THE DIFFERENCE BETWEEN CONSECUTIVE
PRIME NUMBERS. IV

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1. Introduction. Let \( p_n \) denote the \( n \)th prime, and let \( \Theta \) be the
lower bound of all positive numbers \( \sigma_0 \) such that no Dirichlet \( L \)-function
\( L(s, \chi) \) has a zero at \( s = \sigma + it \), where \( \sigma > \sigma_0 \). Write

\[
\ell = \liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n}.
\]

The purpose of this paper is to combine the methods used in two
earlier papers\(^1\) in order to prove the following theorem.

**Theorem.**

(1) \( \ell \leq c(1 + 4\Theta)/5 \),

where \( c < 42/43 \).

The precise definition of \( c \) is given by (18) and (8) below.

The theorem is an improvement on the result \( \ell \leq (1+4\Theta)/5 \) ob-
tained in II. It was shown in III that

(2) \( \ell \leq d = 0.966 \cdots < 57/59 \).

The numbers \( c \) and \( d \) are connected by the relation

\[
1 - d = (3 + 2^{1/2})(1 - c)/4 > 1 - c,
\]

so that (1) is an improvement on (2) only if \( \Theta \) is not too close to
unity, in fact, if \( \Theta < 0.986 \cdots \). In particular, if the "grand Riemann
hypothesis" is true, that is, if \( \Theta = 1/2 \), we have (using the value of \( c \)
given by (18))

\[
\ell \leq 3c/5 < 109/186 = 0.58602 \cdots .
\]

2. Notation. Since (2) is sharper than (1) for \( \Theta = 1 \), we shall assume
that \( \Theta < 1 \), and write

(3) \( \sigma = (1 + 4\Theta)/5 \).

Let \( N \) be a large positive integer, and define

(4) \( X = N^{1-\sigma}(\log N)^{-8(1+\gamma)} \).

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\(^1\) See \([2]\) and \([3]\); I refer to these as II and III respectively. Numbers in brackets
refer to the references cited at the end of the paper.

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as done in equation (6.7) of II; here \( \gamma \) is a positive constant which
may be taken to be 2.

Let \( \tilde{\omega}(\nu) \) be the number of prime pairs \( \nu, \nu' \) which satisfy
\[
N < \nu' \leq \nu \leq 2N, \quad \nu - \nu' = \nu,
\]
where \( \nu \) is an integer. We suppose that \( h, k, m, n \) and \( r \) are integers
such that, for some positive \( A_1, A_2 \) independent of \( N \),
\[
(5) \quad A_1 \log N < 2k/3 < k < A_2 \log N,
\]
\[
(6) \quad 0 \leq m < n < A_2 \log N,
\]
\[ r = k - h. \]

The letters \( a, c, \alpha, \beta \) denote positive numbers independent of \( N \),
and we write
\[
(7) \quad D = \prod_{p > 2} \left( 1 + \frac{1}{p(p - 2)} \right),
\]
and take
\[
(8) \quad \lambda = 7.3566 > e^{-2\gamma \Lambda_0/2},
\]
where \( \gamma \) is Euler’s constant and \( \Lambda_0 \) is Buchsteb’s number 46.67347
(see relation (4) of III).

For \( \mu = 1, 2, \cdots, k \) we define \( a_\mu \) as follows:
\[
(9) \quad a_\mu = \begin{cases} 
\alpha & (0 < \mu \leq r, \ k - r < \mu \leq k), \\
\beta & (r < \mu \leq k - r), 
\end{cases}
\]
and write
\[
(10) \quad \Psi(\theta) = \sum_{\mu=1}^{k} a_\mu e^{i\mu\theta}.
\]
Then
\[
(11) \quad |\Psi(\theta)|^2 = \xi(0) + 2 \sum_{\nu=1}^{k-1} \xi(\nu) \cos 4\pi\nu\theta,
\]
where
\[
\xi(\nu) = \sum_{1 \leq a \leq \nu, 1 \leq i \leq k} a_ia_j.
\]

3. Lemmas.

**Lemma 1.** We have
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\[ \xi(v) = \begin{cases} 
2(r - v)\alpha^2 + 2r\alpha\beta + (k - 2r - v)\beta^2 & (0 \leq v < r), \\
2r\alpha\beta + (k - 2r - v)\beta^2 & (r \leq v < k - 2r), \\
(2r - k + v)\alpha^2 + 2(k - r - v)\alpha\beta & (k - 2r \leq v < k - r), \\
(k - v)\alpha^2 & (k - r \leq v < k), 
\end{cases} \]

and

\[ \sum_{i=1}^{k-1} \xi(v) = \frac{1}{2} \{2r\alpha + (k - 2r)\beta\}^2 + O(\log N). \]

This follows easily from (9) and (11), and we omit the proof. It is in this lemma that the restriction \( k - r = h > 2k/3 \) made in (5) is used.

