DEFORMATION THEORY OF SUBSPACES IN A RIEMANN SPACE

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Summary. This paper deals with the infinitesimal transformation (2,1) of a family \((V_m)\) of subspaces \(V_m\) in a Riemann space. In §§1–4 the transformation (2,1) is applied on internal objects of a \(V_m\), while in §§5 and 6 the characteristic mixed tensors \(K_{\alpha r\cdots a_1}\) (cf. the equation \((1,2; 4)\)) of a \(V_m\) are investigated with respect to (2,1). Finally, some applications of the theory are given in §§7 and 8. In particular the statement expressed by the equation \((8,10)b\) is the generalization of the well known Levi-Civita result for \(m = 1\), while the statement expressed by the equation \((8,10)c\) generalizes the classical result (for \(m = 1, n = 2\)) by Jacobi.

1. Preliminary.

(1) Let \(V_n\) be a \(n\)-dimensional Riemann space \((n \geq 2)\), referred to the real coordinate system \(\xi^i,\) \(g_{\lambda\mu} = g_{\mu\lambda}\) its metric tensor, \(\Gamma^\gamma_{\lambda\mu}\) the corresponding Christoffel symbols and \(\nabla_\mu\) the covariant derivative operator in \(V_n\) with respect to \(\Gamma^\gamma_{\lambda\mu}\). The curvature tensor of \(\Gamma^\gamma_{\lambda\mu}\) will be denoted by

\[
(1,1;1) \quad R^\gamma_{\rho\mu\lambda} = \partial_\rho \Gamma^\gamma_{\mu\lambda} - \partial_\lambda \Gamma^\gamma_{\mu\rho} + \Gamma^\rho_{\alpha\mu} \Gamma^\alpha_{\gamma\lambda} - \Gamma^\rho_{\alpha\lambda} \Gamma^\alpha_{\gamma\mu}
\]

\(\partial_\rho = \frac{\partial}{\partial \xi^\rho}\).

(2) Let \(V_m\) be a \(m\)-dimensional subspace of \(V_n\) \((1 \leq m < n)\) referred to the real parameters\(^1\) \(\eta^a\) and let

\[
(1,2;1) \quad \xi^a = \xi^a(\eta^1, \cdots, \eta^m)
\]

be its parametric equations.\(^2\) Throughout this paper we consider only the case where the matrix of the mixed tensor

\[
(1,2;2) \quad T^a_\alpha = \partial_\alpha \xi^a
\]

is of rank \(m\). Hence the metric tensor\(^3\)

\[
g_{ab} = g_{ba} = g_{\lambda\mu} T^\lambda_{ab}
\]

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\(^1\) Greek (Latin) indices run from 1 to \(n\) (from 1 to \(m\)).

\(^2\) Any function considered in this paper is understood to be a real and continuous one, as well as its derivatives which appear in the discussion.

\(^3\) Here and later on we put \(T^\lambda_{a_1\cdots a_v} = T^{a_1\cdots a_v}_\lambda, v = 2, 3, \cdots \cdots\).
is of rank $m$ and consequently the corresponding Christoffel symbols $\Gamma_{ab}^c = \Gamma_{ba}^c$ exist. The curvature tensor of $V_m$ will be denoted by

$$P_{abcd} = \partial_a \Gamma_{bd}^c - \partial_d \Gamma_{bc}^e + \Gamma_{bc}^e \Gamma_{ed}^h - \Gamma_{bd}^e \Gamma_{ec}^h.$$ Considering the connections $\Gamma_{ab}^c$ and $\Gamma_{bc}^e$ we may introduce three different kinds of covariant derivatives, namely

1. For a tensor field defined over $V_n$,

$$\nabla_{\mu} V^\ldots_{\ldots} = \partial_{\mu} V^\ldots_{\ldots} + \Gamma_{\alpha\mu}^\alpha V^\ldots_{\ldots} + \ldots - \Gamma_{\lambda\mu}^\lambda V^\ldots_{\ldots} - \ldots,$$

2. For a tensor field defined over $V_m$ and

$$\nabla_a V^\ldots_{\ldots} = \partial_a V^\ldots_{\ldots} + \Gamma_{\alpha a}^\alpha V^\ldots_{\ldots} + \ldots - \Gamma_{\lambda a}^\lambda V^\ldots_{\ldots} - \ldots,$$

3. For a mixed tensor field defined over $V_m$. The embedding theory of a $V_m$ in a $V_n$ may be described by means of the mixed tensors $T_a^r$ and

$$K_{a_1\ldots a_r} = D_{a_r}\ldots D_{a_2}T_{a_1}^r, \quad r = 2, \ldots, N,$$

where $N$ is the number of osculating spaces of $V_m$. Incidentally $K_{a b}^r = K_{b a}^r$ lies with its index $\nu$ in the first normal space of $V_m$.

(3) A family $(V_m)$ of a set of $V_m$'s is defined as a set of $V_m$'s such that through any generic point of the $V_n$ there is only one element $V_m$ of $(V_m)$. Each element $V_m$ of the family $(V_m)$ is referred to the parameter system $\eta^a$ and we keep the parameter transformation $\eta^a \leftrightarrow \eta^{a'}$ independent of the coordinate system $\xi^r$. Hence

$$\partial_{\mu} \frac{\partial \eta^{a'}}{\partial \eta^a} = 0, \quad \partial_{\mu} \frac{\partial \eta^a}{\partial \eta^{a'}} = 0.$$ On the other hand, an object $\Omega$ defined over $(V_m)$ is defined also over $V_n$ and consequently may be thought of as expressed either by means of the $\eta^a$ or by means of the $\xi^r$. This is in particular true for $T_a^r, e_{ab}, \Gamma_{ab}^r$. Whenever we apply on such an object the operator $\partial_{\mu}$ it is understood that we consider it expressed by means of the $\xi^r$. Taking in account (1,3;1) we see that $\partial_{\mu} e_{ab}, \partial_{\mu} \Gamma_{ab}^r, \nabla_a T_a^r$ are

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5 $\nabla_a T_a^r = \partial_a T_a^r + \Gamma_{\mu a}^r T_{a \mu}$ according to (1,2;3a), while (cf. (1,2;3b)) $\nabla_b T_a^r = \partial_b T_a^r + \Gamma_{a b}^{a'} T_a^{a'} + \Gamma_{\lambda b} T_{a b}^{a'}$. 
tensors (resp. vectors) not only with respect to the coördinate transformation but also with respect to the parameter transformation. Let $P$ be a generic point of $V_n$ and let $V_m$ be the element of $(V_m)$ containing $P$. If a tensor field $T$ defined over $V_n$ has the property that $T(P)$ is in the tangential space (or in some of the normal spaces) of the $V_m$ already mentioned, then we say that the field $T$ is tangent (normal) to $(V_m)$.

