A NOTE ON INTERPOLATION

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Let $f(x)$ be a function of period 2π and let $I_n(x;f)$ be the trigonometric polynomial of order n coinciding with f at $2n+1$ equidistant points

$$(1)\qquad \{x_0, x_1, \cdots, x_{2n}\} = \{x_0, x_0+h, \cdots, x_0+2nh\}, \quad h = h_n = \frac{2\pi}{2n+1},$$

of the interval $[0, 2\pi]$. The familiar formula of Lagrange for $I_n(x;f)$ can be written in the form

$$(2)\qquad I_n(x;f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_n(x-t)d\omega_{2n+1}(t)$$

where $D_n(u) = 2^{-1}\sum_{\nu=-n}^{n} e^{i\nu u}$ is the Dirichlet kernel and $\omega_{2n+1}(t)$ is a step function constant in the interior of the intervals $[x_0+kh, x_0+(k+1)h]$ and having a jump h at the points x_0+kh.[1] Thus $\omega_{2n+1}(t)$ is determined up to an arbitrary additive constant. It is well known that $I_n(x;f)$ need not tend to $f(x)$ even if f is everywhere continuous.

S. Bernstein[2] pointed out that the situation is considerably improved if we consider the trigonometric polynomial of order n assuming at the points x_k the values

$$(3)\qquad 2^{-1}\{f(x_k - h/2) + f(x_k + h/2)\} \qquad (k = 0, 1, \cdots, 2n).$$

In this case, for every bounded f, the interpolating polynomials are contained between the upper and lower bounds of f and converge to f at every point of continuity, the convergence being uniform over every closed interval of continuity. The same holds if we take the polynomials assuming at the points x_k the values

$$(4)\qquad 2^{-2}\{f(x_k - h) + 2f(x_k) + f(x_k + h)\}$$

or

$$(5)\qquad 2^{-3}\{f(x_k - 3h/2) + 3f(x_k - h/2) + 3f(x_k + h/2) + f(x_k + 3h/2)\}$$

Received by the editors June 21, 1949.

[1] This notation seems to have first been used by J. Marcinkiewicz, *On interpolation polynomials* (in Polish), Wiadomosci Matematyczne vol. 39 (1935) pp. 207–221.

[2] S. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange*, Communications of the Mathematical Society of Kharkov vol. 5 pp. 49–57.

($k=0, 1, \cdots, 2n$), and so on.

It is natural to expect that in the cases (4), (5), and so on, the convergence of the interpolating polynomials to f is stronger than is (3), and we shall show that this is so by proving that then even the differentiated polynomials converge to the derivatives of f, provided the latter exist (a fact which has practical uses). More precisely, our result is as follows:

THEOREM. *Let p be a positive integer and let f be a bounded measurable function of period 2π which has at the point x' a derivative of order $j<p$. Let $T_{n,p}(x; f)$ be the trigonometric polynomial of order n which assumes at the points $x_k=x_0+kh$ ($k=0, 1, \cdots, 2n; h=h_n=2\pi/(2n+1)$) the values*

$$(6)\quad f_k = \frac{1}{2^p}\sum_{i=0}^{p} C_{p,i} f\left(x_k + \left(i - \frac{p-1}{2}\right)h\right), \quad \textit{where } C_{p,i} = \frac{p!}{i!\,(p-i)!}$$

Then

$$(7)\quad \lim_{n\to\infty} \frac{d^j}{dx^j} T_{n,p}(x'; f) = f^{(j)}(x') \cdot$$

In this theorem the number $x_0=x_0^{(n)}$ may depend on n. The derivative here is taken in the generalized sense due to Peano and de la Vallee-Poussin, that is, if for small values of $|t|$,

$$f(x+t) = a_0 + \frac{a_1}{1!}t + \frac{a_2}{2!}t^2 + \cdots + \frac{a_{j-1}}{(j-1)!}t^{j-1} + \frac{(a_j+\epsilon_t)}{j!}t^j$$

where the a's are constants and $\epsilon_t \to 0$ as $t\to 0$, then f is said to have a jth generalized derivative at x, which is equal to a_j.

