THE DIFFERENCE BETWEEN CONSECUTIVE PRIME NUMBERS. IV

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1. **Introduction.** Let p_n denote the nth prime, and let Θ be the lower bound of all positive numbers σ_0 such that no Dirichlet L-function $L(s, \chi)$ has a zero at $s=\sigma+it$, where $\sigma>\sigma_0$. Write

$$l = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}.$$

The purpose of this paper is to combine the methods used in two earlier papers[1] in order to prove the following theorem.

THEOREM.

$$l \leqq c(1 + 4\Theta)/5, \tag{1}$$

where $c<42/43$.

The precise definition of c is given by (18) and (8) below.

The theorem is an improvement on the result $l\leqq(1+4\Theta)/5$ obtained in II. It was shown in III that

$$l \leqq d = 0.966 \cdots < 57/59. \tag{2}$$

The numbers c and d are connected by the relation

$$1 - d = (3 + 2^{3/2})(1 - c)/4 > 1 - c,$$

so that (1) is an improvement on (2) only if Θ is not too close to unity, in fact, if $\Theta<0.986 \cdots$. In particular, if the "grand Riemann hypothesis" is true, that is, if $\Theta=1/2$, we have (using the value of c given by (18))

$$l \leqq 3c/5 < 109/186 = 0.58602 \cdots .$$

2. **Notation.** Since (2) is sharper than (1) for $\Theta=1$, we shall assume that $\Theta<1$, and write

$$\sigma = (1 + 4\Theta)/5. \tag{3}$$

Let N be a large positive integer, and define

$$X = N^{1-\sigma}(\log N)^{-8(1+\gamma)} \tag{4}$$

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[1] See [2] and [3]; I refer to these as II and III respectively. Numbers in brackets refer to the references cited at the end of the paper.

as done in equation (6.7) of II; here γ is a positive constant which may be taken to be 2.

Let $\tilde{\omega}(\nu)$ be the number of prime pairs p, p' which satisfy

$$N < p' \leqq p \leqq 2N, \qquad p - p' = \nu,$$

where ν is an integer. We suppose that h, k, m, n and r are integers such that, for some positive A_1, A_2 independent of N,

$$A_1 \log N < 2k/3 < h < k < A_2 \log N, \tag{5}$$

$$0 \leqq m < n < A_2 \log N, \tag{6}$$

$$r = k - h.$$

The letters a, c, α, β denote positive numbers independent of N, and we write

$$D = \prod_{p>2}\left(1 + \frac{1}{p(p-2)}\right), \tag{7}$$

and take

$$\lambda = 7.3566 > e^{-2\gamma}\Lambda_0/2, \tag{8}$$

where γ is Euler's constant and Λ_0 is Buchstab's number 46.67347 (see relation (4) of III).

For $\mu = 1, 2, \cdots, k$ we define a_μ as follows:

$$a_\mu = \begin{cases} \alpha & (0 < \mu \leqq r,\ k - r < \mu \leqq k), \\ \beta & (r < \mu \leqq k - r), \end{cases} \tag{9}$$

and write

$$\Psi(\theta) = \sum_{\mu=1}^{k} a_\mu e^{4\pi i \mu\theta}. \tag{10}$$

Then

$$|\Psi(\theta)|^2 = \xi(0) + 2\sum_{\nu=1}^{k-1} \xi(\nu) \cos 4\pi\nu\theta, \tag{11}$$

where

$$\xi(\nu) = \sum_{i-j=\nu,\, 1\leqq j\leqq i\leqq k} a_i a_j.$$

3. Lemmas.

LEMMA 1. *We have*

$$\xi(\nu) = \begin{cases} 2(r-\nu)\alpha^2 + 2\nu\alpha\beta + (k-2r-\nu)\beta^2 & (0 \leqq \nu < r), \\ 2r\alpha\beta + (k-2r-\nu)\beta^2 & (r \leqq \nu < k-2r), \\ (2r-k+\nu)\alpha^2 + 2(k-r-\nu)\alpha\beta & (k-2r \leqq \nu < k-r), \\ (k-\nu)\alpha^2 & (k-r \leqq \nu < k), \end{cases}$$

and

$$\sum_{\nu=1}^{k-1} \xi(\nu) = \frac{1}{2}\{2r\alpha + (k-2r)\beta\}^2 + O(\log N).$$

This follows easily from (9) and (11), and we omit the proof. It is in this lemma that the restriction $k-r=h>2k/3$ made in (5) is used.