2. Fundamental definitions. Let $V^r = V^r(\xi)$ be the components of a contravariant vector field given over $V_n$ and let

\[
\xi^r = \xi^r + \epsilon V^r \quad (\epsilon \to 0 \text{ is a constant})
\]

be the infinitesimal transformation of a Lie group with the generator $V^r \partial_r$. If a generic point $P(\xi)$ describes a subspace $V_m$ of the family $(V_m)$ then the point $*P = P(\xi)$ describes a subspace $*V_m$. We shall assume throughout this paper that the set of all $*V_m$ constructed in this way is a family, which we denote by $(*V_m)$. In the following definition $X = X(P)$ denotes a tensor field (a set of tensor fields) defined either over $(V_m)$ or over $V_n$ and $F[X] = F(\xi)$ is a function of $X$.

**Definition (2.1).** $X'$ is the value of $X$ in $*P$, $X' = X(*P)$, $\circ X$ is the tensor at $*P$ which one gets by parallel displacement in $V_n$ of $X(P)$ from $P$ to $*P$. If $X$ is defined over $(V_m)$, then $X$ is the tensor field (a set of tensor fields) defined over $(*V_m)$ in the same way as $X$ is defined over $(V_m)$. If $X$ is defined over $V_n$, then $X = X'$. Furthermore

\[
F = F[*X], \quad F^\circ = F[\circ X], \quad F' = F(\xi)
\]

in the first case and

\[
F = F' = F(\xi), \quad F^\circ = F[\circ X]
\]

in the second case.

We use these symbols in the following definition of the symbols $\tau$, $\omega$, and $\Delta$:

**Definition (2.2).** The operators $\tau$, $\omega$, $\Delta$ are defined by the following equations

\[
\begin{align*}
\text{(a)} & \quad \tau F = \lim_{\epsilon \to 0} \frac{*F - F^\circ}{\epsilon} \quad \text{(translation of $F$)}, \\
\text{(b)} & \quad \omega F = \lim_{\epsilon \to 0} \frac{F' - \circ F}{\epsilon} \quad \text{(variation of $F$)}, \\
\text{(c)} & \quad \Delta F = (\tau - \omega) F \quad \text{(deformation of $F$)}.
\end{align*}
\]

In this paper we shall investigate the application of these operators
on the objects of \((V_m)\).

3. The fields \(T_a^\alpha\) and \(g_{ab}\).

**Theorem (3,1).** We have

\[
\begin{align*}
(3.1) \quad & (a) \quad \tau T_a^\alpha = \nabla_a V^\alpha, \\
& (b) \quad \omega T_a^\alpha = V^\alpha \nabla_a T_a^\alpha, \\
& (c) \quad \Delta T_a^\alpha = L_a^\alpha,
\end{align*}
\]

where

\[
(3.2) \quad L_a^\alpha = \nabla_a V^\alpha - V^\alpha \nabla_a T_a^\alpha = \partial_a V^\alpha - V^\alpha \partial_a T_a^\alpha.
\]

**Proof.** Let \(X = F[X] = T_a^\alpha\). Then

\[
\begin{align*}
*F = \frac{\partial^* \xi}{\partial \eta^a} & = T_a^\alpha + \epsilon \partial_a V^\alpha, \\
\circ F & = T_a^\alpha(\xi) = T_a^\alpha + \epsilon V^\alpha \partial_a T_a^\alpha + \cdots, \quad 6
\end{align*}
\]

and consequently

\[
(3.3) \quad (a) \quad *F - \circ F = T_a^\alpha + \epsilon(\partial_a V^\alpha + \Gamma^\alpha_{\lambda \mu} T_a^\lambda) - T_a^\alpha + \cdots ,
\]

\[
(b) \quad F' - \circ F = T_a^\alpha + \epsilon V^\alpha(\partial_a T_a^\alpha + \Gamma^\alpha_{\lambda \mu} T_a^\lambda) - T_a^\alpha + \cdots.
\]

From (3.3)ab we have (3.1)ab and these equations together with (3.2) lead to (3.1)c.

**Theorem (3,2).** We have

\[
\begin{align*}
(3.4) \quad & (a) \quad \tau g_{ab} = 2 W_{ab}, \\
& (b) \quad \omega g_{ab} = G_{ab}, \\
& (c) \quad \Delta g_{ab} = 2 g_{\lambda \mu} L^\lambda_{(a} T^\mu_{b)}
\end{align*}
\]

where

\[
(3.5) \quad W_{ab} = T_{(a}^\mu \nabla_b V^\mu, \quad G_{ab} = V^\alpha \partial_\mu g_{ab}.
\]

**Proof.** Put \(X = g_{\lambda \mu}\). Then

\[
\begin{align*}
(3.6) \quad & (a) \quad *X \equiv X' = *g_{\lambda \mu} = g_{\lambda \mu} + \epsilon V^\mu \partial_\mu g_{\lambda \mu} + \cdots , \\
& (b) \quad \circ X = \circ g_{\lambda \mu} = g_{\lambda \mu} + \epsilon V^\mu (\Gamma^\mu_{\lambda \rho} g_{\rho \mu} + \Gamma^\mu_{\rho \mu} g_{\lambda \rho}) + \cdots = *g_{\lambda \mu}.
\end{align*}
\]

On the other hand, if we denote by \(X\) the set of tensor fields \(g_{\lambda \mu}, \quad T_a^\alpha\) we have for \(F[X] = g_{\lambda \mu} T_a^\lambda T_b^\mu = g_{ab}\) by virtue of (3,6)

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4 Throughout this paper the dots denote the coefficient of \(\epsilon\).

