary teams,” the organizers wrote in 2022.

Fast-forward to March 2024, when a research team from that 2022 BIG MRC, led by Bill Kay (Pacific Northwest National Laboratory), published its research in *Nature: Scientific Reports*, titled “Community detection in hypergraphs via mutual information maximization.”

The report’s authors include Kay, Jürgen Kritschgau (Carnegie Mellon University), Daniel Kaiser (Indiana University), Oliver Alvarado Rodriguez (New Jersey Institute of Technology), Ilya Amburg (Pacific Northwest National Laboratory), Jessalyn Bolkema (California State University, Dominguez Hills), Thomas Grubb (University of California San Diego), Fangfei Lan (University of Utah), Sepideh Maleki (University of Texas at Austin), and Phil Chodrow (Middlebury College).

“Hypergraphs are networks that model multi-way relationships, like relationships between songs being grouped together by what playlists they appear on, or people being grouped together by what social clubs they participate in,” Kay said. “We used tools from compression and communications to identify community structure in real-world hypergraph data.”

And, he noted, “We are still an active, regularly meeting research team, and the MRC made it possible.”

Originally convening in June 2022 at Beaver Hollow Conference Center in western New York, the report’s research team has a mixed background of mathematics and computer science, with jobs in or aspirations of careers in labs, academia, and industry. “We had some hands-on keyboard programmers and also some chalk-to-chalkboard theoreticians, and all of these components were pretty relevant to the project,” Kay said.
“Our group collaborated very well, with the more mathematical minds focusing on the theoretical underpinnings of our new algorithm and those of us more interested in the computational side focusing on how to best translate the theory to code to be truly usable for real-world problems,” said Alvarado Rodriguez, currently a fourth-year computer science PhD candidate at NJIT.

What pulled these scientists together?

“The main draw to the MRC when I applied was the interdisciplinary/applied nature of the proposed problems. I did my PhD in graph theory from a pure math perspective and was very interested in learning about the network science perspective,” said Kritschgau, currently a postdoc in the Fariborz Maseeh Department of Mathematics and Statistics at Portland State University. “I was also in the middle of my first postdoc looking for good problems to work on.”

Bolkema, an assistant professor at California State University, Dominguez Hills, was looking for opportunities to reengage with research collaborators after starting a tenure-track position in 2020. “During peak pandemic years, when it felt like everyone was in survival mode,” she said. “I had always heard positive things about the MRC model and experience, so when this project/program was announced that seemed like a good fit, I jumped at the chance.”

For Kaiser, a current PhD student in the Department of Informatics at Indiana University, it was the topic of hypergraphs. “I had been toying with utilizing hypergraphs in another project and the thought of getting some hands-on experience working with hypergraphs alongside other researchers was too much to pass up,” he said.

Graph and hypergraph models are employed to understand the kinds of critical systems studied at the national laboratories: computer networks, infrastructure systems, systems biology, and social networks. By addressing several deep theoretical problems posed by the organizers, the BIG MRC participants sought to develop models and analytical methods to enable data-driven analysis of such systems.

Based on his involvement with math summer research programs as a participant and mentor, Kay said it’s impossible to predict which programs will yield publications. But, he said, “I think I can say that pretty quickly in, the team was very committed to getting enough stuff done in the MRC to create sufficient [momentum] to motivate continued work.”

Kaiser credits the MRC environment. “I hadn’t really worked with mathematicians before, mostly physicists turned network scientists. I was nervous I would be out of place as I also lean more towards a ‘scientist’ crowd than a ‘mathematician’ crowd. My experience put that anxiousness to rest and showed me how powerful collaborations like this could be.”

“The collaboration experience was great,” Kritschgau said. “This was my second MRC and the first one was entirely virtual, so I was really looking forward to getting to do an MRC in person. However, I tested positive for COVID the night before the flight. Bill and the rest of the group were very supportive in terms of making a second virtual MRC possible for me.”

Bolkema was similarly afflicted. “We were included very graciously by MRC organizers as Zoom attendees,” she recalled. “My collaboration experience was really positive despite this. We got carried around on laptops to all the different sessions, and the team we worked with really made an effort to keep us in the loop. I sometimes wonder if this hybrid start to our collaboration helped us transition to working remotely after the event more smoothly.”

Post-MRC, the research team maintained regular contact, using the Slack platform whenever developments occurred or decisions about the direction of the work were needed. “There were a few weeks where our chat was very active and people were posting all kinds of ideas, and plots from experiments,” Kritschgau said.

“Jessalyn hosted meetings on Zoom most weeks as well, so we would gather and chat about what we’ve been up to or work through some roadblock together,” Kaiser said.

“Expectations are really flexible—some weeks are well attended, some weeks less so, but we try to keep everyone in the loop,” Bolkema said. “We have managed to meet up at the Joint Meetings, and we got together for an intense research weekend in November to wrap up edits on our paper and kick off the next project.”

“Honestly,” Kay said, “we didn’t know we had a Nature: Scientific Reports paper until the first round of revisions came back.

“We were definitely aware that we had something good. There was strong theory, good empirical results, some novelty with enough tie-in to extant literature. All the factors were there, but we definitely were taking a shot.”

“Publishing original, high-quality research is one of the most influential factors in determining a mathematician’s career advancement,” said Sarah Bryant, AMS director of programs. “We are very proud to be able to provide the opportunity for the kind of research experience that sustains these connections.” She notes that the “BIG” in “BIG MRC” has now expanded to BEGIN (Business, Entrepreneurship, Government, Industry, and Nonprofit).

Bryant, who collects information from past MRC participants, said, “The data show us that MRCs launch many productive collaborations, resulting in journal publications, conference proceedings, and other scholarly works. In
fact, since 2011, when we started collecting this publication data, there have been 320 publications reported.” As of fall 2023, she said, the 2022 cohort alone had produced 21 papers in various stages of preparation.

This BIG MRC group was not the first MRC of its year to publish a paper, but Kay’s team’s success out of the gate was inspiring.

“I really am a big believer in opportunities like the MRC for early-career mathematicians to get a chance to work together and bolster their reputations,” he said. “The team that came together really, truly actualized this paper in phenomenal form.”

“We were very happy to share this new algorithm with everyone interested in it, and hope it will be usable for others, and that it will inspire further research in the hypergraph community detection and compression space,” said Alvarado Rodriguez.

“It’s always exciting when your work is published, of course, but I found this particularly exciting,” Kaiser said. “My MRC collaborators are like a bunch of free friends, and having us all work on something so neat together and it be reviewed and accepted by other scientists is a great feeling.”

Credits
Figure 1 is courtesy of Oliver Alvarado Rodriguez.

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Honoring Aziz: Memorial Session Held at Spring Eastern Sectional Meeting

Elaine Beebe

At an AMS Spring Eastern Sectional Meeting in early April, Zhisheng Shuai of the University of Central Florida presented his work on global stability in infectious disease models.

Shuai spoke at a special session called “Mathematics of Infectious Disease: A Session in Memory of Dr. Abdul-Aziz Yakubu.” And he did so “at Howard University, which makes this really special,” Shuai said.

Yakubu, who died in 2022 at age 64, was a professor of mathematics at Howard, where he chaired the department for a decade. Born in Ghana, Yakubu held visiting positions at Cornell University, North Carolina State University (where he earned his PhD), Ohio State University, and Botswana International University of Science and Technology. He lectured widely in North America, Africa, Asia, and Europe on his research in mathematical applications to the biological sciences and was a devoted mentor who championed greater diversity and inclusion in the field of mathematics.

At the special session, colleagues remembered Yakubu as a very private person who was social with his mathematics. “He also was so dedicated that he worked on math and supporting math colleagues while in the hospital,” recalled Howard mathematics professor Katharine Gurski.

“It was an honor and privilege to have had the opportunity to collaborate with Dr. Yakubu,” said Chadi Saad-Roy, co-organizer of the special session and Miller Research Fellow, University of California, Berkeley. “I am deeply grateful. It was a wonderful experience.”

“Aziz was my close friend and brother,” said Abba Gumel, co-organizer of the session and the Michael and Eugenia Brin Endowed E-Nnovate Chair in Mathematics at the University of Maryland. Gumel, Saad-Roy, and Daniel Cooney (University of Illinois Urbana-Champaign) proposed the special session to organizers of the Sectional Meeting.

