References


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A NOTE ON THE CHARACTERISTIC POLYNOMIALS OF CERTAIN MATRICES

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The purpose of this note is to establish the following theorem on matrices with elements in an arbitrary field $\mathbb{F}$. In particular, $I$ denotes the $n \times n$ identity matrix; 0 is used for the zero matrix, square or rectangular, its dimensions being clear in each case from the context. The determinant of an $n \times n$ matrix $M(\lambda)$ with elements in $\mathbb{F}(\lambda)$, the ring of polynomials in $\lambda$ with coefficients in $\mathbb{F}$, is denoted by $|M(\lambda)|$; in particular, if $M$ is an $n \times n$ matrix with elements in $\mathbb{F}$, then $|\lambda I - M|$ is the characteristic polynomial of $M$.

Theorem. If $A$ is an $n \times m$ matrix, and $D$ is an $n \times n$ matrix, then in order that $|\lambda I - AB| = |\lambda I - AB - D|$ for arbitrary $m \times n$ matrices $B$ it is necessary and sufficient that $D$ be nilpotent and $DA = 0$. In particular, if $C$ is an $m \times n$ matrix, then in order that $|\lambda I - AB| = |\lambda I - A(B + C)|$ for arbitrary $m \times n$ matrices $B$ it is necessary and sufficient that $ACA = 0$.

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1 In the original manuscript of this paper attention was restricted to $\mathbb{C}$ the field of complex numbers. The proof there given of the fact that the nilpotency of $D$ and $DA = 0$ imply $|\lambda I - AB| = |\lambda I - AB - D|$, for arbitrary $B$, was valid for arbitrary fields $\mathbb{F}$. The proof here given of the necessity of $DA = 0$ for $\mathbb{F}$ an arbitrary field was suggested by the referee.
An alternate proof of the sufficiency part of the statement in the last sentence of the theorem has been obtained recently by Parker,\(^2\) in generalizing an earlier result of A. Brauer.\(^3\)

The principal tool used in the proof of the above theorem is the fact that a matrix \(D\) is nilpotent if and only if \(\lambda^n = |\lambda I - D|\). That \(\lambda^n = |\lambda I - D|\) implies the nilpotency of \(D\) follows from the Cayley-Hamilton theorem. On the other hand, if \(D\) is nilpotent of index \(s\), then \(|\lambda I - D^t| = \lambda^n\) for \(r \geq s\); while if \(\lambda^n = |\lambda I - D^t|\) for \(r \geq t + 1 > 1\), then \(|\lambda I - D^t| \cdot |\lambda I + D^t| = |\lambda^2 I - D^{2t}| = |\lambda^2 I| = \lambda^{2n}\), so that \(\lambda I - D^t = \lambda^n\), and by induction \(|\lambda I - D| = \lambda^n|\).

The sufficiency part of the statement in the first sentence of the above theorem is established by noting that if \(D\) is nilpotent and \(D A = 0\), then \(\lambda^n |\lambda I - A B| = (\lambda I - D)(\lambda I - A B)| = |\lambda^2 I - \lambda(A B + D)| = \lambda^n|\lambda I - A B - D|\), and \(|\lambda I - A B| = |\lambda I - A B - D|\) for arbitrary \(m \times n\) matrices \(B\).

On the other hand, suppose that \(|\lambda I - A B| = |\lambda I - A B - D|\) for all \(m \times n\) matrices \(B\). The choice \(B = 0\) gives \(|\lambda I - D| = \lambda^n\), and hence the nilpotency of \(D\). Then \(|\lambda^2 I - \lambda(A B + D)| = \lambda^n|\lambda I - (A B + D)| = |\lambda I - D| \cdot |\lambda I - A B| = |\lambda^2 I - \lambda(A B + D) + D A B|\) for arbitrary \(m \times n\) matrices \(B\). Now if \(D A = K = \{K_{ij}\} (i = 1, \ldots, n; \beta = 1, \ldots, m)\) has a nonzero element \(K_{pq}\), then the choice \(B_0 = ||\delta_{q}^{\alpha} \delta_{p}^\beta K^{-1}_{pq}|| (\alpha = 1, \ldots, m; j = 1, \ldots, n)\); gives \(D A B_0 = G = ||G_{ij}|| (i, j = 1, \ldots, n)\) with \(G_{ij} = 0\) if \(j \neq p\), \(G_{pp} = 1\). If \(A B_0 + D = H = ||H_{ij}|| (i, j = 1, \ldots, n)\), then from the above, \(|\lambda^2 I - \lambda H| = \lambda^2 - \lambda H + G|\). This is impossible, however, since \(|\lambda^2 I - \lambda H + G| = |\lambda^2 I - \lambda H| + |R(\lambda)|\), where \(R_{ij}(\lambda) = \lambda^2 \delta_{ij} - \lambda H_{ij}\) if \(j \neq p\), \(R_{pp}(\lambda) = G_{pp}\), and consequently \(|R(\lambda)|\) is a non-zero polynomial \(\lambda^{2n-2} + \) terms of lower degree. Hence \(D A = 0\), and the statement of the first sentence of the theorem is proved.

The second sentence of the theorem follows from the first part, together with the remark that if \(D = A C\), then \(0 = D A = A C A\) implies \(0 = D A C = D^4\) so that \(D\) is nilpotent.

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