BANACH-HAUSDORFF LIMITS

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1. Introduction. Let m denote the Banach lattice of bounded real sequences $x=(x_0, x_1, x_2, \cdots)$ with $\|x\|=\sup_n|x_n|$ and $x\geqq y$ if $x_n\geqq y_n$ for all n. Let S denote the shift operator $S(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots)$. Banach[1] has established the existence of real-valued functionals $L(x)$ defined over m with the properties

(I) $L(ax+by)=aL(x)+bL(y)$ (a, b real);
(II) $L(1)=1$;
(III) $L(x)\geqq 0$ if $x\geqq 0$;
(IV) $L(Sx)=L(x)$.

The conditions (I)–(IV) imply that

$$L_*(x) = \liminf_n x_n \leqq L(x) \leqq \limsup_n x_n = L^*(x), \tag{1}$$

whence a *Banach limit* of a convergent sequence is the ordinary limit.

Designating the class of regular Hausdorff[2] transformations by $\mathfrak{H}$, we term $L(x)$ a *Banach-Hausdorff* (*B-H*) *functional* or *limit* if in addition to (I)–(IV) it satisfies

(V) $L(Hx) = L(x)$ ($H \in \mathfrak{H}$).

The corresponding regularity property of B-H limits then takes the form: If the bounded sequence x is summable to a by some regular Hausdorff method, then $L(x)=a$ for every B-H functional L.

The obvious problem is the existence of B-H limits. Our solution yields simultaneously the existence and the *domain of uniqueness*—that is, the set of x in m on which all B-H functionals coincide.[3]

2. Hausdorff lore. We recall the following properties of the class $\mathfrak{H}$ of regular Hausdorff transformations: $\mathfrak{H}$ is the convex Abelian semi-group of linear transformations of m into itself defined by the Toeplitz matrices (a_{mn}), where

$$\begin{aligned} a_{mn} &= C_{m,n}\int_0^1 u^n(1-u)^{m-n}d\alpha(u) \qquad && (n \leqq m) \\ &= 0 && (n > m); \end{aligned} \tag{2}$$

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[1] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[2] F. Hausdorff, Math. Zeit, vol. 9 (1921) pp. 74–109, 280–299.

[3] The domain of uniqueness of the ordinary Banach limits has been determined by G. G. Lorentz, Acta Math. vol. 80 (1948) pp. 167–190.

and the $\alpha(u)$ are real functions of bounded variation (BV) in the interval (0, 1) satisfying the end conditions: $\alpha(0)=\alpha(0+)=0$, $\alpha(1)=1$.

H in $\mathfrak{H}$ is termed *definite* or *completely regular*[4] (c.r.) if the generating function $\alpha(u)$ is non-decreasing or, equivalently, if all the matrix elements a_{mn} are non-negative. If H is c.r., $\|H\|=1$ and

$$L_*(x) \leqq L_*(Hx) \leqq L^*(Hx) \leqq L^*(x). \tag{3}$$

The standard decomposition of BV functions yields the canonical resolution of a regular H:

$$H = aH_1 - bH_2 \qquad (H_1, H_2 \text{ c.r.}; a - b = 1, b \geqq 0). \tag{4}$$

Henceforth we need consider only the convex semi-group $\mathfrak{H}_+ \subset \mathfrak{H}$ of c.r. Hausdorff transformations and may replace (V) with the equivalent condition

(V bis) $L(Hx) = L(x)$ $(H \in \mathfrak{H}_+)$.

The following conventions will materially shorten our work:

Definition 1. *C is the class of convergent sequences; C_0 the subclass of sequences converging to* 0 *(null sequences). $x \sim y$ if and only if $x-y \in C_0$.*

Clearly $\sim$ is a congruence relation under addition, scalar multiplication, and the operations H.

Definition 2. $P_+(x) = \inf_{H \in \mathfrak{H}_+} L^*(Hx)$; $P_-(x) = -P_+(-x) = \sup_{H \in \mathfrak{H}_+} L_*(Hx)$.

Our basic results may now be formulated as

Theorem 1. (a) *A n.a.s.c. that the linear functional $L(x)$ on m be a B-H functional is that*

$$P_-(x) \leqq L(x) \leqq P_+(x) \qquad (x \in m). \tag{5}$$

(b) *B-H functionals exist.*

(c) *A n.a.s.c. that all B-H functionals coincide at x is that $P_-(x) = P_+(x)$.*

That B-H functionals must satisfy the inequality (5) is an immediate consequence of (1) and (V bis). The condition $P_-(x) = P_+(x)$ is then obviously sufficient to ensure coincidence at x.

3. **Convexity.** To establish the remaining assertions of Theorem 1, we derive the convexity and other necessary properties of $P_+(x)$, and then adapt the Hahn-Banach[1] extension procedure. The decisive role of the Hölder-Cesàro transformation $H_0x=y$, where $y_n=(n+1)^{-1}$

[4] Cf. G. H. Hardy, *Divergent series*, Oxford, 1949.

