SOME REMARKS ON A THEOREM OF ERDÖS CONCERNING ASYMPTOTIC DENSITY

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In [1] Erdös proves the following:[1]

THEOREM. *If A and B are non-empty sets of positive integers, of asymptotic densities α, β, respectively; and if furthermore*

$$\alpha + \beta \leqq 1, \tag{1}$$

$$1 \in B, \tag{2}$$

$$\alpha \geqq \beta, \tag{3}$$

then for the asymptotic density γ of the sum set $C = A + B$ (consisting of the integers of A, B, and the sums $a+b$, $a \in A$, $b \in B$) we have

$$\gamma \geqq \alpha + \beta/2. \tag{4}$$

More precisely, at least one of the sets

$$\{a + b\}, \qquad \{a, a + 1\} \qquad\qquad a \in A, b \in B \tag{5}$$

must have asymptotic density greater than or equal to $\alpha + \beta/2$.

Due to the asymmetry introduced by the hypothesis (2), the above theorem leaves open the case where $\beta > \alpha$. Actually, the entire theorem as it stands holds without hypothesis (3). This results from the fact that Erdös' proof of the theorem given in [1] uses only the condition $\alpha > \beta/2$. In the contrary case, $2\alpha \leqq \beta$, if we let a_1 be any fixed integer of A, the set $\{b + a_1\}$, $b \in B$, has asymptotic density $\beta \geqq \alpha + \beta/2$; and the theorem follows.

In this note it is proposed to give (using only hypotheses (1) and (2)) a short proof of (4) which does not however yield the more precise information concerning the sets of (5). This proof is based upon the (α, β) hypothesis for Schnirelman density, proved in [2], [3].

If either α or β equals 0, (4) follows trivially. Hence we may assume that both α and β are not 0. Since α is the asymptotic density of A, given any ϵ, $\alpha > \epsilon > 0$, we can find a largest integer m such that

$$A(m) < (\alpha - \epsilon)m. \tag{6}$$

(If $A(z) \geqq (\alpha - \epsilon)z$ for all $z \geqq 1$, we take $m = 0$.) We then consider the

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[1] The definitions of all terms used may be found in [1]. Numbers in brackets refer to the references cited at the end of the paper.

positive terms of the sequence $a_i - m$, $a_i \in A$ and form the set

$$A_\epsilon = \{a_i - m\}.$$

We then have for $x > 0$,

$$A_\epsilon(x) = A(x + m) - A(m)$$
$$\geqq (\alpha - \epsilon)x$$

so that the Schnirelman density of the set A_ϵ is not less than $\alpha - \epsilon$. In particular, then, $1 \in A_\epsilon$, whence $m+1 \in A$.

Similarly, we construct a

$$B_\epsilon = \{b_j - l\}, \qquad b_j \in B, l + 1 \in B,$$

which has Schnirelman density not less than $\beta - \epsilon$. Then, by the (α, β) hypothesis, the Schnirelman density of the set

$$C_\epsilon = \{a_i - m, b_j - l, a_i + b_j - m - l\}$$

is not less than $\alpha + \beta - 2\epsilon$. Thus for any $x > 0$,

$$C_\epsilon(x) \geqq (\alpha + \beta - 2\epsilon)x.$$

Adding $m+l+1$ to the sequence C_ϵ we obtain

$$\tilde{C}_\epsilon = \{a_i + l + 1, b_j + m + 1, a_i + b_j + 1\}$$

where for large x,

$$\tilde{C}_\epsilon(x) \geqq (\alpha + \beta - 2\epsilon)x - (m + l + 1).$$

However, since $l+1 \in B$, $m+1 \in A$,

$$\tilde{C}_\epsilon \subset \{a_i + b_j, a_i + b_j + 1\} = \overline{C}.$$

Hence for large x,

$$\overline{C}(x) \geqq (\alpha + \beta - 2\epsilon)x - (m + l + 1).$$

Next, letting $\nu =$ the number of $a_i + b_j \leqq x$ such that $a_i + b_j + 1 \notin \{a_i + b_j\}$, we consider two cases:

Case 1. $\nu \leqq (\beta/2)x$. Then the number of $\{a_i + b_j\}$ less than or equal to x is greater than or equal to $(\alpha + \beta/2 - 2\epsilon)x - (m + l + 1)$.

Case 2. $\nu > (\beta/2)x$. If $a_i + b_j + 1 \notin \{a_i + b_j\}$, then $a_i + b_j \notin A$ (since $1 \in B$). Hence in this case the set $\{a_i, a_i + b_j\}$ contains (for large x) more than $(\alpha + \beta/2 - \epsilon)x$ integers less than or equal to x.

Thus in any case we obtain that for x large enough the number of integers not greater than x in the set $\{a_i, a_i + b_j\}$ is greater than $(\alpha + \beta/2 - 2\epsilon)x - (m + l + 1)$, which suffices to establish (4).

References

1. P. Erdös, *On the asymptotic density of the sum of two sequences*, Ann. of Math. vol. 43 (1942) pp. 65–68.
2. H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. vol. 43 (1942) pp. 523–527.
3. Emil Artin and Peter Scherk, *On the sum of two sets of integers*, Ann. of Math. vol. 44 (1943) pp. 138–142.

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CERTAIN BEST POSSIBLE RESULTS IN THE THEORY OF SCHNIRELMANN DENSITY

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Let A be a set of distinct non-negative integers, and $A(n)$ the number of positive integers not greater than n in A. Then the Schnirelmann density α of A, referred to below simply as density, is defined as

$$\alpha = \text{g.l.b.} \frac{A(n)}{n} \cdot$$

If A and B are two such sets, the set $C=A+B$ consists of all distinct sums $a+b$ with a in A and b in B. The densities of A, B, and C will be denoted by α, β, and γ respectively.

The $\alpha+\beta$ hypothesis, first proved by Mann [3],[2] states that, if A and B each contain 0, then

$$\gamma \geqq \min (1, \alpha+\beta). \tag{1}$$

Erdös and Niven [2] state that (1) is "a best possible result" without giving details. It is shown below that, under the above conditions, and for any pair (α, β), there exist sets for which the equality sign holds in (1).

Theorem 1. *Let α and β be any real numbers such that*

$$0 \leqq \alpha \leqq 1 \quad \text{and} \quad 0 \leqq \beta \leqq 1.$$

Then there exist sets A and B, each containing 0 *and of densities α and*

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[2] Numbers in brackets refer to the references cited at the end of the paper.