7 Substantially equivalent formulas to (3,1)a and (3,4)a may be found also in the papers in the bibliography, which deal with the theory of deformation of subspaces from a different point of view.
\[(a) \quad \star F = \star g_{ab} \star T_{a}^{\lambda} \star T_{b}^{\mu} = (g_{\lambda\mu} + \epsilon V^{\omega} \partial_{u}g_{\lambda\mu} + \cdots) \\
\quad \cdot \left( T_{a}^{\lambda} + \epsilon \partial_{a}V^{\lambda} + \cdots \right) \left( T_{b}^{\mu} + \epsilon \partial_{b}V^{\mu} + \cdots \right) \\
= g_{ab} + \epsilon \left[ V^{\omega} T_{a}^{\lambda} T_{b}^{\mu} \partial_{u}g_{\lambda\mu} + 2 g_{\lambda\mu} T_{(a)}^{\nu} \partial_{(a)}V^{\nu} \right] + \cdots \\
= g_{ab} + 2 \epsilon T_{(a)}^{\lambda} T_{b}^{\mu} \nabla_{\lambda} V_{\mu} + \cdots,
\]
\[(3,7) \quad (b) \quad F' = \star g_{ab} = \star g_{ab} T_{a}^{\lambda} T_{b}^{\mu} = (g_{\lambda\mu} + \epsilon V^{\omega} \partial_{u}g_{\lambda\mu} + \cdots) \\
\quad \cdot \left( T_{a}^{\lambda} + \epsilon V^{\omega} \partial_{a}T_{a}^{\lambda} + \cdots \right) \left( T_{b}^{\mu} + \epsilon V^{\omega} \partial_{b}T_{b}^{\mu} + \cdots \right) \\
= g_{ab} + \epsilon \left[ V^{\omega} T_{a}^{\lambda} T_{b}^{\mu} \partial_{u}g_{\lambda\mu} + 2 g_{\lambda\mu} V^{\omega} (\partial_{a}T_{(a)}^{\nu}) T_{b}^{\mu} \right] + \cdots \\
= g_{ab} + \epsilon V^{\omega} \partial_{u}g_{ab} + \cdots,
\]
\[(c) \quad \circ F = \circ g_{ab} = \circ g_{ab} \circ T_{a}^{\lambda} T_{b}^{\mu} = [g_{\lambda\mu} + \epsilon V^{\omega} (\Gamma_{\lambda\mu}^{\nu} g_{\nu\rho} + \Gamma_{\mu\nu}^{\rho} g_{\nu\lambda}) + \cdots] \\
\quad \cdot \left( T_{a}^{\lambda} - \epsilon V^{\omega} \Gamma_{\lambda a}^{\nu} T_{\nu}^{\rho} + \cdots \right) \left( T_{b}^{\mu} - \epsilon V^{\omega} \Gamma_{\beta b}^{\rho} T_{\rho}^{\mu} + \cdots \right) \\
= g_{ab} + \epsilon \left[ T_{a}^{\lambda} T_{b}^{\mu} V^{\omega} (\Gamma_{\lambda\mu}^{\nu} g_{\nu\rho} + \Gamma_{\mu\nu}^{\rho} g_{\nu\lambda}) - 2 g_{\lambda\mu} T_{(a)}^{\nu} V^{\omega} T_{b}^{\mu} \right] \\
\quad + \cdots = g_{ab} + \cdots.
\]

Hence
\[(d) \quad \star F - \circ F = \star g_{ab} - \circ g_{ab} = 2 \epsilon T_{(a)}^{\nu} T_{b}^{\mu} \nabla_{\lambda} V_{\mu} + \cdots,
\]
\[(3,7) \quad (e) \quad F' - \circ F = \star g_{ab} - \circ g_{ab} = \epsilon V^{\omega} \partial_{u}g_{ab} + \cdots.
\]

These equations lead to \((3,4)ab\) and from these we have \((3,4)c\).

**Definition (3,1).** The transformation \((2,1)\) is called a rigid translation if \(\Gamma_{ab} = 0\).

**Theorem (3,3a).** Let \(V^{\nu}\) be a tangential vector field to \((V_{m})\). A necessary and sufficient condition that \((2,1)\) be a rigid translation is the (Killing) equation
\[(3,8) \quad D(aV_{b}) = 0.
\]

**Proof.** In our case we have \(V_{a} = V_{\lambda} T_{a}^{\lambda}, \ V_{\lambda} = V_{\lambda} T_{\lambda}^{\mu}\) and consequently (by virtue of \(\Gamma_{\lambda a}^{\nu} = T_{\lambda}^{\nu} \nabla_{\lambda} T_{a}^{\lambda}\))
\[(3,9) \quad D_{b} V_{a} = \partial_{b} V_{\lambda} T_{a}^{\lambda} - \Gamma_{\lambda a}^{\nu} T_{b}^{\lambda} V_{\nu} = (\partial_{b} V_{\lambda}) T_{a}^{\lambda} + V_{\lambda} \partial_{b} T_{a}^{\lambda} - (\nabla_{b} T_{a}^{\lambda}) V_{\lambda}
\]

The proof follows from \((3,9), (3,5),\) and \((3,4)a\).

**Theorem (3,3b).** Let \(V^{\nu}\) be a normal vector field to \((V_{m})\). A necessary and sufficient condition that \((2,1)\) be a rigid translation is: \(V^{\nu}\) is in the \(x\)th normal space of \(V_{m}, \ x = 2, 3, \cdots\).
Proof. In our case we have $V_a T^a_\lambda = 0$ and consequently

$$(3,10) \quad T^\lambda_a T^\mu_b \nabla_V = - T^\lambda_a V^\mu_b \nabla_T = - (D_a T^\mu_b) V^\mu = - K^\mu_a V^\mu.$$ 

The tensor $K^\mu_a$ lies with $\mu$ in the first normal space of $(V_m)$. By virtue of this fact, we get the theorem (3,3b) from (3,4)a and (3,5).

Remark. The theorem (3,3b) is the generalization of the well known fact that the "deformation" of the arc of a curve in a three-dimensional Euclidean space along its binormal is equal to zero.

4. The field $\Gamma^e_{ab}$.

Theorem (4,1). We have

$$\begin{align*}
(4,1) & \quad \tau \Gamma^e_{ab} = W^e_{ab} = W_{ab} g^{dc}, \quad (b) \quad \omega \Gamma^e_{ab} = G^e_{ab}, \\
& \quad (c) \quad \Delta \Gamma^e_{ab} = Q^e_{ab}
\end{align*}$$

where

$$\begin{align*}
(4,2) & \quad W_{ab} = D_a W^d_b + D_b W^d_a - D^d W_{ba}, \\
& \quad G^e_{ab} = V^a \partial_w \Gamma^e_{ab}, \\
& \quad Q^e_{ab} = W_{ab} g^{dc} - G^e_{ab}.
\end{align*}$$

Proof. First of all we have

$$(4.3) \quad \partial_a W^d_b + \partial_b W^d_a = D_{ab} + 2 \Gamma^e_{ab} W^d_e.$$ 

Furthermore, if $X$ is the set of tensors $g^{ad}, g^{cd}$, we get for $F[X] = \Gamma^e_{ab}$ by virtue of (3,7)d and (4,3)

$$\begin{align*}
*F & = * \Gamma^e_{ab} = 1/2 * g^{ed} \left( \partial_a * g^{bd} + \partial_b * g^{ad} - \partial_d * g_{ab} \right) \\
& = 1/2 \left[ \partial_c \partial^e d - 2 \epsilon W^e c \right] \left[ \partial_d (\partial^d g_{bd} + 2 \epsilon W_b d + \cdots) \\
& \quad + \partial_b (\partial^d g_{ad} + 2 \epsilon W_a d + \cdots) \\
& \quad - \partial_d (\partial^e d g_{ab} + 2 \epsilon W_a b + \cdots) \right] \\
& = \partial^e d \left( W_{cd} + 2 \epsilon W d e \right) - 2 \epsilon W c d \Gamma^e_{ba} g^{cd} + \cdots.
\end{align*}$$

