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A PROOF THAT THE GROUP OF ALL HOMEOMORPHISMS OF THE PLANE ONTO ITSELF IS LOCALLY ARCWISE CONNECTED

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Let P be a plane and let $H(P)$ be the group of all homeomorphisms of the plane P onto itself. We topologize $H(P)$ by defining convergence to mean uniform convergence on each compact subset of P. The resulting topology is equivalent to the compact-open topology defined in [1][1] by Fox. It is also known (see [4]) that $H(P)$ is a topological group under this topology. The result obtained in this paper is the following theorem.

THEOREM. *$H(P)$ is locally arcwise connected.*

1. **A metric for $H(P)$.** We assume a rectangular coordinate system for P and let d be the corresponding metric for P. For each positive number r we define $S(r)$ to be the set of all points (u, v) in P such that $\max(|u|, |v|) \leq r$. If f and g are members of $H(P)$ we define

$$\rho(f, g) = \sup_{r>0} \min\left(1/r, \sup_{x \in S(r)} d(f(x), g(x))\right).$$

It is a routine matter to verify that ρ is a distance function which defines an admissible metric for $H(P)$. A metric which is essentially the same as ρ is used by M. Bebutoff in [2]. We shall make use of the fact that $\rho(f, g) < \epsilon$ if and only if $d(f(x), g(x)) < \epsilon$ for all x in $S(1/\epsilon)$.

2. **Isotopy and arcs.** By an isotopy we shall mean a homotopy

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

F such that $F_t \in H(P)$ whenever $0 \leqq t \leqq 1$. It follows from a theorem of Fox in [1] that an isotopy is equivalent to a continuous function on the unit interval into $H(P)$. Since the image of the unit interval under a continuous function is always arcwise connected, we see that members f and g of $H(P)$ can be joined by an arc in a subset U of $H(P)$ if and only if there exists an isotopy F such that $F_0 = f$, $F_1 = g$, and $F_t \in U$ whenever $0 \leqq t \leqq 1$.

3. **Proof of the theorem.** Since $H(P)$ is a topological group, it is sufficient to prove that $H(P)$ is locally arcwise connected at the identity homeomorphism I. To do this it is sufficient to show that corresponding to each positive number ϵ there is a positive number δ such that if $\rho(f, I) < \delta$, where $f \in H(P)$, then there exists an isotopy B such that $B_0 = f$, $B_1 = I$, and $\rho(B_t, I) < \epsilon$ if $0 \leqq t \leqq 1$.

Suppose $\epsilon > 0$. We may assume without loss of generality that $\epsilon < 1$. Choose a positive number δ such that $\delta < \epsilon/7$ and such that $2/\delta = (2n+2)\delta$ for some positive integer n. It is well known that $H(P)$ contains exactly two components, the component containing I consisting of all orientation preserving homeomorphisms. In this connection see [8]. We shall also assume δ small enough so that the δ-neighborhood of I contains only orientation preserving homeomorphisms. Now choose any f such that $f \in H(P)$ and $\rho(f, I) < \delta$.

We define T to be the isotopy for which T_t, $0 \leqq t \leqq 1$, is the translation which increases the first coordinate of each point of P by $t(2/\delta + \delta)$ and leaves the second coordinate invariant. Define C_t, $0 \leqq t \leqq 1$, to be the set $T_t(S(1/\epsilon))$, and then define K to be the set $\bigcup_{0 \leqq t \leqq 1} C_t$. Finally, we define $R_0 = S(1/\delta)$ and $R_1 = T_1(R_0)$.

It will be convenient to prove two lemmas.

LEMMA 1. *There exists $h \in H(P)$ such that $h | C_0 = f | C_0$, $h | C_1 = I | C_1$ and $d(x, h(x)) < \epsilon$ for all x in K.*

PROOF. Let the segment $\overline{ab}$ be the side of the square R_1 consisting of all points of R_1 whose first coordinate is a minimum and let $\overline{\alpha\beta}$ be the side of R_0 consisting of all points of R_0 whose first coordinate is a maximum. We may assume that a and α have the same second coordinate. Define x_0 to be the point on $\overline{ab}$ which is at distance δ from a. If x_i, $0 \leqq i < n$, has been defined, we then define x_{i+1} to be the point on $\overline{ab}$ which is at distance 2δ from x_i and which is between x_i and b. Since $\overline{ab}$ is of length $2/\delta$ and $2/\delta = (2n+2)\delta$, x_n is at distance δ from b.

