A NOTE ON INTERPOLATION

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Let \( f(x) \) be a function of period \( 2\pi \) and let \( I_n(x; f) \) be the trigonometric polynomial of order \( n \) coinciding with \( f \) at \( 2n+1 \) equidistant points

\[
\{x_0, x_1, \ldots, x_{2n}\} = \{x_0, x_0 + h, \ldots, x_0 + 2nh\},
\]

(1)

\[
h = h_n = \frac{2\pi}{2n + 1},
\]

of the interval \([0, 2\pi]\). The familiar formula of Lagrange for \( I_n(x; f) \) can be written in the form

\[
I_n(x; f) = \frac{1}{\pi} \int_0^{2\pi} f(t) D_n(x - t) d\omega_{2n+1}(t)
\]

(2)

where \( D_n(u) = 2^{-1} \sum_{r=-n}^{n} e^{iur} \) is the Dirichlet kernel and \( \omega_{2n+1}(t) \) is a step function constant in the interior of the intervals \([x_0 + kh, \ x_0 + (k+1)h]\) and having a jump \( h \) at the points \( x_0 + kh \).\(^1\) Thus \( \omega_{2n+1}(t) \) is determined up to an arbitrary additive constant. It is well known that \( I_n(x; f) \) need not tend to \( f(x) \) even if \( f \) is everywhere continuous.

S. Bernstein\(^2\) pointed out that the situation is considerably improved if we consider the trigonometric polynomial of order \( n \) assuming at the points \( x_k \) the values

\[
2^{-1} \{f(x_k - h/2) + f(x_k + h/2)\} \quad (k = 0, 1, \ldots, 2n).
\]

(3)

In this case, for every bounded \( f \), the interpolating polynomials are contained between the upper and lower bounds of \( f \) and converge to \( f \) at every point of continuity, the convergence being uniform over every closed interval of continuity. The same holds if we take the polynomials assuming at the points \( x_k \) the values

\[
2^{-2} \{f(x_k - h) + 2f(x_k) + f(x_k + h)\}
\]

(4)

or

\[
2^{-3} \{f(x_k - 3h/2) + 3f(x_k - h/2) + 3f(x_k + h/2) + f(x_k + 3h/2)\}
\]

(5)

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\(^1\) This notation seems to have first been used by J. Marcinkiewicz, On interpolation polynomials (in Polish), Wiadomosci Matematyczne vol. 39 (1935) pp. 207-221.

\(^2\) S. Bernstein, Sur une modification de la formule d’interpolation de Lagrange, Communications of the Mathematical Society of Kharkov vol. 5 pp. 49-57.
It is natural to expect that in the cases (4), (5), and so on, the convergence of the interpolating polynomials to \( f \) is stronger than is (3), and we shall show that this is so by proving that then even the differentiated polynomials converge to the derivatives of \( f \), provided the latter exist (a fact which has practical uses). More precisely, our result is as follows:

**Theorem.** Let \( p \) be a positive integer and let \( f \) be a bounded measurable function of period \( 2\pi \) which has at the point \( x' \) a derivative of order \( j < p \). Let \( T_{n,p}(x'; f) \) be the trigonometric polynomial of order \( n \) which assumes at the points \( x_k = x_0 + kh \ (k = 0, 1, \ldots, 2n; h = h_n = 2\pi/(2n+1)) \) the values

\[
(6) \quad f_k = \frac{1}{2^p} \sum_{i=0}^{p} C_{p,i} f \left( x_k + \left( i - \frac{p-1}{2} \right) h \right), \quad \text{where} \quad C_{p,i} = \frac{p!}{i! (p-i)!}.
\]

Then

\[
(7) \quad \lim_{n \to \infty} \frac{d^j}{dx^j} T_{n,p}(x'; f) = f^{(j)}(x').
\]

In this theorem the number \( x_0 = x_0^{(a)} \) may depend on \( n \). The derivative here is taken in the generalized sense due to Peano and de la Vallee-Poussin, that is, if for small values of \( |t| \),

\[
f(x + t) = a_0 + a_1 t + \frac{a_2}{2!} t^2 + \cdots + \frac{a_{j-1}}{(j-1)!} t^{j-1} + \frac{(a_j + \epsilon_t)}{j!},
\]

where the \( a \)'s are constants and \( \epsilon_t \to 0 \) as \( t \to 0 \), then \( f \) is said to have a \( j \)th generalized derivative at \( x \), which is equal to \( a_j \).

