DEFORMATION THEORY OF SUBSPACES IN A RIEMANN SPACE

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Summary. This paper deals with the infinitesimal transformation (2,1) of a family (V_m) of subspaces V_m in a Riemann space. In §§1–4 the transformation (2,1) is applied on internal objects of a V_m, while in §§5 and 6 the characteristic mixed tensors $K^{\nu}_{a_r \cdots a_1}$ (cf. the equation (1,2; 4)) of a V_m are investigated with respect to (2,1). Finally, some applications of the theory are given in §§7 and 8. In particular the statement expressed by the equation (8,10)b is the generalization of the well known Levi-Civita result for $m=1$, while the statement expressed by the equation (8,10)c generalizes the classical result (for $m=1$, $n=2$) by Jacobi.

1. Preliminary.

(1) Let V_n be a n-dimensional Riemann space ($n \geqq 2$), referred to the real coördinate system ξ^ν,[1] $g_{\lambda\mu}=g_{\mu\lambda}$ its metric tensor, $\Gamma^{\nu}_{\lambda\mu}=\Gamma^{\nu}_{\mu\lambda}$ the corresponding Christoffel symbols and ∇_μ the covariant derivative operator in V_n with respect to $\Gamma^{\nu}_{\lambda\mu}$. The curvature tensor of $\Gamma^{\nu}_{\lambda\mu}$ will be denoted by

$$(1,1;1) \qquad R_{\omega\mu\lambda}^{\cdot\cdot\cdot\nu} \equiv \partial_\mu \Gamma^{\nu}_{\lambda\omega} - \partial_\omega \Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\alpha\mu}\Gamma^{\alpha}_{\lambda\omega} - \Gamma^{\nu}_{\alpha\omega}\Gamma^{\alpha}_{\lambda\mu} \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial \xi^\mu}\right).$$

(2) Let V_m be a m-dimensional subspace of V_n ($1 \leqq m < n$) referred to the real parameters[1] η^a and let

$$(1,2;1) \qquad \xi^\nu = \xi^\nu(\eta^1, \cdots, \eta^m)$$

be its parametric equations.[2] Throughout this paper we consider only the case where the matrix of the mixed tensor

$$(1,2;2) \qquad T^{\nu}_{a} \equiv \partial_a \xi^\nu \qquad \left(\partial_a \equiv \frac{\partial}{\partial \eta^a}\right)$$

is of rank m. Hence the metric tensor[3]

$$g_{ab} = g_{ba} = g_{\lambda\mu} T^{\lambda\mu}_{ab}$$

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[1] Greek (Latin) indices run from $\dot{1}$ to $\dot{n}$ (from 1 to m).

[2] Any function considered in this paper is understood to be a real and continuous one, as well as its derivatives which appear in the discussion.

[3] Here and later on we put $T^{\lambda \cdots \nu}_{a_1 \ldots a_v} \equiv T^{\lambda}_{a_1} \ldots T^{\nu}_{a_v}$, $v=2, 3, \cdots$.

is of rank m and consequently the corresponding Christoffel symbols $\Gamma^c_{ab}=\Gamma^c_{ba}$ exist. The curvature tensor of V_m will be denoted by

$$P_{dcb}^{\ \ \ a} \equiv \partial_c\Gamma_{bd}^a - \partial_d\Gamma_{bc}^a + \Gamma_{hc}^a\Gamma_{bd}^h - \Gamma_{hd}^a\Gamma_{bc}^h.$$

Considering the connections $\Gamma^\nu_{\lambda\mu}$ and Γ^a_{bc} we may introduce three different kinds of covariant derivatives, namely

(1,2;3a) $\quad \nabla_\mu V_{\lambda\cdots}^{\nu\cdots} \equiv \partial_\mu V_{\lambda\cdots}^{\nu\cdots} + \Gamma_{\alpha\mu}^\nu V_{\lambda\cdots}^{\alpha\cdots} + \cdots - \Gamma_{\lambda\mu}^\alpha V_{\alpha\cdots}^{\nu\cdots} - \cdots,$

for a tensor field defined over V_n,

(1,2;3b) $\nabla_a V_{\lambda\cdots}^{\nu\cdots} \equiv \partial_a V_{\lambda\cdots}^{\nu\cdots} + T_a^\mu\{\Gamma_{\alpha\mu}^\nu V_{\lambda\cdots}^{\alpha\cdots} + \cdots - \Gamma_{\lambda\mu}^\alpha V_{\alpha\cdots}^{\nu\cdots} - \cdots\},$

for a tensor field defined over V_m and

(1,2;3c) $\quad D_b V_{a\cdots}^{\nu\cdots} \equiv \partial_b V_{a\cdots}^{\nu\cdots} + \Gamma_{\lambda\mu}^\nu T_b^\mu V_{a\cdots}^{\lambda\cdots} + \cdots - \Gamma_{ab}^c V_{c\cdots}^{\nu\cdots} - \cdots,$

for a mixed tensor field defined over V_m. The embedding theory of a V_m in a V_n may be described by means of the mixed tensors T^ν_a and

(1,2;4) $$K_{a_r\cdots a_1}^{\nu} \equiv D_{a_r}\cdots D_{a_2}T_{a_1}^\nu, \qquad r = 2, \cdots, N,$$

where N is the number of osculating spaces of V_m.[4]

Incidentally $K^\nu_{ab}=K^\nu_{ba}$ lies with its index ν in the first normal space of V_m.

(3) A family (V_m) of a set of V_m's is defined as a set of V_m's such that through any generic point of the V_n there is only one element V_m of (V_m). Each element V_m of the family (V_m) is referred to the parameter system η^a and we keep the parameter transformation $\eta^{a'}\leftrightarrow\eta^a$ independent of the coördinate system ξ^ν. Hence

(1,3;1) $$\partial_\mu \frac{\partial\eta^{a'}}{\partial\eta^a} = 0 \qquad \partial_\mu \frac{\partial\eta^a}{\partial\eta^{a'}} = 0.$$

On the other hand, an object Ω defined over (V_m) is defined also over V_n and consequently may be thought of as expressed either by means of the η^a or by means of the ξ^ν. This is in particular true for T^ν_a, g_{ab}, Γ^c_{ab}. Whenever we apply on such an object the operator ∂_μ it is understood that we consider it expressed by means of the ξ^ν. Taking in account (1,3;1) we see that[5] $\partial_\mu g_{ab}$, $\partial_\mu\Gamma^c_{ab}$, $\nabla_\omega T^\nu_a$ are

[4] Cf. V. Hlavatý: *Embedding theory of a W_m in a W_n* (to be published in Actualités Scientifiques et Industrielles) where the case of a Riemann space is included. Cf. also J. A. Schouten and E. R. van Kampen, *Über die Krümmung einer V_m in V_n; eine Revision der Krümmungstheorie* (Math. Ann. vol. 105, p. 144–159).

[5] $\nabla_\omega T^\nu_a \equiv \partial_\omega T^\nu_a + \Gamma^\nu_{\lambda\omega}T^\lambda_a$ according to (1,2;3a), while (cf. (1,2;3b)) $\nabla_b T^\nu_a \equiv \partial_b T^\nu_a + \Gamma^\nu_{\lambda\omega}T^{\lambda\omega}_{ab}$.

tensors (resp. vectors) not only with respect to the coördinate transformation but also with respect to the parameter transformation. Let P be a generic point of V_n and let V_m be the element of (V_m) containing P. If a tensor field T defined over V_n has the property that $T(P)$ is in the tangential space (or in some of the normal spaces) of the V_m already mentioned, then we say that the field T is tangent (normal) to (V_m).

