PROOF OF RAMANUJAN'S PARTITION CONGRUENCE FOR THE MODULUS 11^3

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1. Introduction. Ramanujan's conjecture, that

$$(1.1)\qquad p(11^\alpha l + \rho) \equiv 0 \pmod{11^\alpha}, \qquad l = 0, 1, 2, \cdots; \alpha = 1, 2, \cdots$$

where $p(n)$ is the number of partitions of n and ρ is the least positive integer[2] such that

$$(1.2)\qquad 24\rho \equiv 1 \pmod{11^\alpha},$$

has never been proved for general values of α. Proofs for $\alpha=1, 2$ were given by Ramanujan and the author.[3] The object of this note is to supply a proof for $\alpha=3$, that is, to establish the congruence

$$(1.3)\qquad p(1331l + 721) \equiv 0 \pmod{11^3}, \qquad l = 0, 1, 2, \cdots.$$

The proof in I for $\alpha=1$ used a method developed by Rademacher (Trans. Amer. Math. Soc. vol. 51 (1942) pp. 609–636). It was effected by constructing an identity which, by analogy, would be of the form

$$(1.4)\qquad \sum_{l=0}^{\infty} p(11^\alpha l + \rho)x^{l+1} = 11^\alpha \sum$$

for general α. Here $\sum$ should be a sum, with integer coefficients, of certain functions having expansions in powers of x with integral coefficients. Now the power series on the left-hand side of (1.4) can be expressed rather simply in terms of the Dedekind function

$$(1.5)\qquad \begin{aligned} \eta(\tau) &= \exp(\pi i\tau/12) \prod_{1}^{\infty} (1 - x^m) \\ &= \exp(\pi i\tau/12) \left\{1 + \sum_{1}^{\infty} p(n)x^n\right\}^{-1}, \end{aligned}$$

where, throughout this paper, $x = \exp 2\pi i\tau$ and $\text{Im }\tau > 0$. In fact, it is

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[2] Actually ρ can be given explicitly. It is $(23\cdot 11^\alpha+1)/24$ for α even and $(13\cdot 11^\alpha+1)/24$ for α odd.

[3] Cf. footnotes 2 and 5 in J. Lehner, *Ramanujan identities involving the partition function for the moduli* 11^α, Amer. J. Math. (1943) pp. 492–520. This paper will be referred to hereafter as I.

easy to deduce (I, (7.32)) from the above definition that

$$(1.6)\quad \prod_{m=1}^{\infty}(1 - x^{\epsilon m})\sum_{l=0}^{\infty} p(11^{\alpha}l + \rho)x^{l+1} = 11^{-\alpha}\sum_{\lambda \bmod 11^{\alpha}} \eta(\epsilon\tau)\eta^{-1}((\tau + 24\lambda)/11^{\alpha}) = L(\tau; 11^{\alpha}),$$

say, where ϵ is 1 for even α and 11 for odd α.

The important fact about $L(\tau; 11^{\alpha})$ is that (I, Theorem 8) it is a modular function on the subgroup[4] $\Gamma_0(11)$ of the full modular group, and has its only singularity at the parabolic vertex $\tau=0$ of the subgroup's fundamental region. As such it must be a polynomial of the form

$$(1.7)\qquad Q_{\alpha}(A, C) = (\gamma_0 + \gamma_1 C + \cdots) + AC(\delta_0 + \delta_1 C + \cdots)$$

in the two basic functions $A(\tau)$ and $C(\tau)$, whose definitions we shall recall in §3. Thus we have

$$(1.8)\quad \prod_{m=1}^{\infty}(1 - x^{\epsilon m})\sum_{l=0}^{\infty} p(11^{\alpha}l + \rho)x^{l+1} = L(\tau; 11^{\alpha}) = Q_{\alpha}(A, C).$$

Now the power series of $A(\tau)$ and $C(\tau)$ in terms of x have integral coefficients, and so Ramanujan's conjecture for 11^{α} follows if we can show that the coefficients of the polynomial Q_{α} are divisible by 11^{α}. This was done in I for $\alpha=1$ by explicit calculation.

