It was proved by Hurewicz\(^1\) that a compact space which is both LC\(^1\) and lc\(^n\) is LC\(^n\). In the present paper the corresponding result for locally compact spaces is proved, (a) for uniform local connection, and (b) for relative local connection.\(^2\) The extension of Hurewicz's theorem to locally compact spaces is included in (b). The main difficulty in extending Hurewicz's methods is that his "Satz 6," on the passage from \(\varepsilon\)-homotopy to true homotopy, cannot be carried over to locally compact spaces without substantial modification, even when uniform local connection is assumed. To overcome this a stronger form of the lc\(^p\) and LC\(^p\) conditions is used, namely (for lc\(^p\)), the existence of a function \(\xi(\delta, x)\) such that, given a compact set \(F\) in the neighbourhood \(U(x, \xi(\delta, x))\) of any point \(x\), there is a compact subset \(F'\) of \(U(x, \delta)\) such that every \(g\)-cycle in \(F\) bounds in \(F'\), for \(0 \leq g \leq p\); and analogously for LC\(^p\). It is shown that these are equivalent to the ordinary lc\(^p\) and LC\(^p\) properties in locally compact (metric) spaces.

1. Definitions. It is assumed once for all that the containing space \(X\) is locally compact, and has metric \(\rho\).\(^3\) Homologies are relative to integral coefficients; cycles in \(X\) are Vietoris-cycles (but finite cycles are chains on some simplicial complex with vertices in \(X\)). The statement that \(\Gamma\) bounds in \(E\) means that \(\Gamma\) bounds in a compact subset of \(E\). \(p\) denotes an integer not less than 0.

The letter \(F\), with various suffixes, always denotes a compact set. If \(G\) is open, the statement "\(F \subseteq G\) with a margin \(\alpha\)" means that \(\alpha > 0\), and \(\text{Cl}(U(F, \alpha))\) is compact and contained in \(G\).\(^4\) The existence of margins for every such \(F\) and \(G\) is ensured by the local compactness of \(X\).

A set \(E_1\) is \(ac^p\) rel. \(E_2\) ("acyclic up to \(p\) rel. \(E_2\)") if every \(g\)-cycle in \(E_1\) bounds in \(E_2\), for \(0 \leq g \leq p\). \(E_1\) is \(as^p\) rel. \(E_2\) ("aspherical up to \(p\) rel. \(E_2\)") if every mapping of the \(g\)-sphere \(S^g\) into \(E_1\) is null-homotopic in \(E_2\), for \(0 \leq g \leq p\). The set \(E_1\) is strongly \(ac^p\) (or strongly \(as^p\)) rel. \(E_2\) if, given any \(F\) in \(E_1\) there is an \(F'\) in \(E_2\) such that \(F\) is \(ac^p\) (or \(as^p\)) rel. \(F'\).
Among the various definition of lcp and ulc\(^p\) that are in the field we choose those which impose the heaviest conditions on the bounding cycles, and therefore the lightest conditions on the space. If \(E_1 \subseteq E_2 \subseteq X\), \(E_1\) is lcp rel. \(E_2\) if there is a positive function \(\eta(x, \delta)\) such that \(E_1 U(x, \eta(x, \epsilon))\) is acp rel. \(E_2 U(x, \epsilon)\) for all points \(x\) of \(E_2\) and any positive \(\epsilon\); and \(X\) is (absolutely) lcp if it is lcp rel. \(X\). The space \(X\) is ulc\(^p\) if there is a positive function \(\eta(\epsilon)\) such that for all points \(x\) of \(X\), \(U(x, \eta(\epsilon))\) is acp rel. \(U(x, \epsilon)\).

2. Homology. We consider chain-realisations of (abstract simplicial) complexes, in the sense of Lefschetz [7] and Begle [1]. The complexes realised all have their vertices in \(X\), and every realisation \(t\) of a complex \(K\) is to satisfy \(t(\sigma^0) = \sigma^0\), for all vertices \(\sigma^0\) of \(K\). If \(C\) is a finite chain on a complex in \(X\), a realisation of \(C\) means a chain \(t(C)\), where \(t\) is a realisation of the carrier complex \(\|C\|\).

If \(X\) is connected, a realisation of any complex can, by an arbitrarily small displacement of the vertices, be so modified that accidental clashes are avoided, that is, any common vertex of \(\|t(\sigma)\|\) and \(\|t(\sigma')\|\) belongs to some \(\|t(\sigma'')\|\), where \(\sigma''\) is a common sub-cell of \(\sigma\) and \(\sigma'\). It will be assumed that this is always arranged. There is, then, for any vertex \(x\) of \(\|t(C)\|\), a unique simplex \(\sigma\) of lowest dimension in \(\|C^p\|\), such that \(x \in \|t(\sigma)\|\). This \(\sigma\) may be called the \(C^p\)-carrier of \(x\).