“Mathematics of Infectious Disease” convened speakers from the US, Mexico, and the UK, who presented the latest advances in the theory and methodology for studying the dynamics of real-world phenomena arising in population biology. They included graduate students, postdocs, up-and-coming faculty, and experts in the field of mathematical biology and in areas from epidemiology to ecology to immunology. Most had ties to Yakubu.

“I took the organizers at their word,” said mathematical epidemiologist John Glasser from the Centers for Disease Control, who opted to present work created with Yakubu: a report commissioned by the government of Jamaica modeling the transmission of SARS-CoV-2, with Zhilan Feng (Purdue University/NSF, also a presenter in attendance).
“That’s what Aziz did; he was an advocate of discrete time models,” Glasser said, adding that in Jamaica during the pandemic, “Aziz had a critical role here. He was the voice of reason.”

Gurski presented “Injectable PrEP and the Emergence of Drug-Resistant HIV Strains in Acute Infections.” “I made a connection between my work and work by Jim Cushing, a longtime friend and collaborator of Aziz’s,” she noted. “The particular paper I was referencing was from a journal volume honoring Aziz.”

“Mathematics of Infectious Disease” was highly interactive. The audience provided engaged and constructive feedback to the speakers for a hearty cross-pollination of ideas.

“We were lucky to have many junior presenters in the session, which honored Dr. Yakubu’s support and advocacy for junior colleagues,” Saad-Roy said. These included Soyoung Park, a PhD student of Gumel’s at the University of Maryland, who delivered her first-ever conference talk, about the effects of vaccination and Pap screening on HPV and cancer.

“Although I had no prior connection to Dr. Yakubu, I was drawn to his work after hearing presentations on it during the session,” Park said. “One of the presenters recommended his paper on Bovine Babesiosis using a PDE model, which I saved to read later.

“The COVID models that incorporated behavior, addressed by several speakers, were also impressive, and I got interested in reading more papers on the topic,” she added. In the works is an edited volume in the AMS Contemporary Mathematics series of research presented at the special session.

“One of the major highlights was seeing the truly diverse, and broad, range of topics that Dr. Yakubu contributed to, from very theoretical work to advising policy,” Saad-Roy said. “Another major highlight were the personal touches that multiple presenters included in honor of Dr. Yakubu,” such as when Shuai advanced his slide deck to show a photo of Yakubu, poised and smiling, then proceeded to cite his late colleague’s work in convergence.

Gumel added, “I was also very impressed with the hospitality the Howard University community, particularly the math department, accorded to us all.” Attending parts of the session were Bourama Toni, professor and chair of Howard’s mathematics department, and Tepper Gill, professor in the Department of Electrical Engineering and Computer Science, as well as several Howard graduate students.

For her part, Gurski and her Howard colleagues are continuing Yakubu’s Math in Medicine program, which connects the math department to Howard University Hospital. “Aziz strongly believed that his math should be directly beneficial to Black Americans and to Africa,” Gurski said. “He also believed that Howard should be a lead in this. He was very dedicated to Howard.”

And at Howard that April weekend, mathematicians appreciated the chance to remember an esteemed colleague through the sharing of mathematics.

“I sincerely thank Abba and his colleagues for organizing this special session to commemorate Aziz and honor his memory,” Shuai said. “It has provided an opportunity for me, along with many others, to reflect on Aziz’s contributions, follow his work, and remember our interactions.”

Gurski agreed. “A math research session in honor of Aziz is a way we could talk about his achievements and how his mathematics and his life impacted all of us.”

Credits
Figure 1, Figure 2, and Figure 3 are courtesy of AMS Communications.
WASHINGTON DELIVERS 2024 EINSTEIN PUBLIC LECTURE

Two data points stood out among the figures presented by Talitha Washington during the AMS Einstein Public Lecture, April 6, 2024, at Howard University.

The first was that the United States will be shorthanded three million workers in STEM fields by 2030. The second: African Americans comprise 12 percent of the US population but only three percent of data analytics professionals.

These numbers are a problem, Washington told the audience. “And as a mathematician, I like to solve problems.”

Washington is director and lead principal investigator of the National Data Science Alliance (NDSA) at Clark Atlanta University, where she is a professor of mathematics.

Funded by the National Science Foundation in 2022, Washington leads a national alliance of Historically Black Colleges and Universities (HBCUs) seeking to increase the number of Black people with expertise in data science by at least 20,000 by 2027.

She also directs the four-year-old Data Science Initiative, Atlanta University Center (AUC) Consortium, which aims to expand data science research and curriculum across HBCUs, stewarded by AUC institutions (Clark Atlanta University, Morehouse College, Morehouse School of Medicine, and Spelman College).

And in 2001, “the year I graduated, six Black women got PhDs in math,” she said. “I think I know all of them.”

The Einstein Public Lecture was featured at the well-attended AMS 2024 Spring Sectional Meeting, hosted at Howard. During her lecture, Washington showed a short film of activities in data science that have drawn students as young as precollege.

She also spoke at length about the need for educational collaboration with industry and entrepreneurs. “We need to bring industry knowledge to classrooms,” Washington said. “For the continuity of the workforce; half of students are going into industry, et cetera, not academia.”

—AMS Communications

CONGRESSIONAL BRIEFING EXPLORES RANDOMIZATION AND COLLECTIVE INTELLIGENCE

On April 18, 2024, the American Mathematical Society (AMS) and the Simons Laufer Mathematical Sciences Institute (SLMath) held a joint Congressional briefing titled “Collective Intelligence: How Local Interactions Determine Global Coordination.” This Capitol Hill presentation was given by Dana Randall, co-executive director of the Institute for Data Engineering and Science, ADVANCE Professor of Computing, and adjunct professor of mathematics at the Georgia Institute of Technology.

Randall explained to Congressional staff and other attendees how the mathematical concept of randomness is a key ingredient connecting the simple, local actions of individuals with the complex, emergent outcomes of intelligent collectives, providing innovative pathways for advancing technology, medicine, and science. She described how randomness may be the key to fire ants forming living bridges, how robot swarms collectively accomplish tasks, and how next-generation materials can be programmed to adapt to environmental cues.

The AMS holds annual Congressional briefings as a means to communicate information to policymakers. Speakers bring science directly to Capitol Hill decision-makers and discuss how federal investment in basic research in math and science pays off for American taxpayers and helps the nation remain a world leader in innovation. More information on these briefings can be found at https://www.ams.org/government/dc-outreach.

—AMS Office of Government Relations
Artificial Intelligence is Theme of JMM 2025

Artificial intelligence—AI—is an official theme of the 2025 Joint Mathematics Meetings, announced the AMS Advisory Group on Artificial Intelligence and the Mathematical Community.

The topic of AI will be reflected in talks, sessions, and panels at JMM 2025, to be held January 8–11 in Seattle.

“AI has become a part of all aspects of our work and our personal lives, and the only reason this has happened is due to the mathematical developments behind AI,” said AMS President Bryna Kra, a member of the advisory group.

“Going forward, it is imperative that the science behind AI continue to be a part of the conversation, and that practitioners of AI be trained in the mathematical tools behind the theory.”

“With the AI theme at the JMM, we will highlight the theory, practice, and implications for our profession.”

In 2023, the AMS formed the ad hoc Advisory Group on Artificial Intelligence and the Mathematical Community to focus on issues at the AI forefront. These include the role of mathematics in the development and deployment of AI; the impact of AI on research in mathematics; the use of AI in publications, education, and research; and the effects of AI on the mathematical community.

The advisory group welcomes your input at https://www.ams.org/artificial-intelligence.

—AMS Communications
Mathematics People

Wallenberg Foundation Funds 18 Mathematicians

Eighteen mathematicians received SEK 29 million (US$2.6 million) in research funding from the Knut and Alice Wallenberg Foundation, the largest private financier of research in Sweden.

Through the foundation’s mathematics program, both younger and more experienced senior mathematicians are recruited to Sweden every year, and young Swedish mathematicians are given the opportunity to travel through international postdoctoral positions. Swedish universities with a mathematics department nominate candidates for the program, which are then evaluated by the Royal Swedish Academy of Sciences.

