NOTE ON PRESERVATION OF MEASURABILITY

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If m measures X (that is, m is a countably subadditive function on subsets of X to the non-negative reals), and if f is a function on subsets of Y to subsets of X, it is natural to inquire into the behavior of the composite function n, where $n(A)=m(f(A))$ for each $A\subset Y$. Rather obvious sufficient conditions are given below for n to measure Y, and for a subset A of Y to be n-measurable.

Several definitions are necessary. A subset C of X is called m-measurable-F if and only if $m(T\cap D)=m(T\cap D\cap C)+m(T\cap D\cap(X-C))$ whenever $T\subset X$ and $D\in F$. A subset B of Y is called a set of m-continuity-f if and only if $m(f(B)\cap f(Y-B))=0$. Suppose henceforth that (*) $m(f(A)\cap(X-\bigcup_{B\in\mathcal{K}}f(B)))=0$ whenever $A\subset Y$ and $\mathcal{K}$ is a countable covering of A by subsets of Y.

THEOREM 1. *If* (*), *then n measures Y.*

PROOF. If $A\subset Y$, and $\mathcal{K}$ is a countable covering of A by subsets of Y, then $n(A)=m(f(A))\leqq m(f(A)\cap\bigcup_{B\in\mathcal{K}}f(B))+m(f(A)\cap(X-\bigcup_{B\in\mathcal{K}}f(B)))=m(\bigcup_{B\in\mathcal{K}}(f(A)\cap f(B)))\leqq\sum_{B\in\mathcal{K}}m(f(A)\cap f(B))\leqq\sum_{B\in\mathcal{K}}m(f(B))=\sum_{B\in\mathcal{K}}n(B)$, so that n measures Y.

LEMMA 1. *If* (*), $B\subset Y$, *and* $C\subset Y$, *then* $n(B\cap C)\leqq m(f(B)\cap f(C))$.

PROOF. Since $B\cap C\subset B$ and $B\cap C\subset C$, $m(f(B\cap C)\cap(X-f(B)\cap f(C)))=m(f(B\cap C)\cap(X-f(B)))\cup(f(B\cap C)\cap(X-f(C)))\leqq m(f(B\cap C)\cap(X-f(B)))+m(f(B\cap C)\cap(X-f(C)))=0$. Hence $n(B\cap C)\leqq m(f(B\cap C)\cap(f(B)\cap f(C)))+m(f(B\cap C)\cap(X-f(B)\cap f(C)))\leqq m(f(B)\cap f(C))$.

THEOREM 2. *If* (*), *A is a set of m-continuity-f, and $f(A)$ is m-measurable-range f, then A is n-measurable.*

PROOF. Suppose $T\subset Y$. By Lemma 1, $n(T\cap A)\leqq m(f(T)\cap f(A))$ and $n(T\cap(Y-A))\leqq m(f(T)\cap f(Y-A))\leqq m(f(T)\cap f(Y-A)\cap f(A))+m(f(T)\cap f(Y-A)\cap(X-f(A)))\leqq m(f(A)\cap f(Y-A))+m(f(T)\cap(X-f(A)))$. By hypothesis, $m(f(A)\cap f(Y-A))=0$ and $m(f(T)\cap f(A))+m(f(T)\cap(X-f(A)))=m(f(T))=n(T)$. Therefore $n(T\cap A)+n(T\cap(Y-A))\leqq n(T)$.

REFERENCE

1. T. J. McMinn, *Restricted measurability*, Bull. Amer. Soc. vol. 54 (1948) p. 1106.

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