Using (2) one easily sees that

\[
T_{n,p}(x; f) = \frac{1}{2^p} \sum_{j=0}^{p} C_{p,j} \frac{1}{\pi} \int_0^{2\pi} f \left( \left( j - \frac{p-1}{2} \right) h \right) D_n(x - t) d\omega_{2n+1}(t)
\]

\[
= \frac{1}{2^p} \sum_{j=0}^{p} C_{p,j} \frac{1}{\pi} \int_0^{2\pi} f(t) D_n \left( x + \left( j - \frac{p-1}{2} \right) h - t \right) d\omega_{2n+1} \left( t - \left( j - \frac{p-1}{2} \right) h \right).
\]

If \( p \) is odd, then \( d\omega_{2n+1}(t - (j - (p-1)/2)h) \) may be replaced by \( d\omega_{2n+1}(t) \), and if \( p \) is even,—by \( d\omega_{2n+1}(t + h/2) \). Setting, therefore,
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\[ \omega_{2n+1}(t) = \omega_{2n+1}(t) \quad \text{if } \ p \text{ is odd}, \]

and

\[ \omega_{2n+1}^*(t) = \omega_{2n+1}(t + h/2) \quad \text{if } \ p \text{ is even}, \]

we see that

\[ \tag{8} T_{n,p}(x; f) = \frac{1}{\pi} \int_0^{2\pi} f(t)D_{n,p}(x - t)d\omega_{2n+1}^*(t) \]

where

\[ D_{n,p}(u) = \frac{1}{2^p} \sum_{j=0}^{p} C_{p,j} D_n \left( u + \left( j - \frac{p-1}{2} \right) h \right) \]

\[ = \frac{1}{2^p} \sum_{j=0}^{p} C_{p,j} \frac{1}{2} \sum_{r=-n}^{n} e^{i\sigma(u+(j-(p-1)/2))h} \]

\[ = \frac{1}{2^p} \sum_{r=-n}^{n} e^{i\sigma u} \cos \frac{v h}{2} = \frac{1}{2} + \sum_{r=1}^{n} \cos \nu u \cos \frac{v h}{2}. \]

In formula (8) we can write \( d\omega_{2n+1}(t) \) instead of \( d\omega_{2n+1}^*(t) \), since this amounts to translating \( x_0 \) by \( h/2 \) for \( p \) even, and as we have already stated our \( x_0 \) is arbitrary. If the \( v \)th partial sum of the polynomial \( I_n \) (see (2)) is denoted by \( I_{n,v} \), we have

\[ \tag{10} I_{n,v}(x; f) = \frac{1}{\pi} \int_0^{2\pi} f(t)D_v(x - t)d\omega_{2n+1}(t). \]

Let us apply summation by parts to the last sum in (9). We get

\[ D_{n,p}(u) = \sum_{r=0}^{n-1} D_r(u) \Delta \cos^p \frac{\nu h}{2} + D_n(u) \cos^p \frac{nh}{2}, \]

and this inserted in (8) gives (see (10))

\[ \tag{11} T_{n,p}(x; f) = \sum_{r=0}^{n-1} I_{n,v}(x; f) \Delta \cos^p \frac{\nu h}{2} + I_n(x; f) \cos^p \frac{nh}{2}. \]

For any sequence \( a_0, a_1, \ldots, \) we use the notation \( \Delta a_r = a_r - a_{r+1}, \)

\( \Delta a = \Delta^{r-1} a_r. \)

