NOTE ON A RESULT OF LEVINE AND LIFSCHITZ

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In a paper some years ago, [1],[1] Levine and Lifschitz considered, among other questions, the relationship between the gaps of a Fourier series and its admissible integral orders of zeros, a problem first treated by S. Mandelbrojt [2]. Let $f(t)$ be a function summable over the interval $(-\pi, \pi)$ and let

$$(1) \qquad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

be its associated Fourier series. Let $\psi(\alpha)$ be a continuous non-negative nondecreasing function defined for $0 \leqq \alpha \leqq 2\pi$. We shall say that $f(t)$ has a left-hand zero of integral order $\psi(\alpha)$ at $t = \pi$ if

$$\phi(\alpha) \equiv \int_{\pi-\alpha}^{\pi} |f(t)| \, dt \leqq \psi(\alpha) \qquad (0 \leqq \alpha \leqq 2\pi).$$

(One could equally well consider such a zero at any other point.) Natural choices for $\psi(\alpha)$ are:

a. $$\psi(\alpha) = \alpha^n \qquad (n = 1, 2, \cdots),$$

b. $$\psi(\alpha) = e^{-(1/\alpha)^\rho} \qquad (0 < \rho < \infty),$$

c. $$\psi(\alpha) \begin{cases} = 0 & (0 \leqq \alpha \leqq \alpha_0), \\ > 0 & (\alpha_0 < \alpha \leqq 2\pi). \end{cases}$$

If $\psi(\alpha) \not\equiv 0$, then there exists a number α_0 such that $\psi(\alpha) = 0$ $(0 \leqq \alpha \leqq \alpha_0)$ and $\psi(\alpha) > 0$ $(\alpha_0 < \alpha \leqq 2\pi)$. Let

$$r = -\log \psi(\alpha)/\alpha \qquad (\alpha_0 < \alpha \leqq 2\pi).$$

It is easily seen that for all r sufficiently large this equation may be inverted to give α as a function $\alpha = \eta(r)$ of r. It is $\eta(r)$ which we shall use as the measure of the zero of $f(t)$. It should be noted that such extreme behavior as a. and c. may be permitted and that our theorems are significant throughout this entire range.

Let $N(t)$ be the number of indices $n < t$ whose coefficients a_n and b_n are not both equal to zero. The density of these indices may be characterized by the function

Presented to the Society, February 26, 1949; received by the editors February 19, 1949.

[1] Numbers in brackets refer to the references cited at the end of the paper

$$\zeta(r) = \log r + 2r^2 \int_0^\infty \frac{N(t)}{t(t^2 + r^2)} dt.$$

This slight correction to the original formula of Levine and Lifschitz was pointed out by S. Mandelbrojt in his review of their paper, see [3].

Levine and Lifschitz proved that if

$$\liminf_{r\to\infty} [3\zeta(2r) - r\eta(r)] = -\infty,$$

then $f(t) = 0$ almost everywhere in $(-\pi, \pi)$. In particular

$$\liminf_{r\to\infty} [c\zeta(r) - r\eta(r)] = -\infty$$

implies $f(x) = 0$ almost everywhere if $c = 12$, since $\zeta(\alpha r) \leqq \alpha^2\zeta(r)$, $\alpha > 1$, r sufficiently large. In the other direction Levine and Lifschitz proved that this does not hold for $c < 1/2$. It is the object of this note to show that it does hold for $c = \pi$.

The method of proof is to construct the function

$$Q(z) = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{n_k^2}\right)$$

where n_k are the values of the indices of the nonzero coefficients in the series (1). $Q(z)$ assumes its maximum modulus for a circle with center the origin on the imaginary axis and

$$\log |Q(iy)| = \zeta(y) \qquad (y > 0).$$

The integral function

$$F(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{-itz}dt$$

has zeros at those integers where $Q(z)$ does not. Thus it is easily proved that the function

$$\Phi(z) = \frac{F(z)Q(z)}{\sin \pi z}$$

is integral and satisfies the bound

$$|\Phi(z)| < De^{\zeta(|z|)}$$

for all z, and for $y > 0$ the bound

$$|\Phi(z)| < De^{\zeta(|z|) - y\eta(y)} \qquad (z = x + iy = re^{i\theta}).$$

For these results we refer to [1].

