ON A CONJECTURE ON SIMPLE GROUPS

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The purpose of this paper is to rephrase a conjecture about simple groups into the language of linear algebra.

Let $G$ be a group of finite order $o(G)$. Then by $\Gamma_p$ we shall mean the group ring of $G$ over a field of characteristic $p$ (for instance the integers modulo $p$). We shall denote the radical of $\Gamma_p$ by $N_p$. If $p = 0$ or $p \mid o(G)$, then it is known that $N_p = (0)$; and if $p \mid o(G)$, $N_p \neq (0)$.

We now consider the following two assertions:

(A) If $G$ is a simple group of odd order, $o(G)$ is a prime.

(B) If $G$ is a group of odd order $o(G)$, then for some prime $p$, $p \mid o(G)$, we can find a $g \in G$, $g \neq 1$, such that $g - 1 \in N_p$.

The theorem which we propose to prove is:

**Theorem.** (A) is equivalent to (B).

1. (B) implies (A).

**Definition.** $U_p = \{ g \in G \mid g - 1 \in N_p \}$.

**Lemma 1.** $U_p$ is a normal $p$-subgroup of $G$.

**Proof.** (a) $U_p$ is a subgroup of $G$, for if $g_1, g_2 \in U_p$, then since $N_p$ is a left-ideal of $\Gamma_p$, $g_1(g_2 - 1) + (g_1 - 1) = g_1g_2 - 1 \in N_p$.

(b) $U_p$ is normal, for if $g - 1 \in N_p$, since $N_p$ is a two-sided ideal of $\Gamma_p$, $h(g - 1)h^{-1} = hgh^{-1} - 1 \in N_p$ for all $h \in G$.

(c) If $g - 1 \in N_p$, then for some integer $s$, $(g - 1)^s = 0 = g^s - 1$. So if $g \in U_p$, $g$ is of order $p^s$ for some $s$. So $U_p$ is a $p$-group.

**Corollary.** (B) implies (A).

**Proof.** By (B), $U_p \neq 1$ for some $p \mid o(G)$. Hence since $G$ is simple, and since $U_p$ is a normal subgroup of $G$, $U_p = G$. Thus $G$ is of order $p^s$, and $G$ being simple, $s = 1$. Hence (B) implies (A).

2. (A) implies (B).

**Lemma 2.** (A) implies that every group of odd order is solvable.

**Proof.** Let $G = G_1 \supset G_2 \supset \cdots \supset G_r = 1$ be a composition series for $G$. Since the $G_i/G_{i+1}$ are simple and of odd order, by (A) they must be of prime order; hence the lemma is proved.

Since a solvable group contains a normal $p$-subgroup [1, p. 25, 1949].
Theorem 20,1 we immediately obtain, using Lemma 2, the following lemma.

**Lemma 3.** (A) implies that if $G$ is of odd order, then it contains a normal $p$-subgroup.

For groups of certain orders the $N_p$ can be completely described. This is true for $p$-groups. If $G$ is of order $p^r$, then for every $g \in G$, $g - 1 \in N_p$ [2, p. 176, Theorem 1.2 or 3, p. 239]. For our case it is sufficient to use the weaker result:

**Lemma 4.** If $G$ is of order $p^r$, then for some $g \neq 1$ in $G$, $g - 1 \in N_p$.

**Proof.** Since $G$ is of order $p^r$, it has a nontrivial center $C$. Let $g \neq 1$ be in $C$. Then since $g - 1$ is in the center of $\Gamma_p$, and since $(g - 1)^{p^r} = g^{p^r} - 1 = 1 - 1 = 0$, $g - 1 \in N_p$.

Suppose that $S$ is the normal $p$-subgroup of Lemma 3. An element of the form $g - 1 \in \Gamma_p$ is in $N_p$ if and only if for every irreducible representation $\phi$ of $G$, $\phi(g) = 1$. Clifford's theorem [4, p. 534, Theorem 1] reduces the irreducible representations of $G$ to irreducible representations (or into ones fully reducible into irreducible components) of $S$. For the $g$ of Lemma 4 in $S$, for every irreducible representation $\phi$ of $S$, $\phi(g) = 1$. So by Clifford's theorem, for every irreducible representation $\phi$ of $G$, $\phi(g) = 1$. Thus $g - 1 \in N_p$. And so we have shown the following lemma.

**Lemma 5.** (A) implies (B).

**Bibliography**

1. A. Speiser, *Gruppen der endlichen Ordnung*.

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1 Numbers in brackets refer to the bibliography at the end of the paper.