2. **Fundamental definitions.** Let $V^\nu = V^\nu(\xi)$ be the components of a contravariant vector field given over V_n and let

$$*\xi^\nu = \xi^\nu + \epsilon V^\nu \qquad (\epsilon \to 0 \text{ is a constant}) \tag{2,1}$$

be the infinitesimal transformation of a Lie group with the generator $V^\lambda \partial_\lambda$. If a generic point $P(\xi)$ describes a subspace V_m of the family (V_m) then the point $*P \equiv P(*\xi)$ describes a subspace $*V_m$. We shall assume throughout this paper that the set of all $*V_m$ constructed in this way is a family, which we denote by $(*V_m)$. In the following definition $X = X(P)$ denotes a tensor field (a set of tensor fields) defined either over (V_m) or over V_n and $F[X] \equiv F(\xi)$ is a function of X.

DEFINITION (2,1). *X' is the value of X in $*P$, $X' \equiv X(*P)$, $°X$ is the tensor at $*P$ which one gets by parallel displacement in V_n of $X(P)$ from P to $*P$. If X is defined over (V_m), then $*X$ is the tensor field (a set of tensor fields) defined over $(*V_m)$ in the same way as X is defined over (V_m). If X is defined over V_n, then $*X \equiv X'$. Furthermore*

$$*F \equiv F[*X], \qquad °F \equiv F[°X], \qquad F' \equiv F(*\xi) \tag{2,2a}$$

in the first case and

$$*F \equiv F' \equiv F(*\xi), \qquad °F \equiv F[°X] \tag{2,2b}$$

in the second case.

We use these symbols in the following definition of the symbols τ, ω, and Δ:

DEFINITION (2,2). *The operators τ, ω, Δ are defined by the following equations*

$$\begin{aligned} &\text{(a)} \quad \tau F \equiv \lim_{\epsilon \to 0} \frac{*F - °F}{\epsilon} && (\textit{translation of } F), \\ &\text{(b)} \quad \omega F \equiv \lim_{\epsilon \to 0} \frac{F' - °F}{\epsilon} && (\textit{variation of } F), \\ &\text{(c)} \quad \Delta F \equiv (\tau - \omega) F && (\textit{deformation of } F). \end{aligned} \tag{2,3}$$

In this paper we shall investigate the application of these operators on the objects of (V_m).

3. **The fields T_a^ν and g_{ab}.**

THEOREM (3,1). *We have*

$$(3,1)\qquad \text{(a)}\quad \tau T_a^\nu = \nabla_a V^\nu, \qquad \text{(b)}\quad \omega T_a^\nu = V^\mu \nabla_\mu T_a^\nu, \qquad \text{(c)}\quad \Delta T_a^\nu = L_a^\nu,$$

where

$$(3,2)\qquad L_a^\nu \equiv \nabla_a^\nu V - V^\mu \nabla_\mu T_a^\nu = \partial_a V^\nu - V^\mu \partial_\mu T_a^\nu.$$

PROOF. Let $X \equiv F[X] \equiv T_a^\nu$. Then

$$*F \equiv \frac{\partial^* \xi^\nu}{\partial \eta^a} = T_a^\nu + \epsilon \partial_a V^\nu, \qquad F' \equiv T_a^\nu(*\xi) = T_a^\nu + \epsilon V^\mu \partial_\mu T_a^\nu + \cdots,^{6}$$

$$°F \equiv °T_a^\nu = T_a^\nu - \epsilon V^\mu \Gamma_{\lambda\mu}^\nu T_a^\lambda + \cdots$$

and consequently

$$(3,3)\qquad \begin{aligned} &\text{(a)}\quad *F - °F = T_a^\nu + \epsilon(\partial_a V^\nu + \Gamma_{\lambda\mu}^\nu T_a^\mu V^\lambda) - T_a^\nu + \cdots, \\ &\text{(b)}\quad F' - °F = T_a^\nu + \epsilon V^\mu(\partial_\mu T_a^\nu + \Gamma_{\lambda\mu}^\nu T_a^\lambda) - T_a^\nu + \cdots. \end{aligned}$$

From (3,3)ab we have (3,1)ab and these equations together with (3,2) lead to (3,1)c.

THEOREM (3,2). *We have*

$$(3,4)\qquad \text{(a)}\quad \tau g_{ab} = 2W_{ab},^{7} \qquad \text{(b)}\quad \omega g_{ab} = G_{ab}, \qquad \text{(c)}\quad \Delta g_{ab} = 2g_{\lambda\mu} L_{(a}^\lambda T_{b)}^\mu$$

where

$$(3,5)\qquad W_{ab} \equiv T_{(ab)}^{\lambda\mu} \nabla_\lambda V_\mu, \qquad G_{ab} \equiv V^\mu \partial_\mu g_{ab}.$$

PROOF. Put $X \equiv g_{\lambda\mu}$. Then

$$(3,6)\qquad \begin{aligned} &\text{(a)}\ *X \equiv X' = *g_{\lambda\mu} = g_{\lambda\mu} + \epsilon V^\omega \partial_\omega g_{\lambda\mu} + \cdots, \\ &\text{(b)}\ °X \qquad = °g_{\lambda\mu} = g_{\lambda\mu} + \epsilon V^\omega(\Gamma_{\lambda\omega}^\nu g_{\nu\mu} + \Gamma_{\mu\omega}^\nu g_{\lambda\nu}) + \cdots = *g_{\lambda\mu}. \end{aligned}$$

On the other hand, if we denote by X the set of tensor fields $g_{\lambda\mu}$, T_a^ν we have for $F[X] \equiv g_{\lambda\mu} T_a^\lambda T_b^\mu = g_{ab}$ by virtue of (3,6)

[6] Throughout this paper the dots denote the coefficient of ϵ^2.

[7] Substantially equivalent formulas to (3,1)a and (3,4)a may be found also in the papers in the bibliography, which deal with the theory of deformation of subspaces from a different point of view.

$$\text{(a) } {}^*F \equiv {}^*g_{ab} = {}^*g_{\lambda\mu}{}^*T_a^{\lambda}{}^*T_b^{\mu} = (g_{\lambda\mu} + \epsilon V^{\omega}\partial_\omega g_{\lambda\mu} + \cdots)$$
$$\cdot(T_a^{\lambda} + \epsilon\partial_a V^{\lambda} + \cdots)(T_b^{\mu} + \epsilon\partial_b V^{\mu} + \cdots)$$
$$= g_{ab} + \epsilon[V^{\omega}T_a^{\lambda}T_b^{\mu}\partial_\omega g_{\lambda\mu} + 2g_{\lambda\mu}T^{\mu}_{(b}\partial_{a)}V^{\lambda}] + \cdots$$
$$= g_{ab} + 2\epsilon T^{\lambda}_{(a}T^{\mu}_{b)}\nabla_\lambda V_\mu + \cdots,$$

$$(3,7)\quad \text{(b) } F' \equiv g'_{ab} = g'_{\lambda\mu}T_a^{\lambda'}T_b^{\mu'} = (g_{\lambda\mu} + \epsilon V^{\omega}\partial_\omega g_{\lambda\mu} + \cdots)$$
$$\cdot(T_a^{\lambda} + \epsilon V^{\rho}\partial_\rho T_a^{\lambda} + \cdots)(T_b^{\mu} + \epsilon V^{\sigma}\partial_\sigma T_b^{\mu} + \cdots)$$
$$= g_{ab} + \epsilon[V^{\omega}T_a^{\lambda}T_b^{\mu}\partial_\omega g_{\lambda\mu} + 2g_{\lambda\mu}V^{\omega}(\partial_\omega T^{\lambda}_{(a})T^{\mu}_{b)}] + \cdots$$
$$= g_{ab} + \epsilon V^{\omega}\partial_\omega g_{ab} + \cdots,$$