$\cdot(x_0+\cdots+x_n)$ corresponds to $\alpha(u)=u$, is due to the following elementary property:

LEMMA 1. $H_0Sx\sim H_0x\sim SH_0x$.

The requisite properties of $P_\pm(x)$ are

LEMMA 2. (a) $P_+(x+y)\leqq P_+(x)+P_+(y)$;
(b) $P_+(ax)=aP_+(x)$ $(a\geqq 0)$;
(c) $P_-(x)\leqq P_+(x)$;
(d) $|P_\pm(x)|\leqq\|x\|$; $P_\pm(x)\geqq 0$ *if* $x\geqq 0$;
(e) *If* $x\in C$, $P_\pm(x)=\lim_n x_n$ *and* $P_\pm(x+y)=\lim_n x_n+P_\pm(y)$; *in particular*, $P_\pm(x)=P_\pm(y)$ *if* $x\sim y$;
(f) $P_\pm(Hx)=P_\pm(x)$ $(H\in\mathfrak{H}_+)$;
(g) $P_\pm(Sx)=P_\pm(x)$;
(h) $P_\pm(Sx-x)=0$;
(i) $P_\pm(Hx-x)=0$ $(H\in\mathfrak{H}_+)$.

PROOF. (a) Given $\epsilon>0$, there exist H_1, H_2 in $\mathfrak{H}_+$ such that $L^*(H_1x)\leqq P_+(x)+\epsilon$, $L^*(H_2y)\leqq P_+(y)+\epsilon$. Set $H_3=H_2H_1=H_1H_2$. Then $L^*(H_3x)=L^*(H_2H_1x)\leqq L^*(H_1x)\leqq P_+(x)+\epsilon$. Similarly $L^*(H_3y)\leqq P_+(y)+\epsilon$. Hence $P_+(x+y)\leqq L^*(H_3(x+y))\leqq L^*(H_3x)+L^*(H_3y)\leqq P_+(x)+P_+(y)+2\epsilon$. Since $\epsilon>0$ was arbitrary, $P_+(x+y)\leqq P_+(x)+P_+(y)$.

(b), (d), and (e) are obvious, and (c) follows on setting $y=-x$ in (a). (f) is an easy consequence of the definition of $P_\pm(x)$ by way of (3) and the semi-group property of $\mathfrak{H}_+$. Lemma 1 and (e), (f) now yield

(g) $P_\pm(Sx)=P_\pm(H_0Sx)=P_\pm(H_0x)=P_\pm(x)$;
(h) $P_\pm(Sx-x)=P_\pm(H_0Sx-H_0x)=0$.

To establish (i) note the ergodic[5] property $\|H_nH-H_n\|\leqq 2(n+1)^{-1}$ of the operators $H_n=(n+1)^{-1}\sum_0^n H^i\in\mathfrak{H}_+$. Thus $|P_\pm(Hx-x)|=|P_\pm((H_nH-H_n)x)|\leqq\|(H_nH-H_n)x\|\leqq 2(n+1)^{-1}\|x\|$. n being an arbitrary integer, $P_\pm(Hx-x)=0$.

We complete the proof of Theorem 1. That any linear L satisfying (5) is a B-H functional is now immediate: (II) follows from the identity $P_\pm(1)=1$; (III) from Lemma 2(d); the identities $L(Sx-x)=0$ (IV) and $L(Hx-x)=0$ (V bis) from Lemma 2, (h) and (i) respectively.

Application of the Hahn-Banach extension theorem to the convex function $P_+(x)$ yields linear functionals L such that $L(x)\leqq P_+(x)$. But then $L(x)=-L(-x)\geqq -P_+(-x)=P_-(x)$, whence the L satisfy (5).

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[5] Cf. W. F. Eberlein, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 43–47; Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240.

Finally, the necessity of the condition $P_-(x) = P_+(x)$ for coincidence of all B-H functionals at x is implicit in the extension procedure: If $P_-(y) < P_+(y)$ for some y in m, the value of $L(y)$ can be chosen arbitrarily in the interval $P_-(y) \leqq L(y) \leqq P_+(y)$. For let $L(x) = \lim_n x_n$ $(x \in C)$, and recall that we may continue L into the subspace $x + ay$ $(x \in C$, a real) by choosing $L(y)$ arbitrarily in the interval

$$\sup_{x \in C} \{P_-(x + y) - L(x)\} \leqq L(y) \leqq \inf_{x \in C} \{P_+(x + y) - L(x)\}.$$

But the second inequality reduces to the first (Lemma 2(e)), and our assertion follows on continuing L into the whole space m.

Further developments will appear in a second note.

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