LEMMA 2.

$$\sum_{1\leqq q\leqq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i\nu f/q}$$
$$= \frac{2}{D} \prod_{p|\nu, p>2} \left(\frac{p-1}{p-2}\right) + O(\nu X^{-1}(\log\log X)^2).$$

Here $\mu(q)$ is the Möbius function, $\phi(q)$ is Euler's function, and D is defined by (7).

It follows from the well known relation

$$\sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i\nu f/q} = \sum_{d|(2\nu,q)} d\mu\left(\frac{q}{d}\right)$$

that

$$\sum_{1\leqq q\leqq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} e^{4\pi i\nu f/q} = \sum_{q\leqq X} \frac{\mu(q)}{\phi^2(q)} \prod_{p|(2\nu,q)} (1-p)$$
$$= \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi^2(q)} \prod_{p|(2\nu,q)} (1-p)$$
$$- \sum_{q>X} \frac{\mu(q)}{\phi^2(q)} \prod_{p|(2\nu,q)} (1-p).$$

By a straightforward reduction the first term on the right may be shown to equal

$$\frac{2}{D} \prod_{p|\nu, p>2} \left(\frac{p-1}{p-2}\right).$$

See, for example, §§3.21–3.23 of [1]. Since $1/\phi(n) = O(n^{-1} \log \log n)$, the second term on the right is

$$O\left\{X^{-1}(\log \log X)^2 \prod_{p \mid 2\nu} (p-1)\right\} = O\{\nu X^{-1}(\log \log X)^2\}.$$

LEMMA 3. *If $s \geqq 0$ and is independent of N, then*

$$\sum_{\nu=m}^{n-1} \nu^s \prod_{p \mid \nu, p>2} \left(\frac{p-1}{p-2}\right) = D \sum_{\nu=m}^{n-1} \nu^s + O\{(\log N)^s (\log \log N)^2\}.$$

The proof is similar to that of the lemma in III. We suppose that d is any product $p_{i_1} p_{i_2} \cdots p_{i_j}$ of distinct odd primes, and let d' be the associated product

$$d' = (p_{i_1} - 2)(p_{i_2} - 2) \cdots (p_{i_j} - 2).$$

We permit d to take the value unity and then put $d' = 1$. Then, clearly,

$$\sum_{d<n} \frac{1}{dd'} = \sum_{d \geqq 1} \frac{1}{dd'} + O\left\{\frac{1}{n} (\log \log 3n)^2\right\}$$
$$= D + O\left\{\frac{1}{n} (\log \log 3n)^2\right\},$$

since $1/d' = O\{d^{-1}(\log \log d)^2\}$ for large[2] d; and

$$\prod_{p \mid \nu, p>2} \left(\frac{p-1}{p-2}\right) = \prod_{p \mid \nu, p>2} \left(1 + \frac{1}{p-2}\right)$$
$$= \sum_{d \mid \nu} \frac{1}{d'}.$$

For a given d', ν is a multiple of d, say $\nu = \mu d$, and, since $m \leqq \nu < n$, μ takes all integer values in the interval $m/d \leqq \mu < n/d$, which we denote by $I(d)$. Clearly

$$\sum_{\mu \in I(d)} (\mu d)^s = \frac{n^{s+1} - m^{s+1}}{(s+1)d} + O(n^s),$$

and

$$\sum_{\nu=m}^{n-1} \nu^s = \frac{n^{s+1} - m^{s+1}}{s+1} + O(n^s).$$

[2] It is possible to dispense with the factor $(\log \log 3n)^2$.

Hence

$$\begin{aligned}\sum_{\nu=m}^{n-1} \nu^s \prod_{p|\nu, p>2} \left(\frac{p-1}{p-2}\right) &= \sum_{d<n} \frac{1}{d'} \sum_{\mu \in I(d)} (\mu d)^s \\ &= \frac{n^{s+1}-m^{s+1}}{s+1} \sum_{d<n} \frac{1}{dd'} + O\left(n^s \sum_{d<n} \frac{1}{d'}\right) \\ &= \frac{n^{s+1}-m^{s+1}}{s+1} \left\{D + O\left(\frac{1}{n} (\log\log 3n)^2\right)\right\} \\ &\quad + O\{n^s(\log\log 3n)^2 \log 2n\} \\ &= D \sum_{\nu=m}^{n-1} \nu^s + O\{n^s(\log 2n)^2\},\end{aligned}$$

and the lemma follows from (6).

