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CERTAIN BEST POSSIBLE RESULTS IN THE THEORY OF SCHNIRELMANN DENSITY

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Let A be a set of distinct non-negative integers, and $A(n)$ the number of positive integers not greater than n in A. Then the Schnirelmann density α of A, referred to below simply as density, is defined as

$$\alpha = \text{g.l.b.} \frac{A(n)}{n} \cdot$$

If A and B are two such sets, the set $C=A+B$ consists of all distinct sums $a+b$ with a in A and b in B. The densities of A, B, and C will be denoted by α, β, and γ respectively.

The $\alpha+\beta$ hypothesis, first proved by Mann [3],[2] states that, if A and B each contain 0, then

$$\gamma \geqq \min (1, \alpha + \beta). \tag{1}$$

Erdös and Niven [2] state that (1) is "a best possible result" without giving details. It is shown below that, under the above conditions, and for any pair (α, β), there exist sets for which the equality sign holds in (1).

THEOREM 1. *Let α and β be any real numbers such that*

$$0 \leqq \alpha \leqq 1 \quad \text{and} \quad 0 \leqq \beta \leqq 1.$$

Then there exist sets A and B, each containing 0 *and of densities α and*

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[2] Numbers in brackets refer to the references cited at the end of the paper.

β respectively, such that the density γ of C is given by

$$\gamma = \min(1, \alpha + \beta). \tag{2}$$

PROOF. For every $n>0$, let a_n be the least positive integer not greater than $\alpha \cdot n!$. Then

$$0 \leqq a_{n+1} - a_n \leqq (n+1)! - n!. \tag{3}$$

Define A as the set consisting of 0, 1, and each of the (possibly vanishing) blocks $n!+1, n!+2, \cdots, n!+a_{n+1}-a_n$. By (3), no two blocks have an element in common; therefore

$$\frac{A(n!)}{n!} = \frac{1 + \sum_{j=1}^{n-1} (a_{j+1} - a_j)}{n!} = \frac{a_n}{n!} \geqq \alpha.$$

Since

$$\lim_{n\to\infty} \frac{a_n}{n!} = \alpha$$

and

$$\frac{A(n)}{n} \geqq \frac{A(m!)}{m!} \geqq \alpha$$

for some m depending upon n, we see that the density of A is α. Define b_n and B similarly in terms of β.

Let $C=A+B$ have density γ, and let $c \leqq n!$ belong to C. Then $c=a+b$ where $a\leqq n!$ and $b\leqq n!$, so that, from the construction of A and B, $a\leqq (n-1)!+a_n-a_{n-1}$ and $b\leqq(n-1)!+b_n-b_{n-1}$. Therefore

$$\begin{aligned} c &\leqq 2(n-1)! + a_n + b_n - a_{n-1} - b_{n-1} \\ &\leqq 2(n-1)! + a_n + b_n \end{aligned}$$

which implies

$$C(n!) \leqq 2(n-1)! + a_n + b_n \tag{4}$$

from which it follows that

$$\lim_{n\to\infty} \frac{C(n!)}{n!} \leqq \alpha + \beta \tag{5}$$

and finally

$$\gamma \leqq \alpha + \beta. \tag{6}$$

Combining (1) and (6), we have (2).[8]

One may ask whether it is possible to weaken the condition that A and B both contain 0 and still to have (1). In this connection, we have the following result.

THEOREM 2. *Let E and F be finite sets of distinct non-negative integers. If, whenever A contains E and B contains F,* (1) *holds, then E and F both contain* 0.

PROOF. Suppose that E does not contain 0. Let N be the largest integer in E or F. Define A as the set of positive integers congruent to any of the integers $1, 2, \cdots, N \pmod{2N+1}$ and B as the set of non-negative integers congruent to any of the integers $0, 1, 2, \cdots, N \pmod{2N+1}$. Then

$$\alpha = \frac{N}{2N+1}, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{2N}{2N+1}$$

so that

$$\gamma < \alpha + \beta < 1$$

which contradicts (1). Similarly, F contains 0.

Finally, we note that (1) cannot be improved by requiring A and B to contain certain finite sets in addition to 0. This follows immediately from Theorem 1, with the observations that the densities of the sets A and B used in the proof would be unchanged by the addition of finite sets, and that (4) would remain valid for sufficiently large n, implying (5) and (6).

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[8] It is possible, by a more detailed argument, to prove (2) directly without the use of (1).