Using (2) one easily sees that

$$\begin{aligned} T_{n,p}(x; f) &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\frac{1}{\pi}\int_0^{2\pi} f\left(t + \left(j - \frac{p-1}{2}\right)h\right) D_n(x-t)\,d\omega_{2n+1}(t) \\ &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\frac{1}{\pi}\int_0^{2\pi} f(t) D_n\left(x + \left(j - \frac{p-1}{2}\right)h - t\right) \\ &\quad \cdot d\omega_{2n+1}\left(t - \left(j - \frac{p-1}{2}\right)h\right). \end{aligned}$$

If p is odd, then $d\omega_{2n+1}(t-(j-(p-1)/2)h)$ may be replaced by $d\omega_{2n+1}(t)$, and if p is even,—by $d\omega_{2n+1}(t+h/2)$. Setting, therefore,

$$\overset{*}{\omega}_{2n+1}(t) = \omega_{2n+1}(t) \qquad \text{if} \quad p \text{ is odd,}$$

and

$$\overset{*}{\omega}_{2n+1}(t) = \omega_{2n+1}(t + h/2) \quad \text{if} \quad p \text{ is even,}$$

we see that

$$T_{n,p}(x; f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_{n,p}(x - t)d\overset{*}{\omega}_{2n+1}(t) \tag{8}$$

where

$$\begin{aligned} D_{n,p}(u) &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}D_n\left(u + \left(j - \frac{p-1}{2}\right)h\right) \\ &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu(u+(j-(p-1)/2)h)} \\ &= \frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu u}\frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}e^{i(j-(p-1)/2)\nu h} \\ &= \frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu u}\cos^p\frac{\nu h}{2} = \frac{1}{2} + \sum_{\nu=1}^{n}\cos \nu u \cos^p\frac{\nu h}{2}. \end{aligned} \tag{9}$$

In formula (8) we can write $d\omega_{2n+1}(t)$ instead of $d\overset{*}{\omega}_{2n+1}(t)$, since this amounts to translating x_0 by $h/2$ for p even, and as we have already stated our x_0 is arbitrary. If the νth partial sum of the polynomial I_n (see (2)) is denoted by $I_{n,\nu}$, we have

$$I_{n,\nu}(x; f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_\nu(x - t)d\omega_{2n+1}(t). \tag{10}$$

Let us apply summation by parts to the last sum in (9). We get

$$D_{n,p}(u) = \sum_{\nu=0}^{n-1} D_\nu(u)\Delta\cos^p\frac{\nu h}{2} + D_n(u)\cos^p\frac{nh}{2},$$

and this inserted in (8) gives (see (10))

$$T_{n,p}(x; f) = \sum_{\nu=0}^{n-1} I_{n,\nu}(x; f)\Delta\cos^p\frac{\nu h}{2} + I_n(x; f)\cos^p\frac{nh}{2}. \tag{11}$$

For any sequence $a_0, a_1, \cdots$, we use the notation $\Delta a_\nu = a_\nu - a_{\nu+1}$, $\Delta^s a = \Delta(\Delta^{s-1}a_\nu)$.

Let $S^k_{n,\nu}$ and $I^k_{n,\nu}$ denote the kth Cesaro sums and the kth Cesaro means, respectively, of the sequence $I_{n,\nu}$ for n fixed. Thus

$$\overset{0}{S}_{n,\nu} = I_{n,\nu},\ \overset{k}{S}_{n,\nu} = \overset{k-1}{S}_{n,0} + \overset{k-1}{S}_{n,1} + \cdots + \overset{k-1}{S}_{n,\nu},$$

$$\overset{k}{I}_{n,\nu} = \frac{\overset{k}{S}_{n,\nu}}{\overset{k}{A}_{\nu}},\ \text{with } \overset{k}{A}_{\nu} = C_{\nu+k,\nu}.$$