**Theorem 1.** If \(X\) is lc\(^p\) and \(\alpha > 0\), and if \(F\) is a compact subset of \(X\), there exists a finite set of \(p\)-cycles \(\Gamma^p_1, \Gamma^p_2, \ldots, \Gamma^p_k\) in \(U(F, \alpha)\) such that every \(\Gamma^p_i\) in \(F\) has \(\sum_{t=1}^k n_i \Gamma^p_i\) in \(U(F, \alpha)\), for suitable integers \(n_i\).

Since \(X\) is 0-lc its components are open sets, and therefore the compact set \(F\) meets only a finite number of them. It is clearly sufficient to prove the theorem for each separate component meeting \(F\), that is, we may assume \(X\) to be connected.

The theorem is proved by combining the following results.

(A). Given a compact set \(F_1\) there exists a positive function \(\lambda(\epsilon)\)

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\(^{a}\) Cf. Eilenberg and Wilder [3] for the corresponding homotopy property.

\(^{b}\) Note that the definition of a partial realisation \(t\) of \(K\) requires \(t(\sigma^0)\) to be defined for all 0-cells \(\sigma^0\) of \(K\). The norm of a full or partial realisation \(t\) of \(K\) is max \(\rho(x, y)\) for \(x \in \|t(\sigma_1)\|, y \in \|t(\sigma_2)\|\), where \(\sigma_1\) and \(\sigma_2\) are subcells of the same cell \(\sigma\) of \(K\). The mesh of \(t\) is max \(\Delta \|\sigma'\|\) for all simplexes \(\sigma'\) of chains \(t(\sigma)\). (\(\Delta E\) = diameter of \(E\)).

\(^{7}\) Wilder [8] (see also Begle [2, Corollary 2.3]) has proved that when homologies are mod \(m\), the conditions of Theorem 1 imply that at most a finite number of cycles of \(F\) are independent in \(U(F, \alpha)\). The analogous result with integral coefficients is not strong enough for present purposes, since it would allow, for example, an infinite base \((\Gamma^p_i)\) with \(\Gamma^p_i \sim 2 \Gamma^p_{i+1}\).
such that any partial chain-realisation \( t_0 \) of a complex of dimension not greater than \( p+1 \) of norm less than \( \lambda(\epsilon) \) in \( F_1 \) can be extended to a full realisation in \( X \) of norm less than \( \epsilon \); and there is a positive function \( \kappa(\delta, \epsilon) \) such that if mesh \( t_0 < \kappa(\delta, \epsilon) \), mesh \( t \) can be made less than \( \delta \).

This is Begle [2], Theorem 2.1.8 (Note that when \( t_0 \) is defined only for vertices the \( \kappa \)-condition is automatically satisfied.)

The \( \alpha \) of the enunciation of Theorem 1 may be supposed such that \( \text{Cl}(U(F, \alpha)) \) is compact. Let \( \sum_{e_n} e_n \) be a positive series with sum less than \( \alpha \), such that \( e_{n+1} < e_n/3 \) and, taking \( F_1 = \text{Cl}(U(F, \alpha)) \) in (A), \( e_{n+1} < \lambda(\epsilon_n) \).

(B) Every finite cycle \( C^n_p \) in \( F \) of mesh less than \( \epsilon_1 \) is the first member of a projection-cycle \( \{ C^n_p \} \) in \( U(F, \alpha) \), the projection \( \phi_n : C^{n+1}_p \to C^n_p \) being an \( \epsilon_n \)-projection.\(^8\)

Let \( \eta_n = \sum_{e_n} \epsilon_n \) and make the inductive hypothesis that a finite cycle \( C^n_p \) of mesh less than \( \epsilon_{n+1} \) is defined in \( U(F, \eta_{n-1}) \) (for \( n = 0 \), in \( F \)). Since \( U(F, \eta_{n-1}) \subseteq U(F, \alpha) \subseteq F_1 \), there is by (A) a chain-realisation \( C^{n+1}_p \) of \( C^n_p \), of mesh less than \( \epsilon_{n+1} \) and norm less than \( \epsilon_n \), and hence contained in \( U(F, \eta_{n-1} + \epsilon_n) = U(F, \eta_n) \). This justifies the recursive definition of \( C^n_p \). For each vertex \( x \) of \( C^{n+1}_p \), take \( \phi_n(x) \) to be any vertex of the \( C^n_p \)-carrier of \( x \) (defined above). Then \( \phi_n \) is an \( \epsilon_n \)-projection.

Let the sequence \( (\epsilon_n) \) satisfy the conditions of (B), and also \( \epsilon_{n+1} < \kappa(\epsilon_{n+2}, \epsilon_n) \) (\( F_1 \) being as before).