To go on to higher values of α we proved that the functions $L(\tau; 11^{\alpha})$ can be given inductively by using the linear operator

$$(1.9)\qquad U_{11}F(\tau) = UF(\tau) = 11^{-1}\sum_{\lambda=0}^{10} F((\tau + \lambda)/11).$$

In fact (I, Theorem 7), if we put $L(\tau; 11^0)=1$, we have

$$(1.10)\qquad L(\tau; 11^{2m+1}) = U\{\Phi(\tau)\cdot L(\tau; 11^{2m})\},$$

$$(1.11)\qquad L(\tau; 11^{2m+2}) = UL(\tau; 11^{2m+1}),$$

for $m=0, 1, 2, \cdots$, where

$$(1.12)\qquad \Phi(\tau) = \Phi(\tau; 121) = \eta(121\tau)\eta^{-1}(\tau).$$

The nature of the inductive step depends on the parity of α.

Now UC^j, $U\Phi C^{j-1}$, UAC^j, $U\Phi AC^j$, $j=1, 2, \cdots$, are modular functions on $\Gamma_0(11)$ whose only singularities are at the parabolic vertex zero (I, Theorem 8). Thus, because U is linear, we can go from

[4] Defined by the condition $11 | c$ in the modular substitution $\tau' = (a\tau+b)/(c\tau+d)$.

one of the identities (1.8) to the next provided we know the expressions for $UC^i, \cdots$, as polynomials in A and C. However, if we are interested only in proving Ramanujan's conjecture for a given α, we can abbreviate this explicit procedure considerably, as was done in I for $\alpha=2$ and will be done below for $\alpha=3$. The basic tool will still be the inductive property (1.10), (1.11).

2. **Method.** We shall find it convenient to replace the rational basis $A(\tau)$, $C(\tau)$ by the basis $C(\tau)$, $D(\tau)$, where

$$11D(\tau) = A(\tau)C(\tau) - 1, \tag{2.1}$$

and $D(\tau)$ has integral coefficients (I, Lemma 4). A modular function on $\Gamma_0(11)$ whose only singularity is at the parabolic vertex $\tau=0$ (for example, $L(\tau;\, 11^\alpha)$) can be expressed as a polynomial in C and D which is of first degree in D.

Now let us use the notation $F(\tau)\equiv G(\tau)$ (mod 11^α) to mean that corresponding coefficients of the expansions of $F(\tau)$ and $G(\tau)$ about $i\infty$ in powers of x are congruent mod 11^α. Then proving Ramanujan's conjecture for $\alpha=3$ is equivalent to proving that

$$L(\tau;\, 11^3) \equiv 0 \pmod{11^3}. \tag{2.2}$$

The operator U affects a power series as follows (I, (4.74)):

$$U\Big(\sum_{n\geqq\nu} a_n x^n\Big) = \sum_{m\geqq\nu/11} a_{11m}x^m. \tag{2.3}$$

Thus we can say that if the coefficients of the expansion of $F(\tau)$ are integral, then so are the coefficients in the expansion of $UF(\tau)$. Hence from $F(\tau)\equiv G(\tau)$ (mod 11^α), we can infer that $UF(\tau)\equiv UG(\tau)$ (mod 11^α).

The identity for $L(\tau;\, 11)$ is explicitly (I, (6.6))

$$L(\tau;\, 11) = 11AC(11C+2) - 11(11^2C^2 + 32C + 2),$$

or in terms of C, D

$$L(\tau;\, 11) = 11^2D(11C+2) - 11(11^2C^2 + 21C). \tag{2.4}$$

Hence

$$L(\tau;\, 11) \equiv 2\cdot 11^2D - 21\cdot 11C \pmod{11^3}. \tag{2.5}$$

Now we apply the second inductive step (1.11) and make use of the remark in the preceding paragraph and the linearity of U to obtain

$$L(\tau;\, 11^2) = UL(\tau;\, 11) \equiv 2\cdot 11^2UD - 21\cdot 11UC \pmod{11^3}. \tag{2.6}$$

In the next section we shall prove

(A) $$UC \equiv 11a_1 C \pmod{11^2}, \qquad a_1 = \text{integer},$$

(B) $$UD \equiv 0 \pmod{11}.$$

From these congruences, the truth of which we assume for the moment, we conclude from (2.6) that

(2.7) $$L(\tau;\, 11^2) \equiv 11^2 a_1 C \pmod{11^3}.$$

Now $\Phi(\tau)$ has an expansion in powers of x with integral coefficients, as we easily verify from (1.12) and (1.5). Hence, if we apply the first inductive step (1.10) to (2.7), we find

(2.8) $$L(\tau;\, 11^3) \equiv 11^2 a_1\, U\Phi C \pmod{11^3}.$$

Therefore, the desired congruence (1.3) will be completely proved if we can establish (A), (B), and

(C) $$U\Phi C \equiv 0 \pmod{11}.$$

3. **Proof of (A), (B), and (C).** In the proof of these equations, we shall make use of the fact, demonstrated in I, §8, that their left members, namely, UC, UD, $U\Phi C$, are entire[5] modular functions on the subgroup $\Gamma_0(11)$ of the modular group, and shall express them as polynomials in the basic functions C, D of this subgroup. We therefore recall the definition of C and D. As before, τ is a complex variable with positive imaginary part, and $x = \exp 2\pi i\tau$. All the functions listed below have Laurent expansions in x with integral coefficients. For further details, cf. I, §4.

