LIMITING VALUES OF SUBHARMONIC FUNCTIONS[1]

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1. **Introduction.** In 1928, Littlewood obtained the following result [1, p. 393].[2]

(1.1) THEOREM. *Let $u(r, \theta)$ be subharmonic in the unit circle, $r<1$, and satisfy*

$$\int_{-\pi}^{\pi} |u(r, \theta)| \, d\theta = O(1), \qquad r < 1. \tag{1.2}$$

Then there exists a finite-valued function $U(\theta)$, $0 \leqq \theta < 2\pi$, such that $\lim_{r\to 1} u(r, \theta) = U(\theta)$ *for almost all values of θ.*

In 1934, Priwaloff published a generalization of Littlewood's result, which turned out to be incorrect. When the domain under consideration is the unit circle, then Priwaloff's generalization consisted in allowing "non-tangential" approaches to the boundary of the disc.

In 1942, during the course of his lectures on *Subharmonic functions* at Brown University, the late Professor J. D. Tamarkin discovered an error in Priwaloff's proof. Then in a letter to Tamarkin in 1943, Professor A. Zygmund described a counter-example to Priwaloff's result.

In this paper, we present several generalizations of Littlewood's result (see §3) as well as several counter-examples to Priwaloff's result (see §4).

2. **Definitions and lemmas.** Points on the unit circle will be denoted by $e^{i\xi}$; the point (1, 0) will be denoted by B. The coordinates (x, y) and (r, θ) will be used interchangeably to denote the point P, and (ρ, ϕ) will be used to denote the point Q; here P and Q are points in the unit disc. The point inverse to P with respect to the unit circle will be denoted by $P'(1/r, \theta)$.

The letter A denotes a positive absolute constant, though not always the same from one occurance to another.

We shall use the notation $s=1-r$ and $\sigma=1-\rho$.

The symbol $L_\psi(\xi)$ will denote the (short) line segment approaching the point $e^{i\xi}$ from the interior of the unit circle and making an angle ψ

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

with the radius vector to the point $e^{i\xi}$. We shall consider ψ positive if the rotation from the radius vector to $L_\psi(\xi)$ is clockwise. For a fixed angle ψ, $|\psi| < \pi/2$, the family of lines $L_\psi(\xi)$, $0 \leqq \xi < 2\pi$, has for its envelope a circle of radius $\rho_\psi = |\sin \psi|$ inside and concentric with the unit circle. We shall let $S_\psi(\xi)$, $0 \leqq \psi < \pi/2$, denote the area in the ring $\rho_\psi \leqq \rho < 1$ between the lines $L_\psi(\xi)$ and $L_{-\psi}(\xi)$. We shall use $\tilde{S}_\psi(\xi)$ to denote the area in the ring $\rho_\psi \leqq \rho < 1$ which is outside the sector $S_\psi(\xi)$.

The phrase "P may approach the unit circumference at the point $e^{i\xi}$ along any non-tangential path" is equivalent to the statement "P may approach the point $e^{i\xi}$ along any path which remains within some sector $S_\psi(\xi)$, $\psi < \pi/2$." For brevity this type of approach shall be designated by $P \to_{S_\psi(\xi)} e^{i\xi}$. In general the nature of the path of approach shall be indicated as a subscript to the symbol $\to$.

A simple geometric argument yields the following result.

(2.1) Lemma. *Let $P(r, \theta)$ be a point on the segment $L_\alpha(0)$, $\alpha < \pi/2$, and let $Q(\rho, \phi)$ be a point on the segment $L_\alpha(\tau)$. Then there exists a constant k_1, $0 < k_1 < \infty$, such that $r > \rho_\alpha + (1 - \rho_\alpha)/2$ implies $PQ \geqq k_1\tau$, $0 \leqq \tau < 2\pi$.*