On the other hand, we see from (3,7)c that

$$\begin{align*}
(4.5) & \quad \partial^e d \Gamma^e_{ba} = \Gamma^e_{ba} + \cdots.
\end{align*}$$

From (4.5), (4.4) we have (4.1)a. The equation (4.1)b follows from

$$(4.6) \quad F' = \Gamma^e_{ab} = \Gamma^e_{ab} + \epsilon V^a \partial_w \Gamma^e_{ab} + \cdots$$

and (4.5). The equation (4.1)c is obvious.
Later on we shall need also the coefficients

\[(4,7) \quad \Gamma^e_{ab} = \frac{1}{2} g^{cd} \left( \partial_d g^e_{bc} + \partial_c g^e_{bd} - \partial_b g^e_{cd} \right)\]

which are related to \( \Gamma^e_{ab} \) by a relation introduced in the following lemma:

**Lemma (4.1).** We have

\[(4,8) \quad \Gamma^e_{ab} - \Gamma^e_{ab} = \frac{\epsilon}{2} g^{cd} \left[ L_a \partial_d g_{eb} + L_b \partial_d g_{ea} - L_d \partial_d g_{ab} \right] + \cdots \]

**Proof.** We get from \((4,7)\) and \((3,7)b\)

\[(4,9) \quad \begin{align*}
\Gamma^e_{ab} &= \frac{1}{2} \left[ \epsilon V^d \partial_d g_{eb} + \epsilon V^e \partial_e g_{bd} + \cdots \right] \\
&\quad + \frac{\epsilon}{2} \left( \partial_d V^e \partial_e g_{bd} + \partial_e V^d \partial_d g_{bd} + \cdots \right) \\
&\quad - \partial_d \left( \epsilon V^e \partial_e g_{ab} + \epsilon V^d \partial_d g_{ab} + \cdots \right) \\
&= \Gamma^e_{ab} + \frac{\epsilon}{2} \left[ 2 \lambda \frac{\partial \lambda}{\partial g_{bd}} V^e \partial_d g_{eb} + g^{cd} \left( \partial_d V^e \partial_e g_{bd} \right. \right. \\
&\quad \left. \left. + \partial_d V^e \partial_e g_{bd} + \partial_e V^d \partial_d g_{bd} - \partial_d \partial_e g_{ab} \right) \right] + \cdots .
\end{align*}\]

On the other hand

\[(4,10) \quad \begin{align*}
\Gamma^e_{ab} &= \Gamma^e_{ab} + \epsilon V^d \partial_d \Gamma^e_{ab} + \cdots = \Gamma^f_{ab} + \epsilon \left[ V^d \left( \partial_d g^e_{bd} \right) \Gamma^h_{ab} \right. \\
&\quad + 1/2 g^{cd} \left( \partial_d \partial_d g_{bd} + \partial_e \partial_e g_{bd} - \partial_d \partial_e g_{ab} \right) \right] + \cdots ,
\end{align*}\]

\[(4,11) \quad \begin{align*}
\left( \partial_a \partial_d \partial_e - \partial_d \partial_a \partial_e \right) g_{bd} &= \left( \Gamma^\lambda_{ab} \partial_e \lambda - \left( \partial_{\lambda} T^\lambda_{ab} \right) \partial_e \lambda - T^\lambda_{ab} \partial_e \lambda \right) \partial_d g_{bd} \\
&= - \left( \partial_{\rho} T^\rho_{ab} \right) \left( \partial_{\lambda} g_{bd} \right).
\end{align*}\]

Comparing \((4,9), (4,10), \) and \((4,11), \) we obtain

\[(4,12) \quad \begin{align*}
\Gamma^e_{ab} - \Gamma^e_{ab} &= \frac{\epsilon}{2} g^{cd} \left[ \left( \partial_d V^e - V^e \partial_d \lambda \right) \partial_d g_{bd} + \left( \partial_e V^d - V^d \partial_e \lambda \right) \partial_d g_{bd} \\
&\quad - \left( \partial_d V^e - V^e \partial_d \lambda \right) \partial_d g_{ab} + \cdots \right]
\end{align*}\]

and this equation gives us at once \((4,8).\)

5. The field \( K^\rho_{ab}.\)

* Throughout §§5 and 6 we shall sometimes make use of a special coordinate (parameter) system and shall symbolize the statement: "in the special coordinate (parameter) system \( \Gamma^\rho_{ab} = 0 (\Gamma^\rho_{ab} = 0) \) at \( P \)" by using the symbol \( \equiv \). \( \Gamma^\rho_{ab} = 0 (\Gamma^\rho_{ab} = 0). \)

An equation with \( \equiv \) has to be understood as valid in a special coordinate (parameter) system at \( P \) where both last equations hold. On the other hand, the sign \( = \) does not impose any restriction whatsoever on the choice of the coordinate (parameter) system.
Theorem (5.1). The following equation holds:

\[ \tau K'_{ba} = D_b D_a V' - V'^\nu T'^\lambda_{ba} R'_{\nu \mu} - W'_{ba} T'_{c}. \]

Proof. First of all we have according to (2,2)b for \( X = g_{\mu \nu}, \ g^{\lambda \nu}, \ F[X] = F(\xi) = \Gamma'_{\mu \nu} \)

\[ \ast \Gamma'_{\mu \nu} = \Gamma'_{\nu \mu} = \epsilon V'^\mu \partial_\mu \Gamma'_{\nu \mu} + \cdots \]

and moreover by virtue of (4,4) and (4,5)

\[ \ast \Gamma'_{\mu \nu} = \epsilon W'_{\nu \mu} + \cdots. \]