Suppose that $\Phi(z)\not\equiv 0$; then $z=0$ is a zero of some finite order m, $0\leqq m<\infty$. Let us regard the circle C: $|z|=r$. On this circle

$$\begin{aligned}\log|\Phi(z)/Dz^m| &\leqq \zeta(r) - m\log r & (\pi\leqq \arg z\leqq 2\pi)\\ &\leqq \zeta(r) - m\log r - r\sin\theta\,\eta(r\sin\theta) & (0\leqq \arg z\leqq \pi)\\ &\leqq \zeta(r) - m\log r - \sin\theta\, r\eta(r),\end{aligned}$$

since $\eta(r)$ is a nonincreasing function of r. Because the harmonic measure at the origin of an infinitesimal arc of length $rd\theta$ on the circle C is $(1/2\pi)d\theta$, we have, by an integrated form of the two constants theorem,

$$\begin{aligned}\log|\Phi(z)/Dz^m|_{z=1} &\leqq \frac{1}{2\pi}\int_0^{2\pi}[\zeta(r)-m\log r]d\theta - \frac{1}{2\pi}\int_0^{\pi} r\eta(r)\sin\theta d\theta\\ &\leqq \frac{1}{\pi}[\pi\zeta(r) - r\eta(r) - \pi m\log r].\end{aligned}$$

Because we have assumed

$$\liminf_{r\to\infty}[\pi\zeta(r) - r\eta(r)] = -\infty,$$

it follows that $[\Phi(z)/z^m]_{z=0}=0$, a contradiction. Thus we must have $\Phi(z)\equiv 0$. Consequently $F(z)\equiv 0$ and $f(t)=0$ almost everywhere in the interval $(-\pi,\pi)$.

$f(t)$ is said to have a two-sided zero of integral order $\psi(\alpha)$ at $t=\pi$ if

$$\phi(\alpha) = \int_{-\pi}^{-\pi+\alpha}|f(t)|\,dt + \int_{\pi-\alpha}^{\pi}|f(t)|\,dt \leqq \psi(\alpha) \qquad (0\leqq\alpha\leqq\pi).$$

In this case a simple adaptation of the above argument shows that the condition

$$\liminf_{r\to\infty}\left[\frac{\pi}{2}\zeta(r) - r\eta(r)\right] = -\infty$$

is sufficient in order to have $f(t)=0$ almost everywhere in $(-\pi,\pi)$.

If in the above conditions we replace lim inf by lim, more precise results may be obtained. Indeed if $f(t)$ has a two-sided zero at $t=\pi$, the condition

$$\lim_{r\to\infty}[(1+\epsilon)\zeta(r) - r\eta(r)] = -\infty \qquad (\epsilon>0)$$

implies that $f(t)=0$ almost everywhere in $(-\pi,\pi)$. This follows from the fact that on the lines $\arg z=\pi/2+\delta$, $\arg z=-\pi/2-\delta$, we have the bound

$$\left|\Phi(z)\right| \leqq De^{\zeta(|z|)-|z|\cos\delta\,\eta(|z|\cos\delta)} \leqq De^{\zeta(|z|)-\cos\delta\,|z|\eta(|z|)}$$

so that if $\cos\delta\geqq(1+\epsilon)^{-1}$, $\Phi(z)$ tends to 0 as we go to infinity on these lines. Then, since $\Phi(z)$ is at most of order one, by the Phragmén-Lindelöf Principle it is bounded in the enclosed angle and this follows similarly for the angle $-\pi/2+\delta\leqq\arg z\leqq\pi/2-\delta$ and the two complementary angles. Thus $\Phi(z)\equiv 0$ and the result is proved.

If $f(t)$ has a one-sided zero at $t=\pi$, the same argument applied to $\Phi(z)\Phi(-z)$ shows that the condition

$$\lim_{r\to\infty}\left[(2+\epsilon)\zeta(r)-r\eta(r)\right]=-\infty \qquad (\epsilon>0)$$

implies that $f(t)=0$ almost everywhere in $(-\pi,\pi)$.

It is easy to give examples which provide lower bounds for the constant c which may occur in the above result. Indeed, let $f_1(t)$ be identically zero from $-\pi+\delta$ to π and arbitrary (but not zero) in the rest of $(-\pi,\pi)$. We see at once that $\zeta(r)$ is asymptotically not greater than πr while for the choice $\psi(\alpha)=\phi(\alpha)$, $r\eta(r)$ is asymptotically equal to $(2\pi-\delta)r$. Thus the constant c cannot be taken less than 2. Similarly the function $f_2(t)$ identically zero outside $(-\delta,\delta)$, arbitrary in this interval, provides an example showing that for a two-sided zero the constant c cannot be taken as less than 1.

It should be mentioned that this does not, of course, supersede the result of Levine and Lifschitz since they showed that c cannot be taken less than 1/2 even when only very restricted classes of lacunary series are admitted.

References

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3. ———, Review of [1], Mathematical Reviews vol. 3 (1942) p. 106.

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