On the Theory of Anisotropic Minimal Surfaces

Antonio De Rosa

1. Introduction

In the 1870s the Belgian physicist Plateau conjectured that every closed wire bounds at least one soap film. Since soap films tend to minimize the surface tension energy, which is proportional to the surface area, Plateau's conjecture can be formulated as follows:

Given any closed Jordan curve γ in $\mathbb{R}^3$, among all surfaces spanning the boundary γ, there exists a surface Σ of least area.

Formalizing and proving Plateau's conjecture turned out to be extremely challenging for mathematicians. Indeed it led to the development of several new theories in analysis and geometry and paved the way for the rise of geometric measure theory. In 1930, Douglas and Radó[1] solved the Plateau problem for surfaces that arise as the image of a 2-dimensional disk in $\mathbb{R}^n$. Thereafter, a variety of new theories were developed to describe and solve the Plateau problem in $\mathbb{R}^n$ for surfaces of general d-dimension and general topology: the theory of finite perimeter sets by De Giorgi, the theory of currents by Federer and Fleming, the theory of varifolds by Almgren and Allard, the theory of homological spanning conditions by Reifenberg, the theory of sliding minimizers by David, the theory of differential chains and linking number spanning conditions by Harrison and Pugh and several others.

Such a wide variety of tools was crucial because, depending on the natural phenomena to model, one has to use a different notion of surface and boundary. For instance, among boundaries of finite perimeter sets, solutions of the Plateau problem in dimension $n \leq 7$ are oriented smooth submanifolds, in contrast to soap films in $\mathbb{R}^3$ that can have 1-dimensional singular sets. One possible approach to model soap films and to capture their 1-dimensional singular sets, is the set-theoretic Plateau problem, which consists in minimizing the d-dimensional Hausdorff measure $\mathcal{H}^d$ within families of d-dimensional sets in $\mathbb{R}^n$ with suitable boundary conditions. The set-theoretic approach was pioneered by Reifenberg, Almgren, and Taylor, and has recently been further developed by David, Harrison and Pugh, De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi, and Fang and Kolasinski.

Since solutions of the Plateau problem are minimizers of the surface area functional, they are in particular critical points of this functional in the sense of the calculus of variations. Critical points of the surface area are referred to as (isotropic) minimal surfaces.

The existence and regularity of a solution Σ of the Plateau problem relies on the study of the so-called tangent cones of Σ, obtained through a standard blow-up procedure, described below. We consider the following one-parameter family of minimal surfaces $\{\Sigma_{x,r}\}_{r>0}$, obtained by dilating Σ around a point $x \in \Sigma$:

$$\Sigma_{x,r} := \frac{\Sigma - x}{r} \quad \text{with} \quad r > 0.$$

In the unitary ball $B(0,1)$, the surface area of this family $\Sigma_{x,r}$ enjoys a powerful *monotonicity formula*, i.e.,

$$r \mapsto \mathcal{H}^d(\Sigma_{x,r} \cap B(0,1)) \qquad \text{is nondecreasing.} \qquad (1)$$

The validity of the monotonicity formula (1) for minimal surfaces implies there is a tangent cone T such that, as $r \to 0^+$, up to subsequences $\Sigma_{x,r} \to T$ in a measure theoretic sense. One can also prove that T minimizes the surface area.

However, for several natural phenomena the use of the surface area functional is just a first approximation. In order to capture microstructures, numerous models in applied sciences employ directionally dependent functionals, known as anisotropic energies.

Since anisotropic energies are not invariant under rigid motions, their critical points do not enjoy the same

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Antonio De Rosa was funded by the European Union: the European Research Council (ERC), through StG "ANGEVA", project number: 101076411. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

Communicated by Notices *Associate Editor Chikako Mese.*

DOI: https://doi.org/10.1090/noti2980

[1] *Given the upper bound of 20 references, we cannot include the references to all cited results herein and this survey should not be intended as an exhaustive literature review on anisotropic minimal surfaces.*

conservation laws of isotropic minimal surfaces. For instance the monotonicity formula (1) is not known to hold for minimizers of general anisotropic energies. Consequently, the study of anisotropic minimal surfaces is significantly more challenging than the study of their isotropic counterparts.

