Tschebyschef's theorem
\[ \alpha \frac{x}{\log x} \leq \pi(x) \leq \gamma \frac{x}{\log x} \]
now follows at once.

Reference


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THE RADICAL OF A NON-ASSOCIATIVE RING

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In this paper a definition is proposed for the radical of a non-associative ring. Our results are somewhat similar to those given for algebras by Albert in [3], but the difficulties that arose in the earlier theory from absolute divisors of zero have been overcome. With slight modifications, the present proofs are applicable to algebras.

A non-associative ring \( R \) is an additive abelian group closed under a product operation with respect to which the two distributive laws hold. Multiplication on the right (left) by a fixed element \( x \in R \) determines an endomorphism \( R_x \) \( (L_x) \) of \( R \) as an additive group. For \( x, y \in R \),
\[ x \cdot y = xR_y = yL_x. \]

The \( R_x \) and \( L_y \) generate an associative ring \( A \) called the transformation ring of \( R \). Clearly \( A \) can be construed as a representation space for \( A \), and this representation is faithful. The two-sided ideals of \( R \), which are defined as for associative rings, are exactly the \( A \)-subspaces of \( R \). The theory of ring homomorphisms goes over intact to the non-associative case.

A nonzero element \( a \in R \) is called an absolute divisor of zero if \( a \cdot x = x \cdot a = 0 \) for all \( x \in R \). If \( A \) has a unit element \( I \) and \( R \) contains no absolute divisors of zero, then the unit element \( I \) is the identity.

Received by the editors March 19, 1949.

1 Numbers in brackets refer to the references cited at the end of the paper.
mapping on \( R \). For if \( aI = b \) where \( a, b \in R \), then \( aI - b = 0 = (aI - b)I = (a - b)I \), \( (a - b)I = 0 \) and so \( a = b \).

We now assume that the minimum condition holds in \( R \). It has been shown in [5, Theorems 3 and 14] that for the case of simple rings this is equivalent to the assumption that \( R \) can be regarded as an algebra of finite dimension over a certain field.

**Theorem 1.** Let \( A \) satisfy the minimum condition and \( R \) contain no absolute divisors of zero. Then \( R = R_1 \oplus R_2 \oplus \cdots \oplus R_n \), where the \( R_i \) are simple if and only if \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) where the \( A_i \) are simple, and conversely.

**Proof.** If \( R = R_1 \oplus R_2 \oplus \cdots \oplus R_n \) where the \( R_i \) are simple, then \( R = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) where \( A_i \) is the transformation ring for \( R_i \). Since \( R_i \) is an irreducible \( A_i \)-space, it follows that \( A_i \) is simple. Conversely, suppose \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) where the \( A_i \) are simple rings generated by pairwise orthogonal idempotents \( E_i \) in the center of \( A \). Clearly \( R = R_1 \oplus R_2 \oplus \cdots \oplus R_n \), where \( R_i = RE_i \). It is easily seen that \( A_i \) is the transformation ring for \( R_i \). If \( R_i \) has a proper ideal \( M_i \), then \( R_i = M_i \oplus N_i \), a direct sum of \( A_i \)-subspaces since \( A_i \) is semisimple. Let \( \mathcal{E} \) be the ideal of \( A_i \) which annihilates \( M_i \). For every nonzero element \( x \) in \( R_i \), \( R_x \) and \( L_x \), at least one of which must be nonzero, are in \( \mathcal{E} \). Clearly \( E_i \) is not in \( \mathcal{E} \) and so \( \mathcal{E} \) is a proper ideal of \( A_i \), which is impossible. Therefore \( R_i \) is simple.

**Corollary.** Under the assumptions of Theorem 1, \( R \) is simple if and only if \( A \) is simple.

The ring \( R \) is defined to be semisimple if the following conditions are satisfied:

(i) \( A \) satisfies the minimum condition.

(ii) \( R \) contains no absolute divisors of zero.

(iii) \( R \) is a direct sum of simple rings.

