INFINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF CERTAIN GROUPS

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Various considerations in the study of unitary representations of groups lead to the following question (cf. [1] and [2]):[1] Given a group G every one of whose elements has only a finite number of distinct conjugates; is every irreducible unitary representation of G finite-dimensional? Attempts have been made to prove that the answer is affirmative; in this note we give examples which settle the problem in the negative. These examples also show that the structure of the unitary representations of even countable discrete groups is more complicated than the so far existing theory (cf. [3], [4], and [5]) would lead one to expect. We shall come back to this point elsewhere.

Let F be any field with a finite number of elements $\alpha, \beta, \cdots$ and $\mathfrak{g}$ the group of all matrices

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$$

with α and β arbitrary elements of F, $\alpha \neq 0$. Let $\mathfrak{a}$ be the subgroup of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathfrak{b}$ the subgroup of matrices

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Let $x(b)$ be an arbitrary character of the commutative group $\mathfrak{b}$, that is, $b \rightarrow x(b)$ is a homomorphism of $\mathfrak{b}$ to the multiplicative group of complex numbers of absolute value one. We shall need the following lemma for which F need not yet be finite.

LEMMA 1. *If there exists an element a of $\mathfrak{a}$, $a \neq$ identity of $\mathfrak{a}$, such that $x(aba^{-1}) = x(b)$ for all elements b of $\mathfrak{b}$, then $x(b) = 1$ for all b.*

PROOF. The proof is obvious.

Let G be the group all infinite sequences $(g_1, g_2, \cdots)$ where each $g_n \in \mathfrak{g}$ and each sequence has only a finite number of elements not

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[1] Numbers in brackets refer to the references cited at the end of the paper.

equal to the identity element e of $\mathfrak{g}$; elements of $\mathbf{G}$ are to be multiplied componentwise. Then it is clear that each element of $\mathbf{G}$ has only a finite number of distinct conjugates.

Let $\mathbf{A}$ be the subgroup of those elements $A=(g_1, g_2, \cdots)$ of $\mathbf{G}$ for which each $g_n \in \mathbf{a}$ and similarly $\mathbf{B}$ be the subgroup of those elements B all of whose components are in $\mathbf{b}$. Since every element g of $\mathfrak{g}$ is a unique product $g=ba$ with $b\in\mathbf{b}$, $a\in\mathbf{a}$, therefore every element G of $\mathbf{G}$ can be written uniquely as a product $G=BA$ with $B\in\mathbf{B}$, $A\in\mathbf{A}$.

For every character $X(B)$ of the commutative group $\mathbf{B}$ we now define a unitary representation of the group $\mathbf{G}$ in a Hilbert space $\mathbf{H}$ as follows. Let $\mathbf{H}$ consist of all those complex-valued functions $f(A)$ defined on $\mathbf{A}$ for which $\|f\|^2=\sum_{A\in\mathbf{A}}|f(A)|^2<\infty$. With every element A_0 of $\mathbf{A}$ we associate the unitary operator $U(A_0): f(A)\to f(AA_0)$, and with every $B_0\in\mathbf{B}$ the operator $U(B_0)$: $f(A)\to X(AB_0A^{-1})f(A)$. If $G=BA$ let $U(G)=U(B)U(A)$, then it is immediately verified that $G\to U(G)$ is a unitary representation of $\mathbf{G}$, that is, a homomorphism of $\mathbf{G}$ into the multiplicative group of unitary operators of $\mathbf{H}$.

Let $X(B)=\prod_{n=1}^{\infty} x_n(b_n)$ where $x_n(b)$ is for each n a character of $\mathbf{b}$ and b_n the nth component of B.

LEMMA 2. *If $X(B)$ is such that $x_n(b)$ is not the constant character for any n, then the above representation is irreducible, that is, there are no proper closed linear subspaces of $\mathbf{H}$ invariant under all the $U(G)$.*

PROOF. Suppose there is a bounded linear operator C in $\mathbf{H}$ which commutes with all $U(G)$. Then $CU(A)=U(A)C$ for all A implies by Lemma 5.3.2 of [6] that there exists a complex-valued function $c(A)$ defined on $\mathbf{A}$ such that

$$(Cf)(A) = \sum_{A'\in\mathbf{A}} c(AA'^{-1})f(A').$$

Now $U(B)C=CU(B)$ for all $B\in\mathbf{B}$ implies

(1) $\quad X(ABA^{-1})c(A) = X(B)c(A)$ for all $A\in\mathbf{A}$ and $B\in\mathbf{B}$.

If there is an A such that $X(ABA^{-1})=X(B)$ for all B then $x_n(a_nba_n^{-1})=x_n(b)$ for all $b\in\mathbf{b}$, a_n being the nth component of this A. By hypothesis $x_n(b)$ is not the constant character of $\mathbf{b}$, therefore Lemma 1 implies that $a_n=e$ for all n, hence $A=E=$identity element of $\mathbf{A}$. Thus for any $A\neq E$ there exists an element B such that $X(ABA^{-1})\neq X(B)$. Therefore equation (1) implies $c(A)=0$ for all $A\neq E$. Hence C is a scalar multiple of the identity transformation of $\mathbf{H}$.

Thus we have constructed infinite-dimensional irreducible unitary representations of $\mathbf{G}$ in a completely elementary manner.

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A NOTE ON THE CHARACTERISTIC POLYNOMIALS OF CERTAIN MATRICES

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The purpose of this note is to establish the following theorem on matrices with elements in an arbitrary field $\mathfrak{F}$.[1] In particular, I denotes the $n\times n$ identity matrix; 0 is used for the zero matrix, square or rectangular, its dimensions being clear in each case from the context. The determinant of an $n\times n$ matrix $M(\lambda)$ with elements in $\mathfrak{F}(\lambda)$, the ring of polynomials in λ with coefficients in $\mathfrak{F}$, is denoted by $|M(\lambda)|$; in particular, if M is an $n\times n$ matrix with elements in $\mathfrak{F}$, then $|\lambda I-M|$ is the characteristic polynomial of M.

THEOREM. *If A is an $n\times m$ matrix, and D is an $n\times n$ matrix, then in order that $|\lambda I-AB|=|\lambda I-AB-D|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that D be nilpotent and $DA=0$. In particular, if C is an $m\times n$ matrix, then in order that $|\lambda I-AB|$ $=|\lambda I-A(B+C)|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that $ACA=0$.*

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[1] In the original manuscript of this paper attention was restricted to $\mathfrak{F}$ the field of complex numbers. The proof there given of the fact that the nilpotency of D and $DA=0$ imply $|\lambda I-AB|=|\lambda I-AB-D|$, for arbitrary B, was valid for arbitrary fields $\mathfrak{F}$. The proof here given of the necessity of $DA=0$ for $\mathfrak{F}$ an arbitrary field was suggested by the referee.