ON A SEMI-GROUP OF SUBSETS OF A LINEAR SPACE

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A family \( \mathcal{J} \) of subsets of a space is defined to be a semi-group of subsets provided that the intersection of any two subsets of \( \mathcal{J} \) is again a subset of \( \mathcal{J} \). In giving a characterization of a Banach space of continuous functions in terms of the geometry of the space, Clarkson\(^1\) considered a cone \( C \) which had the property that the family of all of its translates formed a semi-group. This note is concerned with an investigation of the structure of a subset \( S \) of a linear space \( \mathcal{X} \) such that the family of all translates of \( S \) form a semi-group, that is, for any points \( x, y \) of \( \mathcal{X} \) there exists a \( z \in \mathcal{X} \) such that \( (x+S) \cap (y+S) = z+S \). It will be shown that under certain rather weak restrictions, \( S \) must be a convex cone.

Let \( \mathcal{X} \) be an abstract linear space. The only topology which will be assumed for \( \mathcal{X} \) is that which arises from the topology of the straight lines of \( \mathcal{X} \), that is, if \( u_n = \alpha_n x + \beta_n y, x \neq y, \alpha_n + \beta_n = 1, \alpha_n \to \alpha, \beta_n \to \beta \), then \( u_n \to ax + by \). This will be called the linear topology of \( \mathcal{X} \). We shall assume that all straight lines of \( \mathcal{X} \) are complete in this topology. A point \( x \) is an extreme point of a set \( S \) if \( x \in S \) and there exists no line segment with end points in \( S \) which contains \( x \) in its interior. The theorem to be proved is the following:

THEOREM. Let \( S \) be a subset of \( \mathcal{X} \) which is closed in the linear topology and which has at least one extreme point. Let the family of all translates of \( S \) form a semi-group. Then \( S \) is a convex cone, that is, \( S \) is convex and there exists a point \( v \in S \) such that

\[
S = \bigcup_{\lambda \geq 0} \{ y = v + \lambda x, x \in S \}.
\]

The theorem is proved by four lemmas.

LEMMA 1. If \( x+S = S \), then \( x = 0 \).

PROOF. It is evident that any translate of \( S \) can be used initially instead of \( S \). Thus we may assume that \( S \) contains \( 0 \) without loss of generality. If \( x+S = S \), \( x \in S \). Also \( S = S-x \) and \( -x \in S \). If \( y \in S \), \( x+y \in S \) and \( -x+y \in S \). However, if \( x \neq -x \) this shows that \( y \) is the midpoint of a line segment joining two points of \( S \) and con-

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tradscts the hypothesis that $S$ has an extreme point.

**Lemma 2.** $S$ contains only one extreme point.

**Proof.** Suppose that $S$ contained at least two extreme points. It may be assumed that one is the point $\theta$. Let the second be $v \neq \theta$. Consider the translation $S + v$. Since $\theta$ and $v$ are both extreme points, $2v \in S$ and $S + v$ does not contain $S$. Hence $(S + v) \cap S = y + S$ where $y \neq \theta$ and $y$ is an extreme point of $y + S$. Also

$$\{v + [(S + v) \cap S]\} \cap S$$

$$= (v + y + S) \cap S \supset (v + y + S) \cap (y + S)$$

$$= y + (y + S) = 2y + S.$$

This shows that $v$, $y + v$, and $2y + v$ are in $S$ and $y + S$ contrary to the conclusion that $y$ is an extreme point of $y + S$.