Applying to the right side of (11) repeated summation by parts we easily get

$$\begin{aligned} T_{n,p}(x;f) &= \sum_{\nu=0}^{n-2} \overset{1}{S}_{n,\nu}(x;f)\Delta^{2} \cos^{p}\frac{\nu h}{2} + \overset{1}{S}_{n,n-1}(x;f)\Delta\cos^{p}\frac{(n-1)h}{2} \\ &\quad + \overset{0}{S}_{n,n}\cos^{p}\frac{nh}{2} \\ &= \cdots\cdots\cdots\cdots \\ &= \sum_{\nu=0}^{n-p-1} \overset{p}{S}_{n,\nu}(x;f)\Delta^{p+1}\cos^{p}\frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} \overset{p-k}{S}_{n,n-p+k}(x;f)\Delta^{p-k}\cos^{p}\frac{(n-p+k)h}{2} \\ &= \sum_{\nu=0}^{n-p-1} \overset{p}{I}_{n,\nu}(x;f)\overset{p}{A}_{\nu}\Delta^{p+1}\cos^{p}\frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} \overset{p-k}{I}_{n,n-p+k}(x;f)\overset{p-k}{A}_{n-p+k}\Delta^{p-k}\cos^{p}\frac{(n-p+k)h}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^{i}}{dx^{i}} T_{n,p}(x';f) &= \sum_{\nu=0}^{n-p-1} \frac{d^{i}}{dx^{i}}\overset{p}{I}_{n,\nu}(x';f)\overset{p}{A}_{\nu}\Delta^{p+1}\cos^{p}\frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} \frac{d^{i}}{dx^{i}}\overset{p-k}{I}_{n,n-p+k}(x';f)\overset{p-k}{A}_{n-p+k}\Delta^{p-k}\cos^{p}\frac{(n-p+k)h}{2}. \end{aligned} \tag{12}$$

By way of making clear our reasons for expressing $d^{i}T_{n,p}(x';f)/dx^{i}$ in the form (12), we observe first that if f happens to be a trigonometric polynomial P, then the theorem is true. For then, as seen from (6), $T_{n,p}(x;\ f) = (1/2^{p})\sum_{j=0}^{p} C_{p,j}\ P(x+(j-(p-1)/2)h)$. Differentiating this equation j times and letting $n\to\infty$, we get $T_{n,p}^{(j)}(x';f)\to p^{(j)}(x')$. Furthermore we can find a polynomial P which has the same derivatives as f at the point x'. Subtracting P from f, we can suppose that $f(x'+t) = o(|t|^{j})$ if $f^{(j)}(x')$ exists. Our task is then to show that for such f the expression (12) approaches zero as $n\to\infty$.

To this end we shall evaluate the orders of the various terms on the right side of (12) for such f. In particular we shall show that

$$(13)\qquad \frac{d^j}{dx^j} I^p_{n,\nu}(x'; f) \to 0 \quad \text{as } \nu \to \infty, \nu \leqq n, \text{ for } j = 0, 1, \cdots, p-1.$$

Once (13) is established, it will immediately follow that

$$(14)\qquad \frac{d^j}{dx^j} I^{p-k}_{n,\nu}(x'; f) = o(\nu^k), \quad k = 0, 1, \cdots, p; j = 0, 1, \cdots, p-1.$$

For $k=0$, (14) is the assertion (13). We assume that (14) has been proved for $k=0, 1, \cdots, k_1$. Then

$$\frac{d^j}{dx^j} I^{p-(k_1+1)}_{n,\nu}(x'; f) = -\frac{1}{A_\nu^{p-(k_1+1)}} \Delta A^{p-k_1}_{\nu-1} \frac{d^j}{dx^j} I^{p-k_1}_{n,\nu-1}(x'; f)$$

$$= O\left(\frac{A^{p-k_1}_\nu}{A_\nu^{p-k_1-1}} \frac{d^j}{dx^j} I^{p-k_1}_{n,\nu}(x'; f)\right)$$

$$= O(\nu) \cdot o(\nu^{k_1}) = o(\nu^{k_1+1}).$$

This proves (14), given (13).

