GENERALIZATION OF A THEOREM OF OSGOOD TO THE CASE OF CONTINUOUS APPROXIMATION[1]

A. M. OSTROWSKI

In the historical development of the theory of convergence the following theorem of Osgood[2] has played a very important role: *Suppose that the functions* $f_\nu(x)$ *are continuous in the interval* (a, b) *and with* $\nu \to \infty$ *converge to a limit* $f(x)$ *which is also continuous in* (a, b); *then to any* $\epsilon > 0$ *there exists a subinterval* (a', b') *of* (a, b) *and an integer* n_0 *such that*

$$|f_\nu(x) - f(x)| < \epsilon$$

is valid in (a', b') *for all* $\nu > n_0$.

Although since 1897 a great number of results in this direction have been found, Osgood's theorem has not been as yet completely superseded and provides sometimes a useful means in dealing with convergent sequences. In what follows I prove an analogous theorem for the case of *continuous* convergence.

THEOREM. *Let* $f(t, x)$ *be a continuous function of the point* (t, x) *for* $a < x < b$ *and* $t \geqq T$ *and suppose that we have*

$$\lim_{t \to \infty} f(t, x) = f(x)$$

where $f(x)$ *is also continuous in* (a, b); *then for any* $\epsilon > 0$ *there exists a subinterval* J *of* (a, b) *and a number* T_0 *such that we have for* $x \prec J$, $t \geqq T_0$:

$$|f(t, x) - f(x)| < \epsilon \qquad (t \geqq T_0,\ x \prec J). \tag{1}$$

The proof of our theorem follows easily from the following

LEMMA. *If* $f(t, x)$ *is a continuous function of the point* (t, x) *for* $t \geqq T$ *and* $a'' < x < b''$, *where* T *is a positive integer, then a sequence of positive numbers* t_ν *tending to* ∞ *can be found such that to each* $t \geqq T$ *there corresponds a* t_ν *tending to* ∞ *with* t, *for which we have*

$$|f(t, x) - f(t_\nu, x)| \leqq 1/t.$$

PROOF OF THE LEMMA. Since $f(t, x)$ is *uniformly* continuous in any

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[2] W. F. Osgood, *Non-uniform convergence and the integration of series term by term*, Amer. J. Math. vol. 19 (1897) p. 161, the "Fundamental Theorem."

rectangle $a'' \leqq x \leqq b''$, $n \leqq t \leqq n+1$, for $n \geqq T$, there corresponds to any integer $n \geqq T$ a positive integer N_n such that we have

$$(2) \qquad |f(t, x) - f(t_0, x)| \leqq \frac{1}{n+1}$$

if t and t_0 lie between n and $n+1$ and $|t-t_0| \leqq 1/N_n$, $a'' \leqq x \leqq b''$. We subdivide all intervals $\langle n, n+1\rangle$ with $n \geqq T$ in N_n equal parts and denote the division points from $t_1 = T$ on by $t_1, t_2, \cdots$. Then for each $t \geqq T$ there exists a t_ν from the interval between $[t]$ and $[t]+1$ such that $t_\nu \leqq t < t_{\nu+1}$ and therefore by (2) for all x in $\langle a'', b''\rangle$

$$|f(t, x) - f(t_\nu, x)| \leqq \frac{1}{[t_\nu] + 1} \leqq \frac{1}{t} \cdot$$

Our lemma is proved.

Proof of the theorem. Without loss of generality T in the theorem can be supposed a positive integer. Apply the lemma to a closed interval $\langle a'', b''\rangle$ contained in (a, b) and form for this closed interval and the function $f(t, x)$ the sequence t_ν of the lemma. In applying Osgood's theorem to the sequence of functions $f(t_\nu, x)$, to a given $\epsilon > 0$ we find a subinterval J of $\langle a'', b''\rangle$ and an integer n_0 such that

$$(3) \qquad |f(t_\nu, x) - f(x)| \leqq \epsilon/2 \qquad (\nu \geqq n_0,\ x \prec J).$$

Take then T_0 such that $1/T_0 < \epsilon/2$ and such that for each $t \geqq T_0$, a t_ν corresponding to t by the lemma has an index $\geqq n_0$. Then we have

$$(4) \qquad |f(t, x) - f(t_\nu, x)| \leqq 1/t \leqq 1/T_0 < \epsilon/2$$

for all x from $\langle a'', b''\rangle$ and the assertion (1) follows from the inequalities (3) and (4). Our theorem is proved.

Both the theorem and the lemma proved remain valid if t goes to ∞ through a set M which has a closed intersection with any finite interval. The necessary modifications in the proof are obvious. The case of a finite limiting point for t is reduced to the case treated above by a linear transformation.

National Bureau of Standards and
University of Basle