The Green's function for the unit disc is given by

$$g(P, Q) = \log \frac{rQP'}{QP} \cdot$$

Littlewood showed that

$$0 \leqq g(P, Q) \leqq A\sigma s/PQ^2 \tag{2.2}$$

holds [1, p. 394]. We shall require further estimates of $g(P, Q)$, similar to (2.2). To that end let α be fixed, $|\alpha| < \pi/2$; then we can locate the point $Q(\rho, \phi)$ by the coordinates (σ, τ) where $\sigma = 1 - \rho$, and $L_\alpha(\tau)$ is the segment passing through Q meeting the unit circumference at $e^{i\tau}$. We shall divide the ring $\rho > \rho_0$ into three sets R_1, R_2, and R_3 defined explicitly in terms of the coordinates σ and τ of the point Q as follows:

$$R_1: \quad |\sigma - s| \leqq s/2, \qquad |\tau| \leqq s, \qquad \rho > \rho_0,$$
$$R_2: \quad |\sigma - s| > s/2, \qquad |\tau| \leqq s, \qquad \rho > \rho_0,$$
$$R_3: \quad |\tau| > s, \qquad \rho > \rho_0.$$

In R_1 we shall prove

$$g(P, Q) \leqq \frac{\sigma}{s} \log \frac{As^2}{\tau^2}, \quad \text{for} \quad s < \frac{1}{2} \cdot \tag{2.3}$$

We know that

$$(2.4)\qquad g(P, Q) = \log \frac{r \cdot P'Q}{PQ} = \frac{1}{2} \log \frac{r^2 \cdot P'Q^2}{PQ^2}\,.$$

First note that $r<1$, and for Q in R_1 we have $PQ \leqq As$. Also $PP' = 1/r - r = (1-r)(1+r)/r \leqq 4s$ for $s<1/2$, and $P'Q \leqq PP' + PQ \leqq As$. At this point we use (2.1) to obtain the important fact that $PQ \geqq k_1\tau$ where $0<k_1<\infty$. Thus

$$g(P, Q) \leqq \frac{1}{2} \log \frac{As^2}{k_1^2\tau^2}$$

and (2.3) follows at once from the fact $|\sigma - s| \leqq s/2$ implies $1/2 \leqq \sigma/s \leqq 3/2$.

In R_2 we shall prove

$$(2.5)\qquad g(P, Q) \leqq \frac{A\sigma s}{s^2 + \tau^2}\,.$$

By (2.2) we have that

$$g(P, Q) \leqq \frac{A\sigma s}{PQ^2} = \frac{A\sigma s}{PQ^2/2 + PQ^2/2}\,.$$

But for Q in R_2, $|\sigma - s| > s/2$; and hence $PQ^2/2 \geqq s^2/8$. By (2.1), $PQ^2/2 \geqq k_1\tau^2/2$, and so we have

$$g(P, Q) \leqq \frac{A\sigma s}{s^2/8 + k_1^2\tau^2/2}$$

which implies (2.5).

In R_3 we have

$$(2.6)\qquad g(P, Q) \leqq \frac{A\sigma s}{\tau^2}$$

which follows at once from (2.2) and (2.1).

Finally, we shall need the following simple results on approximations [1, §3].

Suppose that $J^*(\tau)$ is non-negative and monotone increasing to a finite $J^*(\pi)$ in $0 \leqq \tau \leqq \pi$ and satisfies

$$\limsup_{\tau \to +0} \frac{J^*(\tau)}{\tau} \leqq \eta(\rho_0);$$

then

$$\limsup_{s\to+0} s\int_{s-0}^{\pi}\frac{dJ^*(\tau)}{\tau^2} \leqq A\eta(\rho_0), \tag{2.7}$$

$$\limsup_{s\to+0} s\int_{0}^{s}\frac{dJ^*(\tau)}{s^2+\tau^2} \leqq A\eta(\rho_0), \tag{2.8}$$

and

$$\limsup_{s\to+0} \frac{1}{s}\int_{0}^{s}\log\frac{As}{\tau^2}\, dJ^*(\tau) \leqq A\eta(\rho_0). \tag{2.9}$$

