ON UPPER SEMICONTINUOUS FUNCTIONS

E. J. MICKLE AND T. RADÓ

1. **Introduction.** In his important paper [1] (see Bibliography) on the theory of the Lebesgue area of surfaces, L. Cesari established the following interesting result. Let X be the unit cube $0 \leqq x_j \leqq 1$, $j=1, \cdots, m$, of Euclidean m-space and let Y be the unit cube $0 \leqq y_i \leqq 1$, $i=1, \cdots, n$, of Euclidean n-space. Let $F(x_1, \cdots, x_m, y_1, \cdots, y_n)$ be a real-valued, bounded, upper semicontinuous function in the product space $X \times Y$. Then for each point $(x_1, \cdots, x_m) \in X$, F is a bounded, upper semicontinuous function of $(y_1, \cdots, y_n) \in Y$, and hence F attains, for each fixed $(x_1, \cdots, x_m) \in X$, a maximum value $M(x_1, \cdots, x_m)$, while $(y_1, \cdots, y_n)$ varies in Y. According to the theorem of Cesari, there exists then in X Borel measurable functions $y_i = \Phi_i(x_1, \cdots, x_m)$, $i=1, \cdots, n$, such that identically

$$F[x_1, \cdots, x_m, \Phi_1(x_1, \cdots, x_m), \cdots, \Phi_n(x_1, \cdots, x_m)] = M(x_1, \cdots, x_m).$$

The proof given by Cesari is essentially an induction proof. The purpose of this note is to give a direct proof for a more general result (see our Theorem 3.1) which contains the theorem of Cesari as a very special case. In this generalized result, X is any metric space and Y is any compact metric space. In this general setting, the concept of a real-valued Borel measurable function is replaced by the following concept (cf. Kuratowski [**2**, p. 177]). A single-valued transformation T from a metric space S_1 into a metric space S_2 is termed a Borel transformation if for every closed set $F_2 \subset S_2$ the inverse set $T^{-1}(F_2)$ is a Borel set in S_1 (and, hence, if the inverse of every Borel set in S_2 is a Borel set in S_1).

2. **Preliminaries.** Throughout this section X will denote a metric space and I will denote the unit interval $0 \leqq t \leqq 1$. Points in X will be denoted by x.

LEMMA 2.1. *In the product space $X \times I$ let E be a set which satisfies the following conditions.*

(a) *For each t_0 the set*

$$\underset{(x,t)}{E} \; [(x, t) \in E, t = t_0]$$

is a Borel set.

Received by the editors January 10, 1949.

(b) *For each* x_0 *the set*

$$\underset{t}{E}\,[(x_0, t) \in E]$$

is a closed interval containing 0. *Then* E *is a Borel set.*

PROOF. Let $r_1, r_2, \cdots$ be the set of rational numbers in I. For each pair of integers n and m we set

$$E_m^n = \underset{(x,t)}{E}\,[(x, r_n) \in E, 0 \leqq t \leqq r_n + 1/m, t \leqq 1].$$

It follows from condition (a) that each set E_m^n is a Borel set. Thus the sets

$$E_m = \sum_n E_m^n, \qquad m = 1, 2, \cdots,$$

are Borel sets. The reader will verify readily that $E = \prod_m E_m$. Therefore E is a Borel set.

LEMMA 2.2. *In the product space* $X \times I$ *let* E *be a set which satisfies the following conditions.*

(a) *For each* t_0 *the set*

$$\underset{(x,t)}{E}\,[(x, t) \in E, t = t_0]$$

is a Borel set.

(b) *For each* x_0 *the set*

$$\underset{t}{E}\,[(x_0, t) \in E]$$

is a closed interval containing 1. *Then* E *is a Borel set.*

PROOF. The proof is similar to that used in Lemma 2.1.

LEMMA 2.3. *Let* $t = f(x)$ *be a single-valued transformation from* X *into* I *and in the product space* $X \times I$ *let*

$$E_* = \underset{(x,t)}{E}\,[0 \leqq t \leqq f(x)], \qquad E^* = \underset{(x,t)}{E}\,[f(x) \leqq t \leqq 1].$$

A necessary and sufficient condition that $t = f(x)$ *be a Borel transformation is that* E_* *be a Borel set. The same statement holds with* E_* *replaced by* E^*.

PROOF. Assume that $t = f(x)$ is a Borel transformation. Then E_* and E^* clearly satisfy the conditions of Lemma 2.1 and Lemma 2.2 respectively. Hence E_* and E^* are Borel sets.

Assume that E_* is a Borel set. We assert that $f^{-1}(B)$ is a Borel set whenever B is a Borel set in I. By well known results it is sufficient to prove this assertion for the case where

$$B = \underset{t}{E}\,[t_0 \leqq t \leqq 1]. \tag{1}$$

Let B be a Borel set as given in (1). Set

$$X^0 = \underset{(x,t)}{E}\,[(x, t) \in X \times I, t = t_0], \qquad B_* = E_* X^0.$$

Let P: $(x, t) \to x$ be the projection of $X \times I$ onto X. From the definition of E_* and by well known properties of Borel sets we have the following facts concerning the sets B and B_*. (i) $f^{-1}(B) = P(B_*)$. (ii) Since E_* is assumed to be a Borel set, B_* is a Borel subset of X^0. (iii) Since P is a bicontinuous transformation from X^0 onto X and B_* is a Borel subset of X^0, $P(B_*)$ is a Borel set. From (i) and (iii) it follows that $f^{-1}(B)$ is a Borel set.

A similar reasoning applies if E^* is assumed to be a Borel set.

