ON REARRANGEMENTS OF SERIES

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In an interesting paper published in the Bulletin of the American Mathematical Society, R. P. Agnew[1] considers some questions on rearrangement of conditionally convergent series.

He considers the metric space E in which a point x is a permutation $(x_1, x_2, x_3, \cdots)$ of the positive integers and in which the distance $\rho(x, y)$ between two points $x \equiv (x_1, x_2, x_3, \cdots)$ and $y \equiv (y_1, y_2, y_3, \cdots)$ of E is given by the Fréchet formula

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Agnew proves that the space E is of the second category at each of its points.

He also considers the following problem: Let $C_1 + C_2 + C_3 + \cdots$ be a conditionally convergent series of real terms. Denote C_n by $C(n)$. To each $x \in E$ there corresponds a rearrangement $\sum_{n=1}^{\infty} C(x_n)$ of the series $\sum C(n)$ or $\sum C_n$ and also to each rearrangement of the series corresponds a point $x \in E$. Thus the ways in which the series may be rearranged form a set which has the potency of E. It is well known that $x \in E$ exists for which $\sum_n C(x_n)$ converges to any preassigned number, diverges to $+\infty$ or to $-\infty$, or oscillates with prescribed upper and lower limits. The set A of $x \in E$ for which $\sum_{n=1}^{\infty} C(x_n)$ converges is therefore a proper subset of E. Professor Agnew considers the nature of the set A and proves that it is of the first category so that the complementary set $E - A$ is of the second category. In point of fact he proves more than that. He shows that the set of points $x \in E$ for which $\sum_{n=1}^{\infty} C(x_n)$ has unilaterally bounded partial sums is of the first category. His first theorem runs as follows:

For each $x \in E$, except those belonging to a set of the first category,

$$\liminf_{N \to \infty} \sum_{n=1}^{N} C(x_n) = -\infty, \qquad \limsup_{N \to \infty} \sum_{n=1}^{N} C(x_n) = +\infty.$$

It is possible to add something more to the above in regard to the nature of the set A. In fact we can prove the following theorem:

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[1] Ralph Palmer Agnew, *On rearrangement of series*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 797–799.

THEOREM 1. *For each $x \in E$, except those of a set B (F_σ in E) that is an outer-limiting set of sets closed in E, we have*

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty, \qquad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty.$$

Combining the above with the result given in Agnew's paper, we may assert that the set of points x for which

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty \quad \text{and} \quad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty$$

is the inner limiting set of a sequence of everywhere dense sets open in E.

PROOF. We require the following ideas for the proof of our theorem. Consider the series $\sum_{n=1}^{\infty} C(x_n)$. Let $s_1, s_2, s_3, \cdots, s_N, \cdots$, where $s_N = \sum_{n=1}^{N} C(x_n)$, be the partial sums.

Let us denote the maximum of the numbers $(s_1, s_2, \cdots, s_N)$ by U_N. Then $s_1, U_2, U_3, \cdots$ is a monotone nondecreasing sequence of real numbers, and $\lim_{N\to\infty} U_N$ exists either as a finite number or as $+\infty$. We write $\lim_{N\to\infty} U_N$ as $\lim_{N\to\infty}$ upper bound $\sum_{n=1}^{N} C(x_n)$.

In a similar manner if u_N denotes min $(s_1, s_2, \cdots, s_N)$, then $s_1, u_2, u_3, \cdots$ is a monotone nonincreasing sequence of real numbers and $\lim_{N\to\infty} u_N$ exists either as a finite number or as $-\infty$. We denote $\lim_{N\to\infty} U_N$ by the symbol $\lim_{N\to\infty}$ lower bound $\sum_{n=1}^{N} C(x_n)$.

Now, let $h>0$ be large at pleasure. We define B_h to be the set of points of E for which $\lim_{N\to\infty}$ upper bound $\sum_{n=1}^{N} C(x_n) \leqq h$.

Now let $\xi_1, \xi_2, \xi_3, \cdots$ be a convergent sequence of points of E all belonging to B_h. Let their limit point ξ also belong to E. We shall prove that ξ also belongs to B_h.

Let $\xi_n \equiv (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \cdots)$ and $\xi \equiv (x_1, x_2, x_3, \cdots)$. The number complexes $(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \cdots)$ for $n=1, 2, 3, \cdots$ and also the complex $(x_1, x_2, x_3, \cdots)$ are different permutations of the positive integers.

If now ξ does not belong to B_h we must have

$$\lim_{N\to\infty} \text{upper bound} \sum_{n=1}^{N} C(x_r) > h.$$

So, for sufficiently large N, say $N \geqq M'$ where M' is some positive integer depending on ξ and h, we must have $U_N > h$.

In particular, therefore, $U_{M'} > h$. But $U_{M'}$ is the maximum of $(s_1, s_2, s_3, \cdots, s_{M'})$, so there is a positive integer M $(1 \leqq M \leqq M')$ such that $s_M > h$, that is, there is a positive integer M for which

$$S_M = C(x_1) + C(x_2) + \cdots + C(x_M) > h.$$

We may now take a sphere of sufficiently small radius to ensure that the first M elements of all points of E that lie in the interior of the sphere are identical with those of ξ in value and in order. So for all these points the corresponding rearranged series are such that the Mth partial sum of each of them equals $S_M=C(x_1)+C(x_2)+\cdots+C(x_M)$ and so exceeds h in value.