1.1. Anisotropic energies.

We fix $d, n \in \mathbb{N}$ with $1 \leq d < n$ and we consider a smooth anisotropic integrand

$$F : \mathbb{R}^n \times G(d,n) \to (0, \infty),$$

where $G(d,n)$ denotes the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^n$. F is usually required to satisfy an ellipticity condition, which can be thought of as a generalization of a uniform convexity property of F in the Grassmannian variable. We recall that there is a plethora of ellipticity conditions for F, especially in codimension strictly bigger than one, i.e., $d < n - 1$. The appropriate ellipticity conditions to assume on F depend on the types of surfaces and boundaries considered and on the regularity theory one is interested in. To avoid introducing too many conditions, we will sometimes use the generic phrase "F is elliptic."

We define the anisotropic energy of a d-dimensional smooth surface Σ as

$$\mathbf{F}(\Sigma) := \int_{\Sigma} F(x, T_x\Sigma)\, d\mathcal{H}^d(x), \tag{2}$$

where $T_x\Sigma$ is the tangent space of Σ at x. We observe that if $F \equiv 1$, then $\mathbf{F}(\Sigma) = \mathcal{H}^d(\Sigma)$. Hence anisotropic energies subsume the surface area functional.

The anisotropic energy (2) is well defined also for weaker notions of surfaces: namely, the d-recitifiable sets. A set is said to be d-rectifiable if it can be covered, up to an $\mathcal{H}^d$-negligible set, by countably many C^1 manifolds. Since d-rectifiable sets admit a notion of tangent space at $\mathcal{H}^d$-a.e. point, they are well suited to compute anisotropic energies.

The minimal configurations for (2) are clearly not invariant by translation or rotation. This is a common behavior in several problems in materials science. Indeed, anisotropic integrands were introduced by Gibbs to model the surface tension arising at the interface between two different materials. Moreover, Wulff observed that crystal structures are polyhedral, because they are optimal configurations for energies that assign different weights to different tangent planes of the boundary surface. Since the pivotal studies of Almgren, Taylor, and Allard, anisotropic energies have attracted a great interest in the geometric analysis community, leading to important contributions in a variety of applications: crystal structures, capillarity problems, gravitational fields, homogenization problems, and many others.

In solving the Plateau problem with respect to (2) and, more generally, in the study of critical points of (2), which are referred to as anisotropic minimal surfaces, one faces a main obstruction: the lack of the monotonicity formula (1). As observed by Allard, monotonicity (1) is deeply related to the isotropic setting. Nevertheless, solutions to the set-theoretic anisotropic Plateau problem were obtained by several authors, as, for instance, by Harrison and Pugh in [HP17] and by De Philippis, De Rosa, and Ghiraldin in [DPDRG20]. In particular we recall the following theorem:

Theorem 1 ([DPDRG20]). *Let F be elliptic. Let $\Gamma \subset \mathbb{R}^n$ be a closed set and $\mathcal{P}$ be a family of relatively closed d-rectifiable subsets of $\mathbb{R}^n \setminus \Gamma$, such that $\mathcal{P}$ is closed by Lipschitz deformations of $\mathbb{R}^n \setminus \Gamma$. If $\{\Sigma_j\}_{j\in\mathbb{N}} \subset \mathcal{P}$ is a minimizing sequence for $\mathbf{F}$ in $\mathcal{P}$, i.e.,*

$$\liminf_{j\to\infty} \mathbf{F}(\Sigma_j) = \inf\{\mathbf{F}(L) : L \in \mathcal{P}\},$$

then, up to subsequences, the following limit holds in the sense of measures

$$F(\cdot, T_{(\cdot)}\Sigma_j)\mathcal{H}^d \llcorner \Sigma_j \overset{\star}{\rightharpoonup} F(\cdot, T_{(\cdot)}\Sigma)\mathcal{H}^d \llcorner \Sigma \quad \text{in } \mathbb{R}^n \setminus \Gamma,$$

and consequently

$$\liminf_{j\to\infty} \mathbf{F}(\Sigma_j) \geq \mathbf{F}(\Sigma).$$

Moroever Σ is smooth outside of a relatively closed set $S \subset \Sigma$ with $\mathcal{H}^d(S) = 0$.