Starting with the formula for $\eta(\tau)$ in (1.5), we define in turn:

(3.1) $$\psi(\tau) = \{\eta(\tau)\eta(11\tau)\}^2,$$

(3.2) $$g(\tau) - 1/120 = 2\sum_{1}^{\infty} \sigma_3(n)x^n, \qquad \sigma_3(n) = \sum_{d|n} d^3,$$

(3.3) $$G(\tau) = 11^2 g(11\tau) - g(\tau),$$

(3.4) $$\theta(\tau) = \sum_{m,n=-\infty}^{+\infty} x^{Q(m,n)}, \qquad Q = m^2 + mn + 3n^2,$$

(3.5) $$A(\tau) = \theta^2(\tau)/\psi(\tau),$$

(3.6) $$B(\tau) = G(\tau)/\psi^2(\tau),$$

(3.7) $$C(\tau) = 2^{-1}\cdot 11^{-2}\{A^2(\tau) - 10A(\tau) - 22 - B(\tau)\}.$$

[5] An *entire* modular function is one which has no singularities except possibly at the parabolic vertices of the fundamental region, which, in this particular case, are located at 0 and $i\infty$.

The expansions of the following functions of interest at the parabolic vertices $(0, i\infty)$ of $\Gamma_0(11)$ appear in I (references at right of display lines are to equations in I). In order to avoid confusion we write $(UF|\tau')$ to denote the result of replacing τ by τ' in the right member of (1.9). By $P(x^s)$ we shall mean a series in powers of x, with integral coefficients, beginning with a term in x^s.

$$(3.8)\qquad A(\tau) = x^{-1} + 6 + \cdots \qquad (6.41),$$

$$(3.9)\qquad C(\tau) = x + 5x^2 + \cdots \qquad (6.43),$$

$$(3.10)\qquad A(-1/11\tau) = A(\tau) = x^{-1} + 6 + \cdots \qquad (4.53),$$

$$(3.11)\qquad 11^2C(-1/11\tau) = x^{-2} + 2x^{-1} + \cdots \qquad (6.46),$$

$$(3.12)\qquad UC(\tau) = P(x), \qquad (8.2),$$

$$(3.13)\qquad 11^3(UC\,|\,-1/11\tau) = 11^2C(-1/121\tau) + 11^2P(x) = x^{-22} + \cdots \qquad (8.81),\ (9.31).$$

The first terms of the expansions of D can be deduced from (2.1), (3.8)–(3.11):

$$(3.14)\qquad D(\tau) = x + \cdots\,,$$

$$(3.15)\qquad 11^3D(-1/11\tau) = x^{-3} + \cdots\,.$$

By Fermat's Theorem and an elementary theorem on binomial coefficients, we observe that

$$(3.16)\qquad 11^2C(-1/121\tau) = \{11^2C(-1/11\tau)\}^{11} + 11P(x^{-21});$$

consequently, in view of (3.13),

$$(3.17)\qquad 11^3(UC\,|\,-1/11\tau) = \{11^2C(-1/11\tau)\}^{11} + 11P(x^{-21}).$$

The argument in the first paragraph on p. 505 of I shows that an entire modular function on $\Gamma_0(11)$ which vanishes at infinity and has at most a pole of first order at the vertex $\tau = 0$ must be identically zero. We apply this principle to the function

$$\Delta(\tau) = 11^3UC(\tau) - \{11^2C(\tau)\}^{11} - \Big\{11^3\alpha_1C(\tau) + 11^4\beta_0D(\tau) + \sum_{j=2}^{10} 11^{2j+1}\alpha_jC^j(\tau) + 11^3D(\tau)\cdot\sum_{j=1}^{9} 11^{2j+1}\beta_jC^j(\tau)\Big\},$$