Let \( S_{n,v}^k \) and \( I_{n,v}^k \) denote the \( k \)th Cesaro sums and the \( k \)th Cesaro means, respectively, of the sequence \( I_{n,v} \) for \( n \) fixed. Thus
$S_{n,r}^0 = I_{n,r}, S_{n,r}^k = S_{n,0}^k + S_{n,1}^k + \cdots + S_{n,r}^k,$

$I_{n,r}^k = \frac{S_{n,r}^k}{A_r^k},$ with $A_r^k = C_{r+k,r}.$

Applying to the right side of (11) repeated summation by parts we easily get

$$T_{n,p}(x; f) = \sum_{r=0}^{n-2} S_{n,r}(x; f)\Delta^r \cos \frac{p \nu h}{2} + S_{n,n-1}(x; f)\Delta \cos \frac{(n-1)h}{2}$$

$$+ S_{n,n}^0 \cos \frac{nh}{2}$$

$$= \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$= \sum_{r=0}^{n-p-1} S_{n,r}(x; f)\Delta^{r+1} \cos \frac{p \nu h}{2}$$

$$+ \sum_{k=0}^{p} S_{n,n-p+k}(x; f)\Delta^{p-k} \cos \frac{p(n-p+k)h}{2}$$

$$= \sum_{r=0}^{n-1} I_{n,r}(x; f)A_r^{p+1} \Delta^{p+1} \cos \frac{p \nu h}{2}$$

$$+ \sum_{k=0}^{p} I_{n,n-p+k}(x; f)A_{n-p+k}^{p-k} \Delta^{p-k} \cos \frac{p(n-p+k)h}{2}.$$ 

Hence

$$\frac{d^i}{dx^i} T_{n,p}(x'; f) = \sum_{r=0}^{n-p-1} \frac{d^i}{dx^i} I_{n,r}(x'; f)A_r^{p+1} \Delta^{p+1} \cos \frac{p \nu h}{2}$$

$$+ \sum_{k=0}^{p} \frac{d^i}{dx^i} I_{n,n-p+k}(x'; f)A_{n-p+k}^{p-k} \Delta^{p-k} \cos \frac{p(n-p+k)h}{2}.$$ 

By way of making clear our reasons for expressing $d^i T_{n,p}(x'; f)/dx^i$ in the form (12), we observe first that if $f$ happens to be a trigonometric polynomial $P$, then the theorem is true. For then, as seen from (6), $T_{n,p}(x; f) = (1/2\pi) \sum_{j=0}^{\nu} C_{p,j} P(x + (j - (p-1)/2)h).$ Differentiating this equation $j$ times and letting $n \to \infty$, we get $T^{(j)}_{n,p}(x'; f) \to \phi^{(j)}(x').$ Furthermore we can find a polynomial $P$ which has the same derivatives as $f$ at the point $x'$. Subtracting $P$ from $f$, we can suppose that $f(x'+t) = o(|t|)$ if $f^{(j)}(x')$ exists. Our task is then to show that for such $f$ the expression (12) approaches zero as $n \to \infty$. 

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To this end we shall evaluate the orders of the various terms on the right side of (12) for such $f$. In particular we shall show that

$$\frac{d^i}{dx^i} I_{n,r}^p(x'; f) \to 0 \quad \text{as} \quad \nu \to \infty, \; \nu \leq n, \; \text{for} \; j = 0, 1, \cdots, \rho - 1.\quad (13)$$

Once (13) is established, it will immediately follow that

$$\frac{d^i}{dx^i} I_{n,r}^{p-k} (x'; f) = o(\nu), \; k = 0, 1, \cdots, \rho; \; j = 0, 1, \cdots, \rho - 1.\quad (14)$$

For $k = 0$, (14) is the assertion (13). We assume that (14) has been proved for $k = 0, 1, \cdots, k_1$. Then

$$\frac{d^i}{dx^i} I_{n,r}^{p-(k_1+1)} (x'; f) = \frac{1}{A_{r}^{p-(k_1+1)}} \Delta A_{r}^{p-k_1} \frac{d^i}{dx^i} I_{n,r-1}^{p-k_1}(x'; f)$$

$$= O \left( \frac{A_{r}^{p-k_1}}{A_{r}^{p-k-1}} \frac{d^i}{dx^i} I_{n,r}^{p-k_1} (x'; f) \right)$$

$$= O(\nu) \cdot o(\nu^{k_1}) = o(\nu^{k_1+1}).$$

This proves (14), given (13).