Now, since $\lim \xi_n=\xi$, for all sufficiently large n, say $n\geqq P$ where P is a suitable positive integer, ξ_n lies within the sphere referred to above. So

$$\sum_{r=1}^{M} C(x_r^{(n)}) = C(x_1) + C(x_2) + \cdots + C(x_M) > h$$

for $n=P, P+1, P+2, \cdots$. Therefore

$$\lim_{N\to\infty} \text{upper bound} \sum_{r=1}^{N} C(x_r^{(n)}) > h \qquad (n = P, P+1, \cdots),$$

that is, $\xi_p, \xi_{p+1}, \xi_{p+2}, \cdots$ all belong to $E-B_h$ which is contrary to the hypothesis. So ξ belongs to B_h, that is, the set B_h is closed in E.

In a like manner, we may prove that if B_{-h} stands for the set of points of E for which

$$\lim_{N\to\infty} \text{lower bound} \sum_{r=1}^{N} C(x_r) \geqq -h$$

and if $\{\zeta_1, \zeta_2, \zeta_3, \cdots\}$ is a convergent sequence of points of E all belonging to B_{-h} with ζ as the limit which itself belongs to E, the $\zeta\in B_{-h}$. So, B_{-h} is closed in E.

It follows therefore that for any positive integer n, the set B_n+B_{-n} is closed in E, that is, the set of points of E, for which

$$\lim_{N\to\infty} \text{upper bound} \sum_{r=1}^{N} C(x_r) \leqq n$$

or

$$\lim_{N\to\infty} \text{lower bound} \sum_{r=1}^{N} C(x_r) \geqq -n$$

is closed in E.

The set B referred to above is, therefore, given by

$$B = \sum_{n=1}^{\infty} (B_n + B_{-n}).$$

So, B is a set F_σ in E, that is, the outer limiting set of a sequence of sets closed in E.

It follows that the set of points $x \in E$ for which

$$\liminf_{N\to\infty} \sum_{n=1}^{N} C(x_n) = -\infty \quad \text{and} \quad \limsup_{N\to\infty} \sum_{n=1}^{N} C(x_n) = +\infty$$

is G_δ in E.

We next propose to derive some simple properties about the power of E.

THEOREM 2. *The set E has the power c of the continuum.*

PROOF. Let p be the power of the set E. Let us associate the point $x = (x_1, x_2, x_3, \cdots)$ of the set E to the irrational number α which is the value of the simple continued fraction

$$\frac{1}{x_1 +} \frac{1}{x_2 +} \frac{1}{x_3 +} \cdots .$$

Thus to each point $x \in E$ there corresponds an irrational number α in $0 < \alpha < 1$ and to two distinct points x and y of E correspond two distinct irrational numbers in the above range. Thus the set E is equivalent to a proper subset of the real numbers. Therefore

$$p \leqq C.$$

On the other hand, to any real number β $(0 < \beta < 1)$ there corresponds a rearrangement of the conditionally convergent series $\sum_{n=1}^{\infty} C_n$ such that the rearranged series converges to β. In fact there are an infinity of such rearrangements with the sum β. So to any real number correspond an infinity of distinct points of E. Again to any other real number γ, there correspond an infinity of other distinct points of E, each of which is different from any of those corresponding to β. Thus the set of real numbers is equivalent to a proper subset of the aggregate of points of E. Therefore $p \geqq C$. Combining these two, we have $p = c$.

THEOREM 3. *Every point of the metric space E is of degree c (power of the continuum) in E.*[2]

PROOF. Take any point $x = (x_1, x_2, x_3, \cdots)$ of E. Then

$$C(x_1) + C(x_2) + C(x_3) + \cdots$$

is a rearrangement of the given conditionally convergent series. Let N be a positive integer. Let $0 < \alpha < 1$ and let

[2] The proof of Theorem 3 is due to the referee. It may be remarked that a proof of the above modelled on the plan of proof of Theorem 2 may also be easily given.

$$\alpha = .\alpha_1\alpha_2\alpha_3\alpha_4 \cdots$$

be the dyadic expansion of α which may terminate with zeros but not with ones.

Let $x_n^{(\alpha)} = x_n$ $(n = 1, 2, 3, \cdots, N+1)$ and if $\alpha_p = 0$, let

$$x_{N+2p}^{(\alpha)} = x_{N+2p}, \qquad x_{N+2p+1}^{(\alpha)} = x_{N+2p+1}$$

and if $\alpha_p = 1$, let

$$x_{N+2p}^{(\alpha)} = x_{N+2p+1}, \qquad x_{N+2p+1}^{(\alpha)} = x_{N+2p}.$$

Then to different numbers α correspond different points

$$x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)}, \cdots).$$

All points $x^{(\alpha)}$ lie within the sphere with center x and radius 2^{-N}.

And since lim $C(x_n) = 0$, the series

$$C(x_1^{(\alpha)}) + C(x_2^{(\alpha)}) + C(x_3^{(\alpha)}) + \cdots$$

has, for each α, exactly the same limits of oscillation as the series $\sum C(x_n)$.

This proves that the points of E that lie within the sphere $S(x, 2^{-N})$ has the power of the continuum.

It proves more. It easily leads to the theorem:

THEOREM 4. *Every point of E is the limit point of a set of points of E of power c at which the rearrangements of the conditionally convergent series behave in any prescribed manner.*

PROOF. Take any point $x = (x_1, x_2, x_3, \cdots)$. Let l and L be any two real numbers, $-\infty$ and $+\infty$ not excepted, such that $-\infty \leqq l \leqq L \leqq +\infty$. In a sphere with x as center and 2^{-N} as radius there exists a point $X = (x_1, x_2, \cdots, x_{N+1}, X_{N+2}, X_{N+3}, \cdots)$ at which the rearranged series has l and L as its limits of oscillation. Again, by the above, in a sufficiently small sphere with X as center lying entirely within the first sphere, an infinite set of points of the power C exist at which the rearranged series has the same limits of oscillation l and L.

Thus in every neighbourhood of x, there exists a set of points of the power of the continuum, at which the rearranged series has any prescribed limits of oscillation.

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