where the α, β are rational integers chosen so that all terms of the principal part of Δ at $\tau' = -1/11\tau = 0$ drop out except possibly the term in x^{-1}. This choice is clearly possible in view of (3.17), (3.11), and (3.15). Furthermore, all terms of Δ vanish at $\tau = i\infty$, as we see from (3.9), (3.12), (3.14), and therefore Δ itself. It follows that $\Delta(\tau)$

is identically zero, and we infer the truth of the congruence

$$UC(\tau) \equiv \alpha_1 C(\tau) + 11\beta_0 D(\tau) \pmod{11^2}. \tag{3.18}$$

Now writing

$$C(\tau) = \sum_1^\infty c_n x^n, \qquad D(\tau) = \sum_1^\infty d_n x^n, \qquad C^2(\tau) = \sum_2^\infty c_n^{(2)} x^n, \tag{3.19}$$

and remembering (2.3), we obtain the following linear congruences for α_1 and β_0 on matching the coefficients of x and x^2 in the two members of (3.18):

$$\left.\begin{aligned} c_1\alpha_1 + 11d_1\beta_0 &\equiv c_{11} \\ c_2\alpha_1 + 11d_2\beta_0 &\equiv c_{22} \end{aligned}\right\} \pmod{11^2},$$

or using the values in the tables of §4,

$$\left.\begin{aligned} \alpha_1 + 11\beta_0 &\equiv 44 \\ 5\alpha_1 + 66\beta_0 &\equiv 99 \end{aligned}\right\} \pmod{11^2}. \tag{3.20}$$

Therefore, we have

$$\begin{aligned} \alpha_1 &\equiv 6(44) - 99 \pmod{11^2}, \\ \alpha_1 &\equiv 0 \pmod{11}; \end{aligned} \tag{3.21}$$

$$\begin{aligned} 11\beta_0 &\equiv 99 - 5(44) = 11(9 - 20) \pmod{11^2}, \\ \beta_0 &\equiv 0 \pmod{11}. \end{aligned} \tag{3.22}$$

With (3.18), (3.21), and (3.22), we have completed the proof of (A).

The proof of (B) is quite similar. From I, (8.81), and (3.14), (3.15),

$$\begin{aligned} 11^4(UD \mid -1/11\tau) &= 11^3 D(-1/121\tau) + 11^3 P(x) \\ &= x^{-33} + \cdots , \end{aligned} \tag{3.23}$$

and, as before,

$$11^4(UD \mid -1/11\tau) = \{11^3 D(-1/11\tau)\}^{11} + 11P(x^{-32}). \tag{3.24}$$

Now employing the same procedure used for UC, we obtain easily

$$11^4 UD(\tau) \equiv \alpha_1 11^3 C(\tau) + \beta_0 11^4 D(\tau) \pmod{11^5},$$

with integral α_1, β_0. Equating coefficients of x, we have $\alpha_1 + 11\beta_0 \equiv 0 \pmod{11}$, that is, $\alpha_1 = 11\alpha$. Hence,

$$UD \equiv \alpha C + \beta_0 D \pmod{11}, \qquad \alpha,\ \beta_0 \text{ integral}. \tag{3.25}$$

We now again equate coefficients and get the congruences

$$\begin{aligned} \alpha + \beta_0 &\equiv d_{11} \\ 5\alpha + 6\beta_0 &\equiv d_{22} \end{aligned} \pmod{11} \tag{3.26}$$

for the determination of α and β_0. The determinant of the coefficients is seen to be equal to unity. Hence

$$\alpha \equiv \begin{vmatrix} d_{11} & d_1 \\ d_{22} & d_2 \end{vmatrix} \equiv 0 \pmod{11}, \tag{3.27}$$

since $d_{11} \equiv d_{22} \equiv 0 \pmod{11}$. Likewise,

$$\beta_0 \equiv \begin{vmatrix} c_1 & d_{11} \\ c_2 & d_{22} \end{vmatrix} \equiv 0 \pmod{11}. \tag{3.28}$$

This proves (B).