LEMMA 4.

$$\begin{aligned}\sum_{1\leq q\leq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} \sum_{\nu=1}^{k-1} \xi(\nu) e^{4\pi i \nu f/q} &= 2\sum_{\nu=1}^{k-1} \xi(\nu) + O\{\log N(\log\log N)^2\} \\ &= \{2r\alpha + (k-2r)\beta\}^2 + O\{\log N(\log\log N)^2\}.\end{aligned}$$

By Lemma 2, the left-hand member is equal to

$$\frac{2}{D}\sum_{\nu=1}^{k-1} \xi(\nu) \prod_{p|\nu, p>2} \left(\frac{p-1}{p-2}\right) + O\left\{kX^{-1}(\log\log X)^2 \sum_{\nu=1}^{k-1} \xi(\nu)\right\}.$$

It follows from Lemmas 1 and 3 (with $s=0$, 1) that this equals

$$2\sum_{\nu=1}^{k-1} \xi(\nu) + O\{\log N(\log\log N)^2\} + O\{X^{-1}(\log N)^3(\log\log X)^2\},$$

and the lemma follows from (4) and the second part of Lemma 1.

4. **Proof of the theorem.** By considering the expression

$$I(N, k) = \int_0^1 |S(\theta)\Psi(\theta)|^2 d\theta,$$

where

$$S(\theta) = \sum_{N<p\leq 2N} e^{2\pi i\theta p} \log p \qquad (p \text{ prime}),$$

and $\Psi(\theta)$ is defined by (10), it is possible to show that, for large N,

$$
\begin{aligned}
I(N, k) &= \xi(0)N \log N\{1 + o(1)\} \\
&\quad + 2\{1 + o(1)\} \log^2 N \sum_{\nu=1}^{k-1} \xi(\nu)\bar{\omega}(2\nu) \\
&> N\{1 + o(1)\} \sum_{1\leq q\leq X} \frac{\mu^2(q)}{\phi^2(q)} \sum_{f=0,(f,q)=1}^{q-1} \left|\Psi\left(\frac{f}{q}\right)\right|^2 \\
&\quad + o(N \log^2 N) \qquad (12) \\
&= \xi(0)N\{1 + o(1)\} \sum_{1\leq q\leq X} \frac{\mu^2(q)}{\phi(q)} \\
&\quad + 2N\{1 + o(1)\}\Re \sum_{1\leq q\leq X} \frac{\mu^2(q)}{\phi^2(q)} \\
&\quad \cdot \sum_{f=0,(f,q)=1}^{q-1} \sum_{\nu=1}^{k-1} \xi(\nu)e^{4\pi i\nu f/q} + o(N \log^2 N).
\end{aligned}
$$

The proof of this is similar in every respect to the proof of (4.1) and (6.8) of II, except that $\Psi(\xi)$ is defined by (10) and is not $\sum_{\nu=1}^{k-1} e^{2\pi i\nu\theta}$ as in II. The restriction $k<\log N$ made in II is unessential and can be replaced by $k<A_2 \log N$.

It follows from (3), (4), (12), Lemma 4, and Lemma 4 of[3] II that

$$
\begin{aligned}
&2\{1 + o(1)\} \log^2 N \sum_{\nu=1}^{k-1} \xi(\nu)\bar{\omega}(2\nu) \\
&\quad > 4N\{1 + o(1)\} \sum_{\nu=1}^{k-1} \xi(\nu) - \sigma N \log N\{1 + o(1)\}\xi(0) \\
&\quad + o(N \log^2 N) \qquad (13) \\
&\quad = 2N\{2r\alpha + (k - 2r)\beta\}^2\{1 + o(1)\} \\
&\quad - \sigma N \log N\{2r\alpha^2 + (k - 2r)\beta^2\}\{1 + o(1)\} \\
&\quad + o(N \log^2 N).
\end{aligned}
$$

Now, just as in III, it can be shown by Buchstab's form of Brun's method[4] that, for sufficiently large N,

[3] There is a misprint in the enunciation of the lemma which should read: "*If X is large* $\sum_{1\leq q\leq X} \mu^2(q)/\phi(q) \sim \log X$."

