ON LACUNARY DIRICHLET SERIES
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The following theorem is suggested by a result of S. Mandelbrojt [2, p. 101] concerning lacunary Fourier series.

**Theorem 1.** Let \( f(s) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k s} \), where \( 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \), \( \lim_{k \to \infty} \lambda_k = \infty \). We denote by \( \gamma_c \) the abscissa of convergence of this series and by \( \gamma_a \) the abscissa of analyticity. It is assumed that \( \gamma_c < \infty, \gamma_a > -\infty \). Let \( \nu < 1 \) be the exponent of convergence of the \( \lambda_k \)'s and suppose that

\[
|f(s_a + \sigma)| = O[\sigma^{-(1/\nu)^+}]
\]

(\( \sigma \to 0+ \))

where \( s_a = \gamma_a + i\tau_a \). Then if \( \mu > \nu/(1-\nu) \), \( f(s) \equiv 0 \).

We shall in what follows prove a more general theorem including Theorem 1 as a special case. The methods of the present paper are closely related to and in part derived from the work of L. Schwartz [4]. However no appeal is made to other than standard theorems of analysis.

Let \( m(0) = 0 \) and let \( m(\sigma) \) be an increasing function defined for \( 0 \leq \sigma \leq \alpha, \alpha > 0 \). A function \( f(s) \) which we may suppose analytic in the half-plane \( \sigma > \gamma_1 \) is said to have a zero of modular order \( m(\sigma) \) at \( s_1 = \gamma_1 + i\tau_1 \) if

\[
|f(s_1 + \sigma)| \leq m(\sigma)
\]

(\( 0 < \sigma \leq b \))

for some \( b > 0 \). Let us define \( \sigma \) as a function of \( \rho, \sigma = \eta(\rho) \), by the equation \( e^{-\sigma \rho} = m(\sigma) \). Since \( m(\sigma) \) decreases as \( \sigma \) decreases to 0 this definition is effective. It is \( \eta(\rho) \) which we shall use as the measure of the zero of \( f(s) \). As an example, if \( m(\sigma) = \exp[-(\sigma^{-\nu})] \) then \( \eta(\rho) = \rho^{1/(\nu+1)} \).

As the measure of the degree of lacunarity of our Dirichlet series we introduce

\[
\xi(\rho) = \sum_{k=1}^{\infty} \log \left( 1 + \frac{2\rho}{\lambda_k} \right).
\]

If the sequence \( \{\lambda_k\}_1^{\infty} \) has exponent of convergence \( \nu \), then by a standard theorem on integral functions [5, p. 251],

\[
\xi(\rho) = O(\rho^{\nu+1})
\]

(\( \rho \to +\infty \)).

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1 Numbers in brackets refer to the references cited at the end of the paper.

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We may now state our principal theorem.

**Theorem 2.** Let \(0 < \lambda_1 < \lambda_2 < \cdots, \sum_1^\infty \lambda_k^{-1} < \infty\), and let \(\xi(\rho)\) be defined as above. Let

\[
f(s) = \sum_{k=1}^\infty a_k e^{-\lambda_k s}
\]

have abscissa of convergence \(\gamma_c < \infty\) and abscissa of analyticity \(\gamma_a > -\infty\), and let \(\gamma_a + i\tau_a\) be a zero of \(f(s)\) of modular order \(m(\sigma)\). Then if \(\eta(\rho)\) is defined as above and if

\[
\liminf_{\rho \to \infty} \xi(\rho) - \rho \eta(\rho) = -\infty,
\]

we have \(f(s) = 0\).

In the case considered in Theorem 1 we have, as we have seen, \(\xi(\rho) = O(\rho^{\tau+\epsilon}), \eta(\rho) = \rho^{-1/(\sigma+1)}\). The assumption \(\mu > \nu/(1-\nu)\) shows that condition (1) is satisfied. Thus Theorem 2 does include Theorem 1.

It is clearly sufficient to prove our theorem for \(\gamma_a = \tau_a = 0\).

It is immediately verifiable that the function

\[
\frac{1}{(1 + z)} \prod_{k=1, k \neq n}^\infty \left( \frac{\lambda_k + \rho - z}{\lambda_k + \rho + z} \right) = k_n(z, \rho) \quad (\rho > 0)
\]
is analytic for \(x \geq 0\), and that

\[
\int_{-\infty}^\infty |k_n(x + iy, \rho)|^2 dy \leq \pi \quad (x \geq 0).
\]

By a simple extension of Plancherel's theorem, see [3, p. 8], we see that if

\[
\phi(n, \rho, \sigma) = \frac{1}{2\pi i} \int_{-\infty}^{i\sigma} \frac{k_n(z, \rho)}{k_n(\lambda_n + \rho, \rho)} e^{\zeta z} dz \quad (0 \leq \sigma < \infty),
\]

then

\[
\int_0^\infty \phi(n, \rho, \sigma) e^{-\sigma x} d\sigma = \frac{k_n(z, \rho)}{k_n(\lambda_n + \rho, \rho)} \quad (x > 0).
\]