**Lemma 2.**

\[ \sum_{1 \leq q \leq X} \frac{\mu^2(q)}{\Phi^2(q)} \sum_{f=0, (f,q)=1}^{q-1} e^{4\pi i f/q} = \frac{2}{D} \prod_{p \mid r, p \geq 2} \left( \frac{p - 1}{p - 2} \right) + O(\nu X^{-1}(\log \log X)^2). \]

Here \( \mu(q) \) is the Möbius function, \( \phi(q) \) is Euler’s function, and \( D \) is defined by (7).

It follows from the well known relation

\[ \sum_{f=0, (f,q)=1}^{q-1} e^{4\pi i f/q} = \sum_{d \mid (2r,q)} d\mu \left( \frac{q}{d} \right) \]

that

\[ \sum_{1 \leq q \leq X} \frac{\mu^2(q)}{\Phi^2(q)} \sum_{f=0, (f,q)=1}^{q-1} e^{4\pi i f/q} = \sum_{q \leq X} \frac{\mu(q)}{\Phi^2(q)} \prod_{p \mid (2r,q)} (1 - p) \]

\[ = \sum_{q > X} \frac{\mu(q)}{\Phi^2(q)} \prod_{p \mid (2r,q)} (1 - p) \]

\[ - \sum_{q > X} \frac{\mu(q)}{\Phi^2(q)} \prod_{p \mid (2r,q)} (1 - p). \]

By a straightforward reduction the first term on the right may be shown to equal

\[ \frac{2}{D} \prod_{p \mid r, p \geq 2} \left( \frac{p - 1}{p - 2} \right). \]
See, for example, §§3.21–3.23 of [1]. Since $1/\phi(n) = O(n^{-1} \log \log n)$, the second term on the right is

$$O\left\{X^{-1}(\log \log X)^2 \prod_{p \mid 2^r} (p - 1)\right\} = O\{X^{-1}(\log \log X)^2\}.$$

**Lemma 3.** If $s \geq 0$ and is independent of $N$, then

$$\sum_{r=2}^{m-1} \nu^r \prod_{p \mid s^s, p > 2} \left(\frac{p - 1}{p - 2}\right) = D \sum_{r=2}^{m-1} \nu^r + O\{(\log N)^s(\log \log N)^{s}\}.$$

The proof is similar to that of the lemma in III. We suppose that $d$ is any product $p_{i_1}p_{i_2} \cdots p_{i_j}$ of distinct odd primes, and let $d'$ be the associated product

$$d' = (p_{i_1} - 2)(p_{i_2} - 2) \cdots (p_{i_j} - 2).$$

We permit $d$ to take the value unity and then put $d' = 1$. Then, clearly,

$$\sum_{d < n} \frac{1}{dd'} = \sum_{d \geq 1} \frac{1}{dd'} + O\left\{\frac{1}{n} (\log \log 3n)^2\right\}$$

$$= D + O\left\{\frac{1}{n} (\log \log 3n)^2\right\},$$

since $1/d' = O\{d^{-1}(\log \log d)^2\}$ for large $d'$, and

$$\prod_{p \mid s^s, p > 2} \left(\frac{p - 1}{p - 2}\right) = \prod_{p \mid s^s, p > 2} \left(1 + \frac{1}{p - 2}\right)$$

$$= \sum_{d' \geq 1} \frac{1}{d'}.$$

For a given $d'$, $\nu$ is a multiple of $d$, say $\nu = \nu d$, and, since $m \leq \nu < n$, $\mu$ takes all integer values in the interval $m/d \leq \mu < n/d$, which we denote by $I(d)$. Clearly

$$\sum_{\mu \in I(d)} (\nu d)^s = \frac{n^s - m^s + 1}{(s + 1)d} + O(n^s),$$

and

$$\sum_{r=m}^{n-1} \nu^r = \frac{n^s - m^s + 1}{s + 1} + O(n^s).$$

* It is possible to dispense with the factor $(\log \log 3n)^2$. 