Six researchers received international postdoctoral positions and funding for two years after they return to Sweden:

- Jiacheng Xia, Chalmers University of Technology (University of Wisconsin, Madison)
- Josefien Kuijper, Stockholm University (University of Toronto)
- Stefan Reppen, Stockholm University (University of California, Berkeley)
- Linnéa Gyllingberg, Uppsala University (University of California, Los Angeles)
- Robin Stoll, Stockholm University (University of Cambridge, UK)
- Louis Hainault, Stockholm University (University of Chicago)

Five researchers received grants to recruit a foreign researcher for a postdoctoral position in Sweden:

- Eusebio Gardella, University of Gothenburg
- Gustavo Jasso Ahuja, Lund University
- Danijela Damjanović, KTH Royal Institute of Technology
- Christian Johansson, University of Gothenburg
- Klas Modin, Chalmers University of Technology

Seven established researchers from outside Sweden will be visiting professors at Swedish universities (in parentheses):

- Christina Lienstromberg, University of Stuttgart, Germany (Lund University)
- Rita Pardini, University of Pisa, Italy (Stockholm University)
- Jonathan Breuer, Hebrew University of Jerusalem, Israel (KTH Royal Institute of Technology)
- Mario Wüthrich, ETH Zurich, Switzerland (Stockholm University)
- Ezra Getzler, Northwestern University (Uppsala University)
- Marco Martens, Stony Brook University (Uppsala University)
- Steffen Rohde, University of Washington (KTH Royal Institute of Technology)

The mathematics program is a long-term investment from the foundation, granting SEK 650 million (US$59.2 million) to Swedish mathematics research between 2014 and 2030. Including this round, 152 researchers have received funding through the program to date.

—Knut and Alice Wallenberg Foundation

Hicks Receives 2024 Blackwell-Tapia Prize

Illya Hicks, professor and department chair of computational applied mathematics and operations research at Rice University, was awarded the 2024 Blackwell-Tapia Prize. The prize will be presented at the twelfth Blackwell-Tapia Conference and Award Ceremony on November 15-16, 2024, at the Institute for Computational and Experimental Research in Mathematics (ICERM), Brown University.

Hicks is a leading researcher of combinatorial optimization, graph theory, and integer programming. He earned his PhD in computational and applied mathematics from Rice University in 2000. Hicks served as a professor of
Amir Mohammadi, University of California San Diego (UCSD), was awarded the thirteenth Michael Brin Prize in Dynamical Systems on April 6, 2024, at the 2024 Spring Dynamical Systems Conference at the University of Maryland. The prize is given for specific contributions and carries a cash award of US$18,000.

Mohammadi was recognized for his fundamental contributions to effective counting and equidistribution in Teichmüller and homogeneous dynamics, according to a news release.

Mohammadi received his PhD in 2009 from Yale University. He is a former L. E. Dickson Instructor at the University of Chicago, assistant professor at the University of Texas at Austin, and von Neumann Fellow at the Institute for Advanced Study (IAS). He joined the UCSD faculty in 2016 and was named professor in 2020.

—Michael Brin Prize

Three Win 2024 Rollo Davidson Prize

The Rollo Davidson Trustees awarded the Rollo Davidson Prize for 2024 jointly to Pierre-Francois Rodriguez (Imperial College London), Tianyi Zheng (University of California San Diego), and Ilya Chevyrev (University of Edinburgh), according to a news release.

Rodriguez was honored for his outstanding work on percolation of the level sets of the Gaussian free field and the vacant set of random walks and random interlacements and on the fluctuations of the discrete Gaussian model.

Zheng was honored for her deep results and resolution of long-standing conjectures on random walks on groups.

Chevyrev was recognized for his contributions to rough analysis and singular stochastic PDEs, “and in particular to our understanding of the 2D Yang-Mills measure,” according to a press release.

Based at the University of Cambridge, UK, the Rollo Davidson Trust has awarded an annual prize to one or more young probabilists since 1976.

—Rollo Davidson Trust

Krieg Awarded 2024 Traub Prize

David Krieg, University of Passau (Germany), was awarded the 2024 Joseph F. Traub Prize for Achievement in Information-Based Complexity (IBC). Krieg will receive $3,000, sponsored by Elsevier, and a plaque at the Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing Conference (MCQMC) in Waterloo, Canada, in August.

“Krieg works in the fields of information-based complexity, approximation theory, and data science, with a special interest in optimal deterministic or randomized algorithms for high-dimensional problems such as function recovery and numerical integration,” according to a news release.

March 31, 2025, is the nomination deadline for the 2025 Joseph F. Traub Prize for Achievement in IBC. A nomination can be based on work done in a single year, a number of years, or over a lifetime. It can be published in any journal, number of journals, or monographs. Send nominations to Erich Novak at erich.novak@uni-jena.de.

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New Books Offered by the AMS

Algebra and Algebraic Geometry

Categorical, Combinatorial and Geometric Representation Theory and Related Topics

Pramod N. Achar, Louisiana State University, Baton Rouge, LA.
Kailash C. Misra, North Carolina State University, Raleigh, NC, and
Daniel K. Nakano, University of Georgia, Athens, GA, Editors

This book is the third Proceedings of the Southeastern Lie Theory Workshop Series covering years 2015–21. During this time five workshops on different aspects of Lie theory were held at North Carolina State University in October 2015; University of Virginia in May 2016; University of Georgia in June 2018; Louisiana State University in May 2019; and College of Charleston in October 2021.

Some of the articles by experts in the field describe recent developments while others include new results in categorical, combinatorial, and geometric representation theory of algebraic groups, Lie (super) algebras, and quantum groups, as well as on some related topics.

The survey articles will be beneficial to junior researchers. This book will be useful to any researcher working in Lie theory and related areas.

Proceedings of Symposia in Pure Mathematics, Volume 108

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Applications

Quantum Computation and Quantum Information
A Mathematical Perspective

J. M. Landsberg, Texas A&M University, College Station, TX

This book presents the basics of quantum computing and quantum information theory. It emphasizes the mathematical aspects and the historical continuity of both algorithms and information theory when passing from classical to quantum settings.

The book begins with several classical algorithms relevant for quantum computing and of interest in their own right. The postulates of quantum mechanics are then presented as a generalization of classical probability. Complete, rigorous, and self-contained treatments of the algorithms of Shor, Simon, and Grover are given. Passing to quantum information theory, the author presents it as a straightforward adaptation of Shannon’s foundations to information theory. Both Shannon’s theory and its adaptation to the quantum setting are explained in detail. The book concludes with a chapter on the use of representation theory in quantum information theory. It shows how all known entropy inequalities, including the celebrated strong subadditivity of von Neumann entropy, may be obtained from a representation theory perspective.

With many exercises in each chapter, the book is designed to be used as a textbook for a course in quantum computing and quantum information theory. Prerequisites are elementary undergraduate probability and undergraduate algebra, both linear and abstract. No prior knowledge of quantum mechanics or information theory is required.

Graduate Studies in Mathematics, Volume 243
September 2024, 204 pages, Hardcover, ISBN: 978-1-4704-7557-4, LC 2024009659, 2020 Mathematics Subject Classification: 81P45, 81P68, 68Q12, 94A15, 20G05, 94A24,
General Interest

Your Daily Epsilon of Math Wall Calendar 2025
Rebecca Rapoport, Harvard University, Cambridge, MA, and Michigan State University, East Lansing, MI, and Dean Chung, Harvard University, Cambridge, MA, and University of Michigan, Ann Arbor, MI

Keep your mind sharp all year long with Your Daily Epsilon of Math Wall Calendar 2025 featuring a new math problem every day and 13 beautiful math images! Let mathematicians Rebecca Rapoport and Dean Chung tickle the left side of your brain by providing you with a math challenge for every day of the year. The solution is always the date, but the fun lies in figuring out how to arrive at the answer, and possibly discovering more than one method of arriving there.

Problems run the gamut from arithmetic through graduate level math. Some of the most tricky problems require only middle school math applied cleverly. With word problems, math puns, and interesting math definitions added into the mix, this calendar will intrigue you for the whole year.

End the year with more brains than you had when it began with Your Daily Epsilon of Math Wall Calendar 2025.

June 2024, 14 pages, ISBN: 978-1-4704-7800-1, 2020 Mathematics Subject Classification: 00A07, 00A09, 00A06, 00A08, 00A05, List US$20, AMS members US$16, MAA members US$18, Order code MBK/151

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What's Happening in the Mathematical Sciences, Volume 13
Dana Mackenzie and Leila Sloman

The What’s Happening in the Mathematical Sciences series presents a selection of recent discoveries and exciting fields of research in mathematics, explained in depth but in a slow-paced, reader-friendly way.

In the first few months of 2023, artificial “brains” like ChatGPT and GPT-4 were constantly in the news, and they have already turned into big business. One chapter in this book, “Deep Learning: Part Math, Part Alchemy”, explains how math disentangles hype from reality and explains some of the remarkable advances of machine learning. Meanwhile, “Organizing the Chaos Inside the Brain” explores animal brains, and describes how biologists can apply chaos theory to simulate the wanderings of a fly from firing data on neurons within its brain.