$$\text{(c) } {}^\circ F \equiv {}^\circ g_{ab} = {}^\circ g_{\lambda\mu}{}^\circ T_a^{\lambda}{}^\circ T_b^{\mu} = [g_{\lambda\mu} + \epsilon V^{\omega}(\Gamma^{\nu}_{\lambda\omega}g_{\nu\mu} + \Gamma^{\nu}_{\mu\omega}g_{\lambda\nu}) + \cdots]$$
$$\cdot(T_a^{\lambda} - \epsilon V^{\rho}\Gamma^{\lambda}_{\alpha\rho}T_a^{\alpha} + \cdots)(T_b^{\mu} - \epsilon V^{\sigma}\Gamma^{\mu}_{\beta\sigma}T_b^{\beta} + \cdots)$$
$$= g_{ab} + \epsilon[T_a^{\lambda}T_b^{\mu}V^{\omega}(\Gamma^{\nu}_{\lambda\omega}g_{\nu\mu} + \Gamma^{\nu}_{\mu\omega}g_{\lambda\nu}) - 2g_{\lambda\mu}T^{\mu}_{(b}T^{\alpha}_{a)}V^{\omega}\Gamma^{\lambda}_{\alpha\omega}]$$
$$+ \cdots = g_{ab} + \cdots.$$

Hence

$$(3,7)\quad \begin{aligned} &\text{(d) } {}^*F - {}^\circ F \equiv {}^*g_{ab} - {}^\circ g_{ab} = 2\epsilon T^{\mu}_{(b}T^{\lambda}_{a)}\nabla_\lambda V_\mu + \cdots, \\ &\text{(e) } F' - {}^\circ F = g'_{ab} - {}^\circ g_{ab} = \epsilon V^{\omega}\partial_\omega g_{ab} + \cdots. \end{aligned}$$

These equations lead to (3,4)ab and from these we have (3,4)c.

DEFINITION (3,1). *The transformation* (2,1) *is called a rigid translation if* $\tau g_{ab} = 0$.

THEOREM (3,3a). *Let* V^{ν} *be a tangential vector field to* (V_m). *A necessary and sufficient condition that* (2,1) *be a rigid translation is the* (*Killing*) *equation*

$$(3,8)\qquad D_{(a}V_{b)} = 0.$$

PROOF. In our case we have[8] $V_a = V_\lambda T_a^{\lambda}$, $V_\lambda = V_c T_\lambda^{c}$ and consequently (by virtue of $\Gamma^c_{ab} = T^c_\lambda \nabla_b T^\lambda_a$)

$$(3,9)\quad \begin{aligned} D_bV_a &= \partial_b V_\lambda T_a^{\lambda} - T_\lambda^{c}(\nabla_b T_a^{\lambda})V_c = (\partial_b V_\lambda)T_a^{\lambda} + V_\lambda \partial_b T_a^{\lambda} - (\nabla_b T_a^{\lambda})V_\lambda \\ &= T_a^{\nu}(\partial_b V_\nu - \Gamma^{\lambda}_{\mu\nu}T_b V_\lambda^{\mu}) = T_b T_a^{\mu}\nabla_\mu V_\lambda. \end{aligned}$$

The proof follows from (3,9), (3,5), and (3,4)a.

THEOREM (3,3b). *Let* V^{ν} *be a normal vector field to* (V_m). *A necessary and sufficient condition that* (2,1) *be a rigid translation is*: V^{ν} *is in the* x*th normal space of* V_m, $x = 2, 3, \cdots$.

[8] $T_\lambda^{c} \equiv T_a^{\nu} g^{ac} g_{\nu\lambda}$.

PROOF. In our case we have $V_\lambda T^\lambda_a = 0$ and consequently

$$(3,10)\qquad T^\lambda_a T^\mu_b \nabla_\lambda V_\mu = -T^\lambda_a V_\mu \nabla_\lambda T^\mu_b = -(D_a T^\mu_b)V_\mu = -K^\mu_{ab} V_\mu.$$

The tensor K^μ_{ab} lies with μ in the first normal space of (V_m). By virtue of this fact, we get the theorem (3,3b) from (3,4)a and (3,5).

REMARK. The theorem (3,3b) is the generalization of the well known fact that the "deformation" of the arc of a curve in a three-dimensional Euclidean space along its binormal is equal to zero.

4. The field Γ^c_{ab}.

THEOREM (4,1). *We have*

$$(4,1)\qquad \text{(a)}\ \ \tau\Gamma^c_{ab} = W^c_{ab} \equiv W_{abd} g^{dc}, \qquad \text{(b)}\ \ \omega\Gamma^c_{ab} = G^c_{ab}, \qquad \text{(c)}\ \ \Delta\Gamma^c_{ab} = Q^c_{ab}$$

where

$$(4,2)\qquad \begin{aligned} &\text{(a)}\ \ W_{abd} \equiv D_a W_{db} + D_b W_{da} - D_d W_{ba},\\ &\text{(b)}\ \ G^c_{ab} = V^\omega \partial_\omega \Gamma_{ab},\\ &\text{(c)}\ \ Q^c_{ab} \equiv W_{abd} g^{dc} - G^c_{ab}.\end{aligned}$$

PROOF. First of all we have

$$(4,3)\qquad \partial_a W_{db} + \partial_b W_{da} - \partial_d W_{ab} = W_{abd} + 2\Gamma^e_{ba} W_{de}.$$

Furthermore, if X is the set of tensors g_{ab}, g^{cd}, we get for $F[X] \equiv \Gamma^c_{ab}$ by virtue of (3,7)d and (4,3)

$$(4,4)\qquad \begin{aligned} {}^*F \equiv {}^*\Gamma^c_{ab} &= 1/2\,{}^*g^{cd}(\partial_a {}^*g_{bd} + \partial_b {}^*g_{ad} - \partial_d {}^*g_{ab})\\ &= 1/2[{}^\circ g^{cd} - 2\epsilon W^{cd} + \cdots][\partial_a({}^\circ g_{bd} + 2\epsilon W_{bd} + \cdots)\\ &\quad + \partial_b({}^\circ g_{ad} + 2\epsilon W_{ad} + \cdots)\\ &\quad - \partial_d({}^\circ g_{ab} + 2\epsilon W_{ab} + \cdots)]\\ &= {}^\circ\Gamma^c_{ba} + \epsilon\{{}^\circ g^{cd}[W_{abd} + 2\Gamma^e_{ba} W_{de}]\\ &\quad - 2W^{cd}\,{}^\circ\Gamma^e_{ba}\,{}^\circ g_{ed}\} + \cdots .\end{aligned}$$

On the other hand, we see from (3,7)c that

$$(4,5)\qquad {}^\circ\Gamma^c_{ba} = \Gamma^c_{ba} + \cdots .$$

From (4,5), (4,4) we have (4,1)a. The equation (4,1)b follows from

$$(4,6)\qquad F' \equiv \Gamma'^c_{ab} = \Gamma^c_{ab} + \epsilon V^\omega \partial_\omega \Gamma^c_{ab} + \cdots$$

and (4,5). The equation (4,1)c is obvious.

