ON LEBESGUE'S DENSITY THEOREM

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The density theorem of Lebesgue [1] may be stated in the following form: If S is a measurable linear point set, the metric density of S exists and is equal to 0 or 1 almost everywhere. We prove the converse that for every set Z of measure 0 there is a measurable set S whose metric density does not exist at any point of Z. We note, however, that in order for Z to be the set of points for which the metric density of some set S exists but is different from 0 or 1, Z must be both of measure 0 and of first category. As a converse, we show that for every F_σ type set Z of measure 0, thus of first category, there is a measurable set S whose metric density exists but is different from 0 or 1 at every point of Z.

THEOREM 1. *If Z is any set of measure* 0, *there is a measurable set S whose metric density does not exist at any point of Z.*

PROOF. Let $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ be a sequence of open sets each covering Z. Since Z is of measure 0, the sets G_ν may be so chosen that, for every n, the relative measure of G_{n+1} is $1/n$ in each of the disjoint open intervals I_{np}, $p=1, 2, \cdots$, constituting G. Let

$$S = (G_1 - G_2) + (G_3 - G_4) + \cdots + (G_{2k-1} - G_{2k}) + \cdots .$$

S is a measurable set. Let z be any point in Z. For every n, there is a p_n such that z is in I_{np_n}, one of the disjoint open intervals constituting G_n. For odd values of n, the relative measure of S in I_{np_n} is not less than $1-1/n$. For then $S \supset G_n - G_{n+1}$, a set whose relative measure in I_{np_n} is exactly $1-1/n$. For even values of n, $S \subset G_{n+1}$, a set whose relative measure in I_{np_n} is $1/n$. Hence, in this case, the relative measure of S in I_{np_n} is not greater than $1/n$. But the length of I_{np_n} converges to zero, as n increases, both for odd and even values of n. The upper and lower metric densities of S at z are accordingly equal to 1 and 0, respectively, so that the metric density of S does not exist at any point of Z.

Theorem 1 does not assert that the metric density of S exists at every point outside of Z. In fact, if Z is of measure 0 and not measurable Borel there is no set S such that Z is the totality of points at which the metric density of S does not exist. For, the set of points at which the metric density of S does not exist is, as may readily be shown, of type $G_{\delta\sigma}$.

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It is natural to ask whether for every set Z of measure 0 there is a measurable set S whose metric density exists but is different from 0 or 1 at every point of Z. The answer is no, as is shown by the following:

THEOREM 2. *The set of points for which the metric density of a measurable set S exists but is not equal to* 0 *or* 1 *is of measure* 0 *and of first category.*

PROOF. We shall suppose all sets are contained in the open interval $(0, 1)$. Let T be the set of points for which the metric density of S exists, U those points of T for which the metric density of S is 0 or 1, and $Z=T-U$. By the density theorem of Lebesgue, Z is of measure 0 and U is of measure 1. The metric density of S is a function of Baire class 1 on T. For, let x be in T and let $f_n(x)$, $n=1, 2, \cdots$, be the relative measure of S in the interval $(x-1/n, x+1/n)$. For every n, $f_n(x)$ is continuous and is, accordingly, continuous relative to T. Since the metric density of S exists at every point of T, $f(x) =\lim_{n\to\infty} f_n(x)$ exists, is equal to the metric density of S, and is a function of Baire class 1 on T relative to T. Its points of discontinuity must be a set of first category relative to T [2]. On the other hand, U, as a set of measure 1, is everywhere dense in T. Thus, every interval containing a point of Z also contains points of U; that is, points for which $f(x)$ is either 0 or 1. Since for every x in Z, $f(x)$ is different from 0 or 1, Z must be a subset of the set of points of discontinuity of $f(x)$. Z is, accordingly, of first category relative to T and, therefore, relative to $(0, 1)$.

As a partial converse to Theorem 2, we have the following theorem.