We define y_i, $0 \leqq i \leqq n$, to be the point of $f(R_0)$ nearest to x_i which has the same second coordinate as x_i. Now define z_i, $0 \leqq i \leqq n$, to be the point $f^{-1}(y_i)$. Since f transforms each point of R_0 a distance less than δ, it is easy to see that the points z_i all lie on $\overline{\alpha\beta}$ and moreover lie in the same order on $\overline{\alpha\beta}$ as the corresponding points x_i lie on $\overline{ab}$.

We now define h^* to be the homeomorphism which agrees with f on R_0, which agrees with I on R_1, and which transforms each segment $\overline{z_i x_i}$, $0 \leqq i \leqq n$, linearly into the segment $\overline{y_i x_i}$. We now make use of a theorem of Gehman (see [3]) and extend h^* to a homeomorphism h of P onto P.

Let Q_i, $0 \leqq i < n$, be the trapezoid (with interior) which has vertices x_i, x_{i+1}, z_{i+1}, z_i. It is easily seen that for each i, $0 \leqq i < n$, there is a rectangle with sides of length 6δ and 2δ which contains both Q_i and $h(Q_i)$. Thus, if $x \in Q_i$ we obtain $d(x, h(x)) \leqq \delta(40)^{1/2} < 7\delta = \epsilon$. It follows that if $x \in R_0 \cup R_1 \cup \bigcup_{i=0}^{n-1} Q_i$ then $d(x, h(x)) < \epsilon$. Moreover, it is easily seen that $K \subset R_0 \cup R_1 \cup \bigcup_{i=0}^{n-1} Q_i$. We therefore see that the homeomorphism h has the desired properties.

LEMMA 2. *There exists an isotopy G such that $G_0 = f$, $G_1 | C_0 = I | C_0$ and $\rho(G_t, I) < \epsilon$ whenever $0 \leqq t \leqq 1$.*

PROOF. Let h be as in Lemma 1. We then define $G_t = T_t^{-1} h T_t h^{-1} f$ whenever $0 \leqq t \leqq 1$. This clearly defines an isotopy G. Since $T_0 = I$, we readily obtain $G_0 = f$.

Let $x \in C_0$. We obtain $h^{-1}f(x) = x$ since $h | C_0 = f | C_0$. Now, using the fact that $T_1(x) \in C_1$ and $h | C_1 = I | C_1$, we obtain $hT_1 h^{-1} f(x) = hT_1(x) = T_1(x)$. Therefore

$$G_1(x) = T_1^{-1} h T_1 h^{-1} f(x) = T_1^{-1} T_1(x) = x.$$

We have shown that $G_1 | C_0 = I | C_0$.

Suppose $0 \leqq t \leqq 1$ and $x \in C_0$. Then $T_t h^{-1} f(x) = T_t(x) \in K$. Thus $hT_t h^{-1} f(x)$ is within ϵ of $T_t(x)$. Since T_t is a translation, this fact implies that $T_t^{-1} h T_t h^{-1} f(x)$ is within ϵ of $T_t^{-1} T_t(x) = x$. Thus $d(G_t(x), x) < \epsilon$ for each $x \in C_0 = S(1/\epsilon)$ whenever $0 \leqq t \leqq 1$. We therefore obtain $\rho(G_t, I) < \epsilon$ whenever $0 \leqq t \leqq 1$.

We now use the isotopy G defined in Lemma 2 to define the isotopy B. If $x \in P$ we define:

$$B_t(x) = G_{2t}(x) \qquad \text{for } 0 \leqq t \leqq 1/2;$$

$$B_t(x) = G_1(2(1-t)x)/2(1-t) \qquad \text{for } 1/2 < t < 1;$$

and

$$B_1(x) = I(x).$$

It is easily verified that B has the desired properties.

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