3. **Generalizations of Littlewood's theorem.** If $u(P)$ is subharmonic in the unit circle, and if $u(P)$ satisfies (1.2), then $u(P)$ can be represented as

$$u(P) = h(P) - \iint g(P, Q)dF(Q) \tag{3.1}$$

where $h(P)$ is harmonic in the circle $r<1$, and satisfies

$$\int_{-\pi}^{\pi} | h(r, \theta) | d\theta = O(1), \qquad r < 1, \tag{3.2}$$

and $F(Q)$, the generalized mass distribution function associated with $u(P)$, is a non-negative, additive function of sets satisfying

$$\iint (1 - \rho)dF(Q) < + \infty. \tag{3.3}$$

The integrals in (3.1) and (3.3) are Stieltjes-Radon integrals extended over the unit circle, $\rho<1$ [1, Lemma 3].

Since $h(P)$ satisfies condition (3.2), the limit of $h(P)$ exists for almost all values of ξ as P approaches $e^{i\xi}$ along any non-tangential path (see [4, Chapter IV, Theorem 1 and Chapter II, Theorem 7]). Having thus disposed of the harmonic part of (3.1), we have reduced our problem to the consideration of the behavior of the potential of positive mass distribution

$$w(P) = \iint g(P, Q)dF(Q) \tag{3.4}$$

as P approaches the unit circumference.

Littlewood showed that $w(P)$ had radial limit zero at almost all points of the unit circumference [1, Lemma 4]. Priwaloff tried unsuccessfully to show that $w(P)$ had a non-tangential limit zero at almost all points of the unit circumference. Since Littlewood showed that (1.2) and the combination of (3.1) and (3.3) are equivalent restrictions on the subharmonic function $u(P)$, the results in this paper are derived from the consideration of potential functions of the type (3.4) whose associated mass distribution functions $F(Q)$ satisfy (3.3). Such potential functions of positive mass distribution are superharmonic (see [3, 4.34] and [3, 1.1]). Since the negative of a superharmonic function is subharmonic, we may apply the theory of subharmonic functions, with obvious modifications, to our potential functions (3.4) whenever necessary.

We shall use the method of Littlewood to generalize (1.1) from the case of radial approach to the case in which the approach to the boundary of the unit circle is along the rotations of a fixed line segment or a fixed curve.

(3.5) THEOREM. *Let $u(P)$ be subharmonic in the unit circle, $r<1$' and satisfy* (1.2). *Let $L_\alpha(\xi)$ be a straight line segment which approaches the unit circumference at the point $e^{i\xi}$ making a fixed angle $\alpha<\pi/2$ with the radius vector to the point $e^{i\xi}$. Then there exists a finite valued function $U(\xi)$, $-\pi\leqq\xi<\pi$, such that, for almost all values of ξ,*

$$\lim_{P\to_{L_\alpha(\xi)} e^{i\xi}} u(P) = U(\xi).$$

PROOF. In view of the preceding remarks it will be sufficient to prove with the aid of (3.3) that

$$\lim_{P\to_{L_\alpha(\xi)} e^{i\xi}} \iint g(P, Q)dF(Q) = 0 \tag{3.6}$$

for almost all values of ξ, $0\leqq\xi<2\pi$.

Let

$$\epsilon(\rho_0) = \iint_{\rho>\rho_0} (1-\rho)dF(Q) \tag{3.7}$$

and let $\eta(\rho_0)=(\epsilon(\rho_0))^{1/2}$. By (3.3) we have that $\lim_{\rho_0\to 1}\epsilon(\rho_0)=0$ and thus that $\lim_{\rho_0\to 1}\eta(\rho_0)=0$.

Next we shall define

$$\Phi(\xi) = \iint \sigma dF(Q) \tag{3.8}$$

over the part of the ring $\rho>\rho_0$ between and including the segments $L_\alpha(0)$ and $L_\alpha(\xi)$. From the properties of σ, $F(Q)$, and $L_\alpha(\xi)$ it follows that for ρ_0 near to one, $\Phi(\xi)$ is a monotone increasing function of ξ. In view of (3.7), $\Phi(\xi)$ remains less than $2\epsilon(\rho_0)$, $0\leqq\xi\leqq 2\pi$, and $\Phi'(\xi)$ exists for almost all values of ξ.