Let $g(x, t)$ be a real-valued, bounded, upper semicontinuous function defined on the product space $X \times C$ where C is a closed subset of I and C contains 0. Set

$$M(x) = \max g(x, t) \qquad \text{for } x \text{ fixed}, t \in C.$$

Since $g(x, t)$ is an upper semicontinuous function and C is compact, the set

$$E(x) = \underset{t}{E}\,[g(x, t) = M(x), x \text{ fixed}]$$

is a closed subset of C for each x. Set

$$t = f(x) = \min \tau \qquad \text{for } \tau \in E(x). \tag{2}$$

LEMMA 2.4. *Under the above conditions the transformation $t = f(x)$ defined in* (2) *is a Borel transformation from X into C and for each $x \in X$*

$$g[x, f(x)] = M(x). \tag{3}$$

PROOF. The relation (3) follows from the definition of $t = f(x)$ in (2). To prove that $t = f(x)$ is a Borel transformation we set

$$g_1(x, t) = \max g(x, \tau) \qquad \text{for } 0 \leqq \tau \leqq t, \tau \in C.$$

Since 0 is in C, $g_1(x, t)$ is defined on $X \times I$. It is easily shown that $M(x)$ and $g_1(x, t)$ are bounded, upper semicontinuous functions on $X \times I$ and $g_1(x, t) \leqq M(x)$. Thus the function

$$\phi(x, t) = M(x) - g(x, t) \geqq 0$$

is a Borel measurable function. Hence the set

$$E_0 = \underset{(x,t)}{E} \, [\phi(x, t) = 0]$$

is a Borel set. It is easily shown that the set E_0 is equal to the set

$$E^* = \underset{(x,t)}{E} \, [f(x) \leqq t \leqq 1].$$

Therefore, by the result in Lemma 2.3, $t=f(x)$ is a Borel transformation.

The reader can easily verify the following result on Borel transformations.

LEMMA 2.5. *Let Y be a metric space which is the product of n metric spaces $Y_1, \cdots, Y_n$ and let*

$$T: \quad y_i = \Phi_i(x), \qquad i = 1, \cdots, n, \; x \in X, \; y_i \in Y_i,$$

be a single-valued transformation from X into Y. A necessary and sufficient condition that T be a Borel transformation from X into Y is that each of the transformations $y_i = \Phi_i(x)$ be a Borel transformation from X into Y_i.

3. The main results.

THEOREM 3.1. *Let X be a metric space, let Y be a compact metric space, and let $F(x, y)$ be a real-valued, bounded, upper semicontinuous function defined on the product space $X \times Y$. Then there exists a Borel transformation*

$$y = \Phi(x) \tag{4}$$

from X into Y such that for each $x \in X$

$$F[x, \Phi(x)] = M(x), \text{ where } M(x) = \max F(x, y) \text{ for } x \text{ fixed}, \; y \in Y. \tag{5}$$

PROOF. Since Y is compact, Y is the continuous image of a closed subset C of the unit interval I: $0 \leqq t \leqq 1$, where C contains 0 (see Whyburn [3, p. 34]). Let us denote this continuous transformation by

$$y = \phi(t), \qquad t \in C.$$

Then

$$g(x, t) = F[x, \phi(t)] \tag{6}$$

is a real-valued, bounded, upper semicontinuous function defined on the product space $X \times C$ and (see (5))

$$M(x) = \max g(x, t) \qquad \text{for } x \text{ fixed}, \; t \in C.$$

The transformation $t=f(x)$ defined in Lemma 2.4 in terms of $g(xt)$ is a Borel transformation from X into C for which

$$g[x, f(x)] = M(x). \tag{7}$$

Set

$$y = \Phi(x) = \phi[f(x)]. \tag{8}$$

From (6), (7), and (8)

$$F[x, \Phi(x)] = F[x, \phi(f(x))] = g[x, f(x)] = M(x).$$

Thus the transformation defined in (8) satisfies the relation (5). Let B be a Borel set in Y. Then

$$\Phi^{-1}(B) = f^{-1}[\phi^{-1}(B)].$$

Since $y=\phi(t)$ is a continuous transformation from C onto Y, $\phi^{-1}(B)$ is a Borel subset of C, and since $t=f(x)$ is a Borel transformation from X into C, $f^{-1}[\phi^{-1}(B)]$ is a Borel set. Thus the transformation defined in (8) is a Borel transformation from X into Y which satisfies the conditions of the theorem.

If in the preceding theorem Y is the product of n metric spaces $Y_1, \cdots, Y_n$, the Borel transformation (4) can be written in the form

$$y_i = \Phi_i(x), \quad x \in X, \quad y_i \in Y_i, \qquad i = 1, \cdots, n. \tag{9}$$

By Lemma 2.5 we thus have the result:

COROLLARY. *Each transformation* $y_i=\Phi_i(x)$, $i=1, \cdots, n$, *given in* (9) *is a Borel transformation from* X *into* Y_i.

If X coincides with the unit cube $0 \leqq x_j \leqq 1$, $j=1, \cdots, m$, of Euclidean m-space and Y coincides with the unit cube $0 \leqq y_i \leqq 1$, $i=1, \cdots, n$, of Euclidean n-space, then we obtain as a special case the theorem of Cesari stated in the introduction.

BIBLIOGRAPHY

1. L. Cesari, *Caratterizzazione analitica delle superficie continue di area finita secondo Lebesgue*, Annali della R. Scuola Normale Superiore di Pisa (2) vol. 10 (1941) pp. 253–294, and vol. 11 (1942) pp. 1–42.

2. C. Kuratowski, *Topologie*, Warsaw, 1933.

3. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, 1942.

OHIO STATE UNIVERSITY