Hence

\[ \ast K'_{ba} = \ast D_b \ast T'_{a} = \partial_b (T'_{a} + \epsilon \partial_a V' + \cdots) \]

\[ + \epsilon V' \partial_\mu \Gamma'_{\lambda \nu} (T'_{b} + \epsilon \partial_b V' + \cdots) (T'_{a} + \epsilon \partial_a V' + \cdots) \]

\[ - \epsilon W'_{\nu \mu} (T'_{c} + \epsilon \partial_c V' + \cdots) \]

and consequently

\[ \ast K'_{ba} = \partial_b T'_{a} + \epsilon \left\{ \partial_\nu \partial_a V' + \epsilon \partial_\nu \Gamma'_{\lambda \nu} T'^\lambda_{ba} - W'_{ba} T'_{c} \right\} + \cdots \]

\[ = D_b T'_{a} + \epsilon \left\{ D_b D_a V' - V'^\nu \partial_\mu \Gamma'_{\nu \lambda} - \partial_\mu \partial_a V' \right\} T'^\lambda_{ba} - W'_{ba} T'_{c} \]

\[ + \cdots \]

\[ = D_b T'_{a} + \epsilon \left\{ D_b D_a V' - V'^\nu T'^\lambda_{ba} R'_{\nu \mu} - W'_{ba} T'_{c} \right\} + \cdots. \]

On the other hand

\[ \circ K'_{ba} = D_b T'_{a} + \cdots = K'_{ba} + \cdots. \]

The equations (5,4)b and (5,4)c lead to (5,2) with the \( \doteq \) sign. This equation, being a tensor equation valid for a special coordinate (and parameter) system at a generic point \( P \), holds for all coördinate (and parameter) systems at a generic point.

The second theorem concerns the variation of \( K'_{ba} \):

Theorem (5.2). The following equation holds:

\[ \omega K'_{ba} = V'^\nu [D_b \nabla_a T'_{a} - T'^\lambda_{ba} R'_{\nu \mu} + (\nabla_\mu T'_{a})(\nabla_a T'_{b})] - G_{ba} T'_{c}. \]

Proof. First of all we have
\[ \partial \omega \partial b T^r_a = \partial \omega \partial a T^r_a = \partial \omega T^b_b \partial a T^r_a = T^b_b \partial a T^r_a = \langle \partial \omega T^b_b \rangle (\partial a T^r_a) \]
\[ \pm (\nabla_T b_a) - \nabla_T a c \]

Consequently

\[ K_{ba} = (\partial \omega b a) = K_{ba} + \varepsilon \epsilon V^a b a + \cdots \]

\[ \pm K_{ba} = \varepsilon \epsilon V^a [\partial a b \partial T_a^r + (\partial a b \partial T_a^r) T^b_b - (\partial a b \partial T_a^r)] + \cdots \]

\[ = K_{ba} + \varepsilon \epsilon V^a \{ \partial a b\partial T_a^r + T^b_b (\partial a b \partial T_a^r - \partial a b \partial T_a^r) \}
+ (\nabla_T b_a) \nabla_T a c \}

The proof follows from (5,6) and (5,4)c.

**Theorem (5,3).** Put

\[ (5,7)a \quad L_{ba} = D_b L_a + L_b \nabla_T a - Q_{ba T a} \]

Then

\[ (5,7)b \quad L_{ba} = D_b L_a - V^a T^b_b \partial \mu \lambda - Q_{ba T c} \]

and

\[ (5,8) \quad \Delta K_{ba} = L_{ba}. \]

**Proof.** We have from (5,2) and (5,5)

\[ (5,9) \quad \Delta K_{ba} = (\tau - \omega) K_{ba} = D_b D_a V^a - V^a D_b \partial \nabla_T a \]

Since

\[ D_b L_a = D_b [D_a V^a - V^a \nabla_T a] \]

\[ = D_b D_a V^a - (D_b V^a) \nabla_T a - V^a D_b \partial \nabla_T a, \]

the equation (5,9) reduces to

\[ \Delta K_{ba} = D_b L_a + (\nabla_T a c) (\nabla b V^a - V^a \nabla_T b) - Q_{ba T c} \]

which proves (5,7)a and (5,8). The equation (5,7)b follows from (5,7)a and

\[ (5,10) \quad (T^b_b \nabla_T a c - V^a \nabla_T a c) T_a = [L_b \nabla_T a c + V^a T^b_b (\nabla_T a c - \nabla_T a c)] T_a \]

\[ = L_b \nabla_T a c - V^a T^b_b \partial \mu \lambda \]
6. The field $K_{a_{r}, \ldots, a_{1}}$. The last theorem may easily be generalized:

**Theorem (6,1).** Put

\[
L_{a_{r}, \ldots, a_{1}}' = D_{a_{r}}L_{a_{r-1}, \ldots, a_{1}}' + L_{a_{r}}^{\nu}K_{a_{r-1}, \ldots, a_{1}}' - \sum_{1}^{r-1} Q_{a_{s}, a_{r}}^{'*}K_{a_{r-1}, \ldots, a_{s+1}, a_{s-1}, \ldots, a_{1}}'.
\]

(6,1)

Then

\[
L_{a_{r}, \ldots, a_{1}}' = D_{a_{r}}L_{a_{r-1}, \ldots, a_{1}}' + (T_{\alpha}^{\nu}V_{\omega}^{\nu} - V_{\omega}^{\nu}T_{\alpha}^{\nu})K_{a_{r-1}, \ldots, a_{1}}' - \sum_{1}^{r-1} Q_{a_{s}, a_{r}}^{'*}K_{a_{r-1}, \ldots, a_{s+1}, a_{s-1}, \ldots, a_{1}}'.
\]

(6,1b)

and

\[
\Delta K_{a_{r}, \ldots, a_{1}}' = L_{a_{r}, \ldots, a_{1}}' \quad (r = 2, 3, \ldots).
\]

**Proof.** (The theorem has already been proved for $r = 2$.) Let us assume that we have proved it for some $r \geq 2$. Then we have from (6,2)

\[
K_{a_{r}, \ldots, a_{1}}' = K_{a_{r}, \ldots, a_{1}}' + eL_{a_{r}, \ldots, a_{1}}' + \ldots
\]

(6,3)

On the other hand

\[
D_{a_{r+1}}K_{a_{r}, \ldots, a_{1}}' = \partial_{a_{r+1}}K_{a_{r}, \ldots, a_{1}}' + e \left[ V_{\omega}(\partial_{a_{r}}K_{a_{r}, \ldots, a_{1}}')T_{a_{r+1}}^{\lambda}K_{a_{r}, \ldots, a_{1}}' - \sum_{1}^{r} W_{a_{s}, a_{r+1}}^{'*}K_{a_{r}, \ldots, a_{s+1}, a_{s-1}, \ldots, a_{1}}' \right] + \ldots
\]

