NOTE ON PRESERVATION OF MEASURABILITY

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If \( m \) measures \( X \) (that is, \( m \) is a countably subadditive function on subsets of \( X \) to the non-negative reals), and if \( f \) is a function on subsets of \( Y \) to subsets of \( X \), it is natural to inquire into the behavior of the composite function \( n \), where \( n(A) = m(f(A)) \) for each \( A \subseteq Y \). Rather obvious sufficient conditions are given below for \( n \) to measure \( Y \), and for a subset \( A \) of \( Y \) to be \( n \)-measurable.

Several definitions are necessary. A subset \( C \) of \( X \) is called \( m \)-measurable if and only if \( m(T \cap D) = m(T \cap D \cap C) + m(T \cap D \cap (X - C)) \) whenever \( T \subseteq X \) and \( D \subseteq F \). A subset \( B \) of \( Y \) is called a set of \( m \)-continuity-\( f \) if and only if \( m(f(B) \cap f(Y - B)) = 0 \). Suppose henceforth that \((*) \) \( m(f(A) \cap (X - \bigcup_{B \in \mathcal{C}} f(B))) = 0 \) whenever \( A \subseteq Y \) and \( \mathcal{C} \) is a countable covering of \( A \) by subsets of \( Y \).

**Theorem 1.** If \((*)\), then \( n \) measures \( Y \).

**Proof.** If \( A \subseteq Y \), and \( \mathcal{C} \) is a countable covering of \( A \) by subsets of \( Y \), then \( n(A) = m(f(A)) = m(f(A) \cap \bigcup_{B \in \mathcal{C}} f(B)) + m(f(A) \cap (X - \bigcup_{B \in \mathcal{C}} f(B))) = m(\bigcup_{B \in \mathcal{C}} f(A) \cap f(B)) \leq m(\bigcup_{B \in \mathcal{C}} m(f(A) \cap f(B)) = \sum_{B \in \mathcal{C}} m(f(A) \cap f(B)), \) so that \( n \) measures \( Y \).

**Lemma 1.** If \((*)\), \( B \subseteq Y \), and \( C \subseteq Y \), then \( n(B \cap C) \leq m(f(B) \cap f(C)) \).

**Proof.** Since \( B \cap C \subseteq B \) and \( B \cap C \subseteq C \), \( m(f(B \cap C) \cap (X - f(B) \cap f(C))) = m(f(B \cap C) \cap (X - f(B)) \cap (f(B \cap C) \cap (X - f(C))) \leq m(f(B \cap C) \cap (X - f(B))) + m(f(B \cap C) \cap (X - f(C))) = 0 \). Hence \( n(B \cap C) \leq m(f(B \cap C) \cap (f(B) \cap f(C)) + m(f(B \cap C) \cap (X - f(B) \cap f(C))) \leq m(f(B) \cap f(C)). \)

**Theorem 2.** If \((*)\), \( A \) is a set of \( m \)-continuity-\( f \), and \( f(A) \) is \( m \)-measurable-range \( f \), then \( A \) is \( n \)-measurable.

**Proof.** Suppose \( T \subseteq Y \). By Lemma 1, \( n(T \cap A) \leq m(f(T) \cap f(A)) \) and \( n(T \cap (Y - A)) \leq m(f(T) \cap f(Y - A)) \leq m(f(T) \cap f(Y - A) \cap f(A)) + m(f(T) \cap f(Y - A) \cap (X - f(A))) \leq m(f(A) \cap f(Y - A)) + m(f(T) \cap f(A)) + m(f(T) \cap (X - f(A))) = m(f(T)) = n(T) \). Therefore \( n(T \cap A) + n(T \cap (Y - A)) \leq n(T) \).

**Reference**


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