**Lemma 3.** If $v$ is the extreme point of $S$, $S$ is a cone with vertex $v$.

**Proof.** It may be assumed that $v = \theta$. It must be shown that if $x \in S$, then $\lambda x \in S$ for all $\lambda \geq 0$. Since $\lambda x$ is a complete ray, if there exist values of $\lambda$ for which $\lambda x \notin S$, there must exist a smallest value $\lambda_0$ such that $\lambda_0 x \in S$ because of the linear closure of $S$. $(\lambda_0 x + S) \cap S = y + S$ where $y$ is the extreme point of $y + S$. However, since $\lambda_0 x \in (\lambda_0 x + S) \cap S$, $\lambda_0 x$ is an extreme point of $y + S$ and since there is only one extreme point of the set, $y = \lambda_0 x$ by Lemma 1. Then $\lambda_0 x + S \subseteq S$ and $2\lambda_0 x \in S$. By induction it can be seen that if $\lambda_0 x \in S$, then $n\lambda_0 x \in S$ for any positive integer $n$. Also there exists a $z \in S$ such that $(2^{-1}\lambda_0 x + S) \cap S = z + S$. This implies the existence of a point $u \in S$ such that $z = u + 2^{-1}\lambda_0 x$. Also, by the first part of the proof, $u + \lambda_0 x \in S$. Since $z$ is the extreme point of $z + S$, $u + (n/2)\lambda_0 x \in z + S \subseteq S$ for every integer $n$ by the same argument as was used in the first part of the proof. Then $S = S - z$ contains all points of the form $u + (n/2)\lambda_0 x - (u + 2^{-1}\lambda_0 x)$. In particular, $S$ contains the point $2^{-1}\lambda_0 x$, contrary to the assumption that $\lambda_0$ was the smallest nonzero value of $\lambda$ for which $\lambda x \in S$. This shows that if $x \in S$, $\lambda x \in S$ for all $\lambda \geq 0$ and $S$ is a cone with vertex $\theta$.

It is to be noted that if $\theta$ is the vertex, the above proof shows that if $x \in S$, then $\lambda x \in S$ for all non-negative $\lambda$ and $\lambda x + S \subseteq S$. Hence, if $y \in S$, $y + \lambda x \in S$ for all $\lambda \geq 0$. This shows that if $S$ contains a ray through any point, it contains a parallel ray through every point of the set.

**Lemma 4.** $S$ is convex.
Proof. It must be shown that if \( x \) and \( y \) are in \( S \), then \( \alpha x + \beta y \in S \) for \( \alpha, \beta \geq 0, \alpha + \beta = 1 \). If \( S \) has vertex \( \theta \) and if \( y = \lambda x \), this is true by Lemma 3. If \( y \neq \lambda x \), then the rays \( \lambda x, \nu y, \lambda \geq 0, \nu \geq 0 \), are distinct and are in \( S \). If for any point \( \alpha x + \beta y, \alpha, \beta > 0, \alpha + \beta = 1 \), there exists a \( \mu \neq 0 \) such that \( \mu(\alpha x + \beta y) \in S \), then \( \mu(\alpha x + \beta y) \in S \) for all \( \mu \geq 0 \) by Lemma 3. \( S \) contains the ray \( y + \lambda x \) where \( \lambda \geq 0 \). If the existence of positive numbers \( \mu \) and \( \lambda \) can be demonstrated such that \( \mu(\alpha x + \beta y) = y + \lambda x \), the lemma is proved. If this is so, \( (\mu x - \lambda)x + (\mu \beta - 1)y = \theta \) and if \( \mu = 1/\beta, \lambda = \alpha/\beta \), the rays \( \mu(\alpha x + \beta y) \) and \( y + \lambda x \) have a common point. Since all points of the form \( y + \lambda x \) are in \( S \), \( (1/\beta) (\alpha x + \beta y) \in S \) and hence \( \mu(\alpha x + \beta y) \in S \) for all \( \mu \geq 0 \), in particular for \( \mu = 1 \). Hence \( S \) is convex.

By making use of this theorem it is possible to weaken the hypotheses of Clarkson's theorem slightly and to state it in the following form.

Theorem. A necessary and sufficient condition that a Banach space be equivalent to the space of all continuous functions over a bicompact Hausdorff space \( H \) is that there exist a closed set \( S \) with an extreme point, containing \( \theta \) in its interior, such that all translates of \( S \) form a semi-group and such that the unit sphere in the space is the set \( S \cap (-S) \).

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