(6,4)

and consequently by virtue of (6,3)

\[
K_{a_{r+1}, \ldots, a_{1}}' = \partial_{a_{r+1}}K_{a_{r}, \ldots, a_{1}}' + e \left\{ \partial_{a_{r+1}}L_{a_{r}, \ldots, a_{1}}' + (\partial_{a_{r+1}}V_{\omega}^{\nu})\partial_{a_{r}}K_{a_{r}, \ldots, a_{1}}' + V_{\omega}^{\nu}\partial_{a_{r+1}}\partial_{a_{r}}K_{a_{r}, \ldots, a_{1}}' + V_{\omega}(\partial_{a_{r}}K_{a_{r}, \ldots, a_{1}}')T_{a_{r+1}}^{\lambda}K_{a_{r}, \ldots, a_{1}}' - \sum_{1}^{r} W_{a_{s}, a_{r+1}}^{'*}K_{a_{r}, \ldots, a_{s+1}, a_{s-1}, \ldots, a_{1}}' \right\} + \ldots
\]
\[ D_{a_{r+1}}K^r_a, \ldots a_1 + \varepsilon \left\{ D_{a_{r+1}}L^r_a, \ldots a_1 \right\} \]
\[ + (\nabla_{a_{r+1}} V^\rho) \nabla_\rho K^r_a, \ldots a_1 \]
\[ - \sum_1^r W^e_{a_{a_{r+1}}} K^r_a, \ldots a_{a+1}a_{a-1} \ldots a_1 \]
\[ + V^\mu \left[ (\partial_{a_{a_{r+1}}} T^{a_{r+1}}_{a_{r+1}}) K^\lambda_a, \ldots a_1 + \partial_{a_{r+1}} \partial_{a_{a_{r+1}}} K^r_a, \ldots a_1 \right] \}
\[ + \ldots . \]

The tensor \( K^r_{a_{r+1}} \ldots a_1 \) satisfies the following relation
\[ K^r_{a_{r+1}} \ldots a_1 = K^r_{a_{r+1}} \ldots a_1 + \varepsilon V^\mu \partial_{a_{a_{r+1}}} K^r_a, \ldots a_1 + \ldots \]
\[ = K^r_{a_{r+1}} \ldots a_1 + \varepsilon V^\mu \partial_{a_{a_{r+1}}} D_{a_{r+1}} K^r_a, \ldots a_1 + \ldots \]
\[ = K^r_{a_{r+1}} \ldots a_1 + \varepsilon V^\mu \left[ \partial_{a_{a_{r+1}}} D_{a_{r+1}} K^r_a, \ldots a_1 \right] \]

Subtracting (6.5) from (6.4) we have
\[ K^r_{a_{r+1}} \ldots a_1 - K^r_{a_{r+1}} \ldots a_1 = \varepsilon \left\{ D_{a_{r+1}}L^r_a, \ldots a_1 \right\} \]
\[ + (\nabla_{a_{r+1}} V^\rho) \nabla_\rho K^r_a, \ldots a_1 \]
\[ - \sum_1^r Q^e_{a_{a_{r+1}}} K^r_a, \ldots a_{a+1}a_{a-1} \ldots a_1 \]
\[ + V^\mu \left[ (\partial_{a_{a_{r+1}}} \partial_{a_{a_{r+1}}} - \partial_{a_{a_{r+1}}} \partial_{a_{a_{r+1}}}) K^r_a, \ldots a_1 \right] + \ldots . \]

Since
\[ (\partial_{a_{r+1}} \partial_{a_{a_{r+1}}}) - \partial_{a_{a_{r+1}}} \partial_{a_{a_{r+1}}}) K^r_a, \ldots a_1 \]
\[ = [T''_{a_{r+1}} \partial_{a_{a_{r+1}}} - (\partial_{a_{a_{r+1}}} T''_{a_{a_{r+1}}}) \partial_{a_{a_{r+1}}} - T''_{a_{r+1}} \partial_{a_{a_{r+1}}} \partial_{a_{a_{r+1}}}] K^r_a, \ldots a_1 \]
\[ = - (\nabla_{a_{a_{r+1}}} T''_{a_{r+1}}) \nabla_\mu K^r_a, \ldots a_1 , \]

the equation (6.6) reduces to
In this tensor equation we may obviously suppress the dot in \( \dot{=} \) and consequently if we put
\[
L'^{a_{r+1}} \cdots a_1 = D_{a_{r+1}} L'^{a_r} \cdots a_1 + L'^{a_{r+1}} \nabla_\rho K'^{a_r} \cdots a_1 - \sum_{1}^{r} Q_{a_{r+1}} K'^{a_r} \cdots a_{r+1} a_{r+1} \cdots a_1
\]
we have from (6,7)
\[
\Delta K'^{a_{r+1}} \cdots a_r = L'^{a_{r+1}} \cdots a_1.
\]
Hence if (6,2) holds for some \( r \) then it holds also for \( r+1 \). But (6,2) holds for \( r=2 \) and consequently it holds for \( r=2, 3, \ldots \). The remaining part of the theorem (equation (6,1)b) may be proved by means of an equation similar to (5,10)

7. The differential equation \( L'_a = 0 \).

**Theorem (7,1).** The system of differential equations

\[
L'_a = 0 \quad \text{or} \quad \nabla_a V'_a = V_\lambda^\mu T'_a
\]
is completely integrable and its general solution involves \( n \) arbitrary constants.

**Proof.** We have first
\[
\nabla_{[\lambda} V_{\mu]} = \nabla_{[\lambda} T'_a \nabla_\mu V'_a = (\nabla_{[\lambda} T'_a) \nabla_\mu V'_a + T_{[\lambda}^\rho T'_a \nabla_\rho \nabla_\mu V'_a
\]
(7,2)
\[
= - \frac{1}{2} R'^{\lambda}_{\rho \mu \lambda} T'_a^\rho
\]
and by virtue of (7,1)
\[
\nabla_{[\lambda} V_{\mu]} T'_a = (\nabla_{[\lambda} V_\rho) \nabla_{[\mu} T'_a + V'^\rho T'_a \nabla_{[\mu} \nabla_\rho T'_a
\]
(7,3)
\[
= V'^\rho [(\nabla_{[\mu} T'_a)(\nabla_{[\mu} T'_a) + T_{[\mu} T'_a \nabla_{[\mu} T'_a)].
\]
On the other hand
\[
0 = V'^\rho \nabla_{[\mu} T'_a V_{[\mu} T'_a = V'^\rho (\nabla_{[\mu} T'_a V_{[\mu} T'_a) + V'^\rho T_{[\mu} \nabla_{[\mu} T'_a
\]
and consequently (7,3) reduces to

\[(7.4) \quad \nabla_{\{b} V^\lambda \triangledown_{\lambda]} T^r\}_a = V^\mu T^r_{\{b}(\nabla_{\mu} V_{\omega}| - \nabla_{\mu} V_{\lambda}|)T^\lambda\}_a = V^\mu T^r_{\{ba]\} R_{\omega\mu\lambda}.\]

The integrability conditions of (7,1) are obtained by substituting in

\[\nabla_{\{b} V^\lambda \triangledown_{\lambda]} T^r\}_a \]

from (7,2) and (7,4). In doing so we obtain

\[\frac{1}{2} V^\lambda T^r_{ba}( - R^{\omega\mu\lambda} - R_{\lambda\omega\mu} + R_{\lambda\omega\mu})\]

\[= - \frac{1}{2} V^\lambda T^r_{ba}(R^{\omega\mu\lambda} + R_{\lambda\omega\mu} + R_{\lambda\omega\mu}) = 0\]

since

\[R^{\omega\mu\lambda} + R_{\lambda\omega\mu} + R_{\lambda\omega\mu} = 0.\]

Hence the integrability conditions of (7,1) are identically satisfied and consequently (7,1) is completely integrable and its general solution involves \(n\) arbitrary constants.