(C) Let \( \Gamma^p_r = (Z^p_m) \) be, for \( r = 1, 2, \) a \( p \)-cycle in \( X \). Sufficient conditions for \( \Gamma^p_r \sim \Gamma^p_s \) in \( U(F, \alpha) \) are

\[ Z^p_{rn} \sim_{\epsilon_{n+1}} Z^p_{r, n+1} \text{ in } U(F, \eta_n) \quad \text{for } n \geq 0 \]

and

\[ Z^p_{10} \sim_{\epsilon_1} Z^p_{20} \text{ in } U(F, \epsilon_0). \]

Let \( Z^p_{rn+1} - Z^p_{rn} = \beta Y^p_{rn+1} \) (= boundary of \( Y^p_{rn+1} \)), where \( Y^p_{rn+1} \) is a chain of mesh less than \( \epsilon_{n+1} \) in \( U(F, \eta_n) \) (whence mesh \( Z^p_m < \epsilon_{n+1} \)); and let \( Z^p_{10} - Z^p_{20} = \beta D^p_{0+1} \), where \( D^p_{0+1} \) is of mesh less than \( \epsilon_2 \), and \( D^p_{0+1} \subseteq U(F, \epsilon_0) \). Assume inductively that for some \( n \geq 0 \), a finite chain \( D^p_{0+1} \) has been defined so that \( \beta D^p_{0+1} = Z^p_{10} - Z^p_{20} \), mesh \( D^p_{0+1} < \epsilon_{n+2} \), and \( D^p_{0+1} \subseteq U(F, \eta_n) \). Then \( Q^p_n = D^p_{0+1} + Y^p_{rn+1} - Y^p_{rn} \) is a \( (p+1) \)-chain with boundary \( Z^p_{rn+1} - Z^p_{20} \), and \( \|Q^p_n\| \subseteq U(F, \eta_n) \subseteq F_1 \). A partial realisation, \( t_0 \), of \( \|Q^p_n\| \) is determined by putting \( t_0(\sigma) = \sigma \) if \( \sigma \in \|Q^p_n\| \).

\(^8\) Our definitions are slightly different, but the proof is almost exactly similar.

\(^9\) Cf. Begle [1, Lemma 2.4]. The property asserted of \( \phi_n \) means that the projection-prism has mesh less than \( \epsilon_n \), and hence mesh \( C^n_p < \epsilon_{n+1} \).

\(^{10}\) \( \|Q^p_n\| \) = set of vertices of \( \|Q^p_{n+1}\| \), and in general \( K_m = \) set of cells of \( K \) of dimensions not greater than \( m \).
Local connection in locally compact spaces

Mesh $t_0 < \epsilon_{n+4} < \kappa(\epsilon_{n+3}, \epsilon_{n+1})$, and norm $t_0 \leq$ mesh $\|Q^{p+1}_n\| < \epsilon_{n+3} < \lambda(\epsilon_{n+1})$. Hence $t_0$ can be extended to give a realisation $D^{p+1}_{n+1}$ of $Q^{p+1}_n$, of mesh less than $\epsilon_{n+3}$, and norm less than $\epsilon_{n+1}$. Hence

$$D^{p+1}_{n+1} \subseteq U(F, \eta_n + \epsilon_{n+1}) = U(F, \eta_{n+1});$$

and

$$\beta D^{p+1}_{n+1} = t(\beta C^{p+1}_n) = t(Z_{1,n+1} - Z_{2,n+1}) = Z_{1,n+1} - Z_{2,n+1}.$$

The recursive definition of $D^{p+1}_n$ is justified, and (C) is therefore proved.

Proof of Theorem 1. Let $\{U(x_i, \epsilon_4/6)\}$ be a finite covering of $F$, with $x_i \in F$, and let $N$ be the nerve of the covering $\{U(x_i, \epsilon_4/2)\}$, with the points $x_i$ as vertices. Choose a $p$-dimensional basis of homology $C_0^p (i=1, 2, \cdots, k)$ of $N$. Since mesh $N < \epsilon_4$, (B) is applicable, with the series $(\epsilon_0, \epsilon_1, \cdots)$ replaced by $(\epsilon_4, \epsilon_4, \cdots)$, to give a projection-cycle $\Gamma^p_0$ in $U(F, \alpha)$ with first member $C_0^p$. The cycles $\Gamma^p_0$ are the required set. For let $\Gamma^p$ be any $p$-cycle in $F$, and $(Z^p_n)$ $(n=0, 1, \cdots)$ a subsequence of its members satisfying

(a) $Z_{n+1} \sim_{\epsilon_{n+3}} Z_{n+2} \subseteq F$, (b) mesh $Z_{n+1} < \epsilon_4/6$.