THEOREM 3. *If Z is a set of measure* 0 *which is of type F_σ, there is a measurable set S whose metric density exists and is different from* 0 *or* 1 *at every point of Z.*

PROOF. We may again assume that all sets are contained in the interval $(0, 1)$. $Z=\sum_{k=1}^{\infty} Z_k$ where the Z_ν are closed sets of measure 0. We associate four sequences of open sets with Z:

$$G_1 = (0, 1) \supset G_2 \supset \cdots \supset G_n \supset \cdots,$$
$$T_1 \supset T_2 \supset \cdots \supset T_n \supset \cdots,$$
$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots,$$
$$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots;$$

where the G_ν and T_ν are descending sequences of open sets such that, for every positive integer k,

$$G_k \supset Z - (Z_1 + Z_2 + \cdots + Z_{k-1}),$$
$$T_k = G_k - Z_k,$$
$$G_{k+1} \subset T_k,$$

and U_k, V_k are disjoint subsets of T_k to be defined presently:

T_k, as an open set, consists of a sequence $I_{k1}, I_{k2}, \cdots$ of disjoint open intervals. For every j, there is a positive integer N_j such that

$$\frac{2}{N_j + 1} \leqq | I_{kj} | < \frac{2}{N_j} \cdot$$

For every $n > N_j + 1$, let the points in I_{kj} whose distance from an end point of I_{kj} is less than $1/n$ and greater than $2^{-1}(1/(n+1)+1/n)$ belong to $U_k^{(n)}$ and those whose distance from an end point of I_{kj} is less than $2^{-1}(1/(n+1)+1/n)$ and greater than $1/(n+1)$ belong to $V_k^{(n)}$. Divide the open interval of points in I_{kj} whose distance from an end point of I_{kj} exceeds $1/(N_j+1)$ into two open intervals of equal length, letting one belong to $U_k^{N_j}$ and the other to $V_k^{N_j}$. Let

$$U_k = \sum_{\nu=1}^{\infty} U_k^{(\nu)}, \qquad V_k = \sum_{\nu=1}^{\infty} V_k^{(\nu)}.$$

Then, for every k,

$$T_k = U_k + V_k + D_k$$

where D_k is a finite or denumerable set consisting of the end points of the disjoint open intervals of U_k, V_k. Moreover, we restrict the choice of the sets G_k, as we may since Z is of measure 0, so that for every $l = 1, 2, \cdots, k-1$, and for every n, the relative measure of G_k in each of the disjoint open intervals of the sets $U_{k-l}^{(n)}$, $V_{k-l}^{(n)}$ does not exceed $1/n^l$. We now define S by the expression

$$(1) \quad S = (U_1 - G_2) + (U_2 - G_3) + \cdots + (U_n - G_{n+1}) + \cdots .$$

S is a measurable set. We show that the metric density of S exists and is equal to $1/2$ at every point of Z. For, let $z \in Z$. There is a smallest k such that $z \in Z_k \subset G_k$. Let J be an open interval containing z with $|J|$ less than the distance from z to the complement of G_k. Since G_k is open, $J \subset G_k$ and there is a positive integer n such that

$$(2) \qquad \frac{1}{n+1} \leqq |J| \leqq \frac{1}{n} \cdot$$

Since $z \in Z_k$, $z \notin T_k$. So, if an end point x of J is in one of the disjoint open intervals of U_k or V_k, the length of this interval is not greater than $2^{-1}(1/n-1/(n+1))=1/2n(n+1)$. For, the distance of x from an end point of the open interval of T_k in which it would be could not exceed $1/n$. The interval J accordingly consists of a set H composed of disjoint open intervals of the sets $U_k^{(\nu)}$, $V_k^{(\nu)}$, $\nu=n, n+1, \cdots$, the same finite number belonging to $U_k^{(\nu)}$ as to $V_k^{(\nu)}$ for every ν; the finite or denumerable set of end points of these intervals; a subset $J\cdot Z_k$ of Z_k; and two remaining open intervals, possibly empty, at the ends of J, the sum of whose lengths is not greater than $1/n(n+1)$. From (2), the relative measure of the sum of these two intervals in J is not greater than $1/n$. Since the relative measures of U_k and V_k in H are evidently each equal to 1/2, the relative measures of $H\cdot U_k$ and $H\cdot V_k$ in J satisfy