We shall also need the facts

$$(15)\qquad A^p_\nu \Delta^{p+1} \cos^p \frac{\nu h}{2} = O\left(\frac{1}{n}\right)$$

and

$$(16)\qquad A^{p-k}_{n-p-k} \Delta^{p-k} \cos^p \frac{n-p+k}{2} h = O\left(\frac{1}{n^k}\right).$$

These follow from the fact that $A^p_\nu = O(\nu^p)$ and that $\Delta^{p+1} \cos^p (\nu h/2) = O(1/n^{p+1})$, $\Delta^{p-k} \cos^p (((n-p+k)/2)h) = O(1/n^p)$, the last two being consequences of the mean value theorem and the fact that $\lim_{n\to\infty} ((n-p+k)/2)h_n = \pi/2$.

With this information we show that $d^jT_{n,p}(x'; f)/dx^j \to 0$ for f of the prescribed type, by denoting in the expression (12) the two sums by A and B, where

$$A = \sum_{\nu=0}^{n-p-1} \frac{d^j}{dx^j} I^p_{n,\nu}(x'; f) A^p_\nu \Delta^{p+1} \cos^p \frac{\nu h}{2}$$

$$= \sum_{\nu=0}^{n-p-1} o(1) \cdot O\left(\frac{1}{n}\right) = o(1), \qquad \text{by (13) and (15),}$$

and where

$$B = \sum_{k=0}^{p} \frac{d^j}{dx^j} I_{n,n-p+k}^{p-k}(x'; f) A_{n-p+k}^{p-k} \Delta^{p-k} \cos^p \frac{n-p+k}{2} h$$

$$= \sum_{k=0}^{p} o(n^k) \cdot O\left(\frac{1}{n^k}\right) = o(1), \qquad \text{by (14) and (16).}$$

We see therefore that our theorem will be proved if (13) is established, and we now proceed to that.

We note first that

$$I_{n,\nu}^{p}(x; f) = \frac{1}{\pi} \int_0^{2\pi} f(t) K_\nu^p (x - t) d\omega_{2n+1}(t),$$

where $K_\nu^p(u)$ is the pth Cesaro mean of order ν of the Dirichlet kernel $D_n(u)$. Thus it is clear that

$$\frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) = \frac{1}{\pi} \int_0^{2\pi} f(t) \frac{d^j}{dx^j} K_\nu^p (x - t) d\omega_{2n+1}(t)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x' + t) \frac{d^j}{dt^j} K_\nu^p (t) d\omega_{2n+1}(x' + t).$$

Hence we have

$$\left| \frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) \right| \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} \epsilon_t |t|^j \cdot \left| \frac{d^j}{dt^j} K_\nu^p(t) \right| d\omega_{2n+1}(x' + t), \tag{17}$$

where ϵ_t is non-negative and approaches zero with t.

It is shown in Zygmund, *Trigonometrical series*, p. 259, that

$$\left| \frac{d^j}{dt^j} K_\nu^p (t) \right| \leqq C\nu^{j+1} \tag{18}$$

and that $|d^j K_\nu^p(t)/dt^j|$ is less than the sum of the three expressions

$$C_1 \sum_{k=1}^{s} \frac{\nu^{-k}}{|t|^{j+1+k}}, \qquad C_2 \sum_{\mu=0}^{j} \frac{\nu^{\mu-p}}{|t|^{p+1+j-\mu}}, \qquad C_3 \sum_{\mu=0}^{j} \frac{\nu^{-s+\mu}}{|t|^{s+j-\mu+1}},$$

where the C's are constants, s is any integer greater than $p+j$, and $|t| \leqq \pi$. It follows easily from this fact that

$$|t|^j \cdot \left| \frac{d^j}{dt^j} K_\nu^p (t) \right| \leqq C \frac{1}{\nu t^2} \qquad \text{for } \frac{1}{\nu} \leqq |t| \leqq \pi. \tag{19}$$