Let $E(\rho_0)$ be the set of values of ξ, $0 \leqq \xi < 2\pi$, for which

$$0 \leqq \Phi'(\xi) \leqq 2\eta(\rho_0) \tag{3.9}$$

and let C.E. be the complement of E. From the relations

$$2\eta^2(\rho_0) = 2\epsilon(\rho_0) \geqq \Phi(2\pi) \geqq \int_0^{2\pi} \Phi'(\xi)d\xi$$
$$\geqq \int_{\mathrm{C}.E.} \Phi'(\xi)d\xi \geqq m(\mathrm{C}.E.)2\eta(\rho_0)$$

we may conclude that the measure of C.E. does not exceed $\eta(\rho_0)$.

At this set $E(\rho_0)$ whose complement C.E. has measure not exceeding $\eta(\rho_0)$ we shall show

$$\limsup_{P \to L_\alpha(\xi)e^{i\xi}} \iint_{\rho > \rho_0} g(P, Q)dF(Q) \leqq A\eta(\rho_0). \tag{3.10}$$

Since

$$\lim_{P \to e^{i\xi}} \iint_{\rho \leqq \rho_0} g(P, Q)dF(Q) = 0$$

uniformly in ξ, the left side of (3.10) is unaltered when the symbol $\rho > \rho_0$ is suppressed and the integration taken over the whole of $\rho < 1$. But then the left side of (3.10) does not depend on ρ_0; hence since $\eta(\rho_0)$ is arbitrarily small, (3.10) once proved implies (3.6) and establishes the theorem.

Let now

$$J(\xi, t) = \iint \sigma dF(Q) \tag{3.11}$$

over the portion of the ring $\rho > \rho_0$ between and including the segments $L_\alpha(\xi)$ and $L_\alpha(\xi + t)$. If now ξ is a point of $E(\rho_0)$ we have by the characteristic property of E

$$\limsup_{t \to 0} \frac{J(\xi, t)}{|t|} \leqq 2\eta(\rho_0). \tag{3.12}$$

We may assume without loss of generality that B, the point for which $\xi = 0$, is a point of the set $E(\rho_0)$. We then write $J(t)$ for $J(0, t)$ and define

$$J^*(t) = J(t) + |J(-t)|. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\limsup_{t\to+0} \frac{J^*(t)}{t} \leqq 4\eta(\rho_0). \tag{3.14}$$

With the aid of (3.14) we shall prove (3.10) for the case $\xi=0$; namely

$$\limsup_{P\to L_\alpha(0)B} \iint_{\rho>\rho_0} g(P, Q)dF(Q) \leqq A\eta(\rho_0), \tag{3.15}$$

thus establishing the theorem. In order to demonstrate (3.15) we divide the ring $\rho>\rho_0$ into the three regions R_1, R_2, and R_3 described in §2 and establish the theorem by showing that

$$\limsup_{P\to L_\alpha(0)B} \iint_{R_i} g(P, Q)dF(Q) \leqq A\eta(\rho_0), \qquad i = 1, 2, 3. \tag{3.16}$$

We shall need Fubini's theorem and the inequalities quoted in §2.

Let

$$\Lambda_1(P) = \iint_{R_1} g(P, Q)dF(Q). \tag{3.17}$$

By (2.3) we have

$$\Lambda_1(P) \leqq \iint_{R_1} \frac{\sigma}{s} \log \frac{As}{\tau^2} dF(\sigma, \tau);$$

and by (3.11), (3.13), and Fubini's theorem we may write

$$\Lambda_1(P) \leqq \int_0^s \frac{1}{s} \log \frac{As^2}{\tau^2} dJ^*(\tau).$$

But then by (2.9) and (3.14) we have that

$$\limsup_{s\to+0} \Lambda_1(P) \leqq \limsup_{s\to+0} \frac{1}{s} \int_0^s \log \frac{As^2}{\tau^2} dJ^*(\tau) \leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have