This issue of What’s Happening also includes many treats for readers who like pure math—especially those who are interested in geometry. In recent months and years, there have been unexpected discoveries in tiling (“One Stone to Rule Them All”), sphere-packing in more than three dimensions (“A Fascination of Spheres”) and the reconstruction of three-dimensional scenes from two-dimensional images (“Multi-View Geometry: E Pluribus Unum”). The chapter “How to Draw an Alternate Universe” will, as promised, open a door to a completely different, non-Euclidean universe—or several of them. Shakespeare’s words, “something rich and strange”, only begin to describe them.

In “How Mathematicians Unearthed the Stubborn Secrets of Fano Varieties”, readers will learn about one of the building blocks of algebraic geometry, the branch of geometry that deals with surfaces defined by polynomial equations. The chapter “Missing One Digit” addresses a seemingly elementary problem in number theory: how many prime numbers do not have a “7” in them? The answer is easy to guess—but hard to prove. “Fluid Flow: Two Paths to a Singularity” discusses another guess that is hard to prove: can fluids in an enclosed region develop “singularities” akin to a breaking wave? Computer evidence is mounting that they can—including some evidence from machine learning algorithms. (Which brings us full circle back to the “Deep Learning” chapter.)

Dana Mackenzie has written for the What’s Happening series since Volume 6, published in 2006. In this volume he is joined by Leila Sloman, whose name will be familiar to many readers from her work for Quanta Magazine.

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NEW BOOKS

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New in Contemporary Mathematics

Algebra and Algebraic Geometry

Moduli Spaces and Vector Bundles—New Trends
Peter Gothen, Universidade do Porto, Portugal, Margarida Melo, Università Roma Tre, Rome, Italy, and Montserrat Teixidor i Bigas, Tufts University, Medford, MA, Editors
This volume contains the proceedings of the VBAC 2022 Conference on Moduli Spaces and Vector Bundles—New Trends, held in honor of Peter Newstead’s 80th birthday, from July 25–29, 2022, at the University of Warwick, Coventry, United Kingdom.

The papers focus on the theory of stability conditions in derived categories, non-reductive geometric invariant theory, Brill-Noether theory, and Higgs bundles and character varieties. The volume includes both survey and original research articles. Most articles contain substantial background and will be helpful to both novices and experts.

Contemporary Mathematics, Volume 803
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What’s Happening in the Mathematical Sciences, Volume 13
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Math Education

Navigating the Math Major Charting Your Course
Carrie Diaz Eaton, Bates College, Lewiston, ME, Allison Henrich, Seattle University, WA, Steven Klee, Amazon Web Services, Seattle, WA, and Jennifer Townsend, Microsoft, Redmond, WA

Are you a mathematics major or thinking about becoming one? This friendly guidebook is for you, no matter where you are in your studies. For those just starting out, there are: interactive exercises to help you chart your personalized course, brief overviews of the typical courses you will encounter during your studies, recommended extracurricular activities that can enrich your mathematical journey.

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Students thinking about life after graduation will find: advice for seeking jobs outside academia, guidance for applying to graduate programs, a collection of interviews with former mathematics majors now working in a wide variety of careers—they share their experience and practical advice for breaking into their field.

Packed with a wealth of information, Navigating the Math Major is your comprehensive resource to the undergraduate mathematics degree program.

Classroom Resource Materials, Volume 73
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Higher Structures in Topology, Geometry, and Physics

Ralph M. Kaufmann, Purdue University, West Lafayette, IN,
Martin Markl, Czech Academy of Sciences, Prague, Czech Republic,
and Alexander A. Voronov, University of Minnesota, Minneapolis, MN, Editors

This volume contains the proceedings of the AMS Special Session on Higher Structures in Topology, Geometry, and Physics, held virtually on March 26–27, 2022.

The articles give a snapshot survey of the current topics surrounding the mathematical formulation of field theories. There is an intricate interplay between geometry, topology, and algebra which captures these theories. The hallmarks are higher structures, which one can consider as the secondary algebraic or geometric background on which the theories are formulated. The higher structures considered in the volume are generalizations of operads, models for conformal field theories, string topology, open/closed field theories, BF/BV formalism, actions on Hochschild complexes and related complexes, and their geometric and topological aspects.

This item will also be of interest to those working in geometry and topology and mathematical physics.

Contemporary Mathematics, Volume 802
bookstore.ams.org/conm-802

New in Memoirs of the AMS

Algebra and Algebraic Geometry

Stratified Noncommutative Geometry

David Ayala, Montana State University, Bozeman, Montana,
Aaron Mazel-Gee, California Institute of Technology, Palo Alto, California, and Nick Rozenblyum, University of Chicago, Illinois

Memoirs of the American Mathematical Society, Volume 297, Number 1485
bookstore.ams.org/memo-297-1485

Cubical Models of (∞,1)-Categories

Brandon Doherty, University of Western Ontario, London, Ontario, Canada, Krzysztof Kapulkin, University of Western Ontario, London, Ontario, Canada, Zachary Lindsey, and Christian Sattler

Memoirs of the American Mathematical Society, Volume 297, Number 1484
bookstore.ams.org/memo-297-1484

A Plethora of Cluster Structures on GL_n

M. Gekhtman, University of Notre Dame, Indiana, M. Shapiro, Michigan State University, East Lansing, Michigan, and A. Vainshtein, University of Haifa, Israel

Memoirs of the American Mathematical Society, Volume 297, Number 1486
bookstore.ams.org/memo-297-1486

Simple Supercuspidal L-Packets of Quasi-Split Classical Groups

Masao Oi, Kyoto University, Japan

This item will also be of interest to those working in number theory.

Memoirs of the American Mathematical Society, Volume 297, Number 1483
bookstore.ams.org/memo-297-1483
Number Theory

On Refined Conjectures of Birch and Swinnerton-Dyer Type for Hasse–Weil–Artin L-Series

David Burns, King’s College London, United Kingdom, and Daniel Macias Castillo, Universidad Autónoma de Madrid, Spain, and Instituto de Ciencias Matemáticas, Madrid, Spain

Memoirs of the American Mathematical Society, Volume 297, Number 1482

New AMS-Distributed Publications

Algebra and Algebraic Geometry

Conformally Invariant Differential Operators on Heisenberg Groups and Minimal Representations

J. Frahm, Aarhus University, Denmark

For a simple real Lie group $G$ with Heisenberg parabolic subgroup $P$, the author studies the corresponding degenerate principal series representations. For a certain induction parameter the kernel of the conformally invariant system of second order differential operators constructed by Barchini, Kable, and Zierau is a subrepresentation which turns out to be the minimal representation. To study this subrepresentation, the author takes the Heisenberg group Fourier transform in the non-compact picture and shows that it yields a new realization of the minimal representation on a space of $L^2$-functions. The Lie algebra action is given by differential operators of order $\leq 3$ and the author finds explicit formulas for the functions constituting the lowest $K$-type.

These $L^2$-models were previously known for the groups $SO(n,n)$, $E_6(6)$, $E_7(-7)$, and $E_8(8)$ by Kazhdan and Savin, for the group $G_2(2)$ by Gelfand, and for the group $SL(3,\mathbb{R})$ by Torasso, using different methods. This new approach provides a uniform and systematic treatment of these cases and also constructs new $L^2$-models for $E_6(2)$, $E_7(-5)$, and $E_8(-24)$ for which the minimal representation is a continuation of the quaternionic discrete series, and for the groups $SO(p,q)$ with either $p \geq q = 3$ or $p, q \geq 4$ and $p + q$ even.

As a byproduct of our construction, the author finds an explicit formula for the group action of a non-trivial Weyl group element that, together with the simple action of a parabolic subgroup, generates $G$.

This item will also be of interest to those working in differential equations.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 180

Analysis

On Efficient Algorithms for Computing Near-Best Polynomial Approximations to High-Dimensional, Hilbert-Valued Functions from Limited Samples

Ben Adcock, Simon Fraser University, Burnaby, BC, Canada, Simone Brugiapaglia, Concordia University, Montreal, QC, Canada, Nick Dexter, Simon Fraser University, Burnaby, BC, Canada, and Sebastian Moraga, Simon Fraser University, Burnaby, BC, Canada

Sparse polynomial approximation is an important tool for approximating high-dimensional functions from limited samples—a task commonly arising in computational science and engineering. Yet, it lacks a complete theory. There is a well-developed theory of best $\gamma$-term polynomial approximation, which asserts exponential or algebraic rates of convergence for holomorphic functions.