We shall also need the facts

$$A_{r}^p \Delta^{p+1} \cos \frac{\nu h}{2} = O \left( \frac{1}{n} \right)\quad (15)$$

and

$$A_{n-p-k}^{p-k} \Delta^{p-k} \cos \frac{n - p + k}{2} h = O \left( \frac{1}{n^k} \right).\quad (16)$$

These follow from the fact that $A_{r}^p = 0(\nu^p)$ and that $\Delta^{p+1} \cos \nu h/2 = O(1/n^{p+1})$, $\Delta^{p-k} \cos ((n - p + k)/2) h = O(1/n^k)$, the last two being consequences of the mean value theorem and the fact that $\lim \nu \to \infty ((n - p + k)/2) h = \pi/2$.

With this information we show that $d^i T_{n,r}(x'; f)/dx^i \to 0$ for $f$ of the prescribed type, by denoting in the expression (12) the two sums by $A$ and $B$, where

$$A = \sum_{r=0}^{n-p-1} \frac{d^i}{dx^i} I_{n,r}^p(x'; f) A_{r}^p \Delta^{p+1} \cos \frac{\nu h}{2}$$

$$= \sum_{r=0}^{n-p-1} o(1) \cdot O \left( \frac{1}{n} \right) = o(1),$$

by (13) and (15),
and where

\[ B = \sum_{k=0}^{p} \frac{d^i}{dx^i} \int_{n-p+k}^{n} I_{n-p+k}(x'; f) A_{n-p+k}^{p-k} \Delta^{p-k} \cos \frac{n-p+k}{2} h \]

\[ = \sum_{k=0}^{p} o(n^k) \cdot O \left( \frac{1}{n^k} \right) = o(1), \quad \text{by (14) and (16)}. \]

We see therefore that our theorem will be proved if (13) is established, and we now proceed to that.

We note first that

\[ I_{n,v}(x; f) = \frac{1}{\pi} \int_{0}^{2\pi} f(t) K_{v}^{p}(x - t) d\omega_{2n+1}(t), \]

where \( K_{v}^{p}(u) \) is the \( p \)th Cesaro mean of order \( v \) of the Dirichlet kernel \( D_{n}(u) \). Thus it is clear that

\[ \frac{d}{dx} I_{n,v}(x'; f) = \frac{1}{\pi} \int_{0}^{2\pi} f(t) \frac{d}{dx} K_{v}^{p}(x - t) d\omega_{2n+1}(t) \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x' + t) \frac{d}{dt} K_{v}^{p}(t) d\omega_{2n+1}(x' + t). \]

Hence we have

\[ \frac{d}{dx} I_{n,v}(x'; f) = - \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) - f(x') K_{v}^{p}(t) d\omega_{2n+1}(x' + t), \]

where \( \epsilon_{t} \) is non-negative and approaches zero with \( t \).

It is shown in Zygmund, Trigonometrical series, p. 259, that

\[ -\frac{d}{dt} K_{v}^{p}(t) \leq C_{v}^{j+1} \]

and that \( |d^i K_{v}^{p}(t)|/dt^i \) is less than the sum of the three expressions

\[ C_{1} \sum_{k=1}^{i} \frac{\nu^{i-k}}{|t|^{j+1+k}}, \quad C_{2} \sum_{\mu=0}^{i} \frac{\nu^{i-j-\mu}}{|t|^{j+\mu+1}}, \quad C_{3} \sum_{\mu=0}^{j} \frac{\nu^{i-j-\mu}}{|t|^{j+\mu+1}}, \]

where the \( C \)'s are constants, \( s \) is any integer greater than \( p+j \), and \( |t| \leq \pi \). It follows easily from this fact that

\[ |t| \cdot \frac{d}{dt} K_{v}^{p}(t) \leq C \frac{1}{\nu t^2} \quad \text{for } \frac{1}{\nu} \leq |t| \leq \pi. \]