If, for each $x$ of $F$, $\theta(x)$ is a vertex of $N$ in $U(x, \epsilon_4/6)$, and if $x$ and $y$ are vertices of the same cell of $Z_0$, $\rho(\theta x, \theta y) < \epsilon_4/6 + \rho(x, y) + \epsilon_4/6 < \epsilon_4/2$, and hence $\theta$ is a simplicial mapping of $Z'_{n+1} \subseteq N$.

Let $\theta(Z^p_0) = Z'_{n+1} \sim \sum n_i C^p_{i0}$ in $N$, for some $n_i$. The pair of cycles $\Gamma^p$ and $\sum n_i \Gamma^p_i$ satisfy the conditions (1) and (2) of (C). Let $\Gamma^p = \{C_{0i}, C_{1i}, C_{2i}, \cdots \}$. Condition (1). For $\Gamma^p$ this follows from (a) above. The chain $\sum n_i \Gamma^p_i$ is a projection-cycle for which $\phi^p$ is an $\epsilon_{n+3}$-projection, and all vertices of $\| \sum n_i C_{n+i} \|$ belong to $\| \sum n_i C_{n+i} \|$. Condition (1) is satisfied if $\| C_{n+1} \| \subseteq U(F, \eta_n)$. This is so, since $\| C_{n+1} \| \subseteq U(F, \sum \epsilon_3 \epsilon_i)$ (proof of (B), $\epsilon_{n+3}$ replacing $\epsilon_i$), and $\sum \epsilon_3 \epsilon_i < \sum \epsilon_i = \eta_n$. Condition (2). The $\theta$-prism joining to $Z^p_0 Z'^p$ has mesh less than $\epsilon_4/2 < \epsilon_4$, whence $Z^p_0 \sim_{\epsilon_4} Z'^p \subseteq F$; and $Z'^p \sim \sum n_i C^p_{i0}$ in $N$, a complex of mesh less than $\epsilon_4$ in $F$.

Theorem 2. If $X$ is lcp, an open set $G_1$ which is acp rel. an open set $G_2$ is also strongly acp rel. $G_2$.

Suppose $F \subseteq G_1$ with a margin $\alpha$. For $0 \leq q \leq p$, let $\Gamma^q_1, \Gamma^q_2, \cdots, \Gamma^q_{q+1}$ be a basis of $q$-cycles in $U(F, \alpha)$ constructed as in Theorem 1. Let $\Gamma^q_1$

\[11\] By the general rule that $t(\sigma^q) = \sigma^q$, above.
bound in the subset $F_{q,i}$ of $G_2$. Then every $q$-cycle in $F$ bounds in the compact subset

$$F' = \text{Cl}(U(F, \alpha)) \cup \bigcup_{q=0}^{p} \bigcup_{i=1}^{k} F_{q,i}$$

of $G_2$.

3. Homotopy. The relation of homotopy is denoted by $\simeq$ and $\varepsilon$-homotopy by $\simeq$. $S^p$ is the sphere $\sum_0^p \xi^2 = 1$ in $R^{p+1}$; $S^r$ is, for $r < p$, the intersection of $S^{r+1}$ with $\xi_{r+1} = 0$; $c_0$ is the point $(1, 0, \cdots, 0)$ of $R^{p+1}$. A set $E_1$ is $\varepsilon$-as$^p$ rel. $E_2$ if, for $0 \leq q \leq p$, every mapping $f: S^q \to E_1 \simeq_\varepsilon 0$ in $E_2$ rel. $c_0$.

**Theorem 3.** Let the open set $G$ be LC$^{p-1}$ rel. $X$ and suppose that a positive function $\eta_0(\delta, x)$ exists with this property: to any point $x$, and any compact $F$ in $GU(x, \eta_0(\delta, x))$ there corresponds a compact $F'$ in $GU(x, \delta)$ such that $F$ is $\varepsilon$-as$^p$ rel. $F'$ for every positive $\varepsilon$. Then $GU(x, \eta_0(\delta, x))$ is strongly as$^p$ rel. $GU(x, \delta)$; and therefore $G$ is LC$^p$ rel. $X$. (The LC$^{p-1}$ condition is vacuously satisfied if $p = 0$.)

**Corollary.** If $G = X$ and $\eta_0(\delta, x) = \eta_0(\delta)$, independent of $x$, $X$ is $p$-ULC.

This theorem replaces "Satz 6" of Hurewicz [4] in locally compact spaces. Although our proof follows his closely, the many changes of detail make it necessary to give the full proof, which depends on the following lemmas D, E, and F.

(D) $(p \geq 1)$ Given a compact set $F$ in a locally compact $LC^{p-1}$ space $X$, and a positive number $\varepsilon$, there exists a positive $\eta_2 = \eta_2(\varepsilon, F)$ with the following property: if $P$ is a polyhedron, and $Q$ a subpolyhedron of $P^{p-1}$, and if $f_0, f_1$ map $P$ into $F$ and satisfy $\rho(f_0, f_1) < \eta_2$, there exists a mapping $f_0: P \to X$, agreeing with $f_0$ on $Q$, and deformable into $f_1$ within $\varepsilon$, in $X$.