Later on we shall need also the coefficients

$$(4,7)\qquad '\Gamma^c_{ab} = 1/2 g^{cd\prime}(\partial_a g'_{bd} + \partial_b g'_{ad} - \partial_d g'_{ab})$$

which are related to $\Gamma^{c\prime}_{ab}$ by a relation introduced in the following lemma:

LEMMA (4,1). *We have*

$$(4,8)\qquad '\Gamma^c_{ab} - \Gamma^{c\prime}_{ab} = \frac{\epsilon}{2} g^{cd}[L^\omega_a \partial_\omega g_{db} + L^\omega_b \partial_\omega g_{da} - L^\omega_d \partial_\omega g_{ab}] + \cdots$$

PROOF. We get from (4,7) and (3,7)b

$$(4,9)\qquad \begin{aligned} '\Gamma^c_{ab} &= 1/2(g^{cd} + \epsilon V^\omega \partial_\omega g^{cd} + \cdots)[\partial_a(g_{bd} + \epsilon V^\rho \partial_\rho g_{bd} + \cdots) \\ &\quad + \partial_b(g_{ad} + \epsilon V^\rho \partial_\rho g_{ad} + \cdots) \\ &\quad - \partial_d(g_{ab} + \epsilon V^\rho \partial_\rho g_{ab} + \cdots)] \\ &= \Gamma^c_{ab} + \frac{\epsilon}{2}\{2\Gamma^h_{ab} g_{hd} V^\rho \partial_\rho g^{cd} + g^{cd}[(\partial_a V^\rho)\partial_\rho g_{bd} \\ &\quad + (\partial_b V^\rho)\partial_\rho g_{ad} - (\partial_c V^\rho)\partial_\rho g_{ab} \\ &\quad + g^{cd} V^\rho(\partial_a \partial_\rho g_{bd} + \partial_b \partial_\rho g_{ad} - \partial_d \partial_\rho g_{ab})]\} + \cdots . \end{aligned}$$

On the other hand

$$(4,10)\qquad \begin{aligned} \Gamma^{c\prime}_{ab} &= \Gamma^c_{ab} + \epsilon V^\rho \partial_\rho \Gamma^c_{ab} + \cdots = \Gamma^c_{ab} + \epsilon[V^\rho(\partial_\rho g^{cd})\Gamma^h_{ab} g_{hd} \\ &\quad + 1/2 g^{cd} V^\rho(\partial_\rho \partial_a g_{bd} + \partial_\rho \partial_b g_{ad} - \partial_\rho \partial_d g_{ab})] + \cdots, \end{aligned}$$

$$(4,11)\qquad \begin{aligned} (\partial_a \partial_\rho - \partial_\rho \partial_a) g_{bd} &= (T^\lambda_a \partial_\lambda \partial_\rho - (\partial_\rho T^\lambda_a)\partial_\lambda - T^\lambda_a \partial_\rho \partial_\lambda) g_{bd} \\ &= -(\partial_\rho T^\lambda_a)(\partial_\lambda g_{bd}). \end{aligned}$$

Comparing (4,9), (4,10), and (4,11), we obtain

$$\begin{aligned} '\Gamma^c_{ab} - \Gamma^{c\prime}_{ab} &= \frac{\epsilon}{2} g^{cd}[(\partial_a V^\rho - V^\lambda \partial_\lambda T^\rho_a)\partial_\rho g_{bd} + (\partial_b V^\rho - V^\lambda \partial_\lambda T^\rho_b)\partial_\rho g_{ad} \\ &\quad - (\partial_d V^\rho - V^\lambda \partial_\lambda T^\rho_d)\partial_\rho g_{ab} + \cdots] \end{aligned}$$

and this equation gives us at once (4,8).

5. **The field K^ν_{ba}.**[9] The first theorem concerns the translation of the

[9] Throughout §§5 and 6 we shall sometimes make use of a special coördinate (parameter) system and shall symbolize the statement: "in the special coördinate (parameter) system $\Gamma^\nu_{\lambda\mu}=0$ ($\Gamma^c_{ba}=0$) at P" by using the symbol $\doteq$, $\Gamma^\nu_{\lambda\mu}\doteq 0$ ($\Gamma^c_{ab}\doteq 0$). An equation with $\doteq$ has to be understood as valid in a special coördinate (parameter) system at P where both last equations hold. On the other hand, the sign $=$ does not impose any restriction whatsoever on the choice of the coördinate (parameter) system.

tensor

$$K^{\nu}_{ba} = K^{\nu}_{ab} \equiv D_b T^{\nu}_{a}. \tag{5,1}$$

THEOREM (5,1). *The following equation holds:*

$$\tau K^{\nu}_{ba} = D_b D_a V^{\nu} - V^{\omega} T^{\lambda\mu}_{ba} R^{\nu}_{\omega\mu\lambda} - W^{c}_{ba} T^{\nu}_{c}. \tag{5,2}$$

PROOF. First of all we have according to (2,2)b for $X \equiv g_{\lambda\mu}, g^{\lambda\nu}$, $F[X] \equiv F(\xi) = \Gamma^{\nu}_{\lambda\mu}$

$$*\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu\prime}_{\lambda\mu} \doteq \epsilon V^{\omega} \partial_{\omega} \Gamma^{\nu}_{\lambda\mu} + \cdots \tag{5,3)a}$$

and moreover by virtue of (4,4) and (4,5)

$$*\Gamma^{c}_{ba} \doteq \epsilon W^{c}_{ba} + \cdots . \tag{5,3)b}$$

Hence

$$\begin{aligned} *K^{\nu}_{ba} = *D_b * T^{\nu}_{a} &\doteq \partial_b (T^{\nu}_{a} + \epsilon \partial_a V^{\nu} + \cdots) \\ &+ \epsilon V^{\rho} (\partial_{\rho} \Gamma^{\nu}_{\lambda\mu}) (T^{\mu}_{b} + \epsilon \partial_b V^{\mu} + \cdots)(T^{\lambda}_{a} + \epsilon \partial_a V^{\lambda} + \cdots) \\ &- \epsilon W^{c}_{ba} (T^{\nu}_{c} + \epsilon \partial_c V^{\nu} + \cdots) \end{aligned} \tag{5,4)a}$$

and consequently

$$\begin{aligned} *K^{\nu}_{ba} &\doteq \partial_b T^{\nu}_{a} + \epsilon \{ \partial_b \partial_a V^{\nu} + V^{\rho} (\partial_{\rho} \Gamma^{\nu}_{\lambda\mu}) T^{\mu\lambda}_{ba} - W^{c}_{ba} T^{\nu}_{c} \} + \cdots \\ &\doteq D_b T^{\nu}_{a} + \epsilon \{ D_b D_a V^{\nu} - V^{\omega} [\partial_{\mu} \Gamma^{\nu}_{\omega\lambda} - \partial_{\omega} \Gamma^{\nu}_{\mu\lambda}] T^{\mu\lambda}_{ba} - W^{c}_{ba} T^{\nu}_{c} \} \\ &\quad + \cdots \\ &\doteq D_b T^{\nu}_{a} + \epsilon \{ D_b D_a V^{\nu} - V^{\omega} T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} - W^{c}_{ba} T^{\nu}_{c} \} + \cdots . \end{aligned} \tag{5,4)b}$$

On the other hand

$$°K^{\nu}_{ba} \doteq D_b T^{\nu}_{a} + \cdots = K^{\nu}_{ba} + \cdots . \tag{5,4)c}$$

The equations (5,4)b and (5,4)c lead to (5,2) with the $\doteq$ sign. This equation, being a tensor equation valid for a special coördinate (and parameter) system at a generic point P, holds for all coördinate (and parameter) systems at a generic point.

