ON EULER TRANSFORMS
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Let $E_0$ denote the class of non-constant functions satisfying

$$d\phi(x) \geq 0 \quad \text{where } 0 \leq x < \infty.$$  \hspace{1cm} (1)

Then the Euler transform

$$\phi_\lambda(x) = \int_0^\infty (x + t)^{-\lambda} d\phi(t), \quad \text{where } 0 < x < \infty,$$  \hspace{1cm} (2)

is defined for some $\phi$ of class $E_0$. Let $E_\lambda$ denote the class of all functions $\phi_\lambda$ belonging to a fixed $\lambda > 0$ and to some $\phi$ of class $E_0$. It will be shown that

$$E_\lambda \text{ is a (proper) subset of } E_\mu \text{ if } \lambda < \mu.$$  \hspace{1cm} (3)

This implies that

$$E_\infty = \lim_{\lambda \to \infty} E_\lambda \text{ if } E_\infty = \sum_{0 < \lambda < \infty} E_\lambda.$$  \hspace{1cm} (4)

The class $E_\infty$ is closely related to the Hausdorff-Bernstein class, consisting of all functions which are completely monotone for $0 < x < \infty$. Let $E^\infty$ denote the latter class. It will be shown that

$$E_\infty \text{ is a (proper) subset of } E^\infty,$$  \hspace{1cm} (5)

and that, with reference to the “natural” topology on $E^\infty$,

$$E_\infty \text{ is dense on } E^\infty.$$  \hspace{1cm} (6)

It should be noted that $E^\infty$ consists of all functions representable in the form

$$\phi^\infty(x) = \int_0^\infty e^{-xt} d\phi(t), \quad \text{where } 0 < x < \infty,$$  \hspace{1cm} (7)

provided that $\phi$, instead of being subject to both restrictions (1), is subject only to the second of those restrictions and to the assumption that the integral (7) is convergent at every $x > 0$ (but not necessarily at $x = 0$). By the “natural” topology on $E^\infty$ is meant that defined by the Helly convergence of monotone functions.

Proof of (3). It is readily verified from (1) and (2) that, as $x \to \infty$, no $\phi_\lambda(x)$ can tend to 0 as strongly as $x^{-\mu}$, if $\mu > \lambda$. On the

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other hand, (2) shows that \( \phi_\mu(x) = x^{-\mu} \) if \( \phi(t) = \text{sgn} \ t \). Hence, the parenthetical assertion of (3) will need no further proof.

The main assertion of (3) is that, if \( 0 < \lambda < \mu \), there belongs to every \( \phi_\lambda(t) \) of class \( E_\lambda \) some \( \phi^*_\lambda(t) \) of class \( E_0 \) satisfying

\[
\phi_\lambda(x) = \phi^*_\lambda(x), \quad \text{where } 0 < x < \infty.
\]

It will be shown that such a \( \phi^*_\lambda(t) \) is supplied by the absolutely continuous function having the derivative

\[
d\phi^*_\lambda(t)/dt = A \int_0^t (t-s)^{\mu-\lambda-1} \phi(s) \quad (0 < t < \infty, \phi^*(0) = \phi^*(+0)),
\]

where \( A = A(\lambda, \mu) \) is a positive constant.

In view of (2), the assertion of (8) and (9) means that

\[
\int_0^\infty (x+t)^{-\lambda} \phi(t) = A \int_0^\infty (x+t)^{-\mu} \left\{ \int_0^t (t-s)^{\mu-\lambda-1} \phi(s) \right\} dt,
\]

where \( 0 < x < \infty \). It follows therefore from (1), and from (the Stieljes form of) Fubini's theorem, that it is sufficient to verify the identity

\[
\int_0^\infty (x+t)^{-\lambda} \phi(t) = A \int_0^\infty \left\{ \int_t^\infty (x+t)^{-\mu}(t-s)^{\mu-\lambda-1} ds \right\} \phi(s).
\]

But the latter holds for every \( \phi \) of class \( E_\lambda \) if

\[
(x+t)^{-\lambda} = A \int_t^\infty (x+s)^{-\mu}(s-t)^{\mu-\lambda-1} ds
\]

is an identity in \( (x, t) \), where \( x > 0, t > 0 \). Hence, if the integration variable \( s \) is replaced by \( s-t \), and if \( x+t \) is then called \( x \), it follows that it is sufficient to verify the identity

\[
x^{-\lambda} = A \int_0^\infty (x+s)^{-\mu} s^{\mu-\lambda-1} ds,
\]

where \( A \) is independent of \( x \). Finally, the truth of this identity follows by changing \( s \) to \( xs \) (at a fixed \( x > 0 \)).

This proves (3). It also follows that the value of the constant \( A = A(\lambda, \mu) \) which occurs in (9) is given by

\[
1 = A \int_0^\infty (1+s)^{-\mu} s^{\mu-\lambda-1} ds.
\]

Incidentally, the last integral can readily be transformed into the
integral defining \( B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu - \lambda)}{\Gamma(\mu)} \); so that, in (9),

(9 bis) \( A(\lambda, \mu) = 1/B(\lambda, \mu) \).

Proof of (5). It is seen from (1) that \((-1)^n\) times the \(n\)th derivative of the function (2) is non-negative for \(n = 0, 1, 2, \cdots \). This means that every class \( E_\lambda \) is contained of the Hausdorff-Bernstein class, \( E^\omega \). Hence, by (4), \( E_\infty \) is contained in \( E^\omega \). The parenthetical part of (5) follows from the fact that \( e^{-x} \) is in \( E^\omega \), but \( e^{-z} \) is not in \( E_\infty \). For otherwise \( e^{-z} \) would be in \( E_\lambda \) some \( \lambda \), which would imply, for \( x > 0 \) and \( t_0 > 0 \), that

\[
\epsilon^{-z} \geq (x + t_0)^{-\lambda}(\phi(t_0) - \phi(0)),
\]

by (1) and (2). If \( t_0 \) is chosen so that \( \phi(t_0) - \phi(0) > 0 \), the last formula line leads to a contradiction for large \( x \). This contradiction shows that \( e^{-z} \) cannot be in \( E_\infty \) and completes the proof of (5).

Proof of (6). Let \( b > 0 \) and \( \lambda > 0 \). Then it is readily verified that

\[
b^\lambda \int_0^\infty (x + t)^{-\lambda}d \text{ sgn}(t - b) = (1 + x/b)^{-\lambda},
\]

where \( 0 \leq x < \infty \). Clearly, the expression on the left of this identity represents a function of class \( E_\lambda \). On the other hand, the expression on the right tends to \( e^{-az} \) if \( \lambda \to \infty \) and \( b = \lambda/a \), where \( a \) is any positive constant. Accordingly, every function of the form \( e^{-az} \) is a limit, as \( \lambda \to \infty \), of functions contained in \( E_\lambda \). Hence, the same is true of every function of the form

\[
\sum_{k=1}^m c_k e^{-a_kz},
\]

where \( c_k, a_k \) are arbitrary non-negative constants. Since the latter sum is identical with the case

\[
\phi(x) = \sum_{a_k z} c_k
\]

of the transform (7), the assertion of (6) now follows by a standard application of Helly’s theorems on monotone functions.

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