We now eliminate the restriction that \( R \) contain no absolute divisors of zero. Suppose \( R \) has an ideal \( S \). Let \( \mathcal{S} \) be the set consisting of all \( S \in A \) such that \( RS \subseteq \mathcal{S} \). Clearly \( RA_i \subseteq RA \subseteq \mathcal{S} \) and \( RA \subseteq S \subseteq \mathcal{S} \). Furthermore, \( \mathcal{S} \) is an additive subgroup and so an ideal of \( A \). Let \( [x] \) and \( [y] \) be any two residue classes of \( R - S \). Then \( [x] \cdot [y] = [x \cdot y] = [xR_y] = [yL_x] \). If \( [x^U] = [x^V] \) for all \( \{x\} \in R - S \) where \( U, V \in A \), then \( U - V \in \mathcal{S} \) and so right (left) multiplication by a fixed element of \( R - S \) determines a unique residue class of \( A - \mathcal{S} \). Then it is easy to prove that \( A - \mathcal{S} \) is isomorphic to the transformation ring for \( R - S \). It follows immediately that if \( A \) satisfies the minimum
condition, then so does the transformation ring for any difference ring of $\mathcal{R}$.

Let $\mathcal{M}_1$ be the ideal consisting of zero and all absolute divisors of zero in $\mathcal{R}$. Continuing by induction, let $\mathcal{M}_{i+1}$ be the set consisting of all $x \in \mathcal{R}$ such that $a \cdot x$ and $x \cdot a \in \mathcal{M}_1$ for all $a \in \mathcal{R}$. The $\mathcal{M}_i$ are ideals of $\mathcal{R}$ and $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$.

**Lemma 1.** $\mathcal{M}_{i+1}$ is the set consisting of all $x \in \mathcal{R}$ such that $R_x$ and $L_x$ are left-annihilators of $\mathcal{M}_i$.

**Proof.** Suppose $x \in \mathcal{M}_{i+1}$. Then $aR_x$ and $aL_x \subseteq \mathcal{M}_i$ for all $a \in \mathcal{R}$. Each succeeding application of transformation in $\mathcal{A}$ gives rise to an element in the preceding $\mathcal{M}_i$. Clearly $aR_x T_1 T_2 \cdots T_j = aL_x T_1 T_2 \cdots T_j = 0$ for all $a \in \mathcal{R}$, where $T_1$, $T_2$, $\cdots$, $T_j$ are arbitrary elements of $\mathcal{M}_i$. Therefore $R_x T_1 T_2 \cdots T_j = L_x T_1 T_2 \cdots T_j = 0$ and so $R_x$ and $L_x$ are left annihilators of $\mathcal{M}_i$. Conversely, if $aR_x T_1 T_2 \cdots T_j = aL_x T_1 T_2 \cdots T_j = 0$ for all $a \in \mathcal{R}$, where $T_1$, $T_2$, $\cdots$, $T_j$ are arbitrary elements of $\mathcal{M}_i$, then $aR_x = T_1 T_2 \cdots T_j$ and $aL_x = T_1 T_2 \cdots T_j$ are in $\mathcal{M}_i$ by definition of $\mathcal{M}_i$. Continuing in this way, it is easily seen that $aR_x$ and $aL_x$, that is, $a \cdot x$ and $x \cdot a$ are in $\mathcal{M}_i$ for all $a \in \mathcal{R}$ and so $x \in \mathcal{M}_{i+1}$.

**Lemma 2.** If $\mathcal{A}$ satisfies the minimum condition, then there exists a least integer $l$ such that $\mathcal{M}_l = \mathcal{M}_{l+1}$.

**Proof.** Let $k$ be the first integer for which $\mathcal{A}^{k-1} = \mathcal{A}^k$. The existence of $k$ is insured by the minimum condition on $\mathcal{A}$. Suppose $x$ is an element of $\mathcal{M}_{k+1}$ not in $\mathcal{M}_k$. By Lemma 1, $R_x$ and $L_x$ would be left-annihilators of $\mathcal{M}_i$ but not both of $\mathcal{A}^{i-1}$. This is impossible, and so $\mathcal{M}_{k+1} = \mathcal{M}_k$. Therefore $k$ is an upper bound for $l$ and the lemma is proved.

In particular, if $\mathcal{A}$ has a unit element, then $\mathcal{M}_3 = \mathcal{M}_1$, since $\mathcal{A}$ can have no left-annihilators.

**Theorem 2.** Let $\mathcal{A}$ satisfy the minimum condition. Then $\mathcal{R} - \mathcal{M}_1$ has no absolute divisors of zero. Furthermore, $\mathcal{M}_1$ is contained in every ideal $\mathcal{H}$ for which $\mathcal{R} - \mathcal{H}$ has no absolute divisors of zero.