There are also increasingly mature methods such as (weighted) $\ell^2$-minimization for practically computing such approximations. However, whether these methods achieve...
**NEW BOOKS**

the rates of the best s-term approximation is not fully understood. Moreover, these methods are not algorithms per se, since they involve exact minimizers of nonlinear optimization problems.

This paper closes these gaps by affirmatively answering the following question: Are there robust, efficient algorithms for computing sparse polynomial approximations to finite- or infinite-dimensional, holomorphic, and Hilbert-valued functions from limited samples that achieve the same rates as the best s-term approximation?

The authors do so by introducing algorithms with exponential or algebraic convergence rates that are also robust to sampling, algorithmic, and physical discretization errors. Their results involve several developments of existing techniques, including a new restarted primal-dual iteration for solving weighted ℓ1-minimization problems in Hilbert spaces. Their theory is supplemented by numerical experiments demonstrating the efficacy of these algorithms.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Memoirs of the European Mathematical Society, Volume 13**
bookstore.ams.org/emsmem-13

**Differential Equations**

**Massless Phases for the Villain Model in d ≥ 3**

**Paul Dario**, Université Paris-Est Créteil, France, and **Wei Wu**, NYU Shanghai, China

A major open question in statistical mechanics, known as the Gaussian spin wave conjecture, predicts that the low temperature phase of the Abelian spin systems with continuous symmetry behave like Gaussian free fields. In this paper the authors consider the classical Villain rotator model in ℤd, d ≥ 3, at sufficiently low temperature and prove that the truncated two-point function decays asymptotically as |x|−d, with an algebraic rate of convergence.

The authors also obtain the same asymptotic decay separately for the transversal two-point functions. This quantifies the spontaneous magnetization result for the Villain model at low temperatures and constitutes a first step toward a more precise understanding of the spin-wave conjecture. The authors believe that their method extends to finite range interactions and to other Abelian spin systems and Abelian gauge theory in d ≥ 3. They also develop a quantitative perspective on homogenization of uniformly convex gradient Gibbs measures.

This item will also be of interest to those working in probability and statistics.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 447
April 2024, 221 pages, Softcover, ISBN: 978-2-85629-985-2, 2020 Mathematics Subject Classification: 82B20, 82B41, 35B27, 35J08, List US$81, AMS members US$64.80, Order code AST/447
bookstore.ams.org/ast-447

**Math Education**

**Introduction to Number Theory in Mathematics Contests**

**Book 2**

**Titu Andreescu**, University of Texas at Dallas, Richardson, Texas and **Marian Tetiva**

The second volume, Book 2, of Introduction to Number Theory in Mathematics Contests starts with focusing on the most important classical, basically polynomial congruences, and arithmetic functions. It features beautiful problems with unique and interesting results, such as the Erdős-Ginzburg-Ziv theorem (stating that among any 2n − 1 integers, one can find n whose sum is divisible by n), and also some other classical results arising from the Prime Number Theorem.

The important (because of its many applications) “lifting the exponent” lemma is present in the book as well along with the beautiful theorem of Lucas about binomial coefficients modulo a prime, Lagrange’s theorem on the number of solutions of a polynomial congruence modulo a prime, and Gauss’s theorem about the existence/non-existence of primitive roots modulo an arbitrary positive integer.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 53
March 2024, 239 pages, Hardcover, ISBN: 979-8-98905286-8, 2020 Mathematics Subject Classification: 00A05, 00A07,
NEW BOOKS

121 Number Theory Problems for Mathematics Competitions

Titu Andreescu, University of Texas at Dallas, Richardson, Texas, and Alessandro Ventullo, University of Milan, Italy

Even though there are plenty of number theory books for students preparing for mathematics competitions, many are either outdated or actually addressed to experts in the field. Therefore, this book is aimed at those who have little or moderate knowledge of the subject.

The book follows almost the same line of the lectures of the level one number theory course taught by the second author at the AwesomeMath Summer program (AMSP) combined with the vast problem-solving experience and style of the first author. The book contains many examples, proposed problems followed by solutions, and a selection from many recent mathematical contests and Olympiads, journals, as well as tests given to students during recent years at AMSP.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

XYZ Series, Volume 52
March 2024, 237 pages, Hardcover, ISBN: 979-8-9890528-4-4, 2020 Mathematics Subject Classification: 00A05, 00A07, 97U40, 97D50, List US$59.95, AMS members US$47.96, Order code XYZ/53

bookstore.ams.org/xyz-53

Probability and Statistics

Statistical Mechanics of Mean-Field Disordered Systems
A Hamilton–Jacobi Approach

Tomas Dominguez, University of Toronto, ON, Canada, and Jean-Christophe Mourrat, École Normale Supérieure de Lyon, France

The goal of this book is to present new mathematical techniques for studying the behavior of mean-field systems with disordered interactions. The authors mostly focus on certain problems of statistical inference in high dimension and on spin glasses. The techniques they present aim to determine the free energy of these systems, in the limit of large system size, by showing that they asymptotically satisfy a Hamilton–Jacobi equation.

The first chapter is a general introduction to statistical mechanics with a focus on the Curie–Weiss model. The authors give a brief introduction to convex analysis and large deviation principles in Chapter 2 and identify the limit free energy of the Curie–Weiss model using these tools. In Chapter 3, they define the notion of viscosity solution to a Hamilton–Jacobi equation and use it to recover the limit free energy of the Curie–Weiss model. The authors discover technical challenges to applying the same method to generalized versions of the Curie–Weiss model and develop a new selection principle based on convexity to overcome these. They then turn to statistical inference in Chapter 4, focusing on the problem of recovering a large symmetric rank-one matrix from a noisy observation, and they see that the tools developed in the previous chapter apply to this setting as well.

Chapter 5 is preparatory work for a discussion of the more challenging case of spin glasses. The first half of this chapter is a self-contained introduction to Poisson point processes, including limit theorems on extreme values of independent and identically distributed random variables. The authors finally turn to the setting of spin glasses in Chapter 6. For the Sherrington–Kirkpatrick model, they show how to relate the Parisi formula with the Hamilton–Jacobi approach. They conclude with a more informal discussion on the status of current research for more challenging models.

This item will also be of interest to those working in differential equations.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Zurich Lectures in Advanced Mathematics, Volume 32

bookstore.ams.org/emszlec-32
Meetings & Conferences of the AMS
August Table of Contents

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Paid meeting registration is required to submit an abstract to a sectional meeting.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated. The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to https://www.ams.org/meetings/meetings-general for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Meeting in this Issue

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                   | (JMM 2026)                               | 988 |
| March 7–8          | Boise, Idaho                            | 988 |
| April 18–19        | Fargo, North Dakota                     | 988 |

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at https://www.ams.org/welcoming-environment-policy.
IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Palermo, Italy

**July 23–26, 2024**

*Tuesday – Friday*

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

San Antonio, Texas

*University of Texas, San Antonio*

**September 14–15, 2024**

*Saturday – Sunday*

**Meeting #1198**

Central Section

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: Volume 45, Issue 4

Deadlines

For organizers: Expired

For abstracts: July 23, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

**Invited Addresses**


Jason R Schweinsberg, University of California San Diego, *Using coalescent theory to analyze genetic data from growing tumors.*
**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

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**Additive Number Theory and Modular Forms I** (Code: SS 17A), **Debanjana Kundu**, University of Texas - Rio Grande Valley, and **Brandt Kronholm**, University of Texas Rio Grande Valley.


**Advances in Mathematical and Numerical Analysis of Partial Differential Equations for Application-Oriented Computations I** (Code: SS 25A), **Bruce A Wade**, University of Louisiana at Lafayette, **Qin Sheng**, Baylor University, **Abdul Q.M. Khaliq**, Middle Tennessee State University, **JaEun Ku**, Oklahoma State University, and **Xiang-Sheng Wang** and **Yangwen Zhang**, University of Louisiana at Lafayette.

**Applications of Algebraic Geometry I** (Code: SS 23A), **Frank Sottile**, Texas A&M University, **Alperen Ergur**, University of Texas at San Antonio, and **Anne Joyce Shiu**, Texas A&M University.

**Applications of analysis, topology and set theory to model theory I** (Code: SS 33A), **Eduardo Dueñez** and **Jose N Iovino**, The University of Texas at San Antonio.

