ON A THEOREM OF ERDÖS AND TURÁN

ALFRED RÉNYI

Let $p_1=2$, $p_2=3$, $p_3, \cdots, p_n, \cdots$ denote the sequence of primes. P. Erdös and P. Turán proved recently,[1] among a series of similar results, that the sequence $\log p_n$ is neither convex nor concave from some large n onwards, that is, that the sequence

$$p_{n+1}\cdot p_{n-1} - p_n^2$$

changes its sign an infinite number of times. The aim of the present paper is to prove an improvement of this theorem. It shall be proved that the total curvature of the polygonal line having the vertices $n+i\cdot\log p_n$ $(n=1, 2, 3, \cdots)$ in the complex z-plane is infinite. This implies that $p_{n+1}p_{n-1}-p_n^2$ changes its sign an infinite number of times, but our result also contains some quantitative information regarding the oscillations of the sequence $p_{n+1}pp_{n-1}-p_n^2$.

The total curvature G of a finite polygonal line situated in the complex z-plane may be defined as follows:

$$G = \sum_{k=2}^{n-1} \left| \arg \frac{z_{k+1} - z_k}{z_k - z_{k-1}} \right| \tag{1}$$

where $z_1, z_2, \cdots, z_n$ denote the complex numbers corresponding to the vertices of the polygonal line in the complex z-plane. The total curvature of an infinite polygonal line is defined as the limit of the total curvature of its segments, provided this limit exists, and as ∞ in case the total curvature of its segments tends to infinity. It is easy to see that if the polygonal line Π having the vertices $n+i\cdot\log p_n$ were convex or concave from some point onwards, its total curvature would be finite. It will be shown that this is not so, as we shall prove the following theorem.

THEOREM. *The total curvature, as defined by* (1), *of the polygonal line* Π_N *having the vertices* $z_n=n+i\cdot\log p_n$ $(p_n\leqq N)$ *exceeds* $c\cdot\log\log\log N$, *where* c *is a positive constant.*

Let $\pi(n)$ denote the number of primes not exceeding n. For proving our theorem, we shall need only the following propositions of the prime number theory:

Received by the editors October 11, 1948.

[1] P. Erdös and P. Turán, *On some new questions on the distribution of prime numbers*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 371–378.

I. $\pi(n) \sim \dfrac{n}{\log n}$, II. $\log\left(\prod_{p \leq n} p\right) \sim n$,

III. $\lim_{n\to\infty} \dfrac{p_{n+1}}{p_n} = 1$, IV. $p_{n+1} - p_n = O\left(\dfrac{p_n}{\log^2 p_n}\right)$.

I and II are different forms of the prime number theorem, III is a consequence of the prime number theorem, and IV follows from the de La Vallée Poussin estimation of the remainder term of the prime number formula. (In fact by a slight modification of the argument we could dispense with the prime number theorem and use only more elementary results.)

Let G_N denote the total curvature of π_N. We have

$$G_N = \sum_{p_{n+1} \leq N} \left| \arctan \log \frac{p_{n+1}}{p_n} - \arctan \log \frac{p_n}{p_{n-1}} \right|. \tag{2}$$

In what follows $c_1, c_2, \cdots$ denote positive constants. Using III and the identity $\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha\tan\beta)$, we obtain from (2)

$$G_N \geqq c_1 \cdot \sum_{p_{n+1} \leq N} \left| \log \frac{p_{n+1} \cdot p_{n-1}}{p_n^2} \right|. \tag{3}$$

Now we need the following lemma.

LEMMA A.

$$\sum_{n=1}^{\infty} \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2} < \infty.$$

PROOF. According to IV, we have

$$\begin{aligned} &\sum_{N \leq p_n < 2N} \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2} \\ &\qquad \leqq \max_{N \leq p_n < 2N} \left(\frac{p_{n+1} - p_n}{p_n^2} \right) \cdot \sum_{N \leq p_n < 2N} (p_n - p_{n-1}) < \frac{c_2}{\log^2 N}. \end{aligned} \tag{4}$$

Putting $N = 2^k$ in (4) and summing for $k = 1, 2, 3, \cdots$, we obtain Lemma A.

When we apply Lemma A and the identity

$$\frac{p_{n+1} \cdot p_{n-1}}{p_n^2} = 1 + \frac{p_{n+1} - 2p_n + p_{n-1}}{p_n} - \frac{(p_{n+1} - p_n)(p_n - p_{n-1})}{p_n^2}, \tag{5}$$

it follows easily that

$$G_N \geqq c_3 \sum_{p_{n+1} \leqq N} \frac{|p_{n+1} - 2p_n + p_{n-1}|}{p_n} - c_4. \tag{6}$$

Now we prove the following lemma.

