NOTE ON THE ESCALATOR METHOD

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Practical solution of the characteristic vector problem

$$AX = \lambda X, \tag{1}$$

where $A = (a_{ij})$ is a given real nonsingular n-rowed matrix and X is a column $(x_1, \cdots, x_n)'$, has recently been treated anew by J. Morris.[1] His process is called the escalator method. If A_p is a p-rowed principal minor array in the upper left corner of A, then the abbreviated system

$$A_pX_p = \lambda X_p, \qquad p \leqq n, \tag{2}$$

is called the pth stage of (1). As given by Morris, the escalator method is based on expressions for the solutions of the $(p+1)$th stage in terms of those of the pth stage, and depends on the existence of biorthonormal sets of solution vectors for (2) and the transposed system

$$A'_pY_p = \lambda Y_p. \tag{3}$$

But, of course, it is not always possible to find such biorthonormal sets of solution vectors, since the elementary divisors of the pth stage may not be linear. In this note a more general exposition of escalation is given, covering all cases and leading to at least one practical method in cases outside the scope of Morris' formulas. In addition, a result on the transmission of roots from one stage to the next is included.

Let λ_i be a characteristic root of multiplicity n_i of the system (2), and X_{pi} a corresponding characteristic vector. As is well known, there exists a nonsingular P_0 such that

$$A_pP_0 = P_0A_0, \quad \text{where} \quad A_0 = \begin{pmatrix} \lambda_1 & \delta_1 & & & & & & 0 \\ & \lambda_1 & \delta_2 & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \lambda_1 & 0 & & \\ & & & & & \lambda_2 & \delta_{n_1} & \\ & & & & & & \cdot & \cdot \\ 0 & & & & & & & \cdot \end{pmatrix}, \tag{4}$$

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[1] J. Morris, *The escalator method*, Wiley, 1947.

the δ_i's being zero or one. If we set

$$C = (a_{p+1,1}, \cdots, a_{p+1,p}), \quad D = (a_{1,p+1}, \cdots, a_{p,p+1})',$$

$$(5) \qquad A_{p+1} = \begin{pmatrix} A_p & D \\ C & a_{p+1,p+1} \end{pmatrix}, \quad P = \begin{pmatrix} P_0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = P^{-1}A_{p+1}P,$$

then one can easily expand the equation

$$(6) \quad |A_{p+1} - \lambda I_{p+1}| = |B - \lambda I_{p+1}| = \begin{vmatrix} A_0 - \lambda I_p & P_0^{-1}D \\ CP_0 & a_{p+1,p+1} - \lambda \end{vmatrix} = 0,$$

where I_p is the p-rowed identity matrix. This equation is the general escalator equation for the determination of the characteristic roots at the $(p+1)$th stage. When A_0 is diagonal, then (6) coincides with Morris' equation[2] and is especially effective in symmetric systems. Since $A_p' P_0^{-1\prime} = P_0^{-1\prime} A_0'$, the transposed system (3) leads to the same escalator equation $|B' - \lambda I_{p+1}| = 0$.

For a determination of P_0, one may place X_{p1} in the first column, X_{p2} in the (n_2+1)th column, and so on. The remaining coefficients and the δ_i's may then be solved from the system (4).[3]

For a determination of the characteristic vectors of the $(p+1)$th stage,[4] let X^* be a $p+1$-columned matrix of the unknown vectors and let Λ be the diagonal matrix of corresponding characteristic roots of the $(p+1)$th stage as determined from the escalator equation (6). Then from $A_{p+1}X^* = X^*\Lambda$ it follows that $P^{-1}A_{p+1}PP^{-1}X^* = P^{-1}X^*\Lambda$, or $BZ^* = Z^*\Lambda$, where $Z^* = P^{-1}X^*$ and $B = P^{-1}A_{p+1}P$ (as defined by (5)). Because of the simple structure of B, the equations $BZ^* = Z^*\Lambda$ afford an easy determination of Z^*, hence of $X^* = PZ^*$.

As for the transposed system $A_{p+1}'Y^* = Y^*\Lambda$, we note that if Z^* is nonsingular (which is possible whenever the elementary divisors at the $(p+1)$th stage are linear), it follows from $BZ^* = Z^*\Lambda$ that $B'Z^{*-1\prime} = Z^{*-1\prime}\Lambda$, hence $Y^* = P^{-1\prime}Z^{*-1\prime} = X^{*-1\prime}$ is a solution of the transposed system.

If $CP_0 = (P_1, \cdots, P_p)$, $P_0^{-1}D = (Q_1, \cdots, Q_p)'$, then a necessary and sufficient condition for a root λ_r of the pth stage to be a root of the $(p+1)$th stage is $P_rQ_r = 0$ for any pair P_r, Q_r arising from a column of P_0 which is a characteristic vector of the pth stage cor-

[2] The components of CP_0 and $P_0^{-1}D$ are Morris' P_i's and P_i''s.

[3] In connection with (4), the escalator method for simultaneous linear systems as compactly presented by R. A. Frazer, *Note on the escalator method*, Philosophical Magazine vol. 38 (1947) pp. 287–289, is pertinent.

[4] Checks and other special procedures have been developed by R. J. Lambert, Master's Thesis, Iowa State College of Agriculture and Mechanic Arts, 1948.

responding to λ_r.

For the necessity, let x_{p+1} be the last component of X_{p+1} and notice that the systems $A_{p+1}X_{p+1}=\lambda X_{p+1}$ and $A'_{p+1}Y_{p+1}=\lambda Y_{p+1}$ may be written

$$(7)\qquad \begin{aligned}(A_p - \lambda I_p)X_p + Dx_{p+1} &= 0,\\ CX_p + x_{p+1}(a_{p+1,p+1} - \lambda) &= 0,\end{aligned}$$

$$(8)\qquad \begin{aligned}(A_p' - \lambda I_p)Y_p + C'y_{p+1} &= 0,\\ D'Y_p + y_{p+1}(a_{p+1,p+1} - \lambda) &= 0.\end{aligned}$$

Hence if $P_r=CX_{pr}=0$, then (7) is satisfied by $\lambda=\lambda_r$ and $X_{p+1}=(X'_{pr}, 0)'$. Similarly, $Q_r=0$ implies (8) is satisfied by $\lambda=\lambda_r$ and $Y_{p+1}=(Y'_{pr}, 0)'$.

On the other hand, if we left-multiply the p equations $(A_p, D)X_{p+1}=\lambda X_p$ by Y'_{pr} and recall that $A_p' Y_{pr}=\lambda_r Y_{pr}$, then

$$(9)\qquad (\lambda_r - \lambda)Y'_{pr}X_p + Q_r x_{p+1} = 0.$$

Now if (9) is satisfied by $\lambda=\lambda_r$ and a corresponding vector $X_{p+1,r}$, then either $Q_r=0$ or $x_{p+1}=0$. If $x_{p+1}=0$, then the first p components of $X_{p+1,r}$ satisfy the pth stage, hence $P_r=0$. Thus the sufficiency is proved.

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