**Applications of Probability in Biology I** (Code: SS 8A), **Jason R Schweinsberg**, University of California San Diego.

**A Showcase of Algebraic Geometry at Undergraduate Institutions I** (Code: SS 22A), **David Swinarski**, Fordham University, **Julie Rana**, Lawrence University, and **Han-Bom Moon**, Fordham University.

**Commutative algebra and connections to combinatorics I** (Code: SS 28A), **Michael Robert DiPasquale**, **Louiza Fouli**, and **Arvind Kumar**, New Mexico State University.

**Differential Geometry** (Code: SS 1A), **Alvaro Pampano**, Texas Tech University, **Bogdan D. Suceava**, California State University Fullerton, and **Magdalena Daniela Toda**, Texas Tech University and **Prabir Roychowdhury**, School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, and **William R Ott**, University of Houston.

**Enumerative Combinatorics I** (Code: SS 16A), **Brian K. Miceli**, Trinity University, and **Lara Pudwell**, Valparaiso University.

**Geometric Group Theory and Low-Dimensional Topology I** (Code: SS 37A), **George Domat** and **Khanh Le**, Rice University.

**Graph Theory I** (Code: SS 15A), **Youngho Yoo** and **Chun-Hung Liu**, Texas A&M University.

**Harmonic Analysis, Geometric Measure Theory and PDE I** (Code: SS 14A), **Dorina I. Mitrea** and **Marius Mitrea**, Baylor University.

**Homological and combinatorial methods in noncommutative algebra I** (Code: SS 5A), **Amrei Oswald** and **Be’eri Greenfeld**, University of Washington.

**Homological Commutative Algebra I** (Code: SS 13A), **Luigi Ferraro**, University of Texas Rio Grande Valley, and **Alexis Hardesty**, Texas Woman’s University.

**Inquiry Oriented Learning in the Mathematics Classroom I** (Code: SS 34A), **Carolyn Luna**, University of Texas At San Antonio, and **Jennifer Austin**, University of Texas at Austin.

**L-functions and Automorphic Forms I** (Code: SS 24A), **Lea Beneish**, University of North Texas, and **Melissa Emory**, Oklahoma State University.

**Link invariants and surfaces in 4-manifolds I** (Code: SS 21A), **Michael Willis** and **Sherry Gong**, Texas A&M University, **Hansapani Rodrigo**, The University of Texas Rio Grande Valley, and **Lakshmi Roychowdhury** and **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.


**Mathematical Physics and Numerical Methods I** (Code: SS 39A), **Vu Hoang** and **Jose Morales**, University of Texas at San Antonio.
Mathematics of Infectious Disease Emergence, Spread, and Control I (Code: SS 38A), Zhuolin Qu, University of Texas at San Antonio, and Michael Andrew Robert, Virginia Tech.

Mathematics: The gateway to Social Justice I (Code: SS 31A), Juan B. Gutiérrez, University of Texas at San Antonio, James Broda, Washington and Lee University, Funda Gultepe, University of Toledo, Ron Buckmire, Occidental College, Matthew Salomone, Bridgewater State University, Joseph Edward Hibdon, Northeastern Illinois University, and Terrance Pendleton, Drake University.

Methods & Applications of Data-driven Manufacturing I (Code: SS 19A), Kristen Lee Hallas, The University of Texas Rio Grande Valley, and Benjamin Peters and Jianzhi Li, University of Texas Rio Grande Valley.

Modeling and analysis in biological and epidemiological systems I (Code: SS 18A), Michael Lindstrom, The University of Texas Rio Grande Valley, and Erwin Suazo and Zhaosheng Feng, University of Texas Rio Grande Valley.

Non-Archimedean, Algebraic, Tropical Geometry and applications I (Code: SS 30A), Jackson S. Morrow, University of North Texas, and Farbod Shokrieh, University of Washington.

Noncommutative Geometry and Analysis I (Code: SS 20A), Zhizhang Xie, Guoliang Yu, Bo Zhu, and Simone Cecchini, Texas A&M University.

Operator algebras, quantum information and computation I (Code: SS 36A), Jose A Morales Escalante, University of Texas at San Antonio, and Marius Junge, University of Illinois, Urbana and Champaign.

Periodicity in Quantum Systems I (Code: SS 11A), Long Li, Rice University, Wencai Liu, Texas A&M University, and Tal Malinovitch, Rice University.

Quasi-periodic and Disordered Systems I (Code: SS 7A), Alberto Takase, Rice University, Omar Hurtado, University of California, Irvine, and Matthew H Faust, Texas A&M University.

Recent developments on local and nonlocal PDEs I (Code: SS 6A), Fernando Charro, Wayne State University, and Thialita Nascimento, Iowa State University.

Recent studies in topics related to ion channel problems I (Code: SS 29A), Mingji Zhang, New Mexico Institute of Mining and Technology, and Saulo Orizaga, New Mexico Tech.

Recent trends in differential equations applied to biological processes I (Code: SS 32A), Rachidi B. Salako, University of Nevada, Las Vegas, and Markjoe O. Uba and Maria Amarakristi Onyido, Northern Illinois University.


Spectral Theory of Schrödinger Operators and Related Topics I (Code: SS 26A), Christoph Fischbacher, Fritz Gesztesy, and Jon Harrison, Baylor University.

The many scales of mathematical analysis of fluid I (Code: SS 4A), Xin Liu, Texas A&M University, Quyuan Lin, Clemson University, and Cheng Yu, University of Florida.

Theoretical and Numerical Aspects of Nonlinear Dispersive Wave Equations I (Code: SS 35A), Baofeng Feng, University of Texas Rio Grande Valley, and Geng Chen and Yannan Shen, University of Kansas.

Topics in Convexity I (Code: SS 2A), Zokhrab Mustafaev, University of Houston-Clear Lake.

Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), Betsy Stovall, University of Wisconsin-Madison.

Savannah, Georgia

Georgia Southern University

October 5–6, 2024

Saturday – Sunday

Meeting #1199

Southeastern Section

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of Abstracts: Volume 45, Issue 4

Deadlines

For organizers: Expired

For abstracts: August 13, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.
Invited Addresses

Peter Bubenik, University of Florida, Topological Data Analysis: from geometry, algebra and combinatorics to analysis, learning and applications.

Akos Magyar, University of Georgia, To Be Announced.

Sarah Peluse, Princeton/IAS, To Be Announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Topics in Graph Theory and Combinatorics. (Code: SS 4A), Songling Shan, Auburn University, and Zi-Xia Song, University of Central Florida.

Advances in applied algebraic geometry (Code: SS 15A), Kisun Lee and Michael Byrd, Clemson University.

Advances in the theory of integrable partial differential equations (Code: SS 27A), Barbara Prinari, University at Buffalo, and Zechuan Zhang, SUNY Buffalo.

Algebraic, combinatorial and geometric aspects of representation theory. (Code: SS 19A), Cornelius Pillen, University of South Alabama, Aparna Upadhyay, University at Buffalo, SUNY, and Arik Wilbert, University of South Alabama.

Applicable Analysis of Multi-Physics Partial Differential Equations Systems (Code: SS 17A), George Avalos, University of Nebraska-Lincoln, and Justin Thomas Webster, University of Maryland, Baltimore County.

Biological Systems Modeling and Analysis: recent progress and current challenges (Code: SS 16A), Dawit Denu, Georgia Southern University.

Commutative Algebra (Code: SS 1A), Saeed Nasseh, Tricia Muldoon Brown, and Alina C. Iacob, Georgia Southern University.

Control, PDEs and Inverse Problems. (Code: SS 29A), Tien Khai Nguyen, North Carolina State University, Loc Hoang Nguyen, UNC Charlotte, and Thuy T. Le, North Carolina State University.

Convexity, Probability, and Asymptotic Geometric Analysis (Code: SS 12A), Galyna Livshyts, Georgia Institute of Technology, Steven Hoehner, Longwood University, and Stephanie Mui, Georgia Institute of Technology.

Deterministic and Stochastic PDEs: Theoretical and Numerical Analyses (Code: SS 8A), Pelin Guven Geredeli, Clemson University, and Xiang Wan, Loyola University Chicago.

Dynamical Systems and Control Systems with Applications (Code: SS 13A), Yan Wu, Georgia Southern University, and Liancheng Wang, Kennesaw State University.

Ergodic theory and discrete analysis. (Code: SS 23A), Neil Lyall and Tomasz Szarek, University of Georgia.