The proof of (D) is omitted, since only obvious changes are needed in the proofs of H, Sätze 1–3.

Let $G$ be as in Theorem 3. Let $F_0 \subseteq G$ with a margin $\alpha_0$, and let $0 < \delta \leq \alpha_0$. There exists a finite covering $\{ U_i \} = \{ U(x_i, \eta_0(\delta, x_i))/2 \}$

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13 See H, p. 477: $f_1 \sim \epsilon f_2$ if $f_1$ and $f_2$ are connected by an $\epsilon$-chain of points in the space of mappings.

14 This is the homotopy-local-connection introduced by Lefschetz [5]. For relative local connection see Eilenberg and Wilder [3].

15 No distinction is made in terminology between a polyhedral complex $P$ and the polyhedron which is its locus; but the corresponding abstract simplicial complex determined by the vertices of $P$ is to be distinguished from $P$. It is denoted by $\|P\|$.

16 That is, $f_0'$ and $f_1$ are connected in the space $XP$ by an arc of diameter less than $\epsilon$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
of the compact set \( F_0 \), with \( x_i \in F_0 \). Since \( \eta_0(\delta, x_i) \leq \delta \leq \alpha_0 \), \( U \subseteq G \).

Let \( \eta = \min_i \eta_0(\delta, x_i) \).

(E) \( (p \geq 1) \) If \( f_1, f_2 \) map the \( p \)-element \( E^p \) (with boundary \( S^{p-1} \)) into \( F_0 \), and if \( f_1(S^{p-1}) = f_2(S^{p-1}) \) and \( \Delta f_r(E^p) < \eta_0/2 \), for \( r = 1, 2 \), then \( f_1 \simeq f_2 \) with fixed \( S^{p-1} \) in \( U(x_i, \delta) \), for some \( i \).

If \( y \in f_1(S^{p-1}), \) one of the points \( x_i \) satisfies \( \rho(x_i, y) < \eta_0(\delta, x_i)/2 \) and therefore \( f_r(E^p) \subseteq U(x_i, \eta_0(\delta, x_i)/2) \subseteq U(x_i, \eta_0(\delta, x_i)) \) for \( r = 1, 2 \). Hence \( f_1 \simeq f_2 \) in \( U(x_i, \delta) \), by the conditions of Theorem 3 and H, Satz 4.

We suppose that for each compact \( F_0 \) and each positive \( \delta \), such a covering \( \{ U_i \} \) is chosen, and denote the corresponding \( \min_i \eta_0(\delta, x_i) \) by \( \eta_0(\delta, F_0) \). Let \( \mu(x, y) \) be defined as in H. 18

(F) Given any compact \( F_0 \) in \( G \) there exists \( \eta_4(\delta, F_0) > 0 \) such that if \( f_1, f_2 \) map \( S^{p} \) into \( F_0 \) and \( \rho(f_1, f_2) < \eta_4(\delta, F_0) \), then \( \mu(f_1, f_2) < \delta \). (The distances \( \rho(f_1, f_2) \) and \( \beta(f_1, f_2) \) are in the function space \( X^{G_0} \).)

First suppose \( p > 0 \). Let \( \alpha \) be a margin of \( F_0 \) rel. \( G \). We define:

\[
\delta' = \min(\delta, \alpha), \quad F_1 = \text{Cl}(U(F_0, \alpha)),
\]

\[
\xi = -\frac{1}{6} \eta_0(\delta', F_0), \quad \eta_4(\delta, F_0) = \eta_4(\xi, F_0),
\]

the functions \( \eta_2, \eta_4 \) having the meanings given above. Let two mappings \( f_1, f_2 \) of \( S^p \) into \( F_0 \) be given, satisfying \( \rho(f_1, f_2) < \eta_4(\delta, F_0) \). Let \( S^{p} \) be simplicially sub divided into a polyhedron \( \Sigma^p \), so finely that \( \Delta f_r(S^p) \leq \xi \) for each (continuous) \( p \)-simplex \( \sigma^p \) of \( \Sigma^p \), \( r = 1, 2 \). Then by (D), a mapping \( f'_1 : S^p \to X \) exists, agreeing with \( f_1 \) on \( S^{p-1} \), and such that \( f'_1 \subseteq f_1 \) within \( \xi \) in \( G \). Hence \( \mu(f'_1, f_1) \leq \xi < \delta/2 \). Also, since \( \delta' \leq \alpha \), \( f'_1(S^p) \subseteq U(F_0, \alpha/2) \subseteq F_1 \); and \( \rho(f'_1, f_1) \leq \mu(f'_1, f_1) \leq \xi \). Therefore \( \Delta f'_1(\sigma^p) \leq 3\xi = \eta_0(4-\delta', F_1)/2 \) for every \( \sigma^p \) of \( S^p \). By Lemma (E) it follows that \( f'_1(S^p) \subseteq f'_1(S^p) \subseteq f_1(\sigma^p) \), with fixed \( \beta(\sigma^p) \), in a set of diameter \( \delta/2 \), for every positive \( \epsilon \). Since this holds for every \( \sigma^p \) of \( S^p \), \( \rho(f'_1, f_1) < \delta/2 \). Thus \( \mu(f_1, f_2) \leq \mu(f_1, f'_1) + \mu(f'_1, f_2) < \delta \).

