# PROOF OF A THEOREM OF JACOBI

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Jacobi[1] proved the following theorem:

*If $G(z)$ is defined in $[-1, 1]$, then*

$$(1) \qquad \begin{aligned} I_n &\equiv \int_0^\pi G(\cos x) \cos nx\, dx \\ &= \frac{1}{1\cdot 3\cdot 5 \cdots (2n-1)} \int_0^\pi G^{(n)}(\cos x) \sin^{2n} x\, dx. \end{aligned}$$

His first proof, for the case in which $G(z)$ may be expanded in a power series, depends on the formula

$$(2) \qquad \begin{aligned} &\int_0^\pi \cos^p x \cos nx\, dx \\ &\qquad = \frac{p(p-1)\cdots(p-n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} \int_0^\pi \cos^{p-n} x \sin^{2n} x\, dx, \end{aligned}$$

which is itself a special case of (1). His second proof, which assumes nothing about the derivatives of $G(z)$ of order exceeding $n$, depends on the lemma

$$(3) \qquad \frac{d^{n-1}}{dz^{n-1}} (1-z^2)^{(2n-1)/2} = (-1)^{n-1} 1\cdot 3\cdot 5 \cdots (2n-1) \frac{\sin nx}{n},$$

where $z = \cos x$. He points out that (3) may also be deduced from (1).

We offer here a short proof by induction which does not involve previous knowledge of (2) or (3). For $n=1$ the theorem is seen to be true by an integration by parts. Now

$$\begin{aligned} I_{n+1} &= \int_0^\pi G(\cos x) \cos x \cos nx\, dx - \int_0^\pi G(\cos x) \sin x \sin nx\, dx \\ &= \int_0^\pi G(\cos x) \cos x \cos nx\, dx - n \int_0^\pi G_1(\cos x) \cos nx\, dx \end{aligned}$$

by integration by parts, where $G_1(z)$ is an integral of $G(z)$. Applying the induction hypothesis to $F(z) = zG(z) - nG_1(z)$ and observing that $F^{(n)}(z) = zG^{(n)}(z)$, we get

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[1] C. G. J. Jacobi, J. Reine Angew. Math. vol. 15 (1836) pp. 1–26.

$$I_{n+1} = \int_0^\pi F(\cos x) \cos nx\, dx$$

$$= \frac{1}{1\cdot 3\cdot 5 \cdots (2n-1)} \int_0^\pi G^{(n)}(\cos x) \sin^{2n} x \cos x\, dx.$$

Another integration by parts yields

$$I_{n+1} = \frac{1}{1\cdot 3\cdot 5 \cdots (2n+1)} \int_0^\pi G^{(n+1)}(\cos x) \sin^{2n+2} x\, dx,$$

and the theorem is proved.

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