$$\limsup_{P\to L_\alpha(0)B} \Lambda_1(P) \leqq A\eta(\rho_0). \tag{3.18}$$

Let

$$\Lambda_2(P) = \iint_{R_2} g(P, Q)dF(Q). \tag{3.19}$$

By (2.5) we have

$$\Lambda_2(P) \leqq \iint_{R_2} \frac{A\sigma s}{s^2+\tau^2}\, dF(\sigma, \tau);$$

and by (3.11),(3.13), and Fubini's theorem we may write

$$\Lambda_2(P) \leqq \int_0^s \frac{As}{s^2+\tau^2}\, dJ^*(\tau).$$

But then by (2.8) and (3.14) we have that

$$\limsup_{s\to+0} \Lambda_2(P) \leqq \limsup_{s\to+0} As \int_0^s \frac{dJ^*(\tau)}{s^2+\tau^2} \leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have

$$\limsup_{P\to_{L_\alpha(0)}B} \Lambda_2(P) \leqq A\eta(\rho_0). \tag{3.20}$$

Let

$$\Lambda_3(P) = \iint_{R_3} g(P, Q)dF(Q). \tag{3.21}$$

By (2.6) we have

$$\Lambda_3(P) \leqq \iint_{R_3} \frac{A\sigma s}{\tau^2}\, dF(\sigma, \tau);$$

and again using (3.11), (3.13), and Fubini's theorem we may write

$$\Lambda_3(P) \leqq \int_s^\pi \frac{As}{\tau^2}\, dJ^*(\tau).$$

Then by (2.7) and (3.14) we have that

$$\limsup_{s\to+0} \Lambda_3(P) \leqq \limsup_{s\to+0} s \int_{s-0}^{\pi} \frac{AdJ^*(\tau)}{\tau^2} \leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have finally

$$\limsup_{P\to_{L_\alpha(0)}B} \Lambda_3(P) \leqq A\eta(\rho_0). \tag{3.22}$$

Combining the results (3.18), (3.20), and (3.22) gives us (3.16), thus proving the theorem.

(3.23) COROLLARY. *Let $C_\alpha(0)$ be a curve which approaches the unit circumference at B, and is tangent to $L_\alpha(0)$ at B. Let $C_\alpha(\xi)$ be the curve obtained by rotating the curve $C_\alpha(0)$ through an angle ξ about the origin. Then if $u(P)$ is the subharmonic function of Theorem* (3.5), *we have*

$$\limsup_{P\to C_\alpha(\xi)e^{i\xi}} u(P) = U(\xi)$$

for almost all values of ξ, $0\leqq\xi\leqq 2\pi$.

PROOF. The proof goes through exactly as the proof of the theorem. The crucial part of the argument is the demonstration of Lemma (2.1) for P on $C_\alpha(0)$ and Q on $C_\alpha(\tau)$, which is needed to obtain the essential inequalities (2.3), (2.5), and (2.6). Lemma (2.1) may be proved for such curves $C_\alpha(0)$ and $C_\alpha(\tau)$ if they are tangent to the lines $L_\alpha(0)$ and $L_\alpha(\tau)$ respectively. If, however, the curve $C_\alpha(0)$ oscillated rapidly as it approached the unit circumference, (2.1) would obviously be false, and the proof could not be carried out. The counter-examples in the following section need such rapidly oscillating curves.

(3.24) COROLLARY. *For almost all values of* ξ, $0\leqq\xi\leqq 2\pi$,

$$\lim_{P\to L_\alpha(\xi)e^{i\xi}} u(P) = U(\xi)$$

for almost all values of α, $\alpha=\alpha(\xi)$, $0\leqq\alpha<\pi/2$.