In Theorem 1 the set Γ should be interpreted as the assigned boundary of the anisotropic Plateau problem, and $\mathcal{P}$ is the feasible set of the anisotropic Plateau problem. The assumption on $\mathcal{P}$ of being closed by Lipschitz deformations can be weakened to include more cases of interest, see [DPDRG20]. A main step in proving Theorem 1 is to show the d-rectifiability of Σ. In the isotropic setting, this can be obtained by the monotonicity formula (1) and by proving a density lower bound, as these two properties allow one to employ a deep rectifiability theorem of Preiss. However, since the monotonicity formula (1) does not hold for anisotropic minimal surfaces, we cannot apply Preiss's theorem to prove Theorem 1. To overcome this obstacle, in [DPDRG20] it is crucial to utilize the theory of varifolds.

1.2. Varifolds.

The *d-varifolds* are weak notions of d-dimensional surfaces. More precisely a d-varifold is a Radon measure on the Grassmannian bundle $\mathbb{R}^n \times G(d,n)$. Simple examples are the rectifiable d-varifolds, i.e., d-varifolds that can be represented as

$$V = \theta\mathcal{H}^d \llcorner \Sigma \otimes \delta_{T_x\Sigma}, \tag{3}$$

for some d-rectifiable set Σ and a Borel function $\theta : \mathbb{R}^n \to \mathbb{R}^+$. However there are examples of nonrectifiable varifolds, obtained naturally as limits of surfaces. An example is provided in Fig.1, where the limit varifold is

$$\mathcal{H}^1 \llcorner \ell_1 \otimes \frac{1}{2}(\delta_{\ell_2} + \delta_{\ell_3}),$$

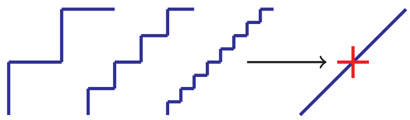

Figure 1. An example of a nonrectifiable varifold.

with ℓ_1 being the oblique line and ℓ_2, ℓ_3 being respectively the horizontal and vertical lines.

The mass $\|V\|$ of a d-varifold V is the Radon measure on $\mathbb{R}^n$ defined by

$$\|V\|(A) := V(A \times G(d,n)) \qquad \text{for all } A \subset \mathbb{R}^n \text{ Borel}.$$

For a d-varifold V we consider (when it exists) its d-dimensional density at $x \in \mathbb{R}^n$:

$$\Theta(x,V) := \lim_{r\to 0^+} \frac{\|V\|(B_r(x))}{\omega_d r^d},$$

where $\omega_d := \mathcal{H}^d(B^d(0,1))$ is the measure of the d-dimensional unit ball in $\mathbb{R}^d$. Note that for a rectifiable d-varifold V, the d-dimensional density $\Theta(x,V)$ is equal to the value $\theta(x)$ in (3) for $\|V\|$-a.e. x.

Given the duality between d-varifolds and $C_c^0(\mathbb{R}^n \times G(d,n))$, the definition of anisotropic energy (2) can be naturally generalized also to d-varifolds V as follows:

$$\mathbf{F}(V) := \int_{\mathbb{R}^n\times G(d,n)} F(x,T)dV(x,T).$$

The anisotropic first variation of V with respect to F is defined as

$$\delta_F V(g) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathbf{F}((\mathrm{id}+\varepsilon g)_\#(V)), \quad \forall g \in C_c^1(\mathbb{R}^n,\mathbb{R}^n),$$

where for a diffeomorphism $\psi \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, we define the push-forward of V with respect to ψ as the varifold $\psi_\# V$ satisfying:

$$\int_{\mathbb{R}^n\times G(d,n)} \Phi(x,T)d(\psi_\# V)(x,T) = \int_{\mathbb{R}^n\times G(d,n)} \Phi(\psi(x), d_x\psi(T))J\psi(x,T)dV(x,T),$$

for every $\Phi \in C_c^0(G(\psi(\Omega)))$. Here $d_x\psi(T)$ is the image of T under the map $d_x\psi$ and

$$J\psi(x,T) := \sqrt{\det\left(\left(d_x\psi|_T\right)^* \circ d_x\psi|_T\right)}$$

denotes the d-Jacobian determinant of the differential $d_x\psi$ restricted to the d-plane T. If $V = \mathcal{H}^d \llcorner \Sigma \otimes \delta_{T_x\Sigma}$ is the d-varifold canonically associated to a smooth d-dimensional submanifold Σ, the pushforward $\psi_\# V$ is simply the image $\psi(\Sigma)$ of Σ under the map ψ.

A d-varifold V is called stationary with respect to F if

$$\delta_F V = 0. \tag{4}$$

We observe that if $V = \mathcal{H}^d \llcorner \Sigma \otimes \delta_{T_x\Sigma}$ for a smooth d-dimensional surface Σ and $\delta_F V = 0$, then Σ is an anisotropic minimal surface.

We say that a d-varifold V has F-mean curvature in L^p if there exists a map $H_F \in L^p(\mathbb{R}^n,\mathbb{R}^n;\|V\|)$ such that

$$\delta_F V(g) = -\int_{\mathbb{R}^n} (H_F(x), g(x))d\|V\|(x), \forall g \in C_c^1(\mathbb{R}^n,\mathbb{R}^n).$$

2. Regularity for Anisotropic Minimal Surfaces

In the study of partial differential equations (PDEs), a typical question is about the regularity of weak solutions to certain PDEs. Weak solutions are very useful as it is often easier to prove their existence. However this generates the new task of investigating how close they are to being classical solutions. This is achieved by deducing information about their regularity from the PDE.

This is precisely the case of the stationarity equation (4) for anisotropic minimal surfaces, which can be regarded as a geometric PDE in integral form. Since stationary d-varifolds are well-defined weak solutions of equation (4), it is crucial to study how close stationary varifolds are to being smooth anisotropic minimal surfaces, by deducing structural and regularity properties from (4). To this aim, our first step in this section is to understand whether V is a rectifiable d-varifold, and the second step is to investigate the smoothness of the support of V. As we detail below, this analysis is useful, for instance, to deduce the regularity of the solution Σ of the anisotropic Plateau problem in Theorem 1, which a priori is just the support of the limiting d-varifold of the minimizing sequence of d-varifolds

$$\{F(\cdot, T_{(\cdot)}\Sigma_j)\mathcal{H}^d \llcorner \Sigma_j \otimes \delta_{T_{(\cdot)}\Sigma_j}\}_{j\in\mathbb{N}}.$$

Rectifiability. Allard's rectifiability theorem [All72] asserts that every d-varifold V in $\mathbb{R}^n$ with locally bounded isotropic first variation is rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive d-dimensional density. This result is essential in many applications, such as to prove a compactness theorem for integral varifolds [All72], to solve the Plateau problem, in the min-max theory, and in geometric flows.

De Philippis, De Rosa, and Ghiraldin proved in [DPDRG18] the same rectifiability result for anisotropic energies, provided F satisfies the atomic condition introduced in [DPDRG18, Definition 1.1]. Since the atomic condition is fairly technical to define, for simplicity of notation we recall below the condition (BC), which was introduced in [DRK20] and proved to be equivalent to the atomic condition.

Definition 1. *An anisotropic integrand F satisfies (BC) if and only if for every $(x,T) \in \mathbb{R}^n \times G(d,n)$, denoting $F_x(y,S) := F(x,S)$ for every $(y,S) \in \mathbb{R}^n \times G(d,n)$, if μ is a probability* *measure over $G(d,n)$ such that*

$$\delta_{F_x}\big[(\mathcal{H}^d \llcorner T)\otimes\mu\big]=0,$$

then $\mu=\delta_T$.