We now use (18) and (19) to prove...
by splitting the integral in (20) into two parts, \( \int_{-1/\nu}^{1/\nu} + \int_R = A + B \), where \( R \) is the complement of \([-1/\nu, 1/\nu]\) in \([-\pi, \pi]\).

\[
A = \frac{1}{\pi} \int_{-1/\nu}^{1/\nu} |t|^i \left| \frac{d^i}{dt^i} K_{\nu}^p(t) \right| d\omega_{2n+1}(x' + t) 
\leq C \nu^{i+1} \int_{-1/\nu}^{1/\nu} |t|^i d\omega_{2n+1}(x' + t).
\]

Hence

\[
A \leq C \nu^{i+1} \frac{1}{\nu^j} \int_{-1/\nu}^{1/\nu} d\omega_{2n+1}(x' + t) = C \nu \cdot \frac{2\pi}{2n + 1} = C.
\]

Furthermore

\[
B = \frac{1}{\pi} \int_R |t|^i \left| \frac{d^i}{dt^i} K_{\nu}^p(t) \right| d\omega_{2n+1}(x' + t),
\]

so that by (19),

\[
B \leq C \int_R \frac{1}{\nu t^2} d\omega_{2n+1}(x' + t) = C \frac{1}{\nu} \sum_{x_k \in R} \frac{1}{(x' + x_k)^2} \frac{2\pi}{2n + 1}.
\]

Now

\[
\sum_{x_k \in R} \frac{1}{(x' + x_k)^2} \leq 2 \sum_{j=0}^{\infty} \frac{1}{(1/\nu + (2\pi/(2n + 1)))^j} = 2 \left[ \nu^2 + \sum_{j=1}^{\infty} \frac{1}{(1/\nu + (2\pi/(2n + 1)))^j} \right] 
\leq 2 \left[ \nu^2 + \frac{2n + 1}{2\pi} \int_0^\infty \frac{du}{(1/\nu + u)^2} \right] 
= 2 \left[ \nu^2 + \frac{2n + 1}{2\pi} \right].
\]

Thus

\[
B \leq C \frac{1}{\nu} \cdot \frac{2\pi}{2n + 1} \cdot 2 \left[ \nu^2 + \frac{2n + 1}{2\pi} \right] \leq C'.
\]

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and (20) is proved.

To see that the integral in (17) can be made arbitrarily small by taking \( v \) sufficiently large, we choose a \( \delta > 0 \) small enough so that \( \epsilon_t < \epsilon \) for \( |t| < \delta \), and split the integral into two parts \( \int_{-\delta}^{\delta} + \int_{S} = A + B \), \( S \) being the complement of \( [-\delta, \delta] \) in \( [-\pi, \pi] \). Since by (19), 
\[
|t|^i |d^i K^p(t)/dt^i| \text{ approaches zero uniformly in } S,
\]
we see that \( B = o(1) \) as \( v \to \infty \). Furthermore
\[
A = \frac{1}{\pi} \int_{-\delta}^{\delta} t |d^i K^p(t)| d\omega_{2n+1}(x' + t)
\]
\[
\leq \frac{\epsilon}{\pi} \int_{-\delta}^{\delta} t |d^i K^p(t)| d\omega_{2n+1}(x' + t) \leq \epsilon C, \text{ by (20)}.
\]

Thus \( A + B \) can be made arbitrarily small by choosing \( v \) sufficiently large.

Thus (13) is established and our theorem is proved.

It is not difficult to see that if \( f \) has a \( j \)th derivative \( (j < p) \) at each point of an interval \( a \leq x \leq b \) and that derivative is bounded (continuous) there, then the expression under the limit sign in (7) is uniformly bounded (convergent) in that interval.

It should be noted that a slight alteration in the proof of (20) yields the fact that if \( f \) has a \( p \)th derivative at the point \( x' \), then
\[
\frac{d^p}{dx^p} T_{n,p}(x';f) = o(\log n).
\]