**Theorem (7,2).** Let \(V_1, \ldots, V_n\) be \(n\) linear independent solutions of (7,1). Then\(^{10}\)

\[(7.5) \quad W^r = V_{[a|} \partial_{\rho} V_{|} \quad \text{(a, e = 1, \ldots, n; a \neq e)}\]

is a vector solution of (7,1).

**Proof.** \(W^r\) is obviously a vector. Hence to prove our theorem we have to show that it satisfies the equation

\[(7.6) \quad \partial_a W^r = W^\lambda \partial_\lambda T^r.\]

From (7,1) we have for

\[(7.7) \quad \partial_a W^r = \partial_a V_{[a|} \partial_{\rho} V_{|} = V_{[a|} (\partial_\rho T_{\lambda}) \partial_\lambda V_{|} + V_{[a|} \partial_\rho \partial_\lambda V_{|}.\]

On the other hand \((\partial_\rho V_{[a|} - \partial_\rho \partial_a) V_{|} = (\partial_\rho \partial_\lambda) V_{|}\) and consequently (7,7) reduces to

\[(7.8) \quad \partial_a W^r = (\partial_\rho T_{\lambda}) (V_{[a|} \partial_\rho V_{|} - V_{[a|} \partial_\rho \partial_\lambda V_{|}) + V_{[a|} \partial_\rho \partial_\lambda V_{|} = V_{[a|} \partial_\rho \partial_\lambda T_{a} = W^\lambda T_{a}.\]

\(^{10}\) It is assumed that in the equations (7,5)-(7,8) the brackets [ ] do not affect the Greek and Latin indices.
 Remark. The theorem (7,2) shows that the set of all generators $V^d\partial_r$ of infinitesimal transformations (2,1) is closed under the operations of taking linear combination and forming commutators. Hence these generators form a Lie algebra.

8. Special cases.

Definition (8,1). We say that a transformation (2,1) reproduces the family $(V_m)$ if

$$(8.1) \quad \Delta K^r_1 \ldots a_1 = 0 \quad (r = 1, 2, \ldots ; K^r_a \equiv T^r_a).$$

Theorem (8,1). A necessary and sufficient condition that $(V_m)$ be reproduced under (2,1) is: $V^r$ is a solution of

$$(8.2) \quad L^r_a = 0.$$

Proof. Let $(V_m)$ be reproduced by (2,1). Then from the first of (8,1) we have (8,2) by virtue of (3,1)c. Conversely, if (8,2) is satisfied, the following equations hold (cf. (3,4)c and (4,8))

$$(8.3) \begin{align*}
(a) & \quad \ast g_{ab} = g'_{ab} + \cdots , \\
(b) & \quad \ast \Gamma^e_{ab} = \Gamma^e_{ab} + \cdots .
\end{align*}$$

Consequently we have from (4,1)c and (8,3)b

$$(8.4) \quad \ast \Gamma^e_{ab} = \Gamma^e_{ab} + \cdots .$$

On the other hand we get by virtue of (8,3)a and (8,3)b

$$(8.3) \begin{align*}
(c) & \quad \ast g_{ab} = g'_{ab} + Q^e_{ab} + \cdots , \\
(d) & \quad \ast \Gamma^e_{ab} = \Gamma^e_{ab} + \cdots .
\end{align*}$$

Comparing (8,3)(c) and (d) we have

$$(8.4) \quad Q^e_{ab} = 0.$$

The equations (8,2), (8,4) together with (6,2) and (6,1)a lead to (8,1) for $r = 2, 3, \ldots$.

In the next theorem we shall consider a family $(V_m)$ of totally geodesic subspaces, that is, a family for which

$$(8.5) \quad K^r_{ab} = 0$$

and we prove first the following lemma.

Lemma (8,1). If (8,5) holds, then $W_{a\dot{b}}$ (cf. the equation (4,2)) satisfies

$11$ This equation imposes some conditions on the structure of the large space $V_n$ (cf. the equation (8,12)).
the following equation

\[(8.6)\]

\[W_{bad} = W_{ab} = P_{\text{bad}}^e W_e + D_a D_b W_d\]

where \(P_{\text{bad}}^e\) is the curvature tensor of \(V_m\) and \(W_e = T^e_c V_c\).

**Proof.** Let \(W^* (N^*)\) be the tangential (normal) component of \(V^*\)

\[(8.7)\]

\[V^* = W^* + N^*, \quad W^* = V^\lambda T^\lambda_c T^c_c, \quad N^*_c T^c_c = 0.\]

From (3.5) we have

\[(8.8)\]

\[W_{ab} = T^\lambda_{(a} T^a_{b)} \nabla_\lambda V^* \quad = T^\lambda_{(a b)} \nabla_\lambda W^* + T^\lambda_{(a b)} \nabla_\lambda N^*_\mu\]

\[= D_{(a} W_{b)} - N^*_\mu \nabla_{(a} T^\mu_{b)}.\]

If (8.5) holds then

\[0 = N^*_\mu D_a T^a_{b} = N^*_\mu (\nabla_a T^a_{b} - \Gamma^a_{b a} T^c_c) = N^*_\mu \nabla_a T^a_{b}\]

and (8.8)a reduces to

\[(8.8)b\]

\[W_{ab} = D_{(a} W_{b)}.\]

Consequently, we get from (4.2)a

\[(8.9)a\]

\[W_{ab} = D_{[a} D_{d]} W_{b} + D_{[b} D_{d]} W_{a} + D_{[b} D_{a]} W_{d} + D_a D_b W_d.\]

Because for any vector \(P_b\) whatsoever

\[D_{[a} D_{d]} P_{b} + D_{[a} D_{b]} P_{a} + D_{[b} D_{a]} P_{d} = 0\]

\[D_{[b} D_{d]} P_{a} = 1/2 P_{\text{bad}}^e P_{e}\]

where \(P_{\text{bad}}^e\) is the curvature tensor of \(V_m\), (8.9)a reduces to (8.6).