The rectifiability theorem in [DPDRG18] then reads:

Theorem 2 ([DPDRG18])**.** *If F satisfies the atomic condition, or equivalently (BC), then for every d-varifold V whose anisotropic first variation $\delta_F V$ is a Radon measure, the varifold*

$$V_* := V\llcorner\{x\in\mathbb{R}^n : \Theta(x,V)>0\}\times G(d,n) \qquad (5)$$

is rectifiable.

Moreover, if F is autonomous, i.e., $F(x,T)\equiv F(T)$, the reverse implication also holds. That is, if F does not satisfy the atomic condition, then there exists a d-varifold V such that $\delta_F V$ is a Radon measure and the associated V_* as in (5) is not rectifiable.

Some ideas for the proof of Theorem 2 were inspired by the "strong constancy lemma" of Allard [All86, Theorem 4]. Theorem 2 was crucial in the proof of the rectifiability of Σ in Theorem 1. Indeed the stationarity of the limit d-varifold follows from $\{\Sigma_j\}_{j\in\mathbb{N}}$ being a minimizing sequence, while the density of the limit d-varifold can be proved to be positive via a set-theoretic analog of the Federer-Fleming deformation theorem due to David and Semmes. Thereafter, Theorem 2 has found several other applications, such as in the solution of the anisotropic Plateau problem in classes of integral and rectifiable varifolds [DR18], and in the anisotropic min-max theory [DPDR24] that we will discuss more in detail in Section 3.

$C^{1,\alpha}$-regularity. A celebrated theorem of Allard [All72] states that, given a rectifiable d-varifold V in $\mathbb{R}^n$ with density greater than or equal to one and isotropic mean curvature H_{Area} bounded in L^p with $p>d$, for every $x\in\mathbb{R}^n$ such that $\Theta(x,V)=1$ the support of V is $C^{1,\alpha}$ in a neigborhood of x. The proof deeply relies on the monotonicity formula (1). Hence, it is unknown whether this result holds for anisotropic energies, i.e., assuming an L^p bound on the anisotropic mean curvature H_F.

In codimension one, i.e., $d=n-1$, the anisotropic counterpart of this regularity theorem has been obtained by Allard under an additional lower density bound assumption [All86]. However, the proof relies on the maximum principle, hence it cannot be extended to general codimension. In general codimension, De Rosa and Tione [DRT22] proved a partial $C^{1,\alpha}$-regularity theorem in the case the d-varifold V is associated to a Lipschitz graph. To this aim, they introduce the uniform scalar atomic condition (USAC), a novel ellipticity condition that can be thought of as a uniform version of the atomic condition or of the equivalent (BC) in Definition 1. USAC allowed them to obtain a Caccioppoli inequality, that was crucial to prove the following theorem:

Theorem 3. *Let $F\in C^2$ be a functional satisfying USAC, let $p>d$, and consider an open bounded set $\Omega\subset\mathbb{R}^d$. Let $u\in Lip(\Omega,\mathbb{R}^{n-d})$ be a map whose graph induces a varifold V with F-mean curvature $H_F\in L^p(\Omega\times\mathbb{R}^{n-d},\mathbb{R}^n;\|V\|)$. Then there exists $\alpha\in(0,1)$ and an open set $\Omega_0\subset\Omega$ such that $\mathcal{H}^d(\Omega_0)=\mathcal{H}^d(\Omega)$ and $u\in C^{1,\alpha}(\Omega_0,\mathbb{R}^{n-d})$.*

It remains a challenging problem to prove the anisotropic counterpart of Allard's regularity theorem [All72], that is, to extend Theorem 3 to all rectifiable varifolds satisfying a lower density bound.