If \( p = 0 \) let \( \eta_4(\delta, F_0) = \eta_0(\delta', F_0)/2 \). (The definition of \( \eta_8 \) remains significant when \( p = 0 \).) It is sufficient in this case to show that if \( x, y \in F_0 \) and \( \rho(x, y) < \eta_4(\delta, F_0) \), then \( x \) and \( y \) are joined by an \( \epsilon \)-chain of points of \( G \), of diameter less than \( \delta \). There exists a point \( x_i \) such that \( \rho(x, x_i) < \eta_0(\delta', x_i)/2 \). Since \( \rho(x, y) < \eta_0(\delta', x_i)/2 \), \( x \) and \( y \) are both in \( U(x_i, \eta_0(\delta', x_i)) \subseteq U(x_i, \alpha) \subseteq G \). Hence the required chain exists, by the conditions of Theorem 3.

Theorem 3 can now be proved. Let a positive \( \delta \), a point \( x \) of \( X \), and a compact \( F \) in \( GU(x, \eta_0(\delta, x)) \) be given, and let \( F' \) be a set as in the

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18 In any metric space \( R \), \( \mu(x, y) \) is the greatest lower bound of diameters of subsets of \( R \) in which \( x \) and \( y \) are \( \epsilon \)-connected for every positive \( \epsilon \).
enunciation of Theorem 3. Then \( F \) is as\( p \) rel. \( F_2 = \text{Cl}(U(F', \beta)) \), where \( \beta \) is any margin of \( F' \) rel. \( GU(x, \delta) \). Let \( \delta_n = \eta_n(\beta/2^n, F_2) \), let \( x_0 \) be a point of \( F \), and let \( f \) map \( S^p \) into \( F \). By the conditions of the theorem, the points \( f \) and \( [x_0] \) (constant function \( x_0 \)) of \( F_2^p \) are joined by a \( \delta_1 \)-chain, \( L_1 \), of points, all lying in \( F'^{1p} \). Assume inductively that \( L_1, \cdots, L_n \) have been determined, \( L_n \) being formed by joining each consecutive pair of points of \( L_{n-1} \) by a \( \delta \)-chain; and that the mappings which are the "points" of \( L_n \) all map \( S^p \) into \( U(F', \sum_{i=1}^{n-1} \beta/2^i) \subset F_2 \) (into \( F' \) when \( n = 1 \)). Two consecutive points \( f_r, f_s \) of \( L_n \) satisfy \( \rho(f_r, f_s) < \delta_n = \eta_n(\beta/2^n, F_2) \); and hence by Lemma \( (F) \), \( \mu(f_r, f_s) < \beta/2^n \), that is, \( f_r \) and \( f_s \) can be joined by a \( \delta_{n+1} \)-chain of total diameter less than \( \beta/2^n \) in the function space. This justifies the recursive definition of \( L_n \).

It now follows, exactly as in H, p. 481, that \( \text{Cl}(U_{i=1}^n L_n) \) is the locus of a continuous path joining \( f \) to \([x_0]\) in \( F_2^p \). The proof of Theorem 3 is thus completed.

4. Homology and homotopy. Theorem 4, general case \((p \geq 2)\). If \( G \) is LC\( i \) rel. \( X \) and lc\( p \) rel. \( X \) there is a positive function \( \xi(\delta, x) \) such that \( GU(x, \xi(\delta, x)) \) is strongly as\( p \) rel. \( GU(x, \delta) \) for all \( x \) of \( X \).

Case 0: Put \( p = 0 \) and omit "LC\( i \) rel. \( X \) and."

Case 1: Put \( p = 1 \) and omit "and lc\( i \) rel. \( X \)."

Immediate corollaries of this theorem are

**Theorem 4.1 \((p \geq 2)\).** If \( G \) is LC\( i \) and lc\( p \) it is LC\( p \), all rel. \( X \).

(When \( G = X \) this is the generalisation of Hurewicz's theorem to locally compact spaces.)

**Theorem 4.2 \((p \geq 0)\).** If \( G \) is LC\( p \) rel. \( X \) there exists a positive function \( \xi(\delta, x) \) such that \( GU(x, \xi(\delta, x)) \) is strongly as\( p \) rel. \( GU(x, \delta) \) for all \( x \) of \( X \).