LEMMA B. *If each of*

$$a,\ a + D,\ a + 2D,\ \cdots,\ a + k \cdot D \tag{7}$$

is a prime, we have

$$D \geqq \exp(c_5 k). \tag{8}$$

PROOF. First we have $k < a$, for if not, $a + aD$, which is composite, would be contained in the sequence (7). Further if p is a prime, $p \leqq k$, p must be a divisor of D, for if not, the numbers a, $a+D$, $a+2D$, $\cdots$, $a+(p-1)\cdot D$ would be all incongruent mod p, and thus one of them would be divisible by p, and not equal to p owing to $p \leqq k < a$. Thus D is divisible by p, and when we apply II, Lemma B follows.

To prove our theorem we need one more lemma. For the sake of brevity let us put

$$\Delta_k = p_{k+1} - 2p_k + p_{k-1}. \tag{9}$$

Let us divide the sequence of primes $p_{k+1} \leqq n$ into blocks $(p_{\nu_r}, p_{\nu_r+1}, p_{\nu_r+2}, \cdots, p_{\nu_r+k_r})$ characterized by $\Delta_{\nu_r} \neq 0$, $\Delta_{\nu_r+\nu} = 0$ for $\nu = 1, 2, \cdots, k_r$. The number of these blocks shall be denoted by $\lambda(n)$, that is, we put

$$\lambda(n) = \sum_{p_{\nu_r+1} \leqq n} 1. \tag{10}$$

We shall prove the following lemma.

LEMMA C.

$$\lambda(n) \geqq \frac{c_6 n}{\log n \cdot \log \log n}.$$

PROOF. According to Lemma B we have

$$\sum_{r=1}^{\lambda(n)} k_r \cdot \exp(c_5 k_r) \leqq \sum_{p_{k+1} \leqq n} (p_{k+1} - p_k) \leqq n. \tag{11}$$

On the other hand

$$\sum_{r=1}^{\lambda(n)} (k_r + 1) \geqq \pi(n) - 1. \tag{12}$$

If $\lambda(n) > \pi(n)/2$, our lemma is proved. Thus we may suppose $\lambda(n) < \pi(n)/2$. In this case it follows from (12) and I that

$$\sum_{r=1}^{\lambda(n)} k_r > \frac{c_7 \cdot n}{\log n}. \tag{13}$$

Applying the inequality of the arithmetic and geometric means as well as the inequality of Cauchy-Schwarz, we obtain, using (11) and (13), that

$$\begin{aligned} n &\geqq \sum_{r=1}^{\lambda(n)} k_r \exp(c_5 k_r) \geqq \left(\sum_{r=1}^{\lambda(n)} k_r\right) \exp\left(c_5 \frac{\sum_{r=1}^{\lambda(n)} k_r^2}{\sum_{r=1}^{\lambda(n)} k_r}\right) \\ &\geqq \frac{c_7 n}{\log n} \cdot \exp\left(\frac{c_5 \sum_{r=1}^{\lambda(n)} k_r}{\lambda(n)}\right). \end{aligned} \tag{14}$$

It follows that

$$\log \log n \geqq \log c_7 + \frac{c_5 c_7 n}{\lambda(n) \log n}, \tag{15}$$

which proves Lemma C.

Now the proof of our theorem can be easily completed. Using (6) and with respect to $\Delta_{\nu_r} \neq 0$ we have evidently

$$G_N \geqq c_3 \sum_{r=1}^{\lambda(N)} \frac{1}{p_{\nu_r}} - c_4. \tag{16}$$

Hence by (10)

$$\begin{aligned} G_N \geqq c_3 \sum_{n=1}^{N} \frac{\lambda(n) - \lambda(n-1)}{n} - c_4 &\geqq c_3 \sum_{n=1}^{N} \frac{\lambda(n)}{n(n+1)} - c_4 \\ &\geqq c_8 \sum_{n=1}^{N} \frac{1}{n \cdot \log n \cdot \log \log n}, \end{aligned} \tag{17}$$

which proves our theorem.

It may be mentioned that the question whether there exist blocks of primes $(p_{\nu_r}, p_{\nu_r+1}, \cdots, p_{\nu_r+k_r})$ with $\Delta_{\nu_r+\nu} = 0$, $\nu = 1, 2, \cdots, k_r$, of any length k_r, and even the question whether there exist an infinity of indices k for which $\Delta_k = 0$ are unsolved.

UNIVERSITY OF BUDAPEST