The second theorem concerns the variation of K^{ν}_{ab}:

THEOREM (5,2). *The following equation holds:*

$$\omega K^{\nu}_{ba} = V^{\omega} [D_b \nabla_{\omega} T^{\nu}_{a} - T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} + (\nabla_{\mu} T^{\nu}_{a})(\nabla_{\omega} T^{\mu}_{b})] - G^{c}_{ba} T^{\nu}_{c}. \tag{5,5}$$

PROOF. First of all we have

$$\partial_\omega\partial_b T^\nu_a - \partial_b\partial_\omega T^\nu_a = \partial_\omega T^\mu_b \partial_\mu T^\nu_a - T^\mu_b \partial_\mu\partial_\omega T^\nu_a = (\partial_\omega T^\mu_b)(\partial_\mu T^\nu_a)$$
$$\doteq (\nabla_\omega T^\mu_b)(\nabla_\mu T^\nu_a),$$
$$D_b \nabla_\omega T^\nu_a \doteq \partial_b\partial_\omega T^\nu_a + T^{\mu\lambda}_{b\ a}(\partial_\mu \Gamma^\nu_{\lambda\omega}).$$

Consequently

$$\begin{aligned} K'^\nu_{ba} &\equiv (D_b T^\nu_a)' = K^\nu_{ba} + \epsilon V^\omega \partial_\omega K^\nu_{ba} + \cdots \\ &\doteq K^\nu_{ba} + \epsilon V^\omega[\partial_\omega\partial_b T^\nu_a + (\partial_\omega \Gamma^\nu_{\lambda\mu}) T^{\mu\lambda}_{ba} - (\partial_\omega \Gamma^c_{ab}) T^\nu_c] + \cdots \\ &= K^\nu_{ba} + \epsilon V^\omega \{ [D_b \nabla_\omega T^\nu_a + T^{\mu\lambda}_{ba}(\partial_\omega \Gamma^\nu_{\lambda\mu} - \partial_\mu \Gamma^\nu_{\lambda\omega}) \\ &\quad + (\nabla_\omega T^\mu_b)(\nabla_\mu T^\nu_a)] - G^c_{ba} T^\nu_c \}. \end{aligned} \tag{5,6}$$

The proof follows from (5,6) and (5,4)c.

THEOREM (5,3). *Put*

$$L^\nu_{ba} \equiv D_b L^\nu_a + L^\mu_b \nabla_\mu T^\nu_a - Q^c_{ba} T^\nu_c. \tag{5,7)a$$

Then

$$\begin{aligned} L^\nu_{ba} &= D_b L^\nu_a - V^\omega T^{\mu\lambda}_{ba} R^\nu_{\omega\mu\lambda} - Q^c_{ba} T_c \\ &\quad + (T^\mu_b \nabla_\mu V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_b \nabla_\mu) T^\nu_a \end{aligned} \tag{5,7)b$$

and

$$\Delta K^\nu_{ba} = L^\nu_{ba}. \tag{5,8}$$

PROOF. We have from (5,2) and (5,5)

$$\begin{aligned} \Delta K^\nu_{ba} = (\tau - \omega) K^\nu_{ba} &= D_b D_a V^\nu - V^\omega D_b \nabla_\omega T^\nu_a \\ &- V^\omega (\nabla_\omega T^\mu_b) \nabla_\mu T^\nu_a - Q^c_{ba} T^\nu_c. \end{aligned} \tag{5,9}$$

Since

$$\begin{aligned} D_b L^\nu_a &= D_b[D_a V^\nu - V^\rho \nabla_\rho T^\nu_a] \\ &= D_b D_a V^\nu - (D_b V^\rho) \nabla_\rho T^\nu_a - V^\omega D_b \nabla_\omega T^\nu_a, \end{aligned}$$

the equation (5,9) reduces to

$$\Delta K^\nu_{ba} = D_b L^\nu_a + (\nabla_\mu T^\nu_a)(\nabla_b V^\mu - V^\rho \nabla_\rho T^\mu_b) - Q^c_{ba} T^\nu_c$$

which proves (5,7)a and (5,8). The equation (5,7)b follows from (5,7)a and

$$\begin{aligned} (T^\mu_b \nabla_\mu V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_b \nabla_\mu) T^\nu_a &= [L^\mu_b \nabla_\mu + V^\omega T^\mu_b (\nabla_\mu \nabla_\omega - \nabla_\omega \nabla_\mu)] T^\nu_a \\ &= L^\mu_b \nabla_\mu T^\nu_a - V^\omega T^\mu_b T^\lambda_a R^\nu_{\mu\omega\lambda} \end{aligned} \tag{5,10}$$

6. **The field $K^{\nu}_{a_r \cdots a_1}$.** The last theorem may easily be generalized:

THEOREM (6,1). *Put*

(6,1)a
$$L^{\nu}_{a_r \cdots a_1} \equiv D_{a_r} L^{\nu}_{a_{r-1} \cdots a_1} + L^{\mu}_{a_r} \nabla_\mu K^{\nu}_{a_{r-1} \cdots a_1} - \sum_{1}^{r-1}{}_s Q^{c}_{a_s a_r} K^{\nu}_{a_{r-1} \cdots a_{s+1} c a_{s-1} \cdots a_1}.$$

Then

(6,1)b
$$L^{\nu}_{a_r \cdots a_1} \equiv D_{a_r} L^{\nu}_{a_{r-1} \cdots a_1} + (T^{\mu}_{a_r} \nabla_\mu V^{\omega} \nabla_\omega - V^{\omega} \nabla_\omega T^{\mu}_{a_r} \nabla_\mu) K^{\nu}_{a_{r-1} \cdots a_1} - V^{\omega} T^{\mu}_{a_r} R^{\nu}_{\omega\mu\lambda} K^{\lambda}_{a_{r-1} \cdots a_1} - \sum_{1}^{r-1}{}_s Q^{c}_{a_s a_r} K^{\nu}_{a_{r-1} \cdots a_{s+1} c a_{s-1} \cdots a_1}$$

and

(6,2)
$$\Delta K^{\nu}_{a_r \cdots a_1} = L^{\nu}_{a_r \cdots a_1} \qquad (r = 2, 3, \cdots).$$

PROOF. (The theorem has already been proved for $r=2$.) Let us assume that we have proved it for some $r \geqq 2$. Then we have from (6,2)

(6,3)
$$\begin{aligned} {}^*K^{\nu}_{a_r \cdots a_1} &= K'^{\nu}_{a_r \cdots a_1} + \epsilon L^{\nu}_{a_r \cdots a_1} + \cdots \\ &= K^{\nu}_{a_r \cdots a_1} + \epsilon [L^{\nu}_{a_r \cdots a_1} + V^{\omega} \partial_\omega K^{\nu}_{a_r \cdots a_1}] + \cdots . \end{aligned}$$

On the other hand

$$ {}^*D_{a_{r+1}} {}^*K^{\nu}_{a_r \cdots a_1} \doteq \partial_{a_{r+1}} {}^*K^{\nu}_{a_r \cdots a_1} + \epsilon \Big[V^{\omega} (\partial_\omega \Gamma^{\nu}_{\lambda\mu}) T^{\mu}_{a_{r+1}} K^{\lambda}_{a_r \cdots a_1} - \sum_{1}^{r}{}_s W^{c}_{a_s a_{r+1}} K^{\nu}_{a_r \cdots a_{s+1} c a_{s-1} \cdots a_1} \Big] + \cdots $$

and consequently by virtue of (6,3)