PROOF. Define the function $X(\alpha, \xi)$ as

$$X(\alpha,\xi) = \begin{cases} 1 & \text{if} \quad \lim_{P\to L_\alpha(\xi)e^{i\xi}} u(P) \quad \text{fails to exist,} \\ 0 & \text{if} \quad \lim_{P\to L_\alpha(\xi)e^{i\xi}} u(P) \quad \text{does exist.} \end{cases}$$

It is not difficult to show that $X(\alpha, \xi)$ is a measurable function of the variables α and ξ. It follows from Theorem (3.5) that for fixed α, $X(\alpha, \xi)=0$ for almost ξ. It follows immediately from Fubini's theorem that for almost any fixed value of ξ, $X(\alpha, \xi)=0$ for almost all α. This proves the corollary.

4. **Counter-examples.** In view of the remarks made at the beginning of §3, a counter-example to Priwaloff's theorem will be any potential function (3.4) whose associated mass distribution function, $F(Q)$, satisfies (3.3) but which fails to have a non-tangential limit at a set of points on the unit circumference of positive measure.

The following counter-example is essentially the one described by Zygmund in a letter to Tamarkin in 1943.

(4.1) EXAMPLE. Consider the potential function

$$w_1(P) = \iint g(P, Q)dF_1(Q). \tag{4.2}$$

Suppose the mass distribution function $F_1(Q)$ is totally discreet; that is, the mass is concentrated at a denumerable set E of points converging to the unit circumference. For example, let

$$E = E_1 + E_2 + E_3 + \cdots + E_n + \cdots$$

where E_n consists of n points equally distributed on the circumference of the circle of radius $\rho_n = 1 - 1/n$ with one of the n points lying on the positive x-axis. At each point of E_n place a mass of $1/2^n$. Then the total mass contained in the set E_n is $n/2^n$ and we have

$$\iint (1 - \rho) dF_1(Q) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{n}{2^n} = 1.$$

Thus (3.3) is satisfied for the mass distribution function $F_1(Q)$.

For ψ such that $\tan^{-1} 2\pi < \psi < \pi/2$, it is not difficult to show that for sufficiently large n at least one point from each of the sets E_n, E_{n+1}, E_{n+2}, $\cdots$ lies within the sector $S_\psi(\xi)$ for any value of ξ, $0 \leqq \xi < 2\pi$.

Choose a path approaching $e^{i\xi}$ which passes through points of the sets E_n, E_{n+1}, $\cdots$ and remains within the sector $S_\psi(\xi)$. At each of these points $w_1(P) = +\infty$. Thus we have that

$$\limsup_{P \to S_\psi(\xi) e^{i\xi}} w_1(P) = +\infty \tag{4.3}$$

for all values of ξ, $0 \leqq \xi < 2\pi$.

Since $w_1(P)$ satisfies (3.3) yet fails to have a finite non-tangential limit at every point of the unit circumference, we have a counter-example to Priwaloff's theorem.

It is interesting to note that

$$\liminf_{P \to S_\psi(\xi) e^{i\xi}} w_1(P) = 0 \tag{4.4}$$

for almost all values of ξ, $0 \leqq \xi < 2\pi$. For Littlewood showed that the radial limit of a potential of the form (4.2) is zero at almost all points of the unit circumference [1, Lemma 4].

If, instead of considering sectorial approaches to the unit circumference, we restrict the path of approach to lie between any two distinct curves tangent to each other at the point $e^{i\xi}$, the limit of the potential function (3.4) may still fail to exist at all points of the unit circumference. It is easy to construct such a function by simply placing a denumerable infinity of masses on each of the circles of radius ρ_n and at the same time preserve (3.3).

The final counter-example is that of a bounded superharmonic function whose associated mass distribution function $F(Q)$ is absolutely continuous. That means, for any Borel measurable subset e of the unit circle

$$F(e) = \iint_e f(Q)dQ \tag{4.5}$$

where $f(Q)$ is a non-negative Lebesgue integrable function called the density function associated with the given superharmonic function (see [3, 4.33]).