3. Existence of Anisotropic Minimal Surfaces: The Min-max Theory

Since Theorem 2 and Theorem 3 are local results, they can be proved with slightly more work in a Riemannian manifold ambient space, rather than just in the Euclidean ambient space $\mathbb{R}^n$. Working in a closed Riemannian manifold, the problem of finding closed anisotropic minimal hypersurfaces becomes meaningful. This problem was posed by Allard [All83, Page 2]. On the contrary, by the maximum principle, in the ambient space $\mathbb{R}^n$ there are no closed anisotropic minimal hypersurfaces.

For the area functional, the problem of finding closed isotropic minimal hypersurfaces has a long literature. In 1917 Birkhoff introduced a min-max method to prove the existence of a closed geodesic on any 2-dimensional Riemannian sphere. Fet and Lyusternik extended this result, proving that every closed Riemannian manifold admits a closed geodesic. In order to generalize the aforementioned results to higher dimension, Almgren introduced the theory of varifolds and developed a min-max theory to prove the existence of stationary integral d-varifolds in any closed n-dimensional Riemannian manifold. In codimension one, that is when $d=n-1$, the regularity of these integral $(n-1)$-varifold was proved by Pitts [Pit81] for $3\le n\le 6$, and by Schoen and Simon [SS81] for every $n\ge 3$:

Theorem 4 ([Pit81, SS81])**.** *Let M be an n-dimensional smooth closed Riemannian manifold. Then there is a nontrivial embedded isotropic minimal hypersurface $\Sigma\subset M$ without boundary, which is smooth outside of a singular set of Hausdorff dimension at most $n-8$.*

Theorem 4 motivated the development of the min-max theory, which has played a major role in proving a number of conjectures in geometry and topology.

Allard [All83, Page 2] was interested in understanding whether a version of Theorem 4 holds for anisotropic energies. De Philippis and De Rosa [DPDR24] settled this question for $n=3$ up to at most one singular point, as a corollary of the following more general result about the existence of closed surfaces with prescribed constant anisotropic mean curvature.

Theorem 5 ([DPDR24]). *Let M be a 3-dimensional smooth closed Riemannian manifold, F be an elliptic integrand and $c \in \mathbb{R}$. Then there is a nontrivial almost embedded (embedded if $c = 0$) closed surface $\Sigma \subset M$ with constant anisotropic mean curvature $H_F \equiv c$ and which is smooth outside of at most one singular point $p \in M$.*

In Theorem 5, the singular point p accounts for the index of Σ, which in general may be an unstable surface, as it is constructed via min-max methods. However, we expect that this singular point p is removable.

It is interesting to observe that we do not have a direct proof of Theorem 5 for the anisotropic minimal surfaces case, i.e., $c = 0$. Indeed the case $c = 0$ is obtained by first constructing a sequence of surfaces Σ_k with constant anisotropic mean curvature $c_k = 1/k$ (each having at most one singular point p_k) and then using the local stability of Σ_k to deduce the smooth convergence of a subsequence of Σ_k away from an accumulation point p of the sequence p_k. The limit surface Σ has then anisotropic mean curvature

$$H_F \equiv \lim_{k\to\infty} c_k = 0,$$

that is, Σ is an anisotropic minimal surface with respect to F. Constructing surfaces with positive constant anisotropic mean curvature is easier because their multiplicity can be shown to be either 1 for $\mathcal{H}^2$-a.e. point or 2 for a 1-dimensional touching set. Having multiplicity 1 is critical when performing the blowup analysis to study the tangent cones in the absence of the monotonicity formula (1).

For a closed Riemannian manifold M of general dimension n, we expect the validity of a similar result to Theorem 5: the existence of a nontrivial almost embedded closed hypersurface $\Sigma \subset M$, with $H_F \equiv c$ and with a singular set of zero $\mathcal{H}^{n-3}$-measure. The latter regularity of Σ agrees with the regularity theory for solutions of the anisotropic Plateau problem in codimension one for finite perimeter sets established by Almgren, Schoen, and Simon [SSA77]. This regularity result is almost sharp, because Morgan [Mor90] constructed an example of a uniformly elliptic anisotropic energy for which the 3-dimensional cone over the Clifford torus is a solution of the codimension one anisotropic Plateau problem in $\mathbb{R}^4$. As this 3-dimensional surface has one singular point, this example confirms that the $\mathcal{H}^{n-4}$-measure of the singular set of an anisotropic minimizer of codimension one in $\mathbb{R}^n$ may be in general positive. However, to date it is not known whether there are 3-dimensional anisotropic minimizers in $\mathbb{R}^4$ with a singular set of fractional dimension within $(0, 1)$.