(6,4)
$$\begin{aligned} {}^*K^{\nu}_{a_{r+1} \cdots a_1} \doteq \partial_{a_{r+1}} K^{\nu}_{a_r \cdots a_1} + \epsilon \Big\{ &\partial_{a_{r+1}} L^{\nu}_{a_r \cdots a_1} \\ &+ (\partial_{a_{r+1}} V^{\omega}) \partial_\omega K^{\nu}_{a_r \cdots a_1} + V^{\omega} \partial_{a_{r+1}} \partial_\omega K^{\nu}_{a_r \cdots a_1} \\ &+ V^{\omega} (\partial_\omega \Gamma^{\nu}_{\lambda\mu}) T^{\mu}_{a_{r+1}} K^{\lambda}_{a_r \cdots a_1} \\ &- \sum_{1}^{r}{}_s W^{c}_{a_s a_{r+1}} K^{\nu}_{a_r \cdots a_{s+1} c a_{s-1} \cdots a_1} \Big\} + \cdots \end{aligned}$$

$$
\begin{aligned}
&\doteq D_{a_{r+1}} K^{\nu}_{a_r\cdots a_1} + \epsilon \Big\{ D_{a_{r+1}} L^{\nu}_{a_r\cdots a_1} \\
&\qquad + (\nabla_{a_{r+1}} V^{\rho}) \nabla_{\rho} K^{\nu}_{a_r\cdots a_1} \\
(6,4) \qquad &\qquad - \sum_1^r {}_s W^{c}_{a_s a_{r+1}} K^{\nu}_{a_r\cdots a_{s+1} c a_{s-1}\cdots a_1} \\
&\qquad + V^{\omega} \big[(\partial_\omega \Gamma^{\nu}_{\lambda\mu}) T^{\mu}_{a_{r+1}} K^{\lambda}_{a_r\cdots a_1} + \partial_{a_{r+1}} \partial_\omega K^{\nu}_{a_r\cdots a_1} \big] \Big\} \\
&\qquad + \cdots .
\end{aligned}
$$

The tensor $K^{\nu}_{a_{r+1}\cdots a_1}$ satisfies the following relation

$$
\begin{aligned}
K'^{\nu}_{a_{r+}\cdots a} &= K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega} \partial_\omega K^{\nu}_{a_{r+1}\cdots a_1} + \cdots \\
&= K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega} \partial_\omega D_{a_{r+1}} K^{\nu}_{a_r\cdots a_1} + \cdots \\
&\doteq K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega} \Big[\partial_\omega \partial_{a_{r+1}} K^{\nu}_{a_r\cdots a_1} \\
(6,5) \qquad &\qquad + (\partial_\omega \Gamma^{\nu}_{\lambda\mu}) T^{\mu}_{a_{r+1}} K^{\lambda}_{a_r\cdots a_1} \\
&\qquad - \sum_1^r {}_s (\partial_\omega \Gamma^{c}_{a_s a_{r+1}}) K^{\nu}_{a_r\cdots a_{s+1} c a_{s-1}\cdots a_1} \Big] + \cdots .
\end{aligned}
$$

Subtracting (6,5) from (6,4) we have

$$
\begin{aligned}
{}^*K^{\nu}_{a_{r+1}\cdots a_1} - K'^{\nu}_{a_{r+1}\cdots a_1} &\doteq \epsilon \Big[D_{a_{r+1}} L^{\nu}_{a_r\cdots a_1} \\
(6,6) \qquad &+ (\nabla_{a_{r+1}} V^{\rho}) \nabla_\rho K^{\nu}_{a_r\cdots a_1} - \sum_1^r {}_s Q^{c}_{a_s a_{r+1}} K^{\nu}_{a_r\cdots a_{s+1} c a_{s-1}\cdots a_1} \\
&+ V^{\omega} (\partial_{a_{r+1}} \partial_\omega - \partial_\omega \partial_{a_{r+1}}) K^{\nu}_{a_r\cdots a_1} \Big] + \cdots .
\end{aligned}
$$

Since

$$
\begin{aligned}
(\partial_{a_{r+1}} \partial_\omega &- \partial_\omega \partial_{a_{r+1}}) K^{\nu}_{a_r\cdots a_1} \\
&= [T^{\mu}_{a_{r+1}} \partial_\mu \partial_\omega - (\partial_\omega T^{\mu}_{a_{r+1}}) \partial_\mu - T^{\mu}_{a_{r+1}} \partial_\omega \partial_\mu] K^{\nu}_{a_r\cdots a_1} \\
&\doteq - (\nabla_\omega T^{\mu}_{a_{r+1}}) \nabla_\mu K^{\nu}_{a_r\cdots a_1},
\end{aligned}
$$

the equation (6,6) reduces to

$$(6,7)\qquad \begin{aligned} {}^{*}K^{\nu}_{a_{r+1}\cdots a_1} \quad K'^{\nu}_{a_{r+1}\cdots a_1} &\doteq \epsilon\Big[D_{a_{r+1}}L^{\nu}_{a_r\cdots a} \\ &+ L^{\rho}_{a_{r+1}}\nabla_\rho K^{\nu}_{a_r\cdots a_1} - \sum_1^r {}_s Q^{c}_{a_s a_{r+1}} K^{\nu}_{a_r\cdots a_{s+1} c a_{s-1}\cdots a_1}\Big] + \cdots . \end{aligned}$$

In this tensor equation we may obviously suppress the dot in $\doteq$ and consequently if we put

$$\begin{aligned} L^{\nu}_{a_{r+1}\cdots a_1} \equiv D_{a_{r+1}}L^{\nu}_{a_r\cdots a_1} &+ L^{\rho}_{a_{r+1}}\nabla_\rho K^{\nu}_{a_r\cdots a_1} \\ &- \sum_1^r {}_s Q^{c}_{a_s a_{r+1}} K^{\nu}_{a_r\cdots a_{s+1} c a_{s-1}\cdots a_1}, \end{aligned}$$

we have from (6,7)

$$\Delta K^{\nu}_{a_{r+1}\cdots a_r} = L^{\nu}_{a_{r+1}\cdots a_1}.$$

Hence if (6,2) holds for some r then it holds also for $r+1$. But (6,2) holds for $r=2$ and consequently it holds for $r=2, 3, \cdots$. The remaining part of the theorem (equation (6,1)b) may be proved by means of an equation similar to (5,10)

7. The differential equation $L^{\nu}_a = 0$.

THEOREM (7,1). *The system of differential equations*

$$(7,1)\qquad L^{\nu}_a = 0 \quad \text{or} \quad \nabla_a V^{\nu} = V^{\lambda}\nabla_\lambda T^{\nu}_a$$

is completely integrable and its general solution involves n arbitrary constants.

PROOF. We have first

$$(7,2)\qquad \begin{aligned} \nabla_{[b}\nabla_{a]}V^{\nu} = \nabla_{[b}T^{\mu}_{a]}\nabla_\mu V^{\nu} &= (\nabla_{[b}T^{\mu}_{a]})\nabla_\mu V^{\nu} + T^{\omega}_{[b}T^{\mu}_{a]}\nabla_\omega\nabla_\mu V^{\nu} \\ &= -\frac{1}{2}R^{\nu}_{\omega\mu\lambda}V^{\lambda}T^{\omega\mu}_{ba} \end{aligned}$$

and by virtue of (7,1)

$$(7,3)\qquad \begin{aligned} \nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]} &= (\nabla_{[b}V^{\lambda})\nabla_{|\lambda|}T^{\nu}_{a]} + V^{\mu}T^{\omega}_{[b}\nabla_{|\omega}\nabla_{\mu|}T^{\nu}_{a]} \\ &= V^{\omega}[(\nabla_\omega T^{\mu}_{[b})(\nabla_{|\mu|}T^{\nu}_{a]}) + T^{\mu}_{[b}\nabla_{|\mu}\nabla_{|\omega}T^{\nu}_{a]}]. \end{aligned}$$

On the other hand

$$0 = V^{\omega}\nabla_\omega T^{\mu}_{[b}\nabla_{|\mu|}T^{\nu}_{a]} = V^{\omega}(\nabla_\omega T^{\mu}_{[b})\nabla_{|\mu|}T^{\nu}_{a]} + V^{\omega}T^{\mu}_{[b}\nabla_{\omega|}\nabla_{\mu|}T^{\nu}_{a]}$$

and consequently (7,3) reduces to

$$(7,4)\qquad \nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]} = V^{\omega}T^{\mu}_{[b}(\nabla_{|\mu}\nabla_{\omega|} - \nabla_{|\omega}\nabla_{\mu|})T^{\nu}_{a]} = V^{\omega}T^{\mu\lambda}_{[ba]}R^{\nu}_{\omega\mu\lambda}.$$

The integrability conditions of (7,1) are obtained by substituting in

$$\nabla_{[b}\nabla_{a]}V^{\nu} = \nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]}$$

from (7,2) and (7,4). In doing so we obtain

$$\frac{1}{2}V^{\lambda}T^{\omega\mu}_{ba}(-R^{\nu}_{\omega\mu\lambda} - R^{\nu}_{\lambda\omega\mu} + R^{\nu}_{\lambda\mu\omega})$$

$$= -\frac{1}{2}V^{\lambda}T^{\omega\mu}_{ba}(R^{\nu}_{\omega\mu\lambda} + R^{\nu}_{\lambda\omega\mu} + R^{\nu}_{\mu\lambda\omega}) \equiv 0$$

since

$$R^{\nu}_{\omega\mu\lambda} + R^{\nu}_{\mu\lambda\omega} + R^{\nu}_{\lambda\omega\mu} \equiv 0.$$

Hence the integrability conditions of (7,1) are identically satisfied and consequently (7,1) is completely integrable and its general solution involves n arbitrary constants.