**Definition (8.2).** If \(A = 0\) is an analytic expression for a property of \((V_m)\), then we say that (2.1) preserves this property if \(\Delta A = 0\).

**Theorem (8.2).** Let \((V_m)\) be a family of totally geodesic subspaces. A necessary and sufficient condition that this property be preserved by (2.1) is: The vector \(V^*\) is a solution of any one of the following equations

\[(a) \quad L^*_{ab} = 0,\]

\[(8.10)\]

\[(b) \quad D_{b} D_{a} V^* - V^* T^\lambda_{ba} R^\lambda_{\text{mpl}} - W^*_{ba} T^c_c = 0,\]

\[(c) \quad D_{b} D_{a} N^* - N^* T^\lambda_{ba} R^\lambda_{\text{mpl}} = 0,\]

where in the last equation

\[N^* = V^*(\delta^*_{\lambda} - T^c_c T^c\lambda).\]
A sufficient but not necessary condition is:

The vector $V'$ is a solution of (8,2). If (8,10) but not (8,2) is satisfied by $V'$, then $(V_m)$ is not reproduced. If (8,2) holds, then (8,10) is satisfied and $(V_m)$ is reproduced.

Proof. According to our assumption the transformation (2,1) carries a family $(V_m)$ in a family $(V'_m)$. Hence in order to prove our theorem we have only to look for conditions that the equation $K'^{ab} = 0$ be satisfied. The necessity and sufficiency of (8,10) follows at once from (5,8). If (8,5) is satisfied, then we have

$$(8,11)a \quad K'^{ab} = K^{ab} + \epsilon V^\lambda \partial_\lambda K'^{ab} + \cdots = 0 + \cdots$$

and consequently

$$(8,11)b \quad \Delta K'^{ab} = (\tau - \omega)K'^{ab} = \tau K'^{ab}.$$  

Hence, we have from (8,11)b, (5,8) and (5,2) that in our case a necessary and sufficient condition for $\Delta K'^{ab} = 0$ is that $V'$ be a solution of (8,10)b and this equation is (in our case) equivalent to (8,10)a. On the other hand, the integrability conditions of (8,5) are

$$(8,12) \quad e^{cda}T_d = T_c^{\rho\lambda}R^{\rho\lambda}_{\omega\mu\lambda}$$

and consequently $T^{\rho\lambda}_{\omega\mu\lambda}R^{\rho\lambda}_{\omega\mu\lambda}$ is in $(V_m)$. If we decompose $V'$ according to (8,7), then we have from (8,12)

$$(8,13)a \quad - V^\omega T^{\rho\lambda}_{\omega\mu\lambda} = - W^\epsilon T^{\rho\lambda}_{\epsilon\epsilon}R^{\rho\lambda}_{\omega\mu\lambda} - N^\omega T^{\rho\lambda}_{\omega\mu\lambda}$$

$$= - W^\epsilon e^{cda}T_d - N^\omega T^{\rho\lambda}_{\omega\mu\lambda}.$$  

On the other hand, if (8,5) holds then

$$(8,13)b \quad D_bD_aV' = D_bD_aW^{\epsilon}T^{\epsilon}_{\epsilon} + D_bD_aN' = (D_bD_aW^{\epsilon})T^{\epsilon}_{\epsilon} + D_bD_aN'.$$

Furthermore, if (8,5) holds we have according to (8,6)

$$(8,13)c \quad - W^\epsilon T^{\epsilon}_{\epsilon} = - P^{\epsilon}_{bda}W^{\epsilon}T^{\epsilon}_{\epsilon} - (D_bD_aW^{\epsilon})T^{\epsilon}_{\epsilon} = (P^{\epsilon}_{bda}W^{\epsilon} - D_bD_aW^{\epsilon})T^{\epsilon}_{\epsilon}.$$  

Comparing (8,10)b with (8,13) we have

$$(8,14)a \quad D_bD_aV' - V'^{\rho\lambda}_{\omega\mu\lambda}R^{\rho\lambda}_{\omega\mu\lambda} - W^\epsilon T_{\epsilon} = T_{\epsilon}[D_bD_aW^d - W^\epsilon P^{d}_{\epsilon} + P^{d}_{\epsilon}W^e - D_bD_aW^d]$$

$$+ D_bD_aN' - N'^{\rho\lambda}_{\omega\mu\lambda}R^{\rho\lambda}_{\omega\mu\lambda} = 0.$$  

We use here the well known identities $P_{bac} = P_{abc} = -P_{abc}$. 


Since
\[(D_bD_a - D_aD_b) W^d = - P^d_{bac} W^e\]
\[= P^d_{abc} W^e = - (P^d_{bca} + P^d_{cab}) W^e\]
we obtain from (8,14)a
\[(8,14)\ b \ D_b D_a V^\star - V^\star T_{bc} R_{\mu \lambda}^\star - W^c_{ba} T^e_c = D_b D_a N^\star - N^\star T_{bc} R_{\mu \lambda}^\star\]
and this equation proves the statement about (8,10)c. The last part of the theorem follows easily from Theorem (8,1).

Remark I. The statement about the equation (8,10)b is a generalization of Levi-Civita's result for \(m = 1\).

Remark II. Let \(m = 1, n = 2\). Then
\[(8,15) \ N^\star T^\lambda_1 R^\lambda_1 R_{\mu \lambda} = - K N^\star T \quad (T = \sum R^\lambda_1 T^\lambda_1).\]
If \(\eta^1 = s\), \(s\) being the arc of \(V_1\), then \(D_1 = T^\lambda_1 \nabla_1\) and \(T = 1\). Moreover, if \(j^\nu\) is the unit vector in the direction of \(D_1 T^\lambda_1\), then
\[(8,16) \ D_1 T^\lambda_1 = j^\nu k, \quad D_1 j^\nu = - k T^\lambda_1,\]
where \(k\) is the curvature of \(V_1\). Because we may put \(N^\nu = y j^\nu\), \(y\) being a factor of proportionality, we have on account of (8,16)
\[(8,17) \ D_1 N^\nu = y j^\nu - y k T^\lambda_1.\]
Consequently
\[(8,18) a \quad j^\nu D_1^2 N^\nu = y'.\]
If \((V_1)\) consists of geodesic lines, we have \(k = 0\) and
\[(8,18) b \quad j^\nu N^\nu T^\lambda_1 R^\lambda_1 R^\star_{\mu \lambda} = - K y.\]
Comparing (8,10)c and (8,18) we have
\[y' + K y = 0,\]
which is the well known equation found by Jacobi.

**Bibliography**