Exploring the Geometry for Teachers (GeT) Course (Code: SS 24A), Tuyin An, Georgia Southern University, and Erin Krupa, North Carolina State University.

Extremal and structural graph theory. (Code: SS 9A), Ruth Luo, University of South Carolina, and Zhiyu Wang, Georgia Institute of Technology.

Extremal Problems of Approximation Theory and Harmonic Analysis (Code: SS 35A), Yuliya Babenko, Kennesaw State University, and Scott Kersey, Georgia Southern University.

Fluids, Waves, and Free Boundaries. (Code: SS 2A), David M. Ambrose, Drexel University, and Michael Siegel, New Jersey Institute of Technology.

Game Theories in Network Security (Code: SS 32A), Zheni Utic, Georgia Southern University.

Geometric Maximal Operators and Related Topics. (Code: SS 3A), Paul Hagelstein, Baylor University, and Alex Stokolos, Georgia Southern University.

Harmonic analysis, fractals, and related topics in memory of Ka-Sing Lau and Robert Strichartz (Code: SS 14A), Sze-Man Ngai, Georgia Southern University, and Alexander Teplyaev, University of Connecticut.

Interactions, Discrepancies, Approximations: From Energy Optimization to Dynamics (Code: SS 26A), Ryan W Matzke, Vanderbilt University, and Ihsan Topaloglu, Virginia Commonwealth University.

Modules over Commutative Rings (Code: SS 10A), Laura Ghezzi, New York City College of Technology and The Graduate Center-Cuny, and Joseph P Brennan, University of Central Florida.

Noncommutative Algebras, Quantum Groups, and Related Topics (Code: SS 31A), Garrett Johnson, North Carolina Central University, Xin Tang, Math & Computer Science, Fayetteville State University, and Xingting Wang, Louisiana State University.

Nonlinear Dispersive Equations (Code: SS 34A), Iryna Petrenko, Florida International University, Justin Holmer, Brown University, and Svetlana Roudenko, Florida International University.
Number theory and additive combinatorics (Code: SS 28A), Sarah Peluse, Princeton/IAS, and Giorgis Petridis, University of Georgia.
Partitions and q-series (Code: SS 25A), Andrew V. Sills, Georgia Southern University, and Robert Schneider, University of Georgia.
Poisson geometry, Diffeology and Singular Spaces. (Code: SS 21A), Yi Lin, Georgia Southern University, Jordan Watts, Central Michigan University, and Francois Ziegler, Georgia Southern University.
Recent Advances in Contact and Symplectic Topology (Code: SS 30A), Nur Saglam, Georgia Tech, and Eduardo Fernández, University of Georgia.
Recent advances in Molecular based Computational and Mathematical Bioscience (Code: SS 18A), Shan Zhao, University of Alabama, and Zhan Chen, Georgia Southern University.
Recent Advances in Theory and Practice of Data Science (Code: SS 33A), Divine Wanduku and Ionut Iacob, Georgia Southern University.
Recent Advances of PDEs in Modern Mathematical Physics: Theory and Applications (Code: SS 7A), Yuanzhen Shao, The University of Alabama, and Yi Hu and Shijun Zheng, Georgia Southern University.
Recent developments in applications of complex analysis. (Code: SS 22A), Ashley Ran Zhang, Vanderbilt University, and Burak Hatinoglu, UC Santa Cruz.
Recent Progress in Numerical Methods for PDEs (Code: SS 11A), Xuejian Li and Leo Rebholz, Clemson University.
Topics in commutative algebra and algebraic geometry (Code: SS 6A), Prashanth Sridhar and Michael Brown, Auburn University.
Topological Data Analysis, Theory and Applications (Code: SS 5A), Peter Bubenik and Kevin P. Knudson, University of Florida.
Trees in many contexts. (Code: SS 20A), Hua Wang, Department of Mathematical Sciences, Georgia Southern University, and Heather Smith Blake, Davidson College.

Albany, New York
University at Albany

October 19–20, 2024
Saturday – Sunday

Meeting #1200
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

Deadlines
For organizers: Expired
For abstracts: August 27, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Jennifer Balakrishnan, Boston University, Title to be announced.
Jose Perea, Northeastern University, Title to be announced.
Richard Rimanyi, UNC, Title to be Announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Agent-Based and Mean-Field Modeling for Complex Social Systems (Code: SS 1A), Daniel Brendan Cooney, University of Illinois Urbana-Champaign, Jeungeun Park, SUNY at New Paltz, and Rebecca Hardenbrook, Dartmouth College.

Applied and Computational Topology (Code: SS 2A), Barbara Giunti and Håvard Bakke Bjerkevik, University at Albany, SUNY, and Justin Michael Curry, University at Albany SUNY.

MEETINGS & CONFERENCES

Ergodic Theory - In Memory of Nathaniel Friedman (1938 - 2020) (Code: SS 4A), Karin B. Reinhold, University at Albany, SUNY, Cesar E. Silva, Williams College, and Terry Adams, University at Albany.


Generalized Schubert Calculus and Recent Progress (Code: SS 6A), Changlong Zhong, SUNY Albany, and Richard Rimanyi, UNC.

Geometric Group Theory (Code: SS 7A), Matt Zaremsky, University at Albany, Emily Stark, Wesleyan University, and Daniel Studenmund, Binghamton University.

Harmonic Analysis, Theory of Function Spaces and Their Applications (Code: SS 8A), Liding Yao, University of Wisconsin-Madison, and Chian Yeong Chuah and Jan Lang, The Ohio State University.

Holomorphic Function Spaces and Operators on Them (Code: SS 9A), Kehe Zhu, University at Albany, SUNY, and Zhijian Wu, University of Nevada, Las Vegas.

Homotopy Theory and Algebraic K-Theory (Code: SS 10A), Marco Varisco, University at Albany, State University of New York, and Brenda Johnson, Union College.

Interactions Between Lie Theory and Combinatorics of Symmetric Functions (Code: SS 11A), Hadi Salmasian, University of Ottawa, and Siddhartha Sahi, Rutgers University, New Brunswick NJ.


Mathematics and the Arts in Memory of Nat Friedman (Code: SS 13A), David A Reimann, Albion College, Ergun Akeleman, Texas A&M University, and Alex Feingold, Binghamton University State University of New York.

Matroids, Quivers, F_{2^n}-geometry, and Connections with Algebra (Code: SS 14A), Jiajung Jun, SUNY New Paltz, Chris Ep playoff, The University of the South, and Alexander Sistko, Manhattan College.

Multivariable Operator Theory (Code: SS 15A), Rongwei Yang, Hyun-Kyoung Kwon, Alea L Wittig, and Kate Howell, University at Albany.

Nonlocal Analysis and Geometric Measure Theory (Code: SS 16A), Cornelia Mihaila, Saint Michael's College, and Brian Seguin, Loyola University Chicago.

Nonsmooth Analysis and Geometry (Code: SS 17A), Matthew Badger, University of Connecticut, Ryan Alvarado, Amherst College, and Lisa Naples, Fairfield University, Fairfield CT USA.

Permutation Patterns (Code: SS 18A), Megan A. Martinez, Ithaca College, and Rebecca Nicole Smith, SUNY Brockport.

Probabilistic and Analytic Aspects in Convexity (Code: SS 19A), Michael Roysdon, Case Western Reserve University, Sergii Myroshnychenko, University of the Fraser Valley, Kateryna Tatarko, University of Waterloo, Yiming Zhao, Syracuse University, and Elisabeth M Werner, Case Western Reserve University.

Quantum Mathematics for Computation (Code: SS 20A), Hanmeng (Harmony) Zhan, Worcester Polytechnic Institute, and Christino Tamon, Clarkson University.

Random Processes and Probability (Code: SS 21A), Martin V. Hildebrand, University at Albany, SUNY.

Recent Advances in Harmonic Analysis (Code: SS 22A), Joshua Brough Isralowitz, University At Albany, SUNY, and David Cruz-Uribe, University of Alabama.


Recent Developments in Automorphic Forms and Representation Theory (Code: SS 24A), Moshe Adrian, Queens College, City University of New York, and Anantharam Raghuram, Fordham University.

Recent Developments in Graph Theory (Code: SS 25A), Nathan Kahl and John T. Saccoman, Seton Hall University, and Kerry E Ojakian, Bronx Community (CUNY).

Recent Developments in Physics Informed Machine Learning for Inverse Problems (Code: SS 26A), Taufiquar Khan and Sudeb Majee, University of North Carolina at Charlotte.