For the proof of (C), we need the expansions

$$\Phi(\tau) = x^5 + \cdots \tag{3.29}$$

$$11\Phi(-1/121\tau) = \Phi^{-1}(\tau) = x^{-5} + \cdots . \tag{3.30}$$

The first is computed immediately from (1.12) and (1.5), whereas (3.30) is a consequence of the well known transformation equation

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau), \tag{3.31}$$

where the square root is positive when τ is on the positive imaginary axis. Then by virtue of (2.3) we may write

$$U\Phi C = P(x). \tag{3.32}$$

We must now find the expansion of $U\Phi C$ at $\tau'=0$, that is, $(U\Phi C|-1/11\tau)$. We cannot use I, (8.81) this time, for ΦC is not a function of $\Gamma_0(11)$. Instead, we have recourse to I, p. 509, in which the function $G(\tau; i, j, k) = U\Phi^i A^j C^k$ is considered. (The last two display lines on p. 509 are in error: the line at the bottom of the page should have the factor i multiplying the right member; the display line above this requires the factor[6] $1/\Phi(\tau/11)$ affecting the sum in the right member.) The case we are interested in is $j=0$, $i=k=1$. If we substitute the value of the sum given at the bottom of the page in the expression for $11G(-1/\tau)$ appearing immediately above, and replace τ by 11τ, we obtain

$$\begin{aligned} 11^4(U\Phi C \mid -1/11\tau) &= \{11\Phi(-1/121\tau)\}\{11^2C(-1/121\tau)\} \\ &\quad - 11^3\Phi^{-1}(\tau) \sum_{m=5}^{\infty} \sum_{p=1}^{\infty} \phi_m c_p \left(\frac{24(m+p)}{11}\right) \exp\{2\pi i(m+p)\tau\}, \end{aligned}$$

[6] This factor is also required multiplying the sum in the right member of (8.31).

with the Legendre-Jacobi symbol

$$\left(\frac{24(m+p)}{11}\right).$$

From the last equation we deduce, in view of (3.29), (3.30), and (3.11),

$$(3.33)\quad \begin{aligned} 11^4(U\Phi C \mid -1/11\tau) = \{11\Phi(-1/121\tau)\}\{11^2C(-1/121\tau)\} \\ + 11^3P(x) = x^{-27} + \cdots . \end{aligned}$$

We therefore obtain in the usual way[7] a congruence

$$(3.34)\quad 11^4U\Phi C = \alpha_1 11^2C + \alpha_2 11^4C^2 + 11^3\beta_0 D \pmod{11^5}$$

with rational integers α, β. We want to show that $11^2|\alpha_1$, $11|\beta_0$. This, however, is a consequence of the following congruences obtained by matching the coefficients of x and x^2 in the two members of (3.34):

$$(3.35)\quad \begin{aligned} \alpha_1 + 11\beta_0 \equiv 0 \\ 5\alpha_1 + 66\beta_0 \equiv 0 \end{aligned} \pmod{11^2}.$$

Therefore, $\alpha_1 = 11^2\alpha$, $\beta_0 = 11\beta$, and we have from (3.34)

$$(3.36)\quad U\Phi C \equiv \alpha C + \alpha_2 C^2 + \beta D \pmod{11}, \qquad \alpha, \alpha_2, \beta \text{ integral.}$$

(3.36), in turn, yields the congruences

$$(3.37)\quad \begin{aligned} \alpha \quad\quad\quad + \quad \beta &\equiv e_{11} \\ 5\alpha + \quad \alpha_2 + \quad 6\beta &\equiv e_{22} \pmod{11}, \\ 19\alpha + 10\alpha_2 + 28\beta &\equiv e_{33} \end{aligned}$$

where we have written

$$(3.38)\quad \Phi(\tau)C(\tau) = \sum_{n=6}^{\infty} e_n x^n.$$

The determinant of the coefficients is -1. Since $e_{11} \equiv e_{22} \equiv e_{33} \equiv 0 \pmod{11}$, we have

$$\alpha \equiv - \begin{vmatrix} e_{11} & 0 & 1 \\ e_{22} & 1 & 6 \\ e_{33} & 10 & 28 \end{vmatrix} \equiv 0 \pmod{11},$$

[7] The analogue of (3.16) and (3.24) is not valid for the function ΦC, since $\Phi(\tau)$ is not a function of $\Gamma_0(11)$. The equation we would like to write, $11\Phi(-1/121\tau) \cdot 11^2C(-1/121\tau) = \{11\Phi(-1/11\tau)\cdot 11^2C(-1/11\tau)\}^{11} + 11P(x^{-26})$, is obviously false, for (3.30) shows that $\Phi(-1/11\tau)$ does not have an expansion in integral powers of x.

and likewise $\alpha_2 \equiv \beta \equiv 0 \pmod{11}$. This establishes (C), and with it, the truth of the congruence (1.3).