(4.6) EXAMPLE. Let the circles $C_{n_i}(P_{n_i}, R_n)$, $i=1, 2, \cdots, n$, be the n circles whose centers are at the points of the set E_n in (4.1) and whose radii are $R_n = 1/2^n$. Consider the following function defined in the unit circle.

$$w_2(P) = \begin{cases} 0 \text{ for } P \text{ outside the circles } C_{n_i}; \\ \qquad\qquad i = 1, 2, \cdots, n;\ n = 1, 2, \cdots, \\ 1 - (R/R_n)^{1/2^n} \quad \text{for } P \text{ within one of the circles } C_{n_i} \\ \qquad\qquad \text{where } P \text{ has polar coordinates } (R, \theta) \\ \qquad\qquad \text{with respect to the point } P_{n_i}. \end{cases} \tag{4.7}$$

Outside the circles C_{n_i}, $w_2(P)$ is harmonic since $\Delta w_2(P) = 0$ trivially. For P within any one of the circles C_{n_i} the Laplacian is

$$\begin{aligned} \Delta w_2(P) = \Delta w_2(R, \theta) &= \frac{\partial^2 w_2}{\partial R^2} - \frac{1}{R}\frac{\partial w_2}{\partial \theta} - \frac{\partial^2 w_2}{\partial \theta^2} \\ &= \frac{-1}{2^{2n} R_n^{1/2^n}} [R^{(1/2^n)-2}]. \end{aligned} \tag{4.8}$$

Since the Laplacian is non-positive in the unit circle, our function (4.7) is superharmonic in the unit circle (see [3, 4.1]).

The superharmonic function (4.7) takes on the value 1 at the points P_{n_i} of E and the value 0 in the space between the non-overlapping circles C_{n_i}. Hence if we choose a path approaching $e^{i\xi}$ through the points of E as in (4.1), we find that $w_2(P)$ will oscillate between the values 0 and 1. Thus at every point of the unit circumference (4.7) fails to have a non-tangential limit.

According to Rado [3, 6.2] the density function associated with a superharmonic function is given by the formula

$$f(Q) = (-1/2\pi)\,\Delta w_2(Q). \tag{4.9}$$

Thus for the superharmonic function (4.10) the density function is given by

$$(4.10)\qquad f(Q) = \begin{cases} 0 \text{ for } Q \text{ outside the circles } C_{n_i}, \\ \dfrac{1}{2\pi}\cdot\dfrac{1}{2^{2n}}\cdot\dfrac{1}{R_n^{1/2^n}}\cdot R^{(1/2^n)-2} \\ \qquad \text{where } Q \text{ is within } C_{n_i} \text{ and has polar coordinates} \\ \qquad\qquad (R, \theta) \text{ with respect to } P_{n_i}. \end{cases}$$

Since $f(Q)$ in (4.10) is summable the mass distribution function, $F(Q)$ is absolutely continuous.

In order to show that (4.7) is our desired counter-example, it remains to show that (3.3) is satisfied by $F(Q)$. But in view of (4.5) we may write (3.3) as

$$(4.11)\qquad \iint (1-\rho)f(Q)dQ < +\infty.$$

Since $f(Q)=0$ outside the circles C_{n_i}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, we may write the integral (4.11) as

$$(4.12)\qquad \iint (1-\rho)f(Q)dQ = \sum_{n=1}^{\infty} n \iint_{C_{n_i}} (1-\rho)f(Q)dQ.$$

For Q in C_{n_1} we have that $1-\rho \leqq 1/n+1/2^n$; hence it follows from (4.10) and (4.12) that

$$\iint (1-\rho)f(Q)dQ$$

$$\leqq \sum_{n=1}^{\infty}\left\{1+\frac{n}{2^n}\right\}\left\{\int_0^{2\pi}\int_0^{R_n}\frac{R^{(1/2^n)-2}}{2\pi 2^n R_n^{1/2^n}}\,RdRd\theta\right\}$$

$$\leqq \sum_{n=1}^{\infty}\left\{1+\frac{n}{2^n}\right\}\left\{\frac{1}{2^n}\right\} < +\infty.$$

Thus (4.11) is satisfied and (4.7) is the desired counter-example.

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