Further

\[
\|\phi(n, \rho, \sigma)\|_2 = \left[ \int_0^\infty \phi(n, \rho, \sigma)^2 d\sigma \right]^{1/2} \leq 2^{1/2} k_n(\lambda_n + \rho, \rho)^{-1}.
\]

We assert that
\[(3) \limsup_{\rho \to \infty} \|\phi(n, \rho, \sigma)\|_{2e^{-\eta(\sigma)}} < \infty.\]

We have
\[
\|\phi(n, \rho, \sigma)\|_{2e^{-\eta(\sigma)}}
\leq \left[ \frac{1}{2^{1/2}} \prod_{k=1, k \neq n}^{\infty} \left( \frac{\lambda_n + \lambda_k + 2\rho}{\lambda_k - \lambda_n} \right) \right] \left[ 1 + \lambda_n + \rho \right] \left[ \prod_{k=1}^{\infty} \left( \frac{\lambda_k}{\lambda_k + 2\rho} \right) \right]
\leq \frac{1}{2^{1/2}} \left[ \prod_{k=1, k \neq n}^{\infty} \left( 1 + \frac{\lambda_n}{\lambda_k + 2\rho} \right) \right] \left[ 1 + \lambda_n + \rho \right] \left[ \prod_{k=1}^{\infty} \left( \frac{\lambda_k}{\lambda_n + 2\rho} \right) \right]
\cdot \left[ \prod_{k=1, k \neq n}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_k} \right) \right]^{-1}
\sim \frac{\lambda_n}{2^{1/2}} \left[ \prod_{k=1, k \neq n}^{\infty} \left( 1 - \frac{\lambda_n}{\lambda_k} \right) \right]^{-1} (\rho \to \infty),
\]
which proves our assertion.

We define
\[F(\rho, s) = e^{-\rho s} f(s).\]

We assert that if
\[\|F(\rho, \sigma)\|_{2e^{-\eta(\sigma)}} \leq \left[ \int_0^{\infty} |F(\rho, \sigma)|^2 d\sigma \right]^{1/2},\]
then
\[(4) \limsup_{\rho \to \infty} \|F(\rho, \sigma)\|_{2e^{-\eta(\sigma)}} < \infty.\]

Let \(M\) be a bound for \(f(\sigma)\) for \(0 < \sigma < \infty\). We have
\[
\|F(\rho, \sigma)\|_2 \leq \left[ \int_0^{\eta(\rho)} e^{-2\rho \sigma} |f(\sigma)|^2 d\sigma \right]^{1/2} + \left[ \int_{\eta(\rho)}^{\infty} e^{-2\rho \sigma} |f(\sigma)|^2 d\sigma \right]^{1/2}
\leq m(\eta(\rho)) + Me^{-\eta(\rho)}
\leq (M + 1)e^{-\eta(\rho)},
\]
which proves relation (4).

It follows from a well known theorem concerning restricted over-convergence of Dirichlet series, see [1, p. 141], that there exists an increasing sequence of integers \(l_k\) for which
\[
\lim_{k \to \infty} \sum_{j=0}^{l_k} a_j e^{-\lambda_j \sigma} = f(\sigma)
\]
uniformly for \( \epsilon \leq \sigma < \infty \) for any \( \epsilon > 0 \). Thus if \( \epsilon > 0 \),
\[
\int_0^\infty \phi(n, \rho, \sigma)F(\sigma + \epsilon, \rho)\,d\sigma = \lim_{k \to \infty} \int_0^\infty \phi(n, \rho, \sigma)e^{-\rho(\sigma+\epsilon)} \sum_{j=1}^{k} a_j e^{-\lambda_j(\sigma+\epsilon)}\,d\sigma.
\]
By equation (2)
\[
\int_0^\infty \phi(n, \rho, \sigma)e^{-\rho(\sigma+\epsilon)} a_j e^{-\lambda_j(\sigma+\epsilon)}\,d\sigma = \begin{cases} 0, & j \neq n, \\ e^{-\rho - \lambda_n a_n}, & j = n. \end{cases}
\]
Hence
\[
\int_0^\infty \phi(n, \rho, \sigma)F(\rho, \sigma + \epsilon)\,d\sigma = e^{-\rho - \lambda_n a_n}.
\]
By Schwarz's inequality
\[
|a_n| \leq e^{\rho + \lambda_n} \|\phi(n, \rho, \sigma)\|_2 \|F(\sigma + \epsilon, \rho)\|_2.
\]
If \( \epsilon \) is allowed to approach zero through positive values, we obtain
\[
|a_n| \leq \|\phi(n, \rho, \sigma)\|_2 \|F(\sigma, \rho)\|_2.
\]
Using equations (3) and (4) we see that
\[
\log |a_n| \leq \xi(\rho) - \eta(\rho) + C
\]
where \( C \) is a constant which depends on \( n \) but not upon \( \rho \). If we allow \( \rho \) to increase without limit, assumption (1) of Theorem 2 implies that
\[
\log |a_n| = -\infty,
\]
that is, \( a_n = 0 \).
Since this holds for each \( n = 1, 2, \cdots, f(s) \equiv 0 \) and our theorem is proved.