THEOREM (7,2). *Let* $V^{\nu}_{1}, \cdots, V^{\nu}_{n}$ *be* n *linear independent solutions of* (7,1). *Then*[10]

$$(7,5)\qquad W^{\nu} = V^{\mu}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]} \qquad (\mathfrak{a}, \mathfrak{e} = 1, \cdots, n;\ \mathfrak{a} \neq \mathfrak{e})$$

is a vector solution of (7,1).

PROOF. W^{ν} is obviously a vector. Hence to prove our theorem we have to show that it satisfies the equation

$$(7,6)\qquad \partial_a W^{\nu} = W^{\lambda}\partial_{\lambda}T^{\nu}_{a}.$$

From (7,1) we have for

$$(7,7)\qquad \partial_a W^{\nu} = \partial_a V^{\mu}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]} = V^{\lambda}_{[\mathfrak{a}}(\partial_{\lambda}T^{\mu}_{a})\partial_{\mu}V^{\nu}_{\mathfrak{e}]} + V^{\mu}_{[\mathfrak{a}}\partial_{a}\partial_{\mu}V^{\nu}_{\mathfrak{e}]}.$$

On the other hand $(\partial_a\partial_\mu - \partial_\mu\partial_a)V^{\nu}_{\mathfrak{e}} = -(\partial_\mu T^{\lambda}_{a})\partial_\lambda V^{\nu}_{\mathfrak{e}}$ and consequently (7,7) reduces to

$$(7,8)\qquad \begin{aligned} \partial_a W^{\nu} &= (\partial_\lambda T^{\mu}_{a})(V^{\lambda}_{[\mathfrak{a}}\partial_\mu V^{\nu}_{\mathfrak{e}]} - V^{\lambda}_{[\mathfrak{a}}\partial_\mu V^{\nu}_{\mathfrak{e}]}) + V^{\mu}_{[\mathfrak{a}}\partial_\mu\partial_a V^{\nu}_{\mathfrak{e}]} \\ &= V^{\mu}_{[\mathfrak{a}}\partial_\mu V^{\lambda}_{\mathfrak{e}]}\partial_\lambda T^{\nu}_{a} = V^{\mu}_{[\mathfrak{a}}(\partial_\mu V^{\lambda}_{\mathfrak{e}]})\partial_\lambda T^{\nu}_{a} = W^{\lambda}\partial_\lambda T^{\nu}_{a}. \end{aligned}$$

[10] It is assumed that in the equations (7,5)–(7,8) the brackets [ ] do not affect the Greek and Latin indices.

REMARK. The theorem (7,2) shows that the set of all generators $V^\lambda\partial_\lambda$ of infinitesimal transformations (2,1) is closed under the operations of taking linear combination and forming commutators. Hence these generators form a Lie algebra.

8. Special cases.

DEFINITION (8,1). *We say that a transformation* (2,1) *reproduces the family* (V_m) *if*

$$\Delta K^{\nu}_{a_r\cdots a_1} = 0 \qquad (r = 1, 2, \cdots ; K^{\nu}_{a} \equiv T^{\nu}_{a}). \tag{8,1}$$

THEOREM (8,1). *A necessary and sufficient condition that* (V_m) *be reproduced under* (2, 1) *is:* V^ν *is a solution of*

$$L^{\nu}_{a} = 0. \tag{8,2}$$

PROOF. Let (V_m) be reproduced by (2,1). Then from the first of (8,1) we have (8,2) by virtue of (3,1)c. Conversely, if (8,2) is satisfied, the following equations hold (cf. (3,4)c and (4,8))

$$\text{(8,3)}\qquad \begin{aligned} &\text{(a)} \quad {}^*g_{ab} = g'_{ab} + \cdots , \\ &\text{(b)} \quad {}'\Gamma^{c}_{ab} = \Gamma^{c\prime}_{ab} + \cdots . \end{aligned}$$

Consequently we have from (4,1)c and (8,3)b

$$\text{(8,3)} \qquad \text{(c)} \quad {}^*\Gamma^{c}_{ab} = {}'\Gamma^{c}_{ab} + \epsilon Q^{c}_{ab} + \cdots .$$

On the other hand we get by virtue of (8,3)a and (8,3)b

$$\text{(8,3)} \qquad \text{(d)} \quad {}^*\Gamma^{c}_{ab} = {}'\Gamma^{c}_{ab} + \cdots .$$

Comparing (8,3)(c) and (d) we have

$$Q^{c}_{ab} = 0. \tag{8,4}$$

The equations (8,2), (8,4) together with (6,2) and (6,1)a lead to (8,1) for $r=2, 3, \cdots$.

In the next theorem we shall consider a family (V_m) of totally geodesic subspaces, that is, a family for which

$$\text{(8,5)}^{11} \qquad K^{\nu}_{ab} = 0$$

and we prove first the following lemma.

LEMMA (8,1). *If* (8,5) *holds, then* W_{abd} (*cf. the equation* (4,2)) *satisfies*

[11] This equation imposes some conditions on the structure of the large space V_n (cf. the equation (8,12)).

the following equation

$$W_{bad} \equiv W_{abd} = P^{c}_{bda} W_c + D_a D_b W_d \tag{8,6}$$

where P^c_{bda} *is the curvature tensor of* V_m *and* $W_c \equiv T^{\lambda}_c V_\lambda$.

PROOF. Let $W^\nu (N^\nu)$ be the tangential (normal) component of V^ν

$$V^\nu = W^\nu + N^\nu, \qquad W^\nu = V^\lambda T^c_\lambda T^\nu_c, \qquad N_\nu T^\nu_c = 0. \tag{8,7}$$

From (3,5) we have

$$\begin{aligned} W_{ab} = T^{\lambda}_{(a} T^{\mu}_{b)} \nabla_\lambda V_\mu &= T^{\lambda\mu}_{(ab)} \nabla_\lambda W_\mu + T^{\lambda\mu}_{(ab)} \nabla_\lambda N_\mu \\ &= D_{(a} W_{b)} - N_\mu \nabla_{(a} T^{\mu}_{b)}. \end{aligned} \tag{8,8)a}$$

If (8,5) holds then

$$0 = N_\mu D_a T^\mu_b = N_\mu (\nabla_a T^\mu_b - \Gamma^c_{ba} T^\mu_c) = N_\mu \nabla_a T^\mu_b$$

and (8,8)a reduces to

$$W_{ab} = D_{(a} W_{b)}. \tag{8,8)b}$$

Consequently, we get from (4,2)a

$$W_{abd} = D_{[a} D_{d]} W_b + D_{[b} D_{d]} W_a + D_{[b} D_{a]} W_d + D_a D_b W_d. \tag{8,9)a}$$

Because for any vector P_b whatsoever

$$D_{[a} D_{d]} P_b + D_{[d} D_{b]} P_a + D_{[b} D_{a]} P_d = 0$$
$$D_{[b} D_{d]} P_a = 1/2 P^{c}_{bda} P_c$$

where P^c_{dba} is the curvature tensor of V_m, (8,9)a reduces to (8,6).