4. **Tables.** The preceeding discussion has utilized the values of certain coefficients, which will be developed in this section. In the computation of these, the following formulae, in addition to those given previously, will be found useful:

$$(4.1) \qquad \prod_{1}^{\infty} (1 - x^m) = \sum_{-\infty}^{+\infty} (-1)^n x^{n(3n+1)/2} \qquad \text{(Euler)},$$

$$(4.2) \qquad \prod_{1}^{\infty} (1 - x^m)^{-1} = 1 + \sum_{1}^{\infty} p(n) x^n \qquad \text{(Euler)},$$

$$(4.3) \qquad \theta(\tau) = \sum_{0}^{\infty} A_Q(n) x^n \qquad \text{(I, (4.61) and footnote 16)}$$

with[8]

$$A_Q(n) = \begin{cases} 2 \sum_{d_1 | n} 1 - 2 \sum_{d_2 | n} 1, & 11 \nmid n, \\ A_Q(n/11), & 11 \mid n, \end{cases}$$

where $d_1 \equiv$ quad. res., $d_2 \equiv$ quad. non-res. (mod 11).

The coefficients of the functions C, D, and ΦC are computed in a straightforward way from the definitions (1.5), (1.12), (3.1)–(3.7), (2.1), with the help of (4.1)–(4.3). The values are shown in the following tables to the highest modulus required for the proofs of the preceding section.

RESIDUE OF THE COEFFICIENTS OF CERTAIN POWER SERIES

Power of x	Function									
	ψ^{-1}	θ^2	A	A^2	ψ^{-2}	G	B	C	D	Φ
	modulo 0									
−2				1	1		1			
−1	1		1	12	4		2			
0	2	1	6	70	14	1	− 12			
1	5	4	17	296	40	− 2	− 116	1	1	
2	10	4	46	1,073	105	−18	− 597	5	6	
3	20	8	116	3,460	252	−56	−2,298	19	28	

[8] The formula for $A_Q(n)$ was printed incorrectly in I.

Power of x	Function									
	ψ^{-1}	θ^2	A	A^2	ψ^{-2}	G	B	C	D	Φ
	modulo $2\cdot 11^4$							mod 11^2	mod 11	
4	36	20	252	10,150	574	− 146	− 7,616	63		
5	65	16	533	27,704	1,240	− 252	−22,396	64		1
6	110	32	1,034	12,528	2,580	− 504	27,114	18		1
7	185	16	1,961	26,858	5,180	− 688	22,010	60		2
8	300	36	3,540	23,282	10,108	− 1,170	12,082	21		3
9	481	28	6,253	27,176	19,212	− 1,514	6,028	71		5
10	754	40	10,654	207	6,427	− 2,268	18,539	89		7
11	1,169	4	17,897	23,232	6,452	− 2,422	9,306	44		0
12	1,780	64	2	25,108	28,404	− 4,088	4,276	86		4
13	2,685	40	17,983	3,456	28,712	− 4,396	770	115		0
14	3,996	64	16,304	20,971	2,641	− 6,192	24,911	36		8
15	5,894	56	30	13,030	19,004	− 7,056	16,602	105		9
16	8,600	68	4,916	1,420	23,952	− 9,362	18,326	90		1
17	12,450	40	12,024	20,348	364	− 9,828	12,154	21		0
18	17,860	100	5,352	609	17,447	−13,626	23,561	47		2
19	25,442	48	5,288	15,722	26,482	−13,720	7,612	57		3
20	6,682	104	9,118	29,184	23,816	10,886	3,102	94		0
21	21,237	80	26,561	28,386	1,102	10,018	3,324	95		0
22	11,926	4	21,362	5,929	17,083	7,484	2,607	99		0
23	9,954	56	8,592	5,160	22,928	4,946	22,574	57		0
	modulo $2\cdot 11^3$							modulo 11		
24	1,792	144	46	2,376	220	1,846	1,190	3		6
25	1,421	68	1,435	1,718	2,174	442	1,888	6		0
26	2,436	88	310	893	1,447	366	2,391	3		0
27	2,544	104	1,118	2,562	138	1,712	94	8		1
28	1,012	128	858	384	1,040	354	758	7		1
29	2,125	72	745			1,798				2
30		176				384				0

$d_{11}\equiv d_{22}\equiv 0 \pmod{11}$, $c_1^{(2)}=0$, $c_2^{(2)}=1$, $c_3^{(2)}=10$

$e_{11}\equiv e_{22}\equiv e_{33}\equiv 0 \pmod{11}$, where $\Phi C=\sum e_n x^n$

$\phi_{31}\equiv 5$, $\phi_{32}\equiv 7 \pmod{11}$

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