Regularity of Nonlinear Equations and Free Boundary Problems (Code: SS 27A), Maria Soria-Carro and Inigo Urtiaga Erneta, Rutgers University, and Daniel Restrepo, Johns Hopkins University.

Singularities in Commutative Algebra (Code: SS 28A), Josh Pollitz and Claudia Miller, Syracuse University, and Jason Howell, University at Albany - State University of New York.

Symmetric Functions and Applications (Code: SS 29A), Olya Mandelshtam, University of Waterloo, and Rosa C. Orel lana, Dartmouth College.

Topics in Recreational Math and Finite Geometry (Code: SS 30A), Lauren L Rose, Bard College, Kelly Isham, Colgate University, and Elizabeth McMahon and Gary Gordon, Lafayette College.
Contributed Paper Sessions

AMS Contributed Paper Session (Code: CP 1A), Steven H. Weintraub, Lehigh University.

Riverside, California
University of California, Riverside

October 26–27, 2024
Saturday – Sunday

Meeting #1201
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: To be announced
Issue of Abstracts: Volume 45, Issue 4

Deadlines
For organizers: Expired
For abstracts: September 3, 2024

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Matthew D. Blair, University of New Mexico, Title to be announced.
Hannah K. Larson, UC Berkeley, Title to be announced.
Tianyi Zheng, University of California San Diego, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Extremal Combinatorics (Code: SS 27A), Emily Heath and Shira Zerbib, Iowa State University.
Advances in Understanding of Student Thinking in Lower Division Mathematics Courses (Code: SS 26A), Sara Lapan, University of California, Riverside, Jeffrey S Meyer, California State University, San Bernardino, and Rasha Issa, University of California, Riverside.
Calculating Probabilities using Matrix Methods with Applications to Markovian, Gaussian or Queueing Models (Code: SS 22A), Alan Krinik, California State Polytechnic University, and Randall J. Swift, California State Polytechnic University, Pomona.
Conformal Geometry, Einstein Metrics, and General Relativity (Code: SS 16A), Andrew K. Waldron and Jaroslaw Kopinski, University of California, Davis.
Dynamical Systems (Code: SS 5A), Agnieszka Zelerowicz and Zhenghe Zhang, UC Riverside.
Dynamics of Solutions to Wave Equations (Code: SS 21A), Michael McNulty and Willie Wong, Michigan State University, and Po-Ning Chen, UC Riverside.
Finite groups, their representations, and related structures (Code: SS 6A), Nariel Monteiro, University of California Santa Cruz, Robert Bolje, University of California, Santa Cruz, and Mandi A. Schaeffer Fry, University of Denver.
Gender Equity in the Mathematical Sciences (GEMS) of Combinatorics (Code: SS 2A), Aleyah Dawkins, George Mason University, Andrés Vindas Meléndez, University of Kentucky, and Katie Waddle, University of Michigan.
Geometric and Categorical Representation Theory (Code: SS 14A), Carl Mautner, UC Riverside, and Tom Gannon, University of California - Los Angeles.
Geometry and Topology of Contact and Symplectic Manifolds (Code: SS 9A), Bahar Acu, Pitzer College, Wenyuan Li, University of Southern California, and Hyunki Min, UCLA.
Geometry, topology and dynamics of character varieties (Code: SS 24A), Filippo Mazzoli, University of Virginia, and Brian Collier, University of California, Riverside.
Graphical Calculus in Representation Theory and Low-Dimensional Topology (Code: SS 19A), Emily McGovern, North Carolina State University, and Agustina Czenky, University of Oregon.
Harmonic Analysis and Applications (Code: SS 7A), Rodolfo H. Torres, University of California, Riverside, and Arpad Benyi, Western Washington University.
Harmonic Analysis, Partial Differential Equations, and Spectral Theory associated with Invited Address by Matthew Blair (Code: SS 4A), Matthew D. Blair, University of New Mexico, and Xiaoqi Huang, Louisiana State University.

Logic in SoCal (Code: SS 10A), Meng-Che Ho, California State University, Northridge, Scott Cramer, California State University, San Bernardino, Sheila Miller Edwards, Arizona State University, and Name Trang, University of North Texas.


Non-commutative Algebras in Representation Theory and Topology (Code: SS 20A), Peter Samuelson and Pallav Goyal, University of California, Riverside, and Boris Tsvetlakhovskiy, UC Riverside.

Non-commutative birational geometry, cluster structures and canonical bases (Code: SS 8A), Jacob Greenstein, University of California Riverside, Vladimir Retakh, Rutgers University, and Arkady Berenstein, University of Oregon Eugene.

Probability and Mathematical Physics (Code: SS 25A), David E Weisbart and Rahul D. Rajkumar, University of California Riverside.

Random matrices, related structures, and applications (Code: SS 18A), John Peca-Medlin, University of Arizona, and Yizhe Zhu, University of California Irvine.

Random walks on groups and dynamics of group actions associated with Invited Address by Tianyi Zheng (Code: SS 29A), Omer Tamuz, California Institute of Technology, Gil Goffer, University of California at San Diego, and Tianyi Zheng, University of California San Diego.

Recent Advances in Modeling and Simulation of Complex Fluids (Code: SS 12A), Yiwei Wang, Illinois Institute of Technology, Weitao Chen, University of California, Riverside, and Siting Liu, University of California, Los Angeles.

Several Complex Variables: New Developments and Trends (Code: SS 23A), Ziming Shi, University of California - Irvine, John N Treuer, Texas A&M University, and Bun Wong, University of California, Riverside.

Structural Features in Mathematical Physics (Code: SS 17A), Adam M. Yassine, Pomona College, and Andrea Stine, University of California, Riverside.

Surfaces, 3-manifolds and hyperbolic geometry (Code: SS 13A), Julien Paupert, Thi Hanh VO, and Puttipong Pongtampaisan, Arizona State University.

Topics in Algebraic Geometry (Code: SS 15A), Javier Gonzalez Anaya, Harvey Mudd College, Courtney George, University of California, Riverside, and Jose Gonzalez, University of California at Riverside.

Topics on Geometric Analysis (Code: SS 1A), Xiaolong Li, Wichita State University, Lihan Wang, California State University, Long Beach, and Qi S Zhang, UC Riverside.


Auckland, New Zealand

December 9–13, 2024

Monday – Friday

Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Seattle, Washington

Seattle Convention Center and the Sheraton Grand Seattle

January 8–11, 2025

Wednesday – Saturday

Meeting #1203

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of Abstracts: Volume 46, Issue 1

Deadlines

For organizers: Expired

For abstracts: September 10, 2024
Clemson, South Carolina
Clemson University

**March 8–9, 2025**
Saturday – Sunday

**Meeting #1204**
Southeastern Section
Associate Secretary for the AMS: Brian D. Boe, University of Georgia

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 46, Issue 2

**Deadlines**
For organizers: August 13, 2024
For abstracts: January 14, 2025

Lawrence, Kansas
University of Kansas

**March 29–30, 2025**
Saturday – Sunday

**Meeting #1205**
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 46, Issue 2

**Deadlines**
For organizers: August 27, 2024
For abstracts: February 4, 2025

Hartford, Connecticut
Hosted by University of Connecticut; taking place at the Connecticut Convention Center and Hartford Marriott Downtown

**April 5–6, 2025**
Saturday – Sunday

**Meeting #1206**
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: September 17, 2024
For abstracts: February 11, 2025

San Luis Obispo, California
California Polytechnic State University, San Luis Obispo

**May 3–4, 2025**
Saturday – Sunday

**Meeting #1207**
Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: Not applicable
Issue of *Abstracts*: Volume 46, Issue 3

**Deadlines**
For organizers: October 1, 2024
For abstracts: March 5, 2025
MEETINGS & CONFERENCES

St. Louis, Missouri

St. Louis University

**October 18–19, 2025**
Saturday – Sunday
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Denver, Colorado

University of Denver

**December 6–7, 2025**
Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: Not applicable

Issue of *Abstracts*: Volume 46, Issue 4

**Deadlines**
For organizers: May 6, 2025
For abstracts: October 14, 2025

Washington, District of Columbia

Walter E. Washington Convention Center and Marriott Marquis Washington DC

**January 4–7, 2026**
Sunday – Wednesday
Associate Secretary for the AMS: Betsy Stovall
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Boise, Idaho

Boise State University

**March 7–8, 2026**
Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michelle Ann Manes, AIM
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Fargo, North Dakota

North Dakota State University

**April 18–19, 2026**
Saturday – Sunday
Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison
Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced
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