**Proof of Theorem 4, case 0.** Let \( \eta(\delta, x) \) be such that \( GU(x, \eta(\delta, x)) \) is ac\( 0 \) rel. \( GU(x, \delta) \), and let \( F \subseteq GU(x, \eta(\delta, x)) \). By Theorem 2 there is an \( F' \) in \( GU(x, \delta) \) such that \( F \) is ac\( 0 \) rel. \( F' \). This implies that for any positive \( e \) any two points of \( F \) are joined by an \( e \)-chain in \( F' \), that is, that \( F \) is \( e \)-as\( 0 \) rel. \( F' \) for every positive \( e \). Thus the conditions of Theorem 3 \((0)\) are satisfied, if \( \eta_0 \) is replaced by \( \eta \).

**Proof of Theorem 4, case 1.** This is contained in the following theorem.

**Theorem 5.** If \( X \) is LC\( 1 \), any open set \( G_1 \) which is as\( 1 \) rel. an open set \( G_2 \) is also strongly as\( 1 \) rel. \( G_2 \).
Let $F \subseteq G_1$ with margin $\alpha_0$. Then there exist positive $\alpha_1$ and $\alpha_2$ such that, for $r=0,1$, if $x \in F$, $U(x, \alpha_{r+1})$ is as $^1$ rel. $U(x, \alpha_r/3)$. Let \{ $U(x_i, \alpha_2/6)$ \} be a finite covering of $F$ ($x_i \in F$), and let $N$ be the nerve of the covering \{ $U(x_i, \alpha_2/2)$ \} realised in $X$, with the $x_i$ as vertices. If then $P$ is a polyhedron abstractly isomorphic with $N$, the mapping $g_0$ of $P^0$ into $N^0$ determined by the isomorphism can be extended to a mapping $g_1$ of $P^1$ into $U(F, \alpha_1/3)$.

Let $f$ be a re-entrant path in $F$, that is, $f: (0, 1) \rightarrow F$ with $f(0) = f(1) = x_0 = g_0(x_0)$, say; and let the points $0 < \tau_1 < \tau_2 < \cdots < \tau_k = 1$ divide $f$ into sub-paths $s_j$ of diameter less than $\alpha_2/6$. Now each $f(\tau_i)$ is in some $U(y_j, \alpha_2/6)$, where $y_j \in N^0$ ($y_0 = y_k = x_0$), and a path $\gamma_j$ of diameter less than $\alpha_1/3$ therefore runs from $y_j$ to $f(\tau_i)$. By the usual process of extruding "tails" the path $f = \sum s_j$ is deformable in $U(F, \alpha_1/3)$ into

$$\sum_{j=1}^{k} (\gamma_{j-1} + s_j - \gamma_j)$$

the $\sum$ and $+$-signs denoting the usual path-summation. Now $\rho(y_{j-1}, y_j) < \alpha_2/6 + \Delta s_j + \alpha_2/6 < \alpha_2/2$ and therefore $y_{j-1}, y_j$ are the $g_1$-images of the ends of a 1-cell $\sigma'_j$ of $P^1$. Since $g_1(\sigma'_j)$ and $\gamma_{j-1} + s_j - \gamma_j$ are both in $U(y_j, \alpha_1)$, the path $\gamma_{j-1} + s_j - \gamma_j$ is deformable in $U(y_j, \alpha_0)$, with fixed end points, into the path $g_1|\sigma'_j$. Thus $f \approx g_1(s)$ in $U(F, \alpha_0)$, where $s$ is a path on $P$ with $s(0) = s(1) = x_0$.

Let $P_1, P_2, \ldots, P_l$ be the components of $P$, and let the paths $a_{1r}, a_{2r}, \ldots, a_{mr}$ in $P^r$ be representatives of a base of the fundamental group of $P^r$. The path $s$ lies in one component, say $P^r$, and

$$g_1s \approx \pm g_1a_{1r} \pm g_1a_{2r} \pm \cdots \pm g_1a_{mr}$$

on $g_1(P^r) \subseteq U(F, \alpha_1) \subseteq G_1$. By hypothesis, for each $i$ and $r$, $g_1a_{ir} \approx 0$ in $G_2$, and therefore in a compact set $F_{ir}$ in $G_2$. Hence

$$f \approx 0 \text{ in } Cl(U(F, \alpha_0)) \cup \bigcup_{i=1}^{l} \bigcup_{r=1}^{m_r} F_{ir}$$

a compact subset of $G_2$ independent of $f$.