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Hence
\[
\sum_{r=m}^{n-1} v^* \prod_{p | r, p > 2} \left( \frac{p - 1}{p - 2} \right) = \sum_{d | n} \frac{1}{d'} \sum_{d' | d} (\mu d)^* \\
= \frac{n^{s+1} - m^{s+1}}{s + 1} \sum_{d | n} \frac{1}{dd'} + O \left( n^* \sum_{d | n} \frac{1}{d'} \right) \\
= \frac{n^{s+1} - m^{s+1}}{s + 1} \left\{ D + O \left( \frac{1}{n} (\log \log 3n)^2 \right) \right\} \\
+ O \{ n^* (\log \log 3n)^2 \log 2n \} \\
= D \sum_{r=m}^{n-1} v^* + O \{ n^* (\log 2n)^2 \},
\]
and the lemma follows from (6).

**Lemma 4.**
\[
\sum_{1 \leq q \leq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f-0, (f, q) = 1} \sum_{r=1}^{r-1} \xi(v) e^{2\pi ifr/q} \\
= 2 \sum_{r=1}^{r-1} \xi(v) + O \{ \log N(\log \log N)^2 \} \\
= \{ 2r\alpha + (k - 2r)\beta \} + O \{ \log N(\log \log N)^2 \}.
\]
By Lemma 2, the left-hand member is equal to
\[
\frac{2}{D} \sum_{r=1}^{r-1} \xi(v) \prod_{p | r, p > 2} \left( \frac{p - 1}{p - 2} \right) + O \{ kX^{-1}(\log \log X)^2 \sum_{r=1}^{r-1} \xi(v) \}.
\]
It follows from Lemmas 1 and 3 (with \( s = 0, 1 \)) that this equals
\[
2 \sum_{r=1}^{r-1} \xi(v) + O \{ \log N(\log \log N)^2 \} + O \{ X^{-1}(\log N)^4(\log \log X)^2 \},
\]
and the lemma follows from (4) and the second part of Lemma 1.

4. **Proof of the theorem.** By considering the expression
\[
I(N, k) = \int_0^1 | S(\theta) \Psi(\theta) |^2 d\theta,
\]
where
\[
S(\theta) = \sum_{N < \phi \leq N} e^{2\pi i \phi \theta} \log \phi \quad (\phi \text{ prime}),
\]
and $\Psi(\theta)$ is defined by (10), it is possible to show that, for large $N$,

$$I(N, k) = \xi(0)N \log N \{1 + o(1)\} + 2\{1 + o(1)\} \log^2 N \sum_{r=1}^{k} \xi(r)\omega(2r)$$

$$> N\{1 + o(1)\} \sum_{1 \leq q \leq x} \frac{\mu^2(q)}{\phi(q)} \sum_{f=0, f \neq q}^{\infty} \left| \frac{f}{\phi(f)} \right|^2$$

$$> N\{1 + o(1)\} \sum_{1 \leq q \leq x} \frac{\mu^2(q)}{\phi(q)} \sum_{f=0, f \neq q}^{\infty} \xi(f) e^{2\pi i f/q} + o(N \log^2 N).$$

(12)

$$= \xi(0)N \{1 + o(1)\} \sum_{1 \leq q \leq x} \frac{\mu^2(q)}{\phi(q)}$$

$$+ 2N\{1 + o(1)\} \sum_{1 \leq q \leq x} \frac{\mu^2(q)}{\phi(q)}$$

$$\cdot \sum_{f=0, f \neq q}^{\infty} \sum_{r=1}^{k} \xi(r) e^{2\pi i f/q} + o(N \log^2 N).$$

The proof of this is similar in every respect to the proof of (4.1) and (6.8) of II, except that $\Psi(\xi)$ is defined by (10) and is not $\sum_{n=1}^{x} e^{2\pi i n/q}$ as in II. The restriction $k < \log N$ made in II is unessential and can be replaced by $k < A_2 \log N$.

It follows from (3), (4), (12), Lemma 4, and Lemma 4 of II that

$$2\{1 + o(1)\} \log^2 N \sum_{r=1}^{k-1} \xi(r)\omega(2r)$$

$$> 4N\{1 + o(1)\} \sum_{r=1}^{k-1} \xi(r) - \sigma N \log N \{1 + o(1)\} \xi(0)$$

(13)

$$+ o(N \log^2 N)$$

$$= 2N\{2\alpha + (k - 2r)\beta\} \{1 + o(1)\}$$

$$- \sigma N \log N \{2\alpha^2 + (k - 2r)\beta^2\} \{1 + o(1)\}$$

$$+ o(N \log^2 N).$$

Now, just as in III, it can be shown by Buchstab's form of Brun's method that, for sufficiently large $N$,