The dimension of the singular set for solutions of the Plateau problem in codimension one for finite perimeter sets is connected to the Bernstein problem, which asks whether critical points in $\mathbb{R}^n$ which are graphs of scalar functions defined on all of $\mathbb{R}^{n-1}$ are necessarily hyperplanes. In the case of the area functional, Bernstein, Fleming, De Giorgi, Almgren, Simons, and Bombieri, De Giorgi, and Giusti showed that the answer is positive if and only if $n \leq 8$. For elliptic integrands, the answer was shown to be positive for $n = 3$ by Jenkins and for $n = 4$ by Simon, while Mooney and Yang [MY24] recently answered negatively for dimensions $n \geq 5$, concluding the solution of the anisotropic Bernstein problem.

We end this section remarking that, when F is close to the area functional in the C^3 topology, the anisotropic regularity theory becomes similar to the one of the area functional. More precisely Simon [Sim77] proved that in this case the Bernstein problem has positive answer up to dimension $n = 8$ and Almgren, Schoen, and Simon [SSA77] showed that minimizers of the anisotropic Plateau problem in codimension one for finite perimeter sets are regular up to dimension $n = 7$. Moreover Chodosh and Li [CL23] recently proved that, if F is close to the area functional in the C^4 topology, then stable critical points in dimension $n = 4$ are flat. It remains an interesting question whether these results can be achieved assuming closeness to the area functional in a weaker topology, as for instance the C^2 topology.

4. Constant Anisotropic Mean Curvature Surfaces and the Anisotropic Isoperimetric Problem

The constant anisotropic mean curvature surfaces constructed in Theorem 5 have an important parallel with anisotropic minimal surfaces. While the latter are critical points of the anisotropic Plateau problem, constant anisotropic mean curvature surfaces are critical points of the anisotropic isoperimetric problem. This problem, also known as the Wulff problem, consists in the minimization of the anisotropic energy of boundaries of finite perimeter sets with a prescribed volume constraint $m > 0$. In short, the Wulff problem reads

$$\inf\{\mathbf{F}(\partial\Omega) : \mathcal{H}^n(\Omega) = m, \quad \Omega \text{ is a finite perimeter set in } \mathbb{R}^n\}. \tag{6}$$

If F is an autonomous integrand in codimension one, i.e.,

$$F : G(n-1, n) \to (0, \infty)$$

does not depend on the x variable, then the Wulff problem (6) has a (unique up to translation) solution $\mathcal{W}$, referred to as Wulff shape, which was constructed by Wulff. Henceforth, for the remainder of this section we will assume F is an autonomous integrand. $\mathcal{W}$ plays a central role in crystallography and its minimality has been proved with different techniques by Taylor, Fonseca and Müller, Brothers and Morgan, Gromov, Figalli, Maggi, and Pratelli. For the area functional the Wulff shape coincides with the

Euclidean ball, whose symmetries reflect the rotation invariance of the area functional.

On the other hand, the investigation of critical points of (6) is more subtle. This corresponds to characterizing finite perimeter sets with finite volume whose boundary has constant anisotropic mean curvature in the sense of varifolds. For the area functional, the Euclidean sphere has been long known to be the only smooth closed connected hypersurface with constant isotropic mean curvature. Several proofs have been provided and one of the most geometric is via the Alexandrov moving plane method. This method involves reflecting the hypersurface across a family of parallel hyperplanes and using the maximum principle to deduce the symmetry of the hypersurface with respect to one of these parallel hyperplanes. However the moving plane method is not well suited for anisotropic energies, as the reflection of a constant anisotropic mean curvature surface changes the tangent planes and hence also the geometric PDE solved by the surface. With different techniques, He, Li, Ma, and Ge characterized the Wulff shape boundary as the only closed connected C^2 hypersurface with constant anisotropic mean curvature. The C^2 assumption was removed for the area functional in an ingenious way by Delgadino and Maggi [DM19], who proved the following theorem.

