Proofof a Theorem of Jacobi

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Jacobi\(^1\) proved the following theorem:

If \( G(z) \) is defined in \([-1, 1]\), then

\[
I_n = \int_0^\pi G(\cos x) \cos nx \, dx
\]

\[
= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \int_0^\pi G^{(n)}(\cos x) \sin^{2n} x \, dx.
\]

His first proof, for the case in which \( G(z) \) may be expanded in a power series, depends on the formula

\[
\int_0^\pi \cos^p x \cos nx \, dx
\]

\[
= \frac{p(p-1) \cdots (p-n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \int_0^\pi \cos^{2n} x \sin^{2n} x \, dx,
\]

which is itself a special case of (1). His second proof, which assumes nothing about the derivatives of \( G(z) \) of order exceeding \( n \), depends on the lemma

\[
\frac{d^{n-1}}{dz^{n-1}} (1 - z^2)^{(2n-1)/2} = (-1)^{n-1} 1 \cdot 3 \cdot 5 \cdots (2n - 1) \frac{\sin nx}{n},
\]

where \( z = \cos x \). He points out that (3) may also be deduced from (1).

We offer here a short proof by induction which does not involve previous knowledge of (2) or (3). For \( n=1 \) the theorem is seen to be true by an integration by parts. Now

\[
I_{n+1} = \int_0^\pi G(\cos x) \cos x \cos nx \, dx - \int_0^\pi G(\cos x) \sin x \sin nx \, dx
\]

\[
= \int_0^\pi G(\cos x) \cos x \cos nx \, dx - n \int_0^\pi G_1(\cos x) \cos nx \, dx
\]

by integration by parts, where \( G_1(z) \) is an integral of \( G(z) \). Applying the induction hypothesis to \( F(z) = zG(z) - nG_1(z) \) and observing that \( F^{(n)}(z) = zG^{(n)}(z) \), we get

\[ I_{n+1} = \int_0^\pi F(\cos x) \cos nx\,dx \]

\[ = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \int_0^\pi G^{(n)}(\cos x) \sin^2 x \cos x\,dx. \]

Another integration by parts yields

\[ I_{n+1} = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n + 1)} \int_0^\pi G^{(n+1)}(\cos x) \sin^{2n+2} x\,dx, \]

and the theorem is proved.

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