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* There is a misprint in the enunciation of the lemma which should read: "If $X$ is large $\sum_{1 \leq n \leq x} \mu^2(n)/\phi(n) \sim \log X."$

* We must consider the interval $(N, 2N)$ instead of the interval $(x/\log x, x)$. Our only reason for considering an interval of the form $(N, 2N)$ is that it enables us to quote results directly from II. As far as Brun's method is concerned, the inequality stated remains true if we replace the left-hand side $\omega(2r)$ by the number of prime pairs $p, p'$ which satisfy $AN < p' \leq p \leq (A + 1)N$, $p - p' = 2r > 0$ where $A \geq 0$. 

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\[ \tilde{\omega}(2\nu) < \frac{4\lambda N}{D(\log N)^2} \prod_{\nu \mid \nu \nu > 2} \left( \frac{\ell - 1}{\ell - 2} \right), \]

where \( \lambda \) is defined by (8). It follows from Lemmas 1 and 3 that

\[
\sum_{\nu \mid \nu \nu > k - r} \xi(\nu) \tilde{\omega}(2\nu) < \frac{4\lambda N}{(\log N)^2} \sum_{\nu \mid \nu \nu > k - r} \xi(\nu) + O \left\{ \frac{N}{(\log N)^2} (\log \log N)^2 \right\} = \frac{2\lambda \alpha^2 r^2 N}{(\log N)^2} + O \left\{ \frac{N}{(\log N)^2} (\log \log N)^2 \right\}
\]

since \( r = O(\log N) \).

Accordingly, if we make the assumption that \( \tilde{\omega}(2\nu) = 0 \) for all \( \nu < k = k - r \) we have, from (13) and (14), that

\[
\left\{ \frac{2\alpha + (k - 2\nu)\beta}{\log N} \right\}^2 \{ 1 + o(1) \}
\]

\[ - \frac{1}{2} \sigma \left\{ \frac{2\alpha^2 + (k - 2\nu)\beta^2}{\log N} \right\} \{ 1 + o(1) \}
\]

\[ < 2\lambda^2 \left( \frac{r}{\log N} \right)^2 \{ 1 + o(1) \}. \]

We now take \( \alpha = 1, \beta = 2^{1/2} \), and put

\[
k = \left[ \frac{1}{2} \sigma \left( 1 + \frac{2^{1/2} a}{\lambda} \right) \log N \right],
\]

\[
h = \left[ \frac{1}{2} \sigma \left( 1 - \frac{a}{\lambda} \right) \log N \right],
\]

where \( a \) is a fixed positive number satisfying

\[ a < 3 - 2^{1/2}. \]

It is easily verified that, with this choice of \( a \), \( h > 2k/3 \).

Then, for large \( N \),

\[ r = \frac{a\sigma}{2\lambda} (2^{1/2} + 1) \log N + O(1), \]

\[ 2r\alpha + (k - 2r)\beta = (2 - 2^{1/2})(k + 2^{1/2}h) = \frac{1}{2^{1/2}} \sigma \log N + O(1), \]

\[ 2r\alpha^2 + (k - 2r)\beta^2 = 2h = \sigma \left( 1 - \frac{a}{\lambda} \right) \log N + O(1). \]
It follows from (15) that

\[ \frac{1}{2} \sigma^2 - \frac{1}{2} \sigma^2 \left( 1 - \frac{a}{\lambda} \right) \leq 2\lambda \left( \frac{\sigma a(2^{1/2} + 1)}{2\lambda} \right)^2, \]

that is, that

\[ a \geq 3 - 2^{3/2}, \]

which contradicts (17).

We conclude, accordingly, that \( \omega(2\nu) > 0 \) for some integer \( \nu \) satisfying

\[ \nu < \frac{1}{2} \sigma \left( 1 - \frac{a}{\lambda} \right) \log N. \]

It follows from this inequality, and from (17), that

\[ l \leq \sigma c \]

where

\[ (18) \quad c = 1 - (3 - 2^{3/2})/\lambda = 0.97667 \cdots < \frac{42}{43}, \]

and this proves the theorem.

Note added 24 April 1949. Professor Erdös has informed me that considerable improvements have recently been made in Brun’s method by A. Selberg and others; this will presumably mean that Buchstab’s constant \( A_0 \), and consequently my constants \( c \) and \( d \) (see equations (2) and (18)), can be replaced by smaller values.

REFERENCES


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