**Proof of Theorem 4, General Case** ($p \geq 2$). We make the inductive assumption that $4(p-1)$ is proved, and may therefore, by 4.1 ($p-1$) and 4.2 ($p-1$), assume that $G$ is LC$^{p-1}$ rel. $X$. Let $\xi'(\delta, x)$ be the function corresponding to $\xi$ in the dimension $p-1$, and let $\eta(\delta, x)$ be such that $GU(x, \eta(\delta, x))$ is ac$^p$ rel. $GU(x, \delta)$ for every $x$. Then $\xi'(\delta, x)$ may be put equal to $\eta(\xi'(\delta, x), x)$. This choice will be justified by Theorem 3 if it is shown that the condition of that theorem is satisfied, with $n_0$ replaced by $\xi'$. 

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Let $F \subseteq GU(x, \xi(\delta, x))$. By Theorem 2, there is an $F_1 \subseteq GU(x, \xi'(\delta, x))$ (with a margin $\alpha$) such that $F$ is acp rel. $F_1$; and by Theorem 4($p-1$) there is an $F' \subseteq GU(x, \delta)$ such that $\text{Cl}(U(F_1, \alpha))$ is asp$^{-1}$ rel. $F'$. This is the set $F'$ required in Theorem 3.

Since $X$ is LCP$^{p-1}$ and $F_1$ is compact, there is$^{17}$ a positive function $\eta_1(\xi)$ such that, given a polyhedron $P^p$, and any subpolyhedron $Q$ containing all its vertices, any mapping $f_0: Q \rightarrow F_1$ whose continuous norm$^{18}$ is less than $\eta_1(\xi)$ can be extended to a mapping $f_1: P^p \rightarrow X$ of continuous norm less than $\xi$. Let a mapping $f: S^p \rightarrow F$ and a positive $\varepsilon < \alpha$ be given. Divide $S^p$ simplicially into a polyhedron $\Sigma^p$, and let $C^p$ be a fundamental $p$-cycle on $\|\Sigma^p\|$. The simplicial division is to be so fine that (a) $\Delta f(\sigma) < \eta_1(\varepsilon/2)$ for every (continuous) simplex $\sigma$ of $\Sigma^p$; and (b) there exist an abstract complex $K^{p+1}$ containing $\|\Sigma^p\|$ as a subcomplex, a chain $C^{p+1}$ on $K^{p+1}$ with boundary $C^p$, and a mapping $f_1: K^{p+1} \rightarrow F_1$ of mesh less than $\eta_1(\varepsilon/2)$, with $f_1| \Sigma^p = f| \Sigma^p$. That this is possible follows from the definition of $F_1$. Let $P^{p+1}$ be a polyhedron such that $\|P^{p+1}\| = K^{p+1}$ and $\Sigma^p$ is a subpolyhedron of $P^{p+1}$. The combination of $f$ in $S^p$ and $f_1$ at the vertices of $K^{p+1}$ determines a continuous mapping of the subpolyhedron $\Sigma^p \cup P^p$ of $P^p$ into $F_1$, of continuous norm less than $\eta_1(\varepsilon/2)$. It can therefore be extended to a mapping $g_1: P^p \rightarrow X$ of continuous norm less than $\varepsilon/2$.

Thus $g_1$ is a mapping into $U(F_1, \varepsilon/2) \subseteq \text{Cl}(U(F_1, \alpha))$. From the definition of $F'$ it follows that if $y_0 = g_1(c_0)$, $g_1| P^{p-1} \simeq [y_0]$ in $F'$. Hence$^{20}$ there exists a mapping $g_2: P^p \rightarrow F'$ such that $g_1 \simeq g_2$ in $F'$, and $g_2| P^{p-1} = [y_0]$.

From this point on, the proof that $f \simeq 0$ (rel. $c_0$) in $F'$ proceeds exactly like the remainder of the proof in $H$ (pp. 484 and 485) that $f \simeq 0$ (rel. $x_0$) in $U$. The proof of Theorem 4($p$) is thereby completed.

From the definition of $\xi$ it is clear that if $\xi'$ and $\eta$ are independent of $x$, so also is $\xi$. The case $G = X$ is then of most interest, and gives the following theorems.

**Theorem 6.** If $X$ is ULC$^1$ and ulc$^p$ it is ULC$^p$, if $p \geq 2$; and if $X$ is ulc$^0$ it is ULC$^0$.

**Theorem 6.1.** ($p \geq 0$). If $X$ is ULC$^p$ there exists a positive function $\xi(\delta)$ such that $U(x, \xi(\delta))$ is strongly asp$^p$ rel. $U(x, \delta)$ for all $x$ of $X$.

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17 Lefschetz [6, p. 120] = H, Satz (1a). The modifications needed to allow for $X$ being only locally compact are obvious in view of the compactness of $F_1$.

18 Continuous norm of $f_0$=least upper bound of $\rho(f_0(x), f_0(y))$ for $x, y$ in the same cell of $P^p$.

19 $c_0$ is the point $(1, 0, \cdots, 0)$ (cf. §3).

20 H, Satz 2.
BIBLIOGRAPHY


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