$$\begin{aligned}\frac{1}{2}-\frac{1}{n} &\leqq \frac{m(H\cdot U_k)}{|J|} \leqq \frac{1}{2},\\ \frac{1}{2}-\frac{1}{n} &\leqq \frac{m(H\cdot V_k)}{|J|} \leqq \frac{1}{2}.\end{aligned} \tag{3}$$

But the relative measure of G_{k+l}, $l=1, 2, \cdots$, is less than $1/\nu^l$ in each of the disjoint open intervals of $U_k^{(\nu)}$, $V_k^{(\nu)}$, and since H consists of disjoint open intervals of sets $U_k^{(\nu)}$, $V_k^{(\nu)}$, $\nu\leqq n$, the relative measure of G_{k+l}, $l=1, 2, \cdots$, is less than $1/n^l$ in $H\cdot U_k$ and in $H\cdot V_k$. Since $U_{k+l}\subset G_{k+l}$, the relative measure of U_{k+l}, $l=1, 2, \cdots$, is also less than $1/n^l$ in $H\cdot U_k$ and in $H\cdot V_k$. Thus, by (3), the relative measure in J of

$$U_k+U_{k+1}+\cdots$$

is not greater than $2^{-1}\sum_{k=0}^{\infty} 1/n^k<1/2+1/n$, and the relative measure in J of

$$U_k-(G_{k+1}+G_{k+2}+\cdots)$$

is not less than $(1/2-1/n)(1-\sum_{k=1}^{\infty} 1/n^k)\geqq 1/2-3/n$. But, by (1)

$$J\cdot\{U_k-(G_{k+1}+G_{k+2}+\cdots)\}\subset J\cdot S\subset J\cdot\{U_k+U_{k+1}+\cdots\}.$$

So, the relative measure of S in J satisfies

$$\frac{1}{2}-\frac{3}{n}\leqq\frac{m(S\cdot J)}{|J|}\leqq\frac{1}{2}+\frac{1}{n}, \tag{4}$$

and since $n\rightarrow\infty$ as $|J|\rightarrow 0$, (4) shows that the metric density of S at z exists and is equal to 1/2.

Bibliography

1. S. Saks, *Theory of the integral*, trans. by L. C. Young, Warsaw, 1937, p. 117.
2. F. Hausdorff, *Mengenlehre*, New York, 1944, p. 253.

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NOTE ON ZEROS OF THE HERMITE POLYNOMIALS AND WEIGHTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA

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The main purpose of this note is to point out that, in most cases, use of the mechanical quadrature formula to compute integrals of the form

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \tag{1}$$

is unsatisfactory. One can easily see why this would be the case. In a *closed* interval one can approximate very closely to a continuous function by a high degree polynomial. However, outside the interval, the high degree polynomial will in most cases diverge very markedly from the given function. In (1), this divergence is eventually brought under control by the exponential, but often not in time to prevent a sizeable error.

We illustrate with a numerical example using coefficients from [1],[1] and from the end of section 22 of [2]. It can be shown that

$$\int_{-\infty}^{\infty} \frac{e^{-y^2}}{1+y^2}\, dy = 2\pi^{1/2}e \left\{\frac{\pi^{1/2}}{2} - \int_0^1 e^{-y^2}dy\right\}$$
$$= 1.34329 \cdots$$

(see [2, Theorem 1–2 and formula (1–17)]). However the quadrature formulas for $n=2$, 10, and 16 give

$$n = 2, \qquad 1.18164,$$
$$n = 10, \qquad 1.34164,$$
$$n = 16, \qquad 1.34313.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.