We now use (18) and (19) to prove

(20) $$\frac{1}{\pi}\int_{-\pi}^{\pi}|t|^j\cdot\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t)\leqq C$$

by splitting the integral in (20) into two parts, $\int_{-1/\nu}^{1/\nu}+\int_R=A+B$, where R is the complement of $[-1/\nu,\,1/\nu]$ in $[-\pi,\,\pi]$.

$$A=\frac{1}{\pi}\int_{-1/\nu}^{1/\nu}|t|^j\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t)$$
$$\leqq C\nu^{j+1}\int_{-1/\nu}^{1/\nu}|t|^j d\omega_{2n+1}(x'+t).$$

Hence

$$A\leqq C\nu^{j+1}\cdot\frac{1}{\nu^j}\int_{-1/\nu}^{1/\nu}d\omega_{2n+1}(x'+t)$$
$$=C\nu\cdot\frac{2\pi}{2n+1}\cdot\frac{2n+1}{2\pi\nu}=C.$$

Furthermore

$$B=\frac{1}{\pi}\int_R|t|^j\cdot\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t),$$

so that by (19),

$$B\leqq C\int_R\frac{1}{\nu t^2}d\omega_{2n+1}(x'+t)=C\frac{1}{\nu}\sum_{x_k\in R}\frac{1}{(x'+x_k)^2}\cdot\frac{2\pi}{2n+1}\cdot$$

Now

$$\sum_{x_k\in R}\frac{1}{(x'+x_k)^2}\leqq 2\sum_{j=0}^{\infty}\frac{1}{(1/\nu+(2\pi/(2n+1))j)^2}$$
$$=2\left[\nu^2+\sum_{j=1}^{\infty}\frac{1}{(1/\nu+(2\pi/(2n+1))j)^2}\right]$$
$$\leqq 2\left[\nu^2+\frac{2n+1}{2\pi}\int_0^\infty\frac{du}{(1/\nu+u)^2}\right]$$
$$=2\left[\nu^2+\frac{2n+1}{2\pi}\nu\right].$$

Thus

$$B\leqq C\frac{1}{\nu}\cdot\frac{2\pi}{2n+1}\cdot 2\left[\nu^2+\frac{2n+1}{2\pi}\nu\right]\leqq C',$$

and (20) is proved.

To see that the integral in (17) can be made arbitrarily small by taking ν sufficiently large, we choose a $\delta>0$ small enough so that $\epsilon_t<\epsilon$ for $|t|<\delta$, and split the integral into two parts $\int_{-\delta}^{\delta}+\int_S=A+B$, S being the complement of $[-\delta, \delta]$ in $[-\pi, \pi]$. Since by (19), $|t|^j|d^jK_\nu^p(t)/dt^j|$ approaches zero uniformly in S, we see that $B=o(1)$ as $\nu\rightarrow\infty$. Furthermore

$$A = \frac{1}{\pi}\int_{-\delta}^{\delta}\epsilon_t|t|^j\cdot\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t)$$

$$\leqq \frac{\epsilon}{\pi}\int_{-\pi}^{\pi}|t|^j\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t) \leqq \epsilon C, \qquad \text{by (20).}$$

Thus $A+B$ can be made arbitrarily small by choosing ν sufficiently large.

Thus (13) is established and our theorem is proved.

It is not difficult to see that if f has a jth derivative $(j<p)$ at each point of an interval $a\leqq x\leqq b$ and that derivative is bounded (continuous) there, then the expression under the limit sign in (7) is uniformly bounded (convergent) in that interval.

It should be noted that a slight alteration in the proof of (20) yields the fact that if f has a pth derivative at the point x', then

$$\frac{d^p}{dx^p}T_{n,p}(x';f) = o(\log n).$$

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