We shall now prove that if assumption (1) is replaced by the weaker assumption
\[
\lim_{\rho \to \infty} \xi(\rho) - \eta(\rho) = -\infty \]
where \( \xi < 2^{a/2} \), then Theorem 2 is false.
Let us take for the constants \( \lambda_k, 0 \leq a < b < 1 < 4 < 9 < \cdots \). As in Titchmarsh [5, p. 271] we find that
\[
\log \left[ \prod_1^\infty \left( 1 + \frac{x}{k^2} \right) \right] \sim \pi x^{1/2} \quad (x \to \infty).
\]
It follows that

\begin{equation}
\zeta(p) = \log \left(1 + \frac{2p}{a}\right) + \log \left(1 + \frac{2p}{b}\right) + \sum_{k=1}^{\infty} \log \left(1 + \frac{2p}{k^2}\right) \sim \pi(2p)^{1/2} \quad (p \to \infty).
\end{equation}

We define

\begin{equation}
f(s) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{zs}}{(1 + \frac{z}{a})(1 + \frac{z}{b}) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^2}\right)} \, dz.
\end{equation}

Deforming the line of integration of this integral to \( \Re z = -(n+1/2)^2 \) and allowing \( n \) to approach infinity we obtain for \( s \) real and positive an expansion of \( f(s) \) into a Dirichlet series with exponents \( a, b, 1, 4, 9, \cdots \) which then converges in the half-plane \( \Re s > 0 \). The details of this are left to the reader. Again deforming the line of integration, this time to \( \Re z = (\pi/2\sigma)^2 \), and using equation (5), we see that if \( \epsilon > 0 \) is fixed, then for \( \sigma > 0 \) sufficiently small we have

\[ |f(\sigma)| \leq \exp \left[ -\frac{1-2\epsilon}{4} \frac{\pi^2}{\sigma} \right] M(\sigma) \]

where

\[ M(\sigma) = \frac{1}{2\pi} \int_{\Re z = (\pi/2\sigma)^2} \left| \frac{ds}{(1 + \frac{z}{a})(1 + \frac{z}{b})} \right| = o(1) \quad (\sigma \to 0^+). \]

Thus we may associate with \( f(s) \) and the origin the modular order \( m(\sigma) = \exp \left[ -((1-2\epsilon)/4)(\pi^2/\sigma) \right] \). We immediately find that

\begin{equation}
\eta(\rho) = \frac{\pi}{2} (1 - 2\epsilon)^{1/2} \rho^{-1/2}.
\end{equation}

Comparing equations (6) and (8), we deduce that Theorem 2 is false if \( \epsilon < 2^{-1/2} \), which is what we wished to show.

\section*{References}


2. S. Mandelbrojt, \textit{Analytic functions and classes of infinitely differentiable func-
ON THE ABSOLUTE CONVERGENCE OF TRIGONOMETRICAL SERIES

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1. Theorem 1. Suppose that a trigonometrical series

$$\sum \rho_n \cos (nx - \alpha_n) \quad (\rho_n \geq 0, n = 1, 2, \ldots)$$

and its conjugate series

$$\sum \rho_n \sin (nx - \alpha_n)$$

are convergent absolutely at \(x = x_0\) and \(x = x_1\), respectively. If

$$x_1 - x_0 = \frac{p\pi}{q} \quad (p/q \text{ irreducible})$$

where \(p\) is an integer positive, negative, or zero, and \(q\) is an odd integer, then

$$\sum \rho_n < \infty.$$ 

We shall see in Theorem 2 that the above theorem is no longer true if \(p/q\) is replaced by \(p'/q'\) with an even \(q'\) and \(p'\neq0\), or by an irrational number.

Proof of Theorem 1. If we put \(x_1 - x_0 = \frac{p\pi}{q} = h\), then by Fatou's theorem\(^1\)

(1) \(\sum \rho_n \left| \cos (n(x_1 - sh) - \alpha_n) \right| < \infty\)

for every odd \(s\). Hence from the identity

$$\sin (nx_1 - \alpha_n) = \cos nsh \sin (n(x_1 - sh) - \alpha_n)$$

$$\quad + \sin nsh \cos (n(x_1 - sh) - \alpha_n)$$

we deduce immediately that

(2) \(\sum \rho_n \left| \cos nsh \sin (n(x_1 - sh)) - \alpha_n \right| < \infty.$$

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\(^1\) See, for example, A. Zygmund, *Trigonometrical series*, p. 134.