Theorem 6 ([DM19]). *Among sets of finite perimeter and finite volume, finite unions of Euclidean balls with equal radii are the unique critical points of the isotropic isoperimetric problem.*

Afterward, De Rosa, Kolasinski, and Santilli [DRKS20] extended Theorem 6 to uniformly elliptic $C^{2,\alpha}$ anisotropic integrands adding an extra assumption, equation (7) below, which corresponds roughly speaking to a density lower bound. More precisely they proved the following theorem.

Theorem 7 ([DRKS20]). *Consider an elliptic integrand $F \in C^{2,\alpha}$, with $\alpha \in (0,1)$. Among sets Ω of finite perimeter and finite volume satisfying*

$$\mathcal{H}^{n-1}(\overline{\partial^*\Omega} \sim \partial^*\Omega) = 0, \tag{7}$$

finite unions of Wulff shapes with equal radii are the unique critical points of the anisotropic isoperimetric problem (6).

Both Theorem 6 and Theorem 7 are proved by means of an Heintze-Karcher inequality for sets of finite perimeter, obtained by refining an argument due to Montiel and Ros. In particular, the Heintze-Karcher inequality they proved in the anisotropic setting reads as follows:

Theorem 8 ([DRKS20]). *Consider an elliptic integrand $F \in C^{2,\alpha}$, with $\alpha \in (0,1)$, and a set of finite perimeter $\Omega \subset \mathbb{R}^n$ such that* (7) *holds, the anisotropic first variation $\delta_F[\partial^*\Omega]$ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial^*\Omega$ and the anisotropic mean curvature H_F is bounded, positive and locally $C^{0,\alpha}$ on the $C^{1,\alpha}$ regular part of $\partial^*\Omega$. Then*

$$\mathcal{H}^n(\Omega) \le \frac{n-1}{n} \int_{\partial^*\Omega} \frac{F(T_x\partial^*\Omega)}{H_F(x)} d\mathcal{H}^{n-1}(x).$$

Equality holds if and only if Ω coincides up to a set of zero $\mathcal{H}^n$-measure with a finite union of disjoint open Wulff shapes with radii not smaller than $\frac{n-1}{\|H_F\|_{L^\infty}}$.

Condition (7) is used to apply Allard $C^{1,\alpha}$-regularity theorem in codimension one [All86]. Condition (7) always holds if Ω satisfies a density lower bound at every point in $\overline{\partial^*\Omega}$. In particular it holds for Lipschitz domains, for local minimizers, and for almost minimizers of problem (6). On the other hand, it is not known whether condition (7) holds for every set of finite perimeter Ω such that $\partial^*\Omega$ has constant anisotropic mean curvature H_F. In the isotropic setting, condition (7) follows from the monotonicity formula (1), which as mentioned in Section 1 is not known to hold for general anisotropic integrands. Hence it remains an interesting question whether Theorem 7 holds without condition (7).

Since the Wulff shape is characterized as the unique minimizer for all positive continuous anisotropic integrands, another very interesting question is whether Theorem 7 holds also for anisotropic integrands that are neither $C^{2,\alpha}$ nor elliptic. Particularly relevant for applications in materials science is the case of crystalline integrands, for which the Wulff shapes are obtained by intersecting a finite amount of half-spaces. Since in this case the integrand is convex, but is not C^1, the notion of anisotropic first variation and critical points are suitably defined by Maggi using the convexity in time of the functional along any prescribed variational flow. Figalli and Maggi have greatly advanced our understanding of crystalline energies in several works, see for instance [FM11], proving quantitative stability results for crystalline Wulff shapes for almost minimizers under the action of potential energies.

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