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A NOTE ON THE CHARACTERISTIC POLYNOMIALS OF CERTAIN MATRICES

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The purpose of this note is to establish the following theorem on matrices with elements in an arbitrary field $\mathfrak{F}$.[1] In particular, I denotes the $n\times n$ identity matrix; 0 is used for the zero matrix, square or rectangular, its dimensions being clear in each case from the context. The determinant of an $n\times n$ matrix $M(\lambda)$ with elements in $\mathfrak{F}(\lambda)$, the ring of polynomials in λ with coefficients in $\mathfrak{F}$, is denoted by $|M(\lambda)|$; in particular, if M is an $n\times n$ matrix with elements in $\mathfrak{F}$, then $|\lambda I-M|$ is the characteristic polynomial of M.

THEOREM. *If A is an $n\times m$ matrix, and D is an $n\times n$ matrix, then in order that $|\lambda I-AB|=|\lambda I-AB-D|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that D be nilpotent and $DA=0$. In particular, if C is an $m\times n$ matrix, then in order that $|\lambda I-AB|=|\lambda I-A(B+C)|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that $ACA=0$.*

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[1] In the original manuscript of this paper attention was restricted to $\mathfrak{F}$ the field of complex numbers. The proof there given of the fact that the nilpotency of D and $DA=0$ imply $|\lambda I-AB|=|\lambda I-AB-D|$, for arbitrary B, was valid for arbitrary fields $\mathfrak{F}$. The proof here given of the necessity of $DA=0$ for $\mathfrak{F}$ an arbitrary field was suggested by the referee.

An alternate proof of the sufficiency part of the statement in the last sentence of the theorem has been obtained recently by Parker,[2] in generalizing an earlier result of A. Brauer.[3]

The principal tool used in the proof of the above theorem is the fact that a matrix D is nilpotent if and only if $\lambda^n = |\lambda I - D|$. That $\lambda^n = |\lambda I - D|$ implies the nilpotency of D follows from the Cayley-Hamilton theorem. On the other hand, if D is nilpotent of index s, then $|\lambda I - D^r| = \lambda^n$ for $r \geqq s$, while if $\lambda^n = |\lambda I - D^r|$ for $r \geqq t+1 > 1$, then $|\lambda I - D^t| \cdot |\lambda I + D^t| = |\lambda^2 I - D^{2t}| = |\lambda^2 I| = \lambda^{2n}$, so that $|\lambda I - D^t| = \lambda^n$, and by induction $|\lambda I - D| = \lambda^n$.

The sufficiency part of the statement in the first sentence of the above theorem is established by noting that if D is nilpotent and $DA = 0$, then $\lambda^n |\lambda I - AB| = |(\lambda I - D)(\lambda I - AB)| = |\lambda^2 I - \lambda(AB + D)| = \lambda^n |\lambda I - AB - D|$, and $|\lambda I - AB| = |\lambda I - AB - D|$ for arbitrary $m \times n$ matrices B.

On the other hand, suppose that $|\lambda I - AB| = |\lambda I - AB - D|$ for all $m \times n$ matrices B. The choice $B = 0$ gives $|\lambda I - D| = \lambda^n$, and hence the nilpotency of D. Then $|\lambda^2 I - \lambda(AB + D)| = \lambda^n |\lambda I - (AB + D)| = |\lambda I - D| \cdot |\lambda I - AB| = |\lambda^2 I - \lambda(AB + D) + DAB|$ for arbitrary $m \times n$ matrices B. Now if $DA = K = \|K_{i\beta}\|$ $(i = 1, \cdots, n; \beta = 1, \cdots, m)$ has a nonzero element K_{pq}, then the choice $B_0 = \|\delta_{\alpha q}\delta_{pj}K_{pq}^{-1}\|$ $(\alpha = 1, \cdots, m; j = 1, \cdots, n)$ gives $DAB_0 = G = \|G_{ij}\|$ $(i, j = 1, \cdots, n)$ with $G_{ij} = 0$ if $j \neq p$, $G_{pp} = 1$. If $AB_0 + D = H = \|H_{ij}\|$ $(i, j = 1, \cdots, n)$, then from the above, $|\lambda^2 I - \lambda H| = |\lambda^2 - \lambda H + G|$. This is impossible, however, since $|\lambda^2 I - \lambda H + G| = |\lambda^2 I - \lambda H| + |R(\lambda)|$, where $R_{ij}(\lambda) = \lambda^2 \delta_{ij} - \lambda H_{ij}$ if $j \neq p$, $R_{ip}(\lambda) = G_{ip}$, and consequently $|R(\lambda)|$ is a nonzero polynomial λ^{2n-2} + terms of lower degree. Hence $DA = 0$, and the statement of the first sentence of the theorem is proved.

The second sentence of the theorem follows from the first part, together with the remark that if $D = AC$, then $0 = DA = ACA$ implies $0 = DAC = D^2$ so that D is nilpotent.

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[2] W. V. Parker, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 115–116.

[3] Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 605–607.