DEFINITION (8,2). *If* $A = 0$ *is an analytic expression for a property of* (V_m), *then we say that* (2,1) *preserves this property if* $\Delta A = 0$.

THEOREM (8,2). *Let* (V_m) *be a family of totally geodesic subspaces. A necessary and sufficient condition that this property be preserved by* (2,1) *is: The vector* V^ν *is a solution of any one of the following equations*

$$\begin{aligned} &\text{(a)} \quad L^\nu_{ab} = 0, \\ &\text{(b)} \quad D_b D_a V^\nu - V^\omega T^{\mu\lambda}_{ba} R^{\ \nu}_{\omega\mu\lambda} - W^c_{ba} T^\nu_c = 0, \\ &\text{(c)} \quad D_b D_a N^\nu - N^\omega T^{\mu\lambda}_{ba} R^{\ \nu}_{\omega\mu\lambda} = 0, \end{aligned} \tag{8,10}$$

where in the last equation

$$N^\nu = V^\lambda (\delta^\nu_\lambda - T^\nu_c T^c_\lambda).$$

A sufficient but not necessary condition is:

The vector V^ν *is a solution of* (8,2). *If* (8,10) *but not* (8,2) *is satisfied by* V^ν, *then* (V_m) *is not reproduced. If* (8,2) *holds, then* (8,10) *is satisfied and* (V_m) *is reproduced.*

PROOF. According to our assumption the transformation (2,1) carries a family (V_m) in a family (V_m^*). Hence in order to prove our theorem we have only to look for conditions that the equation $K^\nu_{ab}=0$ be satisfied. The necessity and sufficiency of (8,10)a follows at once from (5,8). If (8,5) is satisfied, then we have

$$(8,11)\text{a} \qquad K'^\nu_{ab} = K^\nu_{ab} + \epsilon V^\lambda \partial_\lambda K^\nu_{ab} + \cdots = 0 + \cdots$$

and consequently

$$(8,11)\text{b} \qquad \Delta K^\nu_{ab} = (\tau - \omega) K^\nu_{ab} = \tau K^\nu_{ab}.$$

Hence, we have from (8,11)b, (5,8) and (5,2) that in our case a necessary and sufficient condition for $\Delta K^\nu_{ab}=0$ is that V^ν be a solution of (8,10)b and this equation is (in our case) equivalent to (8,10)a. On the other hand, the integrability conditions of (8,5) are

$$(8,12) \qquad P^{\ \ \ d}_{cba} T^\nu_d = T^{\omega\mu\lambda}_{cba} R^{\ \ \ \ \nu}_{\omega\mu\lambda}$$

and consequently $T^{\omega\mu\lambda}_{cba} R^{\nu}_{\omega\mu\lambda}$ is in (V_m). If we decompose V^ν according to (8,7), then we have from (8,12)

$$(8,13)\text{a} \qquad \begin{aligned} - V^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} &= - W^c T^{\omega\mu\lambda}_{cba} R^{\nu}_{\omega\mu\lambda} - N^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} \\ &= - W^c P^{\ \ \ d}_{cba} T^\nu_d - N^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda}. \end{aligned}$$

On the other hand, if (8,5) holds then

$$(8,13)\text{b} \quad D_b D_a V^\nu = D_b D_a W^c T^\nu_c + D_b D_a N^\nu = (D_b D_a W^c) T^\nu_c + D_b D_a N^\nu.$$

Furthermore, if (8,5) holds we have according to (8,6)

$$(8,13)\text{c} \qquad \begin{aligned} - W^c_{\ ba} T^\nu_c &= - P^{\ \ \ e}_{bda} W_e g^{dc} T^\nu_c - (D_a D_b W^c) T^\nu_c \\ &= (P^{\ \ \ c}_{eab} W^e - D_a D_b W^c) T^\nu_c.^{12} \end{aligned}$$

Comparing (8,10)b with (8,13) we have

$$(8,14)\text{a} \qquad \begin{aligned} D_b D_a V^\nu - V^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} - W^c_{\ ba} T^\nu_c & \\ = T^\nu_d [D_b D_a W^d - W^c P^{\ \ \ d}_{cba} + P^{\ \ \ d}_{cab} W^c - D_a D_b W^d] & \\ + D_b D_a N^\nu - N^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} = 0. & \end{aligned}$$

[12] We use here the well known identities $P_{bcae} = P_{aebc} = -P_{eabc}$.

Since

$$(D_bD_a - D_aD_b)W^d = - P^d_{bac}W^c$$
$$= P^d_{abc}W^c = - (P^d_{bca} + P^d_{cab})W^c$$

we obtain from (8,14)a

$$\text{(8,14)b} \quad D_bD_aV^\nu - V^\omega T^{\mu\lambda}_{ba}R^\nu_{\omega\mu\lambda} - W^c_{ba}T^\nu_c = D_bD_aN^\nu - N^\omega T^{\mu\lambda}_{ba}R^\nu_{\omega\mu\lambda}$$

and this equation proves the statement about (8,10)c. The last part of the theorem follows easily from Theorem (8,1).

REMARK I. The statement about the equation (8,10)b is a generalization of Levi-Civita's result for $m=1$.[13]

REMARK II. Let $m=1$, $n=2$. Then

$$\text{(8,15)} \quad R^\nu_{\omega\mu\lambda} = 2Kg_{\lambda[\omega}\delta^\nu_{\mu]},$$
$$N^\omega T^\mu_1 R^\lambda_1 R^\nu_{\omega\mu\lambda} = - KN^\nu T \qquad (T = g_{\lambda\mu}T^\lambda_1 T^\mu_1).$$

If $\eta^1 \equiv s$, s being the arc of V_1, then $D_1 = T^\lambda_1\nabla_\lambda$ and $T=1$. Moreover, if j^ν is the unit vector in the direction of $D_1T^\nu_1$, then

$$\text{(8,16)} \quad D_1T^\nu_1 = j^\nu k, \qquad D_1j^\nu = - kT^\nu_1,$$

where k is the curvature of V_1. Because we may put $N^\nu = yj^\nu$, y being a factor of proportionality, we have on account of (8,16)

$$\text{(8,17)} \quad D_1N^\nu = y'j^\nu - ykT^\nu_1.$$

Consequently

$$D^2_1N^\nu = y''j^\nu - 2y'kT^\nu_1 - yk'T^\nu_1 - yk^2j^\nu.$$

If (V_1) consists of geodesic lines, we have $k=0$ and

$$\text{(8,18)a} \quad j_\nu D^2_1N^\nu = y''.$$

On the other hand, we obtain from (8,15) for $\eta^1 = s$

$$\text{(8,18)b} \quad j_\nu N^\omega T^\mu_1 T^\lambda_1 R^\nu_{\omega\mu\lambda} = - Ky.$$

Comparing (8,10)c and (8,18) we have

$$y'' + Ky = 0,$$

[13] Levi-Civita, T., *Sullo scostamento geodetico* (Bollettino della Unione Matematica Italiana (1926) pp. 1–4) and *Sur l'écart géodésique* (Math. Ann. vol. 97 (1926) pp. 292–320). For more general metric spaces cf. also Hlavatý, V., *Les courbes de la variété générale à n dimensions* (Mémorial des Sciences Mathématiques, no. 42).

which is the well known equation found by Jacobi.

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