[4] We must consider the interval $(N, 2N)$ instead of the interval $(x/\log x, x)$. Our only reason for considering an interval of the form $(N, 2N)$ is that it enables us to quote results directly from II. As far as Brun's method is concerned, the inequality stated remains true if we replace the left-hand side $\bar{\omega}(2\nu)$ by the number of prime pairs p, p' which satisfy $AN<p'\leq p\leq(A+1)N$, $p-p'=2\nu>0$ where $A\geq 0$.

$$\tilde{\omega}(2\nu) < \frac{4\lambda N}{D(\log N)^2} \prod_{p|\nu, p>2} \left(\frac{p-1}{p-2}\right),$$

where λ is defined by (8). It follows from Lemmas 1 and 3 that

$$\begin{aligned} \sum_{\nu=k-r}^{k-1} \xi(\nu)\tilde{\omega}(2\nu) &< \frac{4\lambda N}{(\log N)^2} \sum_{\nu=k-r}^{k-1} \xi(\nu) + O\left\{\frac{N}{\log N}(\log\log N)^2\right\} \\ &= \frac{2\lambda\alpha^2 r^2 N}{(\log N)^2} + O\left\{\frac{N}{\log N}(\log\log N)^2\right\} \end{aligned} \tag{14}$$

since $r = O(\log N)$.

Accordingly, if we make the assumption that $\tilde{\omega}(2\nu) = 0$ for all $\nu < h = k - r$ we have, from (13) and (14), that

$$\begin{aligned} &\left\{\frac{2r\alpha + (k-2r)\beta}{\log N}\right\}^2 \{1 + o(1)\} \\ &\qquad - \frac{1}{2}\sigma\left\{\frac{2r\alpha^2 + (k-2r)\beta^2}{\log V}\right\}\{1 + o(1)\} \\ &\qquad < 2\lambda\alpha^2\left(\frac{r}{\log N}\right)^2\{1 + o(1)\}. \end{aligned} \tag{15}$$

We now take $\alpha = 1$, $\beta = 2^{1/2}$, and put

$$\begin{aligned} k &= \left[\frac{1}{2}\sigma\left(1 + \frac{2^{1/2}a}{\lambda}\right)\log N\right], \\ h &= \left[\frac{1}{2}\sigma\left(1 - \frac{a}{\lambda}\right)\log N\right], \end{aligned} \tag{16}$$

where a is a fixed positive number satisfying

$$a < 3 - 2^{3/2}. \tag{17}$$

It is easily verified that, with this choice of a, $h > 2k/3$.

Then, for large N,

$$r = \frac{a\sigma}{2\lambda}(2^{1/2} + 1)\log N + O(1),$$

$$2r\alpha + (k - 2r)\beta = (2 - 2^{1/2})(k + 2^{1/2}h) = \frac{1}{2^{1/2}}\sigma\log N + O(1),$$

$$2r\alpha^2 + (k - 2r)\beta^2 = 2h = \sigma\left(1 - \frac{a}{\lambda}\right)\log N + O(1).$$

It follows from (15) that

$$\frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2\left(1 - \frac{a}{\lambda}\right) \leqq 2\lambda\left\{\frac{\sigma a(2^{1/2}+1)}{2\lambda}\right\}^2,$$

that is, that

$$a \geqq 3 - 2^{3/2},$$

which contradicts (17).

We conclude, accordingly, that $\bar{\omega}(2\nu) > 0$ for some integer ν satisfying

$$\nu < \frac{1}{2}\,\sigma\left(1 - \frac{a}{\lambda}\right)\log N.$$

It follows from this inequality, and from (17), that

$$l \leqq \sigma c$$

where

$$c = 1 - (3 - 2^{3/2})/\lambda = 0.97667\cdots < \frac{42}{43}, \tag{18}$$

and this proves the theorem.

Note added 24 *April* 1949. Professor Erdös has informed me that considerable improvements have recently been made in Brun's method by A. Selberg and others; this will presumably mean that Buchstab's constant Λ_0, and consequently my constants c and d (see equations (2) and (18)), can be replaced by smaller values.

References

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