**Proof.** If $a \cdot x$ and $x \cdot a \in \mathcal{M}_1$ for all $a \in \mathcal{R}$, then $x \in \mathcal{M}_{i+1} = \mathcal{M}_1$. This proves the first part. Suppose $\mathcal{R} - \mathcal{H}$ has no absolute divisors of zero. Clearly $\mathcal{M}_1 \subseteq \mathcal{H}$. Now assume $\mathcal{M}_1 \subseteq \mathcal{H}$. If $x \in \mathcal{M}_{i+1}$, then both $a \cdot x$ and $x \cdot a \in \mathcal{M}_1 \subseteq \mathcal{H}$ for all $a \in \mathcal{R}$ and so $x \in \mathcal{H}$. Therefore $\mathcal{M}_1 \subseteq \mathcal{H}$.

**Lemma 3.** If $\mathcal{A}$ satisfies the minimum condition and $\mathcal{R} - \mathcal{H}$ is semi-

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*In the case of Lie rings, the $\mathcal{M}_i$ constitute the upper central chain.*
simple, then \( R \subseteq \mathfrak{S} \) where \( R \) is the radical of \( \mathfrak{A} \).

**Proof.** The transformation ring for \( \mathfrak{A} - \mathfrak{S} \) is isomorphic to \( \mathfrak{A} - \mathfrak{G} \) where \( \mathfrak{G} \) is the ideal consisting of all \( S \subseteq \mathfrak{A} \) such that \( \mathfrak{R}S \subseteq \mathfrak{S} \). By Theorem 1, \( \mathfrak{A} - \mathfrak{G} \) is semisimple. Therefore \( \mathfrak{R} \subseteq \mathfrak{S} \), and so \( \mathfrak{R} \subseteq \mathfrak{R} \mathfrak{S} \subseteq \mathfrak{S} \).

We now consider the ideal \( \mathfrak{M} \) for the ring \( \mathfrak{R} - \mathfrak{R} \mathfrak{M} \). The existence of this ideal follows from Lemma 2 since the transformation ring for \( \mathfrak{R} - \mathfrak{R} \mathfrak{M} \) satisfies the minimum condition. Let \( \mathfrak{M} \) be the complete reciprocal image of \( \mathfrak{M} \) under the natural homomorphism \( \mathfrak{R} \rightarrow \mathfrak{R} - \mathfrak{R} \mathfrak{M} \). The ideal \( \mathfrak{M} \) will be called the **radical** of \( \mathfrak{R} \). Since \( \mathfrak{R} - \mathfrak{M} \cong (\mathfrak{R} - \mathfrak{R} \mathfrak{M}) - \mathfrak{M} \), it follows from Theorem 2 that \( \mathfrak{R} - \mathfrak{M} \) has no absolute divisors of zero. The transformation ring for \( \mathfrak{R} - \mathfrak{M} \) is isomorphic to \( \mathfrak{A} - \mathfrak{G} \), where \( \mathfrak{G} \) is the ideal consisting of all \( S \subseteq \mathfrak{A} \) such that \( \mathfrak{R}S \subseteq \mathfrak{R} \). Clearly \( \mathfrak{R} \subseteq \mathfrak{G} \) and so \( \mathfrak{A} - \mathfrak{G} \) is semisimple. Theorem 1 then implies

**Theorem 3.** Let \( \mathfrak{A} \) satisfy the minimum condition. Then \( \mathfrak{R} - \mathfrak{M} \) is semisimple.

That \( \mathfrak{M} \) is the minimal ideal having this property is shown by

**Theorem 4.** If \( \mathfrak{A} \) satisfies the minimum condition and \( \mathfrak{R} - \mathfrak{S} \) is semisimple for some ideal \( \mathfrak{S} \), then \( \mathfrak{M} \subseteq \mathfrak{S} \).

**Proof.** By Lemma 3, \( \mathfrak{R} \mathfrak{S} \subseteq \mathfrak{S} \). Clearly \( (\mathfrak{R} - \mathfrak{M}) - (\mathfrak{S} - \mathfrak{R} \mathfrak{M}) \cong (\mathfrak{R} - \mathfrak{R} \mathfrak{M}) - \mathfrak{M} \) and so by Theorem 2, \( \mathfrak{M} \subseteq \mathfrak{S} - \mathfrak{R} \mathfrak{M} \), which implies \( \mathfrak{M} \subseteq \mathfrak{S} \).

The writer is indebted to D. C. Murdoch for suggesting Lemma 1.

**References**


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